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SEPARABLE FUNCTORS IN CORINGS

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We develop some basic functorial techniques for the study of the categories of comodules over corings. In particular, we prove that the induction functor stemming from every morphism of corings has a left adjoint, called ad-induction functor. This construction generalizes the known adjunctions for the categories of Doi-Hopf modules and entwined modules. The separability of the induction and ad-induction functors are characterized, extending earlier results for coalgebra and ring homomorphisms, as well as for entwining structures.

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1. Introduction. The notion of separable functor was introduced by Năstăsescu et al. [12], where some applications for group-graded rings were done. This notion fits satisfactorily to the classical notion of separable algebra over a commutative ring. Every separable functor between abelian categories encodes a Maschke's theorem, which explains the interest concentrated in this notion within the module-theoretical developments in recent years. Thus, separable functors have been investigated in the framework of coalgebras [8], graded homomorphisms of rings [9, 10], Doi-Koppinen modules [6, 7], or finally, entwined modules [4, 5]. These situations are generalizations of the original study of the separability for the induction and restriction of scalars functors associated to a ring homomorphism done in [12]. It turns out that all the aforementioned categories of modules are instances of comodule categories over suitable corings [3]. In fact, the separability of some fundamental functors relating the category of comodules over a coring and the underlying category of modules has been studied in [3]. Thus, we can expect that the characterizations obtained in [4] of the separability of the induction functor associated to an admissible morphism of entwining structures and its adjoint generalize to the corresponding functors stemming from a homomorphism of corings. This is done in this paper.

To state and prove the separability theorems, a basic theory of functors between categories of comodules has been developed in this paper, making the arguments independent from the Sweedler's *sigma-notation.* The plan here is to use purely categorical methods which could be easily adapted to more general developments of the theory. These methods had been sketched in [1, 2] in the framework of coalgebras over commutative rings and are expounded in Sections 2, 3, and 4. In Section 5, a notion of homomorphism of corings is given, which leads to a pair of adjoint functors (the induction functor and its adjoint, called here ad-induction functor). The morphisms of entwining structures [4] are instances of homomorphisms of corings in our setting. Finally, the separability of these functors is characterized.

We use essentially the categorical terminology of [16], with the exception of the term K-linear category and functor, for K a commutative ring (see, e.g., [15, Section I.O.2]). There are, however, some minor differences: the notation $X \in \mathcal{A}$ for a category $\mathcal{A}$ means that X is an object of $\mathcal{A}$, and the identity morphism attached to any object X is represented by the object itself. The notation $\mathcal{M}_K$ stands for the category of all unital K-modules. The fact that G is a right adjoint to some functor F is denoted by $F \dashv G$. For the notion of separable functor, the reader is referred to [12]. Finally, let $f, g : X \to Y$ be a pair of morphisms of right modules over a ring R, and let $k : K \to X$ be its equalizer (i.e., the kernel of $f - g$). We will say that a left R-module Z preserves the equalizer of (f, g) if $k \otimes_R Z : K \otimes_R Z \to X \otimes_R Z$ is the equalizer of the pair $(f \otimes_R Z, g \otimes_R Z)$. Of course, every flat module ${}_R Z$ preserves all equalizers.

2. Bicomodules and the cotensor product functor. First recall from [17] the notion of coring. The concepts of comodule and bicomodule over a coring are generalizations of the corresponding notions for coalgebras. We briefly state some basic properties of the cotensor product of bicomodules. Similar associativity properties were studied in [11] in the framework of coseparable corings and in [2] for coalgebras over commutative rings.

Throughout, $A, A', A'', \dots$ denote associative and unitary algebras over a commutative ring K.

2.1. Corings. An A-*coring* is a three-tuple $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \epsilon_{\mathfrak{C}})$ consisting of an A-bimodule $\mathfrak{C}$ and two A-bimodule maps

$$\Delta_{\mathfrak{C}} : \mathfrak{C} \longrightarrow \mathfrak{C} \otimes_A \mathfrak{C}, \qquad \epsilon_{\mathfrak{C}} : \mathfrak{C} \longrightarrow A, \tag{2.1}$$

such that the diagrams

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "\Delta_{\mathfrak{C}}"] \arrow[d, "\Delta_{\mathfrak{C}}"'] & \mathfrak{C} \otimes_A \mathfrak{C} \arrow[d, "\mathfrak{C} \otimes_A \Delta_{\mathfrak{C}}"] \\ \mathfrak{C} \otimes_A \mathfrak{C} \arrow[r, "\Delta_{\mathfrak{C}} \otimes_A \mathfrak{C}"] & \mathfrak{C} \otimes_A \mathfrak{C} \otimes_A \mathfrak{C}, \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "\Delta_{\mathfrak{C}}"] \arrow[dr, "\cong"'] & \mathfrak{C} \otimes_A \mathfrak{C} \arrow[d, "\mathfrak{C} \otimes_A \epsilon_{\mathfrak{C}}"] \\ & \mathfrak{C} \otimes_A A \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{C} \arrow[r, "\Delta_{\mathfrak{C}}"] \arrow[dr, "\cong"'] & \mathfrak{C} \otimes_A \mathfrak{C} \arrow[d, "\epsilon_{\mathfrak{C}} \otimes_A \mathfrak{C}"] \\ & A \otimes_A \mathfrak{C} \end{tikzcd} \tag{2.2}$$

commute.

From now on, $\mathfrak{C}, \mathfrak{C}', \mathfrak{C}'', \dots$ will denote corings over $A, A', A'', \dots$, respectively.

2.2. Comodules. A *right* $\mathfrak{C}$-*comodule* is a pair (M, ρ_M) consisting of a right A-module M and an A-linear map $\rho_M : M \to M \otimes_A \mathfrak{C}$ such that the diagrams

$$\begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[d, "\rho_M"'] & M \otimes_A \mathfrak{C} \arrow[d, "M \otimes_A \Delta_{\mathfrak{C}}"] \\ M \otimes_A \mathfrak{C} \arrow[r, "\rho_M \otimes_A \mathfrak{C}"] & M \otimes_A \mathfrak{C} \otimes_A \mathfrak{C} \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[dr, "\cong"'] & M \otimes_A \mathfrak{C} \arrow[d, "M \otimes_A \epsilon_{\mathfrak{C}}"] \\ & M \otimes_A A \end{tikzcd} \tag{2.3}$$

commute. Left $\mathfrak{C}$-comodules are similarly defined; we use the notation λ_M for their structure maps. A *morphism* of right $\mathfrak{C}$-comodules (M,ρ_M) and (N,ρ_N) is an A-linear map $f: M \to N$ such that the following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho_M"] & N \arrow[d, "\rho_N"] \\ M \otimes_A \mathfrak{C} \arrow[r, "f \otimes_A \mathfrak{C}"] & N \otimes_A \mathfrak{C}. \end{tikzcd} \tag{2.4}$$

The K-module of all right $\mathfrak{C}$-comodule morphisms from M to N is denoted by $\mathrm{Hom}_{\mathfrak{C}}(M,N)$. The K-linear category of all right $\mathfrak{C}$-comodules is denoted by $\mathcal{M}^{\mathfrak{C}}$. When $\mathfrak{C} = A$, with the trivial coring structure, the category $\mathcal{M}^A$ is just the category of all right A-modules, which is "traditionally" denoted by $\mathcal{M}_A$.

Coproducts and cokernels in $\mathcal{M}^{\mathfrak{C}}$ do exist and can be already computed in $\mathcal{M}^A$. Therefore, $\mathcal{M}^{\mathfrak{C}}$ has arbitrary inductive limits. If ${}_A\mathfrak{C}$ is a flat module, then $\mathcal{M}^{\mathfrak{C}}$ is easily proved to be an abelian category.

2.3. Bicomodules. Let $\rho_M : M \to M \otimes_A \mathfrak{C}$ be a comodule structure over an $A' - A$-bimodule M, and assume that ρ_M is A'-linear. For any right A'-module X, the right A-linear map $X \otimes_{A'} \rho_M : X \otimes_{A'} M \to X \otimes_{A'} M \otimes_A \mathfrak{C}$ makes $X \otimes_{A'} M$ a right $\mathfrak{C}$-comodule. This leads to an additive functor $- \otimes_{A'} M : \mathcal{M}_{A'} \to \mathcal{M}^{\mathfrak{C}}$. When $A' = A$ and $M = \mathfrak{C}$, the functor $- \otimes_A \mathfrak{C}$ is left adjoint to the forgetful functor $U_A : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^A$ (see [11, Proposition 3.1] and [3, Lemma 3.1]). Now assume that the $A' - A$-bimodule M is also a left $\mathfrak{C}'$-comodule with structure map $\lambda_M : M \to \mathfrak{C}' \otimes_A M$. It is clear that $\rho_M : M \to M \otimes_A \mathfrak{C}$ is a morphism of left $\mathfrak{C}'$-comodules if and only if $\lambda_M : M \to \mathfrak{C}' \otimes_{A'} M$ is a morphism of right $\mathfrak{C}$-comodules. In this case, we say that M is a $\mathfrak{C}' - \mathfrak{C}$-bicomodule. The $\mathfrak{C}' - \mathfrak{C}$-bicomodules are the objects of a K-linear category ${}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ whose morphisms are those $A' - A$-bimodule maps, which are morphisms of $\mathfrak{C}'$-comodules and of $\mathfrak{C}$-comodules. Some particular cases are now of interest. For instance, when $\mathfrak{C}' = A'$, the objects of the category ${}^{A'}\mathcal{M}^{\mathfrak{C}}$ are the $A' - A$-bimodules with a right $\mathfrak{C}$-comodule structure $\rho_M : M \to M \otimes_A \mathfrak{C}$ which is A'-linear.

2.4. The cotensor product. Let $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ and $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$. We consider $M \otimes_A N$ and $M \otimes_A \mathfrak{C} \otimes_A N$ as $\mathfrak{C}' - \mathfrak{C}''$-bicomodules with structure maps

$$M \otimes_A N \xrightarrow{M \otimes_A \rho_N} M \otimes_A N \otimes_{A''} \mathfrak{C}'', \qquad M \otimes_A N \xrightarrow{\lambda_M \otimes_A M} \mathfrak{C}' \otimes_{A'} M \otimes_A N, \tag{2.5}$$

$$M \otimes_A \mathfrak{C} \otimes_A N \xrightarrow{M \otimes_A \mathfrak{C} \otimes_A \rho_N} M \otimes_A \mathfrak{C} \otimes_A N \otimes_{A''} \mathfrak{C}'', \tag{2.6}$$

$$M \otimes_A \mathfrak{C} \otimes_A N \xrightarrow{\lambda_M \otimes_A \mathfrak{C} \otimes_A N} \mathfrak{C}' \otimes_{A'} M \otimes_A \mathfrak{C} \otimes_A N. \tag{2.7}$$

The map

$$M \otimes_A N \xrightarrow{\rho_M \otimes_A N - M \otimes_A \lambda_N} M \otimes_A \mathfrak{C} \otimes_A N \tag{2.8}$$

is then a $\mathfrak{C}' - \mathfrak{C}''$-bicomodule map. Let $M \square_{\mathfrak{C}} N$ denote the kernel of (2.8). If $\mathfrak{C}'_{A'}$ and ${}_{A''}\mathfrak{C}''$ preserve the equalizer of $(\rho_M \otimes_A N, M \otimes_A \lambda_N)$, then $M \square_{\mathfrak{C}} N$ is both a $\mathfrak{C}'$ and a $\mathfrak{C}''$-subcomodule of $M \otimes_A N$ and, hence, it is a $\mathfrak{C}' - \mathfrak{C}''$-bicomodule.

Proposition 2.1. *Assume that for every $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ and $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$, $\mathfrak{C}'_{A'}$ and ${}_{A''}\mathfrak{C}''$ preserve the equalizer of $(\rho_M \otimes_A N, M \otimes_A \lambda_N)$. There is an additive bifunctor*

$$-\square_{\mathfrak{C}}- : {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}} \times {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''} \longrightarrow {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}''}. \tag{2.9}$$

In particular, the cotensor product bifunctor (2.9) is defined when $\mathfrak{C}'_{A'}$ and ${}_{A''}\mathfrak{C}''$ are flat modules or when $\mathfrak{C}$ is a coseparable A-coring in the sense of [11].

