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Research Article

Differential Subordination with Generalized Derivative Operator of Analytic Functions

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Motivated by generalized derivative operator defined by the authors (El-Yagubi and Darus, 2013) and the technique of differential subordination, several interesting properties of the operator $\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta}$ are given.

1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$.

Also let $\mathscr{S}$ be the the subclass of $\mathscr{A}$ consisting of all functions which are univalent in $\mathbb{U}$. We denote by $\mathscr{S}^*(\alpha)$ and $\mathscr{C}(\alpha)(0 \le \alpha < 1)$ the familiar subclasses of $\mathscr{A}$ consisting of functions which are, respectively, starlike of order α and convex of order α in $\mathbb{U}$:.

$$\mathscr{S}^*(\alpha) = \left\{ f \in \mathscr{A} : \Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in \mathbb{U} \right\},$$
$$\mathscr{C}(\alpha) = \left\{ f \in \mathscr{A} : \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, z \in \mathbb{U} \right\}. \quad (2)$$

Let $\mathscr{H}(\mathbb{U})$ be the class of holomorphic function in unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we let

$$\mathscr{H}[a,n] = \{ f \in \mathscr{H}(\mathbb{U}), f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}, \quad (z \in \mathbb{U}). \quad (3)$$

Let two functions given by $f(z) = \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=2}^{\infty} b_n z^n$ be analytic in $\mathbb{U}$. Then the Hadamard product (or convolution) $f * g$ of the two functions f, g is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \quad (4)$$

Recall that the function f is subordinate to g if there exists the Schwarz function ω, analytic in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. We denote this subordination by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the (second order) differential subordination

$$\psi\left(p(z), zp'(z), z^2 p''(z); z\right) \prec h(z), \quad (z \in \mathbb{U}), \quad (5)$$

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (5).

A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (5) is said to be the best dominant of (5) (note that the best dominant is unique up to a rotation of $\mathbb{U}$).

In order to prove the original results we need the following lemmas.

Lemma 1 (see [1]). *Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C} - \{0\}$ be a complex number with $\Re\{\gamma\} \geq 0$. If $p \in H[a;n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \quad (z \in \mathbb{U}), \tag{6}$$

then

$$p(z) \prec q(z) \prec h(z), \quad (z \in \mathbb{U}), \tag{7}$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{(\gamma/n)-1} dt, \quad (z \in \mathbb{U}). \tag{8}$$

The function q is convex and is the best dominant.

Lemma 2 (see [2]). *Let g be a convex function in $\mathbb{U}$ and let*

$$h(z) = g(z) + n\alpha z g'(z), \tag{9}$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots, \quad (z \in \mathbb{U}) \tag{10}$$

is analytic in $\mathbb{U}$ and

$$p(z) + \alpha z p'(z) \prec h(z), \quad (z \in \mathbb{U}), \tag{11}$$

then

$$p(z) \prec g(z), \tag{12}$$

and this result is sharp.

Lemma 3 (see [3]). *Let $f \in \mathscr{A}$; if*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2}, \tag{13}$$

then

$$\frac{2}{z} \int_0^z f(t)\, dt, \quad (z \in \mathbb{U}, z \neq 0) \tag{14}$$

belongs to the class of convex functions.

We now state the following generalized derivative operator [4]:

$$\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) = z + \sum_{n=2}^{\infty} \left[\frac{1 + (\lambda_1 + \lambda_2)(n-1) + b}{1 + \lambda_2(n-1) + b}\right]^m \mathscr{C}(\delta, n)\, a_n z^n, \tag{15}$$

where $\lambda_2 \geq \lambda_1 \geq 0$, $\mathscr{C}(\delta,n) = (\delta+1)_{n-1}/(n-1)!$, for $\delta, m, b \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, and $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1, & n = 0, \\ x(x+1)\cdots(x+n-1), & n = \{1,2,3,\ldots\}. \end{cases} \tag{16}$$

Here $\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)$ can also be written in terms of convolution as

$$\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) = \mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \mathscr{D}^{m,b}_{\lambda_2}(z). \tag{17}$$

To prove our results, we need the following inclusion relation:

$$\begin{aligned}(1+b)\,\mathscr{D}^{m+1,b}_{\lambda_1,\lambda_2,\delta} f(z) &= \left(1 - (\lambda_1+\lambda_2) + b\right)\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right) \\ &\quad + (\lambda_1+\lambda_2)\, z\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right)',\end{aligned} \tag{18}$$

where $\varphi^b_{\lambda_2}(z)$ is analytic function given by $\varphi^b_{\lambda_2}(z) = z + \sum_{n=2}^{\infty}(z^n/(1+\lambda_2(n-1)+b))$.

