Intuitionistic Mereology

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Abstract

Two mereological theories are presented based on a primitive apartness relation along with binary relations of mereological excess and weak excess, respectively. It is shown that both theories are acceptable from the standpoint of constructive reasoning while remaining faithful to the spirit of classical mereology. The two theories are then compared and assessed with regard to their extensional import.

Keywords: Mereology; Intuitionism; Apartness; Excess; Extensionality

1 Introduction

Like any formal theory, classical mereology consists of logical axioms and proper axioms. Over the years, philosophical reasons have motivated interest in mereological theories that depart from the classical framework with regard to some of its proper axioms—initially composition and supplementation axioms (see [47, §§3–4]), more recently ordering axioms such as the antisymmetry of parthood [9, 10] or its transitivity [32]. On the other hand, logical axioms have been challenged, too. For instance, many-valued logics [36, 42], free logic [11, 41, 44], and plural quantification [28] have been considered as sensible alternatives to classical first-oder logic and, more recently, mereological theories based on paraconsistent logic have also been proposed [48]. The notion of a 'non-classical mereology' is thus ambivalent, as the epithet 'classical' may itself be understood with reference to one sort of axioms or the other.[1] Mereologists may go non-classical without leaving the *terra firma* of classical logic, or they may get non-classical for reasons that have little or even nothing to do with their views about specific properties of the part-whole relation as such and stem instead from general concerns about the underlying logic. In the latter case, it is indeed an interesting question whether the shift to a different logical framework calls for a parallel

[1] Actually there is some ambiguity also in what counts as classical mereology *tout court*. Here we are thinking of the familiar theory based on classical first-order logic [47]. Historically, however, this theory came to us in different guises. Leśniewski's 'Mereologia' [26, 27] was based on Ontology and Protothetic; Leonard and Goodman's 'Calculus of Individuals' [25] made use of quantification over classes. Such systems are not elementarily axiomatizable [33] and are, therefore, strictly stronger than their first-order approximation, which is due to Goodman [16].

rethinking of our basic mereological tenets. Can we simply change the logic and leave the rest of classical mereology as is? This is not a question that admits a general answer, let alone a uniform one; it must be addressed on a case-by-case basis. In this paper we aim to do so, if only partially, with regard to a theory that so far has received little attention in the literature: intuitionistic mereology.[2]

Our purpose is twofold. First, we want to outline the salient features of such a theory—a theory of the relations of part to whole, and of part to part within a whole, that is acceptable according to the principles of constructive reasoning. We prove that a constructively acceptable counterpart of classical mereology requires more than a mere revision of the underlying logic; it requires genuine revisions at the level of primitive notions and proper axioms. Importantly, simply adding the proper axioms of classical mereology on top of intuitionistic logic would result in a theory that fails to be fully extensional, i.e. to validate the following general principles:

If x and y are part of each other, then $x = y$ (Extensionality of Parthood);

If x and y overlap the same things, then $x = y$ (Extensionality of Overlap);

If x and y are composite things with the same proper parts, then $x = y$ (Extensionality of Proper Parthood).

Except for Extensionality of Parthood (which is just the antisymmetry axiom), Extensionality of Overlap and of Proper Parthood can be derived only using classical logic.

Our second purpose is to show how such principles can nonetheless be recovered when the shift from classical to intuitionistic logic is accompanied by a corresponding revision of the basic notions. Of course extensionality is by itself a contentious issue. The thought that mereological indiscernibility is sufficient for identity sits well with Goodman's principle of nominalism ("no distinction of entities without a distinction of content" [16, p. 26]), but it isn't intrinsic to our understanding of the part-whole relation and one may want to weaken the underlying logic precisely to relinquish this feature of classical mereology.[3] However this is not to say that an intuitionist is perforce committed to this move. At least in principle, a nominalist-extensionalist stance should be compatible with the demands of constructive *reasoning*, so an intuitionist should be able to preserve the spirit, if not the letter, of the three principles listed above. We argue that this is indeed a viable option, provided we understand the relevant mereological vocabulary in an intuitionistically friendly way. As we change the logic, we also need to revisit the fundamental concepts of the theory—and the proper axioms that govern them—in terms of more primitive, constructively justified notions.

[2] There is a literature on so-called 'Heyting mereologies', as in [12], [29], and [37]. Despite the name, however, such theories are based on classical logic and differ from classical mereology with regard to their proper axioms (they lack Weak Supplementation), so they are non-classical in the first sense introduced above. Essentially, they deliver a parthood relation whose structure is not a Boolean but a Heyting algebra (about which see [22]).

[3] For a review of the arguments, see [24, pt. ii]. For an explicit defense of extensionality on behalf of classical mereology, see [46].

2 Intuitionistic Primitives

Building on previous work in constructive mathematics, we focus on the following three primitive relations:

$a \neq b$ a is apart from b
$a \nleqslant b$ a exceeds b
$a \nless b$ a weakly exceeds b

Apartness was originally introduced by Brouwer [3, 4, 5] (with a different notation) to express inequality between real numbers in the constructive analysis of the continuum: whereas saying that two real numbers a and b are unequal only means that the assumption $a = b$ is contradictory, to say that a and b are apart[4] expresses the constructively stronger requirement that their distance on the real line can be effectively measured, i.e. that $|\, a - b \,| > 0$ has a constructive proof. Classically, inequality and apartness coincide, but intuitionistically two real numbers can be unequal without being apart. An early application of the theory of apartness is Heyting's work on projective geometry, where a basic relation $A \,\omega\, B$ between two points A and B is read as: point A is away from (*entfernt von*) point B (cf. axiom IIa in [18, p. 493]). Later [19, p. 49], Heyting defined apartness as a relation $\#$ on a species S such that, for all a, b in S:

If $a \# b$, then $b \# a$;
If $a \# b$, then not $a = b$;
If not $a \# b$, then $a = b$;
If $a \# b$, then, for any element c of S, either $a \# c$ or $b \# c$.

The notation '$\neq$' to denote the apartness relation was eventually introduced by Scott [39] and adopted by later authors, and here we shall follow suit. It is important, however, that it be treated as primitive (rather than shorthand for the negation of '$=$'.)

The excess relation was introduced by von Plato in [34, 35] to express in a constructive way the negation of a partial order $\leqslant$ and has been further investigated by Negri in [30] using sequent calculi. Its basic axiomatization treats $\nleqslant$ as irreflexive and closed under co-transitivity:[5]

Not $a \nleqslant a$;
If $a \nleqslant b$, then, for all c, either $a \nleqslant c$ or $c \nleqslant b$.

Moreover, apartness can be defined in terms of excess as $a \nleqslant b \vee b \nleqslant a$. Here we are going to treat both $\neq$ and $\nleqslant$ as primitives, but their intended reading will correspond to von Plato's. Intuitively, to say that a exceeds b will mean

[4] In Dutch: 'verwijderd', 'plaatselijk verschillend' [4, §2]; in German: 'entfernt' [5, p. 254], 'örtlich verschieden' [3, p. 3].

[5] Similar axioms for constructive ordering relations may be found in Bridges [2]; cf. also Scott [39, §1].

that a outstrips b of a non-zero quantity that is effectively computable. Thus, in particular, a will exceed b only if a is not (intuitionistically) part of b. However, intuitionistically a may fail to be part of b without exceeding it.

Weak excess is introduced here in a parallel fashion to express the constructive negation of a strict partial order $<$. It is called 'weak' insofar as it covers equality as a limit case: whereas excess requires apartness, weak access does not. Thus, intuitively, a will weakly exceed b whenever a is not (intuitionistically) a proper part of b. In this sense, the relationship between $\nleq$ and $\nless$ may be seen as dual to the relationship between parthood and proper parthood in classical mereology, though only insofar as equality may be understood as intuitionistic non-apartness.

Now, the familiar relations of equality, parthood, and proper parthood need not of course disappear altogether. The idea is rather that such relations—hence any other mereological relation that can be defined in terms of these, such as overlap[6]—are not fundamental. In the following we shall show that they can indeed be recovered from our constructive primitives (suitably axiomatized) in such a way as to satisfy the main properties they have in classical mereology, including extensionality. More precisely, we shall see that a mereological theory based on excess (and apartness) is deductively powerful enough to guarantee Extensionality of Parthood and, given Strong Supplementation, Extensionality of Overlap, though not Extensionality of Proper Parthood. On the other hand, a theory based on weak excess (and apartness) can derive all three forms of extensionality. Thus it is weak excess, we shall argue, that provides the best resources for a natural intuitionistic counterpart of classical mereology.