In the special case $\mathfrak{C}' = A'$ and $\mathfrak{C}'' = A''$, we have the bifunctor

$$-\square_{\mathfrak{C}}- : {}^{A'}\mathcal{M}^{\mathfrak{C}} \times {}^{\mathfrak{C}}\mathcal{M}^{A''} \longrightarrow {}^{A'}\mathcal{M}^{A''} \tag{2.10}$$

and, if we further assume that $A' = A'' = K$, we have the bifunctor

$$-\square_{\mathfrak{C}}- : \mathcal{M}^{\mathfrak{C}} \times {}^{\mathfrak{C}}\mathcal{M} \longrightarrow \mathcal{M}_K. \tag{2.11}$$

2.5. Compatibility between tensor and cotensor. Let $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ and $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$ be bicomodules. For any right A'-module W, consider the commutative diagram

$$\begin{array}{ccccccc} & & W \otimes_{A'} (M \square_{\mathfrak{C}} N) & \longrightarrow & W \otimes_{A'} (M \otimes_A N) & \longrightarrow & W \otimes_{A'} (M \otimes_A \mathfrak{C} \otimes_A N) \\ & & \downarrow \psi & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & (W \otimes_{A'} M) \square_{\mathfrak{C}} N & \longrightarrow & (W \otimes_{A'} M) \otimes_A N & \longrightarrow & (W \otimes_{A'} M) \otimes_A \mathfrak{C} \otimes_A N \end{array} \tag{2.12}$$

where ψ is given by the universal property of the kernel in the second row. This leads to the following.

Lemma 2.2. *It follows from (2.12) that $W_{A'}$ preserves the equalizer of $(\rho_M \otimes_A N, M \otimes_A \lambda_N)$ if and only if $\psi : W \otimes_{A'} (M \square_{\mathfrak{C}} N) \cong (W \otimes_{A'} M) \square_{\mathfrak{C}} N$. In particular, ψ is an isomorphism if $W_{A'}$ is flat.*

Next, we prove a basic fact concerning with the associativity of cotensor product.

Proposition 2.3. *Let $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$, $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}''}$, $L \in {}^{\mathfrak{C}'''}\mathcal{M}^{\mathfrak{C}'}$. Assume that $\mathfrak{C}'_{A'}$, $L_{A'}$, $L \otimes_{A'} \mathfrak{C}'_{A'}$, and ${}_{A''}\mathfrak{C}''$ preserve the equalizer of $(\rho_M \otimes_A N, M \otimes_A \lambda_N)$, and that ${}_A\mathfrak{C}$, ${}_AN$, ${}_A\mathfrak{C} \otimes_A N$, and $\mathfrak{C}'''_{A'''}$ preserve the equalizer of $(\rho_L \otimes_{A'} M, L \otimes_{A'} \lambda_M)$. Then there is a canonical isomorphism of $\mathfrak{C}''' - \mathfrak{C}''$-bicomodules*

$$L \square_{\mathfrak{C}'} (M \square_{\mathfrak{C}} N) \cong (L \square_{\mathfrak{C}'} M) \square_{\mathfrak{C}} N. \tag{2.13}$$

PROOF. Since $\mathfrak{C}'_{A'}$ and ${}_{A''}\mathfrak{C}''$ preserve the equalizer of $(\rho_M \otimes_A N, M \otimes_A \lambda_N)$, we know that it is a $\mathfrak{C}' - \mathfrak{C}''$-subbicomodule. Analogously, $L \square_{\mathfrak{C}'} M$ is a $\mathfrak{C}''' - \mathfrak{C}$-subbicomodule of $L \otimes_{A'} M$. In the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & (L\square_{\mathfrak{C}'}M)\square_{\mathfrak{C}}N \arrow[r] \arrow[d] & (L\otimes_{A'}M)\square_{\mathfrak{C}}N \arrow[r] \arrow[d] & (L\otimes_{A'}\mathfrak{C}'\otimes_{A'}M)\square_{\mathfrak{C}}N \arrow[d] \\ 0 \arrow[r] & (L\square_{\mathfrak{C}'}M)\otimes_A N \arrow[r] & (L\otimes_{A'}M)\otimes_A N \arrow[r] & (L\otimes_{A'}\mathfrak{C}'\otimes_{A'}M)\otimes_A N \end{tikzcd} \tag{2.14}$$

the second row is exact because ${}_AN$ preserves the equalizer of $(\rho_L \otimes_{A'} M, L \otimes_{A'} \lambda_M)$. The exactness of the first row is then deduced by using that ${}_A\mathfrak{C} \otimes_A N$ is assumed to preserve the equalizer of $(\rho_L \otimes_{A'} M, L \otimes_{A'} \lambda_M)$. Now, consider the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & (L\square_{\mathfrak{C}'}M)\square_{\mathfrak{C}}N \arrow[r] & (L\otimes_{A'}M)\square_{\mathfrak{C}}N \arrow[r] & (L\otimes_{A'}\mathfrak{C}'\otimes_{A'}M)\square_{\mathfrak{C}}N \, . \\ 0 \arrow[r] & L\square_{\mathfrak{C}'}(M\square_{\mathfrak{C}}N) \arrow[r] \arrow[u, "\psi_1"] & L\otimes_{A'}(M\square_{\mathfrak{C}}N) \arrow[r] \arrow[u, "\psi_2"] & L\otimes_{A'}\mathfrak{C}'\otimes_{A'}(M\square_{\mathfrak{C}}N) \arrow[u, "\psi_3"] \end{tikzcd} \tag{2.15}$$

Lemma 2.2 gives the isomorphisms ψ_2 and ψ_3, which induce the isomorphism ψ_1. $\square$

3. Functors between comodule categories. This section contains technical facts concerning with K-linear functors between categories of comodules over corings. Part of these tools were first developed for coalgebras over commutative rings in [1, 2]. Roughly speaking, it is proved in this paper an analogue to Watts theorem, which allows to represent good enough functors as cotensor product functors. Also is included a result which states that, under mild conditions, a natural transformation gives a bicomodule morphism at any bicomodule. This will be used in the statement and proof of our separability theorems in Section 5.

Let $\mathfrak{C}$ and $\mathfrak{D}$ be corings over K-algebras A and B, respectively, and consider a K-linear functor

$$F : \mathcal{M}^{\mathfrak{C}} \longrightarrow \mathcal{M}^{\mathfrak{D}}. \tag{3.1}$$

3.1. Let T be a K-algebra. For every $M \in {}^T\mathcal{M}^{\mathfrak{C}}$, consider the homomorphism of K-algebras

$$T \cong \operatorname{End}(T_T) \longrightarrow \operatorname{Hom}_{\mathfrak{C}}(T \otimes_T M, T \otimes_T M) \cong \operatorname{Hom}_{\mathfrak{C}}(M, M) \longrightarrow \operatorname{Hom}_{\mathfrak{D}}(F(M), F(M)) \tag{3.2}$$

which induces a left T-module structure over $F(M)$ such that $F(M)$ becomes a $T - \mathfrak{D}$-bicomodule. We have two K-linear functors

$$- \otimes_T F(-), F(- \otimes_T -) : \mathcal{M}^T \times {}^T\mathcal{M}^{\mathfrak{C}} \longrightarrow \mathcal{M}^{\mathfrak{D}}. \tag{3.3}$$

We construct a natural transformation

$$\Upsilon_{-,-} : - \otimes_T F(-) \longrightarrow F(- \otimes_T -). \tag{3.4}$$

Let $\Upsilon_{T,M}$ be the unique isomorphism of $\mathfrak{D}$-comodules making the following diagram commutative

$$\begin{tikzcd}
T\otimes_T F(M) \arrow[rr, "\Upsilon_{T,M}"] \arrow[dr, "\cong"] & & F(T\otimes_T M) \arrow[dl, "\cong"] \\
& F(M) &
\end{tikzcd} \tag{3.5}$$

To prove that $\Upsilon_{T,M}$ is natural at T, consider a homomorphism $f : T \to T$ of right T-modules and define $g : F(M) \to F(M)$ by $g(x) = f(1)x$ for every $x \in F(M)$. Since g is just the image under (3.2) of $f(1) \in T$, it follows that g is a morphism of right $\mathfrak{D}$-comodules. Moreover, g makes the following diagram commutative:

$$\begin{tikzcd}
F(M) \arrow[r, "g"] & F(M) \\
F(T\otimes_T M) \arrow[u, "\cong"] \arrow[r, "F(f\otimes_T M)"] & F(T\otimes_T M) \arrow[u, "\cong"']
\end{tikzcd} \tag{3.6}$$

In the diagram

$$\begin{tikzcd}
& F(M) \arrow[rr, "g"] & & F(M) \\
T\otimes_T F(M) \arrow[ur, "\cong"] \arrow[rr, "f\otimes_T F(M)"] \arrow[d, "\Upsilon_{T,M}"'] & & T\otimes_T F(M) \arrow[ur, "\cong"] \arrow[d, "\Upsilon_{T,M}"'] & \\
F(T\otimes_T M) \arrow[uur, "\cong"'] \arrow[rr, "F(f\otimes_T M)"] & & F(T\otimes_T M) \arrow[uur, "\cong"'] &
\end{tikzcd} \tag{3.7}$$

the commutativity of the front rectangle, which gives that $\Upsilon_{T,M}$ is natural, follows from the commutativity of the rest of the diagram. From Mitchell's theorem [13, Theorem 3.6.5], we obtain a natural transformation

$$\Upsilon_{-,M} : -\otimes_T F(M) \longrightarrow F(-\otimes_T M). \tag{3.8}$$

Moreover, if F preserves coproducts (resp., direct limits, resp., inductive limits) then $\Upsilon_{X,M}$ is an isomorphism for X_T projective (resp., flat, resp., any right T-module).

Proposition 3.1. *If the functor F preserves coproducts, then $\Upsilon_{-,-} : -\otimes_T F(-) \to F(-\otimes_T -)$ is a natural transformation. Moreover, if F preserves direct limits, then $\Upsilon_{X,-}$ is a natural isomorphism for every flat right T-module X. Finally, if F preserves inductive limits, then $\Upsilon_{-,-}$ is a natural isomorphism.*

Proof. By Section 3.1, $\Upsilon_{-,M}$ is natural for every $M \in {}^{T}\mathcal{M}^{\mathfrak{C}}$. Thus, we have only to show that $\Upsilon_{X,-}$ is natural for every $X \in \mathcal{M}^{T}$. We argue first for $X = T$. Let $f : M \to N$ be a homomorphism in ${}^{T}\mathcal{M}^{\mathfrak{C}}$. From the diagram

$$\begin{tikzcd}
 & F(M) \arrow[rr, "F(f)"] & & F(N) \\
T \otimes_T F(M) \arrow[ur, "\cong"] \arrow[rr, "T\otimes_T F(f)"] \arrow[d, "\Upsilon_{T,M}"'] & & T \otimes_T F(M) \arrow[ur, "\cong"] \arrow[d, "\Upsilon_{T,N}"'] & \\
F(T \otimes_T M) \arrow[uur, "\cong"'] \arrow[rr, "F(T\otimes_T f)"] & & F(T \otimes_T N) \arrow[uur, "\cong"'] &
\end{tikzcd} \tag{3.9}$$

we get that $\Upsilon_{A,-}$ is natural. Now, use a free presentation $T^{(\Omega)} \to X$ to obtain that $\Upsilon_{X,-}$ is natural for a general X_T. The rest of the statements are easily derived from this. $\square$

Lemma 3.2. *Let $\eta : F_1 \to F_2$ be a natural transformation, where $F_1, F_2 : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ are K-linear functors which preserve coproducts.*

(1) *For every $M \in {}^{T}\mathcal{M}^{\mathfrak{C}}$, $\eta_M : F_1(M) \to F_2(M)$ is a $T-\mathfrak{D}$-bicomodule homomorphism.*

(2) *Given $X \in \mathcal{M}^{T}$ and $M \in {}^{T}\mathcal{M}^{\mathfrak{C}}$, the diagram*

$$\begin{tikzcd}
F_1(X \otimes_T M) \arrow[r, "\eta_{X\otimes_T M}"] & F_2(X \otimes_T M) \\
X \otimes_T F_1(M) \arrow[u, "\Upsilon_{X,M}"] \arrow[r, "X\otimes_T \eta_M"] & X \otimes_T F_2(M) \arrow[u, "\Upsilon_{X,M}"']
\end{tikzcd} \tag{3.10}$$

is commutative.

