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Research Article

Semilocal Convergence Theorem for the Inverse-Free Jarratt Method under New Hölder Conditions

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Under the new Hölder conditions, we consider the convergence analysis of the inverse-free Jarratt method in Banach space which is used to solve the nonlinear operator equation. We establish a new semilocal convergence theorem for the inverse-free Jarratt method and present an error estimate. Finally, three examples are provided to show the application of the theorem.

1. Introduction

We consider the following boundary value problem:

$$\begin{gathered} x'' = -\lambda G(x), \\ x(a) = x_a, \qquad x(b) = x_b. \end{gathered} \tag{1}$$

Those are equivalent to the following nonlinear integral equation (see [1, 2]):

$$x(s) = \alpha(s) + \lambda \int_a^b k(s,t)\, G(x(t))\, dt, \tag{2}$$

where $\alpha(s) = (1/(b-a))(x_a(b-s) + x_b(s-a))$ and $G : \Omega \subset C[a,b] \to C[a,b]$ is a twice Fréchet-differentiable operator. $C[a,b]$ is the set of all continuous functions in $[a,b]$; $k(s,t)$ is the Green function:

$$k(s,t) = \begin{cases} \dfrac{(b-s)(t-a)}{b-a}, & t \le s, \\ \dfrac{(s-a)(b-t)}{b-a}, & s \le t. \end{cases} \tag{3}$$

Instead of (2), we can try to solve a nonlinear operator equation $F(s) = 0$, where

$$\begin{gathered} F : \Omega \subset C[a,b] \longrightarrow C[a,b], \\ F(x)(s) = x(s) - \alpha(s) - \lambda \int_a^b k(s,t)\, G(x(t))\, dt. \end{gathered} \tag{4}$$

Solving the nonlinear operator equation is an important issue in the engineering and technology field as these kinds of problems appear in many real-world applications. Economics [3], chemistry [4], and physics [5–8] are some of the examples of the scientific and engineering technology areas applied to solve these type of equations. In this study, we consider to establish a new semilocal convergence theorem of the Jarratt method in Banach space which is used to solve the nonlinear operator equation

$$F(x) = 0, \tag{5}$$

where F is defined on an open convex Ω of a Banach space X with values in a Banach space Y.

There are a lot of methods of finding a solution of equation $F(x) = 0$. Particularly iterative methods are often used to solve this problem (see [1, 2, 9, 10]). If we use the famous Newton method, we can proceed as

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n), \quad (n \ge 0)\ (x_0 \in \Omega). \tag{6}$$

Under a reasonable hypothesis, Newton's method is the second-order convergence.

To improve the convergence order, many modified methods have been presented. The famous Halley's method and the supper-Halley method are the third-order convergence.

References [11–22] give the convergence analysis for these methods. Now, we consider the following Jarratt method (see [23–25]):

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ H(x_n, y_n) &= \frac{3}{2} F'(x_n)^{-1} \left[F'\left(x_n + \frac{2}{3}(y_n - x_n) \right) - F'(x_n) \right], \\ x_{n+1} &= y_n - \frac{1}{2} H(x_n, y_n) \left[I - H(x_n, y_n) \right] (y_n - x_n). \end{aligned} \tag{7}$$

In this paper, we discuss the convergence of (7) for solving nonlinear operator equations in Banach spaces and establish a new semilocal convergence theorem under the following condition (see [20, 21]):

$$\left\| F'''(x) - F'''(y) \right\| \leq \omega(\|x - y\|), \tag{8}$$

where $\omega : [0, +\infty) \rightarrow R$ is a nondecreasing continuous function. Finally, the corresponding error estimate is also given.

2. Main Results

In the section, we establish a new semilocal convergence theorem and present the error estimate. Denote $B(x, r) = \{y \in X \mid \|y - x\| < r\}$ and $\overline{B(x, r)} = \{y \in X \mid \|y - x\| \leq r\}$. Suppose that X and Y are the Banach spaces, Ω is an open convex of the Banach space X, and $F : \Omega \subset X \rightarrow Y$ has continuous Fréchet derivative of the third-order. $F'(x_0)^{-1}$ exists, for some $x_0 \in \Omega$, and F satisfies

$$\begin{aligned} &\text{(A1)} \ \|y_0 - x_0\| = \left\| F'(x_0)^{-1} F(x_0) \right\| \leq \eta; \\ &\text{(A2)} \ \left\| F'(x_0)^{-1} F''(x) \right\| \leq M, \quad x \in \Omega, \ M \geq 0; \\ &\text{(A3)} \ \left\| F'(x_0)^{-1} F'''(x) \right\| \leq N, \quad x \in \Omega, \ N \geq 0; \\ &\text{(A4)} \ \left\| F'(x_0)^{-1} \left[F'''(x) - F'''(y) \right] \right\| \leq \omega(\|x - y\|), \\ &\qquad\qquad x, y \in \Omega. \end{aligned} \tag{9}$$

(A5) $\omega(z)$ is a nondecreasing continuous real function for $z > 0$ such that $\omega(0) \geq 0$, and there exists a positive real number $p \in (0, 1]$ such that $\omega(tz) \leq t^p \omega(z)$ for $t \in [0, 1]$ and $z \in [0, +\infty)$.

(A6) Denote $A = \int_0^1 \int_0^1 t(1 - t)(st)^p \, ds \, dt = (1/(p + 1)(p + 2)(p + 3))$, $B = (1/3) \int_0^1 \int_0^1 (2st/3)^p t \, ds \, dt = (2^p / 3^{p+1}(p + 1)(p + 2))$. Let $a_0 = M\eta$, $b_0 = N\eta^2$, $c_0 = \eta^2 \omega(\eta)$, $a_{n+1} = a_n f^2(a_n) g(a_n, b_n, c_n)$, $b_{n+1} = b_n f^3(a_n) \, g^2(a_n, b_n, c_n)$, $c_{n+1} = f^{3+p}(a_n) g^{2+p}(a_n, b_n, c_n)$, where

$$\begin{aligned} f(x) &= \frac{2}{2 - 2x - x^2 - x^3}, \\ g(x, y, z) &= \frac{5x^3 + 2x^4 + x^5}{8} + \frac{xy}{12} + (A + B) z. \end{aligned} \tag{10}$$

First, we get some lemmas.

