POLYNOMIAL VALUES OF (ALTERNATING) POWER SUMS

ANDRÁS BAZSÓ

ABSTRACT. We prove ineffective finiteness results on the integer solutions x, y of the equations

$$b^k + (a+b)^k + \cdots + (a(x-1)+b)^k = g(y)$$

and

$$b^k - (a+b)^k + (2a+b)^k - \ldots + (-1)^{x-1}(a(x-1)+b)^k = g(y),$$

where $g(x) \in \mathbb{Q}[x]$, $\deg g(x) \geq 3$, and $a \neq 0, b$ are given integers with $\gcd(a,b) = 1$.

1. Introduction and new results

Many diophantine problems have been investigated in the literature concerning power sums of consecutive integers. It is well known that the sum

$$S_k(n) = 1^k + 2^k + \ldots + (n-1)^k \tag{1}$$

can be expressed by the Bernoulli polynomials $B_k(x)$ as

$$S_k(n) = \frac{1}{k+1}\left(B_{k+1}(n) - B_{k+1}\right), \tag{2}$$

where the polynomials $B_k(x)$ are defined by the generating series

$$\frac{t\exp(tx)}{\exp(t)-1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}$$

and $B_{k+1} = B_{k+1}(0)$. Hence S_k can be extended to real values x, i.e., to the polynomial

$$S_k(x) = \frac{1}{k+1}\left(B_{k+1}(x) - B_{k+1}\right). \tag{3}$$

Date: August 11, 2014.
2010 *Mathematics Subject Classification.* 11D41, 11B68.
Key words and phrases. Diophantine equations, Bernoulli polynomials, Euler polynomials.

A classical problem of Lucas [16], from 1875, was the study of square values of $S_k(x)$. Later, in 1956, Schäffer [23] investigated n-th power values, that is, the diophantine equation

$$S_k(x) = y^n \quad \text{in integers } x, y. \tag{4}$$

For $k \geq 1$, $n \geq 2$ he proved an ineffective finiteness result on the solutions x, y of (4) provided that $(k,n) \notin \{(1,2),(3,2),(3,4),(5,2)\}$. In the exceptional cases (k,n) he proved the existence of infinitely many solutions. Moreover, Schäffer proposed a still unproven conjecture which says that if (k,n) is not in the above exceptional set, then the only nontrivial solution of equation (4) is $(k,n,x,y) = (2,2,24,70)$. In 1980, Győry, Tijdeman and Voorhoeve [13] proved effective finiteness for the solutions of (4) in the general case when, in (4), n is also unknown. Several generalizations of (4) have been considered, e.g. in the papers of Voorhoeve, Győry and Tijdeman [28], Brindza [10], Dilcher [11] and Urbanowicz [25, 26, 27]. Schäffer's conjecture has been confirmed only in a few cases: for $n = 2$ and $k \leq 58$ by Jacobson, Pintér and Walsh [15]; and for $n \geq 2$ and $k \leq 11$ by Bennett, Győry and Pintér [5]. For further generalizations of (4) and related results see the survey paper of Győry and Pintér [12] and the references given there.

In [8], Bilu et al. considered the diophantine equations

$$S_k(x) = S_\ell(y), \tag{5}$$

and

$$S_k(x) = y\,(y+1)\,(y+2)\ldots(y+(\ell-1))\,. \tag{6}$$

They proved ineffective finiteness results on the solutions x, y of these equations for $k < l$, moreover, they established effective statements for certain small values of k and l.

For a positive integer $n \geq 2$ and for $a \neq 0, b$ coprime integers, let

$$S_{a,b}^k(n) = b^k + (a+b)^k + (2a+b)^k + \ldots + (a\,(n-1)+b)^k\,. \tag{7}$$

It is easy to see that the above power sum is related to the Bernoulli polynomials $B_k(x)$ in the following way

$$S_{a,b}^k(n) = \frac{a^k}{k+1}\left(\left[B_{k+1}\left(n+\frac{b}{a}\right) - B_{k+1}\right] - \left[B_{k+1}\left(\frac{b}{a}\right) - B_{k+1}\right]\right). \tag{8}$$

Thus we can extend $S_{a,b}^k$ for every real value x as

$$S_{a,b}^k(x) = \frac{a^k}{k+1}\left(B_{k+1}\left(x+\frac{b}{a}\right) - B_{k+1}\left(\frac{b}{a}\right)\right). \tag{9}$$

In [14], using a different approach, Howard also obtained relation (9) via generating functions. In the same paper [14], he showed that the alternating power sum

$$T_{a,b}^{k}(n) = b^k - (a+b)^k + (2a+b)^k - \ldots + (-1)^{n-1}(a(n-1)+b)^k \quad (10)$$

can be expressed by means of Euler polynomials $E_k(x)$ as

$$T_{a,b}^{k}(n) = \frac{a^k}{2}\left(E_k\left(\frac{b}{a}\right) + (-1)^{n-1}E_k\left(n + \frac{b}{a}\right)\right), \quad (11)$$

where the classical Euler polynomials $E_k(x)$ are defined by the generating function

$$\frac{2\exp(xt)}{\exp(t)+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad (|t| < \pi).$$

For the properties of Bernoulli and Euler polynomials which will be often used in this paper, sometimes without special reference, we refer to the paper of Brillhart [9]. Thus, depending on the power of -1 in (11), we can extend $T_{a,b}^{k}(n)$ to a polynomial in the following two ways:

$$\mathtt{T}_{a,b}^{k+}(x) = \frac{a^k}{2}\left(E_k\left(\frac{b}{a}\right) + E_k\left(x + \frac{b}{a}\right)\right),$$

$$\mathtt{T}_{a,b}^{k-}(x) = \frac{a^k}{2}\left(E_k\left(\frac{b}{a}\right) - E_k\left(x + \frac{b}{a}\right)\right).$$

Recently, Bazsó, Kreso, Luca and Pintér [2] generalized the results of Bilu et al. [8] on equation (5) to the equation

$$S_{a,b}^{k}(x) = S_{c,d}^{\ell}(y) \quad (12)$$

where x, y are unknown integers, and k, ℓ, a, b, c, d are given integers with $0 < k < l$, $\gcd(a,b) = \gcd(c,d) = 1$.

In the present paper we study the Diophantine equations

$$S_{a,b}^{k}(x) = g(y), \quad (13)$$

$$\mathtt{T}_{a,b}^{k+}(x) = g(y) \quad (14)$$

and

$$\mathtt{T}_{a,b}^{k-}(x) = g(y), \quad (15)$$

where $g(y)$ is a rational polynomial of degree at least 3. These equations have only been investigated in the literature in the case $(a,b) = (1,0)$. Rakaczki [20] and independently Kulkarni and Sury [19] characterized those pairs $(k, g(y))$ for which equation (13) has infinitely many integer solutions. Recently, Kreso and Rakaczki [22] proved an analogous result for equations (14) and (15). For further related results we refer to the papers of Kulkarny and Sury [17, 18], and of Bennett [4].