3 Mereologies based on excess and apartness

We begin by introducing the first sort of theory, IM1. Its language, L_1, comprises the usual first-order logical operators along with the two binary predicates $\nleq$ (excess) and $\neq$ (apartness) treated as primitives. These predicates are governed by the following proper axioms, where all free variables are tacitly assumed to be universally quantified.

$$\neg\, x \nleq x \tag{1}$$
$$x \nleq y \to x \nleq z \lor z \nleq y \tag{2}$$
$$\neg\, x \neq x \tag{3}$$
$$x \neq y \to y \neq x \tag{4}$$
$$x \neq y \to x \neq z \lor z \neq y \tag{5}$$
$$x \neq y \to x \nleq y \lor y \nleq x \tag{6}$$
$$x \nleq y \to x \neq y \tag{7}$$

[6]Classically, overlap is defined as sharing of a common part, and here we shall go along with that definition. We leave it to future work to investigate the possibility of adopting a notion of overlap with greater constructive appeal, such as the intuitionistic overlap relation developed by Ciraulo *et al.* [7, 8] in the context of Sambin's 'overlap algebra' [38].

In the terminology of constructive orders, axiom (1) and (2) state that $\nleqslant$ is irreflexive and co-transitive, whereas axioms (3)–(5) state that $\neq$ is irreflexive and co-transitive as well as symmetric. Axiom (6) may be thought of as the contrapositive of the antisymmetry principle of classical parthood: it states that $\nleqslant$ is co-antisymmetric or, less clumsily, weakly linear. Finally, (7) states that excess implies apartness.

Now let $\mathsf{AX1} = \mathsf{NI} + (1)$–$(7)$, where NI (also known in the literature as NJ) is Gentzen's natural deduction system of first-order intuitionistic logic [14]. Thus $\mathsf{AX1}$ is a variant of von Plato's excess theory mentioned above, whose only primitive $\nleqslant$ obeys axioms (1) and (2); we shall see below that his definition of $x \neq y$ as $x \nleqslant y \vee y \nleqslant x$ yields a system that is essentially equivalent to the result of adding our axioms (3)–(7). Moreover, in $\mathsf{AX1}$ there is a natural way of defining the familiar mereological relations of parthood, equality, and overlap as well as four distinct relations of proper parthood.

$x \leqslant y$	x is part of y
$x = y$	x is equal to y
$x \circ y$	x overlaps y
$x <_i y$	x is a proper$_i$ part of y, for $i \in \{1, 2, 3, 4\}$

The definitions are as follows:

$$x \leqslant y := \neg\, x \nleqslant y \tag{8}$$
$$x = y := \neg\, x \neq y \tag{9}$$
$$x \circ y := \exists z(z \leqslant x \wedge z \leqslant y) \tag{10}$$
$$x <_1 y := x \leqslant y \wedge \neg\, x = y \tag{11}$$
$$x <_2 y := x \leqslant y \wedge \neg\, y \leqslant x \tag{12}$$
$$x <_3 y := x \leqslant y \wedge x \neq y \tag{13}$$
$$x <_4 y := x \leqslant y \wedge y \nleqslant x \tag{14}$$

(The first two definitions of $x <_i y$ parallel the classical definitions of $<$ in terms of non-equality or non-parthood; the last two are their natural counterparts in terms of $\neq$ and $\nleqslant$.) These definitions will be collectively referred to as $\mathsf{DF1}$. Our first mereological theory, $\mathsf{IM1}$, is the result of adding $\mathsf{DF1}$ to $\mathsf{AX1}$.

Here are some important features of $\mathsf{IM1}$. First, it follows immediately that $\leqslant$, $=$, $\circ$, and each $<_i$ has the properties one would expect.

Theorem 1. *In $\mathsf{IM1}$ (i) $\leqslant$ is reflexive, antisymmetric, and transitive (a partial order); (ii) $=$ is reflexive, symmetric, and transitive (an equivalence relation); (iii) $\circ$ is reflexive and symmetric (a tolerance relation); and (iv) each $<_i$ is irreflexive and transitive (a strict partial order). Moreover, (v) $\leqslant$ and $=$ are stable, i.e. satisfy the biconditionals $x \leqslant y \leftrightarrow \neg\neg\, x \leqslant y$ and $x = y \leftrightarrow \neg\neg\, x = y$.*

Proof. The reflexivity of $\leqslant$, $=$, and $\circ$ follows immediately from axioms (1) and (3) and definitions (8)–(10). The remaining proofs for (i)–(iv) are routine. As an illustration we give a derivation of the antisymmetry of $\leqslant$. To this end, here and below it will be convenient to write out derivations by treating proper axioms and definitions as rules of inference, writing (n) to label an inference step that results from an application of the rule corresponding to axiom or definition (n). For example, in place of definition (8) we shall use the following rules:

$$\frac{x \leqslant y}{\neg\, x \nleqslant y}\ (8) \qquad\qquad \frac{\neg\, x \nleqslant y}{x \leqslant y}\ (8)$$

(Labels corresponding to the applications of purely logical rules will be omitted.) On this understanding, the derivation of $\leqslant$-antisymmetry in IM1 goes as follows.

$$\cfrac{\cfrac{\cfrac{\cfrac{\overset{2}{[x \neq y]}}{x \nleqslant y \vee y \nleqslant x}\ (6) \qquad \cfrac{\overset{1}{[x \nleqslant y]} \qquad \cfrac{\cfrac{\overset{3}{[x \leqslant y \wedge y \leqslant x]}}{x \leqslant y}}{\neg\, x \nleqslant y}\ (8)}{\bot} \qquad \cfrac{\overset{1}{[y \nleqslant x]} \qquad \cfrac{\cfrac{\overset{3}{[x \leqslant y \wedge y \leqslant x]}}{y \leqslant x}}{\neg\, y \nleqslant x}\ (8)}{\bot}}{\bot}\ {}^{1}}{\neg\, x \neq y}\ {}^{2}}{x = y}\ (9)}{x \leqslant y \wedge y \leqslant x \to x = y}\ {}^{3}$$

The other derivations are similar and left to the reader. As for (v), both biconditionals follow immediately from definitions (8) and (9) owing to the intuitionistic validity of $\neg\neg$-introduction and $\neg\neg\neg$-reduction. $\square$

Second, in IM1 the relations $\nleqslant$ and $\neq$ are irreducibly distinct from the negations of $\leqslant$ and $=$. Specifically, the conditionals

$$x \nleqslant y \to \neg\, x \leqslant y \tag{15}$$

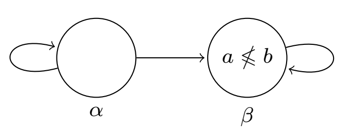
$$x \neq y \to \neg\, x = y \tag{16}$$

are derivable in IM1, but the converse conditionals

$$\neg\, x \leqslant y \to x \nleqslant y \tag{17}$$

$$\neg\, x = y \to x \neq y \tag{18}$$

are not. This can be checked with standard semantic techniques, e.g. Kripke models.[7] For (17) it suffices to consider a model with two worlds α, β and two objects a, b in the domain of α such that (i) β is accessible from α, and (ii) $a \nleqslant b$ holds only at β.



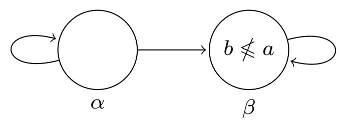
[7]We assume familiarity with Kripke models for intuitionistic logic, referring to Kripke's original article [23] and to [45] for a systematic presentation. Specific applications to intuitionistic theories may be found in [43] and, with special reference to order theories, in [17].

In such a model all the axioms of IM1 are valid. Moreover, we have that $\neg\, a \nleqslant b$ holds at neither world, and hence $\neg\neg\, a \nleqslant b$ holds at α. By (8), this means that $\neg\, a \leqslant b$ holds at α. Yet $a \nleqslant b$ does not hold at α by assumption. Therefore the conditional $\neg\, a \leqslant b \rightarrow a \nleqslant b$ does not hold either, showing the invalidity of (17). The invalidity of (18) is shown in a perfectly similar way, assuming $a \neq b$ holds only at β.

Another principle that is not derivable in IM1 is the following classically looking conditional, which fails for each $i \in \{1, 2, 3, 4\}$.

$$x \leqslant y \rightarrow x <_i y \vee x = y \tag{19}$$

This can be seen by considering slight variants of the previous model, with just two worlds α, β and β accessible from α. For instance, for $i = 4$ it is enough to assume that $b \nleqslant a$ holds only at β.



Then again the axioms of IM1 are valid but $x \leqslant y \rightarrow x <_4 y \vee x = y$ is not. For, on the one hand, since $b \nleqslant a$ does not hold at α, neither does $a \leqslant b \wedge b \nleqslant a$. Hence $a <_4 b$ does not hold at α by (14). Moreover, since $b \nleqslant a$ holds at β, so does $b \neq a$ by (7), from which it follows by (4) that $a \neq b$ also holds at β; and since β is accessible from α, the latter fact implies that $\neg\, a \neq b$ does not hold at α. Hence $a = b$ does not hold at α by (9). So neither $a <_4 b$ nor $a = b$ holds at α, and this is enough to conclude that (i) the disjunction $a <_4 b \vee a = b$ does not hold at α. On the other hand, the model is such that $a \nleqslant b$ holds neither at α nor at any other world accessible from it, which means that $\neg a \nleqslant b$ holds at α. Hence by (8) we also have that (ii) $a \leqslant b$ holds at α. Given (i) and (ii), it follows that $a \leqslant b \rightarrow a <_4 b \vee a = b$ does not hold at α, showing that (19) is not valid for $i = 4$. For $i = 1, 2, 3$, the invalidity of (19) can be shown in similar ways.