Lemma 1. *Suppose that $f(x)$, $g(x, y, z)$ are given by (10). Then*

$\forall x \in (0, 1/2)$, $f(x)$ is increasing and $f(x) > 1$;

$\forall x \in (0, 1/2)$, $y > 0$, $g(x, y, z)$ is increasing;

$\forall \gamma \in (0, 1)$, $x \in (0, 1/2)$, $p > 0$, $f(\gamma x) < f(x)$ and $g(\gamma x, \gamma^2 y, \gamma^{2+p} z) < \gamma^{2+p} g(x, y, z)$.

Lemma 2. *Suppose that $f(x)$, $g(x, y, z)$ are given by (10). If*

$$a_0 \in \left(0, \frac{1}{2} \right), \quad f^2(a_0) \, g(a_0, b_0, c_0) < 1, \tag{11}$$

then

(i) *the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are nonnegative and decreasing;*

(ii) $(1 + (a_n/2)(1 + a_n)) a_n < 1$, $\forall n \geq 0$.

Proof. (i) When $n = 1$,

$$\begin{aligned} 0 \leq a_1 &= a_0 f^2(a_0) \, g(a_0, b_0, c_0) \leq a_0, \\ 0 \leq b_1 &= b_0 f^3(a_0) \, g^2(a_0, b_0, c_0) \leq b_0, \\ 0 \leq c_1 &= c_0 f^{3+p}(a_0) \, g^{2+p}(a_0, b_0, c_0) \leq c_0. \end{aligned} \tag{12}$$

Suppose $a_j \leq a_{j-1}, b_j \leq b_{j-1}$ for $j = 1, 2, \ldots, n$. By Lemma 1, f and g are increasing; then

$$\begin{aligned} a_{n+1} &= a_n f^2(a_n) \, g(a_n, b_n, c_n) \leq a_n f^2(a_0) \, g(a_0, b_0, c_0) \leq a_n, \\ b_{n+1} &= b_n f^3(a_n) \, g^2(a_n, b_n, c_n) \leq b_n f^3(a_0) \, g^2(a_0, b_0, c_0) \leq b_n, \\ c_{n+1} &= c_n f^{3+p}(a_n) \, g^{2+p}(a_n, b_n, c_n) \\ &\leq c_n \left[f^2(a_0) \, g(a_0, b_0, c_0) \right]^{2+p} \leq c_n. \end{aligned} \tag{13}$$

(ii) By (i), $\{a_n\}$ is decreasing and $a_0 \in (0, 1/2)$. So, for all $n \geq 0$,

$$\left(1 + \frac{a_n}{2}(1 + a_n) \right) a_n \leq \left(1 + \frac{a_0}{2}(1 + a_0) \right) a_0 < 1. \tag{14}$$

This completes the proof of Lemma 2. □

Lemma 3. *Suppose that the conditions of Lemma 2 hold. Denote $\gamma = a_1 / a_0 = f^2(a_0) g(a_0, b_0, c_0) < 1$. Then*

(i) $a_n \leq \gamma^{(3+p)^{n-1}} a_{n-1} \leq \gamma^{(((3+p)^n - 1)/(2+p))} a_0$, $b_n \leq (\gamma^{(3+p)^{n-1}})^2 b_{n-1} \leq (\gamma^{(((3+p)^n - 1)/(2+p))})^2 b_0$, $c_n \leq (\gamma^{(3+p)^{n-1}})^{2+p} c_{n-1} \leq \gamma^{(3+p)^n - 1} c_0$ $\forall n \geq 1$;

(ii) $f(a_n) g(a_n, b_n, c_n) \leq \gamma^{(3+p)^n - 1} f(a_0) g(a_0, b_0, c_0) = (\gamma^{(3+p)^n} / f(a_0))$, $\forall n \geq 1$.

Proof. First, by induction, we prove that (i) holds. Because $a_1 = \gamma a_0$ and $f(a_0) > 1$, we have

$$\begin{aligned} b_1 &= b_0 f^3(a_0) \, g^2(a_0, b_0, c_0) \leq \gamma^2 b_0, \\ c_1 &= c_0 f^{3+p}(a_0) \, g^{2+p}(a_0, b_0, c_0) \leq \gamma^{2+p} c_0. \end{aligned} \tag{15}$$

Suppose that (i) holds for $n \geq 1$. Then we get

$$\begin{aligned}
a_{n+1} &= a_n f^2(a_n) g(a_n, b_n, c_n) \\
&\leq \gamma^{(3+p)^{n-1}} a_{n-1} f^2\left(\gamma^{(3+p)^{n-1}} a_{n-1}\right) \\
&\quad \times g\left(\gamma^{(3+p)^{n-1}} a_{n-1}, \left(\gamma^{(3+p)^{n-1}}\right)^2 b_{n-1}, \left(\gamma^{(3+p)^{n-1}}\right)^{2+p} c_{n-1}\right) \\
&\leq \gamma^{(3+p)^{n-1}} a_{n-1} f^2(a_{n-1}) \left(\gamma^{(3+p)^{n-1}}\right)^{2+p} g(a_{n-1}, b_{n-1}, c_{n-1}) \\
&= \gamma^{(3+p)^{n}} a_{n-1} f^2(a_{n-1}) g(a_{n-1}, b_{n-1}, c_{n-1}) = \gamma^{(3+p)^{n}} a_n, \\
a_{n+1} &\leq \gamma^{(3+p)^{n}} a_n \leq \gamma^{(3+p)^{n}} \gamma^{(3+p)^{n-1}} a_{n-1} \\
&\leq \cdots \leq \gamma^{(3+p)^{n}} \gamma^{(3+p)^{n-1}} \cdots \gamma^{(3+p)^{0}} a_0 \\
&= \gamma^{(((3+p)^{n+1}-1)/(2+p))} a_0, \\
b_{n+1} &= b_n f^3(a_n) g^2(a_n, b_n, c_n) \leq b_n \left(\frac{a_{n+1}}{a_n}\right)^2 \leq \left(\gamma^{(3+p)^{n}}\right)^2 b_n \\
&\leq \cdots \leq \left(\gamma^{(3+p)^{n}}\right)^2 \left(\gamma^{(3+p)^{n-1}}\right)^2 \cdots \left(\gamma^{(3+p)^{0}}\right)^2 b_0 \\
&= \left(\gamma^{(((3+p)^{n+1}-1)/(2+p))}\right)^2 b_0, \\
c_{n+1} &= c_n f^{3+p}(a_n) g^{2+p}(a_n, b_n, c_n) \\
&\leq c_n \left[f^2(a_n) g(a_n, b_n, c_n)\right]^{2+p} = c_n \left(\frac{a_{n+1}}{a_n}\right)^{2+p} \\
&\leq \left(\gamma^{(3+p)^{n}}\right)^{2+p} c_n \leq \cdots \leq \gamma^{(3+p)^{n+1}-1} c_0
\end{aligned} \tag{16}$$

