Probabilistic modeling natural way to treat data

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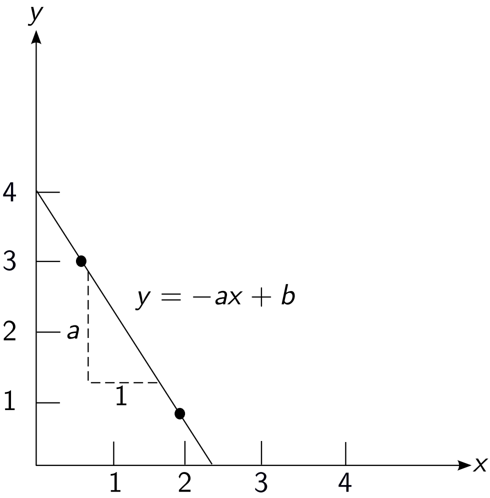
y
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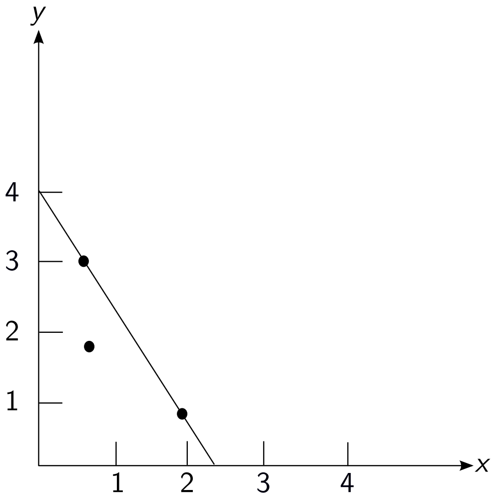
a
1
$y = -ax + b$
1
2
3
4
x


y
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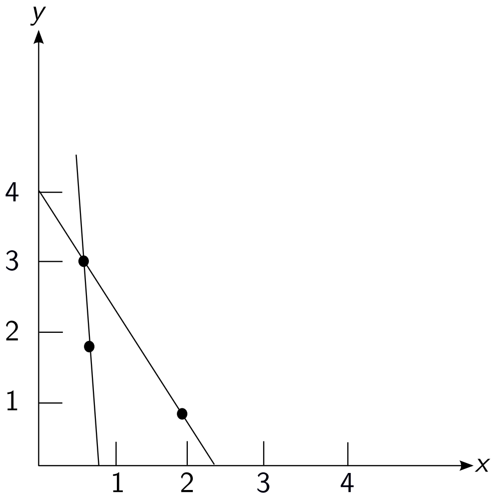

y
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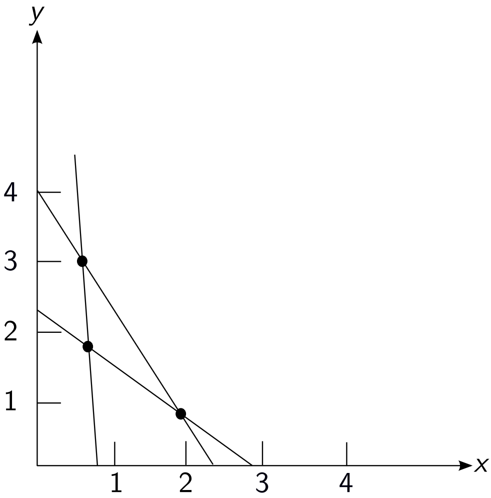
x


y
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x
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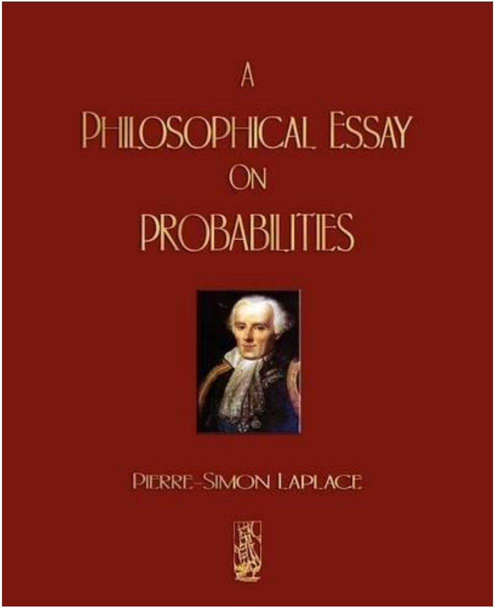

