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Moving Weyl's Theorem from $f(T)$ to T

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Abstract: Schmoeger has shown that if Weyl's theorem holds for an isoloid Banach space operator $T \in \mathcal{B}(X)$ with stable index, then it holds for $f(T)$ whenever $f \in \operatorname{Holo}\sigma(T)$ is a function holomorphic on some neighbourhood of the spectrum of T. In this note we establish a converse.

Key words: Weyl's theorem, Browder's theorem, SVEP.

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1. Introduction

Recall that an operator $T \in \mathcal{B}(X)$ has finite ascent if there is $p \in \mathbb{N}$ for which

$$T^{-p}(0) = T^{-\infty}(0) = \bigcup_{n=1}^{\infty} T^{-n}(0)\,;$$

if in particular

$$T^{-p}(0) = T^{-w}(0) = \left\{ x \in X \ : \ ||T^n x||^{\frac{1}{n}} \to 0 \right\} \qquad \text{(transfinite kernel)}$$

we shall say that T has *finite hyperascent*. The transfinite range of an operator is defined by

$$T^w(X) = \left\{ x \in X \ : \ \begin{array}{c} \text{there exists } k > 0 \text{ and a sequence } \{x_n\} \subset X \\ \text{such that } Tx_1 = x\,,\ Tx_{n+1} = x_n \\ \text{and } \|x_n\| \le k^n \|x\| \text{ for all positive integers } n \end{array} \right\}.$$

If $T \in \mathcal{B}(X)$ has finite ascent, then, in particular, it has the "single valued extension property" (SVEP) at zero, which says [6] that the only holomorphic function f for which $(T - z)f(z) = 0$ for all z in a neighborhood of zero is the zero function. Equivalently, [6], 0 is not in the "local point spectrum" $\pi_{loc}^{left}(T)$. For $T \in \mathcal{B}(X)$, let $N(T)(= T^{-1}(0))$ and $R(T)(= T(X))$ denote, respectively, the null space and the range of the mapping T. Let $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X/R(T)$, if theses spaces are finite dimensional, otherwise let $\alpha(T) = \infty$ and $\beta(T) = \infty$. If the range $R(T)$ of $T \in \mathcal{B}(X)$ is closed and $\alpha(T) < \infty$ (respectively, $\beta(T) < \infty$), then T is said to be an *upper semi-Fredholm* (respectively, a *lower semi-Fredholm*) operator and we denote $T \in \Phi_+(X)$ (respectively $T \in \Phi_-(X)$). If $T \in \Phi_-(X) \cup \Phi_+(X)$ then T is called a *semi-Fredholm* operator (in notation $T \in \Phi_\pm(X)$) and for $T \in \Phi_-(X) \cap \Phi_+(X)$ we say that T is a *Fredholm* operator (in notation $T \in \Phi(X)$). For $T \in \Phi_\pm(X)$, the *index* of T is defined by

$$\operatorname{ind}(T) = \alpha(T) - \beta(T).$$

T has *stable index* if $\operatorname{ind}(T - \mu I)$ is either ≥ 0 or ≤ 0 (exclusive or) for all μ not in the Fredholm spectrum $\sigma_e(T)$ of T; T is *isoloid* if the isolated points of the spectrum of T are eigenvalues of T. Denote with $\pi_{00}(T)$ the set of isolated eigenvalues of T of finite geometric multiplicity, i.e.

$$\pi_{00}(T) = \{\lambda \in \operatorname{iso}\sigma(T) \,:\, 0 < \alpha(T - \lambda) < \infty\}.$$

Similarly, with $\pi_0(T)$ we denote the set of all isolated eigenvalues of T of finite algebraic multiplicity (poles of T). Obviously, $\pi_0(T) \subseteq \pi_{00}(T)$.

Schmoeger, [8], has shown that "if Weyl's theorem holds for an isoloid operator T with stable index", then it holds for $f(T)$ whenever $f \in \operatorname{Holo}\sigma(T)$ (or $f \in \operatorname{Holo}_c\sigma(T)$), the set of all non-trivial holomorphic function on some neighborhood of the spectrum of T (or all function from $\operatorname{Holo}\sigma(T)$ that are not constant on connected component). In this note we address the converse problem. Specifically, we will give conditions under which "Browder's theorem" (respectively "finite hyperascent property") is transmitted back from $f(T)$ to T.

