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Stabilization for Singular Fractional-Order Systems via Static Output Feedback

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ABSTRACT The stabilization problem of singular fractional-order systems with fractional commensurate order $0 < \alpha < 2$ via static output feedback is studied in this paper. For the $0 < \alpha < 1$ case, two methods for the static output feedback control design are provided. In the first method, the controller is designed without decomposing the system matrix, and less variables are within the second method. Furthermore, a method that is similar to the second method of the $0 < \alpha < 1$ case is provided for the $1 \leq \alpha < 2$ case. The controller parameters are computed by solving matrix inequalities, and efficient iterative algorithms are built to solve the resultant matrix inequalities. Numerical examples are provided to show the effectiveness of the proposed results.

INDEX TERMS Singular fractional-order systems, linear matrix inequality (LMI), static output feedback, iterative algorithm.

I. INTRODUCTION

In control theory, the static output feedback stabilization for linear system which is one of the most attractive research topics has important practical and theoretical values. Many researchers have studied static output feedback problems for a variety of types of systems [1]–[5]. Various ways of stabilizing control design have been developed, among which a successful approach is the LMI method. On the basis of LMI, structured Lyapunov matrix method [4], two-step method [6], and iterative algorithm [7] have been advanced.

Many complex dynamical systems, such as electromagnetic systems [8], dielectric polarization [9], viscoelastic systems [10], [11], can be represented by fractional-order systems. Considering the applicability in engineering science, fractional-order systems have been attracted more attention [12]–[14]. The stability and stabilization conditions have been given for a continuous fractional-order systems with fractional-order $0 < \alpha \leq 1$ and $1 \leq \alpha < 2$ [15]–[20].

Recently, some results on stability and stabilization problems of fractional-order control systems have been extended to singular fractional-order systems [21]–[29]. The admissibility of singular fractional-order systems has been investigated in [21]–[24]. For the stabilization of singular fractional-order systems with order $0 < \alpha < 2$, the Weierstrass canonical decomposition method is used in [25]. The normalization problem and the stabilization of singular fractional-order systems as well as the uncertain case are investigated in [26] and [27]. Necessary and sufficient conditions of observer-based stabilization are presented in [28]. By using singular value decomposition, the results of stabilization for uncertain fractional-order systems are derived in [29]. In order to solve the related matrix inequalities more effectively, an iterative algorithm is built in [30].

In this paper, we focus on the stabilization problem via static output feedback for singular fractional-order systems. Based on matrix inequalities, we propose controller design methods without normalizing the singular matrix. We also extend the methods to robust stabilization controller design for uncertain singular fractional-order systems. The static output feedback controllers are designed in terms of bilinear matrix inequalities (BMI), and efficient algorithms are established to deal with these matrix inequalities. Numerical examples are provided to show the advantages of the proposed results.

The paper is organized as follows. In Section 2, we provide the definition of fractional derivative, the problem formulation and useful lemmas. In Section 3, the main results are presented. Section 4 gives numerical examples to illustrate our proposed results. Section 5 is the conclusion.

Notation: Throughout this paper, X^{-1} and X^T denotes, respectively, the inverse and the transpose of X. $A > 0$ ($A < 0$) denotes a positive definite (negative definite)symmetric matrix. $sym\{T\}$ stands for $T + T^T$. The symmetric term in a matrix is denoted by $*$. $\otimes$ is the Kronecker product, $\Theta = \begin{bmatrix} \sin\alpha\frac{\pi}{2} & -\cos\alpha\frac{\pi}{2} \\ \cos\alpha\frac{\pi}{2} & \sin\alpha\frac{\pi}{2} \end{bmatrix}$.

II. PROBLEM FORMULATION

In this paper, we use the Caputo fractional derivative definition.

Definition 2.1 [12]: The Caputo derivative of order α for a function $f(t)$ is defined as:

$$D^\alpha f(t) = \frac{1}{\Gamma(Z-\alpha)} \int_0^t \frac{f^{(Z)}(s)}{(t-s)^{\alpha-Z+1}} ds, \tag{1}$$

where Z is a positive integer satisfying $Z - 1 < \alpha \leq Z$, and $\Gamma(\cdot)$ is the Gamma function.