A third important fact about IM1 regards precisely the plurality of proper parthood predicates defined in (11)–(14). We saw that (11) and (12) are patterned after the classical definitions, using $=$ and $\leqslant$, whereas (13) and (14) characterize proper parthood directly in terms of $\neq$ and $\nleqslant$. As it tuns out, each pair is redundant. Given the co–antisymmetry axiom (6), the former definitions are equivalent to each other (as they are in classical mereology) and are implied by the latter, which are also equivalent to each other.

Theorem 2. *Each the following is derivable in IM1: (i) $x <_1 y \leftrightarrow x <_2 y$; (ii) $x <_3 y \rightarrow x <_1 y$; (ii) $x <_4 y \rightarrow x <_2 y$; (iv) $x <_3 y \leftrightarrow x <_4 y$.*

Proof. We prove explicitly the last claim, considering each direction of the biconditional separately. The other claims are proved similarly. The derivation of left-to-right direction of *(iv)* is as follows.

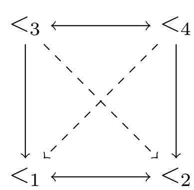
$$
\dfrac{\dfrac{\dfrac{[x \leqslant y \wedge x \neq y]^2}{x \leqslant y} \qquad \dfrac{\dfrac{\dfrac{[x \leqslant y \wedge x \neq y]^2}{x \neq y}}{x \nleqslant y \vee y \nleqslant x}\,(6) \qquad \dfrac{\dfrac{[x \nleqslant y]^1 \qquad \dfrac{\dfrac{[x \leqslant y \wedge x \neq y]^2}{x \leqslant y}}{\neg\, x \nleqslant y}\,(8)}{\bot}}{y \nleqslant x} \qquad [y \nleqslant x]^1}{y \nleqslant x}\,1}{x \leqslant y \wedge y \nleqslant x}}{x \leqslant y \wedge x \neq y \to x \leqslant y \wedge y \nleqslant x}\,2
$$

For the right-to-left direction, the derivation is even simpler.

$$
\dfrac{\dfrac{\dfrac{[x \leqslant y \wedge y \nleqslant x]^1}{x \leqslant y} \qquad \dfrac{\dfrac{\dfrac{[x \leqslant y \wedge y \nleqslant x]^3}{y \nleqslant x}}{y \neq x}\,(7)}{x \neq y}\,(4)}{x \leqslant y \wedge x \neq y}}{x \leqslant y \wedge y \nleqslant x \to x \leqslant y \wedge x \neq y}\,1
$$

□

The content of Theorem 2 is reproduced in the diagram below, where the two additional implications (dashed arrows) follow directly from the others by the transitivity of derivability in NI.



These facts concerning proper parthood are especially important when it comes to the intuitionistic counterparts of other principles of classical mereology, beginning with so-called supplementation principles. Consider Weak Supplementation, which says that whenever something has a proper part, it has another part disjoint from the first. In L_1 we can formulate four variants of this principle, one for each $<_i$.

$$x <_1 y \to \exists z(z \leqslant y \wedge \neg\, z \circ x) \tag{WS$_1$}$$

$$x <_2 y \to \exists z(z \leqslant y \wedge \neg\, z \circ x) \tag{WS$_2$}$$

$$x <_3 y \to \exists z(z \leqslant y \wedge \neg\, z \circ x) \tag{WS$_3$}$$

$$x <_4 y \to \exists z(z \leqslant y \wedge \neg\, z \circ x) \tag{WS$_4$}$$

Given the equivalences in Theorem 2, it's clear that WS_1 and WS_2 coincide, as do WS_3 and WS_4, so really there are only two different ways of recasting Weak Supplementation in IM1. Moreover, the implications in Theorem 2 immediately give us that the equivalent formulations based on $<_1$ and $<_2$, i.e.

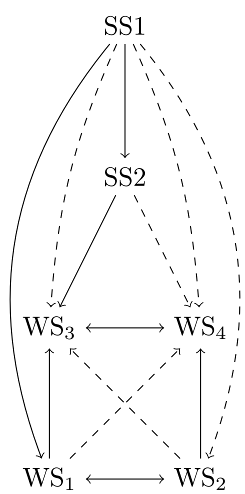
WS_1 and WS_2, entail the other formulations, WS_3 and WS_4. This, in turn, is relevant to the relationship between Weak Supplementation and Strong Supplementation, which again can be formulated either indirectly using $\leqslant$ in the antecedent or directly using $\nleqslant$.

$$\neg\, y \leqslant x \rightarrow \exists z(z \leqslant y \wedge \neg\, z \circ x) \tag{SS1}$$

$$y \nleqslant x \rightarrow \exists z(z \leqslant y \wedge \neg\, z \circ x) \tag{SS2}$$

Since $y \nleqslant x$ implies $\neg\, y \leqslant x$, the first of these principles is stronger than the second; and while it entails both forms of Weak Supplementation, SS2 entails only the weaker form expressed by WS_3 and WS_4. The following theorem and diagram summarize these facts.

Theorem 3. *Each the following is derivable in IM1: (i)* $WS_1 \rightarrow WS_3$*; (ii)* $WS_2 \rightarrow WS_4$*; (iii)* $WS_1 \leftrightarrow WS_2$*; (iv)* $WS_3 \leftrightarrow WS_4$*; (v)* SS1 $\rightarrow WS_1$*; (vi)* SS2 $\rightarrow WS_3$*; (vii)* SS1 $\rightarrow$ SS2.



To be sure, the overall picture is more complex. Classically Weak Supplementation is sometimes formulated using $<$ in the consequent instead of $\leqslant$,[8] so one could do the same in IM1. Each WS_i would have four variants, each one obtained by replacing $\leqslant$ with $<_j$ for some $j \in \{1, 2, 3, 4\}$, for a total of sixteen instances of the following schema:

$$x <_i y \rightarrow \exists z(z <_j y \wedge \neg\, z \circ x) \tag{WS_{ij}}$$

And while the two classical formulations are equivalent, at least so long as $\leqslant$ and $<$ behave classically, one may wonder how things are in IM1. The answer

[8] This is actually Simons' original formulation in [40]; the formulation with $\leqslant$ is from [6]. See also [20] and [47, §3.1].

is that, in light of Theorem 2, the sixteen principles in question reduce to just four principles, namely:

$$x <_1 y \to \exists z(z <_1 y \wedge \neg\, z \circ x) \tag{WS$_{11}$}$$

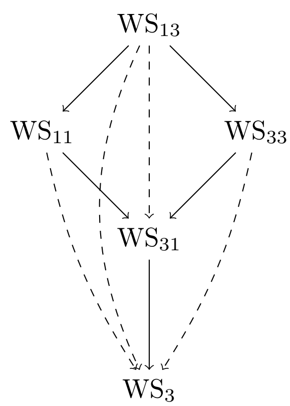
$$x <_1 y \to \exists z(z <_3 y \wedge \neg\, z \circ x) \tag{WS$_{13}$}$$

$$x <_3 y \to \exists z(z <_1 y \wedge \neg\, z \circ x) \tag{WS$_{31}$}$$

$$x <_3 y \to \exists z(z <_3 y \wedge \neg\, z \circ x) \tag{WS$_{33}$}$$

Moreover, since $<_3$ implies $<_1$, these four principle are partially ordered in terms of logical strength, with WS_{13} at the top and WS_{31} at the bottom. And since any $<_i$ implies $\leqslant$, it follows immediately that WS_{31} in turn implies the weakest of the supplementation principles mentioned earlier, namely WS_3 (equivalently: WS_4). These implications are summarized in the following theorem and diagram.

Theorem 4. *Each the following is derivable in IM1: (i)* $\mathrm{WS}_{11} \to \mathrm{WS}_{31}$*; (ii)* $\mathrm{WS}_{13} \to \mathrm{WS}_{33}$*; (iii)* $\mathrm{WS}_{13} \to \mathrm{WS}_{11}$*; (iv)* $\mathrm{WS}_{33} \to \mathrm{WS}_{31}$*; (v)* $\mathrm{WS}_{31} \to \mathrm{WS}_3$.