and from (ii) we get

$$\begin{aligned}
& f(a_n) g(a_n, b_n, c_n) \\
&\quad \leq f\left(\gamma^{(((3+p)^{n}-1)/(2+p))} a_0\right) \\
&\qquad \times g\left(\gamma^{(((3+p)^{n}-1)/(2+p))} a_0, \left(\gamma^{(((3+p)^{n}-1)/(2+p))}\right)^2 b_0, \gamma^{3^n-1} c_0\right) \\
&\quad \leq \gamma^{(3+p)^{n}-1} f(a_0) g(a_0, b_0, c_0) = \frac{\gamma^{(3+p)^{n}}}{f(a_0)}, \quad n \geq 1.
\end{aligned} \tag{17}$$

This completes the proof of Lemma 3. □

Lemma 4. *Suppose that X and Y are Banach spaces, Ω is an open convex of the Banach space X, $F : \Omega \subset X \to Y$ has continuous Fréchet derivative of the second-order, and the sequences $\{x_n\}$, $\{y_n\}$ are generated by (7). Then, for all natural numbers $n \geq 0$, the following formula holds:*

$$\begin{aligned}
& F(x_{n+1}) \\
&\quad = \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)\,dt\,(x_{n+1} - y_n)^2 \\
&\qquad + \left[\int_0^1 F''(x_n + t(y_n - x_n))(1-t)\,dt\right. \\
&\qquad\qquad \left. - \frac{1}{2}\int_0^1 F''\left(x_n + \frac{2}{3}t(y_n - x_n)\right) dt\right](y_n - x_n)^2 \\
&\qquad - \frac{1}{2}\int_0^1 \left[F''(x_n + t(y_n - x_n))\right. \\
&\qquad\qquad \left. - F''\left(x_n + \frac{2}{3}t(y_n - x_n)\right)\right] dt \\
&\qquad\qquad \times (y_n - x_n) H(x_n, y_n)(y_n - x_n) \\
&\qquad + \frac{1}{2}\int_0^1 F''(x_n + t(y_n - x_n))\,dt \\
&\qquad\qquad \times (y_n - x_n) H(x_n, y_n) H(x_n, y_n)(y_n - x_n).
\end{aligned} \tag{18}$$

Proof. Consider

$$\begin{aligned}
& F(y_n) \\
&\quad = F(y_n) - F(x_n) - F'(x_n)(y_n - x_n) \\
&\quad = \int_0^1 F''(x_n + t(y_n - x_n))(1-t)\,dt\,(y_n - x_n)^2, \\
& F'(y_n)(x_{n+1} - y_n) \\
&\quad = -\frac{1}{2}\left[F'(y_n) - F'(x_n)\right] H(x_n, y_n) \\
&\qquad \times [I - H(x_n, y_n)](y_n - x_n) \\
&\qquad - \frac{1}{2} F'(x_n) H(x_n, y_n)[I - H(x_n, y_n)](y_n - x_n) \\
&\quad = -\frac{1}{2}\int_0^1 F''(x_n + t(y_n - x_n))\,dt \\
&\qquad\quad \times (y_n - x_n) H(x_n, y_n)(y_n - x_n) \\
&\qquad + \frac{1}{2}\int_0^1 F''(x_n + t(y_n - x_n))\,dt \\
&\qquad\quad \times (y_n - x_n) H(x_n, y_n) H(x_n, y_n)(y_n - x_n) \\
&\qquad - \frac{1}{2}\int_0^1 F''\left(x_n + \frac{2}{3}t(y_n - x_n)\right) dt\,(y_n - x_n)^2 \\
&\qquad + \frac{1}{2}\int_0^1 F''\left(x_n + \frac{2}{3}t(y_n - x_n)\right) dt \\
&\qquad\quad \times (y_n - x_n) H(x_n, y_n)(y_n - x_n),
\end{aligned}$$

$$\begin{aligned}
&F(x_{n+1})\\
&\quad= F(x_{n+1}) - F(y_n) - F'(y_n)(x_{n+1} - y_n)\\
&\qquad + F(y_n) + F'(y_n)(x_{n+1} - y_n)\\
&\quad= \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)\,dt\,(x_{n+1} - y_n)^2\\
&\qquad + \left[\int_0^1 F''(x_n + t(y_n - x_n))(1-t)\,dt\right.\\
&\qquad\qquad \left. -\frac{1}{2}\int_0^1 F''\left(x_n + \frac{2}{3}t(y_n - x_n)\right)dt\right](y_n - x_n)^2\\
&\qquad - \frac{1}{2}\int_0^1 \left[F''(x_n + t(y_n - x_n))\right.\\
&\qquad\qquad \left. - F''\left(x_n + \frac{2}{3}t(y_n - x_n)\right)\right]dt\\
&\qquad\qquad \times (y_n - x_n) H(x_n, y_n)(y_n - x_n)\\
&\qquad + \frac{1}{2}\int_0^1 F''(x_n + t(y_n - x_n))\,dt\\
&\qquad\qquad \times (y_n - x_n) H(x_n, y_n) H(x_n, y_n)(y_n - x_n).
\end{aligned}\tag{19}$$

This completes the proof of Lemma 4.