A
PHILOSOPHICAL ESSAY
ON
PROBABILITIES
PIERRE-SIMON LAPLACE

Each point can be written as the model+ a corruption:

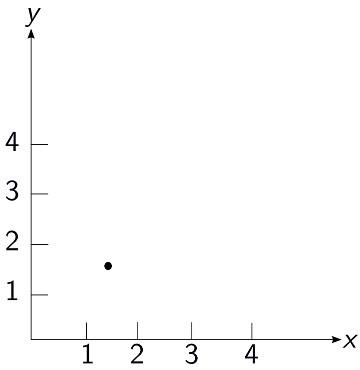
$$y_1 = ax + c + \omega_1$$
$$y_2 = ax + c + \omega_2$$
$$y_3 = ax + c + \omega_3$$

ω is the difference between real world and model which can be presented by a probability distribution.

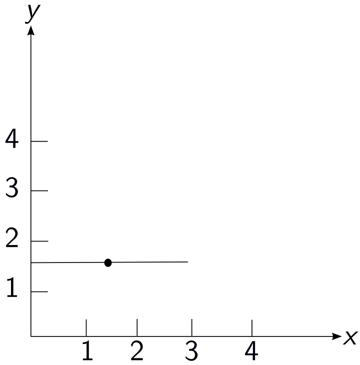
We call ω noise!

What if our observations are less than model parameters?

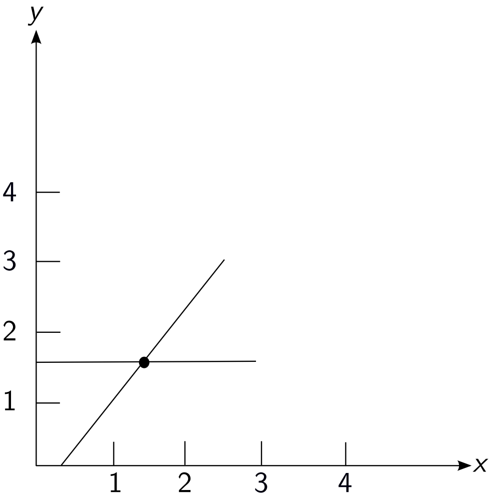
Underdetermined system



How can we fit the $y = ax + b$ line, having only one point?



If b is fixed $\implies a = \frac{y-b}{x}$


y
4
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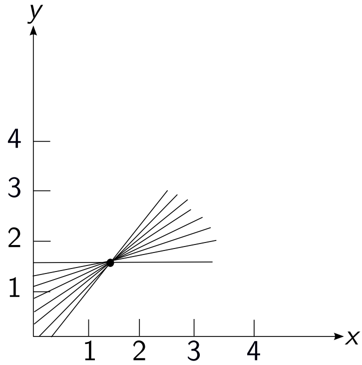
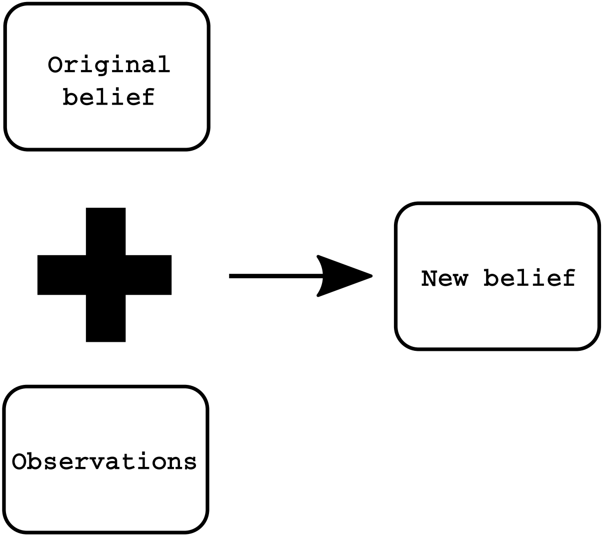
x



