On a natural fuzzification of Boolean logic

R. Bisdorff

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1 Introduction

In this communication we propose two logically sound fuzzification and defuzzification techniques for implementing a credibility calculus on a set of propositional expressions. Both rely on a credibility evaluation domain using the rational interval $[-1, 1]$ where the sign carries a split truth/falseness denotation. The first technique implements the classic min and max operators where as the second technique implements Bochvar-like operators. Main interest in the communication is given to the concept of *natural fuzzification* of a propositional calculus. A formal definition is proposed and the demonstration that both fuzzification techniques indeed verify this definition is provided.

2 Logical fuzzification and polarization: an adjoint pair

2.1 Introducing logical fuzziness

Let P be a set of constants or ground propositions. Let $\neg$, $\vee$ and $\wedge$ denote respectively the contradiction, disjunction and conjunction operators.

The set E of all *well formulated finite expressions* will be generated inductively from the following grammar:

$$\forall p \in P \quad : \quad p \in E, \tag{1}$$

$$\forall x, y \in E \quad : \quad \neg x \mid (x) \mid x \vee y \mid x \wedge y \ \in E. \tag{2}$$

The unary *contradiction* operator $\neg$ has a higher precedence in the interpretation of a formula, but we generally use brackets to control the application range of a given operator and thus to make all formulas have unambiguous semantics. We suppose in the sequel that all other operators such as implication, equivalence, xor etc are derived with the help of these three basic operators: contradiction, conjunction and disjunction.

With these well formulated propositional expressions we associate a rational credibility evaluation $r : E \to [-1, 1]$ where $\forall x, y \in E$, $r_x = 1$ means x is *certainly true*, $r_x = -1$ means that x is *certainly false* and $r_x > r_y$ (resp. $r_x < r_y$) means that propositional expression x is more (resp. less) credible than propositional expression y. Such a credibility domain is called $\mathcal{L}$, and we denote $E^{\mathcal{L}} = \{(x, r_x) \mid x \in E, r_x \in [-1, 1]\}$ a given set of such more or less credible propositional expressions, also called for short $\mathcal{L}$-expressions.

We implement the *contradiction* operator on $\mathcal{L}$-expressions by simply *changing the sign* of the associated credibility evaluation, i.e.

$$\forall (x, r_x) \in E^{\mathcal{L}} : \neg(r, r_x) = (\neg x, -r_x). \tag{3}$$

The sign exchange thus implements an antitone bijection on the rational interval $[-1, 1]$ where the *zero* value appears as contradiction fix-point.

In classical bi-valued logic, it is usual to work syntactically only on the *truth* point of view of the logic, the *untruth* or *falseness* point of view being redundant through the coercion to the excluded middle. For instance, writing "$(a, b) \in R$" implicitly means assuming that this proposition is actually true and its contradiction false, otherwise we would write "$(a, b) \notin R$".

We will also rely syntactically on such an implicit truthfulness point of view and always denote the truthfulness possibly induced from the underlying credibility calculus through a truth projection operator[1] μ, acting as a *positive* domain and range restriction on the credibility operator r.

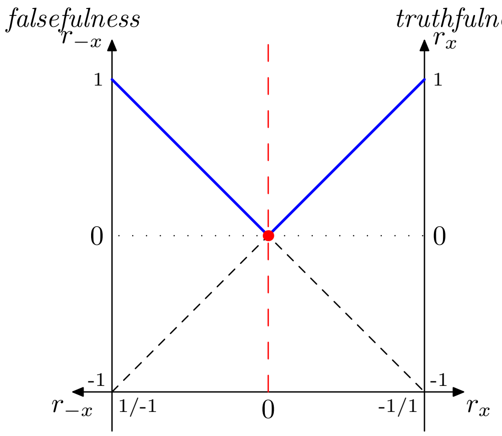

Figure 1: Split Truth/Falseness Semantics

Let $(x, r_x) \in \mathcal{E}^{\mathcal{L}}$ be an $\mathcal{L}$-expression:

$$\mu(x, r_x) = \begin{cases} (x, r_x) \text{ if } r_x \geq r_{\neg x}, \\ (\neg x, r_{\neg x}) \text{ otherwise.} \end{cases} \tag{4}$$

Truthfulness of a given expression x is thus only defined in case the expression's credibility r_x exceeds the credibility $r_{\neg x}$ of its contradiction $\neg x$, otherwise the logical point of view is switched to $\neg x$, i.e the contradicted version of the expression (see Figure 1).