2. Browder's theorem

Recall that T is polaroid if every isolated point λ of the spectrum of T, $\lambda \in \operatorname{iso}\sigma(T)$, is a pole of the resolvent of T. The Browder spectrum $\sigma_b(T)$ and the Weyl spectrum $\sigma_w(T)$ of $T \in \mathcal{B}(X)$ are the sets $\sigma_b(T) = \{\lambda \in \sigma(T) \,:\, T - \lambda \notin \Phi(X) \text{ or } \operatorname{asc}(T-\lambda) \neq \operatorname{des}(T-\lambda)\}$ and $\sigma_w(T) = \{\lambda \in \sigma(T) \,:\, T - \lambda \notin \Phi_0(X)\}$, where $\Phi_0(X)$ denotes the set of all Fredholm operators with index zero. The essential (Fredholm) spectrum is the set $\sigma_e(T) = \{\lambda \in \sigma(T) \,:\, T - \lambda \notin \Phi(X)\}$.

Here $\operatorname{des}(T)$ denotes the *descent* of T, the smallest positive integer n such that $R(T^n) = R(T^{n+1})$ (if no such n exists, then $\operatorname{des}(T) = \infty$), $\operatorname{asc}(T)$ denotes the *ascent* of T, the smallest positive integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ (if no such n exists, then $\operatorname{asc}(T) = \infty$). Browder's theorem holds for T if and only if

$$\sigma_b(T) \subseteq \sigma_w(T), \tag{1}$$

equivalently, [4, Theorem 8.3.1], if and only if T has SVEP on $\sigma(T) \setminus \sigma_w(T)$, equivalently, [6], if and only if

$$\pi_{loc}^{left}(T) \subseteq \sigma_w(T). \tag{2}$$

Remark 2.1. (i) Let $T \in B(X)$ and $f \in \operatorname{Holo}\sigma(T)$. Then Browder's theorem for $f(T)$ implies the spectral mapping theorem for Weyl spectrum:

$$\sigma_w(f(T)) = \sigma_b(f(T)) = f(\sigma_b(T)) \supseteq f(\sigma_w(T)).$$

Since opposite inclusion always holds, we have $\sigma_w(f(T)) = f(\sigma_w(T))$.

(ii) The SVEP property on $\sigma(T) \setminus \sigma_w(T)$ guarantees us even more: Browder's theorem for $f(T)$ for every $f \in \operatorname{Holo}_c \sigma(T)$. Really, let $f \in \operatorname{Holo}_c \sigma(T)$ and $f(\lambda_0) \in \sigma(f(T)) \setminus \sigma_w(f(T))$. Then there is an $r \in \mathbb{N}$, a polynomial h and $g \in \operatorname{Holo}\sigma(T)$ (with no zero in $\sigma(T)$) such that

$$f(z) - f(\lambda_0) \equiv (z - \lambda_0)^r h(z) g(z)$$

with $h(\lambda_0) \neq 0$ and $h(\lambda_0) \notin g(\sigma(T))$. It follows

$$f(T) - f(\lambda_0) = (T - \lambda_0)^r h(T) g(T) \in \Phi_0(X),$$

with $0 \notin \sigma(h(T)g(T))$ and, consequently, $\lambda_0 \notin \sigma_w(T)$. Hence, T has SVEP at λ_0 and, by [1, Theorem 2.39], $f(T)$ has SVEP at $f(\lambda_0)$ that implies Browder's theorem for $f(T)$.

(iii) In the case of $f \in \operatorname{Holo}\sigma(T)$ we need little more, SVEP at all $\lambda \in \sigma(T) \setminus \sigma_e(T)$ or injectivity of f.