Our goal in the sequel is to extend the results of [20, 19, 22] to the general case, i.e, to equations (13)-(15). To do this it will be useful to survey what is known about the decomposition properties of the polynomials involved in the equations under consideration.

By a *decomposition* of a polynomial $F(x)$ over a field $\mathbb{K}$ we mean an equality of the following form

$$F(x) = G_1(G_2(x)) \quad (G_1(x), G_2(x) \in \mathbb{K}[x]),$$

which is *nontrivial* if

$$\deg G_1(x) > 1 \quad \text{and} \quad \deg G_2(x) > 1.$$

Two decompositions $F(x) = G_1(G_2(x))$ and $F(x) = H_1(H_2(x))$ are said to be equivalent if there exists a linear polynomial $\ell(x) \in \mathbb{K}[x]$ such that $G_1(x) = H_1(\ell(x))$ and $H_2(x) = \ell(G_2(x))$. The polynomial $F(x)$ is called *decomposable* over $\mathbb{K}$ if it has at least one nontrivial decomposition over $\mathbb{K}$; otherwise it is said to be *indecomposable.* A detailed discussion on the theory of polynomial decomposition can be found in the monograph of Schinzel [24].

In a recent paper, Bazsó, Pintér and Srivastava [3] proved the following result about the decomposition of the polynomial $S_{a,b}^{k}(x)$ defined above.

Proposition 1. *The polynomial $S_{a,b}^{k}(x)$ is indecomposable over $\mathbb{C}$ for even k. If $k = 2v-1$ is odd, then any nontrivial decomposition of $S_{a,b}^{k}(x)$ over $\mathbb{C}$ is equivalent to the following decomposition:*

$$S_{a,b}^{k}(x) = \widehat{S}_{a,b}^{v}\left(\left(x + \frac{b}{a} - \frac{1}{2}\right)^2\right), \tag{16}$$

where $\widehat{S}_{a,b}^{v}(x)$ is a rational polynomial of degree v.

Proof. This is Theorem 2 of [3]. $\square$

On equation (13), we prove the following.

Theorem 1. *Let $a \neq 0, b \in \mathbb{Z}, \gcd(a,b) = 1$ and let $g(x) \in \mathbb{Q}[x]$, $\deg g(x) \geq 3$. Further, let $\alpha, \beta, c \in \mathbb{Q} \setminus \{0\}$, $\delta(x), q(x) \in \mathbb{Q}[x]$ with $\deg \delta(x) = 1, q(x) \neq 0$. Then, for $k > 3$ equation (13) has only finitely many integer solutions x, y, unless one of the following holds:*

(I) $g(x) = S_{a,b}^{k}(q(x))$

(II) *k is odd and* $g(x) = \widehat{S}_{a,b}^{(k+1)/2}(\delta(x)q(x)^2)$

(III) *k is odd and* $g(x) = \widehat{S}_{a,b}^{(k+1)/2}(c\delta(x)^t)$*, where $t \geq 3$ is an odd integer*

(IV) *k is odd and* $g(x) = \widehat{S}_{a,b}^{(k+1)/2}((\alpha\delta(x)^2 + \beta)q(x)^2)$

(V) *k is odd and* $g(x) = \widehat{S}_{a,b}^{(k+1)/2}(q(x)^2)$

with $\widehat{S}_{a,b}^{v}(x)$ *specified in Proposition 1.*

Our method of proof is based upon Proposition 1 and the general ineffective finiteness criterion of Bilu and Tichy [7] (cf. Proposition 3). Therefore our result is also ineffective.

We note that in the exceptional cases $(I)-(V)$ one can find an equation of the shape (13) having infinitely many integer solutions (see [20] for examples). We further note that for $a=1, b=0, k>3$, our Theorem 1 gives the result of Rakaczki [20, Theorem 1].

The decomposition properties of the polynomials $\mathtt{T}_{a,b}^{k+}(x)$ and $\mathtt{T}_{a,b}^{k-}(x)$ have recently been described in [1] by the present author who proved the following.

Proposition 2. *The polynomials* $T_{a,b}^{k+}(x)$ *and* $T_{a,b}^{k-}(x)$ *are both indecomposable for any odd k. If $k=2m$ is even, then any nontrivial decomposition of* $T_{a,b}^{k+}(x)$ *or* $T_{a,b}^{k-}(x)$ *is equivalent to*

$$T_{a,b}^{k+}(x) = \widehat{T}_{a,b}^{m+}\left(\left(x+\frac{b}{a}-\frac{1}{2}\right)^2\right) \quad \textit{or}$$

$$T_{a,b}^{k-}(x) = \widehat{T}_{a,b}^{m-}\left(\left(x+\frac{b}{a}-\frac{1}{2}\right)^2\right), \quad (17)$$

respectively, where

$$\widehat{T}_{a,b}^{m+}(x) = \frac{a^{2m}}{2}\left(E_{2m}\left(\frac{b}{a}\right)+\tilde{E}_m(x)\right),$$

$$\widehat{T}_{a,b}^{m-}(x) = \frac{a^{2m}}{2}\left(E_{2m}\left(\frac{b}{a}\right)-\tilde{E}_m(x)\right)$$

with

$$\tilde{E}_m(x) = \sum_{n=0}^{m}\binom{2m}{2n}\frac{E_{2n}}{2^{2n}}x^{m-n} \quad \textit{and} \quad E_j = 2^j E_j(1/2).$$

Proof. See [1]. □

Using Proposition 2 and the finiteness criterion from [7] we prove the following two results on equations (14) and (15), which are again ineffective.

Theorem 2. *Let* $a\neq 0, b\in\mathbb{Z}, \gcd(a,b)=1$ *and let* $g(x)\in\mathbb{Q}[x]$, $\deg g(x)\geq 3$. *Further, let* $\alpha,\beta,c\in\mathbb{Q}\setminus\{0\}$, $\delta(x),q(x)\in\mathbb{Q}[x]$ *with* $\deg\delta(x)=1, q(x)\neq 0$. *Then, for $k\geq 7$ equation* (14) *has only finitely many integer solutions x,y, unless one of the following holds:*

(I) $g(x) = T_{a,b}^{k+}(q(x))$
(II) k *is even and* $g(x) = \widehat{T}_{a,b}^{k/2+}(q(x)^2)$
(III) k *is even and* $g(x) = \widehat{T}_{a,b}^{k/2+}(\delta(x)q(x)^2)$
(IV) k *is even and* $g(x) = \widehat{T}_{a,b}^{k/2+}(c\delta(x)^t)$, *where* $t \geq 3$ *is an odd integer*
(V) k *is even and* $g(x) = \widehat{T}_{a,b}^{k/2+}((\alpha\delta(x)^2 + \beta)q(x)^2)$

with $\widehat{T}_{a,b}^{m+}(x)$ *specified in Proposition 2.*

Theorem 3. *Let* $a \neq 0, b \in \mathbb{Z}, \gcd(a,b) = 1$ *and let* $g(x) \in \mathbb{Q}[x]$, $\deg g(x) \geq 3$. *Further, let* $\alpha, \beta, c \in \mathbb{Q} \setminus \{0\}$, $\delta(x), q(x) \in \mathbb{Q}[x]$ *with* $\deg \delta(x) = 1, q(x) \neq 0$. *Then, for* $k \geq 7$ *equation* (15) *has only finitely many integer solutions* x, y, *unless one of the following holds:*

