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Triangular bounded consistency of fuzzy preference relations

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Abstract

There are typically two types of consistency of fuzzy preference relations (FPR), namely additive and multiplicative consistency. They are defined based on the assumption that decision makers are rational and can provide strictly consistent FPRs. To take into consideration the bounded rationality of decision makers, the current study relaxes this assumption and proposes a new measure called triangular bounded consistency. To define triangular bounded consistency, a directed triangle is used to represent three FPRs among any three alternatives, with each directed edge representing an FPR. The condition of restricted max–max transitivity (RMMT) in the directed triangle is quantitatively examined. Under the assumption that the bounded rationality of a decision maker is characterized by their historical FPRs, which are represented by directed triangles that satisfy RMMT, triangular bounded consistency is determined using the historical FPRs. We then illustrate how triangular bounded consistency can be used to verify the consistency of FPRs that are newly provided by decision makers and how to estimate some missing FPRs that are not provided by decision makers. Finally, to demonstrate the application of triangular bounded consistency of FPRs in multi-attribute decision analysis, we investigate a problem that involves selecting areas to market products for a company.

Keywords: Triangular bounded consistency; Fuzzy preference relation; Restricted max–max transitivity; Bounded rationality; Multi-attribute decision analysis

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1. Introduction

Decision makers generally use preference relations to express their preference information when analyzing decision problems (Herrera et al., 2001; Massanet et al., 2016; Wan et al., 2018). The two types of preference relations commonly accepted in the context of pairwise comparison of alternative courses of action (or simply, alternatives) (Genç et al., 2010; Chen et al., 2015; Wan et al., 2017) are multiplicative preference relation (MPR) (Saaty, 1977) and fuzzy preference relation (FPR) (Orlovsky, 1978). As the two types of preference relations can be transformed into each other (Herrera-Viedma et al., 2004), this study focuses on FPR. FPR was first proposed by Orlovsky (1978) to represent the opinion of a decision maker when comparing a set of alternatives. FPR can not only reflect whether an alternative is superior to another but also characterize the degree to which an alternative is preferred (Pan et al., 2017; Zhang et al., 2018). The full definition of FPR is given in Section 2.

Consistency is a prerequisite for using FPRs to model and analyze decision problems (Chen et al., 2014; Li et al., 2018). Verifying whether the required consistency is reached is an important step in the process of analyzing decision problems modeled by FPRs (Ureña et al., 2015; Zhang et al., 2016). Decisions made using inconsistent FPRs may be irrational and of poor quality (Xu et al., 2016; Herowati et al., 2017). A basic requirement for consistency is transitivity. Different types of transitivity, such as weak transitivity, triangular condition, and restricted max–max transitivity (RMMT), have been defined and investigated in existing studies (Liao and Xu, 2015; Liu et al., 2018). RMMT is stricter than weak transitivity and triangular condition (see Section 2) and is

considered as a necessary condition of consistent FPRs (Liu et al., 2014).

To verify the consistency of FPRs, different types of consistency measure of FPR have been developed (Chiclana et al., 2009; Deng et al., 2014; Deng et al., 2015). Representative and commonly accepted types include additive consistency (Pérez et al., 2016; Al Salema and Awasthi, 2018) and multiplicative consistency (Wu and Chiclana, 2014; Krejčí, 2017). Both of these types of consistency satisfy RMMT (Herrera-Viedma et al., 2004). More importantly, they are based on the assumption that decision makers are strictly rational, and thus the FPRs strictly satisfy the given mathematical conditions for the two types of consistency (Herrera-Viedma et al., 2004; Xu et al., 2014). However, it is very difficult or even impossible in practice for decision makers to offer such exact FPRs as they may not have access to all the information required; even if they do, they may not be able to process it properly within a given time limit (Wang and Chen, 2008; Yan and Ma, 2015). This phenomenon is referred to as bounded rationality by Simon (1982). Therefore, in many situations, decision makers may seek to offer "good enough" judgments that may not be perfectly rational or consistent, but reasonably acceptable. Hence, we must consider which judgements are reasonably acceptable and can be used for decision-making.

Therefore, we propose a new consistency concept called triangular bounded consistency to measure the consistency of FPRs. Firstly, the difference between the two commonly accepted types of consistency of FPR, i.e., additive and multiplicative consistency, and then their limitations are analyzed. Secondly, inspired by triangular condition (Dasgupta and Deb, 1996), which is a type of transitivity of FPR, we propose the use of a directed triangle to represent the three FPRs among any three alternatives in which each FPR is represented by a directed edge. Because RMMT is considered as a necessary condition for consistent FPRs, we then examine what the

triangles should look like if the FPRs they represent satisfy RMMT by specifying the range of angles formed by two adjacent edges. Here, adjacent edges represent the two directed edges whose arrows do not both leave or point to the angles they form. That is, the arrow of one directed edge should point to the angle, whereas the arrow of the other edge should point away from the angle. When the collected preference information satisfies RMMT, it is regarded as reasonably acceptable. Otherwise, it should be further revised or ignored. In other words, we assume that all decision makers are at least reasonably rational in the sense that their preferences satisfy RMMT.

Thirdly, we introduce the concept of triangular bounded consistency to measure the consistency of FPRs provided by a decision maker. To estimate triangular bounded consistency of given FPRs, the pairwise preference judgments provided by the decision maker previously, or currently to a set of sample alternatives if no previous judgments are available, should be collected. From the collected information, the preferences satisfying the RMMT can be represented by directed triangles and then used to estimate the lower and upper bounds of triangular bounded consistency of FPRs for this particular decision maker.

One important use of triangular bounded consistency is estimating missing FPRs, as decision makers generally offer only minimal preference information (Sen and Yang, 1994) or an acceptable incomplete FPR matrix (which is defined in Section 2) to avoid inconsistency issues when there are many alternatives under consideration. For example, among alternatives i, j, and k, a decision maker may specify how much they prefer i to j and j to k without mentioning their preference between i and k. When this is the case, we can estimate their preferences between i and k using triangular bounded consistency estimated for them. As the estimated preference is normally a range, to find a point estimate of the missing FPR, an optimization model is

constructed using the least squares method to minimize the difference between the point estimate and the decision maker's usual or historical consistency behavior characterized by the middle point of the range of their triangular bounded consistency.

To demonstrate the application of triangular bounded consistency, we applied this concept to analyze multi-attribute decision analysis (MADA) problems modeled by acceptable incomplete FPR matrices. The main contributions of this study can be summarized as follows: (1) we analyze the difference between additive and multiplicative consistency of FPRs, (2) we specify the value range of the angle formed by two adjacent edges of the directed triangle representing FPRs that satisfy RMMT, (3) we introduce a new consistency concept called triangular bounded consistency to verify the consistency of newly provided FPRs by a decision maker to follow their usual consistency behavior, (4) we apply triangular bounded consistency to find the missing FPRs in an incomplete FPR matrix provided by a decision maker, and (5) we develop a solution process to MADA problems modeled by acceptable incomplete FPR matrices.

The remainder of this paper is organized as follows. Section 2 presents the basics of our proposed consistency concept. Section 3 provides an analysis of additive and multiplicative consistency of FPRs. Section 4 introduces triangular bounded consistency and uses it to estimate missing elements in acceptable incomplete FPR matrices. Section 5 describes the application of triangular bounded consistency to MADA problems modeled by FPRs. This application is demonstrated in Section 6 by analyzing a problem of selecting areas to market the products of a company. Finally, Section 7 presents conclusions and future work to be considered.

2. Preliminaries

In this section, we introduce the basic concepts of FPRs, some representative definitions of consistency and transitivity of FPRs, and the validity judgment on an incomplete FPR matrix with missing elements.

Definition 1. (Orlovsky, 1978) Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of alternatives. $P = (p_{ij})_{n \times n}$ is called an FPR matrix on $X \times X$ such that $p_{ij} \in [0,1]$, $p_{ij} + p_{ji} = 1$, $\forall i, j \in \{1, \ldots, n\}$, where p_{ij} denotes the degree to which alternative x_i is preferred to x_j. Specifically, $p_{ij} = 0.5$ indicates an indifference between x_i and x_j (denoted as $x_i : x_j$), $0.5 < p_{ij} < 1$ indicates that x_i is preferred to x_j (denoted as x_i f x_j), and $p_{ij} = 1$ indicates that x_i is absolutely preferred to x_j.

As mentioned in Section 1, consistency is an important property with respect to FPRs. In previous studies, the two commonly accepted definitions of consistency, namely additive and multiplicative consistency, were used to determine whether an FPR matrix is consistent.

Definition 2. (Tanino, 1984) Given a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$. P is considered additively consistent if it satisfies the following condition:

$$p_{ij} + p_{jk} + p_{ki} = 1.5, \quad \forall i, j, k \in \{1, \ldots, n\}. \tag{1}$$

Definition 3. (Tanino, 1984) Given a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$. P is considered multiplicatively consistent if it satisfies the following condition:

$$p_{ij} \cdot p_{jk} \cdot p_{ki} = p_{ik} \cdot p_{kj} \cdot p_{ji}, \quad \forall i, j, k \in \{1, \ldots, n\}. \tag{2}$$

From the above two definitions, it can be observed that additive and multiplicative consistency are very strict conditions for FPRs to satisfy. Compared with the two types of consistency, transitivity is a weaker condition for FPRs to satisfy. Representative types of transitivity include weak transitivity, RMMT, and triangular condition.

Definition 4. (Tanino, 1984; Herrera-Viedma et al., 2004) Given a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$. P is said to be weakly transitive if it satisfies the following conditions:

(1) if $0.5 \leq p_{ij} \leq 1$ and $0.5 \leq p_{jk} \leq 1$, then $0.5 \leq p_{ik} \leq 1$, $\forall i, j, k \in \{1, \ldots, n\}$, and

(2) if $0 \leq p_{ij} \leq 0.5$ and $0 \leq p_{jk} \leq 0.5$, then $0 \leq p_{ik} \leq 0.5$, $\forall i, j, k \in \{1, \ldots, n\}$.