$$b \sim \pi_1 \implies a \sim \pi_2$$

- This is called Bayesian treatment.
- The model parameters are treated as random variables.

Bayesian perspective



Bayesian formula (inverse probability)

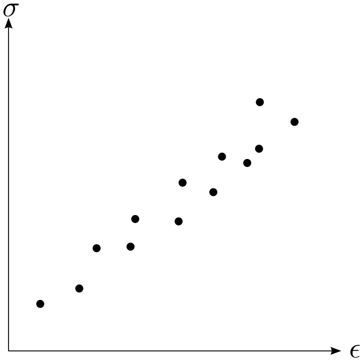
$$\overbrace{\pi(x|y)}^{\textit{posterior}} = \frac{\overbrace{\pi(x)}^{\textit{prior}} \times \overbrace{\pi(y|x)}^{\textit{likelihood}}}{\underbrace{\pi(y)}_{\textit{evidence}}}$$

$y :=$ observation
$x :=$ parameter
$\pi(x) :=$ original belief
$\pi(y|x) :=$ given by the mathematical model that relates y to x
$\pi(y) :=$ is a constant number

Bayesian formula (inverse probability)

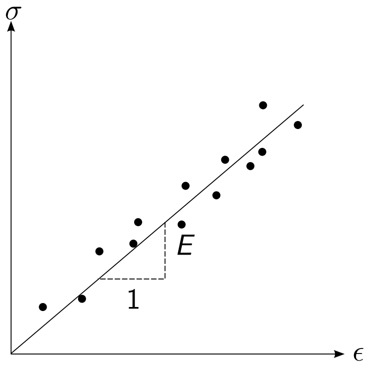
$$\pi(x|y) \propto \pi(x) \times \pi(y|x)$$

BI in computational mechanics



Linear elasticity

$$\sigma = E\epsilon$$

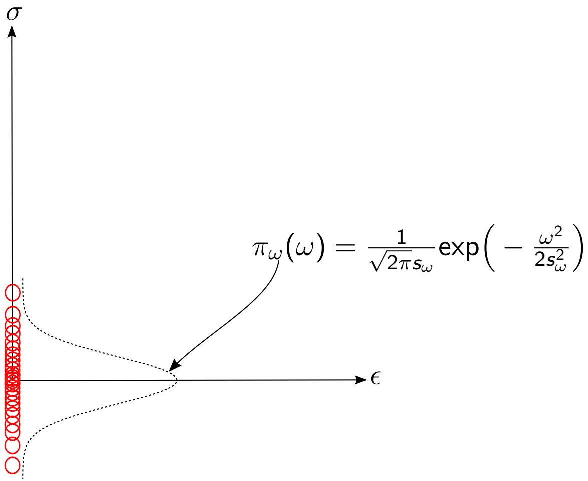


Linear elasticity

$$y = E\epsilon + \omega$$
$$\Omega \sim \pi_\omega(\omega)$$

Capital letters denote a random variable

Linear elasticity



Noise PDF is modeled through calibration test.