By (A1)–(A6), (10), and (11), if $a_0 < 1/2$, then

$$\begin{aligned}
\|H(x_0, y_0)\| &\le M\|y_0 - x_0\|\\
&= M\left\|F'(x_0)^{-1}F'(x_0)\right\|\|y_0 - x_0\| \le a_0,\\
\|x_1 - y_0\| &\le \frac{1}{2}\|H(x_0, y_0)\|\|I - H(x_0, y_0)\|\|y_0 - x_0\|\\
&\le \frac{a_0}{2}(1 + a_0)\|y_0 - x_0\|,\\
\|x_1 - x_0\| &\le \|x_1 - y_0\| + \|y_0 - x_0\|\\
&\le \left[1 + \frac{a_0}{2}(1 + a_0)\right]\|y_0 - x_0\| < R\eta,
\end{aligned}\tag{20}$$

where $R = [1+(a_0/2)(1+a_0)](1/(1-f(a_0)g(a_0, b_0, c_0)))$; hence, $x_1, y_0 \in S(x_0, R\eta)$. Consider

$$\begin{aligned}
&\left\|F'(x_0)^{-1}F'(x_1) - I\right\|\\
&\quad\le M\|x_1 - x_0\| \le \left[1 + \frac{a_0}{2}(1 + a_0)\right]a_0 < 1.
\end{aligned}\tag{21}$$

By Banach lemma, $F'(x_1)^{-1}$ exists, and

$$\left\|F'(x_1)^{-1}F'(x_0)\right\| \le f(a_0) = f(a_0)\left\|F'(x_0)^{-1}F'(x_0)\right\|.\tag{22}$$

□

By Lemma 4, we have

$$\begin{aligned}
&\left\|F'(x_0)^{-1}\right.\\
&\qquad \times \int_0^1 F''(x_0 + t(y_0 - x_0))(1-t)\,dt\\
&\qquad\qquad - \frac{1}{2}F'(x_0)^{-1}\\
&\qquad \times \left.\int_0^1 F''\left(x_0 + \frac{2}{3}t(y_0 - x_0)\right)dt\right\|\\
&\quad= \left\| F'(x_0)^{-1}\int_0^1 F''\left[(x_0 + t(y_0 - x_0))\right.\right.\\
&\qquad\qquad \left. -F''(x_0)\right](1-t)\,dt\\
&\qquad\qquad - \frac{1}{2}F'(x_0)^{-1}\\
&\qquad\qquad \times \left.\int_0^1\left[F''\left(x_0 + \frac{2}{3}t(y_0 - x_0)\right) - F''(x_0)\right]dt\right\|\\
&\quad= \left\|F'(x_0)^{-1}\iint_0^1 F'''\left[(x_0 + st(y_0 - x_0))\right.\right.\\
&\qquad\qquad \left. - F'''(x_0)\right]ds\\
&\qquad\qquad \times t(1-t)\,dt\,(y_0 - x_0)\\
&\qquad\qquad - \frac{1}{3}F'(x_0)^{-1}\\
&\qquad\qquad \times \iint_0^1\left[F'''\left(x_0 + \frac{2}{3}st(y_0 - x_0)\right)\right.\\
&\qquad\qquad \left.\left. -F'''(x_0)\right]ds\,t\,dt\,(y_0 - x_0)\right\|\\
&\quad\le (A + B)\,\omega(\eta)\|(y_0 - x_0)\|,\\
&\left\|F'(x_0)^{-1}F(x_1)\right\|\\
&\quad\le \frac{M}{2}\|x_1 - y_0\|^2 + \frac{N}{12}a_0\|y_0 - x_0\|^3\\
&\qquad + \frac{M}{2}a_0^2\|y_0 - x_0\|^2 + (A + B)\,\omega(\eta)\|y_0 - x_0\|^3,\\
&\|y_1 - x_1\|\\
&\quad\le \left\|F'(x_1)^{-1}F'(x_0)\right\|\left\|F'(x_0)^{-1}F(x_1)\right\|\\
&\quad\le f(a_0)\,g(a_0, b_0, c_0)\|y_0 - x_0\|.
\end{aligned}\tag{23}$$

Hence,

$$\begin{aligned}
\|H(x_1, y_1)\| &\le M\left\|F'(x_1)^{-1}F'(x_0)\right\|\|y_1 - x_1\|\\
&\le Mf^2(a_0)\,g(a_0, b_0, c_0)\|y_0 - x_0\| = a_1,
\end{aligned}$$

$$
\begin{aligned}
&N \left\| F'(x_1)^{-1} F'(x_0) \right\| \left\| y_1 - x_1 \right\|^2 \\
&\quad \le N f^3(a_0)\, g^2(a_0, b_0, c_0) \left\| y_0 - x_0 \right\|^2 = b_1, \\
&\left\| F'(x_1)^{-1} F'(x_0) \right\| \omega(\left\| y_1 - x_1 \right\|) \left\| y_1 - x_1 \right\|^2 \\
&\quad \le f^{3+p}(a_0)\, g^{2+p}(a_0, b_0, c_0)\, \omega(\eta) \left\| y_0 - x_0 \right\|^2 = c_1.
\end{aligned} \tag{24}
$$

Hence,

$$
\begin{aligned}
\left\| x_2 - y_1 \right\| &\le \frac{1}{2} a_1 (1 + a_1) \left\| y_1 - x_1 \right\|, \\
\left\| x_2 - x_1 \right\| &\le \left\| x_2 - y_1 \right\| + \left\| y_1 - x_1 \right\| \\
&\le \left(1 + \frac{1}{2} a_1 (1 + a_1)\right) \left\| y_1 - x_1 \right\|, \\
\left\| x_2 - x_0 \right\| &\le \left\| x_2 - x_1 \right\| + \left\| x_1 - x_0 \right\| \\
&\le \left[1 + \frac{a_0}{2} (1 + a_0)\right] \left[f(a_0)\, g(a_0, b_0, c_0) + 1 \right] \\
&\quad \times \left\| y_0 - x_0 \right\| < R\eta.
\end{aligned} \tag{25}
$$

By

$$
\begin{aligned}
&\left\| F'(x_1)^{-1} F'(x_2) - I \right\| \le M \left\| F'(x_1)^{-1} F'(x_0) \right\| \left\| x_2 - x_1 \right\| \\
&\quad \le a_1 \left[1 + \frac{a_1}{2} (1 + a_1)\right] < 1,
\end{aligned} \tag{26}
$$

hence $F'(x_2)^{-1}F'(x_0)$ exists, and $\|F'(x_2)^{-1}F'(x_0)\| \le f(a_1)\|F'(x_1)^{-1}F'(x_0)\|$. By induction, we can prove that the following Lemma 5 holds.