(I) $g(x) = T_{a,b}^{k-}(q(x))$
(II) k *is even and* $g(x) = \widehat{T}_{a,b}^{k/2-}(q(x)^2)$
(III) k *is even and* $g(x) = \widehat{T}_{a,b}^{k/2-}(\delta(x)q(x)^2)$
(IV) k *is even and* $g(x) = \widehat{T}_{a,b}^{k/2-}(c\delta(x)^t)$, *where* $t \geq 3$ *is an odd integer*
(V) k *is even and* $g(x) = \widehat{T}_{a,b}^{k/2-}((\alpha\delta(x)^2 + \beta)q(x)^2)$

with $\widehat{T}_{a,b}^{m-}(x)$ *specified in Proposition 2.*

The above two theorems extend Theorem 2 of [22] when $\deg g(x) \geq 3$. We further note that in the exceptional cases $(I) - (V)$ of Theorems 2 and 3 a choice of parameters can be found for which equations (14) resp. (15) have infinitely many integer solutions x, y. Such parametric solutions are given in [22] for $a = 1, b = 0$.

2. Proof of Theorem 1

To prove Theorem 1, we need some auxiliary results. First we recall the general ineffective finiteness criterion of Bilu and Tichy [7]. We first define the five kinds of so-called standard pairs of polynomials.

Let α, β be nonzero rational numbers, $\mu, \nu, q > 0$ and $r \geq 0$ be integers, and let $v(x) \in \mathbb{Q}[x]$ be a nonzero polynomial (which may be constant). Denote by $D_\mu(x, \delta)$ the μ-th Dickson polynomial, defined by the functional equation $D_\mu(z + \delta/z, \delta) = z^\mu + (\delta/z)^\mu$ or by the explicit formula

$$D_\mu(x, \delta) = \sum_{i=0}^{\lfloor \mu/2 \rfloor} d_{\mu,i} x^{\mu-2i} \qquad \text{with} \qquad d_{\mu,i} = \frac{\mu}{\mu - i}\binom{\mu - i}{i}(-\delta)^i.$$

Two polynomials $f_1(x)$ and $g_1(x)$ are said to form a *standard pair over* $\mathbb{Q}$ if one of the ordered pairs $(f_1(x), g_1(x))$ or $(g_1(x), f_1(x))$ belongs to the list below. The five kinds of standard pairs are then listed in the following table.

kind	explicit form of $\{f_1(x), g_1(x)\}$	parameter restrictions
first	$(x^q, \alpha x^r v(x)^q)$	$0 \le r < q, (r,q) = 1,$ $r + \deg v(x) > 0$
second	$(x^2, (\alpha x^2 + \beta)v(x)^2)$	-
third	$(D_\mu(x, \alpha^\nu), D_\nu(x, \alpha^\mu))$	$(\mu, \nu) = 1$
fourth	$(\alpha^{\frac{-\mu}{2}} D_\mu(x, \alpha), -\beta^{\frac{-\nu}{2}} D_\nu(x, \beta))$	$(\mu, \nu) = 2$
fifth	$((\alpha x^2 - 1)^3, 3x^4 - 4x^3)$	-

Now we state a special case of the main result of [7], which will be crucial in the proofs of ours.

Proposition 3. *Let $f(x), g(x) \in \mathbb{Q}[x]$ be nonconstant polynomials such that the equation $f(x) = g(y)$ has infinitely many solutions in rational integers x, y. Then $f = \varphi \circ f_1 \circ \lambda$ and $g = \varphi \circ g_1 \circ \mu$, where $\lambda(x), \mu(x) \in \mathbb{Q}[x]$ are linear polynomials, $\varphi(x) \in \mathbb{Q}[x]$, and $(f_1(x), g_1(x))$ is a standard pair over $\mathbb{Q}$.*

We recall the following result concerning Bernoulli polynomials $B_k(x)$ which is due to Brillhart [9].

Lemma 4. *If k is odd, then $B_k(x)$ has no multiple roots. For even k, the only polynomial which can be a multiple factor of $B_k(x)$ over $\mathbb{Q}$ is $x^2 - x - B$, where B is an odd, positive integer.*

The next two lemmas were proved in [2]. Let $c_1, e_1 \in \mathbb{Q} \setminus \{0\}$ and $c_0, e_0 \in \mathbb{Q}$.

Lemma 5. *The polynomial $S^k_{a,b}(c_1 x + c_0)$ is not of the form $e_1 x^q + e_0$ with $q \ge 3$.*

Lemma 6. *The polynomial $S^k_{a,b}(c_1 x + c_0)$ is not of the form*

$$e_1 D_\nu(x, \delta) + e_0,$$

where $D_\nu(x, \delta)$ is the ν-th Dickson polynomial with $\nu > 4, \delta \in \mathbb{Q} \setminus \{0\}$.

For $P(x) \in \mathbb{C}[x]$, a complex number c is said to be an *extremum* if $P(x) - c$ has multiple roots. The *type* of c is defined to be the tuple $(\alpha_1, \dots, \alpha_s)$ of the multiplicities of the distinct roots of $P(x) - c$ in an increasing order. Obviously, $s < \deg P(x)$ and $\alpha_1 + \ldots + \alpha_s = \deg P(x)$.

Proposition 7. *For $a \ne 0$ and $k \ge 3$, $D_\mu(x, \alpha)$ has exactly two extrema $\pm 2\alpha^{\frac{\mu}{2}}$. If μ is odd, then both are of type $(1, 2, 2, \dots, 2)$. If μ is even, then $2\alpha^{\frac{\mu}{2}}$ is of type $(1, 1, 2, \dots, 2)$ and $-2\alpha^{\frac{\mu}{2}}$ is of type $(2, 2, \dots, 2)$.*

Proof. See, for instance [6, Proposition 3.3]. □

Now we are equipped to prove Theorem 1.

Proof of Theorem 1. Let $g(x)$ be a polynomial with rational coefficients and with $\deg g(x) \geq 3$. Suppose that equation (13) has infinitely many solutions in integers x, y. Then by Proposition 3, it follows that there exist $\lambda(x), \mu(x), \varphi(x) \in \mathbb{Q}[x]$ such that

$$S_{a,b}^k(x) = \varphi(f_1(\lambda(x))) \quad \text{and} \quad g(x) = \varphi(g_1(\mu(x))), \tag{18}$$

where $(f_1(x), g_1(x))$ is a standard pair over $\mathbb{Q}$. Proposition 1 implies that

$$\deg \varphi(x) \in \left\{1, \frac{k+1}{2}, k+1\right\}.$$

<u>*First, suppose that* $\deg \varphi(x) = k+1$.</u> Then, by (18), we observe that $\deg f_1(x) = 1$. Thus $S_{a,b}^k(x) = \varphi(t(x))$, where $t(x) \in \mathbb{Q}[x]$ is a linear polynomial. Clearly, $t^{-1}(x) \in \mathbb{Q}[x]$ is also linear. By (18), we obtain $S_{a,b}^k(t^{-1}(x)) = \varphi(t(t^{-1}(x))) = \varphi(x)$. Hence

$$g(x) = \varphi(g_1(\mu(x))) = S_{a,b}^k(t^{-1}(g_1(\mu(x)))) = S_{a,b}^k(q(x)), \tag{19}$$

where $q(x) = t^{-1}(g_1(\mu(x)))$. So, if, in our case, equation (13) has infinitely many solutions, then $g(x)$ is of the form as in Theorem 1 (I).

