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Heat Kernel Measure on Central Extension of Current Groups in any Dimension

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Abstract. We define measures on central extension of current groups in any dimension by using infinite dimensional Brownian motion.

Key words: Brownian motion; central extension; current groups

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1 Introduction

If we consider a smooth loop group, the basical central extension associated to a suitable Kac–Moody cocycle plays a big role in mathematical physics [3, 11, 21, 24]. Léandre has defined the space of L^2 functionals on a continuous Kac–Moody group, by using the Brownian bridge measure on the basis [16] and deduced the so-called energy representation of the smooth Kac–Moody group on it. This extends the very well known representation of a loop group of Albeverio–Hoegh–Krohn [2].

Etingof–Frenkel [13] and Frenkel–Khesin [14] extend these considerations to the case where the parameter space is two dimensional. They consider a compact Riemannian surface Σ and consider the set of smooth maps from Σ into a compact simply connected Lie group G. We call $C_r(\Sigma; G)$ the space of C^r maps from Σ into G and $C_\infty(\Sigma; G)$ the space of smooth maps from Σ into G. They consider the universal cover $\tilde{C}_\infty(\Sigma; G)$ of it and construct a central extension by the Jacobian J of Σ of it $\hat{C}_\infty(\Sigma; G)$ (see [7, 8, 25] for related works).

We can repeat this construction if $r > s$ big enough for $C_r(\Sigma; G)$. We get the universal cover $\tilde{C}_r(\Sigma; G)$ and the central extension by the Jacobian J of Σ of $\tilde{C}_r(\Sigma; G)$ denoted by $\hat{C}_r(\Sigma; G)$.

By using Airault–Malliavin construction of the Brownian motion on a loop group [1, 9], we have defined in [19] a probability measure on $\tilde{C}_r(\Sigma; J)$, and since the Jacobian is compact, we can define in [19] a probability measure on $\hat{C}_r(\Sigma; G)$.

Maier–Neeb [20] have defined the universal central extension of a current group $C_\infty(M; G)$ where M is any compact manifold. The extension is done by a quotient of a certain space of differential form on M by a lattice. We remark that the Maier–Neeb procedure can be used if we replace this infinite dimensional space of forms by the de Rham cohomology groups $H(M : \mathrm{Lie}\, G)$ of M with values in $\mathrm{Lie}\, G$. Doing this, we get a central extension by a finite dimensional Abelian groups instead of an infinite dimensional Abelian group. On the current group $C_r(M; G)$ of C^r maps from M into the considered compact connected Lie group G, we use heat-kernel measure deduced from the Airault–Malliavin equation, and since we get a central extension $\hat{C}_r(M; G)$ by a finite dimensional group Z, we get a measure on the central extension of the current group. Let us recall that studies of the Brownian motion on infinite dimensional manifold have a long history (see works of Kuo [15], Belopolskaya–Daletskii [6, 12], Baxendale [4, 5], etc.).

Let us remark that this procedure of getting a random field by adding extra-time is very classical in theoretical physics, in the so called programme of stochastic-quantization of Parisi–Wu [23], which uses an infinite-dimensional Langevin equation. Instead to use here the Langevin equation, we use the more tractable Airault–Malliavin equation, that represents infinite dimensional Brownian motion on a current group.

2 A measure on the current group in any dimension

We consider $C_r(M;G)$ endowed with its C^r topology. The parameter space M is supposed compact and the Lie group G is supposed compact, simple and simply connected. We consider the set of continuous paths from $[0,1]$ into $C_r(M;G)$ $t \to g_t(\cdot)$, where $S \in M \to g_t(S)$ belongs to $C_r(M;G)$ and $g_0(S) = e$. We denote $P(C_r(M;G))$ this path space.

Let us consider the Hilbert space H of maps h from M into $\operatorname{Lie} G$ defined as follows:

$$\int_\Sigma \langle (\Delta^k + 1)h, h\rangle dS = \|h\|_H^2,$$

where Δ is the Laplace Beltrami operator on M and dS the Riemannian element on M endowed with a Riemannian structure.

We consider the Brownian motion $B_t(\cdot)$ with values in H.