Proof. We need just to prove that (3.10) commutes for $X = T$. In this case, the diagram can be factored out as

$$\begin{tikzcd}
F_1(T \otimes_T M) \arrow[rr, "\eta_{T\otimes_T M}"] \arrow[dr, "\cong"] & & F_2(T \otimes_T M) \arrow[dr, "\cong"] & \\
 & F_1(M) \arrow[rr, "\eta_M"] & & F_2(M). \\
T \otimes_T F_1(M) \arrow[uu, "\Upsilon_{T,M}"] \arrow[ur, "\cong"] \arrow[rr, "T\otimes_T \eta_M"] & & T \otimes_T F_2(M) \arrow[uu, "\Upsilon_{T,M}"'] \arrow[ur, "\cong"'] &
\end{tikzcd} \tag{3.11}$$

Since all trapezia and triangles commute, the back rectangle does, as desired. $\square$

Lemma 3.3. *Let T, S be K-algebras and assume that $F : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ preserves coproducts. Given $X \in \mathcal{M}^S$, $Y \in {}^S\mathcal{M}^T$, and $M \in {}^T\mathcal{M}^{\mathfrak{C}}$, the following formula holds:*

$$\Upsilon_{X,Y\otimes_T M} \circ \left(X \otimes_S \Upsilon_{Y,M}\right) = \Upsilon_{X\otimes_S Y,M}. \tag{3.12}$$

Proof. The equality will be first proved for $X = S$. Consider the diagram

$$\begin{tikzcd}
S\otimes_S F(Y\otimes_T M) \arrow[rr, "\cong"] \arrow[dd, "\Upsilon_{S,Y\otimes_T M}"'] & & F(Y\otimes_T M) \arrow[dd, equal] & \\
& S\otimes_S Y\otimes_T F(M) \arrow[ul, "S\otimes_S \Upsilon_{Y,M}"'] \arrow[dl, "\Upsilon_{S\otimes_S Y,M}"] \arrow[rr, "\cong"] & & Y\otimes_T F(M). \arrow[ul, "\Upsilon_{Y,M}"'] \arrow[dl, "\Upsilon_{Y,M}"] \\
F(S\otimes_S Y\otimes_T M) \arrow[rr, "\cong"] & & F(Y\otimes_T M) &
\end{tikzcd} \tag{3.13}$$

The back rectangle is commutative by definition of $\Upsilon_{S,Y\otimes_T M}$, while the other two parallelograms are commutative because $\Upsilon_{-,-}$ is natural. Therefore, the right triangle is commutative. The equality is now easily extended for $X = S^{(\Omega)}$ and, by using a free presentation $S^{(\Omega)} \to X \to 0$, for any X. □

3.2. Natural transformations and bicomodule morphisms. Let $M \in {}^{\mathfrak{C}'}\mathcal{M}^{\mathfrak{C}}$ be a bicomodule. A functor $F : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ is said to be M-*compatible* if $\Upsilon_{\mathfrak{C}',M}$ and $\Upsilon_{\mathfrak{C}'\otimes_{A'}\mathfrak{C}',M}$ are isomorphisms. By Proposition 3.1, the functor F is M-compatible for every bicomodule M if either the functor F preserves inductive limits or $\mathfrak{C}'_A$ is flat and F preserves direct limits. In case that F is M-compatible, define $\lambda_{F(M)}$ as the unique A'-linear map making the following diagram commutative

$$\begin{tikzcd}
F(M) \arrow[rr, "\lambda_{F(M)}"] \arrow[dr, "F(\lambda_M)"'] & & \mathfrak{C}'\otimes_{A'} F(M) \arrow[dl, "\Upsilon_{\mathfrak{C}',M}"] \\
& F(\mathfrak{C}'\otimes_{A'} M). &
\end{tikzcd} \tag{3.14}$$

Proposition 3.4. *Let F be an M-compatible functor which preserves coproducts. The A'-linear map $\lambda_{F(M)}$ is a left $\mathfrak{C}'$-comodule structure on $F(M)$ such that $F(M)$ becomes a $\mathfrak{C}' - \mathfrak{D}$-bicomodule. Moreover, given $F_1, F_2 : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ M-compatible functors and a natural transformation $\eta : F_1 \to F_2$, the map $\eta_M : F_1(M) \to F_2(M)$ is a $\mathfrak{C}' - \mathfrak{D}$-bicomodule homomorphism.*

Proof. In order to prove that the coaction $\lambda_{F(M)}$ is coassociative, consider the diagram

$$
\begin{tikzcd}
 & & & F(M) \arrow[rrr, "\lambda_{F(M)}"] \arrow[dll, "\lambda_{F(M)}"'] \arrow[ddd, equal] & & & \mathfrak{C}' \otimes_{A'} F(M) \arrow[dll, "\Delta_{\mathfrak{C}'} \otimes_{A'} F(M)"'] \arrow[ddd, "\Upsilon_{\mathfrak{C}',M}"] \\
 & \mathfrak{C}' \otimes_{A'} F(M) \arrow[rrr, "\mathfrak{C}' \otimes_{A'} \lambda_{F(M)}"] \arrow[drr, "\mathfrak{C}' \otimes_{A'} F(M)"] \arrow[ddd, "\Upsilon_{\mathfrak{C}',M}"] & & & \mathfrak{C}' \otimes_{A'} \mathfrak{C}' \otimes_{A'} F(M) \arrow[dl, "\mathfrak{C}' \otimes_{A'} \Upsilon_{\mathfrak{C}',M}"'] \arrow[ddd, "\Upsilon_{\mathfrak{C}' \otimes_{A'} \mathfrak{C}',M}"] & & \\
 & & & \mathfrak{C}' \otimes_{A'} F(\mathfrak{C}' \otimes_{A'} M) \arrow[ddr, "\Upsilon_{\mathfrak{C}',\mathfrak{C}' \otimes_{A'} M}"] & & & \\
 & & & F(M) \arrow[rrr, "F(\lambda_M)"] \arrow[dll, "F(\lambda_M)"'] & & & F(\mathfrak{C}' \otimes_{A'} M) \arrow[dll, "F(\Delta_{\mathfrak{C}'} \otimes_{A'} M)"] \\
 & F(\mathfrak{C}' \otimes_{A'} M) \arrow[rrr, "F(\mathfrak{C}' \otimes_{A'} \lambda_M)"] & & & F(\mathfrak{C}' \otimes_{A'} \mathfrak{C}' \otimes'_{A} M). & &
\end{tikzcd}
\tag{3.15}
$$

We want to see that the top side is commutative. Since F is assumed to be M-compatible, we have just to prove that the mentioned side is commutative after composing with the isomorphism $\Upsilon_{\mathfrak{C}' \otimes_{A'} \mathfrak{C}',M}$. This is deduced by using Lemma 3.3, in conjunction with the naturality of $\Upsilon_{-,M}$ and the very definition of $\lambda_{F(M)}$. The counitary property is deduced from the commutative diagram

$$
\begin{tikzcd}
A' \otimes_{A'} F(M) \arrow[rr, "\Upsilon_{A',M}"] & & F(A' \otimes_{A'} M) \\
 & F(M) \arrow[ul, "\cong"'] \arrow[ur, "\cong"] \arrow[dl, "\lambda_{F(M)}"'] \arrow[dr, "F(\lambda_M)"] & \\
\mathfrak{C}' \otimes_{A'} F(M) \arrow[uu, "\epsilon_{\mathfrak{C}'} \otimes_{A'} F(M)"] \arrow[rr, "\Upsilon_{\mathfrak{C}',M}"] & & F(\mathfrak{C}' \otimes_{A'} M). \arrow[uu, "F(\epsilon_{\mathfrak{C}'} \otimes_{A'} M)"']
\end{tikzcd}
\tag{3.16}
$$

To prove the second statement, consider the diagram

$$
\begin{tikzcd}
F_1(M) \arrow[rr, "\eta_M"] \arrow[dr, "F_1(\lambda_M)"] \arrow[dd, "\lambda_{F_1(M)}"] & & F_2(M) \arrow[dr, "F_2(\lambda_M)"] \arrow[dd, "\lambda_{F_2(M)}"] & \\
 & F_1(\mathfrak{C}' \otimes_{A'} M) \arrow[rr, "\eta_{\mathfrak{C}' \otimes_{A'} M}"] \arrow[dl, "\Upsilon_{\mathfrak{C}',M}"] & & F_2(\mathfrak{C}' \otimes_{A} M). \arrow[dl, "\Upsilon_{\mathfrak{C}',M}"] \\
\mathfrak{C}' \otimes_{A'} F_1(M) \arrow[rr, "\mathfrak{C}' \otimes_{A'} \eta_M"] & & \mathfrak{C}' \otimes_{A'} F_2(M) &
\end{tikzcd}
\tag{3.17}
$$

Both triangles commute by definition of $\lambda_{F_1(M)}$ and $\lambda_{F_2(M)}$, and the upper trapezium is commutative because η is natural. The bottom trapezium commutes by Lemma 3.2. Therefore, the back rectangle is commutative, which just says that η_M is a morphism of left $\mathfrak{C}'$-comodules. This finishes the proof. $\square$

Theorem 3.5. *Assume that $\mathfrak{C}_A$ is flat. If $F : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ is exact and preserves direct limits (e.g., if F is an equivalence of categories), then F is naturally isomorphic to $-\square_{\mathfrak{C}} F(\mathfrak{C})$.*

Proof. Let $\rho_N \to N \otimes_A \mathfrak{C}$ be a right $\mathfrak{C}$-comodule. We have the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N\square_{\mathfrak{C}} F(\mathfrak{C}) \arrow[r] \arrow[d, dashed, "\cong"] & N \otimes_A F(\mathfrak{C}) \arrow[rr, "\rho_N \otimes_A F(\mathfrak{C}) - N \otimes_A \lambda_{F(\mathfrak{C})}"] \arrow[d, "\Upsilon_{N,\mathfrak{C}}", "\cong"'] & & N \otimes_A \mathfrak{C} \otimes_A F(\mathfrak{C}) \arrow[d, "\Upsilon_{N\otimes_A A,\mathfrak{C}}", "\cong"'] \\
0 \arrow[r] & F(N) \arrow[r] & F(N \otimes_A \mathfrak{C}) \arrow[rr, "F(\rho_N \otimes_A \mathfrak{C} - N \otimes_A \Delta_{\mathfrak{C}})"] & & F(N \otimes_A \mathfrak{C} \otimes_A \mathfrak{C})
\end{tikzcd} \tag{3.18}$$

where the desired isomorphism is given by the universal property of the kernel. □

4. Co-hom functors. This section contains a quick study of the left adjoint to a cotensor product functor, if it does exist. The presentation is inspired from the one given in [18, Subsection 1.8] for coalgebras over a field.

Let $\mathfrak{C}$, $\mathfrak{D}$ be corings over K-algebras A and B, respectively.

Definition 4.1. A bicomodule $N \in {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{D}}$ is said to be *quasi-finite* as a right $\mathfrak{D}$-comodule if the functor $- \otimes_A N : \mathcal{M}^A \to \mathcal{M}^{\mathfrak{D}}$ has a left adjoint $\mathrm{h}_{\mathfrak{D}}(N,-) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^A$. This functor is called the *co-hom* functor associated to N.

The natural isomorphism which gives the adjunction is denoted by

$$\Phi_{X,Y} : \mathrm{Hom}_A\left(\mathrm{h}_{\mathfrak{D}}(N,X), Y\right) \longrightarrow \mathrm{Hom}_{\mathfrak{D}}\left(X, Y \otimes_A N\right), \tag{4.1}$$

for $Y \in \mathcal{M}^A$, $X \in \mathcal{M}^{\mathfrak{D}}$.

4.1. Let $\theta : \mathrm{id} \to \mathrm{h}_{\mathfrak{D}}(N,-) \otimes_A N$ be the unit of the adjunction (4.1). Therefore, the isomorphism $\Phi_{X,Y}$ is given by the assignment $f \mapsto (f \otimes_A N)\theta_X$. In particular, the map

$$X \xrightarrow{\theta_X} \mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A N \xrightarrow{\mathrm{id} \otimes_A \lambda_N} \mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A \mathfrak{C} \otimes_A N \tag{4.2}$$

determines an A-linear map

$$\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} : \mathrm{h}_{\mathfrak{D}}(N,X) \longrightarrow \mathrm{h}_{\mathfrak{D}}(N,X) \otimes_A \mathfrak{C} \tag{4.3}$$

such that $(\mathrm{id} \otimes_A \lambda_N)\theta_X = (\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)} \otimes_A \mathrm{id})\theta_X$. The coaction $\rho_{\mathrm{h}_{\mathfrak{D}}(N,X)}$ makes $\mathrm{h}_{\mathfrak{D}}(N,X)$ a right $\mathfrak{C}$-comodule. Therefore we have a functor $\mathrm{h}_{\mathfrak{D}}(N,-) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}$.

