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Towards an entanglement measure for mixed states in CFTs based on relative entropy

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Abstract: Relative entropy of entanglement (REE) is an entanglement measure of bipartite mixed states, defined by the minimum of the relative entropy $S(\rho_{AB}||\sigma_{AB})$ between a given mixed state ρ_{AB} and an arbitrary separable state σ_{AB}. The REE is always bounded by the mutual information $I_{AB} = S(\rho_{AB}||\rho_A \otimes \rho_B)$ because the latter measures not only quantum entanglement but also classical correlations. In this paper we address the question of to what extent REE can be small compared to the mutual information in conformal field theories (CFTs). For this purpose, we perturbatively compute the relative entropy between the vacuum reduced density matrix ρ^0_{AB} on disjoint subsystems $A \cup B$ and arbitrarily separable state σ_{AB} in the limit where two subsystems A and B are well separated, then minimize the relative entropy with respect to the separable states. We argue that the result highly depends on the spectrum of CFT on the subsystems. When we have a few low energy spectrum of operators as in the case where the subsystems consist of finite number of spins in spin chain models, the REE is considerably smaller than the mutual information. However in general our perturbative scheme breaks down, and the REE can be as large as the mutual information.

Keywords: Conformal Field Theory, AdS-CFT Correspondence, Holography and condensed matter physics (AdS/CMT)

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JHEP10(2018)166

Contents

JHEP10(2018)166

1 Introduction and summary

Quantum entanglement is one of the central ideas in modern theoretical physics. It does not only play crucial roles in quantum information theory but also has a broader range of applications, from condensed matter physics to string theory.

When we consider a bipartite pure state $|\Psi\rangle_{AB}$, we call the state does not have any quantum entanglement when it is represented by a direct product state $|\Psi_1\rangle_A \otimes |\Psi_2\rangle_B$. For pure states, the amount of quantum entanglement can correctly be measured by the entanglement entropy (or von Neumann entropy): $S(\rho_A) = S(\rho_B) \equiv -\mathrm{tr}[\rho_A \log \rho_A]$, where $\rho_A \equiv \mathrm{tr}_B|\Psi\rangle\langle\Psi|$ is the reduced density matrix. This is because the entanglement entropy essentially counts the number of Bell pairs which can be distilled from a given pure state $|\Psi\rangle_{AB}$ by local operations and classical communication (LOCC). In LOCC, we can act quantum operations on A and B separately and allow classical communications between A and B at the same time. It is important that the LOCC procedures, which convert a given state into Bell pairs, are reversible for pure states in an asymptotic sense.[1] Namely, after distilling the Bell pairs, one can reproduce the original pure state by performing LOCC on the given Bell pairs. In general, an amount of entanglement quantified by an appropriate entanglement measure has to be always less than the number of Bell pairs necessary to produce a given state by LOCC, and also to be greater than that of Bell pairs distillable from a given state by LOCC. Thus the reversibility guarantees that there is only one measure of quantum entanglement, namely the entanglement entropy [1]. Refer to the reviews [2–6] for studies of entanglement entropy in quantum field theories and holography.

Next let us turn to a bipartite mixed state, which is described by a density matrix ρ_{AB}. A mixed state σ_{AB} has no entanglement if σ_{AB} is separable i.e.

$$\sigma_{AB} = \sum_a p_a \rho_A^a \otimes \rho_B^a, \tag{1.1}$$

where p_a are positive coefficients such that $\sum_a p_a = 1$ and each of $\rho_{A,B}^a$ is a density matrix, which is hermitian and non-negative operator with the unit trace. However, the beautiful story which we find for pure states is missing for mixed states because the LOCC procedures of the conversion between a mixed state and Bell pairs is irreversible in general. Nevertheless, we can define an entanglement measure by a quantity which is monotonically decreasing under LOCC with a few more optional properties such as the asymptotic continuity. We write an entanglement measure for a given bipartite state ρ_{AB} as $E_\#(\rho_{AB})$. Such an entanglement measure is far from unique as is clear from the irreversibility (for entanglement measures of mixed states refer to e.g. [7, 8] for excellent reviews).

So far, few calculations of genuine entanglement measures for mixed states have been performed for quantum field theories. The main reasons for this is that the known entanglement measures, such as the entanglement of formation E_F, the relative entropy of entanglement E_R and the squashed entanglement E_{Sq}, all involve very complicated

[1] Instead of considering a given state itself, one sometimes discusses the procedures on n copies of the original state $\rho_{AB}^{\otimes n}$ followed by the asymptotic ($n \to \infty$) limit. The argument about LOCC reversibility should be correctly taken into account in this regime.

JHEP10(2018)166

minimization procedures. A correlation measure for mixed state, called entanglement of purification [9], involves a slightly simpler minimization procedure, though it is not an entanglement measure. Recently a holographic dual of this quantity has been proposed in [10, 11] and computations of this quantity in field theories and spin chains have been performed in [11, 12] (for more progresses refer to [13–19]). There is another interesting quantity called the logarithmic negativity [20], which does not need any minimizations and thus has been successfully computed in two dimensional CFTs [21–23]. Though this quantity is monotone under LOCC, the asymptotic continuity condition and convexity are not satisfied. Thus it does not coincide with the entanglement entropy $S(\rho_A)$ when the system AB is pure.

The main purpose of this paper is to initiate calculations of a true entanglement measure for mixed state in conformal field theories (CFTs). In particular, we focus on the relative entropy of entanglement $E_R(\rho_{AB})$ [24, 25] among entanglement measures, motivated by recent progresses of computational techniques in CFTs of relative entropies [26–31]. Several bounds for REE in quantum field theories have been obtained in [32, 33] via an algebraic quantum field theory approach[2] (refer to [35] for an excellent review).

The relative entropy of entanglement (REE) is defined as follows. We can measure a distance between two density matrices ρ and σ by the relative entropy:

$$S(\rho||\sigma) = \mathrm{tr}\ \rho\log\rho - \mathrm{tr}\ \rho\log\sigma. \tag{1.2}$$

A basic property of the relative entropy is $S(\rho||\sigma) \geq 0$, where the equality holds iff $\rho = \sigma$.

The REE is defined as the shortest distance in the sense of the relative entropy between a given bipartite state ρ_{AB} and an arbitrary separable state σ_{AB} as follows:

$$E_R(\rho_{AB}) = \inf_{\sigma_{AB}\in \mathrm{Sep}}\ S(\rho_{AB}||\sigma_{AB}), \tag{1.3}$$

where Sep denotes all separable states. It is obvious that $E_R(\rho_{AB}) = 0$ iff ρ_{AB} is separable. Moreover, when ρ_{AB} is pure, $E_R(\rho_{AB})$ coincides with the entanglement entropy $S(\rho_A)$.

In this paper we will study the REE E_R for the vacuum reduced density matrix ρ^0_{AB} of CFTs on two disjoint subsystems $A\cup B(\equiv AB)$ in any dimensions. This REE quantifies how much two subsystems A and B are quantum mechanically entangled in a CFT vacuum. We will analyse the REE assuming the subsystems A and B are far apart in terms of power series of $l/R \ll 1$, where l is the size of A and B, while R is the geometrical distance between A and B.

Another useful measure of correlations between A and B is the mutual information:

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) = S(\rho_{AB}||\rho_A\otimes\rho_B). \tag{1.4}$$

Obviously from the definition of REE, we have the inequality

$$E_R(\rho_{AB}) \leq I(\rho_{AB}). \tag{1.5}$$

[2]In [32], an upper bound of $E_R(\rho_{AB})$ in CFT is given: $E_R(\rho_{AB}) \leq N_O\left(\frac{l}{R}\right)^{2\Delta_O}$, where Δ_O is the conformal dimension of lightest primary operator (except the identity) and N_O is its degeneracy. This follows from Thm 14, Remark 5 of [32]. Note that when $l/R \ll 1$, we can approximate r/R in (235) in [32] by our $(l/R)^2$ via a conformal transformation [34]. Our result in this paper is consistent with this bound and is actually stronger because the REE is at least bounded by the mutual information as in (1.5).

JHEP10(2018)166

This upper bound can also be intuitively understood because the REE measures the amount of quantum entanglement, while the mutual information measures not only quantum entanglement but also classical correlations. When A and B are far apart, the mutual information for a CFT vacuum (its reduced density matrix is written as ρ^0_{AB}) is approximated by the square of vacuum two point function $\langle O_A O_B\rangle$ of the (non-trivial) primary operator O with the lowest conformal dimension Δ (regardless to the positions of operators or the shapes of subsystems):

$$I(\rho^0_{AB}) \simeq (l)^{4\Delta}\frac{\Gamma(\frac{3}{2})\Gamma(2\Delta+1)}{2\Gamma(2\Delta+\frac{3}{2})}\langle O_A O_B\rangle^2 \equiv a_{2\Delta}\left(\frac{l}{R}\right)^{4\Delta}. \tag{1.6}$$

For example, the free massless Dirac fermion CFT in two dimensions corresponds to $\Delta = 1/2$. Thus in our limit $l/R \ll 1$, the REE is at least as small as $(l/R)^{4\Delta}$, as can be seen from its upper bound (1.5). Below we are interested in whether the REE can be much smaller than $(l/R)^{4\Delta}$.

For general mixed states ρ and σ, if $\rho - \sigma$ is very small, the relative entropy becomes symmetric $S(\sigma||\rho) \simeq S(\rho||\sigma)$. Therefore, we will first calculate the relative entropy $S(\sigma_{AB}||\rho_{AB})$ for arbitrary separable density matrices σ_{AB}, and then take the infinitum with respect to the ensemble $\{p_a, \rho^a_A, \rho^a_B\}$. The necessary ingredients for the calculation have been obtained in the previous paper [29] written by the one of the authors, including the vacuum modular Hamiltonian $K_{AB} = -\log\rho_{AB}$ as well as the von Neumann entropy $S(\sigma_{AB})$ for any separable density matrices, assuming $l/R \ll 1$.

In this paper we first compute the contribution of the lightest primary operator to the relative entropy, then minimizing it by assuming it gives the dominant contribution in the large separation limit, as in case of the mutual information. We are able to show that we can make this contribution always vanish by appropriately choosing the separable state at any order of the perturbation. We also give an explanation why the separable state is indistinguishable from ρ^0_{AB} from the viewpoint of local observables.

However, the minimization becomes much more complicated when we include the effects of other operators with higher conformal dimensions. In this case, we find that our perturbative calculation is not enough, since we cannot suppress the expectation value of higher dimensional operators in general.

From these observations we argue that the behavior of REE is highly dependent on the operator spectrum of CFT in the subsystems. For a CFT with few low energy states such as the case where the subsystems consist of finite number of spins in spin chain models, the perturbative analysis is enough and we find that there is tiny quantum entanglement as $E_R(\rho^0_{AB}) \ll I(\rho^0_{AB})$. We can check this statement by having an independent argument in spin chain models.

However, in generic setups our perturbative expansion gets uncontrollable and this implies that the REE can be as large as the mutual information I_{AB}. Especially we expect $E_R(\rho^0_{AB}) \simeq I(\rho^0_{AB})$ for holographic CFTs, as the operator spectrum does not seem to allow us to optimize the minimizations in the definition of REE. On the other hand, since integrable CFTs such as the rational CFTs in two dimensions, have simple operator spectrum

JHEP10(2018)166

and algebra, there might be a chance that the REE can be smaller than the mutual information even when the subsystems are much larger than the lattice spacing. For further investigations, we probably need to develop methods which does not rely on perturbations.