Hence, for $f \in \operatorname{Holo}\sigma(T)$, the passage of Browder's theorem from T to $f(T)$, is not a major problem. More interesting question is how to pass Browder's theorem from $f(T)$ to T. In general, the SVEP and Browder's theorem do not move from $f(T)$ to T. To see this, it is enough consider an operator T without SVEP on $\sigma(T) \setminus \sigma_w(T)$ (in this case no Browder's theorem for T) and $f \equiv c \in \operatorname{Holo}\sigma(T)$ (for more details see [5, p. 227]).

Given $T \in \mathcal{B}(X)$ and $f \in \mathrm{Holo}\,\sigma(T)$, then we define the set

$$A_f(T) = \{\lambda \notin \sigma_w(T) \,:\, f(\lambda) \in \sigma_w(f(T))\} \tag{3}$$

and we say that T has the property S_f if

$$T \text{ has SVEP at every } \lambda \in A_f(T). \tag{S_f}$$

Theorem 2.2. *Let $T \in \mathcal{B}(X)$ be such that Browder's theorem holds for $f(T)$.*

(i) *If $f \in \mathrm{Holo}\,\sigma(T)$ and T has the property (S_f), then Browder's theorem holds for T.*

(ii) *If $f \in \mathrm{Holo}_c\,\sigma(T)$, then Browder's theorem holds for $g(T)$, for any $g \in \mathrm{Holo}_c\,\sigma(T)$.*

Proof. Let $\lambda \in \sigma(T) \setminus \sigma_w(T)$.

Case I: If $f(\lambda) \in f(\sigma(T)) \setminus f(\sigma_w(T))$, then Browder's theorem for $f(T)$ and, consequently, the spectral mapping theorem for Weyl spectrum of T, guarantees us that $f(\lambda)$ is an isolated point of $\sigma(f(T))$ (matter of fact it is a pole). Then λ is an isolated point of the spectrum of T that implies SVEP property for T at λ.

Case II: Let $f(\lambda) \in f(\sigma_w(T))(= \sigma_w(f(T)))$.

(i) Then, by property (S_f), T has SVEP at λ.

(ii) If $f(\lambda) \in \sigma_w(f(T))$, then by injectivity of $f \in \mathrm{Holo}_c\,\sigma(T)$, we have that $\lambda \in \sigma_w(T)$, which is a contradiction to our assumption.

Hence, T has SVEP at all $\lambda \notin \sigma_w(T)$, and Browder's theorem holds for T. Moreover, for any $g \in \mathrm{Holo}_c\,\sigma(T)$, by Remark 2.1 (ii), Browder's theorem holds for $g(T)$, for every $g \in \mathrm{Holo}_c\,\sigma(T)$. ∎

The similar behavior we have in the situation of more general versions of Browder's theorem: the g-Browder, a-Browder or s-Browder theorems. We say that $T \in \mathcal{B}(X)$ obeys

$$g\text{-Browder's theorem if} \qquad \sigma_{bb}(T) \subseteq \sigma_{bw}(T)\,, \tag{4}$$

$$a\text{-Browder's theorem if} \qquad \sigma_{ab}(T) \subseteq \sigma_{aw}(T)\,, \tag{5}$$

$$s\text{-Browder's theorem if} \qquad \sigma_{sb}(T) \subseteq \sigma_{sw}(T)\,, \tag{6}$$

where

$$\sigma_{bb}(T) = \{\lambda \in \sigma(T) : T-\lambda \text{ is not } B\text{-Fredholm or } \mathrm{asc}(T-\lambda) \neq \mathrm{des}(T-\lambda)\},$$
$$\sigma_{bw}(T) = \{\lambda \in \sigma(T) : T-\lambda \text{ is not } B\text{-Fredholm or } \mathrm{ind}(T-\lambda) \neq 0\},$$
$$\sigma_{ab}(T) = \{\lambda \in \sigma_a(T) : T-\lambda \notin \Phi_+(T) \text{ or } \mathrm{asc}(T-\lambda) = \infty\},$$
$$\sigma_{aw}(T) = \{\lambda \in \sigma_a(T) : T-\lambda \notin \Phi_+(T) \text{ or } \mathrm{ind}(T-\lambda) > 0\},$$
$$\sigma_{sb}(T) = \{\lambda \in \sigma_s(T) : T-\lambda \notin \Phi_-(T) \text{ or } \mathrm{des}(T-\lambda) = \infty\},$$
$$\sigma_{sw}(T) = \{\lambda \in \sigma_s(T) : T-\lambda \notin \Phi_-(T) \text{ or } \mathrm{ind}(T-\lambda) < 0\}.$$