Let us also stress that all these supplementation principles involve a notion of overlap that is well-behaved.[9] By itself, the definition in (10) does not prevent $\circ$ from holding trivially because of a "null individual" that is part of everything. However, precisely the existence of such an individual is ruled out (in non-degenerate models) by the principles in question. This is how things are in classical mereology, too. There, the non-existence of a null individual is usually expressed by the following thesis, which follows from Weak Supplementation:

$$\exists x \exists y \neg\, x = y \to \neg \exists z \forall u\, z \leqslant u \tag{20}$$

Here the same idea may be captured by (20) itself, or by its counterpart in terms of apartness:

$$\exists x \exists y\, x \neq y \to \neg \exists z \forall u\, z \leqslant u \tag{21}$$

[9] Thanks to a referee for pressing us on this point.

And it's easy to see that both conditionals can be derived in IM1 from any of the supplementation principles charted above.

Theorem 5. *In IM1, both* (20) *and* (21) *are derivable from* SS1*, from* SS2*, and from each* WS_i *and each* WS_{ij} $(i, j \in \{1, 2, 3, 4\})$.

Proof. Since WS_3 and WS_4 are entailed by all other supplementation principles, and since (21) is an immediate consequence of (20) (because $x \neq y$ implies $\neg\, x = y$), it will suffice to show that (20) can be derived from WS_3 or, equivalently, from WS_4. We give the derivation from WS_3.

$$\dfrac{\dfrac{\dfrac{[\exists z \forall u\, z \leqslant u]^{8} \quad \dfrac{[\exists x \exists y \neg\, x = y]^{9} \quad \dfrac{[\exists y \neg\, x = y]^{6} \quad \dfrac{[\neg\, x = y]^{5} \quad \dfrac{\dfrac{\begin{matrix}[x \neq y]^{4} \quad \mathrm{WS}_3 \quad \forall u\, z \leqslant u \\ \vdots\,\mathscr{D}\,\vdots \\ \bot\end{matrix}}{\neg\, x \neq y}\,4}{x = y}\,(9)}{\bot}}{\bot}\,5}{\bot}\,6}{\bot}\,7}{\neg \exists z \forall u\, z \leqslant u}\,8}{\exists x \exists y \neg\, x = y \to \neg \exists z \forall u\, z \leqslant u}\,9$$

Here $\mathscr{D}$ stands for the following derivation of $\bot$ from $x \neq y$, WS_3, and $\forall u\, z \leqslant u$,

$$\dfrac{\dfrac{x \neq y}{x \neq z \lor z \neq y} \quad \begin{matrix}[x \neq z]^{3} \quad \mathrm{WS}_3 \quad \forall u\, z \leqslant u \\ \vdots\,\mathscr{E}\,\vdots \\ \bot\end{matrix} \quad \begin{matrix}[z \neq y]^{3} \quad \mathrm{WS}_3 \quad \forall u\, z \leqslant u \\ \vdots\,\mathscr{F}\,\vdots \\ \bot\end{matrix}}{\bot}\,3$$

where, in turn, $\mathscr{E}$ and $\mathscr{F}$ are perfectly parallel derivations of $\bot$ from $x \neq z$ and $z \neq y$, respectively. We just give the details for $\mathscr{E}$, where it will be clear that the overall proof depends on the reflexivity of $\leqslant$, as in classical mereology.

$$\dfrac{\dfrac{\forall u\, z \leqslant u}{z \leqslant x} \quad \dfrac{\dfrac{\dfrac{\dfrac{[z \leqslant x]^{2} \quad \dfrac{x \neq z}{z \neq x}}{z <_3 x}\,(13) \quad \mathrm{WS}_3}{\exists v (v \leqslant x \land \neg\, v \circ z)} \quad \dfrac{\dfrac{[v \leqslant x \land \neg\, v \circ z]^{1}}{\neg\, v \circ z} \quad \dfrac{\dfrac{\dfrac{\dfrac{\forall u\, z \leqslant u}{z \leqslant v} \quad \dfrac{\forall z\, z \leqslant z}{z \leqslant z}}{z \leqslant v \land z \leqslant z}}{\exists w (w \leqslant v \land w \leqslant z)}}{v \circ z}\,(10)}{\bot}}{\bot}\,1}{\neg\, z \leqslant x}\,2}{\bot}$$

□

Lastly, it is worth noting that although IM1 treats both $\nleq$ and $\neq$ as primitives, the following biconditional is provable:

$$x \neq y \leftrightarrow x \nleq y \vee y \nleq x \tag{22}$$

From left to right, this is just the co-antisymmetry axiom (6). In the other direction, we have that $x \nleq y \vee y \nleq x \rightarrow x \neq y$ is intuitionistically equivalent to $(x \nleq y \rightarrow x \neq y) \wedge (y \nleq x \rightarrow x \neq y)$, which is an immediate consequence of (7) and (4). It follows, therefore, that in IM1 apartness is definable in terms of $\nleq$. This takes us back to von Plato's treatment of the excess relation, which adopts the definition explicitly along with just (1) and (2), the irreflexivity and co-transitivity axioms for $\nleq$. That axioms (3)–(5) are derivable from (22) is proved in [35, thm. 3.1], and it's easy to verify that (6) and (7) are also derivable. So the two systems are essentially equivalent. It should be noted, however, that von Plato's actual axiomatization, as well as the further developments in [30], do not quite coincide with the system NI + (1) + (2) + (22), since they are based on a quantifier-free logic. By contrast, mereology requires quantification theory. This is clear from the formulation of such theses as (20) and (21), or of any supplementation principle, as well as from the definition of such concepts as overlap. So we need the full strength of L_1.

4 Mereologies based on weak excess and apartness

We now move to the second sort of theory, whose language L_2 has $\nless$ (weak excess) and $\neq$ (apartness) as primitives. The axioms are:

$$x \nless x \tag{23}$$

$$x \nless y \rightarrow x \nless z \vee z \nless y \tag{24}$$

$$\neg\, x \neq x \tag{25}$$

$$x \neq y \rightarrow y \neq x \tag{26}$$

$$x \neq y \rightarrow x \neq z \vee z \neq y \tag{27}$$

$$x \nless y \rightarrow x \nless z \vee z \neq y \tag{28}$$

$$x \nless y \rightarrow x \neq z \vee z \nless y \tag{29}$$

Axioms (23) and (24) state that $\nless$ is reflexive and co-transitive, whereas (25)–(27) coincide with the apartness axioms of AX1, (3)–(5). As for (28) and (29), these axioms may be thought of as contrapositives of the Leibniz principles $x < z \wedge z = y \rightarrow x < y$ and $x = z \wedge z < y \rightarrow x < y$.

Let AX2 = NI + (23)–(29). It turns out that this is quite a strong theory, as it derives two forms of linearity along with instances of the De Morgan laws.

Theorem 6. *Each of the following is derivable in AX2: (i)* $x \nless y \vee y \nless x$*; (ii)* $x \nless y \vee x \neq y$*; (iii)* $(\neg\, x \nless y \vee \neg\, y \nless x) \leftrightarrow \neg(x \nless y \wedge y \nless x)$*; (iv)* $(\neg\, x \nless y \vee \neg\, x \neq y) \leftrightarrow \neg(x \nless y \wedge x \neq y)$*.*

Proof. Regarding (i), consider $x \nless x \to x \nless y \lor y \nless x$. This is an instance of (24). Given (23), $x \nless y \lor y \nless x$ follows by modus ponens. The proof of (ii) is similar. Regarding (iii), one direction is an instance of $\neg A \lor \neg B \to \neg(A \land B)$, which is a theorem of NI. For the other direction we have the following derivation, where A is $\neg(x \nless y \land y \nless x)$ and B is $\neg\, x \nless y \lor \neg\, y \nless x$. We helpfully use (i).

$$\dfrac{\dfrac{x \nless y \lor y \nless x \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{3}{[x \nless y]} \quad \overset{1}{[y \nless x]}}{x \nless y \land y \nless x} \quad \overset{1}{[A]}}{\bot}}{\neg\, y \nless x}}{B}\,{\scriptstyle 1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{2}{[x \nless y]} \quad \overset{3}{[y \nless x]}}{x \nless y \land y \nless x} \quad \overset{4}{[A]}}{\bot}}{\neg\, x \nless y}}{B}\,{\scriptstyle 1}}{B}\,{\scriptstyle 3}}{A \to B}\,{\scriptstyle 4}$$

The proof of (iv) is similar and is left to the reader. □

In AX2 we can again introduce defined predicates for parthood ($\leqslant$), equality ($=$), and overlap ($\circ$) as well as a predicate for proper parthood ($<$), which now can be characterized directly in terms of $\nless$.

$$x < y := \neg\, x \nless y \tag{30}$$
$$x \leqslant y := \neg(x \nless y \land x \neq y) \tag{31}$$
$$x = y := \neg\, x \neq y \tag{32}$$
$$x \circ y := \exists z(z \leqslant x \land z \leqslant y) \tag{33}$$

Of course, given (31) one could still rely on $\leqslant$ to introduce four additional proper$_i$ parthood predicates $<_i$ as in (11)–(14). Alternatively, one could consider splitting $\leqslant$ into four parthood$_j$ predicates

$$x \leqslant_1 y := \neg(x \nless y \land x \neq y) \tag{34}$$
$$x \leqslant_2 y := \neg(x \nless y \land \neg\, x = y) \tag{35}$$
$$x \leqslant_3 y := \neg(\neg\, x < y \land x \neq y) \tag{36}$$
$$x \leqslant_4 y := \neg(\neg\, x < y \land \neg\, x = y) \tag{37}$$

and define a proper$_i$ parthood$_j$ predicate for each $i, j \in \{1, 2, 3, 4\}$. This is just combinatorics and we shall not pursue the details, focusing on the simple predicates $<$ and $\leqslant$. The definitions in (30)–(33) will be collectively referred to as DF2 and we shall take our second theory, IM2, to be the result of adding DF2 to AX2.