2. Main Results

In the present paper, we will use the method of differential subordination to derive certain properties of generalised derivative operator $\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)$. Note that differential subordination has been studied by various authors, and here we follow similar works done by Oros [5] and G. Oros and G. I. Oros [6].

Definition 4. For $\lambda_2 \geq \lambda_1 \geq 0$, $\delta, m, b \in \mathbb{N}_0$, and $0 \leq \alpha < 1$, let $\mathscr{R}^{m,b}_{\lambda_1,\lambda_2,\delta}(\alpha)$ denote the class of functions $f \in \mathscr{A}$ which satisfy the condition

$$\Re\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)\right)' > \alpha, \quad (z \in \mathbb{U}). \tag{19}$$

Also, let $\mathscr{K}^{m,b}_{\lambda_1,\lambda_2,\delta}(\beta)$ denote the class of functions $f \in \mathscr{A}$ which satisfy the condition

$$\Re\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right)' > \beta, \quad (z \in \mathbb{U}). \tag{20}$$

Remark 5. It is clear that $\mathscr{R}^{1,0}_{\lambda_1,0,0}(\alpha) \equiv \mathscr{R}(\lambda_1, \alpha)$, and the class of functions $f \in \mathscr{A}$ satisfy

$$\Re\left(\lambda_1 z f''(z) + f'(z)\right) > \alpha, \quad (z \in \mathbb{U}), \tag{21}$$

studied by Ponnusamy [7] and others.

Theorem 6. *Let*

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad (z \in \mathbb{U}), \tag{22}$$

be convex in $\mathbb{U}$, with $h(0) = 1$ and $0 \leq \alpha < 1$.

If $\lambda_2 \geq \lambda_1 \geq 0$, $\delta, m, b \in \mathbb{N}_0$, and $f \in \mathscr{A}$ and satisfies the differential subordination

$$\left(\mathscr{D}^{m+1,b}_{\lambda_1,\lambda_2,\delta} f(z)\right)' \prec h(z), \quad (z \in \mathbb{U}), \tag{23}$$

then

$$\begin{aligned}\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right)' &\prec q(z) \\ &= 2\alpha - 1 + \frac{2(1-\alpha)(1+b)}{(\lambda_1+\lambda_2)\, z^{(1+b)/(\lambda_1+\lambda_2)}}\sigma\left(\frac{1+b}{\lambda_1+\lambda_2}\right),\end{aligned} \tag{24}$$

where σ is given by

$$\sigma(x) = \int_0^z \frac{t^{x-1}}{1+t} dt, \quad (z \in \mathbb{U}). \tag{25}$$

The function q is convex and is the best dominant.

Proof. By differentiating (18), with respect to z, we obtain

$$\begin{aligned}(1+b)\left(\mathscr{D}^{m+1,b}_{\lambda_1,\lambda_2,\delta} f(z)\right)' &= (1+b)\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right)' \\ &\quad + (\lambda_1+\lambda_2) z \left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right)''.\end{aligned} \tag{26}$$

Using (26) in (23), the differential subordination (23) becomes

$$\begin{aligned}&\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right)' + \frac{(\lambda_1+\lambda_2)}{1+b} z \left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right)'' \\ &\quad \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}.\end{aligned} \tag{27}$$

Let

$$\begin{aligned}p(z) &= \left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right)' \\ &= \left(z + \sum_{n=2}^{\infty} \left(\frac{(1+(\lambda_1+\lambda_2)(n-1)+b)^m}{(1+\lambda_2(n-1)+b)^{m+1}}\right) \mathscr{C}(\delta,n) a_n z^n\right)' \\ &= 1 + p_1 z + p_2 z^2 + \cdots, \quad (p \in \mathscr{H}[1,1], z \in \mathbb{U}).\end{aligned} \tag{28}$$

Using (28) in (27), the differential subordination becomes

$$p(z) + \frac{(\lambda_1+\lambda_2)}{1+b} z p'(z) \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}. \tag{29}$$