We consider the Airault–Malliavin equation (in Stratonovitch sense):

$$dg_t(S) = g_t(S) dB_t(S), \qquad g_0(S) = e.$$

Let us recall (see [17]):

Theorem 1. *If k is enough big, $t \to \{S \to g_t(S)\}$ defines a random element of $P(C_r(M;G))$.*

We denote by μ the heat-kernel measure $C_r(M;G)$: it is the law of the C^r random field $S \to g_1(S)$. It is in fact a probability law on the connected component of the identity $C_r(M;G)_e$ in the current group.

3 A brief review of Maier–Neeb theory

Let us consider $\Pi_2(C_r(M;G)_e)$ the second fundamental group of the identity in the current group for $r > 1$. The Lie algebra of this current group is $C_r(M;\operatorname{Lie} G)$ the space of C^r maps from M into the Lie algebra $\operatorname{Lie} G$ of G [22]. We introduce the canonical Killing form k on $\operatorname{Lie} G$.

$\Omega^i(M;\operatorname{Lie} G)$ denotes the space of C^{r-1} forms of degree i on M with values in $\operatorname{Lie} G$. Following [20], we introduce the left-invariant 2-form Ω on $C_r(M;G)$ with values in the space of forms $Y = \Omega^1(M;\operatorname{Lie} G)/d\Omega^0(M;\operatorname{Lie} G)$ which associates

$$k(\eta, d\eta_1).$$

to (η, η_1), elements of the Lie algebra of the current group.

For that, let us recall that the Lie algebra of the current group is the set of C^r maps η from the manifold into the Lie algebra of G. $d\eta$ is a C^{r-1} 1-form into the Lie algebra of G. Therefore $k(\eta, d\eta_1)$ appears as a C^{r-1} 1-form with values in the Lie algebra of G. Moreover

$$dk(\eta, \eta_1) = k(d\eta, \eta_1) + k(\eta, d\eta_1).$$

This explains the introduction of the quotient in Y. Following the terminology of [20], we consider the period map P_1 which to σ belonging to $\Pi_2(C_r(M;G)_e)$ associates $\int_\sigma \Omega$. Apparently P_1 takes its values in Y, but in fact, the period map takes its values in a lattice L of $H^1(M;\operatorname{Lie} G)$.

It is defined on $\Pi_2(C_r(M;G)_e)$ since Ω is closed for the de Rham differential on the current group, as it is left-invariant and closed and it is a 2-cocycle in the Lie algebra of the current group [20]. We consider the Abelian group $Z = H^1(M; \mathrm{Lie}\, G)/L$. Z is *of finite dimension*.

We would like to apply Theorem III.5 of [20]. We remark that the map P_2 considered as taking its values in Y/L is still equal to 0 when it is considered by taking its values in $H^1(M; \mathrm{Lie}\, G)/L$.

We deduce the following theorem:

Theorem 2. *We get a central extension $\hat{C}_r(M;G)$ by Z of the current group $C_r(M;G)_e$ if $r > 1$.*

Since Z is of finite dimension, we can consider the Haar measure on Z. We deduce from μ a measure $\hat{\mu}$ on $\hat{C}_r(M;G)$.

Remark 1. Instead of considering $C_r(M; \mathrm{Lie}\, G)$, we can consider $W_{\theta,p}(M; \mathrm{Lie}\, G)$, some convenient Sobolev–Slobodetsky spaces of maps from M into $\mathrm{Lie}\, G$. We can deduce a central extension $\hat{C}_{\theta,p}(M;G)$ of the Sobolev–Slobodetsky current group $C_{\theta,p}(M;G)_e$. This will give us an example of Brzezniak–Elworthy theory, which works for the construction of diffusion processes on infinite-dimensional manifolds modelled on M-2 Banach spaces, since Sobolev–Slobodesty spaces are M-2 Banach spaces [9, 10, 18]. We consider a Brownian motion B_t^1 with values in the finite dimensional Lie algebra of Z and $\hat{B}_t = (B_t(\cdot), B_t^1)$ where $B_t(\cdot)$ is the Brownian motion in H considered in the Section 2. Then, following the ideas of Brzezniak–Elworthy, we can consider the stochastic differential equation on $\hat{C}_{\theta,p}(M;G)$ (in Stratonovitch sense):

$$d\hat{g}_t(\cdot) = \hat{g}_t(\cdot) d\hat{B}_t.$$

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