As $r_x \geq r_{\neg x} \Leftrightarrow r_x \geq 0$ it follows from Equation 4 that the sign ($+$ or $-$) of r_x immediately carries the truth functional semantics of $\mathcal{L}$-expressions, in the sense that an $\mathcal{L}$-expression (x, r_x) such that $r_x \geq 0$ may be called *more or less true* ($\mathcal{L}$-true for short) and an expression (x, r_x) such that $r_x \leq 0$ may be called *more or less false* ($\mathcal{L}$-false for short).

[1] In fuzzy set theory, the μ operator generally denotes a fuzzy membership function. We here choose the same μ symbol on purpose as our main $\mathcal{L}$-valued formulas mostly concern $\mathcal{L}$-valued characteristic functions.

Only 0-valued expressions appear to be both $\mathcal{L}$-true and $\mathcal{L}$-false, therefore they are called $\mathcal{L} - undetermined$ [2].

To be able to compute the credibility evaluation associated with any $\mathcal{L}$-expression, we still need to implement $\mathcal{L}$-valued versions of the conjunction and disjunction operators.

The classic min and max operators may be used:
$\forall (x, r_x), (y, r_y) \in \mathcal{E}^{\mathcal{L}}$:

$$(x, r_x) \vee (y, r_y) = (x \vee y, \max(r_x, r_y)) \quad (5)$$
$$(x, r_x) \wedge (y, r_y) = (x \wedge y, \min(r_x, r_y)) \quad (6)$$

The operator triple $< -, \min, \max >$ implements on the rational interval $[-1, 1]$ an ordinal credibility calculus, denoted for short $\mathcal{L}_o$, that gives a first example of what we shall call a *natural fuzzification* of propositional calculus.

To appreciate usefulness of our split truth/falseness semantics, let us look at what happens in the $\mathcal{L}_o$-valued framework with the truthfulness of certain classical tautologies or antilogies.

For instance, truthfulness of the tautology $(x \vee \neg x)$ is always given, as $\max(r_x, -r_x)) \geq 0$ in any case. Tautological $\mathcal{L}_o$-valued propositions thus appear as being $\mathcal{L}_o$-true in any case. Therefore we call them $\mathcal{L}_o$*-tautologies*. On the other hand, truthfulness of the antilogy $(x \wedge \neg x)$ is only defined when $\min(r_x, r_{\neg x}) = 0$. More or less "untruthfulness" of such an expression is however always given. Therefore, we call such propositions $\mathcal{L}_o$*-antilogies*.

Finally, let us investigate an implicative $\mathcal{L}_o$-tautology such as the modus ponens for instance. If we take the classical negative (Kleene-Dienes) definition of the implication, i.e. falseness of the conjunction of $r(x)$ and $\neg r(y)$, we obtain

$$\min(r_x, \max(-r_x, r_y)) \geq 0 \Rightarrow r_y \geq 0,$$

i.e. the following $\mathcal{L}_o$-tautology: " (x, r_x) and $(x, r_x) \Rightarrow (y, r_y)$ being conjointly $\mathcal{L}_o$-true always implies (y, r_y) being $\mathcal{L}_o$-true ".

As a main result of our construction, we recover in this sense all classical tautologies and antilogies as particular limit case if we reduce our $\mathcal{L}_o$-valued credibility calculus to a bi-valued $\{-1, 1\}$ one.

2.2 On natural logical polarization

To explore the formal consequences of our split truth/falseness semantics, we need to formalize the logical *defuzzification* or *polarization* we implicitly operate when applying to $\mathcal{L}$-expressions an $\mathcal{L}$-true or $\mathcal{L}$-false denotation.

Unfortunately, the standard defuzzification technique, denoted in the fuzzy literature as λ-cuts (see Fodor & Roubens [4]), where $\lambda \in [-1, 1]$ represents the level of credibility r_x from which on a given $\mathcal{L}$-expression is considered to be true, is not generally consistent with our split truth/falseness semantics (see Bisdorff [2]).