By using Lemma 1, we have

$$\begin{aligned}p(z) \prec q(z) &= \frac{1+b}{(\lambda_1+\lambda_2) z^{(1+b)/(\lambda_1+\lambda_2)}} \int_0^z h(t) t^{((1+b)/(\lambda_1+\lambda_2))-1} dt \\ &= \frac{1+b}{(\lambda_1+\lambda_2) z^{(1+b)/(\lambda_1+\lambda_2)}} \times \int_0^z \left(\frac{1+(2\alpha-1)t}{1+t}\right) t^{((1+b)/(\lambda_1+\lambda_2))-1} dt \\ &= 2\alpha - 1 + \frac{2(1-\alpha)(1+b)}{(\lambda_1+\lambda_2) z^{(1+b)/(\lambda_1+\lambda_2)}} \sigma\left(\frac{1+b}{\lambda_1+\lambda_2}\right),\end{aligned} \tag{30}$$

where σ is given by (25); that is,

$$\begin{aligned}&\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right)' \\ &\quad \prec q(z) \\ &\quad = 2\alpha - 1 + \frac{2(1-\alpha)(1+b)}{(\lambda_1+\lambda_2) z^{(1+b)/(\lambda_1+\lambda_2)}} \sigma\left(\frac{1+b}{\lambda_1+\lambda_2}\right).\end{aligned} \tag{31}$$

The function q is convex and is the best dominant. The proof is complete. □

Theorem 7. *If $\lambda_2 \geq \lambda_1 \geq 0$, $\delta, m, b \in \mathbb{N}_0$, and $0 \leq \alpha < 1$, then one has*

$$\mathscr{R}^{m+1,b}_{\lambda_1,\lambda_2,\delta}(\alpha) \subset \mathscr{K}^{m,b}_{\lambda_1,\lambda_2,\delta}(\beta), \tag{32}$$

where

$$\beta = 2\alpha - 1 + \frac{2(1-\alpha)(1+b)}{(\lambda_1+\lambda_2)} \sigma\left(\frac{1+b}{\lambda_1+\lambda_2}\right), \tag{33}$$

and σ is given by (25).

Proof. Let $f \in \mathscr{R}^{m+1,b}_{\lambda_1,\lambda_2,\delta}(\alpha)$, and then from (19) we have

$$\Re\left(\mathscr{D}^{m+1,b}_{\lambda_1,\lambda_2,\delta} f(z)\right)' > \alpha, \quad (z \in \mathbb{U}), \tag{34}$$

which is equivalent to

$$\left(\mathscr{D}^{m+1,b}_{\lambda_1,\lambda_2,\delta} f(z)\right)' \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}. \tag{35}$$

Using Theorem 6, we have

$$\begin{aligned}&\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right)' \\ &\quad \prec q(z) \\ &\quad = 2\alpha - 1 + \frac{2(1-\alpha)(1+b)}{(\lambda_1+\lambda_2) z^{(1+b)/(\lambda_1+\lambda_2)}} \sigma\left(\frac{1+b}{\lambda_1+\lambda_2}\right).\end{aligned} \tag{36}$$

Since q is convex and $q(\mathbb{U})$ is symmetric with respect to the real axis, we deduce that

$$\begin{aligned}&\Re\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right)' \\ &\quad > \Re q(1) \\ &\quad = \beta(\alpha, \lambda_1, \lambda_2, b) \\ &\quad = 2\alpha - 1 + \frac{2(1-\alpha)(1+b)}{(\lambda_1+\lambda_2)} \sigma\left(\frac{1+b}{\lambda_1+\lambda_2}\right),\end{aligned} \tag{37}$$

for which we deduce $\mathscr{R}^{m+1,b}_{\lambda_1,\lambda_2,\delta}(\alpha) \subset \mathscr{K}^{m,b}_{\lambda_1,\lambda_2,\delta}(\beta)$. The proof is complete. □

Theorem 8. *Let q be a convex function in $\mathbb{U}$, with $q(0) = 1$, and let*

$$h(z) = q(z) + \frac{(\lambda_1+\lambda_2)}{1+b} z q'(z), \quad (z \in \mathbb{U}). \tag{38}$$

If $\lambda_2 \geq \lambda_1 \geq 0$, $\delta, m, b \in \mathbb{N}_0$, and $f \in \mathcal{A}$ satisfies the differential subordination

$$\left(\mathcal{D}^{m+1,b}_{\lambda_1,\lambda_2,\delta} f(z)\right)' \prec h(z), \tag{39}$$

then

$$\left(\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right)' \prec q(z), \quad (z \in \mathbb{U}), \tag{40}$$

and the result is sharp.