Note that $T \in \mathcal{B}(X)$ is a B-Fredholm operator if for some integer n the range space $R(T^n)$ is closed and $T_n = T|_{R(T^n)}$ is a Fredholm operator. In this case T_m is a Fredholm operator and $\mathrm{ind}(T_m) = \mathrm{ind}(T_n)$ for each $m \geq n$. This enables us to define the index of a B-Fredholm operator T as the index of the Fredholm operator T_n where n is any integer such that $R(T^n)$ is closed and such that T_n is a Fredholm operator.

Let $* \in \{g, a, s\}$. It is known that $*$-Browder's theorem holds for T if T has SVEP at all points $\lambda \notin \sigma_{*w}(T)$ and that $*$-Browder's theorem implies Browder's theorem (matter of fact, g-Browder's theorem is equivalent to Browder's theorem). Moreover, if T has SVEP at all points $\lambda \notin \sigma_{*w}(T)$, then the spectral mapping theorem holds for $\sigma_{*w}(T)$ and the functions from $\mathrm{Holo}_c\,\sigma(T)$. (For more details see [4]).

Let $T \in \mathcal{B}(X)$, then for any $f \in \mathrm{Holo}\,\sigma(T)$ and $* \in \{g, a, s\}$, we define the sets

$$A_f^*(T) = \{\lambda \notin \sigma_{*w}(T) \,:\, f(\lambda) \in \sigma_{*w}(f(T))\},$$

and the property S_f^*

$$(S_f^*) \qquad T \text{ has SVEP at every } \lambda \in A_f^*(T),$$

then we have next theorem:

THEOREM 2.3. *Let $T \in \mathcal{B}(X)$ and $f \in \mathrm{Holo}\,\sigma(T)$. If $*$-Browder's theorem holds for $f(T)$ and T has the property S_f^*, then $*$-Browder's theorem holds for T. Moreover, $*$-Browder's theorem holds for T if and only if it is holds for $g(T)$, for any $g \in \mathrm{Holo}_c\,\sigma(T)$.*

3. WEYL'S THEOREM

If Browder's theorem holds for some $T \in \mathcal{B}(X)$ together with $\pi_0(T) = \pi_{00}(T)$, then we say that T satisfies Weyl's theorem. SVEP alone is not enough for T to satisfy Weyl's theorem: consider, for example, the quasinilpotent operator $Q \in B(\ell^2)$, $Q(x_1, x_2, x_3, \dots) = (\frac{x_2}{2}, \frac{x_3}{3}, \dots)$. A necessary and sufficient condition for T to satisfy Weyl's theorem is that T satisfies Browder's theorem and, for every $\lambda \in \pi_{00}(T)$, $T - \lambda$ has finite hyperascent. Furthermore, if T is polaroid and has SVEP, then both $f(T)$ and $f(T^*)$ satisfy Weyl's theorem for every $f \in \mathrm{Holo}_c\, \sigma(T)$ [3].

For moving Weyl's theorem from $f(T)$ to T we need a variant of (3). Let $T \in \mathcal{B}(X)$ and $f \in \mathrm{Holo}\, \sigma(T)$, then we define the set

$$\Pi_f(T) = \{\lambda \in \pi_{00}(T) \,:\, f(\lambda) \in \sigma_w(f(T))\} \tag{7}$$

and we say that T has the property Π_f if

$$(\Pi_f) \qquad T - \lambda \text{ has a finite hyperascent, for every } \lambda \in \Pi_f(T).$$

Remark 3.1. (i) Let $T \in \mathcal{B}(X)$ and $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \mathbb{C}$ be a finite set of distinct complex numbers. Then, for any polynomial $p(\lambda) = \prod_{i=1}^n (\lambda_i - \lambda)^{m_i}$ we have

$$p(T)^{-1}(0) = \bigoplus_{i=1}^{n} (T - \lambda_i)^{-m_i}(0).$$

Moreover, if $p(\lambda_0) \neq 0$, for some complex number λ_0, then

$$(T - \lambda_0)^{-w}(0) \cap p(T)^{-1}(0) = \{0\}.$$

(ii) If $T, S \in \mathcal{B}(X)$ is a pair of commuting operators, then $T^{-w}(0) \subseteq (TS)^{-w}(0)$. Moreover, if S is an invertible operator, then $T^{-w}(0) = (TS)^{-w}(0)$.