The organization of this paper is as follows: in section 2, we review basic properties of the relative entropy of entanglement. In section 3, after explaining the basic set up, we compute the relative entropy $S(\sigma_{AB}||\rho^0_{AB})$ between the vacuum reduced density matrix ρ^0_{AB} and an arbitrary separable state σ_{AB} in the leading order of the large distance limit $l/R \to 0$, based on results of [29]. In section 4, we minimize the relative entropy with respect to the separable states. We find there alway be a separable state whose relative entropy is vanishing therefore $E_R(\rho^0_{AB}) = 0$ at the quadratic order of perturbative expansions. In section 5 we take into account of higher order perturbative corrections, and argue they do not change our result under certain conditions. In section 6, we discuss the contribution from the next lightest primary, which shows the result of REE is very sensitive to the operator spectrum. In section 7, we will compare our results with other known results and discuss future problems. In the appendix we explain the details of our calculations.

2 Properties of relative entropy of entanglement

The relative entropy of entanglement $E_R(\rho_{AB})$ is defined by (1.3) for a bipartite quantum state ρ_{AB}, i.e. the shortest distance between ρ_{AB} and the set of separable states measured by the relative entropy.

2.1 Properties of REE

The properties of REE is summarized as follows (for more details, refer to [7, 8]):[3]

(i) Faithfulness: $E_R(\rho_{AB}) \geq 0$ and $E_R(\rho_{AB}) = 0$ if and only if ρ_{AB} is separable.

(ii) Monotonicity: $E_R(\rho_{AB})$ is monotonically decreasing under (stochastic) LOCC.

(iii) Convexity: $E_R(\rho_{AB})$ is convex i.e. $E_R(x\rho_{AB} + (1-x)\rho'_{AB}) \leq xE_R(\rho_{AB}) + (1-x)E_R(\rho'_{AB})$ for any $x \in [0,1]$.

(iv) Continuity: $E_R(\rho_{AB})$ is continuous respect to ρ_{AB} i.e. if ρ_{AB} and σ_{AB} are close in trace distance, then the value of $E_R(\rho_{AB})$ approaches that of $E_R(\sigma_{AB})$:[4]

$$||\rho_{AB} - \sigma_{AB}|| \to 0, \text{ then } \frac{|E_R(\rho_{AB}) - E_R(\sigma_{AB})|}{\log \dim \mathcal{H}_{AB}} \to 0, \tag{2.1}$$

where $\mathcal{H}_{AB}$ is the Hilbert space ρ_{AB} and σ_{AB} act on [37].

[3] In this section we deal with the finite dimensional Hilbert space for simplicity. Most of the properties and the inequalities are also proven in the infinite dimensional setup, refer to [32, 36] for recent discussion.

[4] There are many variations of the continuity of entanglement measures. In particular, REE is also asymptotic continuous, which is described by the limit of many copies $\lim_{n\to\infty} \rho^{\otimes n}_{AB}$ and an important property in the axiomatic approach of entanglement measures.

JHEP10(2018)166

(v) Subadditivity: $E_R(\rho_{AB})$ always satisfies the subadditivity $E_R(\rho_{AB}\otimes\rho'_{A'B'}) \leq E_R(\rho_{AB})+E_R(\rho'_{A'B'})$. Note that it does not satisfy the additivity $E_R(\rho_{AB}\otimes\rho'_{A'B'}) = E_R(\rho_{AB}) + E_R(\rho'_{A'B'})$ in general.

(vi) When ρ_{AB} is pure, $E_R(\rho_{AB})$ reduces to the entanglement entropy $S(\rho_A)(=S(\rho_B))$. To see this, consider a pure state $\rho_{AB} = |\psi\rangle\langle\psi|_{AB}$ with the Schmidt decomposition

$$|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}\,|i\rangle_A\,|i\rangle_B\,, \tag{2.2}$$

where $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$. Then it is shown that the closest separable state of ρ_{AB} which reaches the minimization in (1.3) is given by a simple form [25, 38]

$$\sigma_{AB} = \sum_i \lambda_i\,|i\rangle\langle i|_A \otimes |i\rangle\langle i|_B\,. \tag{2.3}$$

Indeed, one can easily check that $S(\rho_{AB}||\sigma_{AB})$ of these states reduces to the entanglement entropy:

$$S(\rho_{AB}||\sigma_{AB}) = -\mathrm{tr}\rho_{AB}\log\sigma_{AB} = -\sum_i \lambda_i\log\lambda_i = S(\rho_A). \tag{2.4}$$

Above properties indicate that REE is a good generalization of entanglement entropy to a genuine entanglement measure for mixed states.

There are several upper/lower bounds for REE: as we have already mentioned, $E_R(\rho_{AB})$ is bounded from above by the mutual information $I(\rho_{AB}) = S(\rho_{AB}||\rho_A\otimes\rho_B)$ as $E_R(\rho_{AB}) \leq I(\rho_{AB})$, which follows directly from the definition of REE. Another upper bound is given the entanglement of formation $E_R(\rho_{AB}) \leq E_F(\rho_{AB})$, which is also a good measure of entanglement for mixed states. On the other hand, a lower bound is given by the distillable entanglement $E_D(\rho_{AB}) \leq E_R(\rho_{AB})$, which counts the number of EPR pairs extractable from a given state ρ_{AB} by LOCC. This bound also leads to an entropic inequality $E_R(\rho_{AB}) \geq \max[S(\rho_A), S(\rho_B)] - S(\rho_{AB})$[5] by virtue of the hashing inequality [39]. It may also be worth noting that there is no generic inequality relationship between REE and the negativity [40].

2.2 Quadratic approximations

In the present paper we will deal with $S(\sigma_{AB}||\rho_{AB})$ rather than $S(\rho_{AB}||\sigma_{AB})$ for technical simplicity, where σ_{AB} represents a separable state. This does not change the main results at the quadratic order of small perturbation of quantum state. Consider the case where ρ and σ are very closed to each other

$$\rho = \sigma + \delta\rho. \tag{2.5}$$

If we expand $S(\rho||\sigma)$ up to the quadratic order of $\delta\rho$, we find (see e.g. [30])

$$S(\rho||\sigma) = \frac{1}{2}\mathrm{tr}\left[\delta\rho\,\frac{d}{dx}\log(\sigma+x\delta\rho)\Big|_{x=0}\right] + O(\delta\rho^3). \tag{2.6}$$

[5]This inequality can be rewritten in terms of conditional entropy $S(B|A) = S(\rho_{AB}) - S(\rho_A)$ as $E_R(\rho_{AB}) \geq \max[-S(A|B), -S(B|A)]$, which was firstly derived in [38].

JHEP10(2018)166

From this expression, it is clear that $S(\rho||\sigma)$ coincides with the reversed one $S(\sigma||\rho)$ up to the quadratic order

$$S(\rho||\sigma) - S(\sigma||\rho) = O(\delta\rho^3). \tag{2.7}$$

One can also understand this symmetry as a consequence from positivity and non-degeneracy of the relative entropy.

As an illustration, consider the case where σ and ρ are 2×2 density matrices, expressed as:

$$\sigma = \begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}, \ \rho = \begin{pmatrix} \alpha+\epsilon & \delta_1 + i\delta_2 \\ \delta_1 - i\delta_2 & 1-\alpha-\epsilon \end{pmatrix}, \tag{2.8}$$

and treat δ_1 and δ_2 as infinitesimally small real parameters. We require $0 < \alpha < 1$ for positivity of density matrix. If we only keep up to quadratic terms of them, we can confirm the equivalence (2.7) explicitly as follows:

$$S(\rho||\sigma) = S(\sigma||\rho) = \frac{\epsilon^2}{2\alpha(1-\alpha)} + \frac{\log\frac{1-\alpha}{\alpha}}{1-2\alpha}(\delta_1^2 + \delta_2^2). \tag{2.9}$$

In [41], an entanglement measure so-called the reversed REE was introduced in the same spirit of REE with reversed components:

$$E_{RR}(\rho_{AB}) = \inf_{\sigma_{AB}\in \text{Sep,LI}} \ S(\sigma_{AB}||\rho_{AB}), \tag{2.10}$$

where the minimization is restricted to a class of separable states locally identical to ρ_{AB} i.e. $\text{tr}_B(\sigma_{AB}) = \rho_A$, $\text{tr}_A(\sigma_{AB}) = \rho_B$. This quantity also satisfies many properties of a good entanglement measure, especially the additivity. However, when ρ_{AB} is pure, $E_{RR}(\rho_{AB})$ generically diverges (or trivially vanishes) and thus it can not be regarded as an appropriate generalization of entanglement entropy for mixed states.

3 The calculation of the relative entropy

3.1 Set up

We begin with a CFT on a d dimensional flat space $\mathbb{R}^d$, and two ball shaped regions A and B, with the radius l and the distance R. In this section we estimate the relative entropy $S(\sigma_{AB}||\rho^0_{AB})$ between the vacuum reduced density matrix on $A \cup B$ defined by,

$$\rho^0_{AB} = \text{tr}_{(AB)^c}|0\rangle\langle 0| \tag{3.1}$$

and an arbitrary separable density matrix σ_{AB}, in the large distance limit $l/R \to 0$.[6]

[6]Precisely speaking, in the actual computation we regard this set up as a particular limit of the system on a cylinder $\mathbb{R} \times S^{d-1}$. Let L be the radius of the spacial sphere S^{d-1}, then the large distance limit in $\mathbb{R}$ is equivalent to the double scaling limit on the cylinder,

$$\frac{l}{L} \to 0, \quad \frac{l}{R} \to 0. \tag{3.2}$$

JHEP10(2018)166

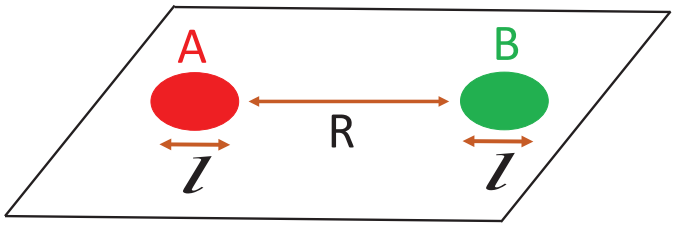

Figure 1. The choice of subsystem A and B to define the REE $E_R(\rho_{AB})$.

It is convenient to split the relative entropy into two parts:

$$S(\sigma_{AB}||\rho^0_{AB}) = -S(\sigma_{AB}) + \text{tr}\, \sigma_{AB} K^0_{AB}, \tag{3.3}$$

where $S(\sigma_{AB})$ is the von Neumann entropy of the separable density matrix and K_{AB} is the modular Hamiltonian of ρ^0_{AB},

$$K^0_{AB} = -\log \rho^0_{AB}. \tag{3.4}$$

3.2 The calculation of $S(\sigma_{AB})$

In this subsection we explain how to compute the von Neumann entropy, $S(\sigma_{AB})$ for a separable state σ_{AB}. This is a slight generalization of the previous calculation done in [28, 29]. Here we only outline the calculation, and leave details in appendix A.