Lemma 5. *Under the hypotheses of Lemma 2, the following items are true for all $n \ge 1$:*

(I) $F'(x_n)^{-1}F'(x_0)$ *exists and* $\|F'(x_n)^{-1}F'(x_0)\| \le f(a_{n-1})\|F'(x_{n-1})^{-1}F'(x_0)\|$;

(II) $\|y_n - x_n\| \le f(a_{n-1})g(a_{n-1}, b_{n-1}, c_{n-1})\|y_{n-1} - x_{n-1}\|$;

(III) $H(x_n, y_n) \le M\|F'(x_n)^{-1}F'(x_0)\|\|y_n - x_n\| \le a_n$;

(IV) $N\|F'(x_n)^{-1}F'(x_0)\|\|y_n - x_n\|^2 \le b_n$;

(V) $\|F'(x_n)^{-1}F'(x_0)\|\omega(\|y_n - x_n\|)\|y_n - x_n\|^2 \le c_n$;

(VI) $\|x_{n+1} - y_n\| \le (a_n/2)(1 + a_n)\|y_n - x_n\|$;

(VII) $\|x_{n+1} - x_n\| \le [1 + (a_n/2)(1 + a_n)]\|y_n - x_n\|$;

(VIII) $\|x_{n+1} - x_0\| \le R\eta$, *where* $R = [1 + (a_0/2)(1 + a_0)]\,(1/(1 - f(a_0)g(a_0, b_0, c_0)))$.

Theorem 6. *Let X and Y be two Banach spaces and $F : \Omega \subset X \to Y$ has continuous Fréchet derivative of the third-order on a nonempty open convex Ω. One supposes that $\Gamma_0 = F'(x_0)^{-1} \in L(Y, X)$ exists for some $x_0 \in \Omega$ and conditions (A1)–(A6) and (11) hold. If $\overline{S(x_0, R\eta)} \subset \Omega$, then the sequence $\{x_n\}$ generated by (7) is well defined and converges to a unique solution x^* of (2) in $S(x_0, (2/M) - R\eta) \cap \Omega$. Furthermore, the following error estimate is obtained:*

$$
\begin{aligned}
&\left\| x^* - x_n \right\| \\
&\quad \le \left[1 + \frac{\gamma^{(((3+p)^n - 1)/(2+p))} a_0}{2} \left(1 + \gamma^{(((3+p)^n - 1)/(2+p))} a_0\right)\right] \\
&\quad \times \frac{1}{1 - \gamma^{(3+p)^n} \Delta} \gamma^{(((3+p)^n - 1)/(2+p))} \Delta^n \eta,
\end{aligned} \tag{27}
$$

where $\gamma = f^2(a_0)g(a_0, b_0, c_0) = a_1/a_0$ and $\Delta = 1/f(a_0)$, $R = (1 + (a_0/2)(1 + a_0))\,(1/(1 - \gamma\Delta))$.

Proof. Firstly, we prove that the sequence $\{x_n\}$ is a Cauchy one. From (II) and by Lemma 3, we have

$$
\begin{aligned}
\left\| y_n - x_n \right\| &\le f(a_{n-1})\, g(a_{n-1}, b_{n-1}, c_{n-1}) \left\| y_{n-1} - x_{n-1} \right\| \\
&\le \cdots \le \left(\prod_{i=0}^{n-1} f(a_i)\, g(a_i, b_i, c_i) \right) \eta \\
&\le \left(\prod_{i=0}^{n-1} \gamma^{(3+p)^i} \Delta \right) \eta = \gamma^{(((3+p)^n - 1)/(2+p))} \Delta^n \eta.
\end{aligned} \tag{28}
$$

For $n \ge 0$, $m \ge 1$,

$$
\begin{aligned}
&\left\| x_{n+m} - x_n \right\| \\
&\quad \le \left\| x_{n+m} - x_{n+m-1} \right\| + \left\| x_{n+m-1} - x_{n+m-2} \right\| \\
&\qquad + \cdots + \left\| x_{n+1} - x_n \right\| \\
&\quad \le \left[1 + \frac{a_n}{2} (1 + a_n)\right] \\
&\qquad \times \left(\left\| y_{n+m-1} - x_{n+m-1} \right\| + \cdots + \left\| y_{n+1} - x_n \right\| \right) \\
&\quad \le \left[1 + \frac{a_n}{2} (1 + a_n)\right] \\
&\qquad \times \left(\gamma^{(((3+p)^{n+m-1} - 1)/(2+p))} \Delta^{n+m-1} \right. \\
&\qquad \qquad \left. + \cdots + \gamma^{(((3+p)^n - 1)/(2+p))} \Delta^n \right) \eta \\
&\quad = \left[1 + \frac{a_n}{2} (1 + a_n)\right] \gamma^{(((3+p)^n - 1)/(2+p))} \Delta^n \eta \\
&\qquad \times \left(\gamma^{((3+p)^n [(3+p)^{m-1} - 1]/(2+p))} \Delta^{m-1} + \cdots + 1 \right).
\end{aligned} \tag{29}
$$

By the Bernoulli inequality $(1 + x)^k - 1 > kx$, so $(3 + p)^k - 1 > k(2 + p)$. Hence, we have

$$
\begin{aligned}
&\left\| x_{n+m} - x_n \right\| \\
&\quad < \left[1 + \frac{a_n}{2} (1 + a_n)\right] \frac{1}{1 - \gamma^{(3+p)^n} \Delta} \gamma^{(((3+p)^n - 1)/(2+p))} \Delta^n \eta.
\end{aligned} \tag{30}
$$

Hence, $\{x_n\}$ is a Cauchy sequence and $x^* = \lim_{n\to\infty} x_n$. Obviously, $x_m \in B(x_0, R\eta)$, for all $m \geq 1$, as if $n = 0$ in (30); we obtain

$$\|x_m - x_0\| < \left(1 + \frac{a_0}{2}(1 + a_0)\right) \frac{1}{1 - \gamma\Delta}\eta = R\eta. \tag{31}$$

Following a similar procedure, we have $y_n \in B(x_0, R\eta)$, for all $n \geq 0$.