Proof. Using (28) in (26), the differential subordination (39) becomes

$$p(z) + \frac{(\lambda_1+\lambda_2)}{1+b} z p'(z) \prec h(z) = q(z) + \frac{(\lambda_1+\lambda_2)}{1+b} z q'(z) \quad (z \in \mathbb{U}). \tag{41}$$

Using Lemma 2, we have

$$p(z) \prec q(z), \quad (z \in \mathbb{U}); \tag{42}$$

that is,

$$\left(\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * \varphi^b_{\lambda_2}(z)\right)' \prec q(z), \quad (z \in \mathbb{U}), \tag{43}$$

and the result is sharp. The proof of Theorem 8 is complete. □

Example 9. For $m = 1$, $\delta = 0$, $\lambda_2 \geq \lambda_1 \geq 0$, $b \in \mathbb{N}_0$, $q(z) = (1+z)/(1-z)$, $f \in \mathcal{A}$, and $z \in \mathbb{U}$, by applying Theorem 8, we have

$$\begin{aligned} h(z) &= \frac{1+z}{1-z} + \frac{(\lambda_1+\lambda_2)}{1+b} z \left(\frac{1+z}{1-z}\right)' \\ &= \frac{(1+b) + 2(\lambda_1+\lambda_2) z - (1+b) z^2}{(1+b)(1-z)^2}. \end{aligned} \tag{44}$$

By using equality (18) we find that

$$\begin{aligned} &(1+b)\,\mathcal{D}^{1,b}_{\lambda_1,\lambda_2,0} \\ &= (1-(\lambda_1+\lambda_2)+b) \\ &\quad \times \left(f(z) * \varphi^b_{\lambda_2}(z)\right) + (\lambda_1+\lambda_2) z \left(f(z) * \varphi^b_{\lambda_2}(z)\right)'. \end{aligned} \tag{45}$$

Now,

$$\begin{aligned} &(1+b)\left(\mathcal{D}^{1,b}_{\lambda_1,\lambda_2,0} * \varphi^b_{\lambda_2}(z)\right) \\ &= (1-(\lambda_1+\lambda_2)+b)\left(z + \sum_{n=2}^{\infty} \frac{a_n z^n}{(1+\lambda_2(n-1)+b)^2}\right) \\ &\quad + (\lambda_1+\lambda_2)\left(z + \sum_{n=2}^{\infty} \frac{n a_n z^n}{(1+\lambda_2(n-1)+b)^2}\right). \end{aligned} \tag{46}$$

A straightforward calculation gives the following:

$$\begin{aligned} &\left(\mathcal{D}^{1,b}_{\lambda_1,\lambda_2,0} f(z) * \varphi^b_{\lambda_2}(z)\right)' \\ &= 1 + \sum_{n=2}^{\infty} \left(\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{(1+\lambda_2(n-1)+b)^2}\right) n a_n z^{n-1} \\ &= \left(z + \sum_{n=2}^{\infty} \left(\frac{(n(1+(\lambda_1+\lambda_2)(n-1)+b))}{(1+\lambda_2(n-1)+b)^2}\right) a_n z^n\right) \\ &\quad \times (z)^{-1} \\ &= \left(\left(f(z) * \varphi^b_{\lambda_2}(z)\right)\right. \\ &\qquad \left. * \left(z + \sum_{n=2}^{\infty} \left(\frac{n(1+(\lambda_1+\lambda_2)(n-1)+b)}{(1+\lambda_2(n-1)+b)}\right) a_n z^n\right)\right) \\ &\quad \times (z)^{-1}. \end{aligned} \tag{47}$$