Consider the following unforced linear singular fractional-order system:

$$ED^\alpha x(t) = Ax(t), \quad 0 < \alpha < 2, \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the pseudo semi-state vector, $A \in \mathbb{R}^{n\times n}$ is the system matrix, $E \in \mathbb{R}^{n\times n}$ is the system singular matrix and $rank(E) = m \leq n$, D^α denotes the Caputo derivative operator. We denote system (2) with (E, A, α).

Definition 2.2 [23]: The triple (E, A, α) is called regular if $det(s^\alpha E - A) \not\equiv 0$. The triple (E, A, α) is called impulse free if $deg(det(sE - A)) = rank(E)$. The triple (E, A, α) is called stable if all the roots of $det(s^\alpha E - A) = 0$ satisfy $|arg(spec(E, A, \alpha))| > \alpha\frac{\pi}{2}$, where $spec(E, A, \alpha)$ is the spectrum of $det(s^\alpha E - A) = 0$. The system (2) is called admissible if (E, A, α) is regular, impulse-free and stable.

The following lemmas are useful for the development.

Lemma 2.1 [22]: Suppose the triple (E, A, α) is regular. Then fractional-order singular system $ED^\alpha x(t) = Ax(t) with$ $0 < \alpha < 1$ is admissible, if and only if there exists $P = kX + Y, X, Y \in \mathbb{R}^{n\times n}$, $\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0$, $k = \tan(\frac{\alpha\pi}{2})$ and $Q \in \mathbb{R}^{(n-m)\times n}$ such that

$$sym\{APE^T + AE_0Q\} < 0, \tag{3}$$

where $E_0 \in \mathbb{R}^{n\times(n-m)}$ is of full column rank and satisfies $EE_0 = 0$.

Regardless of the regularity of the triple (E, A, α), we can always find nonsingular matrices M and N such that

$$MEN = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}, \quad MAN = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}. \tag{4}$$

Lemma 2.2 [23]: The system $ED^\alpha x(t) = Ax(t)$ with $0 < \alpha < 1$ is admissible if and only if there exist matrices $X_1, X_2 \in \mathbb{R}^{m\times m}$, $X_3 \in \mathbb{R}^{(n-m)\times m}$ and $X_4 \in \mathbb{R}^{(n-m)\times(n-m)}$ such that

$$\begin{bmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{bmatrix} > 0, \tag{5}$$

$$sym\{aMANX - bMANY\} < 0, \tag{6}$$

where

$$X = \begin{bmatrix} X_1 & 0 \\ X_3 & X_4 \end{bmatrix}, \qquad Y = \begin{bmatrix} X_2 & 0 \\ 0 & 0 \end{bmatrix}, \tag{7}$$

$m = rank(E)$, $a = \sin(\frac{\alpha\pi}{2})$, $b = \cos(\frac{\alpha\pi}{2})$, $M, N \in \mathbb{R}^{n\times n}$ are arbitrary nonsingular matrices satisfying(4).

Lemma 2.3: The system $ED^\alpha x(t) = Ax(t)$, $1 \leq \alpha < 2$ is admissible if and only if there exist matrices $P_1 \in \mathbb{R}^{m\times m}$, $P_1 > 0$, $P_2 \in \mathbb{R}^{(n-m)\times m}$ and $P_3 \in \mathbb{R}^{(n-m)\times(n-m)}$ such that

$$sym\{\Theta \otimes MANP\} < 0, \tag{8}$$

where

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \tag{9}$$

$m = rank(E), M, N \in \mathbb{R}^{n\times n}$ are arbitrary nonsingular matrices satisfying(4).

Proof: The course of proof is similar to Lemma2.2 in [23], so it is omitted here.

Lemma 2.4 [31]: The following statements in (10) and (11) are equivalent, and matrices Ω, F, Π, and W have appropriate dimensions,

$$\begin{bmatrix} \Omega & * \\ \Pi + WF^T & -W - W^T \end{bmatrix} < 0, \tag{10}$$

$$\Omega < 0, \Omega + F\Pi + \Pi^T F^T < 0. \tag{11}$$

Lemma 2.5 [30]: There exists matrix $G > 0$ such that the following statements in (12) and (13) are equivalent for matrices Ω, F, and Υ with appropriate dimensions,

$$\begin{bmatrix} \Omega - FGF^T & * \\ GF^T + \Upsilon & -G \end{bmatrix} < 0, \tag{12}$$

$$\Omega + F\Upsilon + \Upsilon^T F^T < 0. \tag{13}$$

Consider the following singular fractional-order system:

$$\begin{cases} ED^\alpha x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t), \end{cases} \tag{14}$$

and uncertain singular fractional-order system:

$$\begin{cases} ED^\alpha x(t) = (A + \Delta A)x(t) + Bu(t) \\ y(t) = Cx(t), \end{cases} \tag{15}$$

where $0 < \alpha < 2$ is the fractional commensurate order, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^l$ and $y(t) \in \mathbb{R}^p$ are the state vector, the control input vector and output vector, respectively. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{p\times n}$ are constant matrices, $E \in \mathbb{R}^{n\times n}$ is the system singular matrix with $rank(E) = m \leq n$. ΔA is uncertainty and it is assumed that $\Delta A = RFS$, where R and S are known constant matrices, F is unknown matrix satisfying $FF^T \leq I$.

In this paper, we consider the static output feedback controller for systems (14) and (15) as follows:

$$u(t) = Ky(t), \tag{16}$$

where $K \in \mathbb{R}^{l\times p}$ is the gain matrix. Then the closed-loop control systems can be expressed, respectively, as

$$ED^{\alpha}x(t) = (A + BKC)x(t), \tag{17}$$

and

$$ED^{\alpha}x(t) = (A + BKC + RFS)x(t). \tag{18}$$

Our purpose is to find static output feedback controllers such that the closed-loop control systems (17)–(18) are, respectively, admissible and robust admissible.

III. MAIN RESULT

Here, we present the main results.

Theorem 3.1: If there exist matrix $P \in \{kX + Y, X \in \mathbb{R}^{n\times n}, Y \in \mathbb{R}^{n\times n}, \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0, k = \tan(\frac{\alpha\pi}{2})\}$ and matrices $K(i)$, ΔK, U, W and Q with appropriate dimensions, such that

$$\Sigma(i) \triangleq \begin{bmatrix} \Sigma_{11} & * & * \\ \Sigma_{21} & -U & * \\ \Sigma_{31} & \Delta K^T & -W - W^T \end{bmatrix} < 0, \tag{19}$$

where

$$\Sigma_{11} = sym\left\{APE^T + AE_0Q\right\} - BUB^T, \tag{20}$$
$$\Sigma_{21} = UB^T + K(i)(CPE^T + CE_0Q) + \Delta KC, \tag{21}$$
$$\Sigma_{31} = CPE^T + CE_0Q - W^TC, \tag{22}$$

and A, B, C, E_0 are fixed matrices, then the closed-loop system (17) with order $0 < \alpha < 1$ is admissible. And the output feedback gain matrix can be obtained by

$$K = K(i) + \Delta KW^{-T}. \tag{23}$$

Proof: If $\Sigma(i) < 0$, by Lemma 2.4, it is equivalent to

$$\begin{bmatrix} sym\left\{APE^T + AE_0Q\right\} - BUB^T & * \\ UB^T + K(i)(CPE^T + CE_0Q) + \Delta KC & -U \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ \Delta KW^{-T} \end{bmatrix}\left[CPE^T + CE_0Q - W^TC \;\; 0\right]$$
$$+ \left[CPE^T + CE_0Q - W^TC \;\; 0\right]^T \begin{bmatrix} 0 \\ \Delta KW^{-T} \end{bmatrix}^T < 0,$$

that is

$$\begin{bmatrix} sym\left\{APE^T + AE_0Q\right\} - BUB^T & * \\ UB^T + K(CPE^T + CE_0Q) & -U \end{bmatrix} < 0.$$

By Lemma 2.5, we obtain

$$sym\{(A + BKC)PE^T + (A + BKC)E_0Q\} < 0.$$

By Lemma 2.1, (17) is admissible, which completes the proof.

Remark 3.1: The methods in [26] and [27] normalize the singular matrix firstly, then design controllers for the normal systems. Theorem 3.1 presents a stabilizing design method for the fractional-order singular system without normalizing the singular matrix.

Remark 3.2: To deal with BMI in Theorem 3.1, we let $K = K(i) + \Delta KW^{-T}$, and introduce the slack matrix W to reduce the conservatism. When $K(i)$ is fixed, $\Sigma(i)$ becomes LMI. Next we will introduce an iterative method in which, in each iteration, a new $K(i)$ is generated to be the initial value for the next iteration. This is an effective way to get the gain matrix.

