

ON THE REIDEMEISTER TORSION OF RATIONAL HOMOLOGY SPHERES

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ABSTRACT. We prove that the $\text{mod } \mathbb{Z}$ reduction of the Reidemeister torsion of a rational homology 3-sphere is naturally a \mathbb{Q}/\mathbb{Z} -valued quadratic function uniquely determined by a \mathbb{Q}/\mathbb{Z} -constant and the linking form.

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1. Introduction. Recently, V. Turaev has proved in [3, Theorem 4.3.1] a certain identity involving the Reidemeister torsion of a rational homology sphere M . In this paper, we suitably interpret this identity as a second-order finite difference equation satisfied by the torsion. Roughly speaking this identity states that the finite difference Hessian of the torsion coincides with the linking form of M . This allows us to prove a general structure result for the $\text{mod } \mathbb{Z}$ reduction of the torsion. More precisely, in Proposition 3.3 we prove that the $\text{mod } \mathbb{Z}$ reduction of the torsion is completely determined by three data.

- a certain canonical spin^c -structure σ_0 ,
- the linking form \mathbf{lk} of M ,
- a constant $c \in \mathbb{Q}/\mathbb{Z}$.

By fixing the spin^c -structure σ_0 , we have a natural choice of Euler structure and thus, we can identify the Reidemeister torsion with a \mathbb{Q} -valued function on $H := H_1(M, \mathbb{Z})$. Its $\text{mod } \mathbb{Z}$ reduction is a function $\tau : H \rightarrow \mathbb{Q}/\mathbb{Z}$ of the form

$$\tau(h) = c - \widehat{\mathbf{lk}}(h), \quad (1.1)$$

where $\widehat{\mathbf{lk}}$ denotes a *quadratic form* on H such that

$$\widehat{\mathbf{lk}}(h_1 + h_2) - \widehat{\mathbf{lk}}(h_1) - \widehat{\mathbf{lk}}(h_2) = \mathbf{lk}(h_1, h_2). \quad (1.2)$$

As a consequence, the constant c is a \mathbb{Q}/\mathbb{Z} -valued invariant of the rational homology sphere. Experimentations with lens spaces suggest this invariant is as powerful as the torsion itself.

2. The Reidemeister torsion. We review briefly a few basic facts about the Reidemeister torsion of a rational homology 3-sphere. For more details and examples we refer to [1, 3].

Suppose that M is a rational homology sphere. We set $H := H_1(M, \mathbb{Z})$ and use the multiplicative notation to denote the group operation on H . To remove the sign ambiguities in the definition of torsion, we equip $H_*(M, \mathbb{R})$ with the canonical orientation described in [3].

Denote by $\text{Spin}^c(M)$ the set of isomorphism classes of spin^c -structure on M . It is an H -torsor, that is, the group H acts freely and transitively on $\text{Spin}^c(M)$,

$$H \times \text{Spin}^c(M) \ni (h, \sigma) \longmapsto h \cdot \sigma \in \text{Spin}^c(M). \quad (2.1)$$

We denote by \mathcal{F}_M the space of functions

$$\phi : H \longrightarrow \mathbb{Q}. \quad (2.2)$$

The group H acts on \mathcal{F}_M by

$$H \times \mathcal{F}_M \ni (g, \phi) \longmapsto g \cdot \phi, \quad (2.3)$$

where

$$(g \cdot \phi)(h) = \phi(hg). \quad (2.4)$$

We denote by \int_H the augmentation map

$$\mathcal{F}_M \longrightarrow \mathbb{Q}, \quad \int_H \phi := \sum_{h \in H} \phi(h). \quad (2.5)$$

According to [3], the Reidemeister torsion is an H -equivariant map

$$\tau : \text{Spin}^c(M) \longrightarrow \mathcal{F}_M, \quad \text{Spin}^c(M) \ni \sigma \longmapsto \tau_\sigma = \tau_{M, \sigma} \in \mathcal{F}_M \quad (2.6)$$

such that

$$\int_H \tau_\sigma = 0. \quad (2.7)$$

In particular, if M is an integral homology sphere we have $\tau_{M, \sigma} = 0$. Denote by \mathbf{lk}_M the linking form of M ,

$$\mathbf{lk}_M : H \times H \longrightarrow \mathbb{Q}/\mathbb{Z}. \quad (2.8)$$