Now, let $n \to \infty$ in (28). It follows that $\|F'(x_n)^{-1} F(x_n)\| \to 0$. Besides $\|F(x_n)\| \to 0$, since $\|F(x_n)\| \leq \|F'(x_n)\| \|F'(x_n)^{-1}F(x_n)\|$ and $\{\|F'(x_n)\|\}$ is a bounded sequence, therefore $F(x^*) = 0$ by the continuity of F in $\overline{S(x_0, R\eta)}$.

By letting $m \to \infty$ in (30), we obtain error estimate (28).

To show uniqueness, let us assume that there exists a second solution y^* of (2) in $S(x_0, (2/M) - R\eta) \cap \Omega$. Then

$$\begin{aligned} &\int_0^1 \left\|F'(x_0)^{-1}\left[F'(x^* + t(y^* - x^*)) - F'(x_0)\right]\right\| dt \\ &\leq M \int_0^1 \|x^* + t(y^* - x^*) - x_0\| dt \\ &\leq M \int_0^1 \left[(1-t)\|x^* - x_0\| + t\|y^* - x_0\|\right] dt \\ &< \frac{M}{2}\left(R\eta + \frac{2}{M} - R\eta\right) = 1. \end{aligned} \tag{32}$$

By Banach lemma, we can obtain that the inverse of the linear operator $\int_0^1 F'(x^* + t(y^* - x^*))dt$ exists and

$$\int_0^1 F'(x^* + t(y^* - x^*))\,dt\,(y^* - x^*) = F(y^*) - F(x^*) = 0. \tag{33}$$

We get that $x^* = y^*$.

This completes the proof of Theorem 6. □

3. Application

In this section, we apply the convergence theorem and show three numerical examples.

Example 1. Consider the root of the equation $F(x) = x^{10/3} + x^{7/2} - x - 1 = 0$ on $x \in (0, +\infty)$. Then, we easily get that

$$F'''(x) = \frac{280}{27}x^{1/3} + \frac{105}{8}x^{1/2} \tag{34}$$

does not satisfy (K, p) Hölder condition

$$\|F'''(x) - F'''(y)\| \leq K\|x - y\|^p \tag{35}$$

because, for all $p \in (0, 1]$,

$$\sup_{x,y\in(0,+\infty)} \frac{(280/27)|x-y|^{1/3} + (105/8)|x-y|^{1/2}}{|x-y|^p} = +\infty. \tag{36}$$

Let

$$\omega(z) = \frac{280}{27}z^{1/3} + \frac{105}{8}z^{1/2}, \quad z > 0; \tag{37}$$

then $\omega(tz) \leq t^{1/3}\omega(z)$ for $t \in [0, 1]$ and $z \in [0, +\infty)$;

$$\|F'''(x) - F'''(y)\| \leq \omega\|x - y\|. \tag{38}$$

Let us consider a particular case of (2) from the operator given by the following nonlinear integral equation of mixed Hammerstein type (see [26]):

$$x(s) = \alpha(s) - \sum_{i=1}^{m} \int_a^b k(s,t)\varphi_i(x(t))\,dt, \tag{39}$$

where $-\infty < a < b < +\infty$, u, φ_i, for $i = 1, 2, \ldots, m$, are known functions and x is a solution to be determined. If φ''' is (L_i, p_i) Hölder continuous in Ω, for $i = 1, 2, \ldots, m$, the corresponding operator $F : \Omega \subseteq C[a, b] \to C[a, b]$,

$$\begin{aligned} [F(x)](s) = x(s) + \sum_{i=1}^{m}\int_a^b k(s,t)\varphi_i(x(t))\,dt - \alpha(s), \\ s \in [a, b], \end{aligned} \tag{40}$$

does not satisfy (K, p) Hölder condition; for instance, the max-norm is considered. In this case,

$$\begin{gathered} \|F'''(x) - F'''(y)\| \leq \sum_{i=1}^{m} L_i\|x - y\|^{p_i}, \\ L_i > 0, \quad p_i \in (0, 1], \quad x, y \in \Omega. \end{gathered} \tag{41}$$

To solve this type of equations, we can consider

$$\|F'''(x) - F'''(y)\| \leq \omega(\|x - y\|), \quad x, y \in \Omega, \tag{42}$$

where $\omega(z) = \sum_{i=1}^{m} L_i z^{p_i}$ satisfy $\omega(tz) \leq t^q\omega(z)$, where $q = \min\{p_i, p_2, \ldots, p_m\}$.

Remark 7. Observe that if F''' is Lipschitz continuous in Ω, we can choose $\omega(z) = Kz$, $K > 0$, so that Jarratt's method is of R-order, at least four order. If F''' is (L, p) Hölder continuous in Ω, then we can choose $\omega(z) = Lz^p$, $L < 0$, $p \in (0, 1]$, and Jarratt's method is of R-order, at least $3 + p$.

Example 2. Consider the case as follows:

$$\begin{aligned} x(s) = 1 &+ \frac{1}{32}\int_0^1 k(s,t)\,x(t)^{16/5}\,dt \\ &+ \frac{1}{30}\int_0^1 k(s,t)\,x(t)^{10/3}\,dt, \end{aligned} \tag{43}$$

where the space is $X = C[0, 1]$ with the norm

$$\begin{gathered} \|x\| = \max_{0\leq s\leq 1}|x(s)|, \\ k(s,t) = \begin{cases} t(1-s), & t \leq s, \\ s(1-t), & s \leq t. \end{cases} \end{gathered} \tag{44}$$

This equation arises in the theory of the radiative transfer and neutron transport and in the kinetic theory of gasses. Let us define the operator F on X by

$$F(x) = x(s) - \frac{1}{32}\int_0^1 k(s,t)\, x(t)^{16/5}\, dt - \frac{1}{30}\int_0^1 k(s,t)\, x(t)^{10/3}\, dt - 1. \tag{45}$$