4.2. The following is a basic tool in our investigation.

Proposition 4.2. *Let N be a $\mathfrak{C}-\mathfrak{D}$-bicomodule. Assume that ${}_B\mathfrak{D}$ preserves the equalizer of $(\rho_Y \otimes_A N, Y \otimes_A \lambda_N)$ for every right $\mathfrak{C}$-comodule Y (e.g., ${}_B\mathfrak{D}$ is flat or $\mathfrak{C}$ is a coseparable A-coring in the sense of* [11]*).*

(1) *If N is quasi-finite as a right $\mathfrak{D}$-comodule, then the natural isomorphism (4.1) restricts to an isomorphism $\mathrm{Hom}_{\mathfrak{C}}(\mathrm{h}_{\mathfrak{D}}(N,X),Y) \cong \mathrm{Hom}_{\mathfrak{D}}(X, Y \square_{\mathfrak{C}} N)$. Therefore, $\mathrm{h}_{\mathfrak{D}}(N,-)$ is left adjoint to $-\square_{\mathfrak{C}} N$.*

(2) *Conversely, if $-\square_{\mathfrak{C}} N : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ has a left adjoint, then N is quasi-finite as a right $\mathfrak{C}$-comodule.*

Proof. (1) We need to prove that if $f \in \mathrm{Hom}_A(\mathrm{h}_{\mathfrak{D}}(N,X),Y)$, then f is $\mathfrak{C}$-colinear if and only if the image of $(f \otimes_A N)\theta_X$ is included in $Y \square_{\mathfrak{C}} N$. But these are straightforward computations in view of the definition of the contensor product.

(2) Since the inclusion $\mathfrak{C} \square_{\mathfrak{C}} N \subseteq \mathfrak{C} \otimes_A N$ splits-off, we get from Section 2.5 the natural isomorphism $(-\otimes_A \mathfrak{C}) \square_{\mathfrak{C}} N \cong - \otimes_A (\mathfrak{C} \square_{\mathfrak{C}} N) \cong - \otimes_A N$. Now, the functor $-\otimes_A \mathfrak{C}$ is right adjoint [3, Lemma 3.1] to the forgetful functor $U_A : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^A$ and, by hypothesis, $-\square_{\mathfrak{C}} N$ is right adjoint to the functor $\mathrm{h}_{\mathfrak{D}}(N,-) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}$. This implies that $-\otimes_A N$ is right adjoint to $U_A \mathrm{h}_{\mathfrak{D}}(N,-)$, as desired. $\square$

Example 4.3. Let $N \in {}^{\mathfrak{C}}\mathcal{M}^B$. Then N_B is quasi-finite if and only if $-\otimes_A N : \mathcal{M}^A \to \mathcal{M}^B$ has a left adjoint, that is, if and only if ${}_A N$ is finitely generated and projective. In such a case, the left adjoint is $-\otimes_B \mathrm{Hom}_A(N,A) : \mathcal{M}^B \to \mathcal{M}^{\mathfrak{C}}$. Notice that taking $B = K$, we obtain a canonical structure of right $\mathfrak{C}$-comodule on $N^* = \mathrm{Hom}_A(N,A)$ for every left $\mathfrak{C}$-comodule N such that N is finitely generated and projective as a left A-module.

Example 4.4. Given an A-coring $\mathfrak{C}$ and a K-algebra homomorphism $\rho : A \to B$, we can consider the functor $-\otimes_A \mathfrak{C} : \mathcal{M}^B \to \mathcal{M}^{\mathfrak{C}}$, which is already the composite of the restriction of scalars functor $(-)_\rho : \mathcal{M}^B \to \mathcal{M}^A$ and the functor $-\otimes_A \mathfrak{C} : \mathcal{M}^A \to \mathcal{M}^{\mathfrak{C}}$. Since these functors have both left adjoints, given, respectively, by the induction functor $-\otimes_A B : \mathcal{M}^A \to \mathcal{M}^B$ and the underlying functor $U_A : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^A$, we get that the composite functor $-\otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^B$ is a left adjoint to $-\otimes_A \mathfrak{C} : \mathcal{M}^B \to \mathcal{M}^{\mathfrak{C}}$. Clearly, $-\otimes_A \mathfrak{C} \cong -\otimes_B (B \otimes_A \mathfrak{C})$ and, thus, $B \otimes_A \mathfrak{C}$ becomes a quasi-finite comodule.

5. Separable homomorphisms of corings. We propose a notion of homomorphism of corings which generalizes both the concept of morphism of entwining structures [4] and the coring maps originally considered in [17]. An induction functor is constructed, which is shown to have a right adjoint, called ad-induction functor. The separability of these two functors is characterized in terms which generalize both the previous results on rings [12] and on coalgebras [8]. Our approach rests on the fundamental characterization of the separability of adjoint functors given in [10, 14].

Consider an A-coring $\mathfrak{C}$ and a B-coring $\mathfrak{D}$, where A and B are K-algebras.

Definition 5.1. A *coring homomorphism* is a pair (φ, ρ), where $\rho : A \to B$ is a homomorphism of K-algebras and $\varphi : \mathfrak{C} \to \mathfrak{D}$ is a homomorphism of A-bimodules, and such that the following diagrams are commutative

$$
\begin{tikzcd}
\mathfrak{C} \arrow[rr, "\Delta_{\mathfrak{C}}"] \arrow[dd, "\varphi"] & & \mathfrak{C}\otimes_A\mathfrak{C} \arrow[dr, "\varphi\otimes_A\varphi"] & \\
 & & & \mathfrak{D}\otimes_A\mathfrak{D} \arrow[dl, "\omega_{\mathfrak{D},\mathfrak{D}}"] \\
\mathfrak{D} \arrow[rr, "\Delta_{\mathfrak{D}}"] & & \mathfrak{D}\otimes_B\mathfrak{D} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{C} \arrow[r, "\epsilon_{\mathfrak{C}}"] \arrow[d, "\varphi"] & A \arrow[d, "\rho"] \\
\mathfrak{D} \arrow[r, "\epsilon_{\mathfrak{D}}"] & B
\end{tikzcd}
\tag{5.1}
$$

where $\omega_{\mathfrak{D},\mathfrak{D}} : \mathfrak{D}\otimes_A\mathfrak{D} \to \mathfrak{D}\otimes_B\mathfrak{D}$ is the canonical map induced by $\rho : A \to B$.

Throughout this section, we consider a coring homomorphism $(\varphi,\rho) : \mathfrak{C} \to \mathfrak{D}$. We define the induction and ad-induction functors connecting the categories of comodules $\mathcal{M}^{\mathfrak{C}}$ and $\mathcal{M}^{\mathfrak{D}}$.

5.1. We start with some unavoidable technical work. For every B-bimodule X, denote by $\sigma_X : B\otimes_A X \to X\otimes_B B$ the B-bimodule morphism given by $b\otimes_A x \mapsto bx\otimes_B 1$. Given B-bimodules X, Y, a straightforward computation shows that the diagram

$$
\begin{tikzcd}
B\otimes_A X\otimes_A Y \arrow[r, "\sigma_X\otimes_A Y"] \arrow[d, "B\otimes_A\omega_{X,Y}"] & X\otimes_B B\otimes_A Y \arrow[d, "X\otimes_B\sigma_Y"] \\
B\otimes_A X\otimes_B Y \arrow[r, "\sigma_{X\otimes_B Y}"] & X\otimes_B Y\otimes_B B
\end{tikzcd}
\tag{5.2}
$$

commutes, where $\omega_{X,Y} : X\otimes_A Y \to X\otimes_B Y$ is the obvious map. We have as well that, for every homomorphism of B-bimodules $f : X \to Y$, the following diagram is commutative:

$$
\begin{tikzcd}
B\otimes_A X \arrow[r, "B\otimes_A f"] \arrow[d, "\sigma_X"] & B\otimes_A Y \arrow[d, "\sigma_Y"] \\
X\otimes_B B \arrow[r, "f\otimes_B B"] & Y\otimes_B B
\end{tikzcd}
\tag{5.3}
$$

5.2. The induction functor. Let $\lambda_M : M \to \mathfrak{C}\otimes_A M$ be a left $\mathfrak{C}$-comodule. Define $\widetilde{\lambda}_M : M \to \mathfrak{D}\otimes_A M$ as the composite map

$$
\begin{tikzcd}
M \arrow[rr, "\widetilde{\lambda}_M"'] \arrow[dr, "\lambda_M"'] & & \mathfrak{D}\otimes_A M \\
 & \mathfrak{C}\otimes_A M \arrow[ur, "\varphi\otimes_A M"'] &
\end{tikzcd}
\tag{5.4}
$$

and $\lambda_{B\otimes_A M} : B\otimes_A M \to \mathfrak{D}\otimes_B B\otimes_A M$ as

$$
\begin{tikzcd}
B\otimes_A M \arrow[rr, "\lambda_{B\otimes_A M}"] \arrow[dr, "B\otimes_A\widetilde{\lambda}_M"'] & & \mathfrak{D}\otimes_B B\otimes_A M \\
 & B\otimes_A\mathfrak{D}\otimes_A M \arrow[ur, "\sigma_{\mathfrak{D}}\otimes_A M"'] &
\end{tikzcd}
\tag{5.5}
$$

Proposition 5.2. *The homomorphism of left B-modules $\lambda_{B\otimes_A M}$ endows $B\otimes_A M$ with a structure of left $\mathfrak{D}$-comodule. This gives a functor $B\otimes_A - : {}^{\mathfrak{C}}\mathcal{M}\to{}^{\mathfrak{D}}\mathcal{M}$.*

Proof. Consider the diagram

$$
\begin{tikzcd}
M \arrow[rrrr, "\tilde{\lambda}_M"] \arrow[drr, "\lambda_M"] \arrow[ddr, "\lambda_M"'] \arrow[dddd, "\tilde{\lambda}_M"'] & & & & \mathfrak{D}\otimes_A M \arrow[d, "\mathfrak{D}\otimes_A\lambda_M"] \arrow[ddl, "\mathfrak{D}\otimes_A\tilde{\lambda}_M"'] \\
& & \mathfrak{C}\otimes_A M \arrow[urr, "\varphi\otimes_A M"] \arrow[dd, "\mathfrak{C}\otimes_A\lambda_M"] \arrow[dl, phantom, "(2)"] \arrow[dr, phantom, "(3)"] & & \mathfrak{D}\otimes_A\mathfrak{C}\otimes_A M \arrow[dl, "\mathfrak{D}\otimes_A\varphi\otimes_A M"] \\
& \mathfrak{C}\otimes_A M \arrow[dr, "\Delta_{\mathfrak{C}}\otimes_A M"] \arrow[ddl, "\varphi\otimes_A M"'] & & \mathfrak{D}\otimes_A\mathfrak{D}\otimes_A M \arrow[ddr, "\omega_{\mathfrak{D},\mathfrak{D}}\otimes_A M"'] & \\
& & \mathfrak{C}\otimes_A\mathfrak{C}\otimes_A M \arrow[ur, "\varphi\otimes_A\varphi\otimes_A M"] \arrow[d, phantom, "(1)"] & & \\
\mathfrak{D}\otimes_A M \arrow[rrrr, "\Delta_{\mathfrak{D}}\otimes_A M"'] & & {} & & \mathfrak{D}\otimes_B\mathfrak{D}\otimes_A M.
\end{tikzcd}
\tag{5.6}
$$