Definition 5. (Herrera-Viedma et al., 2004) Given a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$. P is said to have RMMT if it satisfies the following conditions:

(1) if $0.5 \leq p_{ij} \leq 1$ and $0.5 \leq p_{jk} \leq 1$, then $max\{p_{ij}, p_{jk}\} \leq p_{ik} \leq 1$, $\forall i, j, k \in \{1, \ldots, n\}$, and

(2) if $0 \leq p_{ij} \leq 0.5$ and $0 \leq p_{jk} \leq 0.5$, then $0 \leq p_{ik} \leq min\{p_{ij}, p_{jk}\}$, $\forall i, j, k \in \{1, \ldots, n\}$.

Definition 6. (Dasgupta and Deb, 1996; Herrera-Viedma et al., 2004) Given a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$. P is said to satisfy triangular condition if

$$p_{ij} + p_{jk} \geq p_{ik}, \quad \forall i, j, k \in \{1, \ldots, n\}. \tag{3}$$

RMMT is stricter than weak transitivity and triangular condition, and it is generally regarded as a necessary condition for a consistent FPR matrix. An FPR matrix satisfying additive or multiplicative consistency implies that RMMT is satisfied in the matrix, which is formally described in the following theorems.

Theorem 1. (Herrera-Viedma et al., 2004) Given a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$. If P satisfies multiplicative consistency, i.e., if it satisfies

$p_{ij} \cdot p_{jk} \cdot p_{ki} = p_{ik} \cdot p_{kj} \cdot p_{ji}$, $\forall i, j, k \in \{1, \ldots, n\}$, $i \neq j \neq k$, then P must satisfy RMMT.

Theorem 2. (Herrera-Viedma et al., 2004; Liao and Xu, 2014) Given a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$. If P satisfies additive consistency, i.e., if it

satisfies $p_{ij} + p_{jk} + p_{ik} = 1.5$, $\forall\, i, j, k \in \{1, \ldots, n\}$, $i \neq j \neq k$, then P must satisfy RMMT.

When some elements in an FPR matrix P cannot be provided by a decision maker, P is called an incomplete FPR matrix.

Definition 7. (Ureña et al., 2015) Given a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$. If some elements other than the diagonal ones in P cannot be provided, then P is called an incomplete FPR matrix, which satisfies

$$p_{ij} \in [0,1],\ p_{ij} + p_{ji} = 1,\ \forall\, i, j \in \{1, \ldots, n\}. \tag{4}$$

Next, we discuss whether an incomplete FPR matrix is acceptable or not.

Definition 8. (Xu et al., 2014) Given a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, let $P = (p_{ij})_{n \times n}$ be an incomplete FPR matrix on $X \times X$. If the missing elements of P can be determined by its provided elements, then P is called an acceptable incomplete FPR matrix; otherwise, P is not an acceptable incomplete FPR matrix.

Theorem 3. (Xu et al., 2014) Given a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$, let $P = (p_{ij})_{n \times n}$ be an incomplete FPR matrix on $X \times X$. Only when there exists at least one provided element in each row or column of P, with exception of the diagonal elements ($p_{ii} = 0.5$, $i = 1, \ldots, n$), the matrix can be considered acceptable.

3. Analysis of existing consistency of FPRs

Section 2 presents the two commonly accepted types of consistency of FPRs, i.e., additive and multiplicative consistency. In the following, they are analyzed to show their difference and possible limitations in real applications.

Given an MPR matrix $A = (a_{ij})_{n \times n}$ on $X \times X$ with $a_{ij} \in [1/9, 9]$, where $X = \{x_1, x_2, \ldots, x_n\}$ denotes a set of alternatives, it is said to be consistent if we have $a_{ij} \cdot a_{jk} = a_{ik}$, $\forall\, i, j, k$. Owing to the

transformation of an MPR matrix to an FPR matrix, which is demonstrated by Proposition B.1 of Section B.1 in Appendix B of the supplementary material, a consistent MPR matrix can result in a consistent FPR matrix satisfying the condition in Definition 2 presented in Section 2. It can be inferred from the equivalence between the consistency of an MPR matrix and the additive consistency of an FPR matrix that a decision maker is said to be perfectly rational when they provide a consistent FPR matrix. However, multiplicative consistency of an FPR matrix cannot reflect the perfect rationality of a decision maker because a difference exists between additive and multiplicative consistency of an FPR matrix. To facilitate a quantitative measurement of the deviation between additive and multiplicative consistency, the following proposition is given first.

Proposition 1. Suppose that $f(x, y) = \dfrac{xy}{1 - x - y + 2xy} - (x + y - 0.5)$ is a two-variable function with $0 \leq x, y \leq 1$. Then, the function satisfies

(1) when $x, y \in [0.5, 1]$, $f(x, y)$ is a monotonously decreasing function with respect to x or y.

(2) when $x, y \in [0, 0.5]$, $f(x, y)$ is a monotonously decreasing function with respect to x or y.

(3) when $(x - 0.5) \cdot (y - 0.5) \leq 0$, the lower bound of $f(x, y)$ tends to -0.5 if $(x, y) = (0, 1^-)$ or $(x, y) = (1^-, 0)$ while the upper bound of $f(x, y)$ tends to 0.5 if $(x, y) = (0^+, 1)$ or $(x, y) = (1, 0^+)$.

(4) when $(x - 0.5) \cdot (y - 0.5) \geq 0$, $f(x, y)$ reaches the maximum value, i.e., -0.5, if $(x, y) = (1, 1)$, while $f(x, y)$ reaches the minimum value, i.e., 0.5, if $(x, y) = (0, 0)$.

The proof of Proposition 1 is presented in Section A.1 of Appendix A in the supplementary material. It can be deduced from Definitions 2 and 3 presented in Section 2 that $p_{ik} = p_{ij} + p_{jk} - 0.5$ ($\forall i, j, k \in \{1, \ldots, n\}$) holds in an additively consistent FPR matrix, and $p_{ik} = \dfrac{p_{ij} p_{jk}}{1 - p_{ij} - p_{jk} + 2 p_{ij} p_{jk}}$ ($\forall i, j, k \in \{1, \ldots, n\}$) holds in a multiplicatively consistent FPR matrix.

Assume that $p_{ik}^{a} = p_{ij} + p_{jk} - 0.5$ ($\forall i, j, k \in \{1, \ldots, n\}$) is an additively consistent FPR matrix,

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and $p_{ik}^m = \dfrac{p_{ij}p_{jk}}{1-p_{ij}-p_{jk}+2p_{ij}p_{jk}}$ ($\forall i, j, k \in \{1, \ldots, n\}$) is a multiplicatively consistent FPR matrix. Then, the function $f(x, y)$ in Proposition 1 can represent the deviation between p_{ik}^m and p_{ik}^a given that p_{ij} and p_{jk} are two variables. Proposition 1 also reveals that $-0.5 \le p_{ik}^m - p_{ik}^a \le 0.5$ when $(p_{ij} - 0.5)\cdot(p_{jk} - 0.5) \ge 0$, and $-0.5 < p_{ik}^m - p_{ik}^a < 0.5$ when $(p_{ij} - 0.5)\cdot(p_{jk} - 0.5) \le 0$. It can be easily concluded that multiplicative consistency of an FPR matrix is different from additive consistency of the matrix, particularly when $(p_{ij} - 0.5)\cdot(p_{jk} - 0.5) \le 0$ ($\forall i, j, k \in \{1, \ldots, n\}$). As such, multiplicative consistency is incapable of characterizing the perfect rationality of a decision maker.

Example 1. For a set of alternatives $X = \{x_1, \ldots, x_5\}$, suppose that $P = (p_{ij})_{5\times5}$ is an FPR matrix on $X \times X$ with $p_{12} = 0.7$, $p_{23} = 0.6$, $p_{34} = 0.98$, and $p_{45} = 0.1$. The deviations between p_{13}^m and p_{13}^a and between p_{35}^m and p_{35}^a are calculated in the following.

As for p_{12} and p_{23}, we have $p_{13}^a = p_{12} + p_{23} - 0.5 = 0.8$ and $p_{13}^m = \dfrac{p_{12}p_{23}}{1-p_{12}-p_{23}+2p_{12}p_{23}} = 0.78$; therefore, the deviation between p_{13}^m and p_{13}^a is -0.02. When p_{34} and p_{45} are used to consistently decide p_{35}, it can be similarly calculated that $p_{35}^a = p_{34} + p_{45} - 0.5 = 0.58$ and $p_{35}^m = \dfrac{p_{34}p_{45}}{1-p_{34}-p_{45}+2p_{34}p_{45}} = 0.84$, which implies that $p_{35}^m - p_{35}^a = 0.26$. The results in these two situations are clearly limited to the lower and upper bounds given in Proposition 1.

Specifically, $p_{12} = 0$ and $p_{23} = 0.999$ lead to $p_{13}^m - p_{13}^a = -0.499$, and $p_{34} = 0.001$ and $p_{45} = 1$ result in $p_{35}^m - p_{35}^a = 0.499$. This verifies Proposition 1 under the condition that $(p_{ij} - 0.5)\cdot(p_{jk} - 0.5) \le 0$. As a whole, a clear difference can be noted between multiplicative and additive consistency of an FPR matrix from perfect rationality.

In addition, another important difference is noted between multiplicative and additive consistency of an FPR matrix. That is, given p_{ij} and p_{jk}, the domain of p_{ik} for multiplicative

consistency is different from the domain of p_{ik} for additive consistency. To facilitate such comparison, the following proposition is given first.

Proposition 2. Suppose that $f(x, y) = \dfrac{xy}{1 - x - y + 2xy}$ is a two-variable function with $0 \le x, y \le 1$. Then, the function is monotonously increasing with respect to x or y.

The proof of Proposition 2 is presented in Section A.2 of Appendix A. This proposition indicates that p_{ik} generated by multiplicative consistency is always limited to the interval [0, 1] given p_{ij} and p_{jk} such that $0 \le p_{ij}, p_{jk} \le 1$. In detail, it can be determined from Proposition 2 that p_{ik} = 0 when $(p_{ij}, p_{jk}) = (0, 0)$, and $p_{ik} = 1$ when $(p_{ij}, p_{jk}) = (1, 1)$ under the condition that multiplicative consistency is satisfied. On the other hand, when p_{ij} and p_{jk} such that $0 \le p_{ij}, p_{jk} \le 1$ are given to determine p_{ik} using additive consistency, the range of p_{ik} is limited to the interval [−1.5, 1.5]. When $(p_{ij}, p_{jk}) = (0, 0)$, p_{ik} is equal to −1.5, whereas p_{ik} becomes 1.5 when $(p_{ij}, p_{jk}) = (1, 1)$. As a result, the range of p_{ik} from additive consistency is clearly different from the range of multiplicative consistency given p_{ij} and p_{jk} such that $0 \le p_{ij}, p_{jk} \le 1$.