V. Turaev has proved in [3] that τ_σ satisfies the identity

$$\tau_\sigma(g_1 g_2) - \tau_\sigma(g_1) - \tau_\sigma(g_2) + \tau_\sigma(1) = -\mathbf{lk}_M(g_1, g_2) \pmod{\mathbb{Z}} \quad (2.9)$$

for all $g_1, g_2 \in H$, $\sigma \in \text{Spin}^c(M)$. In the above identity, we replace σ by $h \cdot \sigma$ for an arbitrary $h \in H$ and using the H -equivariance of $\sigma \mapsto \tau_\sigma$, we deduce

$$\tau_\sigma(g_1 g_2 h) - \tau_\sigma(g_1 h) - \tau_\sigma(g_2 h) + \tau_\sigma(h) = -\mathbf{lk}_M(g_1, g_2) \pmod{\mathbb{Z}} \quad (2.10)$$

for all $g_1, g_2, h \in H$, $\sigma \in \text{Spin}^c(M)$.

3. A second-order differential equation. The identity (2.10) admits a more suggestive interpretation. To describe it, we need a few more notation.

Denote by \mathcal{S}_M the space of functions $H \rightarrow \mathbb{Q}/\mathbb{Z}$. Each $g \in H$ defines a first-order differential operator

$$\Delta_g : \mathcal{S}_M \longrightarrow \mathcal{S}_M, \quad (\Delta_g u)(h) := u(gh) - u(h), \quad \forall u \in \mathcal{S}_M, h \in H. \quad (3.1)$$

If $\Xi = \Xi_\sigma$ denotes the mod \mathbb{Z} reduction of τ_σ , then we can rewrite (2.10) as

$$(\Delta_{g_1} \Delta_{g_2} \Xi)(h) = -\mathbf{lk}_M(g_1, g_2). \quad (3.2)$$

Note that the second-order differential operator $\Delta_{g_1} \Delta_{g_2}$ can be regarded as a sort of Hessian.

We prove uniqueness and existence results for this equation. We begin with the (almost) uniqueness part.

LEMMA 3.1. *The second-order linear differential equation (3.2) determines Ξ up to an “affine” function, that is, the sum between a character of H and a \mathbb{Q}/\mathbb{Z} -constant.*

PROOF. Suppose that Ξ_1, Ξ_2 are two solutions of the above equation. Set $\Psi := \Xi_1 - \Xi_2$, Ψ satisfies the equation

$$\Delta_{g_1} \Delta_{g_2} \Psi = 0. \quad (3.3)$$

Now, observe that any function $F \in \mathcal{S}_M$ satisfying the second-order equation

$$\Delta_u \Delta_v F = 0, \quad \forall u, v \in H \quad (3.4)$$

is affine, that is, it has the form

$$F = c + \lambda, \quad (3.5)$$

where $c \in \mathbb{Q}/\mathbb{Z}$ is a constant and $\lambda : H \rightarrow \mathbb{Q}/\mathbb{Z}$ is a character. Indeed, the condition

$$\Delta_u (\Delta_v F) = 0, \quad \forall u \quad (3.6)$$

implies $\Delta_v F$ is a constant depending on v , $c(v)$. Thus

$$F(vh) - F(h) = c(v), \quad \forall h. \quad (3.7)$$

The function $\lambda = F - F(1)$ satisfies the same differential equation

$$\lambda(vh) - \lambda(h) = c(v) \quad (3.8)$$

and the additional condition $\lambda(1) = 0$. If we set $h = 1$ in the above equation, we deduce

$$\lambda(v) = c(v). \quad (3.9)$$

Hence,

$$\lambda(vh) = \lambda(h) + \lambda(v), \quad \forall v, h \quad (3.10)$$

so that λ is a character of H and $F = F(1) + \lambda$. Thus, the differential equation (3.2) determines Ξ up to a constant and a character. \square