THEOREM 3.2. *Let $T \in \mathcal{B}(X)$ and $f \in \mathrm{Holo}_c\, \sigma(T)$. If Weyl's theorem holds for $f(T)$ and T has the property Π_f, then Weyl's theorem holds for T.*

Proof. By Theorem 2.2, Browder's theorem holds for T, hence we have to show that $T - \lambda$ has a finite hyperascent, for every $\lambda \in \pi_{00}(T)$ (see [4, Theorem 8.4.5 (vi)]). Let $\lambda_0 \in \pi_{00}(T)$; then $f(\lambda_0) \in f(\sigma(T)) = \sigma(f(T)) = \sigma_w(f(T)) \cup \pi_{00}(f(T))$.

Case I: $f(\lambda_0) \in \pi_0(f(T))$. Since Weyl's theorem holds for $f(T)$, $f(T) - f(\lambda_0)$ has a finite hyperascent, i.e., there exists a positive integer $p \in \mathbf{N}$ such that $(f(T) - f(\lambda_0))^{-w}(0) = (f(T) - f(\lambda_0))^{-p}(0)$.

Since $\lambda \in \mathrm{iso}\, \sigma(T)$, X splits into the direct sum of the transfinite kernel $(T - \lambda I)^{-w}(0)$ and the transfinite range $(T - \lambda I)^w X$, both hyperinvariant under T. If we write S_0 and S_1 for the restriction of $S \in \mathrm{comm}(T)$ to the kernel and the range respectively, then

$$\sigma(S_0) = \{\lambda_0\} \nsubseteq \sigma(S_1).$$

Let $r \in \mathbb{N}$, the polynomial h and $g \in \mathrm{Holo}_c\,\sigma(T)$ (with no zero in $\sigma(T)$) be such that

$$f(z) - f(\lambda_0) \equiv (z - \lambda_0)^r h(z) g(z)$$

with $h(\lambda_0) \neq 0$ and $h(\lambda_0) \notin g(\sigma(T))$. It follows

$$f(T) - f(\lambda_0) = (T - \lambda_0)^r h(T) g(T)$$

with $0 \notin \sigma(g(T))$.

Then

$$(T - \lambda_0)^{-1}(0) \subseteq (T - \lambda_0)^{-r}(0) \subseteq (f(T) - f(\lambda_0))^{-1}(0) \tag{8}$$

and, by Remark 3.1 (i),

$$\begin{aligned} (f(T) - f(\lambda_0))^{-1}(0) &= ((T - \lambda_0)^r h(T))^{-1}(0) \\ &= (T - \lambda_0)^{-r}(0) \bigoplus h(T)^{-1}(0). \end{aligned} \tag{9}$$

Since $f(T_0) - f(\lambda_0)$ has hyperascent $\leq p$, by Remark 3.1 (ii), we have

$$\begin{aligned} (T - \lambda_0)^{-w}(0) &\subseteq (f(T) - f(\lambda_0))^{-w}(0) \\ &= (f(T) - f(\lambda_0))^{-p}(0) = (T - \lambda_0)^{-pr}(0) \oplus h(T)^{-p}(0). \end{aligned}$$

Again, by $(T - \lambda_0)^{-w}(0) \cap h(T)^{-p}(0) = \{0\}$ and Remark 3.1, we have

$$(T - \lambda_0)^{-w}(0) \subseteq (T - \lambda_0)^{-pr}(0).$$

Since the opposite inclusion is always valid, we have that $T - \lambda_0$ has finite hyperascent.