Similarly, using (18), we see that

$$\begin{aligned} &(1+b)\,\mathcal{D}^{2,b}_{\lambda_1,\lambda_2,0} f(z) \\ &\quad = (1-(\lambda_1+\lambda_2)+b) \\ &\qquad \times \left(\mathcal{D}^{1,b}_{\lambda_1,\lambda_2,0} f(z) * \varphi^b_{\lambda_2}(z)\right) \\ &\qquad + (\lambda_1+\lambda_2) z \left(\mathcal{D}^{1,b}_{\lambda_1,\lambda_2,0} f(z) * \varphi^b_{\lambda_2}(z)\right)', \end{aligned} \tag{48}$$

and then

$$\begin{aligned} &(1+b)\left(\mathcal{D}^{2,b}_{\lambda_1,\lambda_2,0} f(z)\right)' \\ &\quad = (1+b)\left(\mathcal{D}^{1,b}_{\lambda_1,\lambda_2,0} f(z) * \varphi^b_{\lambda_2}(z)\right)' \\ &\qquad + (\lambda_1+\lambda_2) z \left(\mathcal{D}^{1,b}_{\lambda_1,\lambda_2,0} f(z) * \varphi^b_{\lambda_2}(z)\right)''. \end{aligned} \tag{49}$$

By using (47) we have

$$\begin{aligned} &\left(\mathcal{D}^{1,b}_{\lambda_1,\lambda_2,0} f(z) * \varphi^b_{\lambda_2}(z)\right)'' \\ &\quad = \sum_{n=2}^{\infty} \left(\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{(1+\lambda_2(n-1)+b)^2}\right) n(n-1) a_n z^{n-2}; \end{aligned} \tag{50}$$

we deduce

$$\begin{aligned} &(1+b)\left(\mathcal{D}^{2,b}_{\lambda_1,\lambda_2,0} f(z)\right)' \\ &= (1+b)\left(1 + \sum_{n=2}^{\infty} \left(\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{(1+\lambda_2(n-1)+b)^2}\right) n a_n z^{n-1}\right) \\ &\quad + (\lambda_1+\lambda_2) \\ &\quad \times \sum_{n=2}^{\infty} \left(\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{(1+\lambda_2(n-1)+b)^2}\right) n(n-1) a_n z^{n-1}; \end{aligned} \tag{51}$$

that is,

$$\begin{aligned}&\left(\mathscr{D}_{\lambda_1,\lambda_2,0}^{2,b} f(z)\right)' \\ &= 1 + \sum_{n=2}^{\infty}\left(\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right)^2 n a_n z^{n-1} \\ &= \left(\left(f(z) * \varphi_{\lambda_2}^{b}(z)\right)\right. \\ &\quad \left. * \left(z + \sum_{n=2}^{\infty}\left(\frac{n(1+(\lambda_1+\lambda_2)(n-1)+b)^2}{(1+\lambda_2(n-1)+b)}\right)a_n z^n\right)\right) \\ &\times (z)^{-1}.\end{aligned} \tag{52}$$

From Theorem 8 we get

$$\begin{aligned}&\left(\left(f(z) * \varphi_{\lambda_2}^{b}(z)\right)\right. \\ &\quad \left. * \left(z + \sum_{n=2}^{\infty}\left(\frac{n(1+(\lambda_1+\lambda_2)(n-1)+b)^2}{(1+\lambda_2(n-1)+b)}\right)a_n z^n\right)\right) \times (z)^{-1} \\ &\quad \prec \frac{(1+b)+2(\lambda_1+\lambda_2)z-(1+b)z^2}{(1+b)(1-z)^2},\end{aligned} \tag{53}$$

which implies that

$$\begin{aligned}&\left(\left(f(z) * \varphi_{\lambda_2}^{b}(z)\right)\right. \\ &\quad \left. * \left(z + \sum_{n=2}^{\infty}\left(\frac{n(1+(\lambda_1+\lambda_2)(n-1)+b)}{(1+\lambda_2(n-1)+b)}\right)a_n z^n\right)\right) \times (z)^{-1} \\ &\quad \prec \frac{1+z}{1-z}, \quad (z \in \mathbb{U}).\end{aligned} \tag{54}$$

Theorem 10. *Let q be a convex function in $\mathbb{U}$, with $q(0) = 1$, and let*

$$h(z) = q(z) + zq'(z), \quad (z \in \mathbb{U}). \tag{55}$$

If $\lambda_2 \geq \lambda_1 \geq 0$, $\delta, m, b \in \mathbb{N}_0$, and $f \in \mathscr{A}$ satisfies the differential subordination

$$\left(\mathscr{D}_{\lambda_1,\lambda_2,\delta}^{m,b} f(z)\right)' \prec h(z), \tag{56}$$

then

$$\frac{\mathscr{D}_{\lambda_1,\lambda_2,\delta}^{m,b} f(z)}{z} \prec q(z), \quad (z \in \mathbb{U}), \tag{57}$$

and the result is sharp.