[2] "*... I have long felt that it is a serious defect in existing logic that it takes no heed of the* limit *between two realms. I do not say that the Principle of Excluded Middle is downright* false*; but I* do *say that in every field of thought whatsoever there is an intermediate ground between* positive assertion *and* negative assertion *which is just as Real as they. ...* "(C. S. Peirce, Letter from February 29, 1909 to William James)

What we need is an extended three-valued cut operator (see Bisdorff & Roubens [1]). Let $E^{\mathcal{L}}$ be a set of $\mathcal{L}$-expressions and let $\mathcal{L}^3$ denote the restriction of $\mathcal{L}$ to the three credibility values $\{-1, 0, 1\}$. $\pi : E^{\mathcal{L}} \rightarrow E^{\mathcal{L}^3}$ represents a logical polarization operator defined as follows:
$\forall (x, r_x) \in E^{\mathcal{L}}$:

$$\pi(x, r_x) = \begin{cases} (x, 1) & \Leftrightarrow \quad r_x > 0 \\ (x, -1) & \Leftrightarrow \quad r_x < 0 \\ (x, 0) & \Leftrightarrow \quad r_x = 0 \end{cases}$$

That π operator indeed implements our split truth/falseness semantics may be summarized by stating the following categorical equation.

$$\mu \circ \pi \; = \; \pi \circ \mu. \tag{7}$$

and a credibility calculus $\mathcal{L}$ verifying Equation 7 is called *natural*.

For instance, we may show that $\mathcal{L}_o$ implements a such natural credibility calculus. For this we must proof that the π operation gives a natural transformation of $\mathcal{L}_o$-valued expressions. Following the general inductive construction of $E^{\mathcal{L}}$ it is sufficient to show naturality of $\mathcal{L}_o$ for each of the basic logical operators.

$\mathcal{L}_o$-valued contradiction: for any $(x, r_x) \in E^{\mathcal{L}_o}$, if $r_x > 0$, $\mu(\pi(x, r_x)) = \mu(x, 1) = (x, 1) = \pi(\mu(x, r_x))$; if $r_x < 0$, $\mu(\pi(x, r_x)) = \mu(x, -1) = (\neg x, 1) = \pi(\neg x, -r_x) = \pi(\mu(x, r_x))$; and if $r_x = 0$, $\mu(\pi(x, r_x)) = \mu(x, 0) = (x, 1) == \pi(x, r_x) = \pi(\mu(x, r_x))$.

$\mathcal{L}_o$-valued disjubction: for any $(x, r_x), (y, r_y) \in E^{\mathcal{L}_o}$, if $r_x > 0$ or $r_y > 0$, $\mu(\pi(x \vee y, \max(r_x, r_y))) = \mu(x \vee y, 1) = (x \vee y, 1) = \pi(x \vee y, \max(r_x, r_y)) = \pi(\mu(x \vee y, \max(r_x, r_y)))$; if $r_x < 0$ and $r_y < 0$, $\mu(\pi(x \vee y, \max(r_x, r_y))) = \mu(x \vee y, -1) = (\neg(x \vee y), 1) = \pi(\neg(x \vee y), \min(-r_x, -r_y)) = \pi(\mu(x \vee y, \max(r_x, r_y)))$.

Finally, $\mathcal{L}_o$-valued conjunction: for any $(x, r_x), (y, r_y) \in E^{\mathcal{L}_o}$, if $r_x > 0$ and $r_y > 0$, $\mu(\pi(x \wedge y, \min(r_x, r_y))) = \mu(x \wedge y, 1) = (x \wedge y, 1) = \pi(x \wedge y, \min(r_x, r_y)) = \pi(\mu(x \wedge y, \min(r_x, r_y)))$; if $r_x < 0$ or $r_y < 0$, $\mu(\pi(x \wedge y, \min(r_x, r_y))) = \mu(x \wedge y, -1) = (\neg(x \wedge y), 1) = \pi(\neg(x \wedge y), \max(-r_x, -r_y)) = \pi(\mu(x \wedge y, \min(r_x, r_y)))$.