Although the range of p_{ik} from additive consistency given p_{ij} and p_{jk} such that $0 \le p_{ij}, p_{jk} \le 1$ can be mapped into the interval [0,1] in some way (Herrera-Viedma et al., 2004), the decision maker providing the FPRs is assumed to be perfectly rational. Meanwhile, the assumption of perfect rationality is relaxed when multiplicative consistency is applied. In real applications, it is very difficult or even impossible for a decision maker to be perfectly rational and provide an additively consistent FPR matrix. More importantly, both additive and multiplicative consistency cannot reflect the real preference of a decision maker with bounded rationality. In other words, for any pair of p_{ij} and p_{jk}, p_{ik} can be generated directly by additive or multiplicative consistency without consideration of the preference of a decision maker. As such, the resulting p_{ik} cannot be

guaranteed to be the one that the decision maker wishes to provide. In the context of decision-making, the direct application of additive and multiplicative consistency is doubtful, owing to a lack of consideration of the preferences of a decision maker with bounded rationality. For these reasons, we develop a new consistency measure of an FPR matrix based on triangular condition in consideration of the preference information of a decision maker with bounded rationality, as described in the next section.

4. Triangular bounded consistency

In this section, starting from triangular condition in Definition 6 presented in Section 2, we quantitatively explain the relationship among three edges of a valid triangle and the RMMT in a triangular context. Then, we propose triangular bounded consistency of an FPR matrix with coverage of the preferences of a decision maker.

4.1 Quantitative description of triangular condition

In Definition 6, the basic condition of a triangle is used to geometrically explain the consistency condition of an FPR matrix. In the following, we quantitatively describe the relationship among three edges of a directed triangle with the angle between two edges.

From an abstract angle, suppose that x, y, and z with $0 \leq x, y, z \leq 2$ denote the three edges of a directed triangle $\triangle ABC$, i.e., edges $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CA}$ in Fig. 1. The angle between x and y is denoted by θ.
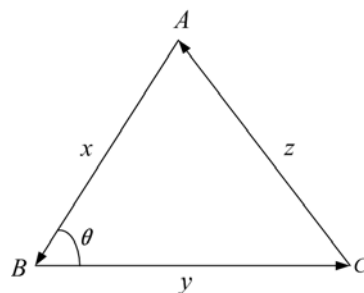
Fig. 1. Directed triangle $\triangle ABC$.

When x, y, and z form a triangle, they must satisfy the following conditions:

(1) $x + y \geq z$,

(2) $x + z \geq y$, and

(3) $y + z \geq x$.

From the above conditions, we can deduce that $|x - y| \leq z \leq x + y$. The angle between x and y

can be constructed from x, y, and z as $\theta = \arccos \dfrac{x^2 + y^2 - z^2}{2xy}$. The relationship between θ and z

and the relationship between θ and x (or y) are revealed in the following.

Property 1. Suppose that x, y, and z with $0 \leq x, y, z \leq 2$ are the lengths of three edges of the

triangle $\triangle ABC$ plotted in Fig. 1 with the angle between edges $\overrightarrow{AB}$ and $\overrightarrow{BC}$ denoted by θ. Then,

we have

(1) θ is monotonously increasing with respect to z, and

(2) θ is monotonously decreasing with respect to x or y when $z \geq \sqrt{\left|x^2 - y^2\right|}$.

The proof of Property 1 is presented in Section A.3 of Appendix A. Given x and y such that $0 \leq$

$x, y \leq 2$ and $x > y$, it can be determined that $x - y \leq z \leq x + y$. In this situation, we can deduce from

Property 1 that $0 \leq \theta \leq \pi$. This is formally presented in the following.

Property 2. Suppose that x, y, and z, such that $0 \leq x, y, z \leq 2$ and $x > y$, are the lengths of three

edges of the triangle $\triangle ABC$ plotted in Fig. 1 with the angle between edges $\overrightarrow{AB}$ and $\overrightarrow{BC}$

denoted by θ. Then, we have

(1) $0 \leq \theta \leq \pi$,

(2) $\theta = 0$ when $z = x - y$, and

(3) $\theta = \pi$ when $z = x + y$.

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4.2 Description of RMMT in directed triangles of three FPRs

As demonstrated in Section 2, although RMMT is covered by additive and multiplicative consistency, it is stricter than weak transitivity and triangular condition. In the developed triangular bounded consistency of an FPR matrix, it is also intended to satisfy RMMT.

Given an FPR matrix $P = (p_{ij})_{n\times n}$ on $X \times X$, where $X = \{x_1, x_2, \ldots, x_n\}$ denotes a set of alternatives, assume that x_i, x_j, and x_k are the three vertices of a triangle, and FPRs p_{ij}, p_{jk}, and p_{ik} represent the lengths of directed edges from vertexes x_i to x_j, x_j to x_k, and x_i to x_k, respectively. The directed edge from vertex x_i to x_j means that alternative x_i is superior or equivalent to x_j, which results in $p_{ij} \geq 0.5$. In general, there are two situations where $(p_{ij} - 0.5) \cdot (p_{jk} - 0.5) \geq 0$ and $(p_{ij} - 0.5) \cdot (p_{jk} - 0.5) \leq 0$, as plotted in Fig. 2. In the first situation, alternative x_i is superior or equivalent to x_j, and x_j is superior or equivalent to x_k; thus, x_i is certainly superior or equivalent to x_k. In other words, $p_{ij} \geq 0.5$ and $p_{jk} \geq 0.5$ must lead to $p_{ik} \geq 0.5$. In the second situation, alternative x_i is superior or equivalent to x_j, and x_j is inferior or equivalent to x_k; whether x_i is superior or equivalent to x_k depends on whether p_{ij} is larger than or equal to p_{kj} (i.e., $1 - p_{jk}$). Situation (*b*) in Fig. 2 indicates that $p_{ij} \geq p_{kj}$. In the following, we discuss RMMT in these two situations.

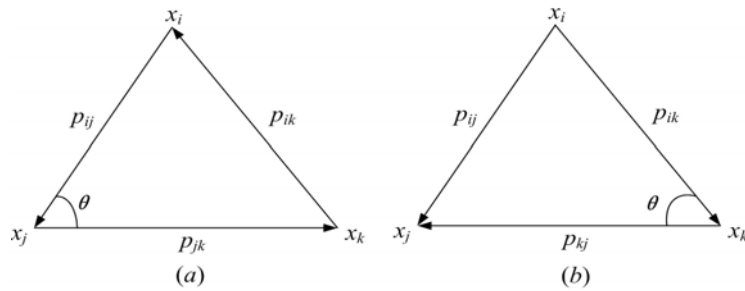

Fig. 2. Two situations of directed triangle $\triangle x_i x_j x_k$.

4.2.1 RMMT in the first situation

In the situation of Fig. 2 (*a*), as demonstrated in Section 4.1, given p_{ij} and p_{jk}, if the combination

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of p_{ik} with p_{ij} and p_{jk} can form a triangle, then we can determine that $|p_{ij} - p_{jk}| \leq p_{ik} \leq p_{ij} + p_{jk}$. This condition covers RMMT.

Proposition 3. Let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$, where $X = \{x_1, x_2, \dots, x_n\}$ denotes a set of alternatives. For any three alternatives x_i, x_j, and x_k ($i \neq j \neq k$), when p_{ij} (the degree to which x_i is superior to x_j) and p_{jk} (the degree to which x_j is superior to x_k) are known, if p_{ij}, p_{jk}, and p_{ik} satisfy RMMT, then the three edges with lengths of p_{ij}, p_{jk}, and p_{ik} must form a triangle.

Given p_{ij} and p_{jk} such that $p_{ij} \geq 0.5$ and $p_{jk} \geq 0.5$ in Fig. 2 (a), under the assumption that RMMT is satisfied, Proposition 3 clearly holds because $|p_{ij} - p_{jk}| < max\{p_{ij}, p_{jk}\}$ and $1 < p_{ij} + p_{jk}$. In such a triangle composed of p_{ij}, p_{jk}, and p_{ik}, the angle between edge $\overrightarrow{x_i x_j}$ with length of p_{ij} and edge $\overrightarrow{x_j x_k}$ with length of p_{jk} is determined by

$$\theta = \arccos \frac{p_{ij}^2 + p_{jk}^2 - p_{ik}^2}{2 p_{ij} p_{jk}}. \tag{5}$$

When p_{ik} is limited to $[max\{p_{ij}, p_{jk}\}, 1]$, the range of θ can be determined.

Property 3. Let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$, where $X = \{x_1, x_2, \dots, x_n\}$ denotes a set of alternatives. For any three alternatives x_i, x_j, and x_k ($i \neq j \neq k$), if their pairwise comparisons p_{ij}, p_{jk}, and p_{ik} satisfy $0.5 \leq p_{ij}, p_{jk} \leq 1$ and $max\{p_{ij}, p_{jk}\} \leq p_{ik} \leq 1$, then we can determine that the angle between edge $\overrightarrow{x_i x_j}$ with length p_{ij} and edge $\overrightarrow{x_j x_k}$ with length p_{jk} in triangle $\triangle x_i x_j x_k$ satisfies

$$\arccos \frac{min\{p_{ij}, p_{jk}\}}{2 max\{p_{ij}, p_{jk}\}} \leq \theta \leq \arccos \frac{p_{ij}^2 + p_{jk}^2 - 1}{2 p_{ij} p_{jk}}, \tag{6}$$

$$\pi/3 \leq \arccos \frac{min\{p_{ij}, p_{jk}\}}{2 max\{p_{ij}, p_{jk}\}} \leq 0.4196\pi, \text{ and} \tag{7}$$

$$\pi/3 \leq \arccos \frac{p_{ij}^2 + p_{jk}^2 - 1}{2 p_{ij} p_{jk}} \leq \pi. \tag{8}$$

The proof of Property 3 is presented in Section A.4 of Appendix A. In order to clearly

demonstrate Property 3, Example B.1 is given in Section B.3 of Appendix B in the supplementary material. In triangle $\triangle x_i x_j x_k$, where the lengths of edges $\overrightarrow{x_i x_j}$ and $\overrightarrow{x_j x_k}$ (p_{ij} and p_{jk}, respectively) are provided, and the length of edge $\overrightarrow{x_i x_k}$ (p_{jk}) is unknown, when $p_{ij} \geq 0.5$ and $p_{jk} \geq 0.5$, p_{ik} can be calculated under the assumption that RMMT is satisfied.