<u>*Next we assume that* $\deg \varphi(x) = 1$.</u> Then there exist $\varphi_1, \varphi_0 \in \mathbb{Q}$ with $\varphi_1 \neq 0$ such that $\varphi(x) = \varphi_1 x + \varphi_0$. We study now the five kinds of standard pairs. In view of our assumptions on k and $\deg g(x)$, it follows that the standard pair $(f_1(x), g_1(x))$ cannot be of the second kind.

If it is of the third or fourth kind, we then have $S_{a,b}^k(\lambda^{-1}(x)) = e_1 D_\mu(x, \delta) + e_0$ for some $e_0 \in \mathbb{Q}, e_1, \delta \in \mathbb{Q} \setminus \{0\}$, which contradicts Lemma 6 since $k = \mu - 1 > 3$.

Now consider the case when, in (18), $(f_1(x), g_1(x))$ is a standard pair of the first kind over $\mathbb{Q}$. Then we have either

(i) $S_{a,b}^k(\lambda^{-1}(x)) = \varphi_1 x^t + \varphi_0$, or

(ii) $S_{a,b}^k(\lambda^{-1}(x)) = \varphi_1 \alpha x^r q(x)^t + \varphi_0$, where $0 \leq r < t, (r,t) = 1$ and $r + \deg q(x) > 0$.

In the first case (i), we get a contradiction by Lemma 5 since $t = k+1 \geq 5$.

In the second case (ii), we have $g(\mu^{-1}(x)) = \varphi_1 x^t + \varphi_0$. Suppose that $t = \deg g(x) > 3$. Then the polynomial $S_{a,b}^k(\lambda^{-1}(x)) - \varphi_0$ has a root with multiplicity at least 4 (since $q(x)^t$ divides it), which is impossible by Lemma 4 unless $q(x)$ is a constant polynomial. We obtain $r \leq 3$ and $q(x) \equiv Q \in \mathbb{Q} \setminus \{0\}$. It follows that

$$S_{a,b}^k(\lambda^{-1}(x)) = \varphi x^r + \varphi_0 \text{ with } \varphi := \varphi_1 \alpha Q^t \in \mathbb{Q} \setminus \{0\}. \tag{20}$$

Lemma 5 implies that $r = k + 1 = 2$ which contradicts $k > 3$.

If, in (ii), $t = 3$, then we have

$$S^k_{a,b}(x) = \varphi_1 \alpha \lambda(x)^r q(\lambda(x))^3 + \varphi_0, \tag{21}$$

where $r \in \{1, 2\}$. If $\deg q(x) = 0$, we get back to (20). We can thus assume that $q(x)$ is nonconstant. Using (9), from (21), we derive that

$$a^k B_k\left(x + \frac{b}{a}\right) = \frac{d}{dx} S^k_{a,b}(x) =$$
$$= \varphi_1 \alpha \lambda(x)^{r-1} \lambda'(x) q(\lambda(x))^2 \left(r q(\lambda(x)) + 3\lambda(x) q'(\lambda(x))\right), \tag{22}$$

whence we infer that $q(\lambda(x - b/a))$ is a multiple factor of $B_k(x)$ over $\mathbb{Q}[x]$. Then, by Lemma 4, k is even and $q(\lambda(x - b/a)) = x^2 - x - B$ for an odd, positive integer B. We obtain from (21), that $k = 6$ and $r = 1$. But in this case $S^6_{a,b}(x) - \varphi_0$ has a root of multiplicity 3, thus, by (22), the sixth Bernoulli polynomial $B_6(x)$ has a double root. However it is impossible since the discriminant of $B_6(x)$ is nonzero.

Finally, suppose that $(f_1(x), g_1(x))$ is a standard pair of the fifth kind. Now (18) implies either

(a) $S^k_{a,b}(\lambda^{-1}(x)) = \varphi_1(\alpha x^2 - 1)^3 + \varphi_0$, or
(b) $S^k_{a,b}(\lambda^{-1}(x)) = \varphi_1(3x^4 - 4x^3) + \varphi_0$.

The second case is impossible, since then we get $k = 3$ contradicting our assumption $k > 3$.

In the first case (a) we infer that $k = 5$ and that $S^5_{a,b}(\lambda^{-1}(x)) - \varphi_0$ has a root with multiplicity at least 3. But the number of roots as well as their multiplicities of a polynomial remain unchanged if we replace the variable x by a linear polynomial of it. Hence we obtain that $S^5_{a,b}(x) - \varphi_0$ also has a root with multiplicity at least 3. But then, by

$$\frac{d}{dx}(S^5_{a,b}(x) - \varphi_0) = \frac{d}{dx} \frac{a^5 B_6\left(x + \frac{b}{a}\right)}{6} = a^5 B_5\left(x + \frac{b}{a}\right), \tag{23}$$

the fifth Bernoulli polynomial $B_5(x)$ would have a multiple root, which is a contradiction by Lemma 4.

Let us consider the remaining case $\deg \varphi(x) = (k + 1)/2$. Clearly, k is then odd, and from (18) we know that $\deg f_1(x) = 2$. Hence it follows that, in (18), $(f_1(x), g_1(x))$ cannot be a standart pair of the fifth kind. Further, we obtain a nontrivial decomposition of $S^k_{a,b}(x)$, which by Proposition 1 implies that there exists a linear polynomial $\ell(x) = \ell_1 x + \ell_0$ over $\mathbb{Q}$ such that

$$\varphi(x) = \widehat{S}^{(k+1)/2}_{a,b}(\ell(x)) \quad \text{and} \quad \ell(f_1(\lambda(x))) = \left(x + \frac{b}{a} - \frac{1}{2}\right)^2. \tag{24}$$

Again, we study the unexcluded kinds of standard pairs over $\mathbb{Q}$.