Case II: If $f(\lambda_0) \in f(\sigma_w(f(T))$, since T has a property Π_f, follows that $T - \lambda_0$ has finite hyperascent. ∎

Remark 3.3. A slight modification of the proofs of Theorem 2.2 and Theorem 3.2 give us the conditions for moving Weyl's theorem form $f(T)$, $f \in \mathrm{Holo}\,\sigma(T)$, to T. For this, beside the property Π_f, we need to suppose that f is an injective function.

If we replace the condition Π_f with stronger condition

$$\lambda \in \operatorname{iso}\sigma(T) \Longrightarrow f(T) - f(\lambda) \text{ has finite hyperascent,} \tag{10}$$

then slight modification of part of the proof of Theorem 3.2 give us that T is polaroid. In this case we can extend Weyl's theorem on $g(T)$, for all $g \in \operatorname{Holo}_c \sigma(T)$.

THEOREM 3.4. *Let $T \in B(X)$ and $f \in \operatorname{Holo}_c \sigma(T)$ such that Weyl's theorem holds for $f(T)$ and T has property (10). Then Weyl's theorem holds for $g(T)$ and $g(T^*)$ for all $g \in \operatorname{Holo}_c \sigma(T)$.*

Proof. In view of the hypothesis Theorem 2.2 implies that T (so also, T^*) satisfies Browder's theorem. Recall now from Theorem 3.2 that if $f(T)$ satisfies condition (10), then T is polaroid (which, in turn, implies that T^* is polaroid); hence T and T^* satisfy Weyl's theorem. Browder's theorem for T implies that T has SVEP at points in $\sigma(T) \setminus \sigma_w(T)$ and by [3, Theorem 2.4] $g(T)$ and $g(T^*)$ satisfy Weyl's theorem for every $g \in \operatorname{Holo}_c \sigma(T)$. ∎

4. APPLICATIONS

A Banach space operator $T \in \mathcal{B}(X)$ is *hereditarily polaroid*, $T \in \mathcal{HP}$, if every part of T (i.e., its restriction to an invariant subspace) is polaroid. The class of $\mathcal{HP}$ operators is large. It contains amongst others the following classes of operators. (We refer the interested reader to [2] for further, but by no means exhaustive, list of $\mathcal{HP}$ operators.)

(a) $H(p)$ operators (operators $T \in \mathcal{B}(X)$ such that $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ for some integer $p = p(\lambda) \geq 0$ and all complex λ). This class of operators contains next well known classes:

(a-i) Hilbert space operators $T \in \mathcal{B}(H)$ which are either *hyponormal* ($|T^*|^2 \leq |T|^2$), or *p-hyponormal* ($|T^*|^{2p} \leq |T|^{2p}$) for some $0 < p \leq 1$ or *(p,k)-quasihyponormal* ($T^{*k}(|T|^{2p} - |T^*|^{2p})T^k \geq 0$) for some integer $k \geq 1$ and $0 < p \leq 1$.

(a-ii) *w-hyponormal* ($|\tilde{T}^*| \leq |T| \leq |\tilde{T}|$, where, for the polar decomposition $T = U|T|$ of T, $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$).

(a-iii) *M-hyponormal* ($||(T-\lambda)^*||^2 \leq M||T-\lambda||^2$ for some $M \geq 1$ and all complex λ) or *class $\mathcal{A}$* ($|T|^2 \leq |T^2|$).

(b) *Paranormal operators* $T \in \mathcal{B}(X)$ ($||Tx||^2 \leq ||T^2x||$ for all unit vectors $x \in X$).

(c) *Totally paranormal operators* $T \in \mathcal{B}(X)$ ($||(T-\lambda)x||^2 \leq ||(T-\lambda)^2x||$ for all unit vectors $x \in X$ and complex λ).

The classes consisting of paranormal operators and $H(p)$ operators are substantial. Thus, the classes consisting of hyponormal or p-hyponormal or $(p,1)$-quasihyponormal or $(1,1)$-quasihyponormal and class $\mathcal{A}$ Hilbert space operators are proper subclasses of the class of paranormal operators; the class $H(p)$ contains in particular the classes consisting of operators which are either totally paranormal or *generalized scalar* or *subscalar* or *multipliers of commutative semi-simple Banach algebras* [1, p. 175].