This completes the demonstration.

The $\mathcal{L}_o$ credibility calculus is however not the only possible natural credibility calculus we may define on E.

3 A Bochvar-like fuzzification of propositional expressions

A second example is given by a multiplicative fuzzification of the classic three-valued Bochvar logic. We shall denote $\mathcal{L}_b$ such a credibility calculus where the operator triple is denoted $< -, \curlyvee, \curlywedge >$.

We keep the traditional sign exchange as $\mathcal{L}_b$-valued contradiction.

The *multiplicative conjunction* operator $\curlywedge$ on a set $E^{\mathcal{L}}$ of $\mathcal{L}$-expressions is defined as follows:

$$\forall x, y \in E : r_{x \wedge y} = r_x \curlywedge r_y = \begin{cases} \mid r_x \times r_y \mid & \text{if } (r_x > 0) \; \wedge \; r_y > 0), \\ - \mid r_x \times r_y \mid & \text{otherwise.} \end{cases}$$

In Figure 2, we may notice that the $\curlywedge$-operator, when restricted to a $\{-1, 1\}$-valued domain, is isomorphic to the classic Boolean conjunction operator.

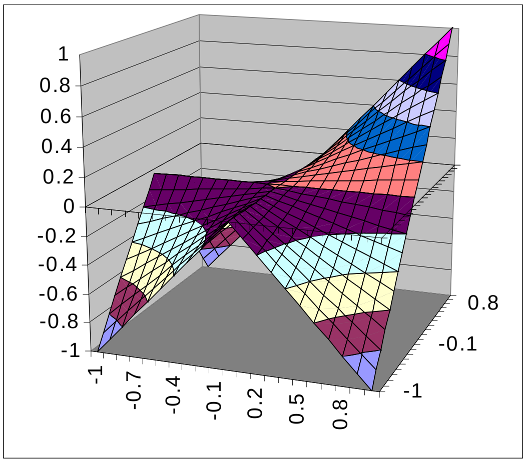

Figure 2: Graphical representation of the multiplicative conjunctive operator

Similarly, we define the *multiplicative disjunction* operator $\curlyvee$ as follows:

$$\forall x, y \in P : r_{x \vee y} = r_x \curlyvee r_y = \begin{cases} - \mid r_x \times r_y \mid & \text{if } (r_x < 0) \ \wedge \ (r_y < 0), \\ \mid r_x \times r_y \mid & \text{otherwise.} \end{cases}$$

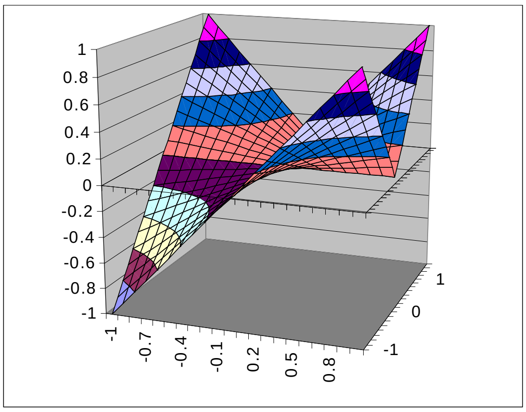

Figure 3: Graphical representation of the multiplicative disjunctive operator

Again, we may notice in Figure 3 that we recover in the limit, when restricted to only $-1, 1$-valued expressions, the classic Boolean disjunction operator.

First, we may verify that the De Morgan duality properties are verified in $\mathcal{L}_b$. Indeed, we easily see that:

$$\forall (x, r_x), (y, r_y) \in E^{\mathcal{L}_b} : r_{x \wedge y} = r_{(\neg(\neg x \vee \neg y))}.$$

Indeed, if $r_x, r_y > 0$, $r_x \curlywedge r_y = r_x \times r_y$. At the same time, $r_{\neg x} \curlyvee r_{\neg y} = (r_{\neg x} \times r_{\neg y}) = -(r_x \times r_y)$. On the contrary, if $r_x, r_y < 0, r_x \curlywedge r_y = -(r_x \times r_y)$, then $r_{\neg x} \curlyvee r_{\neg y}) = (r_{\neg x} \times r_{\neg y}) = (-r_x \times -r(_y) = r_x \times r_y$. If either $r_x > 0$ and $r_y < 0$ or vice versa, the duality relation is equally verified.