As with IM1, it's easy to verify that in IM2 $\leqslant$ is still a partial order, $=$ an equivalence relation, and $\circ$ a tolerance relation. Moreover $<$ is a strict partial order and is stable, like $\leqslant$ and $=$.

Theorem 7. *In IM2 (i) $\leqslant$ is reflexive, antisymmetric, and transitive; (ii) $=$ is reflexive, symmetric, and transitive; (iii) $\circ$ is reflexive and symmetric; (iv) $<$ is irreflexive and transitive; (v) $\leqslant$, $=$, and $<$ are stable.*

Proof. Again, the proofs are routine. We give derivations only for the reflexivity and antisymmetry of $\leqslant$, which will be used later. Here is reflexivity.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{[x \nless x \wedge x \neq x]}}{x \neq x} \qquad \dfrac{}{\neg\, x \neq x}\,(25)}{\bot}}{\neg(x \nless x \wedge x \neq x)}\,1}{x \leqslant x}\,(31)$$

For the proof of antisymmetry, we use again the linearity property $x \nless y \vee y \nless x$ established in Theorem 6(i), which we cite as L for space limitations.

$$\dfrac{\dfrac{\dfrac{\dfrac{L \qquad \dfrac{\dfrac{\overset{1}{[x \nless y]} \quad \overset{2}{[x \neq y]}}{x \nless y \wedge x \neq y} \quad \dfrac{\dfrac{\overset{3}{[x \leqslant y \wedge y \leqslant x]}}{x \leqslant y}}{\neg(x \nless y \wedge x \neq y)}\,(31)}{\bot} \qquad \dfrac{\dfrac{\overset{1}{[y \nless x]} \quad \dfrac{\overset{2}{[x \neq y]}}{y \neq x}\,(26)}{y \nless x \wedge y \neq x} \quad \dfrac{\dfrac{\overset{3}{[x \leqslant y \wedge y \leqslant x]}}{y \leqslant x}}{\neg(y \nless x \wedge y \neq x)}\,(31)}{\bot}}{\bot}\,1}{\neg\, x \neq y}\,2}{x = y}\,(32)}{x \leqslant y \wedge y \leqslant x \to x = y}\,3$$

□

We also note that IM2 is strong enough to prove the following conditional, whose $<_i$-counterpart (19) was not derivable in IM1 for any i.

$$x \leqslant y \to x < y \vee x = y \tag{38}$$

For a proof, assume $x \leqslant y$. By definition (31), this amounts to $\neg(x \nless y \wedge x \neq y)$. By Theorem 6($iv$), the latter is equivalent to $\neg\, x \nless y \vee \neg\, x \neq y$, and hence to $x < y \vee x = y$ by definitions (30) and (32). Thus, $x \leqslant y \to x < y \vee x = y$.

Given the relation of proper parthood defined in (30), both Weak and Strong Supplementation can be formulated in IM2 as in classical mereology, and it is easy to see that, as in classical mereology, the latter principle implies the former by the ordering properties of $<$ (using (38) to derive $\neg\, y \leqslant x$ from $x < y$).

$$x < y \to \exists z(z \leqslant y \wedge \neg\, z \circ x) \tag{WS}$$

$$\neg\, y \leqslant x \to \exists z(z \leqslant y \wedge \neg\, z \circ x) \tag{SS}$$

We leave it to the reader to check that both principles also rule out the existence of a null individual, as expressed by (20) or (21).[10]

5 Excess vs. weak excess

What exactly is the relationship between excess and weak excess? In Section 2 we said it is analogous to the relationship between parthood and proper parthood in classical mereology, and we know those relations are interdefinable. But this

[10] This result also follows from Theorem 8 below.

should not suggest that $\nleqslant$ and $\nless$ are themselves interdefinable intuitionistically. In fact they are not, at least not straightforwardly. We can prove that the system based on $\nless$ allows for a natural way to define a relation $\nleqslant$ satisfying the axioms of AX1. However, in the system based on $\nleqslant$ it is hard to find an equally natural definition of $\nless$ that yields the axioms of AX2 as theorems.

We start with the first claim. The natural way to define $\nleqslant$ in AX2 is to think of excess as 'proper' weak excess: $x \nleqslant y$ iff $x \nless y \wedge x \neq y$. Under this definition, we can show that AX1 is indeed interpretable in AX2. More precisely, let τ be a translation from L_1 to L_2 such that:

$$
\begin{array}{lcl}
\tau(x \nleqslant y) & := & x \nless y \wedge x \neq y \\
\tau(x \neq y) & := & x \neq y \\
\tau(\bot) & := & \bot \\
\tau(A \wedge B) & := & \tau(A) \wedge \tau(B) \\
\tau(A \vee B) & := & \tau(A) \vee \tau(B) \\
\tau(A \to B) & := & \tau(A) \to \tau(B) \\
\tau(\forall x A) & := & \forall x\, \tau(A) \\
\tau(\exists x A) & := & \exists x\, \tau(A)
\end{array}
$$

Recall that intuitionistic negation is defined as $\neg A := A \to \bot$, so we also have $\tau(\neg A) = \neg\, \tau(A)$. On this basis, the main result can be states as follows.

Theorem 8. *For every L_1-formula A derivable in AX1, $\tau(A)$ is derivable in AX2.*

Proof. Both systems are based on NI, so it will suffice to prove the result for formulas that are AX1-axioms. In fact, the translation of the three apartness axioms (3)–(5) coincides with the corresponding axioms of AX2, (25)–(27), so we only need consider the other four axioms of AX1, viz. (1), (2), (6), and (7). The case for (1) is straightforward, since $\tau(\neg x \nleqslant x) := \neg(x \nless x \wedge x \neq x)$ is an immediate consequence of (25). The case for (7) is even easier, since $\tau(x \nleqslant y \to x \neq y) := x \nless y \wedge x \neq y \to x \neq y$ is a theorem of NI. The other two cases, however, involve disjunctions and require more care, so we give detailed proofs.

Proof of (2) – We have $\tau(x \nleqslant y \to x \nleqslant z \vee z \nleqslant y) := x \nless y \wedge x \neq y \to (x \nless z \wedge x \neq z) \vee (z \nless y \wedge z \neq y)$. Let A be the antecedent, $x \nless y \wedge x \neq y$, and let B and C be the two disjuncts in the consequent, $x \nless z \wedge x \neq z$ and $z \nless y \wedge z \neq y$. Consider the following derivation $\mathscr{D}$ of $B \vee C$ from A and $x \nless z$:

$$
\dfrac{
\dfrac{\dfrac{A}{x \nless y}}{x \neq z \vee z \nless y}\,(29)
\qquad
\dfrac{\dfrac{x \nless z \quad [x \neq z]^{2}}{B}}{B \vee C}
\qquad
\dfrac{
\dfrac{\dfrac{A}{x \neq y}}{x \neq z \vee z \neq y}\,(27)
\qquad
\dfrac{\dfrac{x \nless z \quad [x \neq z]^{1}}{B}}{B \vee C}
\qquad
\dfrac{\dfrac{[z \nless y]^{2} \quad [z \neq y]^{1}}{C}}{B \vee C}
}{B \vee C}\,1
}{B \vee C}\,2
$$

In a perfectly similar fashion, we may construct a derivation $\mathscr{E}$ of $B \vee C$ from A and $z \nless y$. Using $\mathscr{D}$ and $\mathscr{E}$, we obtain a derivation of $A \to B \vee C$ as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{2}{[A]}}{x \not< y}}{x \not< z \vee z \not< y}\,(24) \qquad \begin{matrix} \overset{2}{[A]} \quad \overset{1}{[x \not< z]} \\ \ddots\ \mathscr{D}\ \ddots \\ \vdots \\ B \vee C \end{matrix} \qquad \begin{matrix} \overset{2}{[A]} \quad \overset{1}{[z \not< y]} \\ \ddots\ \mathscr{E}\ \ddots \\ \vdots \\ B \vee C \end{matrix}}{B \vee C}\,1}{A \to B \vee C}\,2
$$