The pentagon labeled as (1) commutes since φ is a homomorphism of corings. The four-edged diagram (2) is commutative since M is a left comodule. The commutation of the quadrilateral (3) follows easily from the displayed decomposition of $\mathfrak{D}\otimes_A\tilde{\lambda}_M$. Therefore, the pentagon with bold arrows commutes. Now consider the diagram

$$
\begin{tikzcd}
B\otimes_A M \arrow[rr, bend left, "\lambda_{B\otimes_A M}"] \arrow[dddd, bend right, "\lambda_{B\otimes_A M}"'] \arrow[dd, "B\otimes_A\tilde{\lambda}_M"'] \arrow[dr, "B\otimes_A\tilde{\lambda}_M"] & & \mathfrak{D}\otimes_B B\otimes_A M \arrow[dd, "\mathfrak{D}\otimes_B B\otimes_A\tilde{\lambda}_M"] \arrow[dddd, bend left, "\mathfrak{D}\otimes_B\lambda_{B\otimes_A M}"] \\
& B\otimes_A\mathfrak{D}\otimes_A M \arrow[ur, "\sigma_{\mathfrak{D}}\otimes_A M"] \arrow[d, "B\otimes_A\mathfrak{D}\otimes_A\tilde{\lambda}_M"] \arrow[r, phantom, "(5)"] & {} \\
B\otimes_A\mathfrak{D}\otimes_A M \arrow[dr, "B\otimes_A\Delta_{\mathfrak{D}}\otimes_A M"'] \arrow[dd, "\sigma_{\mathfrak{D}\otimes_A M}"'] \arrow[ur, phantom, "(4)"] & B\otimes_A\mathfrak{D}\otimes_A\mathfrak{D}\otimes_A M \arrow[r, "\sigma_{\mathfrak{D}}\otimes_A\mathfrak{D}\otimes_A M"] \arrow[d, "B\otimes_A\omega_{\mathfrak{D},\mathfrak{D}}\otimes_A M"] & \mathfrak{D}\otimes_B B\otimes_A\mathfrak{D}\otimes_A M \arrow[dd, "\mathfrak{D}\otimes_B\sigma_{\mathfrak{D}}\otimes_A M"] \\
& B\otimes_A\mathfrak{D}\otimes_B\mathfrak{D}\otimes_A M \arrow[dr, "\sigma_{\mathfrak{D}\otimes_B\mathfrak{D}}\otimes_A M"] \arrow[r, phantom, "(6)"] \arrow[dl, phantom, "(7)"] & {} \\
\mathfrak{D}\otimes_B B\otimes_A M \arrow[rr, bend right, "\Delta_{\mathfrak{D}}\otimes_B B\otimes_A M"'] & & \mathfrak{D}\otimes_B\mathfrak{D}\otimes_B B\otimes_A M
\end{tikzcd}
\tag{5.7}
$$

We have proved before that (4) is commutative. Moreover, (5) is obviously commutative and (6) and (7) commute by (5.2) and (5.3). It follows that the outer curved diagram commutes, which gives the pseudo-coassocitative property for the coaction $\lambda_{B\otimes_A M}$. To check the counitary property, let $\iota_M : M \to A \otimes_A M$ be the canonical isomorphism. We get from the commutative diagram

$$\begin{tikzcd}
M \arrow[rr, "\tilde{\lambda}_M"] \arrow[dr, "\lambda_M"] \arrow[dd, "\iota_M"'] & & \mathfrak{D} \otimes_A M \arrow[dd, "\epsilon_{\mathfrak{D}}\otimes_A M"] \\
& \mathfrak{C} \otimes_A M \arrow[ur, "\varphi\otimes_A M"'] \arrow[dl, "\epsilon_{\mathfrak{C}}\otimes_A M"] & \\
A \otimes_A M \arrow[rr, "\rho\otimes_A M"'] & & B \otimes_A M
\end{tikzcd} \tag{5.8}$$

that the diagram

$$\begin{tikzcd}
B \otimes_A M \arrow[rr, "\lambda_{B\otimes_A M}"] \arrow[dr, "B\otimes_A\tilde{\lambda}_M"] \arrow[dd, "B\otimes_A\iota_M"] \arrow[ddrr, bend right, "\iota_{B\otimes_A M}"] & & \mathfrak{D} \otimes_B B \otimes_A M \arrow[dd, "\epsilon_{\mathfrak{D}}\otimes_B B\otimes_A M"] \\
& B \otimes_A \mathfrak{D} \otimes_A M \arrow[ur, "\sigma_{\mathfrak{D}}\otimes_A M"] \arrow[d, "B\otimes_A\epsilon_{\mathfrak{D}}\otimes_A M"'] & \\
B \otimes_A A \otimes_A M \arrow[r, "B\otimes_A\rho\otimes_A M"'] & B \otimes_A B \otimes_A M \arrow[r, "\sigma_B\otimes_A M"'] & B \otimes_B B \otimes_A M
\end{tikzcd} \tag{5.9}$$

commutes, where $\iota_{B\otimes_A M} : B \otimes_A M \to B \otimes_B B \otimes_A M$ denotes the canonical isomorphism. Therefore, $\lambda_{B\otimes_A M} : B \otimes_A M \to \mathfrak{D} \otimes_B B \otimes_A M$ is a left $\mathfrak{D}$-comodule structure map. In order to show that the assignment $M \mapsto B \otimes_A M$ is functorial, we will prove that $B \otimes_A f$ is a homomorphism of left $\mathfrak{D}$-comodules for every morphism $f : M \to N$ in ${}^{\mathfrak{C}}\mathcal{M}$. So, we have to show that the outer rectangle in the following diagram is commutative

$$\begin{tikzcd}
B \otimes_A M \arrow[rrr, "B\otimes_A f"] \arrow[dr, "B\otimes_A\tilde{\lambda}_M"] \arrow[dd, "\lambda_{B\otimes_A M}"'] & & & B \otimes_A N \arrow[dl, "B\otimes_A\tilde{\lambda}_N"'] \arrow[dd, "\lambda_{B\otimes_A N}"] \\
& B \otimes_A \mathfrak{D} \otimes_A M \arrow[r, "B\otimes_A\mathfrak{D}\otimes_A f"] \arrow[dl, "\sigma_{\mathfrak{D}}\otimes_A M"] & B \otimes_A \mathfrak{D} \otimes_A N \arrow[dr, "\sigma_{\mathfrak{D}}\otimes_A N"'] & \\
\mathfrak{D} \otimes_B B \otimes_A M \arrow[rrr, "\mathfrak{D}\otimes_B B\otimes_A f"'] & & & \mathfrak{D} \otimes_B B \otimes_M .
\end{tikzcd} \tag{5.10}$$

From the definition of $\tilde{\lambda}_M$, $\tilde{\lambda}_N$ and the fact that f is a morphism of $\mathfrak{C}$-comodules, it follows that the upper trapezium commutes. The lower trapezium is commutative by (5.3). Since the two triangles commute by definition, we get that the outer rectangle is commutative and $B \otimes_A f$ is a morphism in ${}^{\mathfrak{D}}\mathcal{M}$. $\square$

5.3. Proposition 5.2 also implies by symmetry that for every right $\mathfrak{C}$-comodule $\rho_M : M \to M \otimes_A \mathfrak{C}$, the right B-module $M \otimes_A B$ is endowed with a right $\mathfrak{D}$-comodule structure $\rho_{M\otimes_A B} : M \otimes_A B \to M \otimes_A B \otimes_B \mathfrak{D}$ given by $\rho_{M\otimes_A B} = (M \otimes_A \delta_{\mathfrak{D}})(\widetilde{\rho}_M \otimes_A B)$, where $\widetilde{\rho}_M = (M \otimes_A \varphi)\rho_M$ and $\delta_{\mathfrak{D}} : \mathfrak{D} \otimes_A B \to B \otimes_B \mathfrak{D}$ is the obvious map. We can already state the following.

Proposition 5.3. *The assignment $M \mapsto M \otimes_A B$ establishes a functor $- \otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$.*

5.4. The ad-induction functor. Consider the left $\mathfrak{D}$-comodule structure

$$\lambda_{B\otimes_A\mathfrak{C}} : B \otimes_A \mathfrak{C} \longrightarrow \mathfrak{D} \otimes_B B \otimes_A \mathfrak{C} \tag{5.11}$$

defined on $B \otimes_A \mathfrak{C}$ in Section 5.2.

We have, as well, a canonical structure of right $\mathfrak{C}$-comodule

$$B \otimes_A \Delta_{\mathfrak{C}} : B \otimes_A \mathfrak{C} \longrightarrow B \otimes_A \mathfrak{C} \otimes_A \mathfrak{C}. \tag{5.12}$$

Proposition 5.4. *The $B-A$-bimodule $B \otimes_A \mathfrak{C}$ is a $\mathfrak{D}-\mathfrak{C}$-bicomodule, which is quasi-finite as a right $\mathfrak{C}$-comodule. Therefore, if ${}_A\mathfrak{C}$ preserves the equalizer of $(\rho_Y \otimes_B B \otimes_A \mathfrak{C}, Y \otimes_B \lambda_{B\otimes_A\mathfrak{C}})$ for every right $\mathfrak{D}$-comodule Y, then the functor*

$$-\square_{\mathfrak{D}}(B \otimes_A \mathfrak{C}) : \mathcal{M}^{\mathfrak{D}} \longrightarrow \mathcal{M}^{\mathfrak{C}} \tag{5.13}$$

is right adjoint to

$$- \otimes_A B : \mathcal{M}^{\mathfrak{C}} \longrightarrow \mathcal{M}^{\mathfrak{D}}. \tag{5.14}$$

Proof. Since the comultiplication $\Delta_{\mathfrak{C}}$ is coassociative, we get that the diagram

$$\begin{tikzcd}
\mathfrak{C} \arrow[rr, "\widetilde{\lambda}_{\mathfrak{C}}"] \arrow[dr, "\Delta_{\mathfrak{C}}"] \arrow[ddd, "\Delta_{\mathfrak{C}}"'] & & \mathfrak{D}\otimes_A\mathfrak{C} \arrow[ddd, "\mathfrak{D}\otimes_A\Delta_{\mathfrak{C}}"] \\
& \mathfrak{C}\otimes_A\mathfrak{C} \arrow[ur, "\varphi\otimes_A\mathfrak{C}"'] \arrow[d, "\mathfrak{C}\otimes_A\Delta_{\mathfrak{C}}"] & \\
& \mathfrak{C}\otimes_A\mathfrak{C}\otimes_A\mathfrak{C} \arrow[dr, "\varphi\otimes_A\mathfrak{C}\otimes_A\mathfrak{C}"] & \\
\mathfrak{C}\otimes_A\mathfrak{C} \arrow[ur, "\Delta_{\mathfrak{C}}\otimes_A\mathfrak{C}"] \arrow[rr, "\widetilde{\lambda}_{\mathfrak{C}\otimes_A\mathfrak{C}}"'] & & \mathfrak{D}\otimes_A\mathfrak{C}\otimes_A\mathfrak{C}
\end{tikzcd} \tag{5.15}$$

is commutative. This implies that the left trapezium of the following diagram is commutative:

$$\begin{tikzcd}
B \otimes_A \mathfrak{C} \arrow[rr, "\lambda_{B\otimes_A \mathfrak{C}}"] \arrow[dddd, "B\otimes_A \Delta_{\mathfrak{C}}"'] \arrow[dr, "B\otimes_A \tilde{\lambda}_{\mathfrak{C}}"'] & & \mathfrak{D} \otimes_B B \otimes_A \mathfrak{C} \arrow[dddd, "\mathfrak{D}\otimes_B B\otimes_A \Delta_{\mathfrak{C}}"] \\
& B \otimes_A \mathfrak{D} \otimes_A \mathfrak{C} \arrow[ur, "\sigma_{\mathfrak{D}}\otimes_A \mathfrak{C}"'] \arrow[d, "B\otimes_A \mathfrak{D}\otimes_A \Delta_{\mathfrak{C}}"] & \\
& B \otimes_A \mathfrak{D} \otimes_A \mathfrak{C} \otimes_A \mathfrak{C} \arrow[ddr, "\sigma_{\mathfrak{D}}\otimes_A \mathfrak{C}\otimes_A \mathfrak{C}"] & \\
& & \\
B \otimes_A \mathfrak{C} \otimes_A \mathfrak{C} \arrow[rr, "\lambda_{B\otimes_A \mathfrak{C}\otimes_A \mathfrak{C}}"] \arrow[uur, "B\otimes_A \tilde{\lambda}_{\mathfrak{C}}\otimes_A \mathfrak{C}"] & & \mathfrak{D} \otimes_B B \otimes_A \mathfrak{C} \otimes_A \mathfrak{C}
\end{tikzcd} \tag{5.16}$$

Since we know that the rest of inner diagrams are also commutative, we obtain that the outer diagram commutes, too. This proves that $B \otimes_A \mathfrak{C}$ is a $\mathfrak{D}-\mathfrak{C}$-bicomodule.