Moving Weyl's theorem from $f(T)$ to T, for $f \in \mathrm{Holo}_c\,\sigma(T)$ and $T \in H(p)$, is possible applying Theorem 3.4. This fact is known and we can find more details in [7]. We have:

THEOREM 4.1. *If $f(T) \in H(p)$ for some $T \in \mathcal{B}(X)$ and $f \in \mathrm{Holo}_c\,\sigma(T)$, then T satisfies Weyl's theorem. Moreover, $g(T)$ and $g(T^*)$ satisfy Weyl's theorem for every $g \in \mathrm{Holo}_c\,\sigma(T)$.*

More is true. Recall, [1, p. 177], that a Banach space operator T satisfies a-Weyl's theorem if $\sigma_a(T) \setminus \sigma_{aw}(T) = \pi_{00}^a(T)$, where $\sigma_a(A)$ is the approximate point spectrum of T, $\pi_{00}^a(T) = \{\lambda \in \mathrm{iso}\,\sigma_a(T) : 0 < \dim(T-\lambda I)^{-1}(0) < \infty\}$ and $\sigma_{aw}(T) = \{\lambda \in \sigma_a(T) : T-\lambda$ is not lower semi–Fredholm or $\mathrm{ind}(A-\lambda) \not\leq 0\}$ is the Weyl essential approximate point spectrum of A. If T has SVEP, then $\sigma(T) = \sigma_a(T^*)$, $\sigma_w(T) = \sigma_{aw}(T^*)$ and $\pi_{00}(T) = \pi_{00}^a(T^*)$. Since T satisfies Weyl's theorem if and only if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$ [1, p. 166], we have the following:

COROLLARY 4.2. *If $f(T) \in H(p)$ for some $T \in \mathcal{B}(X)$ and $f \in \mathrm{Holo}_c\,\sigma(T)$, then $g(T^*)$ satisfies a-Weyl's theorem for every $g \in \mathrm{Holo}_c\,\sigma(T)$.*

Proof. Since $f(T)$ has SVEP implies T has SVEP implies $g(T)$ has SVEP [1], $\sigma_a(g(T^*)) \setminus \sigma_{aw}(g(T^*)) = \sigma(g(T)) \setminus \sigma_w(g(T))$. This, since $g(T)$ satisfies Weyl's theorem (see Theorem 4.1), implies $\sigma_a(g(T^*) \setminus \sigma_{aw}(g(T^*)) = \pi_0(g(T)) = \pi_0^a(g(T^*))$. ∎

$\mathcal{HP}$ operators have SVEP [2, Theorem 2.8], so that if $f(T) \in \mathcal{HP}$, for some $f \in \mathrm{Holo}_c\,\sigma(T)$, then $g(T)$ and $g(T^*)$ satisfy Browder's theorem for every $g \in \mathrm{Holo}\,\sigma(T)$. However, since isolated points of $\sigma(T)$ may not survive passage from $\sigma(T)$ to $\sigma(f(T))$, $f \in \text{Holo}_c\, \sigma(T)$, $\mathcal{HP}$ operators do not in general satisfy condition (5). (There is no such problem with $H(p)$ operators.) Now, using that $\lambda \in \text{iso}\, \sigma(T)$ if and only if $f(\lambda) \in \text{iso}\, \sigma(f(T))$, the condition (10) and, that, $f(T)$ has SVEP implies T has SVEP, way we have new version of [2, Theorem 3.6].

Theorem 4.3. *Suppose that $f(T) \in \mathcal{HP}$ for some $T \in \mathcal{B}(X)$ and $f \in \text{Holo}_c\, \sigma(T)$. If f preserves isolated points of $\sigma(T)$, then T satisfies Weyl's theorem. Moreover, $g(T)$ satisfies Weyl's theorem and $g(T^*)$ satisfies a-Weyl's theorem for every $g \in \text{Holo}_c\, \sigma(T)$.*

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