To deal with the BMI in Theorem 3.1, the following LMI-based algorithm is constructed.

Algorithm 3.1:

Step 1: Set $K(i) = 0$. Solve the following optimization problem OP1 with respect to matrices P, ΔK, U, W, Q and ϵ.

$$(OP1): \;\; min\ \epsilon$$
$$s.t.\ \Sigma(i) < \epsilon,$$

Let $K = \Delta KW^{-T}$. If $\epsilon < 0$, stop, and K is the gain matrix. Else, set $i = 0$, $K(0) = K$.

Step 2: Solve OP1 with respect to matrices P, ΔK, U, W, Q and ϵ.

Step 3: Set $K(i+1) = K(i) + \Delta KW^{-T}$.

If $\epsilon < 0$, stop, and $K = K(i+1)$ is the gain matrix. Else, if ϵ is smaller than ε (a prescribed tolerance), that means the algorithm fails finding out the gain matrix. Else, let $i = i+1$, and go to Step 2.

On the basis of Lemma 2.2, we propose the following theorem with less variables than Theorem 3.1.

Theorem 3.2: For the singular fractional system (14), there exists an output feedback controller (16) such that the closed-loop system (17) with order $0 < \alpha < 1$ is admissible, if there exist $K(i)$, ΔK, U and W with appropriate dimensions, and $X, Y \in \mathbb{R}^{n\times n}$, such that

$$\Omega(i) \triangleq \begin{bmatrix} \Omega_{11} & * & * \\ \Omega_{21} & -U & * \\ \Omega_{31} & \Delta K^T & -W - W^T \end{bmatrix} < 0, \tag{24}$$

where

$$\Omega_{11} = sym\{MAN(aX - bY)\} - MBUB^TM^T, \tag{25}$$
$$\Omega_{21} = UB^TM^T + K(i)CN(aX - bY) + \Delta KC, \tag{26}$$
$$\Omega_{31} = CN(aX - bY) - W^TC, \tag{27}$$

and matrices X, Y are constructed as in Lemma 2.2, the gain matrix $K = K(i) + \Delta KW^{-T}$.

Proof: The proof is analogous to that of Theorem 3.1, and it is omitted here.

To deal with the BMI in Theorem 3.2, we have the following Algorithm 3.2.

Algorithm 3.2:

Step 1: Set $K(i) = 0$. Solve the following optimization problem (OP2) with respect to matrices X, Y, ΔK, U, W and ϵ.

$$(OP2) \;\; min\ \epsilon$$
$$s.t.\ \Omega(i) < \epsilon,$$

Let $K = \Delta KW^{-T}$. If $\epsilon < 0$, stop, and K is the gain matrix. Else, set $i = 0$, $K(0) = K$.

Step 2: Solve OP2 with respect to matrices X, Y, ΔK, U, W and ϵ.

Step 3: Set $K(i+1) = K(i) + \Delta K W^{-T}$.

If $\epsilon < 0$, stop, and $K = K(i+1)$ is the gain matrix. Else, if ϵ is smaller than ε (a prescribed tolerance), that means the algorithm fails finding out the gain matrix. Else, let $i = i+1$, and go to Step 2.

Remark 3.3: The matrices M and N satisfying (4) can be easily obtained by using the function *reff* in MATLAB. The inequalities in Theorem 3.2 contain less variables than those in Theorem 3.1, so its related computational complexity is reduced.

In order to study the static output feedback stabilization for the uncertain singular fractional-order system, we have the following theorem.

Theorem 3.3: If there exist matrices $X, Y, K(i), \Delta K, U$ and W with appropriate dimensions and real scalar $\varepsilon > 0$, such that

$$\Xi(i) \triangleq \begin{bmatrix} \Xi_{11} & * & * & * \\ \Xi_{21} & -U & * & * \\ \Xi_{31} & \Delta K^T & -W - W^T & * \\ \Xi_{41} & 0 & 0 & -\varepsilon I \end{bmatrix} < 0, \quad (28)$$

where

$$\begin{aligned} \Xi_{11} &= sym\{MAN(aX - bY)\} - MBUB^T M^T \\ &\quad + \varepsilon MRR^T M^T, & (29) \\ \Xi_{21} &= UB^T M^T + K(i)CN(aX - bY) + \Delta KC, & (30) \\ \Xi_{31} &= CN(aX - bY) - W^T C, & (31) \\ \Xi_{41} &= SN(aX - bY), & (32) \end{aligned}$$