LEMMA 3.2. *Suppose that $b : H \times H \rightarrow \mathbb{Q}/\mathbb{Z}$ is a nonsingular, symmetric, bilinear form on H . Then there exists a quadratic form $q : H \rightarrow \mathbb{Q}/\mathbb{Z}$ such that*

$$\mathcal{H}q = b, \quad (3.11)$$

where

$$(\mathcal{H}q)(u, v) := q(uv) - q(u) - q(v). \quad (3.12)$$

PROOF. Let us briefly recall the terminology in this lemma. b is nonsingular if the induced map $H \rightarrow H^\# := \text{Hom}(H, \mathbb{Q}/\mathbb{Z})$ is an isomorphism. A quadratic form is a function $q : H \rightarrow \mathbb{Q}/\mathbb{Z}$ such that

$$q(1) = 0, \quad q(u^k) = k^2 q(u), \quad \forall u \in H, k \in \mathbb{Z} \quad (3.13)$$

and $\mathcal{H}q$ is a bilinear form.

Suppose that b is a nonsingular, symmetric, bilinear form $H \times H \rightarrow \mathbb{Q}/\mathbb{Z}$. Then, according to [4, Section 7], b admits a resolution. This is a nondegenerate, symmetric, bilinear form

$$B : \Lambda \times \Lambda \longrightarrow \mathbb{Z} \quad (3.14)$$

on a free abelian group Λ such that the induced monomorphism $J_B : \Lambda \rightarrow \Lambda^* := \text{Hom}(\Lambda, \mathbb{Z})$ is a resolution of H ,

$$0 \hookrightarrow \Lambda \xrightarrow{J_B} \Lambda^* \xrightarrow{\pi} H \longrightarrow 0 \quad (3.15)$$

and b coincides with the induced bilinear form on $\Lambda^*/(J_B\Lambda)$ ($n := \#H$),

$$b(\pi(u), \pi(v)) = \frac{1}{n^2} B(J_B^{-1}(nu), J_B^{-1}(nv)) \bmod \mathbb{Z}, \quad \forall u, v \in \Lambda^*. \quad (3.16)$$

Now, set

$$q(\pi(u)) = \frac{1}{2n^2} B(J_B^{-1}(nu), J_B^{-1}(nu)) \bmod \mathbb{Z}. \quad (3.17)$$

This quantity is well defined, that is,

$$\frac{1}{2n^2} B(J_B^{-1}(nu), J_B^{-1}(nu)) = \frac{1}{2n^2} B(J_B^{-1}(nv), J_B^{-1}(nv)) \bmod \mathbb{Z} \quad (3.18)$$

if $v = u + J_B\lambda$, $\lambda \in \Lambda$. Clearly, $\mathcal{H}q = b$. \square

Denote by Q the space of solutions of the equation (3.11), that is, the space of quadratic forms q on H satisfying $\mathcal{H}q = -\mathbf{1k}_M$. Q consists of more than one element. It is a G -torsor, where $G = \text{Hom}(H, \mathbb{Z}_2)$ and the G action is given by

$$(Q \times G) \ni (q, \mu) \longmapsto q + \mu. \quad (3.19)$$

Using the linking form on M we can identify G with the 2-torsion subgroup of H . Denote by Ξ_σ the reduction mod \mathbb{Z} of τ_σ .