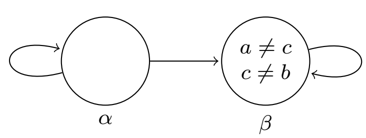
Proof of (6) – Here we have $\tau(x \neq y \to x \not\leq y \vee y \not\leq x) := x \neq y \to (x \not< y \wedge x \neq y) \vee (y \not< x \wedge y \neq x)$. Recall once more the linearity property from Theorem 6(i), namely $x \not< y \vee y \not< x$, and let A and B abbreviate $x \not< y \wedge x \neq y$ and $y \not< x \wedge y \neq x$, respectively. Then we have the following derivation:

$$
\dfrac{\dfrac{x \not< y \vee y \not< x \qquad \dfrac{\dfrac{\overset{1}{[x \not< y]} \quad \overset{2}{[x \neq y]}}{A}}{A \vee B} \qquad \dfrac{\dfrac{\overset{1}{[y \not< x]} \quad \dfrac{\overset{2}{[x \neq y]}}{y \neq x}\,(26)}{B}}{A \vee B}}{A \vee B}\,1}{x \neq y \to A \vee B}\,2
$$

□

Theorem 8 answers one half of our question. It tells us that AX1 can be fully recovered within AX2, which means that the excess relation can effectively be recast in terms of weak excess. Turning now to the other half, it would be nice to have a parallel result—a way of defining in AX1 a binary relation of weak excess that obeys the axioms characteristic of AX2. Unfortunately things are not so easy.

The natural option would be to define $x \not< y$ as $x \not\leq y \vee \neg\, x \neq y$, dualizing the definition of $\not\leq$ in AX2. Such a definition, however, would not deliver the intended result. Particularly, it would not secure the derivability in AX1 of the co-transitivity of $\not<$, which is axiom (24) of AX2. This is shown by the following Kripke model, with two worlds α, β and three objects a, b, c in the domain of α such that (i) β is accessible from α, (ii) $a \neq c$ and $c \neq b$ hold at β, and (iii) $a \not\leq c$ does not hold at α.



In such a model, all axioms of AX1 are valid but $\not<$ as defined above is not co-transitive, i.e. $x \not< y \to x \not< z \vee z \not< y$ is not valid. For notice that $\neg\, a \neq b$ holds at α, since $a \neq b$ does not hold at any world accessible from α. Hence the disjunction $a \not\leq b \vee \neg\, a \neq b$ also holds at α. According to the definition, this means that $a \not< b$ holds at α. On the other hand, since $a \neq c$ holds at the accessible world β, $\neg\, a \neq c$ does not hold at α. Moreover $a \not\leq c$ doesn't hold at α by assumption. Hence the disjunction $a \not\leq c \vee \neg\, a \neq c$, which would amount to

$a \nless c$, does not hold either. Similarly we can see that the definiens of $c \nless b$ does not hold at α. It follows that according to the definition the disjunction $a \nless c \vee c \nless b$ does not hold at α even though $a \nless b$ does, which means that we have a counterexample to the validity of $x \nless y \to x \nless z \vee z \nless y$.

Are there any alternatives? We do not have a general answer, but we conjecture that any attempt to define $\nless$ in terms of $\nleq$ will suffer from similar defects, short of adding (24) *ad hoc*. Weak excess appears to be intuitionistically irreducible to excess. It is nonetheless noteworthy that the above definition *would* work if we allowed some classical reasoning in AX1. Consider the following translation function σ from L_2 to L_1.

$$
\begin{aligned}
\sigma(x \nless y) &:= x \nleq y \vee \neg\, x \neq y \\
\sigma(x \neq y) &:= x \neq y \\
\sigma(\bot) &:= \bot \\
\sigma(A \wedge B) &:= \sigma(A) \wedge \sigma(B) \\
\sigma(A \vee B) &:= \neg(\neg\, \sigma(A) \wedge \neg\, \sigma(B)) \\
\sigma(A \to B) &:= \sigma(A) \to \sigma(B) \\
\sigma(\forall x A) &:= \forall x\, \sigma(A) \\
\sigma(\exists x A) &:= \exists x\, \sigma(A)
\end{aligned}
$$

Again, we have that $\sigma(\neg A) = \neg\, \sigma(A)$. But notice that translating $A \vee B$ as $\neg(\neg\, \sigma(A) \wedge \neg\, \sigma(B))$ betrays a classical, non-constructive understanding of disjunction. If this is accepted, then we have the following general result.

Theorem 9. *For every L_2-formula A derivable in AX2, $\sigma(A)$ is derivable in AX1.*

Proof. As with Theorem 8, we may ignore the pure apartness axioms (25)–(27) and focus on the remaining axioms of AX2, namely (23), (24), (28), and (29). Of these, the first is obviously derivable in AX1, since $\sigma(x \nless x) := x \nleq x \vee \neg\, x \neq x$ is an immediate consequence of the irreflexivity of apartness (3). The derivability of the other three axioms calls for detailed proofs.

Consider (24). We have $\sigma(x \nless y \to x \nless z \vee z \nless y) := x \nleq y \vee \neg\, x \neq y \to \neg(\neg(x \nleq z \vee \neg\, x \neq z) \wedge \neg(z \nleq y \vee \neg\, z \neq y))$. Since $\neg(B \vee C)$ and $\neg B \wedge \neg C$ are intuitionistically equivalent, the consequent of this conditional can be rewritten as $\neg(\neg\, x \nleq z \wedge \neg\neg\, x \neq z \wedge \neg\, z \nleq y \wedge \neg\neg\, z \neq y)$. Now let A abbreviate $\neg\, x \nleq z \wedge \neg\neg\, x \neq z \wedge \neg\, z \nleq y \wedge \neg\neg\, z \neq y$ and consider the following derivation $\mathscr{D}$ of $\bot$ from A, $\neg\, x \neq y$, and $z \nleq x$.

$$
\dfrac{\dfrac{z \nleq x}{z \nleq y \vee y \nleq x}\,(2) \qquad \dfrac{\overset{1}{[z \nleq y]} \quad \dfrac{A}{\neg\, z \nleq y}}{\bot} \qquad \dfrac{\dfrac{\dfrac{\overset{1}{[y \nleq x]}}{y \neq x}\,(7)}{x \neq y}\,(4) \quad \neg\, x \neq y}{\bot}}{\bot}\,1
$$

Using $\mathscr{D}$, we obtain the following derivation $\mathscr{E}$ of $\bot$ from A and $\neg\, x \neq y$.

$$\dfrac{\dfrac{A}{\neg\neg x \neq z} \qquad \dfrac{\dfrac{\dfrac{\overset{2}{[x \neq z]}}{x \nleqslant z \vee z \nleqslant x}\,(6) \qquad \dfrac{\overset{1}{[x \nleqslant z]} \quad \dfrac{A}{\neg x \nleqslant z}}{\bot} \qquad \begin{matrix} A \quad \overset{1}{[z \nleqslant x]} \quad \neg x \neq y \\ \ddots \;\; \mathscr{D} \;\; \cdot^{\cdot^{\cdot}} \\ \vdots \\ \bot \end{matrix}}{\bot}\,{\scriptstyle 1}}{\neg x \neq z}\,{\scriptstyle 2}}{\bot}$$

We now use $\mathscr{E}$ to construct a derivation of the desired conditional.

$$\dfrac{\dfrac{\dfrac{\overset{4}{[x \nleqslant y \vee \neg x \neq y]} \qquad \dfrac{\dfrac{\overset{2}{[x \nleqslant y]}}{x \nleqslant z \vee z \nleqslant y}\,(2) \qquad \dfrac{\overset{1}{[x \nleqslant z]} \quad \dfrac{\overset{3}{[A]}}{\neg x \nleqslant z}}{\bot} \qquad \dfrac{\overset{1}{[z \nleqslant y]} \quad \dfrac{\overset{3}{[A]}}{\neg z \nleqslant y}}{\bot}}{\bot}\,{\scriptstyle 1} \qquad \begin{matrix} \overset{3}{[A]} \quad \overset{2}{[\neg x \neq y]} \\ \ddots \;\; \mathscr{E} \;\; \cdot^{\cdot^{\cdot}} \\ \vdots \\ \bot \end{matrix}}{\dfrac{\bot}{\neg A}\,{\scriptstyle 3}}\,{\scriptstyle 2}}{x \nleqslant y \vee \neg x \neq y \to \neg A}\,{\scriptstyle 4}$$

This completes the proof for axiom (24). The proofs for axioms (28) and (29) are similar and left to the reader. □

6 Extensionality

We are now in a position to go back to our initial point concerning mereological extensionality. Recall the three general principles mentioned in Section 1, which may be stated formally as follows.

$$x \leqslant y \wedge y \leqslant x \to x = y \tag{EP}$$

$$\forall z(z \circ x \leftrightarrow z \circ y) \to x = y \tag{EO}$$

$$\exists z\, z < x \vee \exists z\, z < y \to (\forall z(z < x \leftrightarrow z < y) \to x = y) \tag{EPP}$$

In classical mereology, with $\leqslant$ as a primitive, EP (Extensionality of Parthood) is typically assumed as an axiom whereas EO (Extensionality of Overlap) and EPP (Extensionality of Proper Parthood) are derived from EP with the help of Strong Supplementation.[11] We said that such derivations require classical reasoning that is not constructively admissible, and we can now be more precise.