matrices A, B, C, R, L are fixed, X, Y are constructed as in Lemma 2.2, then the closed-loop system (18) with order $0 < \alpha < 1$ is robust admissible. And the output feedback gain matrix can be obtained by

$$K = K(i) + \Delta K W^{-T}. \quad (33)$$

Proof: By Lemma2.2, the system (18) is robust admissible if and only if there exist matrices X and Y are constructed as in Lemma 2.2, such that

$$sym\{M(A + BKC + \Delta A)N(aX - bY)\} < 0. \quad (34)$$

By applying the inequality

$$H^T V + V^T H \leq \varepsilon H^T H + \frac{1}{\varepsilon} V^T V, \quad (35)$$

H and V are arbitrary matrices and $FF^T \leq I$,

$$\begin{aligned} &sym\{M \Delta AN(aX - bY)\} \\ &\leq \varepsilon MRR^T M^T + \frac{1}{\varepsilon}(aX - bY)^T N^T S^T SN(aX - bY). \quad (36) \end{aligned}$$

That is, if

$$\begin{aligned} &sym\{M(A + BKC)N(aX - bY)\} + \varepsilon MRR^T M^T \\ &\quad + \frac{1}{\varepsilon}(aX - bY)^T N^T S^T SN(aX - bY) < 0, \quad (37) \end{aligned}$$

then the closed-loop system (18) is robust admissible. By Lemma 2.4, Lemma 2.5 and Schur complement, the inequality (37) is equivalent to (28).

To deal with the BMI in Theorem 3.3, we propose the following Algorithm 3.3.

Algorithm 3.3:

Step 1: Set $K(i) = 0$. Solve the following optimization problem (OP3) with respect to matrices X, Y, ΔK, U, W and τ, ϵ.

$$\begin{aligned} (OP3) \quad & min\ \epsilon \\ & s.t.\ \Xi(i) < \epsilon, \end{aligned}$$

Let $K = \Delta K W^{-T}$. If $\epsilon < 0$, stop, and K is the gain matrix. Else, set $i = 0$, $K(0) = K$.

Step 2: Solve OP3 with respect to matrices X, Y, ΔK, U, W and τ, ϵ.

Step 3: Set $K(i+1) = K(i) + \Delta K W^{-T}$.

If $\epsilon < 0$, stop, and $K = K(i+1)$ is the gain matrix. Else, if ϵ is smaller than ε (a prescribed tolerance), that means the algorithm fails finding out the gain matrix. Else, let $i = i+1$, and go to Step 2.

Remark 3.4: For solving static output feedback stabilization problem, the controller design in [29] requires additional constraints, that increases the conservatism, as explained in Example 4.2.

For the $1 \leq \alpha < 2$ case, with the stable condition in Lemma 2.4, we have the following theorem.

Theorem 3.4: If there exist matrices P, ΔK, U, W with appropriate dimensions, such that

$$\Pi(i) \triangleq \begin{bmatrix} \Pi_{11} & * & * \\ \Pi_{21} & -I \otimes U & * \\ \Pi_{31} & I \otimes \Delta K^T & I \otimes (-W - W^T) \end{bmatrix} < 0, \quad (38)$$

where

$$\begin{aligned} \Pi_{11} &= sym\{\Theta \otimes MANP\} - I \otimes MBUB^T M^T, & (39) \\ \Pi_{21} &= \Theta^T \otimes UB^T M^T + I \otimes [K(i)CNP + \Delta KC], & (40) \\ \Pi_{31} &= I \otimes (CNP - W^T C), & (41) \end{aligned}$$

and matrix P is constructed as in Lemma 2.3, then closed-loop system (17) is admissible, the gain matrix $K = K(i) + \Delta K W^{-T}$.

The proof course is similar to that of Theorem 3.1, so it is omitted here. The corresponding algorithm which solves the BMI in Theorem 3.4 is similar to Algorithm 3.1

For the uncertain singular fractional-order system with $1 \leq \alpha < 2$, we have the following corollary.