Proof. Let

$$p(z) = \frac{\mathscr{D}_{\lambda_1,\lambda_2,\delta}^{m,b} f(z)}{z}, \quad (z \in \mathbb{U}). \tag{58}$$

Differentiating (58), with respect to z, we obtain

$$\left(\mathscr{D}_{\lambda_1,\lambda_2,\delta}^{m,b} f(z)\right)' = p(z) + zp'(z), \quad (z \in \mathbb{U}). \tag{59}$$

Using (58), the differential subordination (56) becomes

$$p(z) + zp'(z) \prec h(z) = q(z) + zq'(z), \quad (z \in \mathbb{U}). \tag{60}$$

Using Lemma 2, we deduce that

$$p(z) \prec q(z), \quad (z \in \mathbb{U}). \tag{61}$$

By using (58), we have

$$\frac{\mathscr{D}_{\lambda_1,\lambda_2,\delta}^{m,b} f(z)}{z} \prec q(z), \quad (z \in \mathbb{U}). \tag{62}$$

The proof of Theorem 10 is complete. □

Example 11. For $\delta = 0$, $m = 1$, $\lambda_2 \geq \lambda_1 \geq 0$, $b \in \mathbb{N}_0$, $q(z) = 1/(1-z)$, $f \in \mathscr{A}$, and $z \in \mathbb{U}$, from Theorem 10 we obtain

$$h(z) = \frac{1}{1-z} + z\left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2}. \tag{63}$$

From Example 9, we have

$$\begin{aligned}&(1+b)\,\mathscr{D}_{\lambda_1,\lambda_2,0}^{1,b} f(z) \\ &\quad = (1-(\lambda_1+\lambda_2)+b)\left(f(z) * \varphi_{\lambda_2}^{b}\right) \\ &\quad + (\lambda_1+\lambda_2)\,z\left(f(z) * \varphi_{\lambda_2}^{b}\right)',\end{aligned} \tag{64}$$

and then

$$\begin{aligned}(1+b)\left(\mathscr{D}_{\lambda_1,\lambda_2,0}^{1,b} f(z)\right)' &= (1+b)\left(f(z) * \varphi_{\lambda_2}^{b}\right)' \\ &\quad + (\lambda_1+\lambda_2)\,z\left(f(z) * \varphi_{\lambda_2}^{b}\right)''.\end{aligned} \tag{65}$$

From Theorem 10 we deduce that

$$\left(f(z) * \varphi_{\lambda_2}^{b}\right)' + \frac{(\lambda_1+\lambda_2)}{(1+b)} z\left(f(z) * \varphi_{\lambda_2}^{b}\right)'' \prec \frac{1}{(1-z)^2} \tag{66}$$

implies that

$$\begin{aligned}&\frac{(1-(\lambda_1+\lambda_2)+b)\left(f(z) * \varphi_{\lambda_2}^{b}\right) + (\lambda_1+\lambda_2)\,z\left(f(z) * \varphi_{\lambda_2}^{b}\right)'}{z(1+b)} \\ &\quad \prec \frac{1}{1-z}.\end{aligned} \tag{67}$$

Theorem 12. *Let h be a convex function in $\mathbb{U}$, with $h(0) = 1$, $0 \le \alpha < 1$, and let*

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad (z \in \mathbb{U}). \tag{68}$$

If $\lambda_2 \ge \lambda_1 \ge 0$, $\delta, m, b \in \mathbb{N}_0$, and $f \in \mathscr{A}$ satisfies the differential subordination

$$\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)\right)' \prec h(z), \tag{69}$$

then

$$\frac{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)}{z} \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}, \quad (z \in \mathbb{U}). \tag{70}$$

The function q is convex and is the best dominant.