Typically, the derivation of EO in classical mereology makes use of the following Overlap Principle:

$$\forall z(z \circ x \to z \circ y) \to x \leqslant y \tag{OP}$$

[11] See [47, §3.2]. A notable exception is Leonard and Goodman's Calculus of Individuals [25], which is based on a primitive of mereological disjointness and includes EO (or, rather, its equivalent formulation in terms of disjointness) as an axiom. See also [31] for a systematic overview of classical mereology based on overlap, following Goodman [16].

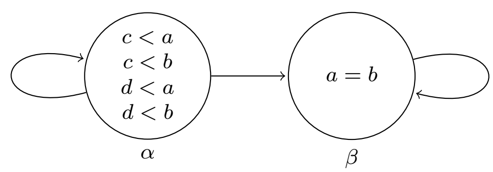
This principle, in turn, is usually established through a proof by contradiction: given the antecedent of OP, one further assumes $\neg\, x \leqslant y$ to derive by modus ponens the consequent of Strong Supplementation, $\exists z(z \leqslant x \wedge \neg\, z \circ y)$, whence a contradiction quickly follows from $\forall z(z \circ x \to z \circ y)$.[12] The last step is intuitionistically valid, but then one would need classical logic to drop the double negation from $\neg\neg\, x \leqslant y$ to obtain $x \leqslant y$.

Similarly, the derivation of EPP typically relies on the following theorem of classical mereology, known from [40] as the Proper Parts Principle:

$$\exists z\, z < x \to (\forall z(z < x \to z < y) \to x \leqslant y) \tag{PPP}$$

As above, to prove this theorem one begins by assuming $\neg\, x \leqslant y$ in order to obtain $\exists z(z \leqslant x \wedge \neg\, z \circ y)$ from Strong Supplementation and then reach a contradiction from this formula together with the antecedents of PPP. Thus, again, the proof eventually requires an application of $\neg\neg$-elimination, which is not intuitionistically available.

The following NI-model shows that it is indeed possible for $\leqslant$ to obey the axioms of a strongly supplemented partial order while violating both OP and PPP.



In the model we have a and b with the same proper parts, c and d, with $a = b$ holding only at β. The partial order axioms for $\leqslant$ are satisfied at both worlds, as are the strict order properties of $<$. (Recall that such models satisfy the inclusion requirement, so all mereological relations that hold at α continue to hold at the accessible worlds, hence at β.) Moreover, $a \leqslant b$ does not hold at α even though $\neg\neg\, a \leqslant b$ does, since $a \leqslant b$ holds at β and hence $\neg\, a \leqslant b$ holds at neither world (and similarly for $b \leqslant a$, $\neg\, b \leqslant a$, and $\neg\neg\, b \leqslant a$). It is also easy to check that Strong Supplementation is satisfied at both worlds (vacuously so with regard to a and b). Yet OP and PPP fail to hold at α, since the relevant antecedents $\forall z(z \circ a \to z \circ b)$, $\exists z\, z < a$, and $\forall z(z < a \to z < b)$ hold despite the fact that the consequent $a \leqslant b$ doesn't (and similarly for $b \leqslant a$ etc.). Clearly EO and EPP fail as well.

All of this confirms that a *purely logical* intuitionistic counterpart of classical mereology, i.e. an intuitionistic mereology obtained simply by revising the logic while adopting the proper axioms of classical mereology verbatim, is *not* going to be fully extensional. We can assume EP along with Strong Supplementation, but EO and EPP will not follow.

There is, to be sure, a sense in which this verdict may be appealed. After all, there are several ways of embedding classical logic into intuitionistic logic by means of translation functions that preserve classical equivalence, and any such

[12] Some authors actually identify Strong Supplementation with OP; see e.g. [21, p. 187].

function would yield a complete intuitionistic counterpart of any classical theory, including the whole of classical extensional mereology.[13] A case in point is the so-called Gödel-Gentzen negative translation (after [15] and [13]), which is obtained by appending double negations to atomic formulas, disjunctions, and existentially quantified formulas. Effectively, this amounts to associating each formula A with a formula $g(A)$ defined inductively as follows:

$$\begin{aligned}
g(x \leqslant y) &:= \neg\neg\, x \leqslant y \\
g(x = y) &:= \neg\neg\, x = y \\
g(\bot) &:= \bot \\
g(A \wedge B) &:= g(A) \wedge g(B) \\
g(A \vee B) &:= \neg(\neg g(A) \wedge \neg g(B)) \\
g(A \to B) &:= g(A) \to g(B) \\
g(\forall x A) &:= \forall x\, g(A) \\
g(\exists x A) &:= \neg \forall x \neg\, g(A)
\end{aligned}$$

This translation has the property that, for any set of formulas Σ in the language, A follows from Σ classically iff $g(A)$ follows from $\{g(B) \colon B \in \Sigma\}$ intuitionistically. (For details, see e.g. [45, ch. 2].) Thus, when Σ comprises the axioms of classical extensional mereology, the translation yields an intuitionistically acceptable counterpart of every theorem thereof, including EP along with EO and EPP. This is telling, since these counterparts are classically equivalent to the originals. However it is hardly what we wanted. The constructive extensionalist isn't just interested in asserting some logically sanitized rendering of classical extensionality; she wants to assert extensionality *tout court*.

So the verdict stands: one way or the other, a purely logical revamping of classical mereology on intuitionistic grounds is not going to be fully extensional. Is the picture any different when it comes to intuitionistic mereologies that, like IM1 and IM2, involve a substantive revision of the proper axioms of classical mereology along with its logical axioms? We said the answer is in the affirmative. Let us finally see why and to what extent.

To begin with, we know from Theorems 1(i) and 7(i) that $\leqslant$ is antisymmetric in IM1 as well as in IM2 (under the relevant definitions), so both theories will be extensional with regard to their parthood relations. While EP is not assumed as an axiom in either theory, it is derivable as a theorem in both.

Importantly, the same will be true of EO as soon as we assume Strong Supplementation. Theorems 1(v) and 7(v) tell us that in both theories $\leqslant$ is a stable relation. Thus, although $\neg\neg$-elimination is not intuitionistically available as a general rule of inference, in IM1 and IM2 the special case corresponding to the inference from $\neg\neg\, x \leqslant y$ to $x \leqslant y$ is legitimate. Both theories may therefore rely on the standard *reductio* argument mentioned above to obtain a derivation of the Overlap Principle from Strong Supplementation, which can certainly be added to the relevant set of proper axioms. Given OP, EO will then follow in

[13] Here, again, we are indebted to a referee for bringing this point to our attention.

both cases by the antisymmetry of $\leqslant$, i.e., effectively, by EP. Of course, strictly speaking there is more than one way of understanding Strong Supplementation intuitionistically, depending on the primitive we choose: in IM1 we have two distinct formulations, corresponding to SS1 and SS2; in IM2 we can recast the standard formulation SS. As it turns out, however, OP will follow equally from each of these principles.

Theorem 10. *(i) SS1 $\to$ OP and SS2 $\to$ OP are derivable in IM1; (ii) SS $\to$ OP is derivable in IM2.*

Proof. With reference to (i), we know from Theorem 4(vii) that SS1 $\to$ SS2 is derivable in IM1. Thus, SS1 $\to$ OP follows directly from SS2 $\to$ OP, for which we have the following derivation. (We assume as given a derivation $\mathscr{D}$ of $z \circ x$ from $z \leqslant x$, which is straightforward given the reflexivity of $\leqslant$; see Theorem 1(i).)

$$
\begin{prooftree}
\AxiomC{$[x \nleqslant y]^{2}$}
\RightLabel{(SS2)}
\UnaryInfC{$\exists z(z \leqslant x \wedge \neg\, z \circ y)$}
\AxiomC{$[z \leqslant x \wedge \neg\, z \circ y]^{1}$}
\UnaryInfC{$z \leqslant x$}
\noLine
\UnaryInfC{$\vdots\ \mathscr{D}$}
\noLine
\UnaryInfC{$z \circ x$}
\AxiomC{$[\forall z(z \circ x \to z \circ y)]^{3}$}
\UnaryInfC{$z \circ x \to z \circ y$}
\BinaryInfC{$z \circ y$}
\AxiomC{$[z \leqslant x \wedge \neg\, z \circ y]^{1}$}
\UnaryInfC{$\neg\, z \circ y$}
\BinaryInfC{$\bot$}
\RightLabel{1}
\BinaryInfC{$\bot$}
\RightLabel{2}
\UnaryInfC{$\neg\, x \nleqslant y$}
\RightLabel{(8)}
\UnaryInfC{$x \leqslant y$}
\RightLabel{3}
\UnaryInfC{$\forall z(z \circ x \to z \circ y) \to x \leqslant y$}
\end{prooftree}
$$