Corollary 3.5: If there exist matrices P, ΔK, U, W with appropriate dimensions real scalar $\varepsilon > 0$, P is constructed as in Lemma 2.3, such that

$$\Upsilon(i) \triangleq \begin{bmatrix} \Upsilon_{11} & * & * & * \\ \Upsilon_{21} & -I \otimes U & * & * \\ \Upsilon_{31} & \Upsilon_{32} & \Upsilon_{33} & * \\ \Upsilon_{41} & 0 & 0 & -\varepsilon I \end{bmatrix} < 0, \quad (42)$$

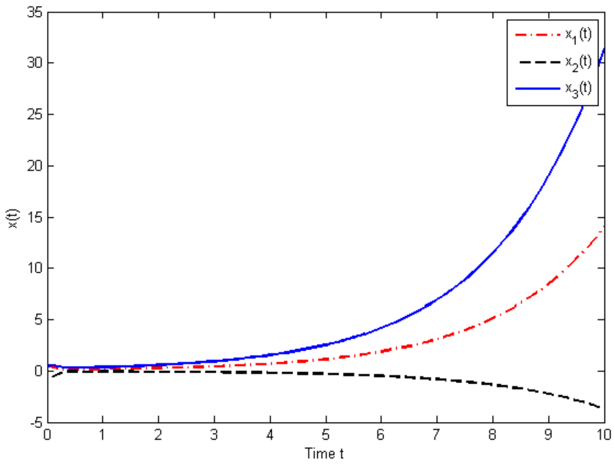

FIGURE 1. State curves for Example 4.1 without controller.

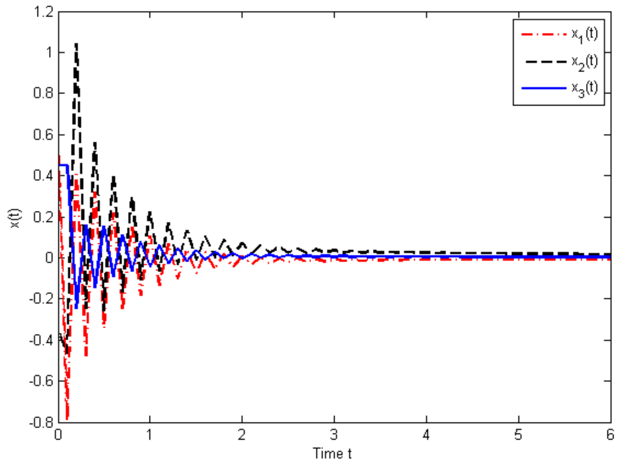

FIGURE 2. State curves for Example 4.1 with controller.

where

$$\Upsilon_{11} = sym\{\Theta \otimes MANP\} \tag{43}$$
$$+I \otimes (-MBUB^T M^T + \varepsilon MRR^T M^T), \tag{44}$$
$$\Upsilon_{21} = \Theta^T \otimes UB^T M^T + I \otimes [K(i)CNP + \Delta KC], \tag{45}$$
$$\Upsilon_{31} = I \otimes (CNP - W^T C), \quad \Upsilon_{32} = I \otimes \Delta K^T, \tag{46}$$
$$\Upsilon_{33} = I \otimes (-W - W^T), \quad \Upsilon_{41} = I \otimes SNP, \tag{47}$$

then the closed-loop system (18) is robust admissible and the gain matrix $K = K(i) + \Delta K W^{-T}$.

IV. NUMERICAL EXAMPLES

In this section, three numerical examples are presented to verify our proposed results.

Example 4.1: Consider the system (14) with parameters: $\alpha = 0.5$, and

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1.8 & -1 & 1 \\ 1 & 0.5 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \tag{48}$$

Obviously, the system is not admissible. Setting the initial states $x_0 = (0.6, -0.35, 0.45)$, the simulation result is shown in Figure 1. The method in [27] fails to find the controller. In order to use Theorem 3.1, let $E_0 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ satisfying $EE_0 = 0$. By Algorithm 3.1, after 3 iterations, the gain matrix can be found as

$$K = \begin{bmatrix} -1.0935 & -0.0857 \\ -0.9231 & -2.9448 \end{bmatrix}. \tag{49}$$

The simulation result (Figure 2) shows that the closed-loop control system (17) with controller (16) is asymptotically stable. Furthermore, the method in Theorem 3.2 is also feasible to design the controller.