First, we assume $(f_1(x), g_1(x))$ to be a standard pair of the first kind. If $(f_1(x), g_1(x)) = (x^t, \alpha x^r p(x)^t)$ with $r < t, (r,t) = 1$ and $r + \deg p(x) > 0$, then by $\deg f_1(x) = 2$, the corresponding standard pair is of the form $(f_1(x), g_1(x)) = (x^2, \alpha x p(x)^2)$. If $\lambda(x) = \lambda_1 x + \lambda_0$, then (24) takes the form $\ell((\lambda_1 x + \lambda_0)^2) = (x + b/a - 1/2)^2$, whence one can deduce that $\ell(x) = x/\lambda_1^2$. Substituting this to (18), we obtain

$$g(x) = \widehat{S}_{a,b}^{(k+1)/2}(\ell(g_1(\mu(x)))) = \widehat{S}_{a,b}^{(k+1)/2}\left(\frac{\alpha\mu(x)p(\mu(x))^2}{\lambda_1^2}\right) \tag{25}$$

So $g(x)$ is of the form as in Theorem 1 (II) with $\delta(x) = \alpha\mu(x)/\lambda_1^2$ and $q(x) = p(\mu(x))$.

In the switched case $(f_1(x), g_1(x)) = (\alpha x^r p(x)^t, x^t)$, where $r < t$, $(r,t) = 1$ and $r + \deg p(x) > 0$, we obtain from $\deg f_1(x) = 2$ that one of the following cases occurs:

(A) $r = 0,\ t = 1$ and $\deg p(x) = 2$, or
(B) $r = 2,\ t > 2$ is odd and $p(x)$ is constant.

In case (A) we have $g_1(x) = x$ which together with (18) and (24) implie

$$g(x) = \widehat{S}_{a,b}^{(k+1)/2}(\ell(g_1(\mu(x)))) = \\ = \widehat{S}_{a,b}^{(k+1)/2}(\ell(\mu(x))) = \widehat{S}_{a,b}^{(k+1)/2}(\delta(x)q(x)^2), \tag{26}$$

where $\delta(x) = \ell(\mu(x))$ and $q(x) \equiv 1$. Thus $g(x)$ is again of the form as in Theorem 1 (II).

If case (B) holds, then we can write $f_1(x) = \beta x^2$, with $\beta = \alpha p(x)^t \in \mathbb{Q} \setminus \{0\}$. Substituting this to (24), we deduce that $\ell(x) = x/(\beta\lambda_1^2)$, whence, by (18), we get

$$g(x) = \widehat{S}_{a,b}^{(k+1)/2}(\ell(g_1(\mu(x)))) = \\ = \widehat{S}_{a,b}^{(k+1)/2}\left(\frac{\mu(x)^t}{\beta\lambda_1^2}\right) = \widehat{S}_{a,b}^{(k+1)/2}(c\delta(x)^t), \tag{27}$$

where $c = 1/(\beta\lambda_1^2),\ \delta(x) = \mu(x)$ and $t > 2$ is odd. This is case (III) in Theorem 1.

Next suppose that, in (18), the standard pair $(f_1(x), g_1(x))$ is of the second kind. If $(f_1(x), g_1(x)) = (x^2, (\alpha x^2 + \beta)v(x)^2)$, then a calculation from (24) leads to $\ell(x) = x/\lambda_1^2$, and from (18) we obtain

$$g(x) = \widehat{S}_{a,b}^{(k+1)/2}(\ell(g_1(\mu(x)))) = \\ = \widehat{S}_{a,b}^{(k+1)/2}\left(\frac{(\alpha x^2 + \beta)v(\mu(x))^2}{\lambda_1^2}\right) = \widehat{S}_{a,b}^{(k+1)/2}((\alpha\delta(x)^2 + \beta)q(x)^2), \tag{28}$$

where $\delta(x) = \mu(x)$ and $q(x) = v(\mu(x))/\lambda_1$. So we are in case (IV) of Theorem 1.

If $(f_1(x), g_1(x)) = ((\alpha x^2 + \beta)v(x)^2, x^2)$, then since $\deg f_1(x) = 2$, $v(x)$ is a constant polynomial and we have

$$g(x) = \widehat{S}_{a,b}^{(k+1)/2}(\ell(g_1(\mu(x)))) = \\ = \widehat{S}_{a,b}^{(k+1)/2}(\ell_1\mu(x)^2 + \ell_0) = \widehat{S}_{a,b}^{(k+1)/2}((\ell_1\delta(x)^2 + \ell_0)q(x)^2), \quad (29)$$

with $\delta(x) = \mu(x)$ and $q(x) \equiv 1$. Again, we arrived at case (IV) of Theorem 1.

Now, if the standard pair $(f_1(x), g_1(x))$ is of the third kind, then $(f_1(x), g_1(x)) = (D_2(x, \alpha^t), D_t(x, \alpha^2))$ with t being odd. Let us substitute $f_1(x) = x^2 - 2\alpha^t$ into (24) to deduce that $\ell(x) = (x + 2\alpha^t)/\lambda_1^2$, whence

$$g(x) = \widehat{S}_{a,b}^{(k+1)/2}(\ell(g_1(\mu(x)))) = \widehat{S}_{a,b}^{(k+1)/2}\left(\frac{D_t(\mu(x), \alpha^2) + 2\alpha^t}{\lambda_1^2}\right). \quad (30)$$

It follows from Proposition 7 that $-2\alpha^t/\lambda_1^2$ is an extremum of the polynomial $D_t(\mu(x), \alpha^2)/\lambda_1^2$, which is of type $(1, 2, \ldots, 2)$ as t is odd. This implies that $(D_t(\mu(x), \alpha^2) + 2\alpha^t)/\lambda_1^2 = \delta(x)q(x)^2$ for some $\delta(x), q(x) \in \mathbb{Q}[x]$ with $\deg \delta(x) = 1$. Hence $g(x)$ is of the form as in Theorem 1 (II).

Finally, consider the case when $(f_1(x), g_1(x))$ is a standard pair of the fourth kind. Then

$$(f_1(x), g_1(x)) = \left(\frac{D_2(x, \alpha)}{\alpha}, \frac{D_t(x, \beta)}{\beta^{(t/2)}}\right),$$

where t is even. Substituting this into (24), it is easy to calculate that $\ell(x) = (\alpha x + 2\alpha)/\lambda_1^2$. Hence, by (18), we obtain

$$g(x) = \widehat{S}_{a,b}^{(k+1)/2}(\ell(g_1(\mu(x)))) = \\ = \widehat{S}_{a,b}^{(k+1)/2}\left(\frac{\alpha\beta^{-t/2}D_t(\mu(x), \beta) + 2\alpha}{\lambda_1^2}\right). \quad (31)$$

Now Proposition 7 implies that

$$-\frac{2\beta^{t/2}\alpha\beta^{-t/2}}{\lambda_1^2} = -\frac{2\alpha}{\lambda_1^2}$$

is one of the two extrema of the polynomial $\alpha\beta^{-t/2}D_t(\mu(x), \beta)/(\lambda_1^2)$ and it is of type $(2, 2, \ldots, 2)$ as t is even. It follows that

$$\frac{\alpha\beta^{-t/2}D_t(\mu(x), \beta) + 2\alpha}{\lambda_1^2} = q(x)^2$$

for some $q(x) \in \mathbb{Q}[x]$. Thus $g(x)$ is of type (V) in Theorem 1. This completes the proof. $\square$

3. Proofs of Theorems 2 and 3

We discuss the proofs of Theorems 2 and 3 jointly by introducing the following notation. Let $\mathtt{T}_{a,b}^{k\pm}(x) \in \left\{\mathtt{T}_{a,b}^{k+}(x), \mathtt{T}_{a,b}^{k-}(x)\right\}$ and similarly, let $\widehat{T}_{a,b}^{m\pm}(x) \in \left\{\widehat{T}_{a,b}^{m+}(x), \widehat{T}_{a,b}^{m-}(x)\right\}$. Now equations (14) and (15) can be written in the common form

$$\mathtt{T}_{a,b}^{k\pm}(x) = g(y), \tag{32}$$

where $g(y)$ is a rational polynomial of degree at least 3.