Fix a spin^c structure σ_0 on M . We deduce that for every $q \in Q$ there exists a constant $k = k(q)$ and a character $\lambda = \lambda_q$ of H

$$\Xi_{\sigma_0}(h) = k(q) + \lambda_q(h) + q(h), \quad \mathcal{H}q = -\mathbf{1k}_M. \quad (3.20)$$

In particular,

$$\begin{aligned} \Xi_{g \cdot \sigma_0}(h) &:= \Xi_{\sigma}(gh) = k(q) + \lambda_q(gh) + q(gh) \\ &= \underbrace{(k(q) + \lambda_q(g) + q(g))}_{c(g,q)} + \underbrace{(\lambda_q(h) + (\mathfrak{L}q)(g, h))}_{\lambda_{q,g}(h)} + q(h) \end{aligned} \quad (3.21)$$

where $\lambda_{q,g}(\bullet) = \lambda_q(\bullet) - \mathbf{lk}_M(g, \bullet)$. Since the linking form is nondegenerate we can find a *unique* $g = g(q)$ such that $\lambda_{q,g} = 0$. We set $\bar{\sigma}(q) = g(q) \cdot \sigma_0$ and $c(q) = c(g(q), q)$. The above computation also shows that for every $\mu \in G$ we have

$$c(q + \mu) - c(q) = q(\mu), \quad \bar{\sigma}(q + \mu) = \mu \cdot \bar{\sigma}(q). \quad (3.22)$$

We have thus proved the following result.

PROPOSITION 3.3. *Suppose M is a rational homology sphere. Then there exist functions*

$$c : Q \rightarrow \mathbb{Q}/\mathbb{Z}, \quad \bar{\sigma} : Q \rightarrow \text{Spin}^c(M) \quad (3.23)$$

so that

$$\tau_{\bar{\sigma}(q)}(h) := q(h) + c(q) \pmod{\mathbb{Z}}, \quad \forall h \in H. \quad (3.24)$$

Moreover,

$$c(q + \mu) - c(q) = q(\mu), \quad \bar{\sigma}(q + \mu) = \mu \cdot \bar{\sigma}(q), \quad \forall \mu \in G. \quad (3.25)$$

REMARK 3.4. (a) Note that $q(\mu) \in (1/4)\mathbb{Z}$, $\forall q \in Q$, $\mu \in \mathbb{Z}$ so that $4c(q)$ is *independent* of q . It is a topological invariant of M !

(b) One can show that the image of the one-to-one map $\bar{\sigma}$ is $\text{Spin}(M)$, the set spin^c structures induced by the spin structures on M . We can thus regard c as a map $c : \text{Spin}(M) \rightarrow \mathbb{Q}/\mathbb{Z}$.

4. Examples. We want to show on some simple examples that the invariant c is nontrivial. First, we need some notation.

We denote by \mathbb{Z}_n the cyclic group with n elements. The functions $f : \mathbb{Z}_n \rightarrow \mathbb{Q}$ can be conveniently described as polynomials $f \in \mathbb{Q}[x]$, where $x^n = 1$. Given two such polynomials f, g , we define the equivalence relation \sim by

$$f \sim g \iff \exists m \in \mathbb{Z} : f = \pm x^m g. \quad (4.1)$$

We will not keep track of Euler structures and/or homology orientations and that is why in the sequel only the \sim -equivalence class of the torsion will be well defined. In particular, the map c constructed in the previous section will be defined only up to a sign.

(a) Suppose that $M = L(8, 3)$. Then its torsion is (see [2])

$$T_{8,3} \sim -\frac{9}{32}x^7 - \frac{3}{32}x^6 - \frac{9}{32}x^5 + \frac{5}{32}x^4 + \frac{7}{32}x^3 - \frac{3}{32}x^2 + \frac{7}{32}x + \frac{5}{32}, \quad (4.2)$$

where $x^8 = 1$ is a generator of \mathbb{Z}_8 . Then

$$q(x^n) = \frac{-3n^2}{16}. \quad (4.3)$$

The set of possible values $(-3m^2/16) \bmod \mathbb{Z}$ is

$$A := \left\{ 0, \frac{-3}{16}, \frac{4}{16}, \frac{5}{16} \right\}. \quad (4.4)$$

The set of possible values of $\Xi(h)$ is

$$B := \left\{ -\frac{9}{32}, -\frac{3}{32}, \frac{5}{32}, \frac{7}{32} \right\}. \quad (4.5)$$