For this purpose, we employ the usual replica trick,

$$S(\sigma_{AB}) = \lim_{n\to 1} \frac{1}{1-n} \log \text{tr}\, \sigma^n_{AB}. \tag{3.5}$$

This Rényi entropy can be expanded as

$$\begin{aligned} \text{tr}\, \sigma^n_{AB} &= \sum_{\{a_k\}} \prod_{k=0}^{n-1} p_{a_k} \text{tr}\left[\left(\rho^{a_1}_A \otimes \rho^{a_1}_B\right) \cdots \left(\rho^{a_n}_A \otimes \rho^{a_n}_B\right)\right] \\ &= \sum_{\{a_k\}} \prod_{k=0}^{n-1} p_{a_k} \text{tr}\left[\rho^{a_1}_A \cdots \rho^{a_n}_A\right] \text{tr}\left[\rho^{a_1}_B \cdots \rho^{a_n}_B\right]. \end{aligned} \tag{3.6}$$

We first compute the right hand side of (3.6) for reduced density matrices of global excitations, $|X_a\rangle$, $|Y_a\rangle$ ($a = 0$ corresponds to the vacuum: $|X_0\rangle = |Y_0\rangle = |0\rangle$)

$$\rho^a_A = \text{tr}_{A^c} |X_a\rangle\langle X_a|, \quad \rho^a_B = \text{tr}_{B^c} |Y_a\rangle\langle Y_a|, \tag{3.7}$$

on cylinder $\mathbb{R} \times S^{d-1}$ with the metric,

$$ds^2 = dt^2 + d\theta^2 + \sin^2\theta d\Omega^2_{d-2}. \tag{3.8}$$

JHEP10(2018)166

We then read off the result for arbitrary ρ_A^a, ρ_B^a from it. We take both subsystems A, B to be isomorphic to the ball shaped region on the spatial sphere S^{d-1},

$$A, B : [0, l/2] \times S^{d-2}. \tag{3.9}$$

Also it is important to notice that in this calculation we do not need to specify the distance between two regions.

State operator correspondence allows us to write the quantities in the right hand side in terms of the 2n point correlation functions on the covering space $\Sigma_n = S_n^1 \times H^{d-1}$ [28],

$$\mathrm{tr}[\rho_A^{a_1} \cdots \rho_A^{a_n}] = \frac{\langle \prod_{k=0}^{n-1} X_{a_k}(w_k) X_{a_k}(\hat{w}_k) \rangle_{\Sigma_n}}{\prod_{k=0}^{n-1} \langle X_{a_k}(w_0) X_{a_k}(\hat{w}_0) \rangle_{\Sigma_1}} \cdot \frac{Z_A^{(n)}}{(Z_A^{(1)})^n}, \tag{3.10}$$

where $X_{a_k}(w_k)$ is the local operator corresponding to the global state $|X_{a_k}\rangle$ and there is a similar relation for the subsystem B and the global state $|Y_{a_k}\rangle$; also $Z_A^{(n)}$ denotes the vacuum partition function on Σ_n. The correlation functions are normalized such that $\langle 1 \rangle_{\Sigma_n} = 1$.

The covering space Σ_n is equipped with the metric,

$$ds^2_{\Sigma_n} = d\tau^2 + du^2 + \sinh^2 u d\Omega^2_{d-2}, \quad \tau \sim \tau + 2\pi n, \tag{3.11}$$

and the locations of the local operators are given by

$$w_k : (\tau_k, u_k) = \left(2\pi \left(k + \frac{1}{2} \right) + \frac{l}{2}, 0 \right), \quad \hat{w}_k : (\tau_k, u_k) = \left(2\pi \left(k + \frac{1}{2} \right) - \frac{l}{2}, 0 \right). \tag{3.12}$$

The small subsystem size limit $l \to 0$ corresponds to choose the particular channel $w_k \to \hat{w}_k$ of these correlation functions. There one can expand them by OPE. By picking up the contribution of the lightest primary operator O with the conformal dimension Δ. By taking the analytic continuation $n \to 1$ of the Rényi entropy, we finally obtain[7]

$$\begin{aligned} -S(\sigma_{AB}) = & -\sum_a p_a \left(\langle K_A^0 \rho_A^a \rangle + \langle K_B^0 \rho_B^a \rangle \right) \\ & + a_\Delta \, (l)^{2\Delta} \left[\left(\sum_a p_a \langle \rho_A^a O \rangle \right)^2 + \left(\sum_a p_a \langle \rho_B^a O \rangle \right)^2 \right] \\ & - C_{OOO} b_\Delta l^{3\Delta} \left[\left(\sum_a p_a \langle \rho_A^a O \rangle \right)^3 + \left(\sum_a p_a \langle \rho_B^a O \rangle \right)^3 \right] \\ & + a_{2\Delta} \, (l)^{4\Delta} \left[\sum_a p_a \langle \rho_A^a O_A \rangle \langle \rho_B^a O_B \rangle - \left(\sum_a p_a \langle \rho_A^a O \rangle \right) \left(\sum_a p_a \langle \rho_B^a O \rangle \right) \right]^2 \end{aligned} \tag{3.14}$$

[7] We choose the $a = 0$ component to be reduced density matrices of the vacuum, ie

$$\rho_A^0 = \mathrm{tr}_{\bar{A}} |0\rangle\langle 0|, \quad \rho_B^0 = \mathrm{tr}_{\bar{B}} |0\rangle\langle 0| \, . \tag{3.13}$$

JHEP10(2018)166

where K_A^0 is the vacuum modular Hamiltonian on the region A. In a CFT vacuum on a ball shaped region, K_A^0 is given by a simple integral of stress tensor. We do not need its precise form, as it is always canceled with other contributions in the relative entropies.

Meanwhile, the von Neumann entropy of a reduced density matrix ρ_A on the single subsystem A is given by (see for example [28])

$$S(\rho_A) = \mathrm{tr}\left[K_A^0 \rho_A\right] - a_\Delta l_A^{2\Delta}\, \mathrm{tr}\left[\rho_A O\right]^2 + C_{OOO} b_\Delta (l_A)^{3\Delta} \mathrm{tr}\left[\rho_A O\right]^3 + \cdots \tag{3.15}$$

with

$$a_\Delta = \frac{\Gamma(\frac{3}{2})\Gamma(\Delta+1)}{2\Gamma(\Delta+\frac{3}{2})}, \quad b_\Delta = \frac{2\sqrt{\pi}}{3\Gamma(\frac{3\Delta+3}{2})}, \tag{3.16}$$

and C_{OOO} is the OPE coefficient of the primary O.

Our result indicates the von Neumann entropy of σ_{AB} gets factorized

$$S(\sigma_{AB}) = S(\sigma_A) + S(\sigma_B), \quad \sigma_A = \sum_a p_a \rho_A^a,\ \sigma_B = \sum_a p_a \rho_B^a \tag{3.17}$$

up to $l^{3\Delta}$ order, and the effect of the classical correlation first enters at $l^{4\Delta}$ order. If we write the correlation part in terms of original separable density matrix σ_{AB}

$$S(\sigma_A) + S(\sigma_B) - S(\sigma_{AB}) = a_{2\Delta}\,(l)^{4\Delta}\left[\langle \sigma_{AB} O_A O_B\rangle - \langle \sigma_A O_A\rangle\langle \sigma_B O_B\rangle\right]^2, \tag{3.18}$$

therefore this part is basically the square of the connected part of the two point function $\langle O_A O_B\rangle$ evaluated on σ_{AB}.

This can be compared with the mutual information $I_{AB}(\rho_{AB}^0)$ of a reduced density matrix ρ_{AB}^0 at this $l^{4\Delta}$ order [29],

$$I_{AB}(\rho_{AB}^0) = a_{2\Delta}\,(l)^{4\Delta}\left[\langle \rho_{AB} O_A O_B\rangle - \langle \rho_A O_A\rangle\langle \rho_B O_B\rangle\right]^2 = a_{2\Delta}\left(\frac{l}{R}\right)^{4\Delta}, \tag{3.19}$$

and the two results are related by the exchange $\sigma_{AB} \leftrightarrow \rho_{AB}^0$. Indeed, as is clear from the discussion in the appendix B, the derivations of the two results are identical to each other, once we identify the two correlation functions $\langle \sigma_{AB} O_A O_B\rangle \leftrightarrow \langle \rho_{AB}^0 O_A O_B\rangle$.

3.3 Modular Hamiltonian and calculation of tr $\sigma_{AB} K_{AB}^0$

Having calculated the von Neumann entropy part, let us move on to the modular Hamiltonian part,

$$\mathrm{tr}\ \sigma_{AB} K_{AB}^0, \quad K_{AB}^0 = -\log \rho_{AB}^0. \tag{3.20}$$

It was shown in [29], K_{AB} takes following form,

$$K_{AB}^0 = K_A^0 + K_B^0 + \tilde{K}_{AB}^0, \tag{3.21}$$

and in the large distance limit $\frac{l}{R} \to 0$, we have

$$\tilde{K}_{AB}^0 = -2a_{2\Delta}\, l^{4\Delta} \langle O_A O_B\rangle O_A O_B + I_{AB}. \tag{3.22}$$

JHEP10(2018)166

This was obtained by starting from the expression of von Neumann entropy $S(\rho_{AB})$ for a generic state ρ_{AB} which is related to the mutual information (3.19), and applying the "first law trick", which will be reviewed in section 5. More details of the discussion can be again found in [29]. I_{AB} in (3.22) denotes the constant part of the modular Hamiltonian. We need this part in order to make sure the relation

$$S_{AB} = \langle \rho_{AB} K_{AB} \rangle \tag{3.23}$$

and I_{AB} coincides with the value of the vacuum mutual information (1.6). Then,

$$\begin{aligned} \mathrm{tr}\left[\sigma_{AB} K^0_{AB}\right] = & \sum_a p_a \left[\langle \rho^a_A K^0_A \rangle + \langle \rho^a_B K^0_B \rangle\right] \\ & - 2a_{2\Delta} \left(\frac{l_A}{R}\right)^{4\Delta} \sum_a p_a \left[\langle \rho^a_A O_A \rangle \langle \rho^a_B O_B \rangle\right] + I_{AB}. \end{aligned} \tag{3.24}$$

3.4 Net result

Combining (3.14) (3.24), the relative entropy we would like to minimize is

$$\begin{aligned} S(\sigma_{AB}||\rho_{AB}) = & \, a_\Delta \, (l)^{2\Delta} \left[\left(\sum_a p_a \langle \rho^a_A O \rangle \right)^2 + \left(\sum_a p_a \langle \rho^a_B O \rangle \right)^2 \right] \\ & - C_{OOO} b_\Delta l^{3\Delta} \left[\left(\sum_a p_a \langle \rho^a_A O \rangle \right)^3 + \left(\sum_a p_a \langle \rho^a_B O \rangle \right)^3 \right] \\ & + a_{2\Delta} \, (l)^{4\Delta} \left[\sum_a p_a \langle \rho^a_A O_A \rangle \langle \rho^a_B O_B \rangle - \left(\sum_a p_a \langle \rho^a_A O \rangle \right) \left(\sum_a p_a \langle \rho^a_B O \rangle \right) \right]^2 \\ & - 2a_{2\Delta} (l)^{2\Delta} \left(\frac{l}{R}\right)^{2\Delta} \sum_a p_a \left[\langle \rho^a_A O_A \rangle \langle \rho^a_B O_B \rangle\right] + I_{AB}. \end{aligned} \tag{3.25}$$

Notice that there are higher order corrections. We will discuss on this in section 5.