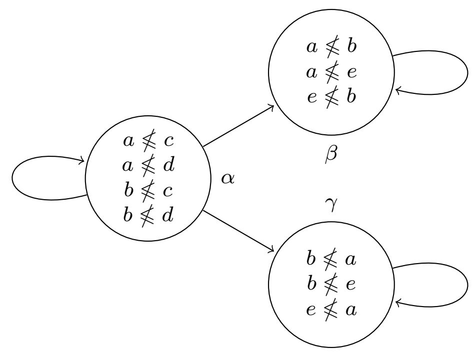
Note that the second-last step, from $\neg\, x \nleqslant y$ to $x \leqslant y$, is precisely the special case of $\neg\neg$-elimination licensed by definition (8). As for (ii), the proof is similar (using definition (31)) and left to the reader. □

With regard to EPP, IM1 and IM2 behave differently. The standard *reductio* argument to obtain the Proper Parts Principle does not only rely on the inference from $\neg\neg\, x \leqslant y$ to $x \leqslant y$, which is admissible in both theories; it also makes use of the classical conditional $x \leqslant y \to x < y \vee x = y$, which, as we saw, is available only in IM2 (see our earlier discussion of (19) and (38)). This difference happens to be crucial.

On the one hand, it turns out that IM1 is indeed too weak to derive PPP from SS1 (and hence, *a fortiori*, from SS2). More precisely, there are four ways of recasting PPP in IM1, one for each proper$_i$ parthood predicate $<_i$:

$$\exists z\, z <_i x \to (\forall z(z <_i x \to z <_i y) \to x \leqslant y) \qquad (\mathrm{PPP}_i)$$

In view of Theorem 2, this really gives us two non-equivalent principles in the language of IM1, and neither turns out to be derivable. Here, for instance, is a model that illustrates the point for $i = 4$.



Given definition (14), this is a model of IM1 where a has exactly two proper$_4$ parts at world α, namely c and d, both of which are also proper$_4$ parts of b. Yet a is not, at α, part of b, since $a \nleqslant b$ holds at β and therefore $\neg\, a \nleqslant b$, which amounts to $a \leqslant b$ by definition (8), holds at neither world. It follows that α violates the conditional $\exists z\, z <_4 a \to (\forall z(z <_4 a \to z <_4 b)) \to a \leqslant b)$, which is an instance of PPP$_4$. (A perfectly parallel story applies to $b \leqslant a$.) However, it's easy to check that SS1 does hold at α. In particular, since $\neg\, a \nleqslant b$ holds at γ, $\neg\neg\, a \nleqslant b$ does not hold at α and hence $\neg\, a \leqslant b$ does not hold either (again by definition (8)), so the conditional $\neg\, a \leqslant b \to \exists z(z \leqslant a \wedge \neg\, z \circ b)$ holds vacuously. It is also easy to check that, thanks to e, SS1 may hold (non-vacuously) at β and at γ as well, e.g. when e is atomic and disjoint from c and d. Then SS1 will hold at every world, and since SS1 implies SS2 by Theorem 4(*vii*), the same is true of SS2. The model will satisfy SS1 and SS2 but not PPP$_4$. Similar models will establish the same result for the other PPP$_i$'s.

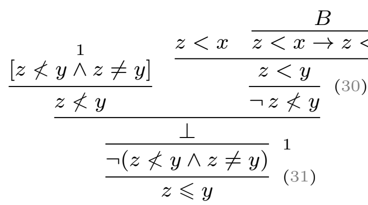
By contrast, IM2 turns out to be strong enough to warrant the derivation: PPP follows directly from Strong Supplementation, as in classical mereology.

Theorem 11. *SS → PPP is derivable in IM2.*

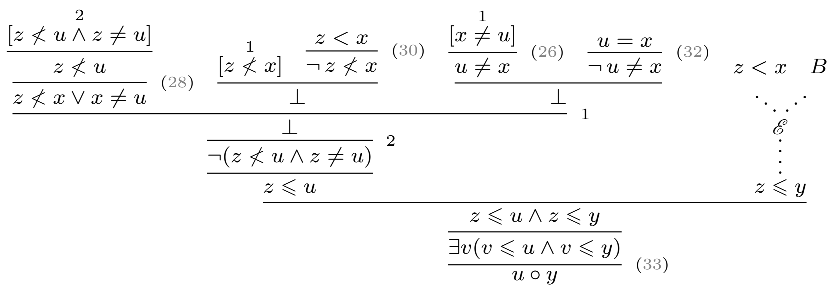
Proof. The derivation is a bit long, so we shall split it into four steps. Let A be $\exists z\, z < x$ and let B be $\forall z(z < x \to z < y)$. We want to construct a derivation of $A \to (B \to x \leqslant y)$. We begin with a derivation $\mathscr{D}$ of $u \circ y$ from B and $u < x$.

$$
\dfrac{
  \dfrac{
    u \leqslant u
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{\overset{1}{[u \nless y \wedge u \neq y]}}{u \nless y}
          \qquad
          \dfrac{\dfrac{u < x \qquad \dfrac{B}{u < x \to u < y}}{u < y}}{\neg\, u \nless y}\,(30)
        }{\bot}
      }{\neg(u \nless y \wedge u \neq y)}\,1
    }{u \leqslant y}\,(31)
  }{
    \dfrac{u \leqslant u \wedge u \leqslant y}{\exists v(v \leqslant u \wedge v \leqslant y)}
  }
}{u \circ y}\,(33)
$$

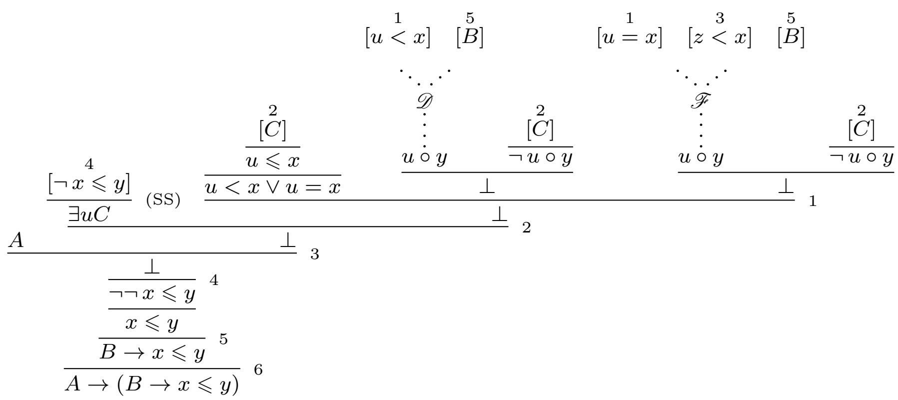
(Here the left-most leaf, $u \leqslant u$, comes from the reflexivity of $\leqslant$ established in Theorem 7(*i*).) Next, consider a derivation $\mathscr{E}$ of $z \leqslant y$ from B and $z < x$.



We now use $\mathscr{E}$ to construct a derivation $\mathscr{F}$ of $u \circ y$ from B, $z < x$, and $u = x$.



Finally, using $\mathscr{D}$ and $\mathscr{F}$, and abbreviating $u \leqslant x \wedge \neg\, u \circ y$ as C, we obtain the following derivation of the entire Proper Parts Principle $A \to (B \to x \leqslant y)$ from the relevant instance of Strong Supplementation, $\neg x \leqslant y \to \exists u C$.



Notice that the inference step from $u \leqslant x$ to $u < x \vee u = x$ in this final derivation requires an instance of the conditional $x \leqslant y \to x < y \vee x = y$. This is precisely the IM2-theorem (38) noted above, whose $<_i$-counterparts do not hold in IM1 for any i. □

Given Theorem 11, the derivability of EPP in IM2 + SS is now an immediate consequence of the antisymmetry of $\leqslant$, i.e., effectively, of EP, exactly as with the derivation of EO. This is how the three extensionality principles are related also in classical mereology. Since IM1 + SS1 and IM1 + SS2 fail to derive PPP, and EPP with it, we conclude that it is the theory of weak excess that provides the best resources for a natural intuitionistic counterpart of classical extensional mereology.[14] In IM1 we would achieve full extensionality only by assuming PPP explicitly, as a further axiom along with SS1 or SS2. While perfectly legitimate in its own right, such a move would defy the classical way of understanding extensionality—as a property of any parthood relation that is antisymmetric and strongly supplemented.

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