Before starting the proof we need the following auxiliary results besides the ones from the previous sections.

The first one is a deep result of Rakaczki [21] concerning the root structure of shifted Euler polynomials.

Proposition 8. *Let $m \geq 7$ be an integer. Then the shifted Euler polynomial $E_m(x) + b$ has at least three simple zeros for arbitrary complex number b.*

The following result is Lemma 11 in [22]. Let $c_1, e_1 \in \mathbb{Q} \setminus \{0\}$ and $c_0, e_0 \in \mathbb{Q}$.

Lemma 9. *The polynomial $E_k(c_1x+c_0)$ is neither of the form $e_1x^q+e_0$ with $q \geq 3$, nor of the form $e_1D_\nu(x,\delta)+e_0$, where $D_\nu(x,\delta)$ is the ν-th Dickson polynomial with $\nu > 4, \delta \in \mathbb{Q} \setminus \{0\}$.*

The next lemma is a simple consequence of the previous one. Further, it is an analogue of Lemmas 5 and 6 from the preceding section.

Lemma 10. *None of the polynomials $T_{a,b}^{k+}(c_1x+c_0)$ and $T_{a,b}^{k-}(c_1x+c_0)$ are either of the form $e_1x^q+e_0$ with $q \geq 3$, or of the form $e_1D_\nu(x,\delta)+e_0$, where $D_\nu(x,\delta)$ is the ν-th Dickson polynomial with $\nu > 4, \delta \in \mathbb{Q} \setminus \{0\}$.*

Proof. We detail the proof only for the 'positive' case. For the 'negative' case the argument is essentially the same.

Since $\mathtt{T}_{a,b}^{k+}(c_1x+c_0) = a^k/2\,(E_k(b/a) + E_k(c_1x+c_0+b/a))$, we have

$$E_k\left(c_1x+c_0+\frac{b}{a}\right) = \frac{2}{a^k}\mathtt{T}_{a,b}^{k+}(c_1x+c_0) - E_k\left(\frac{b}{a}\right). \tag{33}$$

Put $c_0' = c_0 + b/a$, $e_1' = (2e_1)/(a^k)$ and $e_0' = (2e_0)/(a^k) - E_k(b/a)$.

Now if $\mathtt{T}_{a,b}^{k+}(c_1x+c_0) = e_1x^q+e_0$ for some $q \geq 3$, then we obtain

$$E_k(c_1x+c_0') = e_1'x^q+e_0'.$$

This contradicts Lemma 9.

Similarly, if $\mathtt{T}^{k+}_{a,b}(c_1x + c_0) = e_1D_\nu(x, \delta) + e_0$ for some $\nu > 4$ and $\delta \in \mathbb{Q} \setminus \{0\}$, then by (33) we have

$$E_k(c_1x + c_0') = e_1'D_\nu(x, \delta) + e_0',$$

contradicting again Lemma 9. □

Proof of Theorems 2 and 3. Suppose that equation (32) has infinitely many solutions in integers x, y. Then by Proposition 3, there exist $\varphi(x) \in \mathbb{Q}[x]$ and linear polynomials $\lambda(x), \mu(x) \in \mathbb{Q}[x]$ such that

$$\mathtt{T}^{k\pm}_{a,b}(x) = \varphi(f_1(\lambda(x))) \quad \text{and} \quad g(x) = \varphi(g_1(\mu(x))), \tag{34}$$

where $(f_1(x), g_1(x))$ is a standard pair over $\mathbb{Q}$. From $\deg \mathtt{T}^{k\pm}_{a,b}(x) = k$ and from Proposition 2 we infer that

$$\deg \varphi(x) \in \left\{1, \frac{k}{2}, k\right\}.$$

<u>*Suppose first that* $\deg \varphi(x) = k$.</u> Then (34) implies that $\deg f_1(x) = 1$. Therefore $\mathtt{T}^{k\pm}_{a,b}(x) = \varphi(t(x))$, for a linear polynomial $t(x) \in \mathbb{Q}[x]$. Clearly, $t^{-1}(x) \in \mathbb{Q}[x]$ is also linear. Thus, by (34), we get that $\mathtt{T}^{k\pm}_{a,b}(t^{-1}(x)) = \varphi(t(t^{-1}(x))) = \varphi(x)$, whence

$$g(x) = \varphi(g_1(\mu(x))) = \mathtt{T}^{k\pm}_{a,b}(t^{-1}(g_1(\mu(x)))) = \mathtt{T}^{k\pm}_{a,b}(q(x)), \tag{35}$$

where $q(x) = t^{-1}(g_1(\mu(x)))$. So, if equation (32) has infinitely many solutions, then $g(x)$ is of the form (I) in Theorem 2 or 3, respectively.

<u>*Next we assume that* $\deg \varphi(x) = 1$.</u> Then there exist $\varphi_1, \varphi_0 \in \mathbb{Q}$ with $\varphi_1 \neq 0$ such that $\varphi(x) = \varphi_1x + \varphi_0$. We study now the five kinds of standard pairs over $\mathbb{Q}$. In view of $k \geq 7$ and $\deg g(x) \geq 3$, we see that, in (34), the standard pair $(f_1(x), g_1(x))$ cannot be of the second or the fifth kind.

Now in (34), let $(f_1(x), g_1(x))$ assumed to be a standard pair of the first kind. Then we have either

(i) $\mathtt{T}^{k\pm}_{a,b}(\lambda^{-1}(x)) = \varphi_1x^t + \varphi_0$, or

(ii) $\mathtt{T}^{k\pm}_{a,b}(\lambda^{-1}(x)) = \varphi_1\alpha x^r q(x)^t + \varphi_0$, where $0 \leq r < t$, $(r, t) = 1$ and $r + \deg q(x) > 0$.

In the first case (i), we obtain a contradiction by Lemma 10 since $t = k \geq 7$.

In the case (ii), since

$$\mathtt{T}^{k\pm}_{a,b}(\lambda^{-1}(x)) - \varphi_0 = \frac{a^k}{2}\left(E_k\left(\lambda^{-1}(x) + \frac{b}{a}\right) \pm E_k\left(\frac{b}{a}\right) - \frac{2\varphi_0}{a^k}\right),$$

and since the root structure of a polynomial remains the same if the variable of the polynomial is replaced by a linear polynomial of it, we infer by Proposition 8 that $\mathtt{T}_{a,b}^{k\pm}(\lambda^{-1}(x)) - \varphi_0$ has at least three simple zeros. By the assumptions on r and t, this implies that $r = 0$ and $t = \deg g(x) = 1$, which contradicts $\deg g(x) \geq 3$.