4 Minimization

In the previous section we computed the relative entropy $S(\sigma_{AB}||\rho^0_{AB})$ between the vacuum reduced density matrix and an arbitrary separable density matrix σ_{AB} in the large distance limit $\frac{l}{R} \to 0$ keeping only the contributions from the lightest primary operator. In this section, we would like to find the separable density matrix that minimizes the relative entropy and compute the relative entropy of entanglement $E_R(\rho^0_{AB})$. We choose the separable state σ_{AB} to be in the form:

$$\sigma_{AB} = (1-\varepsilon)\rho^0_A \otimes \rho^0_B + \varepsilon \rho^1_A \otimes \rho^1_B, \tag{4.1}$$

where ϵ is a small parameter and $\rho^0_A = \mathrm{tr}_B \rho^0_{AB}$. In addition, $\rho^1_{A,B}$ are arbitrary density matrices with non-vanishing one-point function of the primary O, which is defined to be

$$\mathrm{tr}[\rho^1_A O_A] = \mathrm{tr}[\rho^1_B O_B] = l^{-\Delta} x, \quad (x > 0). \tag{4.2}$$

JHEP10(2018)166

We would like to keep only quadratic perturbations to $S(\sigma_{AB}||\rho^0_{AB})$ so that we have $S(\sigma_{AB}||\rho^0_{AB}) \simeq S(\rho^0_{AB}||\sigma_{AB})$ as in (2.7). To implement this, we define the small perturbations $\delta\rho^0$ and $\delta\rho^1$ by

$$\delta\rho^0 = \rho^0_{AB} - \rho^0_A \otimes \rho^0_B, \quad \delta\rho^1 = \epsilon(\rho^1_A \otimes \rho^1_B - \rho^0_A \otimes \rho^0_B), \tag{4.3}$$

such that

$$\rho^0_{AB} - \sigma_{AB} = \delta\rho^0 - \delta\rho^1. \tag{4.4}$$

Our perturbations are parameterized by the following two small parameters:

$$\begin{aligned} W &\equiv l^{2\Delta}\mathrm{tr}[\delta\rho^0 O_A O_B] = l^{2\Delta}\langle O_A O_B\rangle = \left(\frac{l}{R}\right)^{2\Delta} \ll 1, \\ Z &\equiv l^{2\Delta}\mathrm{tr}[\delta\rho^1 O_A O_B] = l^{2\Delta}\sum_a p_a(\mathrm{tr}\rho^a_A O_A)(\mathrm{tr}\rho^a_B O_B) = \epsilon x^2 \ll 1. \end{aligned} \tag{4.5}$$

It will be useful to note that the mutual information (3.19) when A and B are far apart is at the quadratic order. Indeed, we have

$$I(\rho^0_{AB}) = S(\rho^0_{AB}|\rho^0_A \otimes \rho^0_B) \simeq S(\rho^0_A \otimes \rho^0_B|\rho^0_{AB}) \simeq a_{2\Delta}W^2. \tag{4.6}$$

In this parametrization, our result in the small interval expansion (3.25) is expresses as follows up to the quadratic order of Z and W:

$$S(\sigma_{AB}||\rho^0_{AB}) = \left(a_{2\Delta} + \frac{2a_\Delta}{x^2}\right)Z^2 - 2a_{2\Delta}ZW + a_{2\Delta}W^2. \tag{4.7}$$

By varying Z (or equally ϵ) to minimize the relative entropy, we obtain

$$\mathrm{Min}_Z\left[S(\sigma_{AB}||\rho^0_{AB})\right] = \left(\frac{2a_\Delta a_{2\Delta}}{2a_\Delta + a_{2\Delta}x^2}\right)W^2, \tag{4.8}$$

at $Z = \frac{a_{2\Delta}x^2}{2a_\Delta + a_{2\Delta}x^2}W$.

Next we vary the choice of the state $\rho^1_{A,B}$ so that the one-point function (4.2) gets larger such that $Z = \epsilon x^2$ is still very small. It is obvious that we can define such a state with an arbitrary large x in the continuous limit of field theories. In the limit,

$$x \to \infty, \quad \epsilon \to 0, \quad \text{with} \quad \epsilon x^2 \simeq \left(\frac{l}{R}\right)^{2\Delta} \ll 1, \tag{4.9}$$

we find that the infimum of the relative entropy is vanishing

$$\mathrm{inf}_{Z,x}\left[S(\sigma_{AB}||\rho^0_{AB})\right] = 0, \tag{4.10}$$

up to the quadratic order. Note that at this infimum, the separable state is locally vacuum on the region A and B, i.e. $\mathrm{tr}_{A,B}\sigma_{AB} = \mathrm{tr}_{A,B}\rho_{AB}$.

Finally, by employing the relation (2.7) up to the quadratic order of our perturbation (4.4), we obtain the estimation of REE:

$$E_R(\rho^0_{AB}) = 0 \cdot \left(\frac{l}{R}\right)^{4\Delta} + \text{higher orders of } (l/R). \tag{4.11}$$

JHEP10(2018)166

This manifestly shows that the REE is much smaller than the mutual information

$$\frac{E_R(\rho^0_{AB})}{I(\rho^0_{AB})} \to 0, \tag{4.12}$$

in the limit $(l/R) \to 0$ where A and B are far apart. However, notice again that in this calculation we only keep contributions from the lightest primary operator.

4.1 An interpretation

There is an intuitive way to understand why the separable density matrix σ_{AB} is indistinguishable from the vacuum reduced density matrix ρ^0_{AB}.

It is useful to write the separable density matrix,

$$\sigma_{AB} = \lim_{x\to\infty}\left[\left(1 - \frac{l^{2\Delta}\langle O_A O_B\rangle}{x^2}\right)\rho^0_A\rho^0_B + \frac{l^{2\Delta}\langle O_A O_B\rangle}{x^2}\rho^1_A\rho^1_B\right]. \tag{4.13}$$

Notice that this separable density matrix σ_{AB} reproduces all correlation functions of ρ_{AB} on the disjoint region $A \cup B$, as it should be. In our small subsystem limit, if we truncate the spectrum to the lightest primary operator, we only need to reproduce one and two point functions of $\{1, O\}$:

$$\mathrm{tr}\left[\rho^0_{AB}O_A O_B\right], \quad \mathrm{tr}\left[\rho^0_A O_A\right] = \mathrm{tr}\left[\rho^0_B O_B\right] = 0. \tag{4.14}$$

We can easily see that this is indeed the case,

$$\mathrm{tr}\left[\rho^0_{AB}O_A O_B\right] = \mathrm{tr}\left[\sigma_{AB}O_A O_B\right], \quad \left[\sigma_A O_A\right] = \mathrm{tr}\left[\sigma_B O_B\right] = 0. \tag{4.15}$$

As we will see in the final section, this result corresponds to a critical spin chain example where the subsystem A and B consist of finite number of spins.

Furthermore, this observation makes it clear that for m disjoint subsystems $A_1 \cup \cdots A_m$ the separable density matrix which minimize the analogous relative entropy is given by

$$\sigma_{A_1,\cdots A_m} = \left(1 - \sum_{k=1}^{m}\sum_{\{i_1\cdots i_k\}} P^{(k)}_{\{i_1\cdots i_k\}}\right)\rho^0_{A_1}\cdots\rho^0_{A_n} + \sum_{k=1}^{m}\sum_{\{i_1\cdots i_k\}} P^{(k)}_{\{i_1\cdots i_k\}}\rho_{i_1,i_2\cdots i_k} \tag{4.16}$$

with

$$P^{(k)}_{\{i_1\cdots i_k\}} = \lim_{x\to\infty}\frac{l^{k\Delta}\langle O_{A_{i_1}}O_{A_{i_2}}\cdots O_{A_{i_n}}\rangle}{x^k}, \quad \rho_{i_1\cdots i_k} = \rho^0_{A_1}\cdots\rho^1_{A_{i_1}}\cdots\rho^1_{A_{i_k}}\cdots\rho^0_{A_{i_n}}. \tag{4.17}$$

One can easily see that the density matrix reproduce all k ($\leq m$)point functions of O

4.2 An example of the separable density matrix σ_{AB} in 2d CFT

One can indeed construct a one parameter family of density matrices $\{\rho_\beta\}$, $\beta \to 0$ of which realizes the infimum in a class of two dimensional conformal field theory. Suppose that the lightest primary operator of the 2d CFT in question is the stress tensor $O = T_{zz}$. The we can take ρ^1_A defined by

$$\rho^1_A = \mathrm{tr}_{A^c}|\psi_\beta\rangle\langle\psi_\beta|, \quad \psi_\beta = \frac{e^{-\beta H}}{\sqrt{N}}|B\rangle \tag{4.18}$$

JHEP10(2018)166

where $|B\rangle$ is a boundary state of the CFT, and N is the normalization factor. Then its stress tensor expectation value is

$$x_\beta = l^2 \langle \psi_\beta | T_{zz} | \psi_\beta \rangle = \frac{cl^2}{24\beta^2}, \tag{4.19}$$

and $x_\beta \to \infty$ when $\beta \to 0$.

This implies that if we define ρ_β by

$$\rho_\beta = (1 - \varepsilon_0(x_\beta))\rho_A^0 \otimes \rho_B^0 + \varepsilon_0(x_\beta)\rho_A^1 \otimes \rho_B^1, \tag{4.20}$$

then the density matrix,

$$\sigma_{AB} \equiv \lim_{\beta \to 0} \rho_\beta \tag{4.21}$$

is indistinguishable from the vacuum reduced density matrix ρ_{AB}^0, at least in the $\left(\frac{l}{R}\right)^8$ order.

If we consider a discretized lattice model such as spin chains and introduce the lattice spacing a, then the minimum possible value of the parameter β is $O(a)$. In more general, we expect that for a generic operator with the dimension Δ, the maximal value of x will behave like

$$x_{\max} \sim \left(\frac{l}{a}\right)^\Delta. \tag{4.22}$$

5 Next leading order

In the previous section we found the relative entropy of entanglement $E_R(\rho_{AB}^0)$ is vanishing up to $\left(\frac{l}{R}\right)^{4\Delta}$ order. It is natural to ask whether higher order corrections can modify this result or not. Motivated by this question, in this section we compute $S(\sigma_{AB}||\rho_{AB}^0)$ up to $\left(\frac{l}{R}\right)^{6\Delta}$ by again assuming the lightest primary plays still a dominant role at this order. We also use the fact that the one point functions of the separable state σ_{AB} must be vanishing,

$$\mathrm{tr}\,[\sigma_{AB} O_A] = \mathrm{tr}\,[\sigma_{AB} O_B] = 0, \tag{5.1}$$

in order to reproduce the vacuum one point functions. Restricting σ_{AB} to be in this class of states drastically simplifies the computation below. Notice that from (3.15) this in particular implies that

$$S\left(\sum_a p_a \rho_A^a\right) = S(\rho_A^0), \quad S\left(\sum_a p_a \rho_B^a\right) = S(\rho_B^0). \tag{5.2}$$

5.1 $S(\sigma_{AB})$

The von Neumann entropy $S(\sigma_{AB})$ can be computed along the line of section 3.2 by further expanding the correlator (3.6), in particular allowing 3 O s to propagate in the internal lines of it. The final result of the cubic order is given by (see appendix A for more details):

$$S(\sigma_{AB})\Big|_{l^{6\Delta}} = (l)^{6\Delta} \left(\sum_a p_a \langle \rho^a O \rangle^2\right)^3 C_{OOO}^2 \frac{\Gamma(\frac{1+2\Delta}{2})^3}{12\pi\Gamma(\frac{3+6\Delta}{2})}, \tag{5.3}$$

JHEP10(2018)166

and we can write

$$S(\sigma_{AB})\Big|_{l^{6\Delta}} = \left(d_\Delta C_{OOO}^2\right) Z^3, \tag{5.4}$$

where $d_\Delta \equiv 2^{6\Delta} \frac{\Gamma(\frac{1+2\Delta}{2})^3}{12\pi\Gamma(\frac{3+6\Delta}{2})}$.