Definition 9. Let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$, where $X = \{x_1, x_2, \ldots, x_n\}$ denotes a set of alternatives. For any three alternatives x_i, x_j, and x_k ($i \neq j \neq k$), assume that their pairwise comparisons p_{ij} and p_{jk} satisfy $0.5 \leq p_{ij}, p_{jk} \leq 1$ and RMMT is satisfied. Then, in triangle $\triangle x_i x_j x_k$ composed of three edges with lengths of p_{ij}, p_{jk}, and p_{ik}, p_{ik} is determined by

$$p_{ik} = \sqrt{p_{ij}^2 + p_{jk}^2 - 2 p_{ij} p_{jk} cos\theta} , \tag{9}$$

where θ represents the angle between edges $\overrightarrow{x_i x_j}$ and $\overrightarrow{x_j x_k}$ in triangle $\triangle x_i x_j x_k$ such that

$$arccos \frac{min\{p_{ij}, p_{jk}\}}{2 max\{p_{ij}, p_{jk}\}} \leq \theta \leq arccos \frac{p_{ij}^2 + p_{jk}^2 - 1}{2 p_{ij} p_{jk}} .$$

4.2.2 RMMT in the second situation

In the situation of Fig. 2 (*b*), where $p_{ij} \geq 0.5$, $p_{jk} \leq 0.5$, and $p_{ij} \geq 1 - p_{jk} = p_{kj}$, the problem changes to one considering the angle between edges $\overrightarrow{x_i x_k}$ and $\overrightarrow{x_k x_j}$ in triangle $\triangle x_i x_j x_k$ under the assumption that RMMT is satisfied. First, the situation where the triangle exists is formally presented.

Proposition 4. Let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$, where $X = \{x_1, x_2, \ldots, x_n\}$ denotes a set of alternatives. For any three alternatives x_i, x_j, and x_k ($i \neq j \neq k$), when their pairwise comparisons p_{ij}, p_{jk}, and p_{ik} satisfy $p_{ij} \geq 0.5$, $p_{jk} \leq 0.5$, and $p_{ij} \geq 1 - p_{jk} = p_{kj}$, if p_{ij}, p_{kj}, and p_{ik} satisfy RMMT and $p_{ij} - p_{kj} \leq p_{ik}$, then three edges with lengths of p_{ij}, p_{kj}, and p_{ik} must form a triangle.

Given p_{ij} and p_{jk} such that $p_{ij} \geq 0.5$, $p_{jk} \leq 0.5$, and $p_{ij} \geq p_{kj}$ in Fig. 2 (*b*), under the assumption that RMMT is satisfied and $p_{ij} - p_{kj} \leq p_{ik}$, Proposition 4 clearly holds because $max\{p_{ik}, p_{kj}\} \leq p_{ij}$ can be

used to directly deduce that $p_{ik} \leq p_{ij} + p_{kj}$. In such a triangle, the range of angles between edge $\overrightarrow{x_i x_k}$ with length p_{ik} and $\overrightarrow{x_k x_j}$ with length p_{kj} can be determined.

Property 4. Let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$, where $X = \{x_1, x_2, \ldots, x_n\}$ denotes a set of alternatives. For any three alternatives x_i, x_j, and x_k ($i \neq j \neq k$), if their pairwise comparisons p_{ij}, p_{jk}, and p_{ik} satisfy $p_{ij} \geq 0.5$, $p_{jk} \leq 0.5$, $p_{ij} \geq 1 - p_{jk} = p_{kj}$, and $0.5 \leq p_{ik} \leq p_{ij}$, then we can determine that the angle between edge $\overrightarrow{x_i x_k}$ with length p_{ik}, and edge $\overrightarrow{x_k x_j}$ with length p_{kj} in triangle $\triangle x_i x_j x_k$ satisfies

$$\arccos \frac{min\{p_{ij}, p_{kj}\}}{2 max\{p_{ij}, p_{kj}\}} \leq \theta \leq \arccos \frac{0.5^2 + p_{kj}^2 - p_{ij}^2}{2 \cdot 0.5 \cdot p_{kj}}, \tag{10}$$

$$\pi/3 \leq \arccos \frac{min\{p_{ij}, p_{kj}\}}{2 max\{p_{ij}, p_{kj}\}} \leq 0.4196\pi, \text{ and} \tag{11}$$

$$\pi/3 \leq \arccos \frac{0.5^2 + p_{kj}^2 - p_{ij}^2}{2 \cdot 0.5 \cdot p_{kj}} \leq \pi. \tag{12}$$

The proof of Property 4 is presented in Section A.5 of Appendix A. Example B.2 in Section B.3 clearly demonstrates Property 4. In triangle $\triangle x_i x_j x_k$, where the lengths of edges $\overrightarrow{x_i x_j}$ and $\overrightarrow{x_k x_j}$ (p_{ij} and p_{kj}, respectively) are provided and the length of edge $\overrightarrow{x_i x_k}$ (p_{ik}) is unknown, when $p_{ij} \geq p_{kj}$ ≥ 0.5, p_{ik} can be calculated under the assumption that RMMT is satisfied.

Definition 10. Let $P = (p_{ij})_{n \times n}$ be an FPR matrix on $X \times X$, where $X = \{x_1, x_2, \ldots, x_n\}$ denotes a set of alternatives. For any three alternatives x_i, x_j, and x_k ($i \neq j \neq k$), assume that their pairwise comparisons p_{ij} and p_{kj} satisfy $p_{ij} \geq p_{kj} \geq 0.5$ and RMMT is satisfied. Then, in triangle $\triangle x_i x_j x_k$ composed of three edges with lengths of p_{ij}, p_{kj}, and p_{ik}, p_{ik} is determined by

$$p_{ik} = p_{kj} \cdot cos\theta + \sqrt{p_{ij}^2 - p_{kj}^2 sin^2\theta}, \tag{13}$$

where θ represents the angle between edges $\overrightarrow{x_i x_k}$ and $\overrightarrow{x_k x_j}$ such that $\arccos \frac{min\{p_{ij}, p_{kj}\}}{2 max\{p_{ij}, p_{kj}\}} \leq$

$$\theta \leq \; \arccos \frac{0.5^2 + p_{kj}^2 - p_{ij}^2}{2 \cdot 0.5 \cdot p_{kj}} .$$

Eq. (13) is inferred from $p_{ij} = \sqrt{p_{ik}^2 + p_{kj}^2 - 2 p_{ik} p_{kj} cos\theta}$ presented in Eq. (9).

4.3 Introduction of triangular bounded consistency

By using directed triangles to represent the FPRs for any three alternatives, the RMMT of an FPR matrix is reconstructed in Section 4.2. Taking this as an important foundation, this section introduces a new consistency of an FPR matrix called triangular bounded consistency, which is developed depending on the bounded rationality of a decision maker. Thus, triangular bounded consistency is clearly different from additive and multiplicative consistency. The developed consistency is then applied to estimate the missing elements in an acceptable incomplete FPR matrix.

4.3.1 Concept of triangular bounded consistency

As demonstrated in Section 3, when a decision maker provides an FPR matrix, which is additively consistent, they are assumed to be perfectly rational. In contrast, multiplicative consistency relaxes this assumption to guarantee that all FPRs are limited to [0, 1]. In practice, however, the two types of consistency cannot characterize the bounded rationality of a decision maker. This negatively influences the applicability of the two types of consistency in real cases. To address such a problem, a new consistency of an FPR matrix called triangular bounded consistency is proposed under the condition that RMMT is satisfied.

Triangular bounded consistency is defined under the assumption that a decision maker may not have access to all relevant information or may not be able to process the information consistently and rationally in an efficient way (Yan and Ma, 2015). This guarantees that preferences of the decision maker are reasonably acceptable but not perfectly consistent. The basic idea of checking

triangular bounded consistency of a known triangle denoted by $\triangle x_i x_j x_k$ as plotted in Fig. 2 (*a*) is comprised of four steps:

(1) collecting preferences of a decision maker as directed triangles,

(2) using each directed triangle to determine the third edge $\overset{\llcorner\lrcorner\llcorner\lrcorner}{x_i x_k}$ of $\triangle x_i x_j x_k$ from the two edges $\overset{\llcorner\lrcorner\llcorner\lrcorner}{x_i x_j}$ and $\overset{\llcorner\lrcorner\llcorner\lrcorner}{x_j x_k}$,

(3) creating a consistency interval based on the mean and standard deviation of the resulting third edges, and

(4) judging whether the known third edge is limited to the created consistency interval to confirm the consistency of the known triangle.

These steps are described in detail below.

(1) Collection of preferences

There are typically two ways to collect the preferences of a decision maker. One is to choose them from historical evaluations represented by FPRs. The other is to obtain the preferences of the decision maker by asking them to compare given alternatives in pairs in a similar field they are very familiar with. Suppose that the preferences of a decision maker are gathered and expressed by h triangles $\{(p_a^1,\ p_b^1,\ p_c^1), ..., (p_a^h,\ p_b^h,\ p_c^h)\}$. Because the triangles in the second situation presented in Section 4.2 can be transformed into those in the first situation in accordance with the basic characteristics of FPRs in Definition 1 presented in Section 2, the h triangles are assumed in the first situation, where $p_a^m \geq 0.5$, $p_b^m \geq 0.5$, and $p_c^m \geq \max\{p_a^m, p_b^m\}$ (m = 1, ..., h) are satisfied. The angle between the two edges associated with p_a^m and p_b^m is denoted by θ_m, which is limited to $[\theta_m^-, \theta_m^+] = [\arccos \dfrac{min\{p_a^m, p_b^m\}}{2max\{p_a^m, p_b^m\}}, \ \arccos \dfrac{\left(p_a^m\right)^2 + \left(p_b^m\right)^2 - 1}{2 p_a^m p_b^m}]$, as indicated by Property 3 in Section 4.2.