Finally, suppose that $(f_1(x), g_1(x))$ is a standard pair of the third or the fourth kind over $\mathbb{Q}$. Then we obtain

$$\mathtt{T}_{a,b}^{k\pm}(\lambda^{-1}(x)) = \varphi_1' D_k(x, \delta) + \varphi_0,$$

where $\varphi_1' \in \left\{\varphi_1, a^{k/2}\varphi_1\right\}$ and $\delta \in \mathbb{Q} \setminus \{0\}$. This is a contradiction by Lemma 10 since $k \geq 7$.

The case $\deg \varphi(x) = k/2$. Clearly, k is then even, and from (34) we observe that $\deg f_1(x) = 2$. Hence it follows that, in (34), $(f_1(x), g_1(x))$ cannot be a standart pair of the fifth kind. Further, we obtain a nontrivial decomposition of $\mathtt{T}_{a,b}^{k\pm}(x)$, which by Proposition 2 implies that there exists a linear polynomial $\ell(x) = \ell_1 x + \ell_0$ over $\mathbb{Q}$ such that

$$\varphi(x) = \widehat{T}_{a,b}^{k/2\pm}(\ell(x)) \quad \text{and} \quad \ell(f_1(\lambda(x))) = \left(x + \frac{b}{a} - \frac{1}{2}\right)^2. \tag{36}$$

Again, we study the remaining kinds of standard pairs.

First, we consider the case when, in (34), $(f_1(x), g_1(x))$is a standard pair of the first kind. If $f_1(x) = x^t$, then by $\deg f_1(x) = 2$, we have $(f_1(x), g_1(x)) = (x^2, \alpha x p(x)^2)$. Putting $\lambda(x) = \lambda_1 x + \lambda_0$, (36) takes the form $\ell((\lambda_1 x + \lambda_0)^2) = (x + b/a - 1/2)^2$, whence an easy calculation gives $\ell(x) = x/\lambda_1^2$. Substituting this to (34), we obtain

$$g(x) = \widehat{T}_{a,b}^{k/2\pm}(\ell(g_1(\mu(x)))) = \widehat{T}_{a,b}^{k/2\pm}\left(\frac{\alpha\mu(x)p(\mu(x))^2}{\lambda_1^2}\right) \tag{37}$$

So $g(x)$ is of the form (III) with $\delta(x) = \alpha\mu(x)/\lambda_1^2$ and $q(x) = p(\mu(x))$.

In the switched case $(f_1(x), g_1(x)) = (\alpha x^r p(x)^t, x^t)$, where $r < t$, $(r, t) = 1$ and $r + \deg p(x) > 0$, $\deg f_1(x) = 2$ implies that one of the following cases occurs:

(A) $r = 0,\ t = 1$ and $\deg p(x) = 2$, or
(B) $r = 2,\ t > 2$ is odd and $p(x)$ is a constant polynomial.

In case (A) we have $g_1(x) = x$, whence from (34) and (36) we obtain

$$g(x) = \widehat{T}_{a,b}^{k/2\pm}(\ell(g_1(\mu(x)))) = \\ = \widehat{T}_{a,b}^{k/2\pm}(\ell(\mu(x))) = \widehat{T}_{a,b}^{k/2\pm}(\delta(x)q(x)^2), \tag{38}$$

where $\delta(x) = \ell(\mu(x))$ and $q(x) \equiv 1$. Thus $g(x)$ is again of type (III).

In the second case (B), we can write $f_1(x) = \beta x^2$, with $\beta = \alpha p(x)^t \in \mathbb{Q} \setminus \{0\}$. Substituting this to (36), we deduce that $\ell(x) = x/(\beta\lambda_1^2)$, whence, by (34), we get

$$g(x) = \widehat{T}_{a,b}^{k/2\pm}(\ell(g_1(\mu(x)))) = \\ = \widehat{T}_{a,b}^{k/2\pm}\left(\frac{\mu(x)^t}{\beta\lambda_1^2}\right) = \widehat{T}_{a,b}^{k/2\pm}(c\delta(x)^t), \quad (39)$$

where $c = 1/(\beta\lambda_1^2)$, $\delta(x) = \mu(x)$ and $t > 2$ is odd. This is case (IV) in Theorems 2 or 3, respectively.

Next let, in (34), be a the standard pair $(f_1(x), g_1(x))$ of the second kind. If $(f_1(x), g_1(x)) = (x^2, (\alpha x^2 + \beta)v(x)^2)$, then a calculation from (36) yields $\ell(x) = x/\lambda_1^2$, and by (34) we have

$$g(x) = \widehat{T}_{a,b}^{k/2\pm}(\ell(g_1(\mu(x)))) = \\ = \widehat{T}_{a,b}^{k/2\pm}\left(\frac{(\alpha x^2 + \beta)v(\mu(x))^2}{\lambda_1^2}\right) = \widehat{T}_{a,b}^{k/2\pm}((\alpha\delta(x)^2 + \beta)q(x)^2), \quad (40)$$

where $\delta(x) = \mu(x)$ and $q(x) = v(\mu(x))/\lambda_1$. So we are in case (V) of our Theorems.

In the switched case $(f_1(x), g_1(x)) = ((\alpha x^2 + \beta)v(x)^2, x^2)$, since $\deg f_1(x) = 2$, $v(x)$ is a constant polynomial and

$$g(x) = \widehat{T}_{a,b}^{k/2\pm}(\ell(g_1(\mu(x)))) = \\ = \widehat{T}_{a,b}^{k/2\pm}((\ell_1\mu(x)^2 + \ell_0)q(x)^2), \quad (41)$$

where $q(x) \equiv 1$. Thus, we arrived again at case (V) with $\delta(x) = \mu(x)$ and $\alpha = \ell_1, \beta = \ell_0$.

Now, if the standard pair $(f_1(x), g_1(x))$ is of the third kind over $\mathbb{Q}$, then $(f_1(x), g_1(x)) = (D_2(x, \alpha^t), D_t(x, \alpha^2))$ with t being odd. Let us substitute $f_1(x) = x^2 - 2\alpha^t$ into (36) to deduce that $\ell(x) = (x + 2\alpha^t)/\lambda_1^2$, whence

$$g(x) = \widehat{T}_{a,b}^{k/2\pm}(\ell(g_1(\mu(x)))) = \widehat{T}_{a,b}^{k/2\pm}\left(\frac{D_t(\mu(x), \alpha^2) + 2\alpha^t}{\lambda_1^2}\right). \quad (42)$$

It follows from Proposition 7 that $-2\alpha^t/\lambda_1^2$ is an extremum of the polynomial $D_t(\mu(x), \alpha^2)/\lambda_1^2$, which is of type $(1, 2, \ldots, 2)$ as t is odd. Hence $(D_t(\mu(x), \alpha^2) + 2\alpha^t)/\lambda_1^2 = \delta(x)q(x)^2$ for some $\delta(x), q(x) \in \mathbb{Q}[x]$ with $\deg\delta(x) = 1$. We deduce, that $g(x)$ is of type (III).