5.2 tr $\sigma_{AB} K^0_{AB}$

Next let us compute the expectation value of the modular Hamiltonian at this order. First of all, the von Neumann entropy of a reduced density matrix ρ_{AB} satisfying the locally vacuum condition (5.1), (5.2) (but not necessary a separable state) is directly related to its mutual information,

$$S(\rho_{AB}) = S(\rho^0_A) + S(\rho^0_B) - I_{AB}(\rho_{AB}), \tag{5.5}$$

where $\rho^0_{A,B}$ is the vacuum reduced density matrix on the region A, B respectively.

This mutual information can be computed either directly by a correlator with twist operators in the replica trick or indirectly from $S(\sigma_{AB})$ by the replacement in (5.4)[8]

$$I_{AB}(\rho_{AB})\Big|_{l^{6\Delta}} = -\left(d_\Delta C_{OOO}^2\right) W(\rho_{AB})^3, \tag{5.6}$$

where $W(\rho_{AB}) = l^{2\Delta}\mathrm{tr}[\rho_{AB} O_A O_B]$.

We can use this expression of mutual information for ρ_{AB} satisfying the locally vacuum condition to read off the form of vacuum modular Hamiltonian K^0_{AB} at $l^{6\Delta}$ order, by using the first law trick. Imagine starting from the vacuum reduced density matrix ρ^0_{AB} , and slightly deform it $\rho^0_{AB} \to \rho_{AB} = \rho^0_{AB} + \delta\rho^0_{AB}$, then the value of mutual information $I(\rho_{AB})$ as well as entanglement entropy $I(\rho_{AB})$ are changed by the deformation. In particular the first order change satisfies the first law. If we know the form of $S(\rho_{AB})$ for any ρ_{AB}, we can read off the form of modular Hamiltonian from the above equation. In our current case it goes like,

$$\delta S\Big|_{l^{6\Delta}} = -\delta I_{AB}\Big|_{l^{6\Delta}} = +3\left(d_\Delta C_{OOO}^2\right) W^2 \mathrm{tr}\left[\delta\rho_{AB} O_A O_B\right], \tag{5.7}$$

with $W = W(\rho^0_{AB})$. Since this is true for any $\delta\rho_{AB}$ satisfying the locally vacuum condition, we derive the form of modular Hamiltonian at this order

$$K^0_{AB}\Big|_{l^{6\Delta}} = +3\left(d_\Delta C_{OOO}^2\right) W(\rho^0_{AB})^2 O_A O_B + a_{AB}, \tag{5.8}$$

where a_{AB} is the constant part of the modular Hamiltonian, fixed by the relation $S(\rho^0_{AB}) = \mathrm{tr}\left[\rho^0_{AB} K^0_{AB}\right]$. In this case,

$$a_{AB} = -2\left(d_\Delta C_{OOO}^2\right) W^3. \tag{5.9}$$

By plugging these expressions, we get

$$\mathrm{tr}\,\sigma_{AB} K^0_{AB}\Big|_{l^{6\Delta}} = (d_\Delta C_{OOO}^2)(3ZW^2 - 2W^3) \tag{5.10}$$

[8]For the detail of this replacement, see appendix B.

JHEP10(2018)166

Again notice that the form of $S(\rho_{AB})$ is not generic, and valid only when ρ_{AB} satisfies the locally vacuum condition. Therefore the form of modular Hamiltonian we derive from the expression is only true when it is acted on the space of reduced density matrix satisfying the condition. However it is sufficient for our purpose of computing the expectation value of vacuum modular Hamiltonian with respect to a separable σ_{AB} which satisfies the condition.

A more rigorous argument is as follows. Again consider the change of the density matrix $\rho^0_{AB} \to \sigma_{AB} = \rho^0_{AB} + \delta\rho^0_{AB}$, then

$$\begin{aligned}\delta S_{AB} &\equiv \mathrm{tr}\left[K^0_{AB}(\sigma_{AB} - \rho^0_{AB})\right] + O(\delta\rho^2) \\ &= 3(l)^{6\Delta}\mathrm{tr}\left[(\sigma_{AB} - \rho^0_{AB})O_A O_B\right] W^2(d_\Delta C^2_{OOO}) + O(\delta\rho^2), \\ &= 3d_\Delta C^2_{OOO} W^2(Z - W) + O(\delta\rho^2). \end{aligned} \tag{5.11}$$

From this we can read off the value which we want as follows

$$\begin{aligned}\mathrm{tr}\ \sigma_{AB}K^0_{AB} &= \mathrm{tr}\left[K^0_{AB}(\sigma_{AB} - \rho^0_{AB})\right] + \mathrm{tr}\ \rho^0_{AB}K^0_{AB} \\ &= (d_\Delta C^2_{OOO})(3ZW^2 - 2W^3) \end{aligned} \tag{5.12}$$

in the derivation we do not need to use the precise form of the modular Hamiltonian.

5.3 Minimization

Combining these results, (5.4) and (5.12), we obtain the expression of relative entropy up to this order $l^{6\Delta}$

$$\begin{aligned}S(\sigma_{AB}||\rho^0_{AB}) &= a_{2\Delta}(W^2 - 2WZ + Z^2) \\ &\quad - d_\Delta C^2_{OOO}(2W^3 - 3W^2Z + Z^3). \end{aligned} \tag{5.13}$$

This function again has a minima at $Z = W$, where $S(\sigma_{AB}||\rho^0_{AB})$ is vanishing.

One may worry that this relative entropy negatively diverges in $Z \to \infty$ limit. Of course this is just an artifact of our truncation the perturbative expansion, and the local minima $Z = W$ should be the global minima, as is clear from the argument found in section 4.1.

As long as we assume that only the primary operator O is relevant, the above argument of vanishing $S(\sigma_{AB}||\rho^0_{AB})$ at $Z = W$ continues to be true in all orders in the perturbative expansion with respect to Z and W. First, in this expansion the von Neumann entropy $S(\sigma_{AB})$ is expressed as

$$S(\sigma_{AB}) = \sum_n b_n Z^n, \tag{5.14}$$

where b_n s are unknown coefficients depending on Δ and C_{OOO}, though we do not need their precise values in the argument below. The modular Hamiltonian expectation value $\mathrm{tr}\ \sigma_{AB}K^0_{AB}$ can again be read off from the mutual information of locally vacuum state, which is related to (5.14) by replacing Z to the corresponding two point function,

$$\mathrm{tr}\ \sigma_{AB}K^0_{AB} = \sum_n b_n\left[nW^{n-1}Z - (n-1)W^n\right] \tag{5.15}$$

JHEP10(2018)166

Finally the relative entropy is given by

$$S(\sigma_{AB}||\rho_{AB}) = -\sum_n b_n \left[Z^n - nW^{n-1}Z + (n-1)W^n\right]. \tag{5.16}$$

By taking derivative with respect to Z, we see that each term in the expansion has the minimum at $W = Z$ where the relative entropy vanishes.

In this section we have shown that under the assumption that the primary O, which as the lowest conformal dimension, gives dominant contributions in each order of $\left(\frac{l}{R}\right)$ expansions, the minimum of relative entropy $S(\sigma_{AB}||\rho^0_{AB})$ vanishes. Even though we cannot use the relation (2.7) for perturbations higher than quadratic order, the vanishing relative entropy shows that the vacuum reduced density matrix ρ^0_{AB} is very closed to the separable states at each order of perturbation. Therefore our result here suggests that the reversed one $S(\rho^0_{AB}||\sigma_{AB})$ and the REE $E_R(\rho^0_{AB})$ vanishes in each perturbative order.

6 Contribution from the next lightest primary

So far, we have been discussing possible higher order corrections due to the exchanges of the lightest primary operator. There is another type of corrections to the relative entropy, which is coming from exchanges of heavier operators. To get some intuitions for this, here we study the effect of the next lightest primary O_{NL} with the conformal dimension Δ_{NL}.

If we assume the locally vacuum condition, the contribution of O_{NL} to the relative entropy first enters at $l^{2\Delta+2\Delta_{NL}}$ order. From the replica calculation we find the expression of $S(\sigma_{AB})$, up to this order,

$$-S(\sigma_{AB}) = a_{2\Delta}Z^2 + 2a_{(\Delta+\Delta_{NL})}Z_1^2, \quad Z_1 \equiv l^{\Delta+\Delta_{NL}}\sum_a p_a \langle \rho_A^a O\rangle\langle \rho_A^a O_{NL}\rangle. \tag{6.1}$$

Similarly the mutual information of generic ρ_{AB} up to this order is

$$I_{AB}(\rho_{AB}) = a_{2\Delta}W(\rho_{AB})^2 + a_{(\Delta+\Delta_{NL})}l^{2(\Delta+\Delta_{NL})}\left(\mathrm{tr}[\rho_{AB}O_AO_{B,NL}] + \mathrm{tr}[\rho_{AB}O_BO_{A,NL}]\right) \tag{6.2}$$

Notice however the second term vanishes once we set $\rho_{AB} = \rho^0_{AB}$ thus the modular Hamiltonian part does not receive correction at this order.

The net result of the relative entropy up to this order is therefore

$$S(\sigma_{AB}||\rho^0_{AB}) = a_{2\Delta}(W-Z)^2 + 2a_{(\Delta+\Delta_{NL})}Z_1^2. \tag{6.3}$$

We then minimize this relative entropy. If we can regard second term of (6.3) as a perturbative correction to the first term of order $l^{2\Delta_{NL}}$, then the first order correction to the minimum value of the relative entropy is evaluated just by substituting the separable density matrix (4.13) that minimizes the relative entropy at the leading order. The value of Z_1 for this separable state is given by

$$Z_1 = \langle O_AO_B\rangle\frac{x_{NL}}{x}, \quad x_{NL} = l^{\Delta_{NL}}\langle \rho^1 O_{NL}\rangle \tag{6.4}$$

In order for this to work, we need to require $x \gg x_{NL}$. However it seems difficult to find such ρ_1 in general especially when we need to take x to be large. If we naively

JHEP10(2018)166

construct such ρ_1 with large x, we fail. This is because the maximal value of x and x_{NL} scales as in (4.22) in terms of the lattice spacing a: $x \sim (l/a)^{\Delta}$ and $x_{NL} \sim (l/a)^{\Delta_{NL}}$. Thus we generically expect $x_{NL} \gg x$, assuming $l \gg a$.

From the above analysis of the contribution from the next lightest operator, it does not seem to be possible to reduce the relative entropy $S(\sigma_{AB}||\rho^0_{AB})$ in generic CFTs, by fine-tuning the separable state σ_{AB} as far as we assume our perturbative analysis.

7 Conclusions and discussions

In this paper, we considered the relative entropy of entanglement (REE) $E_R(\rho^0_{AB})$ for CFT vacua. We focus on the case where the subsystem A and B are largely separated compared with their sizes. In this limit we can employ the OPE expansions in terms of operators localized in A and B.