(2) Determination of the third edge

Given that a known triangle $\triangle x_i x_j x_k$ in the first situation is provided, where the lengths of edges $\overline{x_i x_j}$ and $\overline{x_j x_k}$ (p_{ij} and p_{jk}, respectively) are larger than or equal to 0.5, a direct method of using the h triangles to calculate the length of edge $\overline{x_i x_k}$ (p_{ik}) is to regard θ_m as the angle θ between edges $\overline{x_i x_j}$ and $\overline{x_j x_k}$. We can determine from Eq. (6) that $\theta \in [\theta^-, \theta^+] = [\arccos \dfrac{min\{p_{ij}, p_{jk}\}}{2max\{p_{ij}, p_{jk}\}},$

$\arccos \dfrac{p_{ij}^2 + p_{jk}^2 - 1}{2 p_{ij} p_{jk}}]$. For the h triangles, however, θ_m cannot be always guaranteed to be limited to this interval. For example, given (p_a^m, p_b^m, p_c^m) = (0.6, 0.7, 0.85) and (p_{ij}, p_{jk}) = (0.8, 0.8), $\theta_m =$ 0.4515π and $[\theta^-, \theta^+] = [0.3333\pi, 0.4298\pi]$ can be obtained using Eqs. (5) and (6), respectively. It is clear that $\theta_m \notin [\theta^-, \theta^+]$, and a direct method is unavailable in this situation.

Therefore, a ratio method is developed to determine θ in the triangle $\triangle x_i x_j x_k$. The idea of such a method is that the ratio of $\theta_m - \theta_m^-$ to $\theta_m^+ - \theta_m^-$ is equal to the ratio of $\theta - \theta^-$ to $\theta^+ - \theta^-$, as plotted in Fig. 3. Following this idea, we can easily determine that

$$\theta = \theta^- + \frac{\theta_m - \theta_m^-}{\theta_m^+ - \theta_m^-} \cdot (\theta^+ - \theta^-). \tag{14}$$

The value of p_{ik} is then determined using Eq. (9). When all h triangles are applied, p_{ik}^m ($m = 1, \ldots, h$) is generated.
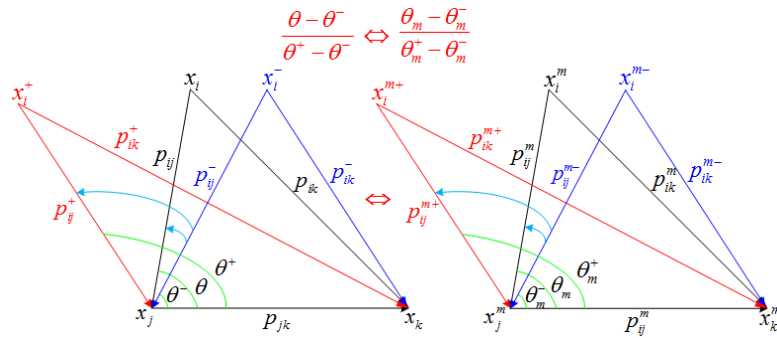

Fig. 3. Equivalent transformation between triangle $\triangle x_i x_j x_k$ and the triangle (p_a^m, p_b^m, p_c^m) provided by the decision maker.

It should be noted that $\theta_m^+ = \arccos \dfrac{\left(p_a^m\right)^2 + \left(p_b^m\right)^2 - 1}{2 p_a^m p_b^m}$ and $\theta^+ = \arccos \dfrac{p_{ij}^2 + p_{jk}^2 - 1}{2 p_{ij} p_{jk}}$ are assumed

in the above analysis to follow the basic property of FPRs that the maximum value of an FPR is 1.

This property is not ensured by additive consistency; thus, it inspires multiplicative consistency. In

theory, the length of the third edge in a triangle may be larger than 1 if the other two edges have

lengths larger than 0.5. From this perspective, additive consistency rather than multiplicative

consistency covers all possible situations that may not follow the basic characteristics of FPRs.

From the viewpoint of the triangle, θ^+ should be set to π to cover all possible situations. Otherwise,

some counterintuitive results may be obtained, which is illustrated by the following example.

Example 2. For a set of alternatives $X = \{x_1, \ldots, x_5\}$, suppose that $P = (p_{ij})_{5 \times 5}$ is an FPR matrix on

$X \times X$ with $p_{12} = 0.61$, $p_{23} = 0.75$, $p_{34} = 0.66$, and $p_{45} = 0.75$. For simplicity, given $(\, p_a^1,\ \ p_b^1,\ \ p_c^1\,) =$

$(0.6,\quad 0.7,\quad 0.78)$, p_{13} and p_{35} are calculated under the condition that $\theta_1^+ =$

$\arccos \dfrac{\left(p_a^1\right)^2 + \left(p_b^1\right)^2 - 1}{2 p_a^1 p_b^1}$ and $\theta^+ = \arccos \dfrac{p_{ij}^2 + p_{jk}^2 - 1}{2 p_{ij} p_{jk}}$ as well as $\theta_1^+ = \pi$ and $\theta^+ = \pi$.

1) $\theta_1^+ = \arccos \dfrac{\left(p_a^1\right)^2 + \left(p_b^1\right)^2 - 1}{2 p_a^1 p_b^1}$ and $\theta^+ = \arccos \dfrac{p_{ij}^2 + p_{jk}^2 - 1}{2 p_{ij} p_{jk}}$

In this situation, it can be calculated that $[\ \theta_1^-,\ \theta_1^+\ ] = [\ \arccos \dfrac{min\left\{p_a^1, p_b^1\right\}}{2 max\left\{p_a^1, p_b^1\right\}},$

$\arccos \dfrac{\left(p_a^1\right)^2 + \left(p_b^1\right)^2 - 1}{2 p_a^1 p_b^1}] = [0.359\pi,\ 0.5571\pi]$, and $\theta_1 = \arccos \dfrac{\left(p_a^1\right)^2 + \left(p_b^1\right)^2 - \left(p_c^1\right)^2}{2 p_a^1 p_b^1} = 0.4071\pi$.

When p_{12} and p_{23} are considered, we can determine that $[\theta^-, \theta^+] = [\ \arccos \dfrac{min\left\{p_{12}, p_{23}\right\}}{2 max\left\{p_{12}, p_{23}\right\}},$

$\arccos \dfrac{p_{12}^2 + p_{23}^2 - 1}{2 p_{12} p_{23}}] = [0.3667\pi,\ 0.5228\pi]$. From $[\ \theta_1^-, \theta_1^+\ ]$ and $[\ \theta^-, \theta^+\ ]$, it can be calculated that θ

$= 0.4046\pi$ using Eq. (14), and it can be further deduced from Eq. (9) that $p_{13}^1 = 0.8151$. As for p_{34}

and p_{45}, in a similar way, we can determine that $[\ \theta^-,\ \theta^+\ ] = [\ \arccos \dfrac{min\left\{p_{34}, p_{45}\right\}}{2 max\left\{p_{34}, p_{45}\right\}},$

$\arccos \dfrac{p_{34}^2 + p_{45}^2 - 1}{2 p_{34} p_{45}}] = [0.355\pi, 0.5006\pi]$, $\theta = 0.3903\pi$, and $p_{35}^1 = 0.8147$. The conclusion that

$p_{13}^1 > p_{35}^1$ is clearly counterintuitive because $p_{12} < p_{34}$ and $p_{23} = p_{45}$.

2) $\theta_1^+ = \pi$ and $\theta^+ = \pi$

When θ_1^+ and θ^+ are changed to π, with respect to p_{12} and p_{23}, θ is changed to 0.4142π, in

which it can be further deduced that $p_{13}^1 = 0.8313$. Under the same condition, θ between p_{34}

and p_{35}^1 and between p_{45} and p_{35}^1 are converted into 0.4034π and 0.838, respectively. The

conclusion that $p_{13}^1 < p_{35}^1$ makes sense because it is in line with the given conditions.

Example 2 reveals that $[\theta_m^-, \theta_m^+] = [\arccos \dfrac{min\{p_a^m, p_b^m\}}{2 max\{p_a^m, p_b^m\}}, \pi]$ for the h triangles and $[\theta^-, \theta^+] =$

$[\arccos \dfrac{min\{p_{ij}, p_{jk}\}}{2 max\{p_{ij}, p_{jk}\}}, \pi]$ because the triangle $\triangle x_i x_j x_k$ should be specified to guarantee the

rationality of p_{ik}^m calculated based on the h triangles. Note that $\theta^+ = \pi$ may result in $p_{ik}^m \geq 1$. As

shown in Definition 1 presented in Section 2, $p_{ik}^m = 1$ means that the alternative x_i is absolutely

preferred to x_k. With this consideration, the situation of $p_{ik}^m > 1$ can be reduced to the situation of

$p_{ik}^m = 1$ without information loss. From this, Property 3 presented in Section 4.2 is also satisfied.

(3) Creation of consistency interval

Each of the h triangles represents the preference of a decision maker and none of them can be

considered to be more representative than another without further information provided by the

decision maker. In this situation, a feasible method of considering the preferences contained in the

h triangles is to find a consistency interval to cover p_{ik}^m ($m = 1, \ldots, h$). Because the mean and

standard deviation are two statistics commonly used to measure the central tendency and

dispersion among a set of discrete numbers, the mean and standard deviation of p_{ik}^m ($m = 1, \ldots,$

h), denoted by

$$\bar{p}_{ik} = \frac{1}{h}\sum_{m=1}^{h} p_{ik}^{m} \quad \text{and} \tag{15}$$

$$S_{ik} = \sqrt{\frac{1}{h-1}\sum_{m=1}^{h}\left(p_{ik}^{m} - \bar{p}_{ik}\right)^{2}} \tag{16}$$

respectively, are applied to construct the consistency interval $[\,\bar{p}_{ik} - S_{ik},\ \ \bar{p}_{ik} + S_{ik}\,]$.

(4) Triangular bounded consistency

Using the constructed consistency interval derived from the h triangles provided by a decision

maker, whether a triangle $\triangle x_i x_j x_k$ is of triangular bounded consistency is judged as follows.

Definition 11. Suppose that h triangles denoted by $\{(\,p_a^1,\ p_b^1,\ p_c^1),\ \dots,\ (\,p_a^h,\ p_b^h,\ p_c^h)\}$ are

provided by a decision maker to characterize their preferences, where $p_a^m \geq 0.5$, $p_b^m \geq 0.5$, and

$p_c^m \geq \max\{p_a^m, p_b^m\}$ ($m = 1, \dots, h$). Given a triangle $\triangle x_i x_j x_k$ with lengths p_{ij}, p_{jk}, and p_{ik} such that p_{ij}

≥ 0.5, $p_{jk} \geq 0.5$, and $p_{ik} \geq \max\{p_{ij}, p_{jk}\}$, assume that p_{ik}^m ($m = 1, \dots, h$) is generated by using Eqs.