Linear elasticity

Bayes' formula:

$$\pi(E|y) = \frac{\pi(E)\pi(y|E)}{\pi(y)} = \frac{\pi(E)\pi(y|E)}{k}$$

$$\pi(E|y) \propto \pi(E)\pi(y|E)$$

Linear elasticity

$$y = E\epsilon + \omega$$
$$\Omega \sim N(0, s_\omega^2)$$

Linear elasticity

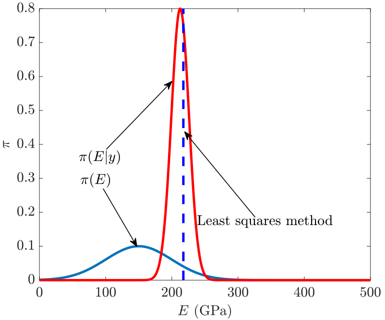
$$\pi(y|E) = \frac{1}{\sqrt{2\pi}s_\omega}\exp\Big(-\frac{(y-E\epsilon)^2}{2s_\omega^2}\Big)$$

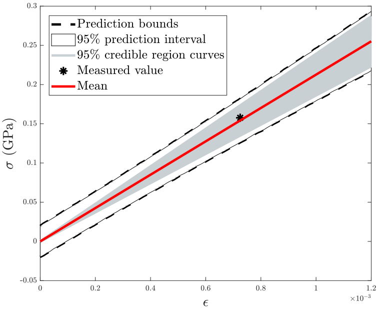
Linear elasticity

Posterior:

$$\pi(E|y) \propto \exp\Big(-\frac{(E-\overline{E})^2}{2s_E^2}\Big)\exp\Big(-\frac{(y-E\epsilon)^2}{2s_\omega^2}\Big)$$

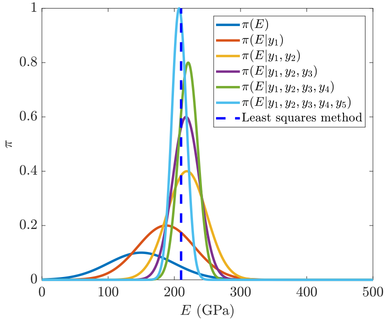
Linear elasticity

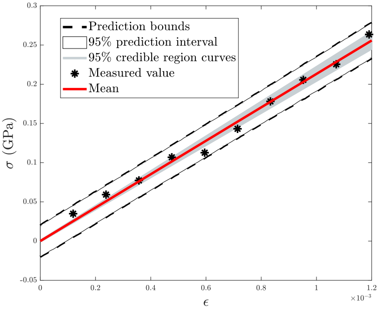




- Prediction interval: An estimate of an interval in which an observation will fall, with a certain probability.
- Credible region: A region of a distribution in which it is believed that a random variable lie with a certain probability.

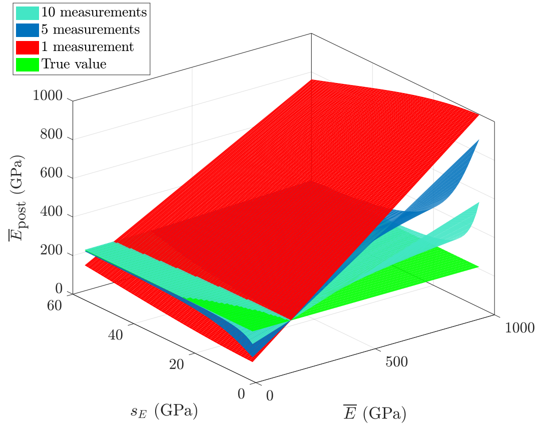
Linear elasticity





- Increase in number of observations/measurements makes us more sure of identification result.

Prior effect



- Increase in number of observations/measurements decreases the effect of prior.

Conclusion

- Probability is the natural way of dealing with uncertainties/unknowns (what Laplace calls it our ignorance).

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- Probability is the natural way of dealing with uncertainties/unknowns (what Laplace calls it our ignorance).
- From Bayesian perspective (inverse probability) the parameters are treated as random variables.
- The same logic can be used to model other kinds of uncertainties/unknowns e.g. model uncertainties and material variability.
- In Bayesian paradigm our assumptions are clearly stated (e.g. the prior, model and ...).
- As the number of observation/measurements increases we become more sure of our identification results.

Acknowledgement

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