7.1 Lightest operator dominant case and spin chain example

In the first part of this paper, we assumed that the lightest primary operator gives the dominant contribution. Under this assumption we were able to show that $E_R(\rho^0_{AB})$ gets much smaller than the mutual information $I(\rho_{AB})$ as in (4.11) and (4.12). This means that the vacuum reduced density matrix ρ_{AB} is an almost separable state. Moreover, under the assumption that the lightest primary is always dominant, we showed that $S(\sigma_{AB}||\rho^0_{AB})$ for a certain separable state σ_{AB}, is vanishing at each order of power expansions of $\left(\frac{l}{R}\right)$ and this strongly suggests that the REE $E_R(\rho^0_{AB})$ also vanishes in the same way. Thus we find that the correlations between A and B are classical in this case.

We expect that the assumption of taking into account only the lightest primary can be justified when we consider a critical spin chain model and the subsystems consist of finite numbers of spins. For this, let us consider a $S = 1/2$ spin chain at a quantum critical point and choose the subsystem A and B to be the p-th and $(p+R)$-th spin, denoted by σ^A_i and σ^B_i, where $i = 1, 2, 3$ i.e. the Pauli matrices, which satisfy the relation $\mathrm{Tr}[\sigma_i\sigma_j] = 2\delta_{ij}$. The correlation function looks like

$$\langle \sigma^A_i \sigma^B_j \rangle \simeq \delta_{ij}|R|^{-2\Delta} \equiv \gamma \cdot \delta_{ij}. \tag{7.1}$$

where Δ is the dimension of the spin operator. Note that when the distance R between two spins are large the magnitude γ gets very small.

In this setup, the reduced density matrix for AB is given by

$$\rho_{AB} = \frac{I_{AB}}{4} + \frac{\gamma}{4}\sum_{i=1}^{3}\left(\sigma^A_i \otimes \sigma^B_i\right). \tag{7.2}$$

In the 4×4 matrix form this reads

$$\rho_{AB} = \begin{pmatrix} 1+\gamma & 0 & 0 & 0 \\ 0 & 1-\gamma & 2\gamma & 0 \\ 0 & 2\gamma & 1-\gamma & 0 \\ 0 & 0 & 0 & 1+\gamma \end{pmatrix}. \tag{7.3}$$

JHEP10(2018)166

The requirement of positivity of density matrix is expresses as $-1 < \gamma < 1/3$. If γ is small as we consider, this condition is clearly satisfied.

Since the dimension $H_A \otimes H_B$ is less than six, we know that the condition of separability is equivalent to the PPT criterion (positivity under partial transposition) [42]. The density matrix under the partial transposition (transposition w.r.t. B) reads

$$(\rho_{AB})^{T_B} = \begin{pmatrix} 1+\gamma & 0 & 0 & 2\gamma \\ 0 & 1-\gamma & 0 & 0 \\ 0 & 0 & 1-\gamma & 0 \\ 2\gamma & 0 & 0 & 1+\gamma \end{pmatrix}. \tag{7.4}$$

In this case the PPT criterion says that ρ_{AB} is separable if and only if $-\frac{1}{3} < \gamma < 1$.

In summary ρ_{AB} is separable when $-1/3 < \gamma < 1/3$ and is not separable (i.e. is entangled) when $-1 < \gamma < -1/3$. Thus, in our spin chain example, when the distance R between A and B are large (i.e. γ is very small), we can conclude that ρ_{AB} is separable and the logarithmic negativity defined by $\mathcal{E} = \log |(\rho_{AB})^{T_B}|$ is vanishing, where T_B is transposition only for B (called partial transposition).

For a larger spin $S \geq 1$, or for larger subsystems A and B, the PPT criterion and separability are not equivalent. However, still it is known that the state (in a finite dimensional Hilbert space) which is very closed to the maximally mixed state $\rho = \frac{I_N}{N}$ is separable [43]. Therefore if two spins are far apart and their correlation functions are small, we can apply this theorem to find that ρ_{AB} is separable.

Indeed, the above results for spin chains are consistent with our field theoretic result that the REE is vanishing in our perturbation theory.

7.2 Generic cases and holographic CFTs

In the later part of this paper, we estimated the contribution from the next lightest primary. This analysis tells us that the higher dimensional operators can give substantial contributions to the relative entropy in general, which violates our perturbation theory. The main reason for this is that if we want to choose a state ρ^1 with a very large expectation value of the lightest primary, then the expectation value of a heavier operator for the same state also inevitably gets larger.

For example, if we consider holographic CFTs, the lightest primary is typically a single trace operator. The double trace operator has the contribution $x_{\text{double}} = x^2_{\text{single}}$ and thus cannot be negligible. This suggests that in holographic CFT, we have $E_R(\rho^0_{AB}) \simeq I_{AB}(\rho^0_{AB})$, i.e. the correlations between A and B origin from quantum entanglement.[9]

Computations of the REEs for integrable CFTs, such as rational CFTs in two dimensions, will need careful treatments. Interestingly, in [21–23], the logarithmic negativity in

[9]The analysis of holographic entanglement entropy [44] shows that the holographic mutual information satisfies the monogamy as shown in [45]. This suggests that the leading order part $O(N^2)$ (i.e. classical gravity part) of holographic entanglement entropy originates from quantum entanglement. In our analysis we take the large separation limit between A and B and thus such a classical gravity contribution is vanishing. Thus, in this paper, we are interested in the higher order part $O(1)$, which is dual to quantum effects in gravity.

JHEP10(2018)166

the same setup as ours was computed in two dimensional CFTs and spin chains and was shown to be much smaller than any powers of l/R for rational CFTs. The logarithmic negativity is known to be monotone under LOCC and is vanishing for all separable states, though can be zero even for non-separable states. In this sense, the relation between the REE and logarithmic negativity is not straightforward. However, this result strongly implies that the quantum entanglement is highly reduced. In our analysis of REE, since the primary operator spectrum and its OPE algebra are simple, it might be possible that the argument for generic CFTs in the above cannot be applied. If so, the REE can be smaller. To completely answer this question, we need to develop calculations of relative entropy beyond our perturbation theory, which is an interesting future problem.

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A Calculation of $S(\sigma_{AB}) = S(\sum_a p_a \rho_A^a \otimes \rho_B^a)$

In this section we calculate $S(\sum_a p_a \rho_A^a \otimes \rho_B^a)$ perturbatively in the small subsystem size expansion.

For a moment we consider the density matrices coming from tracing out global excited states $|X_a\rangle$, $|Y_a\rangle$ on cylinder (3.7), so that their Rényi entropies are computed by (after applying several conformal mappings) the corresponding correlation function on n sheet covering space $\Sigma_n = S^1_n \times H^{d-1}$,

$$\mathrm{tr}\rho_A^{a_1}\cdots\rho_A^{a_n} = \frac{\langle\prod_{k=0}^{n-1} X_{a_k}(w_k)X_{a_k}(\hat{w}_k)\rangle_{\Sigma_n}}{\prod_{k=0}^{n-1}\langle X_{a_k}(w_0)X_{a_k}(\hat{w}_0)\rangle_{\Sigma_1}} \cdot \frac{Z_A^{(n)}}{(Z_A^{(1)})^n}, \tag{A.1}$$

where the locations of these operators $w_k, \hat{w}_k$ are defined in (3.12). Note also that the correlation functions are normalized such that $\langle 1\rangle_{\Sigma_n} = 1$.

In the small subsystem size limit $2l \to 0$, $w_k \to \hat{w}_k$. Also we have

$$Z_A^{(n)} = \mathrm{tr}\ (\rho_A^0)^n, \quad \rho_A^0 = \mathrm{tr}|0\rangle\langle 0|. \tag{A.2}$$

JHEP10(2018)166

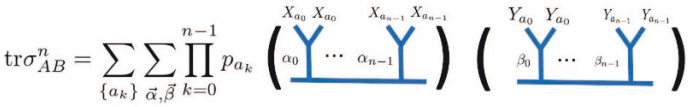

Figure 2. A graphical representation of the expansion the Renyi entropy.

From this we have an expression of the Rényi entropy in terms of correlation functions,

$$\begin{aligned}\mathrm{tr}\ \sigma_{AB}^n &= \sum_{\{a_k\}}\prod_{k=0}^{n-1} p_{a_k}\mathrm{tr}\left[\left(\rho_A^{a_1}\otimes\rho_B^{a_1}\right)\cdots\left(\rho_A^{a_n}\otimes\rho_B^{a_n}\right)\right] \qquad (A.3)\\ &= \sum_{\{a_k\}}\prod_{k=0}^{n-1} p_{a_k}\left(\frac{\langle\prod_{k=0}^{n-1}X_{a_k}(w_k)X_{a_k}(\hat{w}_k)\rangle_{\Sigma_n}}{\prod_{k=0}^{n-1}\langle X_{a_k}(w_k)X_{a_k}(\hat{w}_k)\rangle_{\Sigma_1}}\right)\left(\frac{\langle\prod_{k=0}^{n-1}Y_{a_k}(w'_k)Y_{a_k}(\hat{w}'_k)\rangle_{\Sigma_n}}{\prod_{k=0}^{n-1}\langle Y_{a_k}(w'_0)Y_{a_k}(\hat{w}'_0)\rangle_{\Sigma_1}}\right)\frac{Z_A^{(n)}Z_B^{(n)}}{(Z_A^{(1)}Z_B^{(1)})^n}.\end{aligned}$$

$(w'_k,\hat{w}'_k)$ are again the locations of the local operators for the subsystem B. The strategy to calculate the right hand side of (A.3) is as usual, expanding the correlation functions by using OPEs

$$\frac{X_{a_k}(w_k)X_{a_k}(\hat{w}_k)}{\langle X_{a_k}(w_0)X_{a_k}(\hat{w}_0)\rangle_{\Sigma_1}} = \frac{\langle X_{a_k}(w_k)X_{a_k}(\hat{w}_k)\rangle_{\Sigma_n}}{\langle X_{a_k}(w_0)X_{a_k}(\hat{w}_0)\rangle_{\Sigma_1}}\sum_{\alpha_k} C_{X_{a_k}X_{a_k}\alpha_k}\,(2l)^{\Delta_{\alpha_k}}\alpha_k(w_k) \qquad (A.4)$$

where α_ks are the operators propagating the internal line, and by Δ_{α_k} we denote the scaling dimension of α_k. We also have similar expansion of Y's

$$\frac{Y_{a_k}(w'_k)Y_{a_k}(\hat{w}'_k)}{\langle Y_{a_k}(w_0)Y_{a_k}(\hat{w}_0)\rangle_{\Sigma_1}} = \frac{\langle Y_{a_k}(w'_k)Y_{a_k}(\hat{w}'_k)\rangle_{\Sigma_n}}{\langle Y_{a_k}(w'_0)Y_{a_k}(\hat{w}'_0)\rangle_{\Sigma_1}}\sum_{\alpha_k} C_{Y_{a_k}Y_{a_k}\beta_k}\,(2l)^{\Delta_{\beta_k}}\beta_k(w'_k). \qquad (A.5)$$

Using these formulae

$$\begin{aligned}&\mathrm{tr}\ \sigma_{AB}^n\cdot\frac{(Z_A^{(1)}Z_B^{(1)})^n}{Z_A^{(n)}Z_B^{(n)}} \qquad (A.6)\\ &= \sum_{\{\alpha_0,\cdots\alpha_{n-1}\},\{\beta_0,\cdots\beta_{n-1}\}}\left(\prod_{k=0}^{n-1}J_{\alpha_k\beta_k}\right)\langle\alpha_0(w_0)\cdots\alpha_{n-1}(w_{n-1})\rangle\langle\beta_0(w'_0)\cdots\beta_{n-1}(w'_{n-1})\rangle,\end{aligned}$$