Example 4.2: Consider system (15) with parameters: $\alpha = 0.8$, and

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1.1 & 0 & 1 \\ 0.9 & -1 & 1.1 \\ 4.8 & 1.2 & -2.3 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 \\ 0.9 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1.1 & 0 & 2 \\ 1.1 & -1 & 0.9 \\ 1 & 1 & 1 \end{bmatrix},$$
$$R = \begin{bmatrix} 0.19 & 0.42 & 0.5 \\ 0.28 & 0.21 & 0.1 \\ 0.19 & 0.38 & 0.58 \end{bmatrix},$$
$$S = \begin{bmatrix} 0.28 & 0.31 & 0.15 \\ 0.11 & 0.32 & 0.43 \\ 0.11 & 0.28 & 0.42 \end{bmatrix}. \tag{50}$$

We can't find a feasible solution for Example 4.2 by using the method in [29]. In order to use Theorem 3.3, we can get M and N satisfying (4) by using the function *reff* in MATLAB.

$$M = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}. \tag{51}$$

Solving the LMI in Algorithm 3.3, we obtain the gain matrix

$$K = \begin{bmatrix} -0.0951 & -0.2992 & -0.7455 \end{bmatrix}, \tag{52}$$

which stabilizes the system in Example 4.2. Setting the initial states $x_0 = (-0.35, 0.18, -0.19)$, when $F = diag\{sin(0.1\pi), cos(0.2\pi), sin(0.1\pi)\}$, the simulation result (Figure 3) shows that the closed-loop control system (18) with controller (16) is asymptotically stable.

Example 4.3: Consider a normal system (14) with $E = I$, parameters: $\alpha = 1.2$, and

$$A = \begin{bmatrix} 0 & -3 \\ 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{53}$$

This system is open-loop unstable. The static output feedback stabilization method in of [16, Th. 3.7] requires that $CP = \bar{P}C$, $P > 0$ for some $\bar{P}$, and this restriction leads to the

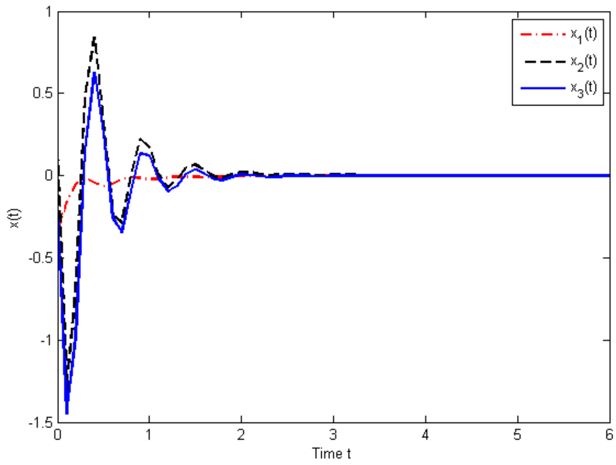

FIGURE 3. State curves for Example 4.2 with controller.

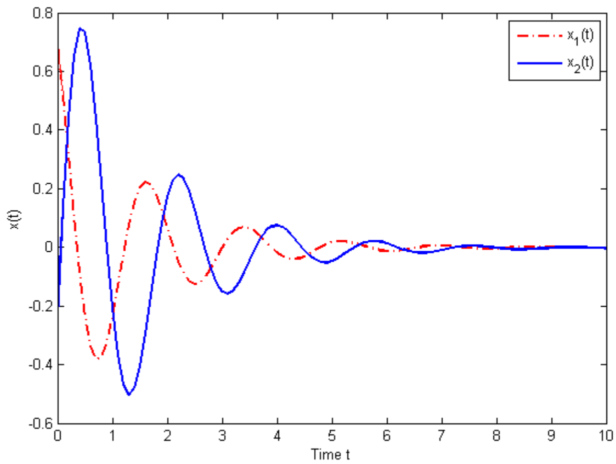

FIGURE 4. State curves for Example 4.3 with controller.

matrix P being a diagonal matrix. The method is infeasible for this system because $\Phi_{11} = \Phi_{22}$ therein are not negative definite. By the result in Theorem 3.4, the system can be stabilized with static output feedback control and the gain matrix $K = -4.0009$. Simulation result (Figure 4) with the initial states $x_0 = (0.8, -0.6)$ confirms the effectiveness of the method.

V. CONCLUSION

The stabilization problem via static output feedback control of the singular fractional-order systems has been investigated. The output feedback gain matrix can be obtained by solving matrix inequalities. Effective algorithms are built to compute the relevant matrix inequalities. Numerical examples are provided to show the effectiveness of the proposed results.

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