(9) and (14) and the mean $\bar{p}_{ik}$ and the standard deviation S_{ik} of p_{ik}^m ($m = 1, \dots, h$) are known.

Then, p_{ij}, p_{jk}, and p_{ik} are considered to satisfy triangular bounded consistency if they satisfy the

following condition:

$$\bar{p}_{ik} - S_{ik} \ \leq p_{ik} \leq\ \bar{p}_{ik} + S_{ik}. \tag{17}$$

Based on Definition 11, triangular bounded consistency of an FPR matrix is defined.

Definition 12. Given a set of alternatives $X = \{x_1, x_2, \dots, x_n\}$, let $P = (p_{ij})_{n \times n}$ be an FPR matrix on

$X \times X$. Suppose that h triangles denoted by $\{(\,p_a^1,\ p_b^1,\ p_c^1),\ \dots,\ (\,p_a^h,\ p_b^h,\ p_c^h)\}$ are provided by a

decision maker to characterize their preferences, where $p_a^m \geq 0.5$, $p_b^m \geq 0.5$, and

$p_c^m \geq \max\{p_a^m, p_b^m\}$ ($m = 1, \dots, h$). Under the given conditions, P is said to have triangular bounded

consistency if p_{ij}, p_{jk}, and p_{ik} ($\forall i, j, k \in \{1, \dots, n\}$) are verified to be of triangular bounded

consistency.

Note that for any three alternatives, we can always find a triangle in the first situation shown in

Section 4.2, which is similar to the situation of the h triangles and potentially assumed in Definition 12.

4.3.2 Estimation of missing FPRs

In real-world applications, it could be a burden on a decision maker to have to compare all alternatives in pairs and guarantee consistency of the comparisons. To relieve this burden, the decision maker is allowed to provide comparisons between some alternatives rather than all alternatives. This gives rise to the question of how to estimate the missing comparisons in an acceptable incomplete comparison matrix as defined in Definition 8 presented in Section 2. This question is addressed using triangular bounded consistency of the matrix for each of the two situations presented in Section 4.2.

(1) Analysis of the first situation

Assume that the h triangles in the first situation, denoted by $\{(p_a^1, p_b^1, p_c^1), \ldots, (p_a^h, p_b^h, p_c^h)\}$, are provided by a decision maker. Under the condition that p_{ij} and p_{jk} with $p_{ij} \geq 0.5$ and $p_{jk} \geq 0.5$ are provided and p_{ik} is unknown, p_{ik}^m ($m = 1, \ldots, h$) limited to [0.5, 1], $\bar{p}_{ik}$, and S_{ik} can be obtained in accordance with Subsection 4.3.1. When p_{ik} is limited to $[\bar{p}_{ik} - S_{ik}, \bar{p}_{ik} + S_{ik}]$, p_{ij}, p_{jk}, and p_{ik} can be considered to have triangular bounded consistency.

(2) Analysis of the second situation

Suppose that p_{ij} and p_{kj} with $p_{ij} \geq p_{kj} \geq 0.5$ (or $p_{kj} \geq p_{ij} \geq 0.5$) are provided; then, the angle between edges $\overleftrightarrow{x_i x_k}$ and $\overleftrightarrow{x_k x_j}$ in triangle $\triangle x_i x_j x_k$ represented by θ is limited to $[\theta^-, \theta^+] =$

$[\arccos \dfrac{min\{p_{ij}, p_{kj}\}}{2max\{p_{ij}, p_{kj}\}}, \pi]$. The reason that the upper bound is π instead of $\arccos \dfrac{0.5^2 + p_{kj}^2 - p_{ij}^2}{2 \cdot 0.5 \cdot p_{kj}}$

is similar to that in the first situation, which is explained in Example 2, so is not discussed here.

For each reference triangle (p_a^m, p_b^m, p_c^m), θ is also determined from Eq. (14) and is used to

calculate p_{ik}^m ($m = 1, \ldots, h$) in accordance with Eq. (13). Note that $p_{ik}^m \geq 0.5$ is specified by

Property 4; thus, $p_{ik}^m = 0.5$ is set when $p_{ik}^m < 0.5$ because $\theta^+ = \pi$. Based on p_{ik}^m ($m = 1, \ldots, h$)

limited to [0.5, 1], the consistency interval [$\bar{p}_{ik} - S_{ik}$, $\bar{p}_{ik} + S_{ik}$] is created from Eqs. (15) and (16).

Similar to the first situation, any point in the interval can be given to p_{ik} to ensure that p_{ij}, p_{kj}, and

p_{ik} are of triangular bounded consistency.

The above analyses indicate that any value in the consistency interval [$\bar{p}_{ik} - S_{ik}$, $\bar{p}_{ik} + S_{ik}$] can

be considered as the estimation of a missing FPR. An important issue is how to find an estimation

in which the information contained in p_{ik}^m ($m = 1, \ldots, h$) is considered as much as possible. To

address this issue, a fair rule is designed to identify such an estimation, which is nearly or even

completely equidistant from p_{ik}^m ($m = 1, \ldots, h$). In other words, the estimation selected from the

consistency interval [$\bar{p}_{ik} - S_{ik}$, $\bar{p}_{ik} + S_{ik}$] is fair for the given h triangles. To find this estimation,

an optimization model is constructed as described below.

$$\text{MIN} \quad \sum_{m=1}^{h}\left(p_{ik}^* - p_{ik}^m\right)^2 \tag{18}$$

$$\text{s.t.} \quad \bar{p}_{ik} - S_{ik} \leq p_{ik}^* \leq \bar{p}_{ik} + S_{ik}, \tag{19}$$

where p_{ik}^* symbolizes a decision variable to differentiate it from the optimized p_{ik}.

Suppose that $f(p_{ik}^*) = \sum_{m=1}^{h}\left(p_{ik}^* - p_{ik}^m\right)^2$; then, solving the optimization model shown in Eqs.

(18) and (19) changes to finding the minimum value of the function $f(p_{ik}^*)$ within the range

[$\bar{p}_{ik} - S_{ik}$, $\bar{p}_{ik} + S_{ik}$]. For this purpose, the first derivative of $f(p_{ik}^*)$ is calculated as $\dfrac{df\left(p_{ik}^*\right)}{dp_{ik}^*} =$

$2\sum_{m=1}^{h}\left(p_{ik}^* - p_{ik}^m\right)$. It can be inferred from $\dfrac{df\left(p_{ik}^*\right)}{dp_{ik}^*} = 0$ that $p_{ik}^* = \dfrac{1}{h}\sum_{m=1}^{h}p_{ik}^m = \bar{p}_{ik}$ may be an

extreme point. To confirm this, the second derivative of $f(p_{ik}^*)$ is obtained as $\dfrac{d^2 f\left(p_{ik}^*\right)}{d(p_{ik}^*)^2} = 2h > 0$,

which indicates that $\dfrac{df\left(p_{ik}^{*}\right)}{dp_{ik}^{*}} < 0$ when $p_{ik} < \bar{p}_{ik}$ and $\dfrac{df\left(p_{ik}^{*}\right)}{dp_{ik}^{*}} > 0$ when $p_{ik} > \bar{p}_{ik}$. That is, $f(p_{ik}) >$

$f(p_{ik}^{*})$ with $p_{ik}^{*} = \bar{p}_{ik}$ always holds whether $p_{ik} < \bar{p}_{ik}$ or $p_{ik} > \bar{p}_{ik}$. As a result, it can be

concluded that $p_{ik}^{*} = \bar{p}_{ik}$ is the extreme point that minimizes the function $f(p_{ik}^{*})$. In other words,

$\bar{p}_{ik}$ is the specific estimation selected from the consistency interval $[\bar{p}_{ik} - S_{ik},\ \bar{p}_{ik} + S_{ik}]$

following the fair rule. Such estimation can be used to generate recommended values of missing

FPRs in an acceptable incomplete FPR matrix under the condition that Theorem 3 in Section 2 is

satisfied.

More importantly, for triangle $\triangle x_i x_j x_k$ with $p_{ij} \geq 0.5$ and $p_{jk} = 0.5$, the value of p_{ik} can be

estimated for both situations described in Section 4.2. When $p_{jk} = 0.5^{+}$, i.e., it is very close to but

larger than 0.5, p_{ik} can be inferred from the first situation and its value is denoted by p_{ik}^{+}.

However, when $p_{jk} = 0.5^{-}$, i.e., it is very close to but less than 0.5, p_{ik} can be obtained from the

second situation and its value is denoted by p_{ik}^{-}. Without loss of generality, p_{jk} reaching 0.5^{-} and

p_{jk} reaching 0.5^{+} are equally likely. As a result, the value of p_{ik} can be estimated by the equality p_{ik}

$= (p_{ik}^{+} + p_{ik}^{-})/2$.

A numerical example and its discussion are presented in Section B.4 of Appendix B to

demonstrate the issue of how to decide whether the known elements in an acceptable incomplete

FPR matrix satisfy triangular bounded consistency; the question of how to estimate the missing

elements in the matrix by using triangular bounded consistency is also discussed there. It should

be noted that when three FPRs provided by a decision maker do not satisfy triangular bounded

consistency, they must be revised.

5. Application of triangular bounded consistency of FPRs in MADA

In this section, we describe how triangular bounded consistency of FPRs is applied to MADA.

5.1 Modeling and analysis of MADA problems with FPRs

An MADA problem arises when a decision maker must compare several alternative courses of action by considering multiple factors to select one or more desirable ones. Pairwise comparison between alternatives is one commonly used approach for MADA owing to its simplicity. Because FPR is one method of recording the decision makers' preferences in pairwise comparisons, in this application, we will apply FPR to analyze the following MADA problem.