We will see that $B \otimes_A \mathfrak{C}$ is a quasi-finite right $\mathfrak{C}$-comodule, that is, that $-\otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^B$ is left adjoint to $-\otimes_B B \otimes_A \mathfrak{C} : \mathcal{M}^B \to \mathcal{M}^{\mathfrak{C}}$. Now, these functors fit in the commutative diagrams

$$\begin{tikzcd}
\mathcal{M}^A \arrow[r, "-\otimes_A \mathfrak{C}"] & \mathcal{M}^{\mathfrak{C}} \\
\mathcal{M}^B \arrow[u, "-\otimes_B B"] \arrow[ur, "-\otimes_B B\otimes_A \mathfrak{C}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}^{\mathfrak{C}} \arrow[r, "U_A"] \arrow[dr, "-\otimes_A B"'] & \mathcal{M}^A \arrow[d, "-\otimes_A B"] \\
& \mathcal{M}^B
\end{tikzcd} \tag{5.17}$$

where U_A denotes the forgetful functor. Since $-\otimes_A B$ is left adjoint to $-\otimes_B B$ and U_A is left adjoint to $-\otimes_A \mathfrak{C}$, we get the desired adjunction. The rest of the proposition follows from Proposition 4.2. □

Remark 5.5. This proposition applies in the case that ${}_A\mathfrak{C}$ is flat or when $\mathfrak{D}$ is a coseparable B-coring in the sense of [11].

5.5. The unit. Let $\bar{\Delta}_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C} \otimes_A B \otimes_B B \otimes_A \mathfrak{C}$ be the composite map

$$\mathfrak{C} \xrightarrow{\Delta_{\mathfrak{C}}} \mathfrak{C} \otimes_A \mathfrak{C} \to \iota\, \mathfrak{C} \otimes_A B \otimes_B B \otimes_A \mathfrak{C}, \tag{5.18}$$

where ι maps $c \otimes c' \in \mathfrak{C} \otimes_A \mathfrak{C}$ to $c \otimes 1 \otimes 1 \otimes c'$. This map is a homomorphism of $\mathfrak{C}$-bicomodules. The unit θ of the adjunction $-\otimes_A B \dashv -\otimes_B B \otimes_A \mathfrak{C}$ at a right $\mathfrak{C}$-comodule M is given by the composite map

$$M \xrightarrow{\rho_M} M \otimes_A \mathfrak{C} \xrightarrow{\varsigma_M \otimes_A \mathfrak{C}} M \otimes_A B \otimes_B B \otimes_A \mathfrak{C}, \tag{5.19}$$

where $\varsigma_M : M \to M \otimes_A B \otimes_B B$ maps $m \in M$ to $m \otimes_A 1 \otimes_B 1$. We see, in particular, that $\theta_{\mathfrak{C}} = \bar{\Delta}_{\mathfrak{C}}$. By Proposition 4.2, θ_M factorizes throughout $(M \otimes_A B)\square_{\mathfrak{D}}(B \otimes_A \mathfrak{C})$ and, therefore, it gives the unit θ_M for the adjunction $-\otimes_A B \dashv -\square_{\mathfrak{D}}(B \otimes_A \mathfrak{C})$ at M. So, the multiplication $\Delta_{\mathfrak{C}}$ finally induces a map,

$$\bar{\Delta}_{\mathfrak{C}} : \mathfrak{C} \longrightarrow (\mathfrak{C} \otimes_A B)\square_{\mathfrak{D}}(B \otimes_A \mathfrak{C}), \tag{5.20}$$

which is a homomorphism of $\mathfrak{C}$-bicomodules.

We are now ready to state the characterization of the separability of the induction functor.

Theorem 5.6. *Assume that $\mathfrak{C}_A$ preserves the equalizer of $(\rho_Y \otimes_B B \otimes_B \otimes_A \mathfrak{C}, Y \otimes_B \lambda_{B\otimes_A \mathfrak{C}})$ for every $Y \in \mathcal{M}^{\mathfrak{D}}$, and that X_A preserves the equalizer of $(\rho_{\mathfrak{C}\otimes_A B} \otimes_B B \otimes_A \mathfrak{C}, \mathfrak{C} \otimes_A B \otimes_B \lambda_{B\otimes_A \mathfrak{C}})$ for every $X \in \mathcal{M}^{\mathfrak{C}}$. The functor $- \otimes_A B : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^{\mathfrak{D}}$ is separable if and only if there is a homomorphism if $\mathfrak{C}$-bicomodules*

$$\omega_{\mathfrak{C}} : (\mathfrak{C} \otimes_A B) \square_{\mathfrak{D}} (B \otimes_A \mathfrak{C}) \longrightarrow \mathfrak{C} \tag{5.21}$$

such that $\omega_{\mathfrak{C}} \bar{\Delta}_{\mathfrak{C}} = \mathfrak{C}$.

Proof. Assume that $- \otimes_A B$ is separable. By [10, Theorem 4.1] or [14, Theorem 1.2], the unit of the adjunction

$$\theta : 1_{\mathcal{M}^{\mathfrak{C}}} \longrightarrow (- \otimes_A B) \square_{\mathfrak{D}} (B \otimes_A \mathfrak{D}) \tag{5.22}$$

is split-mono, that is, there is a natural transformation $\omega : (- \otimes_A B) \square_{\mathfrak{D}} (B \otimes_A \mathfrak{C}) \to 1_{\mathcal{M}^{\mathfrak{C}}}$ such that $\omega\theta = 1_{\mathcal{M}^{\mathfrak{C}}}$. By Lemma 2.2 the functor $(- \otimes_A B) \square_{\mathfrak{D}} (B \otimes_A \mathfrak{C})$ is $\mathfrak{C}$-compatible. By Proposition 3.4, $\omega_{\mathfrak{C}}$ is a $\mathfrak{C}$-bicomodule map. Obviously, $\omega_{\mathfrak{C}} \bar{\Delta}_{\mathfrak{C}} = \mathfrak{C}$.

To prove the converse, we need to construct a natural transformation ω from the bicomodule map $\omega_{\mathfrak{C}}$. Given a right $\mathfrak{C}$-comodule M, consider the diagram

$$\begin{tikzcd}
M \otimes_A B \otimes_B \mathfrak{D} \otimes_B B \otimes_A \mathfrak{C} \arrow[r, "\rho_M \otimes_A B \otimes_B \mathfrak{D} \otimes_B B \otimes_A \mathfrak{C}"] & M \otimes_A \mathfrak{C} \otimes_A B \otimes_B \mathfrak{D} \otimes_B B \otimes_A \mathfrak{C} \\
M \otimes_A B \otimes_B B \otimes_A \mathfrak{C} \arrow[u, shift left, "\rho_{M\otimes_A B} \otimes_B B \otimes_A \mathfrak{C}"] \arrow[u, shift right, "M \otimes_A B \otimes_B \lambda_{B\otimes_A \mathfrak{C}}"'] \arrow[r, "\rho_M \otimes_A B \otimes_B B \otimes_A \mathfrak{C}"] & M \otimes_A \mathfrak{C} \otimes_A B \otimes_B B \otimes_A \mathfrak{C} \arrow[u, shift left, "M \otimes_A \rho_{\mathfrak{C}\otimes_A B} \otimes_B B \otimes_A \mathfrak{C}"] \arrow[u, shift right, "M \otimes_A \mathfrak{C} \otimes_A B \otimes_B \lambda_{B\otimes_A \mathfrak{C}}"'] \\
(M \otimes_A B) \square_{\mathfrak{D}} (B \otimes_A \mathfrak{C}) \arrow[u] \arrow[r, dotted, "\kappa_M"] & M \otimes_A ((\mathfrak{C} \otimes_A B) \square_{\mathfrak{D}} (B \otimes_A \mathfrak{C})) \arrow[u]
\end{tikzcd} \tag{5.23}$$

where the vertical are equalizer diagrams (here, we are using the definition of the cotensor product and the fact that M_A preserves the equalizer of $(\rho_{\mathfrak{C}\otimes_A B} \otimes_B B \otimes_A \mathfrak{C}, \mathfrak{C} \otimes_A B \otimes_B \lambda_{B\otimes_A \mathfrak{C}})$). If we prove that the top rectangle commutes, then there is a unique dotted arrow κ_M making the bottom rectangle commute. The identity

$$\left(M \otimes_A \mathfrak{C} \otimes_A B \otimes_B \lambda_{B\otimes_A \mathfrak{C}}\right)\left(\rho_M \otimes_A B \otimes_B B \otimes_A \mathfrak{C}\right) = \left(\rho_M \otimes_A B \otimes_B \mathfrak{D} \otimes_B B \otimes_A \mathfrak{C}\right)\left(M \otimes_A B \otimes_B \lambda_{B\otimes_A \mathfrak{C}}\right) \tag{5.24}$$

is obvious; so we need just to prove that

$$\begin{aligned} &\left(M \otimes_A \rho_{\mathfrak{C}\otimes_A B} \otimes_B B \otimes_A \mathfrak{C}\right)\left(\rho_M \otimes_A B \otimes_B B \otimes_A \mathfrak{C}\right) \\ &\quad = \left(\rho_M \otimes_A B \otimes_B B \otimes_B \mathfrak{D} \otimes_B B \otimes_A \mathfrak{C}\right)\left(\rho_{M\otimes_A B} \otimes_B B \otimes_A \mathfrak{C}\right), \end{aligned} \tag{5.25}$$

which is equivalent to

$$\left(M \otimes_A \rho_{\mathfrak{C}\otimes_A B}\right)\left(\rho_M \otimes_A B\right) = \left(\rho_M \otimes_A B \otimes_B \mathfrak{D}\right)\rho_{M\otimes_A B}, \tag{5.26}$$

and this last identity is easy to check. Now, consider the diagram

$$
\begin{tikzcd}
M \arrow[rrr, "\theta_M"] \arrow[ddd, "\rho_M"'] \arrow[dr, equal] & & & M \otimes_A B \otimes_B B \otimes_A \mathfrak{C} \arrow[ddd, "\rho_M \otimes_A B \otimes_B B \otimes_A \mathfrak{C}"'] \\
& M \arrow[r, "\theta_M"] \arrow[d, "\rho_M"] & (M \otimes_A B) \square_{\mathfrak{D}} (B \otimes_A \mathfrak{C}) \arrow[d, "\kappa_M"] \arrow[ur, hook] & \\
& M \otimes_A \mathfrak{C} \arrow[r, "M \otimes_A \bar{\Delta}_{\mathfrak{C}}"] \arrow[dl, equal] & M \otimes_A ((\mathfrak{C} \otimes_A B) \square_{\mathfrak{D}} (B \otimes_A \mathfrak{C})) \arrow[dr, hook] & \\
M \otimes_A \mathfrak{C} \arrow[rrr, "M \otimes_A \bar{\Delta}_{\mathfrak{C}}"] & & & M \otimes_A \mathfrak{C} \otimes_A B \otimes_B B \otimes_A \mathfrak{C}
\end{tikzcd}
\tag{5.27}
$$

By Lemma 3.2, we have that $M \otimes_A \theta_M = \theta_{M \otimes_A \mathfrak{C}}$. Since θ is natural, and $\theta_M = \bar{\Delta}_{\mathfrak{C}}$, this implies that the external rectangle commutes. We know that the four trapezia commute, whence the internal rectangle commutes as well. Define $\omega_M = (M \otimes_A \epsilon_{\mathfrak{C}})(M \otimes_B \omega_{\mathfrak{C}})\kappa_M$, which gives a natural transformation $\omega : (- \otimes_A B) \square_{\mathfrak{D}} (B \otimes_A \mathfrak{C}) \to 1_{\mathcal{M}^{\mathfrak{C}}}$. Moreover,

$$
\begin{aligned}
\omega_M \theta_M &= (M \otimes_A \epsilon_{\mathfrak{C}})(M \otimes_A \omega_{\mathfrak{C}})\kappa_M \theta_M \\
&= (M \otimes_A \epsilon_{\mathfrak{C}})(M \otimes_A \omega_{\mathfrak{C}})(M \otimes_A \theta_M)\rho_M \\
&= (M \otimes_A \epsilon_M)\rho_M = M.
\end{aligned}
\tag{5.28}
$$

Therefore, $\omega\theta = 1_{\mathcal{M}^{\mathfrak{C}}}$ and, by [14, Theorem 1.2], the functor $- \otimes_A B$ is separable. □

The counit map $\epsilon_{\mathfrak{C}} : \mathfrak{C} \to A$ is a homomorphism of A-corings, where we consider on A the canonical A-coring structure. When applied to $\epsilon_{\mathfrak{C}}$, Theorem 5.6 boils down to the following corollary.