It is most interesting to notice that in the case where both $\mathcal{L}_b$-expressions are $\mathcal{L}_b$-true, respectively $\mathcal{L}_b$-false, both operators $\curlywedge$ and $\curlyvee$ give the same $\mathcal{L}_b$-credibility. The operators diverge in their result only when contradictory $\mathcal{L}_b$-truth assessments are to be combined. The conjunctive operator aligns the $\mathcal{L}_b$-false part where as the disjunctive operator sustains the $\mathcal{L}_b$-true part of the pair of propositions.

We may furthermore notice that the negational fix-point, the zero value, figures as logical "*black hole*" as is usual in the three-valued Bochvar logic, absorbing all possible logical determinism through any of both binary operators.

$$\forall (x, r_x) \in E^{\mathcal{L}_b} : r_x \curlywedge 0 = r_x \curlyvee 0 = 0.$$

Let us denote $E^{\mathcal{L}_b}_{/-1;1}$ the equivalence classes of all certainly true or false $\mathcal{L}_b$-expressions. The restriction of the $\mathcal{L}_b$ credibility calculus to $E^{\mathcal{L}_b}_{/-1:1}$ gives a classic Boolean algebra.

It is remarkable however, that such a priori obvious properties as impotency of conjunction and disjunction, are only satisfied in this limit Boolean case. Indeed in general, the natural logical consequence of combining more and more fuzzy propositions will sooner or later necessarily end up with a completely undetermined proposition. The same is true when combining conjunctively or disjunctively a number of times the same fuzzy proposition. Indeed, $\forall (x, r_x), (y, r_y) \in E^{\mathcal{L}_b}$ such that $r_x \neq 0$ we have:

$$\begin{aligned} | \, r_x \, | &> | \, r_x \curlywedge r_y \, |, \\ | \, r_x \, | &> | \, r_x \curlyvee r_y \, | \,. \end{aligned}$$

We recover here a similar situation as in classic error propagation. The more we operate with imprecise numbers, we more we increase the imprecision of the out-coming result, and this imprecision is essentially related to the imprecision of the initial inputs.

Finally, to validate now the naturality property of the $\mathcal{L}_b$ calculus, we must show that the curly operators $\curlyvee$ and $\curlywedge$ verify Equation 7. In order to do so, it is again sufficient to show that for any $(x, r_x), (y, r_y) \in E^{\mathcal{L}_b}$ and both the curly operators we have:

$$\begin{aligned} \mu(\pi(x \vee y, r_x \curlyvee r_y)) = \pi(\mu(x \vee y, r_x \curlyvee r_y)), \\ \mu(\pi(x \wedge y, r_x \curlywedge r_y)) = \pi(\mu(x \wedge y, r_x \curlywedge r_y)). \end{aligned}$$

Indeed, for any $(x, r_x), (y, r_y) \in E^{\mathcal{L}_o}$, if $r_x > 0$ or $r_y > 0$, $\mu(\pi(x \vee y, r_x \curlyvee r_y))) = \mu(x \vee y, 1) = (x \vee y, 1) = \pi(x \vee y, r_x \curlyvee r_y) = \pi(\mu(x \vee y, r_x \curlyvee r_y)$; if $r_x < 0$ and $r_y < 0$, $\mu(\pi(x \vee y, r_x \curlyvee r_y)) = \mu(x \vee y, -1) = (\neg(x \vee y), 1) = \pi(\neg(x \vee y), r_x \curlywedge r_y) = \pi(\mu(x \vee y, r_x \curlyvee r_y))$.

And for any $(x, r_x), (y, r_y) \in E^{\mathcal{L}_o}$, if $r_x > 0$ and $r_y > 0$, $\mu(\pi(x \wedge y, r_x \curlywedge r_y)) = \mu(x \wedge y, 1) = (x \wedge y, 1) = \pi(x \wedge y, r_x \curlywedge r_y) = \pi(\mu(x \wedge y, r_x \curlywedge r_y)$; if $r_x < 0$ or $r_y < 0$, $\mu(\pi(x \wedge y, r_x \curlywedge r_y)) = \mu(x \wedge y, -1) = (\neg(x \wedge y), 1) = \pi(\neg(x \wedge y), r_x \curlyvee r_y) = \pi(\mu(x \wedge y, r_x \curlywedge r_y))$.