Suppose that in an MADA problem M alternatives denoted by a_l (l = 1, ..., M) are compared in pairs on L attributes denoted by e_i (i = 1, ..., L). The comparison between alternatives a_l and a_m is expressed by FPR to form an FPR matrix on attribute e_i denoted by $P(e_i(a_{lm})) = (p_i(a_{lm}))_{M \times M}$, where $0 \leq p_i(a_{lm}) \leq 1$, $p_i(a_{lm}) + p_i(a_{ml}) = 1$ for $l, m \in \{1, \ldots, M\}$, and $p_i(a_{ll}) = 0.5$. The relative weights of the L attributes are denoted by $w = (w_1, w_2, \ldots, w_L)$, where $0 \leq w_i \leq 1$ for i = 1, ..., L and $\sum_{i=1}^{L} w_i = 1$ are set to balance the contribution of $p_i(a_{lm})$ to the overall comparison between alternatives a_l and a_m. The weights can be determined by subjective judgments of a decision maker or can be determined based on the performances of the alternatives (Chin et al., 2015; Fu and Chin, 2014; Fu and Xu, 2016). Then, the weighted $P(e_i(a_{lm}))$ can be combined to generate the overall preferred degree of alternative a_l over a_m, denoted by $p(a_{lm})$. The simple additive weighting method (Kaliszewski and Podkopaev, 2016) is one method of carrying out such a combination:

$$p(a_{lm}) = \sum_{i=1}^{L} w_i \cdot p(e_i(a_{lm})). \tag{20}$$

From the aggregated FPR matrix $P(a_{lm}) = (p(a_{lm}))_{M \times M}$, desirable alternatives can be identified using a specific method, such as the net flow method (Fodor and Roubens, 1994). In this method, the leaving flow and entering flow of alternative a_l are calculated from $\phi^+(a_l) = \sum_{m=1, m \neq l}^{M} p(a_{lm})$ and $\phi^-(a_l) = \sum_{m=1, m \neq l}^{M} p(a_{ml})$, respectively. Then, the net flow of alternative a_l is calculated by $\phi(a_l) = \phi^+(a_l) - \phi^-(a_l)$. An alternative with a larger net flow is more preferred; thus, all

alternatives can be compared by their net flows to generate a ranking order of the M alternatives and desirable ones can be identified. The details of this can be found in Section B.2 of Appendix B.

5.2 Process of finding solutions to MADA problems with FPRs based on triangular bounded consistency

The above discussion is about finding solutions to MADA problems with complete FPR matrices. This may not be feasible in practice because offering consistent FPR matrices $P(a_{lm})_{M \times M}$ is very difficult for a decision maker, particularly when M and L are extremely large. To relieve the burden on the decision maker, the decision maker can offer $(M-1) \cdot L$ FPRs between adjacent alternatives, from which the remaining FPRs can be derived (Sen and Yang, 1994). Following this idea, we design a process of analyzing MADA problems based on triangular bounded consistency, which is plotted in Fig. 4.
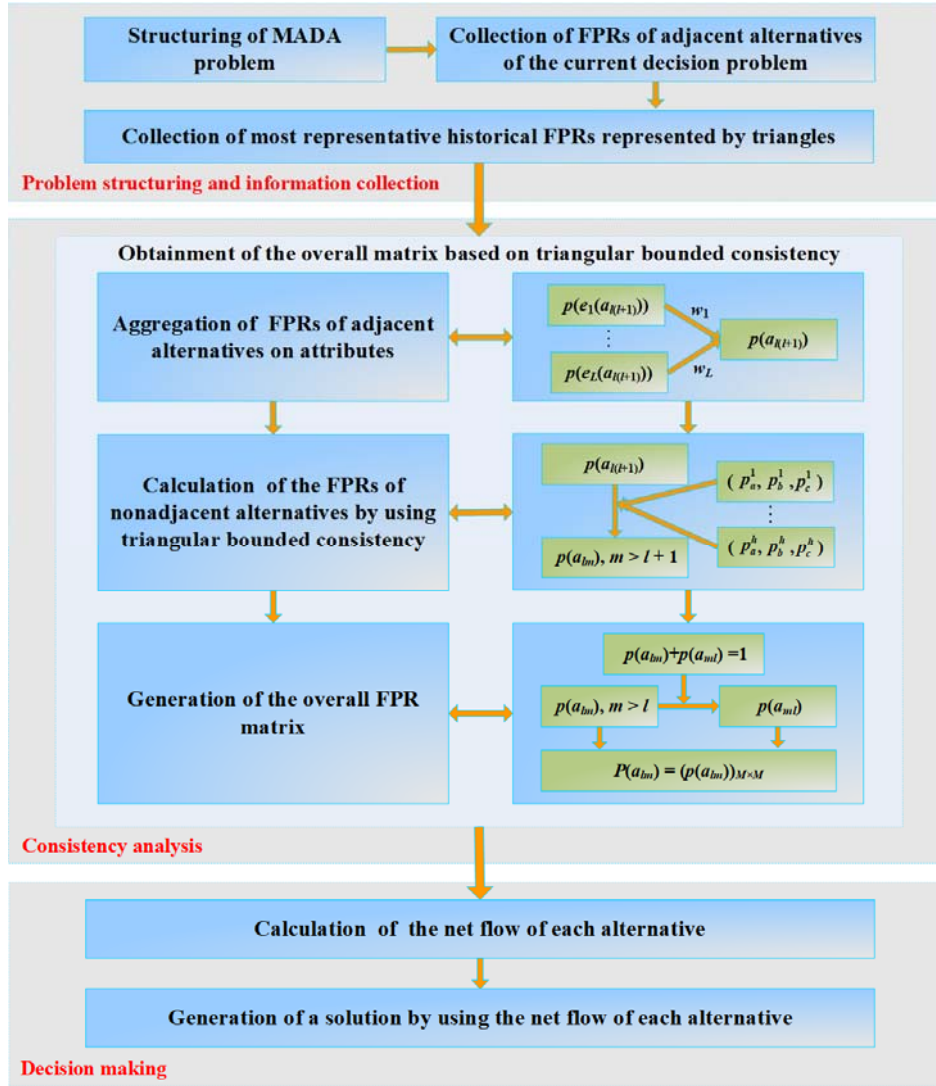
Fig. 4. Process of analyzing MADA problems based on triangular bounded consistency.

Fig. 4 shows the three stages in the process, namely problem structuring and information collection, consistency analysis, and decision-making. In the problem structuring and information collection stage, only $(M-1){\cdot}L$ FPRs between adjacent alternatives denoted by $p(e_i(a_{l(l+1)}))$ ($l =$ 1, ..., $M-1$) are required, which is beneficial for relieving the burden on the decision maker and avoiding inconsistencies in the provided FPRs. This is clearly different from Section 5.1, in which $M{\cdot}(M-1){\cdot}L$ FPRs are required. To estimate FPRs between nonadjacent alternatives, it is necessary to use historical triangles of FPRs obtained from a similar field with which the decision maker is familiar. Suppose that there are h triangles $\{(p_a^1,\ p_b^1,\ p_c^1),\ ...,\ (p_a^h,\ p_b^h,\ p_c^h)\}$ available from

historical FPRs.

In the consistency analysis stage, through Eq. (20), FPRs between adjacent alternatives on each attribute are combined to generate the overall FPR between adjacent alternatives, denoted by $p(a_{l(l+1)})$ ($l = 1, ..., M-1$). Then, FPRs between nonadjacent alternatives are estimated under the condition that triangular bounded consistency is guaranteed. By considering the h triangles as benchmarks, the FPRs between nonadjacent alternatives $p(a_{lm})$ ($\forall l, m \in \{1, ..., M\}, m > l + 1$) in the upper triangle of the FPR matrix are derived from $p(a_{l(l+1)})$ using triangular bounded consistency, as shown in Definition 11 presented in Subsection 4.3.1 and the optimization model in Eqs. (18) and (19). The FPRs $p(a_{ml})$ ($\forall l, m \in \{1, ..., M\}, m > l$) in the lower triangle of the FPR matrix are then obtained in accordance with the basic constraint on FPRs in Definition 1 presented in Section 2, i.e., $p(a_{lm}) + p(a_{ml}) = 1$ ($\forall l, m \in \{1, ..., M\}$).

In the decision-making stage, from the complete FPR matrix, the net flow of each alternative is calculated to generate a solution to the MADA problem.

From the above discussion, it can be observed that the process outlined in Fig. 4 is different from the process of analyzing MADA problems discussed in Section 5.1 in both the problem structuring and information collection stage and the consistency analysis stage. The former involves the estimation of FPRs between nonadjacent alternatives in accordance with triangular bounded consistency. The latter requires complete FPR matrices, which may or may not be consistent.

6. Case study

In this section, a problem of selecting areas to market products, i.e., a market selection problem, for a company located in Qingyang County of Anhui Province in China is investigated to

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demonstrate the application of triangular bounded consistency of FPRs in a MADA problem.

6.1 Description of the market selection problem

We focus on how a company located in Qingyang County of Anhui province, with limestone powder as their main product, selects markets. In the current situation, the potential markets of the company are the five areas of Anhui, Jiangsu, Shanghai, Belgium, and Japan. Selling limestone powder to these five areas simultaneously imposes a heavy burden on the company because this requires a large amount of operating capital and inventory. To relieve this burden, the company's board decides to select the most appropriate two areas to sell in by considering current cash flow difficulties. The manager of the company acts as the decision maker responsible for the alternative selection with the help of five experts from the sales, planning, finance, and research and development departments and a collaborative university (the second author is the expert from the university). The five alternatives are evaluated in pairs on seven attributes: capital inputs, market demands, process loss, capital cycle, research and development inputs, capabilities of market profits, and capabilities of sustainable development.

Suppose that the five alternatives are denoted by A_l ($l = 1, …, 5$) and the seven attributes are denoted by e_i ($i = 1, …, 7$). After analyzing the influence of each attribute on the overall comparison between alternatives, the decision maker uses a method discussed by Ölçer and Odabaşi (2005) to specify that w_i ($i = 1, …, 7$) = (0.15, 0.25, 0.07, 0.1, 0.1, 0.18, 0.15). Based on the analysis of documents and data related to the five alternatives, the decision maker provides comparisons between adjacent alternatives using MPRs, as presented in Table C.1 of Section C.1 in Appendix C of the supplementary material. The MPRs in Table C.1 are then transformed into FPRs (shown in Table 1) using Proposition B.1.

Table 1

FPRs between adjacent alternatives on seven attributes.