Corollary 5.7 (see [3, Theorem 3.5]). For an A-coring $\mathfrak{C}$, the forgetful functor $U_A : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}^A$ is separable if and only if there is an A-bimodule map $\gamma : \mathfrak{C} \otimes_A \mathfrak{C} \to A$ such that $\gamma\Delta_{\mathfrak{C}} = \epsilon_{\mathfrak{C}}$ and, in Sweedler's sigma notation,

$$
c_{(1)}\gamma(c_{(2)} \otimes_A c') = \gamma(c \otimes_A c'_{(1)})c'_{(2)} \quad \forall c, c' \in \mathfrak{C}. \tag{5.29}
$$

Proof. Obviously, the forgetful functor coincides with $- \otimes_A A$, so that we get from Theorem 5.6 a characterization of the separability of this functor in terms of the existence of a $\mathfrak{C}$-bicomodule map $\omega_{\mathfrak{C}} : \mathfrak{C} \otimes_A \mathfrak{C} \to \mathfrak{C}$ such that $\omega_{\mathfrak{C}}\Delta_{\mathfrak{C}} = \mathfrak{C}$. Now, notice that the adjointness isomorphism

$$
\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{C} \otimes_A \mathfrak{C}, \mathfrak{C}) \cong \mathrm{Hom}_A(\mathfrak{C} \otimes_A \mathfrak{C}, A) \tag{5.30}
$$

transfers faithfully the mentioned properties of $\omega_{\mathfrak{C}}$ to the desired properties of $\gamma = \epsilon_{\mathfrak{C}}\omega_{\mathfrak{C}}$. $\square$

5.6. The counit of the adjunction. Let $\hat{\epsilon}_{\mathfrak{C}}$ be the homomorphism of B-bimodules that makes commute the following diagram

$$\begin{tikzcd} B\otimes_A \mathfrak{C}\otimes_A B \arrow[rr, "\hat{\epsilon}_{\mathfrak{C}}"] \arrow[d, "B\otimes_A\epsilon_{\mathfrak{C}}\otimes_A B"] & & B \\ B\otimes_A A\otimes_A B \arrow[r, "B\otimes_A\rho\otimes_A B"] & B\otimes_A B\otimes_A B \arrow[ur, "m"'] & \end{tikzcd} \tag{5.31}$$

where $m : B\otimes_A B\otimes_A B \to B$ is the obvious multiplication map. Define, for every right B-module Y, $\chi_Y = \mu_Y(Y\otimes_A\hat{\epsilon}_{\mathfrak{C}})$, where $\mu_Y : Y\otimes_B B \to Y$ is the canonical isomorphism. This natural transformation χ gives the counit of the adjunction $-\otimes_A B \dashv -\otimes_B(B\otimes_A\mathfrak{C})$. By Section 4.2, the counit of the adjunction $-\otimes_A B \dashv -\square_{\mathfrak{D}}(B\otimes_A\mathfrak{C})$ is given by the restriction of χ to $(-\square_{\mathfrak{D}}(B\otimes_A\mathfrak{C}))\otimes_A B$. We use the same notation for this counit. Now, define $\hat{\varphi}$ as the B-bimodule map completing the diagram

$$\begin{tikzcd} B\otimes_A\mathfrak{C}\otimes_A B \arrow[r, "\hat{\varphi}"] \arrow[d, "B\otimes_A\varphi\otimes_A B"'] & \mathfrak{D} \\ B\otimes_A\mathfrak{D}\otimes_A B \arrow[ur, "m_{\mathfrak{D}}"'] & \end{tikzcd} \tag{5.32}$$

where $m_{\mathfrak{D}} : B\otimes_A\mathfrak{D}\otimes_A B \to \mathfrak{D}$ is the obvious multiplication map given by the B-bimodule structure of $\mathfrak{D}$. We claim that $\hat{\varphi}$ is a $\mathfrak{D}$-comodule map. To prove this, we first show that the diagram

$$\begin{tikzcd} (\mathfrak{D}\square_{\mathfrak{D}}(B\otimes_A\mathfrak{C}))\otimes_A B \arrow[r] & \mathfrak{D}\otimes_B B\otimes_A\mathfrak{C}\otimes_A B \arrow[d, "\chi_{\mathfrak{D}}"] \\ B\otimes_A\mathfrak{C}\otimes_A B \arrow[u, "\lambda_{B\otimes_A\mathfrak{C}\otimes_A B}"] \arrow[r, "\hat{\varphi}"] & \mathfrak{D} \end{tikzcd} \tag{5.33}$$

is commutative. This is done by the following computation, where $\widetilde{m} : B\otimes_B B\otimes_A B \to B$ denotes the obvious multiplication map

$$\begin{aligned} \chi_{\mathfrak{D}}\left(\lambda_{B\otimes_A\mathfrak{C}\otimes_A B}\right) &= \mu_{\mathfrak{D}}\left(\mathfrak{D}\otimes_A\hat{\epsilon}_{\mathfrak{C}}\right)\left(\sigma_{\mathfrak{D}}\otimes_A\mathfrak{C}\otimes_A B\right)\left(B\otimes_A\widetilde{\lambda}_{\mathfrak{C}}\otimes_A B\right) \\ &= \mu_{\mathfrak{D}}\left(\mathfrak{D}\otimes_B m\right)\left(\mathfrak{D}\otimes_B B\otimes_A\rho\otimes_A B\right)\left(\mathfrak{D}\otimes_B B\otimes_A\epsilon_{\mathfrak{C}}\otimes_A\mathfrak{D}\right) \\ &\quad\circ\left(\sigma_{\mathfrak{D}}\otimes_A\mathfrak{C}\otimes_A B\right)\left(B\otimes_A\varphi\otimes_A\mathfrak{C}\otimes_A B\right)\left(B\otimes_A\Delta_{\mathfrak{C}}\otimes_A B\right) \\ &= \mu_{\mathfrak{D}}\left(\mathfrak{D}\otimes_B m\right)\left(\mathfrak{D}\otimes_B B\otimes_A\epsilon_{\mathfrak{D}}\otimes_A B\right)\left(\mathfrak{D}\otimes_B B\otimes_A\varphi\otimes_A B\right) \\ &\quad\circ\left(\sigma_{\mathfrak{D}}\otimes_A\mathfrak{C}\otimes_A B\right)\left(B\otimes_A\varphi\otimes_A\mathfrak{C}\otimes_A B\right)\left(B\otimes_A\Delta_{\mathfrak{C}}\otimes_A B\right) \\ &= \mu_{\mathfrak{D}}\left(\mathfrak{D}\otimes_B m\right)\left(\mathfrak{D}\otimes_B B\otimes_A\epsilon_{\mathfrak{D}}\otimes_A B\right)\left(\sigma_{\mathfrak{D}}\otimes_A\varphi\otimes_A B\right) \\ &\quad\circ\left(B\otimes_A\varphi\otimes_A\mathfrak{C}\otimes_A B\right)\left(B\otimes_A\Delta_{\mathfrak{C}}\otimes_A B\right) \end{aligned}$$

$$\begin{aligned}
&= \mu_{\mathfrak{D}}\left(\mathfrak{D}\otimes_B m\right)\left(\mathfrak{D}\otimes_B B\otimes_A \epsilon_{\mathfrak{D}}\otimes_A B\right)\left(\sigma_{\mathfrak{D}}\otimes_A \mathfrak{D}\otimes_A B\right)\\
&\quad\circ\left(B\otimes_A \varphi\otimes_A \varphi\otimes_A B\right)\left(B\otimes_A \Delta_{\mathfrak{C}}\otimes_A B\right)\\
&= \mu_{\mathfrak{D}}\left(\mathfrak{D}\otimes_B \widetilde{m}\right)\left(\mathfrak{D}\otimes_B B\otimes_B \epsilon_{\mathfrak{D}}\otimes_A B\right)\left(\sigma_{\mathfrak{D}}\otimes_B \mathfrak{D}\otimes_A B\right)\left(B\otimes_A \omega_{\mathfrak{D},\mathfrak{D}}\otimes_A B\right)\\
&\quad\circ\left(B\otimes_A \varphi\otimes_A \varphi\otimes_A B\right)\left(B\otimes_A \Delta_{\mathfrak{D}}\otimes_A B\right)\\
&= \mu_{\mathfrak{D}}\left(\mathfrak{D}\otimes_B \widetilde{m}\right)\left(\mathfrak{D}\otimes_B B\otimes_B \epsilon_{\mathfrak{D}}\otimes_A B\right)\left(\sigma_{\mathfrak{D}}\otimes_B \mathfrak{D}\otimes_A B\right)\\
&\quad\circ\left(B\otimes_A \Delta_{\mathfrak{D}}\otimes_A B\right)\left(B\otimes_A \varphi\otimes_A B\right)\\
&= \mu_{\mathfrak{D}}\left(\mathfrak{D}\otimes_B \widetilde{m}\right)\left(\sigma_{\mathfrak{D}}\otimes_B \epsilon_{\mathfrak{D}}\otimes_A B\right)\left(B\otimes_A \Delta_{\mathfrak{D}}\otimes_A B\right)\left(B\otimes_A \varphi\otimes_A B\right)\\
&= \mu_{\mathfrak{D}}\left(\mathfrak{D}\otimes_B \widetilde{m}\right)\left(\sigma_{\mathfrak{D}}\otimes_B B\otimes_A B\right)\left(B\otimes_A \mathfrak{D}\otimes_B \epsilon_{\mathfrak{D}}\otimes_A B\right)\\
&\quad\circ\left(B\otimes_A \Delta_{\mathfrak{D}}\otimes_A B\right)\left(B\otimes_A \varphi\otimes_A B\right)\\
&= \mu_{\mathfrak{D}}\left(\mathfrak{D}\otimes_B \widetilde{m}\right)\left(\sigma_{\mathfrak{D}}\otimes_B B\otimes_A B\right)\left(B\otimes_A \varphi\otimes_A B\right)\\
&= \mu_{\mathfrak{D}}\left(\mathfrak{D}\otimes_B m\right)\left(\sigma_{\mathfrak{D}}\otimes_A B\otimes_A B\right)\left(B\otimes_A \varphi\otimes_A B\right) = \hat{\varphi}.
\end{aligned} \tag{5.34}$$

We know that $\lambda_{B\otimes_A \mathfrak{C}}\otimes_A B$ is a homomorphism of $\mathfrak{D}$-bicomodules and that, by Section 3.2, $\chi_{\mathfrak{D}}$ is $\mathfrak{D}$-bicolinear, too. This proves that $\hat{\varphi} : B\otimes_A \mathfrak{C}\otimes_A B \to \mathfrak{D}$ is a homomorphism of $\mathfrak{D}$-bicomodules.

We are now in a position to prove our separability theorem for the ad-induction functor.