The first, the second, and the third derivatives of F are defined by

$$\begin{aligned} F'(x)\,u(s) &= u(s) - \frac{1}{10}\int_0^1 k(s,t)\, x(t)^{11/5}\, u(t)\, dt \\ &\quad - \frac{1}{9}\int_0^1 k(s,t)\, x(t)^{7/3}\, u(t)\, dt, \quad u \in X, \\ F''(x)(uv)(s) &= -\frac{11}{50}\int_0^1 k(s,t)\, x(t)^{6/5}\, u(t)\, v(t)\, dt \\ &\quad - \frac{7}{27}\int_0^1 k(s,t)\, x(t)^{\frac{4}{3}}\, u(t)\, v(t)\, dt, \\ &\qquad u \in X, \\ F'''(x)(uvw)(s) &= -\frac{66}{250}\int_0^1 k(s,t)\, x(t)^{1/5}\, u(t)\, v(t)\, w(t)\, dt \\ &\quad - \frac{28}{81}\int_0^1 k(s,t)\, x(t)^{1/3}\, u(t)\, v(t)\, w(t)\, dt, \end{aligned} \tag{46}$$

and we have

$$\begin{aligned} &\left\| \left[F'''(x) - F'''(y) \right] uvw \right\| \\ &\quad \le \frac{66}{250} \max_{s\in[0,1]} \\ &\qquad \times \int_0^1 k(s,t) \left| \left(x(t)^{1/5} - y(t)^{1/5} \right) u(t)\, v(t)\, w(t) \right| dt \\ &\qquad + \frac{28}{81} \max_{s\in[0,1]} \\ &\qquad \times \int_0^1 k(s,t) \left| \left(x(t)^{1/3} - y(t)^{1/3} \right) u(t)\, v(t)\, w(t) \right| dt \\ &\quad \le \frac{66}{250} \times \frac{1}{8} \left\| x - y \right\|^{1/5} \left\| uvw \right\| \\ &\qquad + \frac{28}{81} \times \frac{1}{8} \left\| x - y \right\|^{1/3} \left\| uvw \right\|. \end{aligned} \tag{47}$$

To apply Theorem 6, we choose $x_0 = x_0(s) = 1$ and we look for a domain in the form

$$\Omega = B(1,2) \subseteq C([0,1]). \tag{48}$$

In this case, we have

$$\left\| I - F'(x_0) \right\| \le \frac{19}{720} < 1 \tag{49}$$

and from the Banach lemma, we obtain

$$\begin{gathered} \left\| F'(x_0)^{-1} \right\| \le \frac{720}{701}, \\ \left\| F'(x_0)^{-1} F(x_0) \right\| \le \frac{720}{701} \times \frac{1}{8}\left(\frac{1}{32} + \frac{1}{30} \right) = \eta = \frac{93}{11216}, \\ M = 0.148766\cdots, \qquad N = 0.0948511\cdots, \\ \omega(z) = \frac{33}{100 z^{1/5}} + \frac{7}{162 z^{1/3}}, \qquad p = \frac{1}{5}. \end{gathered} \tag{50}$$

Then $a_0 = M\eta = 0.00123353 < 1/2$, $b_0 = 6.52127 \times 10^{-6}$, $c_0 = 1.47132 \times 10^{-6}$, $\gamma = f^2(a_0) g(a_0, b_0, c_0) = 3.48167 \times 10^{-7} < 1$, $\Delta = 0.998766\cdots$, and $R = 1.00062\cdots$. This means that the hypothesis of Theorem 6 is satisfied. Then, the error bound becomes

$$\begin{aligned} &\left\| x^* - x_n \right\| \\ &\quad \le \left[1 + \frac{\gamma^{(((3.2)^n - 1)/2.2)} a_0}{2} \left(1 + \gamma^{(((3.2)^n - 1)/2.2)} a_0 \right) \right] \\ &\qquad \times \frac{1}{1 - \gamma^{(3.2)^n} \Delta} \gamma^{(((3.2)^n - 1)/2.2)} \Delta^n \eta. \end{aligned} \tag{51}$$

For $n = 1, 2, 3, 4$, we get

$$\begin{gathered} \left\| x_1 - x^* \right\| \le 4.28944 \times 10^{-10}, \\ \left\| x_2 - x^* \right\| \le 5.76451 \times 10^{-16}, \\ \left\| x_3 - x^* \right\| \le 6.63209 \times 10^{-23}, \\ \left\| x_4 - x^* \right\| \le 2.86064 \times 10^{-32}. \end{gathered} \tag{52}$$

Example 3. Let us consider the system of equations $F(u, v) = 0$, where

$$F(u,v) = \left(u^{7/2} - uv - v^{10/3} + 1,\ u^{7/2} + uv - v^{10/3} - 1 \right)^T. \tag{53}$$

Then, we have

$$F'(u,v) = \begin{pmatrix} \frac{7}{2}u^{5/2} - v & -\frac{10}{3}v^{7/3} - u \\ \frac{7}{2}u^{5/2} + v & -\frac{10}{3}v^{7/3} + u \end{pmatrix},$$

$$F'(u,v)^{-1} = \frac{1}{(14/2)\,u^{7/2} + (20/3)\,v^{10/3}} \times \begin{pmatrix} -\frac{10}{3}v^{\frac{7}{3}} + u & \frac{10}{3}v^{\frac{7}{3}} + u \\ -\frac{7}{2}u^{\frac{5}{2}} - v & \frac{7}{2}u^{\frac{5}{2}} - v, \end{pmatrix},$$

$$F''(u,v) = \left(\begin{array}{cc} \frac{35}{4}u^{3/2} & -1 \\ -1 & -\frac{70}{9}v^{4/3} \\ \hline \frac{35}{4}u^{3/2} & 1 \\ 1 & -\frac{70}{9}v^{4/3}. \end{array}\right),$$

$$F'''(u,v)(s,t)^3 = \begin{pmatrix} \frac{105}{8}u^{1/2} & \frac{280}{27}v^{1/3} \\ \frac{105}{8}u^{1/2} & \frac{280}{27}v^{1/3} \end{pmatrix}\begin{pmatrix} s^3 \\ t^3 \end{pmatrix}. \tag{54}$$

Now, we choose $x_0 = (u_0, v_0) = (1.5, 1.5)$ and $\Omega = \{x \mid \|x - x_0\| \le 1.5\}$. We take the max-norm in R^2 and the norm $\|A\| = \max\{|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\}$ for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. We define the norm of a bilinear operator B on R^2 by

$$\|B\| = \sup_{\|u\|=1} \max_i \sum_{j=1}^{2} \left| \sum_{k=1}^{2} b_i^{jk} u_k \right|, \tag{55}$$

where $u = (u_1, u_2)^T$ and $B = \begin{pmatrix} b_1^{11} & b_1^{12} \\ b_1^{21} & b_1^{22} \\ b_2^{11} & b_2^{12} \\ b_2^{21} & b_2^{22} \end{pmatrix}$.