This concludes the demonstration that $\mathcal{L}_b$ does indeed implements a natural credibility calculus.

4 Moving on

In order to situate now the whole family of natural credibility calculus one may define on propositional expressions, let us explore two directions for further investigations.

Following the general properties of the $\mathcal{L}_o$ calculus, we may want to consider the t-norm concept as potential generalization. Unfortunately, the split truth/falseness semantics is not quite compatible with the formal properties of a t-norm. Indeed, let us recall that a t-norm T defined on the interval $[-1;1]$ should verify the following four axioms:

$$\begin{aligned} T(1, r_x) &= r_x, \forall r_x \in [-1;1] && (8) \\ T(r_x, r_y) &= T(r_y, r_x), \forall r_x, r_y \in [-1;1] && (9) \\ T(r_x, r_y) \leq T(r_u, r_v) &\text{ if } -1 \leq r_x \leq r_u \leq 1, -1 \leq r_y \leq r_v \leq 1 && (10) \\ T(r_x, T(r_y, r_z)) &= T(T(r_x, r_y), r_z), \forall r_x, r_y, r_z \in [-1;1]. && (11) \end{aligned}$$

It is easily verified that the multiplicative conjunctive operator $\curlywedge$ verifies three of these axioms, i.e. all except the third one. This is not astonishing, as this axiom is not so "naturally" a logical axiom but rather a geometrical axiom underlying the "triangularity" heritage of the t-norm concept.

What axiom could advantageously replace the "triangular" t-norm condition in order to make fit conceptually the t-norm to a natural credibility calculus on the rational interval $[-1, 1]$?

A possibility might be the following:

$$\mid T(r_x, r_y) \mid \leq \mid T(r_u, r_v) \mid \text{ if } 0 \leq \mid r_x \mid \leq \mid r_u \mid \leq 1, 0 \leq \mid r_y \mid \leq \mid r_v \mid \leq 1.$$

In some sense we would recover the triangular axiom in some absolute terms. But this idea has still to be further explored.

Finally, more following the semiotical intuitions of C.S. Peirce, we may interpret the classic ordinal $\mathcal{L}_o$ credibility calculus and the above introduced Bochvar-like $\mathcal{L}_b$ credibility calculus as some limit constructions of a same semiotical foundation of logical fuzziness. Indeed, the $\mathcal{L}_o$ calculus to be applicable in a practical setting supposes a same closed universal semiotical reference for all ground propositions $p \in P$ as is usual in a mathematical logic context for instance, where as the multiplicative model apparently supposes shared semiotical references for all determined parts and disjoint semiotical references for the logically undetermined parts of each proposition $p \in P$ as is usual for instance in repetitive physical measures with error propagation.

These general considerations leave open the case where each ground expression $p \in P$ is completely supported by a different semiotical reference. In this last case we would get as third limit case some kind of aggregational logic (see Bisdorff [3]) as implemented by the concordance principle in the multicriteria approach to preference aggregation for instance.

References

[1] Bisdorff, R. and Roubens, M. (1996), On defining fuzzy kernels from $\mathcal{L}$-valued simple graphs, in: *Proceedings Information Processing and Management of Uncertainty, IPMU'96*, Granada, 593–599.

[2] Bisdorff, R. (2000), Logical foundation of fuzzy preferential systems with application to the Electre decision aid methods, *Computers & Operations Research* **27** 673–687.

[3] Bisdorff, R. (2002), Logical Foundation of Multicriteria Preference Aggregation. Essay in *Aiding Decisions with Multiple Criteria*, D. Bouyssou et al. (editors), Kluwer Academic Publishers, pp. 379-403.

[4] Fodor, J. and Roubens, M., *Fuzzy preference modelling and multi-criteria decision support*. Kluwer Academic Publishers (1994)