Theorem 5.8. *Assume that ${}_AB$ and ${}_A\mathfrak{C}$ preserve the equalizer of $(\rho_M\otimes_B B\otimes_A \mathfrak{C}, M\otimes_B \lambda_{B\otimes_A \mathfrak{C}})$ for every right $\mathfrak{D}$-comodule M (e.g., ${}_AB$ and ${}_A\mathfrak{C}$ are flat or $\mathfrak{D}$ is a coseparable B-coring in the sense of* [11]*). The functor $-\square_{\mathfrak{D}}(B\otimes_A \mathfrak{C}) : \mathcal{M}^{\mathfrak{D}} \to \mathcal{M}^{\mathfrak{C}}$ is separable if and only if there exists a $\mathfrak{D}$-bicomodule homomorphism $\hat{\nu}_{\mathfrak{D}} : \mathfrak{D} \to B\otimes_A \mathfrak{C}\otimes_A B$ such that $\hat{\varphi}\hat{\nu}_{\mathfrak{D}} = \mathfrak{D}$.*

Proof. If $-\square_{\mathfrak{D}}(B\otimes_A \mathfrak{C})$ is separable then, by [14, Theorem 1.2], there exists a natural transformation

$$\nu : 1_{\mathcal{M}^{\mathfrak{D}}} \longrightarrow \left(-\square_{\mathfrak{D}}(B\otimes_A \mathfrak{C})\right)\otimes_A B \tag{5.35}$$

such that $\chi\nu = 1_{\mathcal{M}^{\mathfrak{D}}}$. In particular, $\chi_{\mathfrak{D}}\nu_{\mathfrak{D}} = \mathfrak{D}$ and, by Proposition 3.4, $\nu_{\mathfrak{D}}$ is a bicomodule map (in fact, we easily get that the functor $(-\square_{\mathfrak{D}}(B\otimes_A \mathfrak{C}))\otimes_A B$ is $\mathfrak{D}$-compatible from the fact that $\mathfrak{D}\square_{\mathfrak{D}}(B\otimes_A \mathfrak{C})$ is a direct summand of $\mathfrak{D}\otimes_B B\otimes_A \mathfrak{C}$ as a left B-module). The map $\lambda_{B\otimes_A \mathfrak{C}}\otimes_A B$ gives an isomorphism of $\mathfrak{D}$-bicomodules $B\otimes_A \mathfrak{C}\otimes_A B \cong (\mathfrak{D}\square_D(B\otimes_A \mathfrak{C}))\otimes_A B$, which implies, after (5.33), that

$$\mathfrak{D} = \chi_{\mathfrak{D}}\nu_{\mathfrak{D}} = \hat{\varphi}\left(\lambda_{B\otimes_A \mathfrak{C}}\otimes_A B\right)^{-1}\nu_{\mathfrak{D}}\ . \tag{5.36}$$

Thus, $\hat{\nu}_{\mathfrak{D}} = (\lambda_{B\otimes_A \mathfrak{C}}\otimes_A B)^{-1}\nu_{\mathfrak{D}}$ is the desired $\mathfrak{D}$-bicomodule map.

For the converse, assume that there is a $\mathfrak{D}$-bicomodule map $\hat{\nu}_{\mathfrak{D}} : \mathfrak{D} \to B\otimes_A \mathfrak{C}\otimes_A B$ such that $\hat{\varphi}\hat{\nu}_{\mathfrak{D}} = \mathfrak{D}$. For each right $\mathfrak{D}$-comodule M, we prove that the homomorphism of the right $\mathfrak{D}$-comodules

$$M \xrightarrow{\rho_M} M\otimes_B \mathfrak{D} \xrightarrow{M\otimes_B \hat{\nu}_{\mathfrak{D}}} M\otimes_B B\otimes_A \mathfrak{C}\otimes_A B \tag{5.37}$$

factorizes throughout $(M\square_{\mathfrak{D}}(B\otimes_A\mathfrak{C}))\otimes_A B$. Since ${}_AB$ preserves the equalizer of $(\rho_M\otimes_B B\otimes_A\mathfrak{C}, M\otimes_B\lambda_{B\otimes_A\mathfrak{C}})$, we know that

$$(M\square_{\mathfrak{D}}(B\otimes_A\mathfrak{C}))\otimes_A B\cong M\square_{\mathfrak{D}}(B\otimes_A\mathfrak{C}\otimes_A B). \tag{5.38}$$

Therefore, we need just to check the equality

$$(\rho_M\otimes_B B\otimes_A\mathfrak{C}\otimes_A B)(M\otimes_B\hat{\nu}_{\mathfrak{D}})\rho_M=(M\otimes_B\lambda_{B\otimes_A\mathfrak{C}\otimes_A B})(M\otimes_B\hat{\nu}_{\mathfrak{D}})\rho_M. \tag{5.39}$$

This is done by the following computation:

$$\begin{aligned}
&(M\otimes_B\lambda_{B\otimes_A\mathfrak{C}\otimes_A B})(M\otimes_B\hat{\nu}_{\mathfrak{D}})\rho_M\\
&\quad=(M\otimes_B\mathfrak{D}\otimes_B\hat{\nu}_{\mathfrak{D}})(M\otimes_B\Delta_{\mathfrak{D}})\rho_M=(M\otimes_B\mathfrak{D}\otimes_B\hat{\nu}_{\mathfrak{D}})\rho_M\\
&\quad=(\rho_M\otimes_B\hat{\nu}_{\mathfrak{D}})\rho_M=(\rho_M\otimes_B B\otimes_A\mathfrak{C}\otimes_A B)(M\otimes_B\hat{\nu}_{\mathfrak{D}})\rho_M.
\end{aligned}\tag{5.40}$$

Thus, we have proved that the natural transformation given in (5.37) factorizes throughout a natural transformation

$$\nu_M: M\longrightarrow (M\square_{\mathfrak{D}}(B\otimes_A\mathfrak{C}))\otimes_A B. \tag{5.41}$$

This means that we have a commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\rho_M"] \arrow[drr, "\nu_M"] & M\otimes_B\mathfrak{D} \arrow[r, "M\otimes_B\hat{\nu}_{\mathfrak{D}}"] & M\otimes_B B\otimes_A\mathfrak{C}\otimes_A B \arrow[dr, "\chi_M"] & \\
 & & (M\square_{\mathfrak{D}}(B\otimes_A\mathfrak{C}))\otimes_A B \arrow[u] \arrow[r, "\chi_M"'] & M
\end{tikzcd} \tag{5.42}$$

Finally, we show that ν_M splits off χ_M by means of the following computation:

$$\begin{aligned}
\rho_M\chi_M\nu_M&=\rho_M\chi_M\big(M\otimes_B\hat{\nu}_{\mathfrak{D}}\big)\rho_M\\
&=\chi_{M\otimes_B\mathfrak{D}}\big(\rho_M\otimes_B B\otimes_A\mathfrak{C}\otimes_A B\big)\big(M\otimes_B\hat{\nu}_{\mathfrak{D}}\big)\rho_M\quad(\chi\ \text{ is natural})\\
&=\big(M\otimes_B\chi_{\mathfrak{D}}\big)\big(\rho_M\otimes_B B\otimes_A\mathfrak{C}\otimes_A B\big)\big(M\otimes_B\hat{\nu}_{\mathfrak{D}}\big)\rho_M\quad(\text{by Lemma 3.2})\\
&=\big(M\otimes_B\chi_{\mathfrak{D}}\big)\big(M\otimes_B\lambda_{B\otimes_A\mathfrak{C}\otimes_A B}\big)\big(M\otimes_B\hat{\nu}_{\mathfrak{D}}\big)\rho_M\quad(\hat{\nu}_{\mathfrak{D}}\ \text{ is bicolinear})\\
&=\big(M\otimes_B\hat{\varphi}\big)\big(M\otimes_B\hat{\nu}_{\mathfrak{D}}\big)\rho_M=\rho_M.
\end{aligned}\tag{5.43}$$

Since ρ_M is a monomorphism, we get $\chi_M\nu_M=M$. By [14, Theorem 1.2], $-\square_D(B\otimes_A\mathfrak{C})$ is a separable functor. □

By applying the stated theorem to $\epsilon_{\mathfrak{C}}:\mathfrak{C}\to A$ we obtain the following corollary.

Corollary 5.9 (see [3, Theorem 3.3])**.** Let $\mathfrak{C}$ be an A-coring. Then the functor $-\otimes_A\mathfrak{C}$ is separable if and only if there exists an invariant $e\in\mathfrak{C}$ (i.e., $e\in\mathfrak{C}$ satisfying $ae=ea$ for every $a\in A$) such that $\epsilon_{\mathfrak{C}}(e)=1$.

Proof. This follows from Theorem 5.8 taking that the A-bimodule homomorphisms from A to $\mathfrak{C}$ correspond bijectively with the invariants of $\mathfrak{C}$ into account. □

5.7. Final remarks. Brzeziński showed [3, Proposition 2.2] that if $(A,C)_\psi$ is an entwining structure over K, then $A \otimes_K C$ can be endowed with an A-coring structure in such a way that the category $\mathcal{M}^{A\otimes_K C}$ is isomorphic with the category $\mathcal{M}_A^C(\psi)$ of entwined $(A,C)_\psi$-modules. It turns out that every morphism of entwining structures $(f,g) : (A,C)_\psi \to (B,D)_\gamma$ in the sense of [4] gives a coring morphism $(f \otimes_K g, f) : A \otimes_K C \to B \otimes_K D$. Some straightforward computations show that the statements of the separability theorem [4, Theorem 3.4] correspond to our Theorems 5.6 and 5.8. In fact, we can give the notions of totally integrable and totally cointegrable morphism of entwining structures to the framework of morphisms of corings $(\phi,\rho) : \mathfrak{C} \to \mathfrak{D}$ by requiring the existence of the splitting bicomodule maps $\omega_{\mathfrak{C}}$ and $\hat{\nu}_{\mathfrak{D}}$, respectively. In the first case, the existence of $\omega_{\mathfrak{C}}$ can be transferred, if desired, to the existence of certain bimodule map with extra properties by means of the adjointness isomorphism

$$\operatorname{Hom}_{\mathfrak{C}}\left((\mathfrak{C}\otimes_A B)\square_{\mathfrak{D}}(B\otimes_A\mathfrak{C}),\mathfrak{C}\right) \cong \operatorname{Hom}_A\left((\mathfrak{C}\otimes_A B)\square_{\mathfrak{D}}(B\otimes_A\mathfrak{C}),A\right). \tag{5.44}$$

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References

[1] K. Al-Takhman, *Äquivalenzen zwischen komodul-kategorien von koalgebren über ringen*, Ph.D. thesis, Universität Düsseldorf, 1999.

[2] ______, *Equivalences of comodule categories for coalgebras over rings*, preprint, 2001.

[3] T. Brzeziński, *The structure of corings: induction functors, Maschke-type theorem, and Frobenius and Galois-type properties*, to appear in Algebras and Repr. Theory.

[4] ______, *Coalgebra-Galois extensions from the extension theory point of view*, Hopf Algebras and Quantum Groups (S. Caenepeel and F. V. Oystaeyen, eds.), Lecture Notes Pure Appl. Math., vol. 209, Dekker, New York, 2000, pp. 47–68.

[5] T. Brzeziński, S. Caenepeel, G. Militaru, and S. Zhu, *Frobenius and Maschke type theorems for Doi-Hopf modules and entwined modules revisited: a unified approach*, Ring Theory and Algebraic Geometry (A. Granja, J. H. Alonso, and A. Verschoren, eds.), Lecture Notes Pure Appl. Math., vol. 221, Dekker, New York, 2001, pp. 1–31.

[6] S. Caenepeel, B. Ion, G. Militaru, and S. Zhu, *Separable functors for the category of Doi-Hopf modules. II*, Hopf Algebras and Quantum Groups (S. Caenepeel and F. Van Oystaeyen, eds.), Lecture Notes Pure Appl. Math., vol. 209, Dekker, New York, 2000, pp. 69–103.

[7] S. Caenepeel, G. Militaru, B. Ion, and S. Zhu, *Separable functors for the category of Doi-Hopf modules, applications*, Adv. Math. **145** (1999), 239–290.

[8] F. Castaño, J. Gómez-Torrecillas, and C. Năstăsescu, *Separable functors in coalgebras. Applications*, Tsukuba J. Math. **21** (1997), 329–344.

[9] ______, *Separable functors in graded rings*, J. Pure Appl. Algebra **127** (1998), 219–230.

[10] A. del Río, *Categorical methods in graded ring theory*, Publ. Mat. **36** (1992), 489–531.

[11] F. Guzmán, *Cointegrations, relative cohomology for comodules, and coseparable corings*, J. Algebra **126** (1989), 211–224.

[12] C. Năstăsescu, M. van den Bergh, and F. van Oystaeyen, *Separable functors applied to graded rings*, J. Algebra **123** (1989), 397–413.

[13] N. Popescu, *Abelian Categories with Applications to Rings and Modules*, Academic Press, London, 1973.

[14] M. D. Rafael, *Separable functors revisited*, Comm. Algebra **18** (1990), 1445–1459.

[15] R. N. Saavedra, *Catégories Tannakiennes*, Lecture Notes in Mathematics, vol. 265, Springer-Verlag, Berlin, 1972.

[16] B. Stenström, *Rings of Quotients*, Springer-Verlag, New York, 1975.

[17] M. Sweedler, *The predual theorem to the Jacobson-Bourbaki theorem*, Trans. Amer. Math. Soc. **213** (1975), 391–406.

[18] M. Takeuchi, *Morita theorems for categories of comodules*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), 629–644.

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