JHEP10(2018)166

where

$$J_{\alpha_k\beta_k} = \sum_{a_k} p_{a_k} \left(\frac{\langle X_{a_k}(w_k) X_{a_k}(\hat{w}_k)\rangle_{\Sigma_n}}{\langle X_{a_k}(w_0) X_{a_k}(\hat{w}_0)\rangle_{\Sigma_1}} \right) \times \left(\frac{\langle Y_{a_k}(w'_k) Y_{a_k}(\hat{w}'_k)\rangle_{\Sigma_n}}{\langle Y_{a_k}(w'_0) Y_{a_k}(\hat{w}'_0)\rangle_{\Sigma_1}} \right) C_{X_{a_k} X_{a_k} \alpha_k} C_{Y_{a_k} Y_{a_k} \beta_k} (2l)^{\Delta_{\alpha_k} + \Delta_{\beta_k}}. \tag{A.7}$$

When the subsystem size l is small, α_k can only be either identity 1 or the first non trivial primary O with the scaling dimension Δ, $\alpha_k \in \{1, O\}$, and similarly, $\beta_k \in \{1, O\}$. This implies that we have the following expansion of tr σ_{AB}^n (A.6) in terms of l^Δ,

$$\text{tr}\, \sigma_{AB}^n \cdot \frac{(Z_A^{(1)} Z_B^{(1)})^n}{Z_A^{(n)}\, Z_B^{(n)}} = L_0^{(n)} + L_2^{(n)}\, (l)^{2\Delta} + L_3^{(n)}\, (l)^{3\Delta} + L_4^{(n)}\, (l)^{4\Delta} + \cdots \tag{A.8}$$

In the next few subsections we calculate these coefficients.

A.1 $L_0^{(n)}$: the first law part

Only the trivial operator configuration can contribute to the coefficient

$$\{\alpha_0, \cdots \alpha_{n-1}\} = \{1, \cdots 1\}, \quad \{\beta_0, \cdots \beta_{n-1}\} = \{1, \cdots 1\} \tag{A.9}$$

therefore $L_0^{(n)} = J_{11}^n$, and

$$\begin{aligned} \frac{\partial}{\partial n} L_0^{(n)}\big|_{n=1} &= \sum_a p_a \left[\frac{\partial}{\partial n} \left(\frac{\langle X_a(w_0) X_a(\hat{w}_0)\rangle_{\Sigma_n}}{\langle X_a(w_0) X_a(\hat{w}_0)\rangle_{\Sigma_1}} \right) + \frac{\partial}{\partial n} \left(\frac{\langle Y_a(w'_0) Y_a(\hat{w}'_0)\rangle_{\Sigma_n}}{\langle Y_a(w'_0) Y_a(\hat{w}'_0)\rangle_{\Sigma_1}} \right) \right] \big|_{n=1} \\ &= -\sum_a p_a \left(\langle K_A^0 (\rho_A^a - \rho_A^0)\rangle + \langle K_B^0 (\rho_B^a - \rho_B^0)\rangle \right), \end{aligned} \tag{A.10}$$

where K_A^0, K_B^0 is vacuum modular Hamiltonian of region A and B respectively. This part is just an analog of the first law part of excited state entanglement entropy.

A.2 $L_1^{(n)}$

Configurations in which only one non trivial operator is present are not allowed because every vacuum one point function vanishes. Therefore $L_1^{(n)} = 0$.

A.3 $L_2^{(n)}$

In this case two types of operator configuration can contribute to the coefficient. One is

$$\{\alpha_0, \cdots \alpha_{n-1}\} = \{1, \cdots O_{q_1}, \cdots O_j, \cdots 1\}, \quad \{\beta_0, \cdots \beta_{n-1}\} = \{1, \cdots 1\},\ q_1 < j \tag{A.11}$$

and

$$\{\alpha_0, \cdots \alpha_{n-1}\} = \{1, \cdots 1\}, \quad \{\beta_0, \cdots \beta_{n-1}\} = \{1, \cdots O_{q_2}, \cdots O_k, \cdots 1\},\ q_2 < k \tag{A.12}$$

In both cases there are two non trivial operators.

$$\begin{aligned} L_2^{(n)} (l)^{2\Delta} = &\frac{J_{11}^{n-2}}{2} \sum_{q_1=0}^{n-1} \sum_{j=0\neq q_1}^{n-1} J_{O_{q_1} 1} J_{O_j 1} \langle O(w_{q_1}) O(w_j)\rangle \\ &+ \frac{J_{11}^{n-2}}{2} \sum_{q_2=0}^{n-1} \sum_{k=0\neq q_2}^{n-1} J_{1 O_{q_2}} J_{1 O_k} \langle O(w_{q_2}) O(w_k)\rangle. \end{aligned} \tag{A.13}$$

JHEP10(2018)166

We are only interested in $n \to 1$ limit. In this case we can set $n = 1$ in J_{O1} as the sum of two point function $\sum_j \langle O(w_{q_1}) O(w_j)\rangle$ is already proportional to $n-1$ [46],

$$f(\Delta, n) = \sum_{j=1}^{n-1} \langle O(w_0) O(w_j)\rangle = \sum_{k=1}^{n-1} \frac{1}{\left(2n \sin \frac{\pi k}{n}\right)^{2\Delta}} \to (n-1) \frac{\Gamma(3/2)\Gamma(\Delta+1)}{2^{2\Delta}\Gamma(\Delta+3/2)}, \quad n \to 1, \tag{A.14}$$

therefore

$$J_{O1} = (2l)^{\Delta} \sum_a p_a C_{X_a X_a O}, \quad J_{1O} = (2l)^{\Delta} \sum_a p_a C_{Y_a Y_a O}, \quad J_{11} = 1. \tag{A.15}$$

Combining them, we conclude,

$$(l)^{2\Delta} \frac{\partial}{\partial n} L_2^{(n)}|_{n=1} = \frac{\Gamma(3/2)\Gamma(\Delta+1)}{2\Gamma(\Delta+3/2)} \left[\left(\sum_a p_a C_{X_a X_a O} \right)^2 + \left(\sum_a p_a C_{Y_a Y_a O} \right)^2 \right] (l)^{2\Delta}. \tag{A.16}$$

A.4 $L_3^{(n)}$

In this term again we have two types of contributions

$$\{\alpha_0, \cdots \alpha_{n-1}\} = \{1, \cdots O_{q_1}, \cdots O_{q_2}, \cdots O_{q_3} \cdots 1\}, \quad \{\beta_0, \cdots \beta_{n-1}\} = \{1, \cdots 1\}, \; q_1 < q_2 < q_3 \tag{A.17}$$

and

$$\{\alpha_0, \cdots \alpha_{n-1}\} = \{1, \cdots 1\}, \quad \{\beta_0, \cdots \beta_{n-1}\} = \{1, \cdots O_{p_1}, \cdots O_{p_2}, \cdots O_{p_3} \cdots 1\}, \; p_1 < p_2 < p_3. \tag{A.18}$$

As in the case of $L_2^{(n)}$, the first contribution generates the cubic order of the von Neuman entrpy on region A, $S(\sigma_A)$ which was explained in (3.15) , and similarly the second contribution generates the cubic order of $S(\sigma_B)$. Therefore we conclude,

$$l^{3\Delta} \frac{\partial}{\partial n} L_3^{(n)}|_{n=1} = C_{OOO} b_\Delta l^{3\Delta} \left[\left(\sum_a p_a C_{X_a X_a O} \right)^3 + \left(\sum_a p_a C_{Y_a Y_a O} \right)^3 \right].$$

A.5 $L_4^{(n)}$

In this case we have

$$\{\alpha_0, \cdots \alpha_{n-1}\} = \{1, \cdots O_{q_1}, \cdots O_j, \cdots 1\}, \quad \{\beta_0, \cdots \beta_{n-1}\} = \{1, \cdots O_{q_2}, \cdots O_k, \cdots 1\} \tag{A.19}$$

and

$$(l)^{4\Delta} L_4^{(n)} = \frac{1}{4} \sum_{q_1=0}^{n-1} \sum_{j=0 \neq q_1}^{n-1} \sum_{q_2=0}^{n-1} \sum_{k=0 \neq q_2}^{n-1} I_{q_1,q_2}^{j,k}. \tag{A.20}$$

The precise form of $I_{q_1,q_2}^{j,k}$ highly depends on the value of the indices. For example, when $(j = q_2, k = q_1)$,

$$I_{q_1,q_2}^{q_2,q_1} = J_{OO}^2 C(q_1 - q_2)^2, \quad C(q_1 - q_2) \equiv \langle O(w_{q_1}) O(w_{q_2})\rangle \tag{A.21}$$

JHEP10(2018)166

with

$$J_{OO} = (2l)^{2\Delta} \sum_a p_a C_{X_a X_a O} C_{Y_a Y_a O}. \tag{A.22}$$

We can compare this expression to (51) of [29] . They can be identified by the replacement $\langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \to J_{OO}^2$.

When $\{q_1 \neq q_2 \neq j \neq k\}$.

$$I^{j,k}_{q_1,q_2} = J_{O1}^2 J_{1O}^2 C(q_1 - j) C(q_2 - k) \tag{A.23}$$

Again this can be compare to (59) of [29], and they are identified by $\langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle \to J_{O1} J_{1O}$.

The strategy to calculate the sum (A.20) is almost same as the calculation of appendix A of [29] ie, first computing the sum with respect to j, k with fixed q_1, q_2,

$$I_{q_1,q_2} = \sum_{j=0\neq q_1}^{n-1} \sum_{k=0\neq q_2}^{n-1} I^{j,k}_{q_1,q_2}, \tag{A.24}$$

then performing the sum with respect to q_1, q_2.

Indeed, we can easily convince ourself that we can derive the result of the sum (A.20) from (69) of appendix A of [29] , just by the replacing $\langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle$ in [29] to J_{OO} and $\langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle$ to $J_{O1} J_{1O}$. Therefore the final result is

$$\begin{aligned}
&(l)^{4\Delta} \frac{\partial}{\partial n} L_4^{(n)} \\
&= \frac{\Gamma(3/2)\Gamma(2\Delta+1)}{2^{4\Delta+1}\Gamma(2\Delta+3/2)} \left(J_{OO} - J_{O1} J_{O1}\right)^2 \\
&= \frac{\Gamma(3/2)\Gamma(2\Delta+1)}{2\Gamma(2\Delta+3/2)} \left[\sum_a p_a C_{X_a X_a O} C_{Y_a Y_a O} - \left(\sum_a p_a C_{X_a X_a O}\right)\left(\sum_a p_a C_{Y_a Y_a O}\right)\right]^2 (l)^{4\Delta}.
\end{aligned} \tag{A.25}$$

A.6 $L_5^{(n)}$

We similarly have $L_5^{(n)}$ term. This term can be relevant in section 5 in which we compute the relative entropy up to $l^{6\Delta}$ term by assuming the locally vacuum condition. However if we assume this condition, $L_5^{(n)}$ term is vanishing, therefore we can ignore this term.

A.7 $L_6^{(n)}$

We can also compute the one more higher term L_6^n once we assume the locally vacuum condition 5.1.