Then, we get the following results: $\eta = \|F'(x_0)^{-1}F(x_0)\| = 0.09598\cdots$, $M = 9.20456\cdots$, $N = 10.7635\cdots$, and $p = 1/3$.

We get that the hypotheses of Theorem 6 are satisfied.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

The authors have made the same contribution. All authors read and approved the final paper.

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References

[1] J. A. Ezquerro, D. González, and M. A. Hernández, "A modification of the classic conditions of Newton-Kantorovich for Newton's method," *Mathematical and Computer Modelling*, vol. 57, no. 3-4, pp. 584–594, 2013.

[2] J. A. Ezquerro, D. González, and M. A. Hernández, "A variant of the Newton-Kantorovich theorem for nonlinear integral equations of mixed Hammerstein type," *Applied Mathematics and Computation*, vol. 218, no. 18, pp. 9536–9546, 2012.

[3] I. K. Argyros, "On the solution of undetermined systems of nonlinear equations in Euclidean spaces," *Pure Mathematics and Applications*, vol. 4, no. 2, pp. 199–209, 1993.

[4] A. Holstad, "Numerical solution of nonlinear equations in chemical speciation calculations," *Computational Geosciences*, vol. 3, no. 3-4, pp. 229–257, 1999.

[5] K. Kowalski and K. Jankowski, "Towards complete solutions to systems of nonlinear equations of many-electron theories," *Physical Review Letters*, vol. 81, no. 6, pp. 1195–1198, 1998.

[6] B. M. Barbashov, V. V. Nesterenko, and A. M. Chervyakov, "General solutions of nonlinear equations in the geometric theory of the relativistic string," *Communications in Mathematical Physics*, vol. 84, no. 4, pp. 471–481, 1982.

[7] S. R. Bickham, S. A. Kiselev, and A. J. Sievers, "Stationary and moving intrinsic localized modes in one-dimensional monatomic lattices with cubic and quartic anharmonicity," *Physical Review B*, vol. 47, no. 21, pp. 14206–14211, 1993.

[8] G. Yuan and X. Lu, "A new backtracking inexact BFGS method for symmetric nonlinear equations," *Computers & Mathematics with Applications*, vol. 55, no. 1, pp. 116–129, 2008.

[9] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, NY, USA, 1970.

[10] R. Lin, Y. Zhao, Z. Smarda, Y. Khan, and Q. Wu, "Newton-Kantorovich and SMAle uniform type convergence theorem for a deformed Newton method in Banach spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 923898, 8 pages, 2013.

[11] L. Kantorovich, "On Newton method," *Trudy Matematicheskogo Instituta imeni VA Steklova*, vol. 28, pp. 104–144, 1949 (Russian).

[12] X. H. Wang and C. Li, "Convergence of Newton's method and uniqueness of the solution of equations in Banach spaces II," *Acta Mathematica Sinica*, vol. 19, no. 2, pp. 405–412, 2003.

[13] I. K. Argyros, "On the Newton-Kantorovich hypothesis for solving equations," *Journal of Computational and Applied Mathematics*, vol. 169, no. 2, pp. 315–332, 2004.

[14] I. K. Argyros, "On the comparison of a weak variant of the Newton-Kantorovich and Miranda theorems," *Journal of Computational and Applied Mathematics*, vol. 166, no. 2, pp. 585–589, 2004.

[15] M. A. Hernandez, "A modification of the classical Kantorovich conditions for Newton's method," *Journal of Computational and Applied Mathematics*, vol. 137, no. 1, pp. 201–205, 2001.

[16] S. Amat, S. Busquier, D. El kebir, and J. Molina, "A fast Chebyshev's method for quadratic equations," *Applied Mathematics and Computation*, vol. 148, no. 2, pp. 461–474, 2004.

[17] J. M. Gutierrez and M. A. Hernandez, "An acceleration of Newton's method: super-Halley method," *Applied Mathematics and Computation*, vol. 117, no. 2-3, pp. 223–239, 2001.

[18] J. A. Ezquerro and M. A. Hernández, "On the *R*-order of the Halley method," *Journal of Mathematical Analysis and Applications*, vol. 303, no. 2, pp. 591–601, 2005.

[19] J. A. Ezquerro and M. A. Hernández, "Fourth-order iterations for solving Hammerstein integral equations," *Applied Numerical Mathematics*, vol. 59, no. 6, pp. 1149–1158, 2009.

[20] J. A. Ezquerro and M. A. Hernández, "Halley's method for operators with unbounded second derivative," *Applied Numerical Mathematics*, vol. 57, no. 3, pp. 354–360, 2007.

[21] J. A. Ezquerro, D. Gonzalez, and M. A. Hernandez, "A variant of the Newton-Kantorovich theorem for nonlinear integral equations of mixed Hammerstein type," *Applied Mathematics and Computation*, vol. 218, no. 18, pp. 9536–9546, 2012.

[22] R. Lin, Y. Zhao, Q. Wu, and J. Hu, "Convergence theorem for a family of new modified Halley's method in Banach space," *Journal of Applied Mathematics*, vol. 2014, Article ID 468694, 8 pages, 2014.

[23] I. K. Argyros, D. Chen, and Q. Qian, "The Jarratt method in Banach space setting," *Journal of Computational and Applied Mathematics*, vol. 51, no. 1, pp. 103–106, 1994.

[24] V. Kanwar, S. Kumar, and R. Behl, "Several new families of Jarratt's method for solving systems of nonlinear equations," *Applications and Applied Mathematics*, vol. 8, no. 2, pp. 701–716, 2013.

[25] I. K. Argyros, "A new convergence theorem for the Jarratt method in Banach space," *Computers & Mathematics with Applications*, vol. 36, no. 8, pp. 13–18, 1998.

[26] M. Ganesh and M. C. Joshi, "Numerical solvability of Hammerstein integral equations of mixed type," *IMA Journal of Numerical Analysis*, vol. 11, no. 1, pp. 21–31, 1991.