Finally, consider the case when $(f_1(x), g_1(x))$ is a standard pair of the fourth kind over $\mathbb{Q}$. Then

$$(f_1(x), g_1(x)) = \left(\frac{D_2(x, \alpha)}{\alpha}, \frac{D_t(x, \beta)}{\beta^{(t/2)}}\right),$$

with an even t. Substituting this into (36), an easy calculation yields $\ell(x) = (\alpha x + 2\alpha)/\lambda_1^2$, whence, by (34), we obtain

$$g(x) = \widehat{T}_{a,b}^{k/2\pm}(\ell(g_1(\mu(x)))) = \\ = \widehat{T}_{a,b}^{k/2\pm}\left(\frac{\alpha\beta^{-t/2}D_t(\mu(x),\beta) + 2\alpha}{\lambda_1^2}\right). \quad (43)$$

Now from Proposition 7 we infer that

$$-\frac{2\beta^{t/2}\alpha\beta^{-t/2}}{\lambda_1^2} = -\frac{2\alpha}{\lambda_1^2}$$

is one of the two extrema of the polynomial $\alpha\beta^{-t/2}D_t(\mu(x),\beta)/(\lambda_1^2)$ and it is of type $(2,2,\dots,2)$ as t is even. Therefore we have

$$\frac{\alpha\beta^{-t/2}D_t(\mu(x),\beta) + 2\alpha}{\lambda_1^2} = q(x)^2$$

for some $q(x) \in \mathbb{Q}[x]$. Thus $g(x)$ is of type (II), and the proof is complete. □

Acknowledgements

Research was supported by the Hungarian Academy of Sciences and by the OTKA grant NK104208.

References

1. A. Bazsó, On alternating power sums of arithmetic progressions, *Integral Transforms Spec. Funct.*, **24** (2013), 945-949.
2. A. Bazsó, D. Kreso, F. Luca and Á. Pintér, On equal values of power sums of arithmetic progressions, *Glas. Mat. Ser. III*, **47** (2012), 253–263.
3. A. Bazsó, Á. Pintér and H. M. Srivastava, On a refinement of Faulhaber's Theorem concerning sums of powers of natural numbers, *Appl. Math. Letters*, **25** (2012), 486–489.
4. M. A. Bennett, A superelliptic equation involving alternating sums of powers, *Publ. Math. Debrecen*, **79** (2011), 317–324.
5. M. A. Bennett, K. Győry and Á. Pintér, On the Diophantine equation $1^k + 2^k + \cdots + x^k = y^n$, *Compos. Math.*, **140** (2004), 1417–1431.
6. Y. F. Bilu, Quadratic factors of $f(x) - g(y)$, *Acta Arith.*, **90** (1999), 341–355.
7. Y. F. Bilu and R. F. Tichy, The Diophantine equation $f(x) = g(y)$, *Acta Arith.*, **95** (2000), 261–288.
8. Y. F. Bilu, B. Brindza, P. Kirschenhofer, Á. Pintér and R. F. Tichy, Diophantine equations and Bernoulli polynomials (with an Appendix by A. Schinzel), *Compositio Math.*, **131** (2002), 173–188.
9. J. Brillhart, On the Euler and Bernoulli polynomials, *J. Reine Angew. Math.*, **234** (1969), 45–64.
10. B. Brindza, On some generalizations of the diophantine equation $1^k + 2^k + \ldots + x^k = y^z$, *Acta Arith.*, **44** (1984), 99-107.

11. K. Dilcher, On a Diophantine equation involving quadratic characters, *Compositio Math.* **57** (1986), 383–403.
12. K. Győry and Á. Pintér, On the equation $1^k + 2^k + \ldots + x^k = y^n$, *Publ. Math. Debrecen*, **62** (2003), 403–414.
13. K. Győry, R. Tijdeman and M. Voorhoeve, On the equation $1^k + 2^k + \ldots + x^k = y^z$, *Acta Arith.*, **37** (1980), 234–240.
14. F. T. Howard, Sums of powers of integers via generating functions, *Fibonacci Quart.*, **34** (1996), 244–256.
15. M. Jacobson, Á. Pintér and P. G. Walsh, A computational approach for solving $y^2 = 1^k + 2^k + \cdots + x^k$, *Math. Comp.*, **72** (2003), 2099–2110.
16. É. Lucas, Problem 1180, *Nouvelles Ann. Math.*, **14** (1875), 336.
17. M. Kulkarni and B. Sury, Diophantine equations with Bernoulli polynomials, *Acta Arith.*, **116** (2005), 25–34.
18. M. Kulkarni and B. Sury, A class of Diophantine equations involving Bernoulli polynomials, *Indag. Math. (N.S.)*, **16** (2005), 51–65.
19. M. Kulkarni and B. Sury, On the Diophantine equation $1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} = g(y)$., In: Diophantine equations, 121-134, *Tata Inst. Fund. Res. Stud. Math.*, Mumbai, 2008.
20. Cs. Rakaczki, On the Diophantine equation $S_m(x) = g(y)$, *Publ. Math. Debrecen*, **65** (2004), 439–460.
21. Cs. Rakaczki, On the simple zeros of shifted Euler polynomials, *Publ. Math. Debrecen*, **79** (2011), 623–636.
22. Cs. Rakaczki and D. Kreso, Diophantine equations with Euler polynomials, *Acta Arith.*, **161** (2013), 267–281.
23. J. J. Schäffer The equation $1^p + 2^p + 3^p + \cdots + n^p = m^q$ *Acta Math.*, **95** (1956), 155–189.
24. A. Schinzel, Polynomials with special regard to reducibility, *Cambridge University Press*, 2000.
25. J. Urbanowicz, On the equation $f(1)1^k + f(2)2^k + \ldots + f(x)x^k + R(x) = by^z$, *Acta Arith.*, **51** (1988), 349-368.
26. J. Urbanowicz, On diophantine equations involving sums of powers with quadratic characters as coefficients, I., *Compositio Math.*, **92** (1994), 249-271.
27. J. Urbanowicz, On diophantine equations involving sums of powers with quadratic characters as coefficients, II., *Compositio Math.*, **102** (1996), 125-140.
28. M. Voorhoeve, K. Győry and R. Tijdeman, On the diophantine equation $1^k + 2^k + \ldots + x^k + R(x) = y^z$, *Acta Math.*, **143** (1979), 1-8; Corr. **159** (1987), 151-152.

András Bazsó
Institute of Mathematics
MTA-DE Research Group "Equations Functions and Curves"
Hungarian Academy of Sciences and University of Debrecen
P.O. Box 12, H-4010 Debrecen, Hungary
E-mail address: bazsoa@science.unideb.hu