From the OPE expansion (A.6) and the condition 5.1, the result is,

$$(l)^{6\Delta} L_6^{(n)} = J_{OO}^3 \left[\frac{1}{6} \sum_{\{q_1,q_2,q_3\}} \langle O_{q_1} O_{q_2} O_{q_3} . \rangle^2_{\Sigma_n}\right]. \tag{A.26}$$

It is hard to directly perform the sum in right hand side and analytically continue the result in n. However we can read off the outcome from (5.15) of [31] where they computed the entangle entropy of an excited state at cubic order,

$$\lim_{n\to 1} \frac{1}{n-1} \sum_{\{q_1,q_2,q_3\}} \langle O_{q_1} O_{q_2} O_{q_3} \rangle_{\Sigma_n} = -C_{OOO} \frac{\Gamma(\frac{1+\Delta}{2})^3}{12\pi\Gamma(\frac{3+3\Delta}{2})}. \tag{A.27}$$

JHEP10(2018)166

In our case (A.26) we have

$$\lim_{n\to 1}\frac{1}{n-1}\sum_{\{q_1,q_2,q_3\}}\langle O_{q_1}O_{q_2}O_{q_3}\rangle^2_{\Sigma_n}=-C^2_{OOO}\frac{\Gamma(\frac{1+2\Delta}{2})^3}{12\pi\Gamma(\frac{3+6\Delta}{2})}. \tag{A.28}$$

Therefore

$$-S\left(\sum_a p_a\,\rho^a_A\otimes\rho^a_B\right)\Big|_{l^{6\Delta}}=-(l)^{6\Delta}\left(\sum_a p_a\langle\rho^a O\rangle^2\right)^3 C^2_{OOO}\frac{\Gamma(\frac{1+2\Delta}{2})^3}{12\pi\Gamma(\frac{3+6\Delta}{2})}. \tag{A.29}$$

By defining

$$Z\equiv(l)^{2\Delta}\sum_a p_a\langle\rho^a O\rangle^2,\quad d_\Delta\equiv 2^{6\Delta}\frac{\Gamma(\frac{1+2\Delta}{2})^3}{12\pi\Gamma(\frac{3+6\Delta}{2})} \tag{A.30}$$

we write

$$-S\left(\sum_a p_a\,\rho^a_A\otimes\rho^a_B\right)\Big|_{l^{6\Delta}}=-\left(d_\Delta C^2_{OOO}\right)Z^3. \tag{A.31}$$

A.8 The final result

By plugging (A.10), (A.16), (A.25) we obtain the expression of the von Neumann entropy up to $l^{4\Delta}$ order,

$$\begin{aligned}
-S\left(\sum_a p_a\,\rho^a_A\rho^a_B\right)&=\frac{\partial}{\partial n}\left[\left(L^{(n)}_0+L^{(n)}_2(l)^{2\Delta}+L^{(n)}_3(l)^{3\Delta}+L^{(n)}_4(l)^{4\Delta}+\cdots\right)Z^{(n)}_A Z^{(n)}_B\right]\Big|_{n=1}\\
&=-\sum_a p_a\left(\langle K^0_A\rho^a_A\rangle+\langle K^0_B\rho^a_B\rangle\right)\\
&\quad+(l)^{2\Delta}a_\Delta\left[\left(\sum_a p_a C_{X_aX_aO}\right)^2+\left(\sum_a p_a C_{Y_aY_aO}\right)^2\right]\\
&\quad-C_{OOO}b_\Delta l^{3\Delta}\left[\left(\sum_a p_a C_{X_aX_aO}\right)^3+\left(\sum_a p_a\langle C_{Y_aY_aO}\rangle\right)^3\right]\\
&\quad+(l)^{4\Delta}a_{2\Delta}\left[\sum_a p_a C_{X_aX_aO}C_{Y_aY_aO}-\left(\sum_a p_a C_{X_aX_aO}\right)\left(\sum_a p_a C_{Y_aY_aO}\right)\right]^2.
\end{aligned} \tag{A.32}$$

We can see that up to the order of $l^{2\Delta}$ the entropy splits, $S=S(\sum p_a\rho^a_A)+S(\sum p_a\rho^a_B)$. However this no longer holds at the $l^{4\Delta}$ order.

JHEP10(2018)166

It can also be written in terms of the reduced density matrices $\{\rho_A^a, \rho_B^a\}$.

$$
\begin{aligned}
-S\left(\sum_a p_a\,\rho_A^a\rho_B^a\right) = &-\sum_a p_a\left(\langle K_A^0\rho_A^a\rangle + \langle K_B^0\rho_B^a\rangle\right)\\
&+a_\Delta\,(l)^{2\Delta}\left[\left(\sum_a p_a\langle\rho_A^a O\rangle\right)^2+\left(\sum_a p_a\langle\rho_B^a O\rangle\right)^2\right]\\
&-C_{OOO}b_\Delta l^{3\Delta}\left[\left(\sum_a p_a\langle\rho_A^a O\rangle\right)^3+\left(\sum_a p_a\langle\rho_B^a O\rangle\right)^3\right]\\
&+a_{2\Delta}\,(l)^{4\Delta}\left[\sum_a p_a\langle\rho_A^a O_A\rangle\langle\rho_B^a O_B\rangle-\left(\sum_a p_a\langle\rho_A^a O\rangle\right)\left(\sum_a p_a\langle\rho_B^a O\rangle\right)\right]^2.
\end{aligned}
\tag{A.33}
$$

The second term is

$$
\begin{aligned}
\mathrm{tr}\left[\sum_a p_a\;\rho_A^a\;\rho_B^i K_{AB}^0\right] = &\sum_i p_i\left[\langle K_A^0\rangle_i + \langle K_B^0\rangle_i\right]\\
&-2a_{2\Delta}(l)^{2\Delta}\left(\frac{l}{R}\right)^{2\Delta}\sum_a p_a\left[\langle\rho_A^a O_A\rangle\langle\rho_B^a O_B\rangle\right]+I_{AB}.
\end{aligned}
\tag{A.34}
$$

The net result is

$$
\begin{aligned}
S(\sigma_{AB}||\rho_{AB}) = &\;a_\Delta\,(l)^{2\Delta}\left[\left(\sum_a p_a\langle\rho_A^a O\rangle\right)^2+\left(\sum_a p_a\langle\rho_B^a O\rangle\right)^2\right]\\
&-C_{OOO}b_\Delta l^{3\Delta}\left[\left(\sum_a p_a\langle\rho_A^a O\rangle\right)^3+\left(\sum_a p_a\langle\rho_B^a O\rangle\right)^3\right]\\
&+a_{2\Delta}\,(l)^{4\Delta}\left[\sum_a p_a\langle\rho_A^a O_A\rangle\langle\rho_B^a O_B\rangle-\left(\sum_a p_a\langle\rho_A^a O\rangle\right)\left(\sum_a p_a\langle\rho_B^a O\rangle\right)\right]^2\\
&-2a_{2\Delta}(l)^{2\Delta}\left(\frac{l}{R}\right)^{2\Delta}\sum_a p_a\left[\langle\rho_A^a O_A\rangle\langle\rho_B^a O_B\rangle\right]+I_{AB}.
\end{aligned}
\tag{A.35}
$$

B On a replacement rule

In the body of the paper, we used the fact that $S(\rho_{AB})$ is related to $S(\sigma_{AB})$ by the replacement,

$$
W(\rho_{AB}) = \mathrm{tr}\left[\rho_{AB}O_AO_B\right] \to \left[\sigma_{AB}O_AO_B\right] = Z(\sigma_{AB}). \tag{B.1}
$$

In this appendix we prove this prescription. For simplicity we consider the case where ρ_{AB} is the reduced density matrix of a pure state,

$$
\rho_{AB} = \mathrm{tr}_{(AB)^c}|V\rangle\langle V|, \tag{B.2}
$$

and for σ_{AB}, (3.7).

JHEP10(2018)166

The Rényi entropy $\mathrm{tr}\rho_{AB}^n$ has an expression in terms of a correlation function of the twist defect D_n [47],

$$\mathrm{tr}\rho_{AB}^n = \langle V(\infty)^{\otimes n} D_n(A) D_n(B) V(0)^{\otimes n} \rangle, \tag{B.3}$$

the correlation function is evaluated on the cyclic orbifold $(CFT)^{\otimes n}/Z_n$ of the original CFT. Here we take $\langle V(\infty)V(0)\rangle = 1$. In the small subsystem size limit $|A|, |B| \to 0$ one can expand the twist defect in terms of local operators,

$$D_n(A) = \sum_{\{O_k\}} l^{\sum_{k=0}^{n-1} \Delta_k} \langle \prod_{k=0}^{n-1} O_k(A) \rangle_{\Sigma_n} \prod_{k=0}^{n-1} O_k(A), \tag{B.4}$$

here $\langle \cdots \rangle_{\Sigma_n}$ indicates that we evaluate the correlation function on the branched space Σ_n, with a cut on the region A. By plugging this expansion (B.4) into (B.3), we get

$$\mathrm{tr}\rho_{AB}^n = \sum_{\{O_k^A, \tilde{O}_k^B\}} l^{\sum_{k=0}^{n-1} (\Delta_k + \tilde{\Delta}_k)} \langle \prod_{k=0}^{n-1} O_k(A) \rangle_{\Sigma_n} \langle \prod_{k=0}^{n-1} \tilde{O}_k(B) \rangle_{\Sigma_n} \prod_{k=0}^{n-1} \langle V(\infty) O_k^A \tilde{O}_k^B V(0) \rangle, \tag{B.5}$$

notice in general $O_k^A \neq \tilde{O}_k^B$. On the other hand from (A.6),

$$\mathrm{tr}\ \sigma_{AB}^n = \sum_{\{O_k^A, \tilde{O}_k^B\}} \langle \prod_{k=0}^{n-1} O_k(A) \rangle_{\Sigma_n} \langle \prod_{k=0}^{n-1} \tilde{O}_k(B) \rangle_{\Sigma_n} \prod_{k=0}^{n-1} J_{O_k \tilde{O}_k}, \tag{B.6}$$

with (A.7)

$$\begin{aligned} J_{O_k \tilde{O}_k} = & \sum_{a_k} p_{a_k} \left(\frac{\langle X_{a_k}(w_k) X_{a_k}(\hat{w}_k) \rangle_{\Sigma_n}}{\langle X_{a_k}(w_0) X_{a_k}(\hat{w}_0) \rangle_{\Sigma_1}} \right) \\ & \times \left(\frac{\langle Y_{a_k}(w_k') Y_{a_k}(\hat{w}_k') \rangle_{\Sigma_n}}{\langle Y_{a_k}(w_0') Y_{a_k}(\hat{w}_0') \rangle_{\Sigma_1}} \right) C_{X_{a_k} X_{a_k} O_k} C_{Y_{a_k} Y_{a_k} \tilde{O}_k} (2l)^{\Delta_{O_k} + \Delta_{\tilde{O}_k}}. \end{aligned} \tag{B.7}$$

In the $n \to 1$ limit, these two expressions (B.5), (B.6) are related by the identification,

$$\begin{aligned} \mathrm{tr}\left[\rho_{AB} O_k(A) \tilde{O}_k(B) \right] = \langle V(\infty) O_k^A \tilde{O}_k^B V(0) \rangle & \leftrightarrow \sum_{a_k} p_{a_k} C_{X_{a_k} X_{a_k} O_k^A} C_{Y_{a_k} Y_{a_k} \tilde{O}_k^B} \\ & = \mathrm{tr}\left[\sigma_{AB} O_k \tilde{O}_k \right]. \end{aligned} \tag{B.8}$$

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