Proof. Let

$$\begin{aligned} p(z) &= \frac{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)}{z} \\ &= \left(z + \sum_{n=2}^{\infty}\left(\frac{(1+(\lambda_1+\lambda_2)(n-1)+b)^m}{(1+\lambda_2(n-1)+b)^m}\right) \times \mathscr{C}(\delta, n)\, a_n z^n\right) \times (z)^{-1} \\ &= 1 + p_1 z + p_2 z^2 + \cdots, \quad (p \in \mathscr{H}[1,1],\ z \in \mathbb{U}). \end{aligned} \tag{71}$$

Differentiating (71), with respect to z, we obtain

$$\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)\right)' = p(z) + zp'(z), \quad (z \in \mathbb{U}). \tag{72}$$

Using (72), the differential subordination (69) becomes

$$p(z) + zp'(z) \prec h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad (z \in \mathbb{U}). \tag{73}$$

Using Lemma 1, we deduce that

$$\begin{aligned} p(z) \prec q(z) &= \frac{1}{z}\int_0^z h(t)\,dt \\ &= \frac{1}{z}\int_0^z \left(\frac{1+(2\alpha-1)t}{1+t}\right) dt \\ &= \frac{1}{z}\left(\int_0^z \frac{1}{1+t}dt + (2\alpha-1)\int_0^z \frac{t}{1+t}dt\right) \\ &= 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \end{aligned} \tag{74}$$

By using (71), we have

$$\frac{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)}{z} \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \tag{75}$$

The proof of Theorem 12 is complete. □

Corollary 13. *If $f \in \mathscr{R}^{m,b}_{\lambda_1,\lambda_2,\delta}(\alpha)$, then*

$$\Re\left(\frac{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)}{z}\right) > (2\alpha - 1) + 2(1-\alpha)\ln 2, \quad (z \in \mathbb{U}). \tag{76}$$

Proof. Since $f \in \mathscr{R}^{m,b}_{\lambda_1,\lambda_2,\delta}(\alpha)$, from Definition 4 we have

$$\Re\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta}(\alpha)\right)' > \alpha, \quad (z \in \mathbb{U}), \tag{77}$$

which is equivalent to

$$\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta}(\alpha)\right)' \prec h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}. \tag{78}$$

Using Theorem 12, we obtain

$$\frac{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)}{z} \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \tag{79}$$

Since q is convex and $q(\mathbb{U})$ is symmetric with respect to the real axis, we have that

$$\Re\left(\frac{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)}{z}\right) > \Re q(1) = (2\alpha - 1) + 2(1-\alpha)\ln 2, \quad (z \in \mathbb{U}). \tag{80}$$

□

Theorem 14. *Let $h \in \mathscr{H}(\mathbb{U})$, with $h(0) = 1$, $h'(0) \ne 0$, which satisfies the inequality*

$$\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad (z \in \mathbb{U}). \tag{81}$$

If $\lambda_2 \ge \lambda_1 \ge 0$, $\delta, m, b \in \mathbb{N}_0$, and $f \in \mathscr{A}$ satisfies the differential subordination

$$\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)\right)' \prec h(z), \tag{82}$$

then

$$\frac{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)\,dt. \tag{83}$$

Proof. Let

$$\begin{aligned} p(z) &= \frac{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)}{z} \\ &= \left(z + \sum_{n=2}^{\infty}\left(\frac{(1+(\lambda_1+\lambda_2)(n-1)+b)^m}{(1+\lambda_2(n-1)+b)^m}\right) \times \mathscr{C}(\delta, n)\, a_n z^n\right) \times (z)^{-1} \\ &= 1 + p_1 z + p_2 z^2 + \cdots, \quad (p \in \mathscr{H}[1,1],\ z \in \mathbb{U}). \end{aligned} \tag{84}$$

Differentiating (84), with respect to z, we obtain

$$\left(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)\right)' = p(z) + zp'(z), \quad (z \in \mathbb{U}). \tag{85}$$

Using (85), the differential subordination (82) becomes

$$p(z) + zp'(z) \prec h(z), \quad (z \in \mathbb{U}). \tag{86}$$

Using Lemma 1, we deduce that

$$p(z) \prec q(z) = \frac{1}{z}\int_0^z h(t)\,dt; \tag{87}$$

by using (84), we have

$$\frac{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)\,dt. \tag{88}$$

From Lemma 3, we see that the function q is convex, and from Lemma 1, q is the best dominant for subordination (82). The proof of Theorem 14 is complete. □

Note that other work related to differential operators and differential subordination can be seen in [8–13].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

Entisar El-Yagubi and Maslina Darus read and approved the final paper.

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