We need to find a constant $c \in \mathbb{Q}/\mathbb{Z}$ such that

$$B \pm c = A. \quad (4.6)$$

Equivalently, we need to figure out orderings $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ of A and B such that $b_i - a_i \bmod \mathbb{Z}$ is a constant independent of i . A little trial and error shows that

$$\vec{A} = \left(0, -\frac{3}{16}, \frac{4}{16}, \frac{5}{16} \right), \quad \vec{B} = \left(-\frac{3}{32}, -\frac{9}{32}, \frac{5}{32}, \frac{7}{32} \right) \quad (4.7)$$

and the constant $c = -3/32$. This is the coefficient of x^2 . We deduce that (modulo \mathbb{Z})

$$F := T_{8,3}(x) + \frac{3}{32} \sim -\frac{3}{16}x^7 - 0 \cdot x^6 - \frac{3}{16}x^5 + \frac{1}{4}x^4 + \frac{1}{4}x^3 - 0 \cdot x^2 + \frac{1}{4}x + \frac{1}{4}. \quad (4.8)$$

The translation of F by x^{-2} is

$$x^{-2} \left(T_{8,3} + \frac{3}{32} \right) = \frac{1}{4}x^7 + \frac{1}{4}x^6 - \frac{3}{16}x^5 - \frac{3}{16}x^3 + \frac{1}{4}x^2 + \frac{1}{4}x. \quad (4.9)$$

(b) Suppose that $M = L(7, 2)$. Then, its torsion is (see [2])

$$T_{7,2} \sim -\frac{2}{7}x^6 + \frac{1}{7}x^5 + \frac{2}{7}x^3 + \frac{1}{7}x - \frac{2}{7}, \quad (4.10)$$

where $x^7 = 1$ is a generator of \mathbb{Z}_7 . We see that in this form $T_{7,2}$ is symmetric, that is, the coefficient of x^k is equal to the coefficient of x^{6-k} . The constant c in this case must be the coefficient of the middle monomial x^3 , which is $2/7$.

(c) Suppose that $M = L(7, 1)$. Then

$$T_{7,1} \sim \frac{2}{7}x^6 + \frac{1}{7}x^5 - \frac{1}{7}x^4 - \frac{4}{7}x^3 - \frac{1}{7}x^2 + \frac{1}{7}x + \frac{2}{7}. \quad (4.11)$$

This is again a symmetric polynomial and the coefficient of the middle monomial is $-4/7$. We see that this invariant distinguishes the lens spaces $L(7, 1)$ and $L(7, 2)$. It is known that these two spaces are homotopic but nonhomeomorphic lens spaces. Thus, the invariant c distinguishes their homeomorphism types, just as the torsion does.

(d) For $M = L(9, 2)$, we have

$$T_{9,2} \sim -\frac{10}{27}x^8 + \frac{2}{27}x^7 - \frac{1}{27}x^6 + \frac{8}{27}x^5 + \frac{2}{27}x^4 + \frac{8}{27}x^3 - \frac{1}{27}x^2 + \frac{2}{27}x - \frac{10}{27}. \quad (4.12)$$

Again, this is a symmetric function, that is, the coefficient of x^k is equal to the coefficient of x^{8-k} , $x^9 = 1$. The constant is the coefficient of x^4 , which is $2/27$. We deduce that mod \mathbb{Z} , we have

$$T_{9,2} - \frac{2}{27} = -\frac{2}{3}x^8 - \frac{2}{9}x^7 - \frac{1}{3}x^6 - \frac{2}{9}x^7. \quad (4.13)$$

(e) Finally, when $M = L(9,7)$ we have

$$T_{9,7} \sim -\frac{8}{27}x^8 - \frac{2}{27}x^7 + \frac{10}{27}x^6 + \frac{1}{27}x^5 - \frac{2}{27}x^4 + \frac{1}{27}x^3 + \frac{10}{27}x^2 - \frac{2}{27}x - \frac{8}{27} \quad (4.14)$$

the polynomial is again symmetric so that the constant c is the coefficient of x^4 which is $-2/27$.

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