Attributes	$p(e_i(A_{12}))$	$p(e_i(A_{23}))$	$p(e_i(A_{34}))$	$p(e_i(A_{45}))$
e_1	0.5415	0.5766	0.6461	0.6577
e_2	0.6577	0.5217	0.6794	0.5597
e_3	0.7717	0.5000	0.6338	0.5923
e_4	0.4188	0.3838	0.4492	0.4492
e_5	0.4492	0.6070	0.6688	0.6070
e_6	0.3423	0.2915	0.3423	0.2915
e_7	0.4492	0.3423	0.4188	0.4760

To estimate the FPRs between nonadjacent alternatives, the decision maker is required to choose the triangles of FPRs that he is most confident with from historical evaluations he made for a past decision problem. The problem is to select the most appropriate city to construct a new factory from five candidates: Tongling (TL), Chizhou (CZ), Hefei (HF), Shanghai (SH), and Taizhou (TZ). The five candidates are evaluated on six attributes, which are production cost, manpower resource, regional conditions, service facility, competitor, and political stability. The decision maker is most confident with the FPRs he made when comparing three candidates on each attribute to form the following six triangles: {(CZ/TL, TL/SH, CZ/SH), (HF/SH, SH/TL, HF/TL), (SH/TZ, TZ/CZ, SH/CZ), (SH/HF, HF/TL, SH/TL), (TZ/CZ, CZ/HF, TZ/HF), (TL/TZ, TZ/SH, TL/SH)} = {(0.59, 0.68, 0.72), (0.36, 0.81, 0.70), (0.55, 0.76, 0.78), (0.64, 0.82, 0.90), (0.15, 0.92, 0.83), (0.25, 0.29, 0.21)}. As is stated in Subsection 4.3.1, the triangles in the second situation can be transformed into those in the first situation in accordance with the basic

characteristics of FPRs in Definition 1 presented in Section 2. For this reason, the above six

triangles are uniformly expressed by six ones in the first situation: {(0.59, 0.68, 0.72), (0.64, 0.70,

0.81), (0.55, 0.76, 0.78), (0.64, 0.82, 0.90), (0.83, 0.85, 0.92), (0.75, 0.71, 0.79)}.

6.2 Generation of a solution to the market selection problem

To find a solution to the market selection problem, the FPRs between adjacent alternatives

$p(e_i(A_{l(l+1)}))$ ($i = 1, …, 7, l = 1, ..., 4$) given in Table 1, which are presented in Section 6.1, are

combined using Eq. (20) and w_i ($i = 1, …, 7$) to generate the aggregated FPRs between adjacent

alternatives $p(A_{l(l+1)})$ ($l = 1, ..., 4$), as presented in Table 2.

Table 2

Aggregated FPRs between the five alternatives.

Areas	A_1	A_2	A_3	A_4	A_5
A_1	0.5	0.5155	0.497	0.561	0.6036
A_2	0.4845	0.5	0.4485	0.5039	0.557
A_3	0.503	0.5515	0.5	0.5621	0.6083
A_4	0.439	0.4961	0.4379	0.5	0.5116
A_5	0.3964	0.443	0.3917	0.4884	0.5

Then, the overall FPR matrix based on triangular bounded consistency must be obtained, as

outlined in the consistency analysis stage of Fig. 4 in Section 5.2. Suppose that the FPRs between

nonadjacent alternatives A_l and A_m in the upper triangle of an FPR matrix are denoted by $p(A_{lm})$ (m

$> l + 1$); they must be subsequently estimated. By considering the six triangles in the first situation

discussed in Subsection 4.2.1 as benchmarks, six estimations of $p(A_{lm})$, denoted by $p^r(A_{lm})$ ($r =$

1, …, 6), can be obtained. Owing to space limitation, the resulting $p^r(A_{lm})$ are not provided in

this paper. The mean and standard deviation of $p^r(A_{lm})$, denoted by $\bar{p}(A_{lm})$ and $\bar{S}(A_{lm})$,

respectively, can be further calculated using Eqs. (15) and (16) to form the consistency interval of the FPR $p(A_{lm})$ and make triangular bounded consistency indicated in Definition 11, which is presented in Subsection 4.3.1. Here, we have $\overline{S}(A_{lm})$ $(l = 1, …, 3, m > l + 1) = (0.003, 0.0171, 0.0175, 0.003, 0.0171, 0.0174)$ for FPRs in the upper triangle of the overall FPR matrix. As presented in Subsection 4.3.2, $\overline{p}(A_{lm})$ is the solution to the optimization model given by Eqs. (18) and (19) when the fair rule is adopted; therefore, the aggregated FPRs between nonadjacent alternatives in the upper triangle of the overall FPR matrix are obtained and presented in Table 2. Note that only $\overline{p}(A_{lm})$ is presented in Table 2 to avoid repetition. From the FPRs between adjacent and nonadjacent alternatives in the upper triangle of the FPR matrix, all FPRs in the lower triangle are obtained in accordance with the basic constraint on FPRs in Definition 1 presented in Section 2, which can be found in Table 2.

From the results presented in Table 2, the leaving and entering flows of the five alternatives are calculated as $\phi^{+}(A_l)$ $(l = 1, …, 5) = (2.1771, 1.9939, 2.2249, 1.8846, 1.7195)$ and $\phi^{-}(A_l)$ $(l = 1, …, 5) = (1.8229, 2.0061, 1.7751, 2.1154, 2.2805)$. Then, $\phi(A_l)$ is derived from $\phi^{+}(A_l)$ and $\phi^{-}(A_l)$ to obtain $(0.3542, -0.0122, 0.4498, -0.2308, -0.561)$. A ranking order of the five alternatives is generated from $\phi(A_l)$, which is A_3 f A_1 f A_2 f A_4 f A_5. Based on the ranking order, the most appropriate two areas are Shanghai (A_3) and Anhui (A_1) with the preferential order of A_3 f A_1. Note that the solution is based on the current situation of the company, and it does not mean that the company will not be willing to sell limestone powder to other areas in the future. When the company overcomes its cash flow difficulties, the decision maker may prefer to sell limestone powder to other areas for long-term and sustainable development.

In the following, we analyze Anhui and Shanghai from real perspectives to demonstrate the above solution. The company has constructed favorable marketing channels in Anhui owing to geographical convenience and thus can satisfy market requirements more rapidly. To sell limestone powder in Anhui, there will be less losses in the transportation process. Owing to the long-term cooperation relationship between the company and consumers in Anhui, the company is capable of achieving a rapid cycle of capital. Despite this, because of the level of economic development in Anhui, the sale market of limestone powder in Anhui is not normative. This negatively contributes to the long-term and sustainable development of the company. Meanwhile, the company has limited capabilities of gaining market profits in Anhui for the same reason.

The situation in Shanghai is different from that in Anhui. For reasons of geographical distance, there will be more loss in the transportation process when the company sells limestone powder to Shanghai. Note that under the given conditions, the company must spend more capital on transportation to guarantee the quality of limestone powder. Owing to the high level of economic development, Shanghai has a normative market and thus provides a good opportunity for the company to gain more market profits. Moreover, Shanghai provides a good business environment, which is beneficial for the capital cycle and the sustainable development of the company. As a whole, the solution above is generated by considering the performances of Anhui and Shanghai on the seven attributes and their weights simultaneously and by conforming to the real situations of Anhui and Shanghai.

6.3 Influence of historical preferences of the decision maker on solutions

As shown in Fig. 4, which is presented in Section 5.2, the historical preferences of a decision maker in the problem structuring and information collection stage influence the consistency analysis stage as well as the decision-making stage. They are considered as benchmarks for

carrying out triangular bounded consistency analysis, and this is reflected in solutions to MADA problems. To examine the potential influence of historical preferences of a decision maker on solutions, we take a given preference represented by a triangle (0.55, 0.76, 0.78) in the market selection problem as an example to conduct a sensitivity analysis.

To facilitate this examination, we fix the first edge and adapt the second or third edge. First, under the condition that the length of the third edge is changed to 0.9, suppose that the length of the second edge changes from 0.5 to 0.9 with a step of 0.01 to satisfy RMMT. For the 41 triangles, the net flows of the five alternatives are calculated and plotted in Fig. 5.
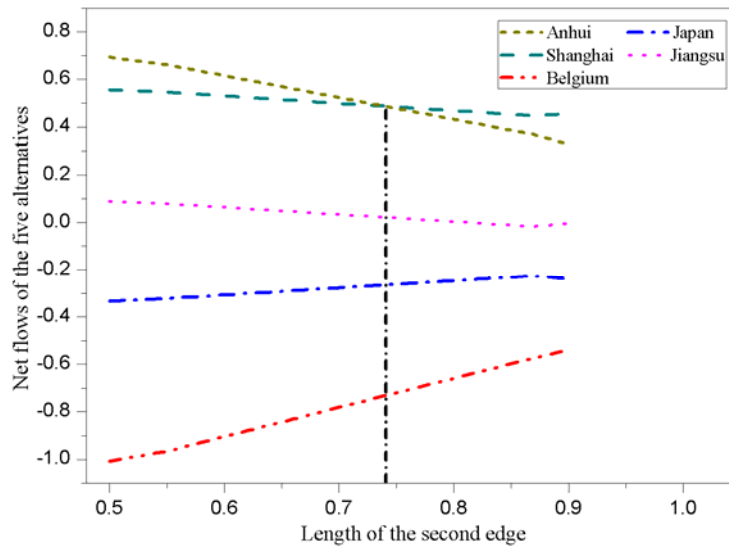

Fig. 5. Movement of net flows of the five alternatives when the length of the second edge changes from 0.5 to 0.9.

Fig. 5 shows that the rankings of Jiangsu (A_2), Belgium (A_4), and Japan (A_5) remain unchanged with an increase in the length of the second edge, and the rankings of Anhui (A_1) and Shanghai (A_3) alter when the length of the second edge approaches 0.74. Before and after 0.74, Anhui (A_1) changes from the best area to the second-best area.

Next, when the length of the second edge is set to 0.6, suppose that the length of the third edge changes from 0.6 to 1 to satisfy RMMT. For the 41 triangles, the net flows of the five alternatives are calculated and plotted in Fig. 6. It is shown in Fig. 6 that the rankings of Jiangsu (A_2), Belgium (A_4)

(A_4), and Japan (A_5) remain unchanged with an increase in the length of the third edge, and the rankings of Anhui (A_1) and Shanghai (A_3) alter when the length of the third edge approaches 0.76. Before and after 0.76, Shanghai (A_3) changes from the best area to the second-best area.
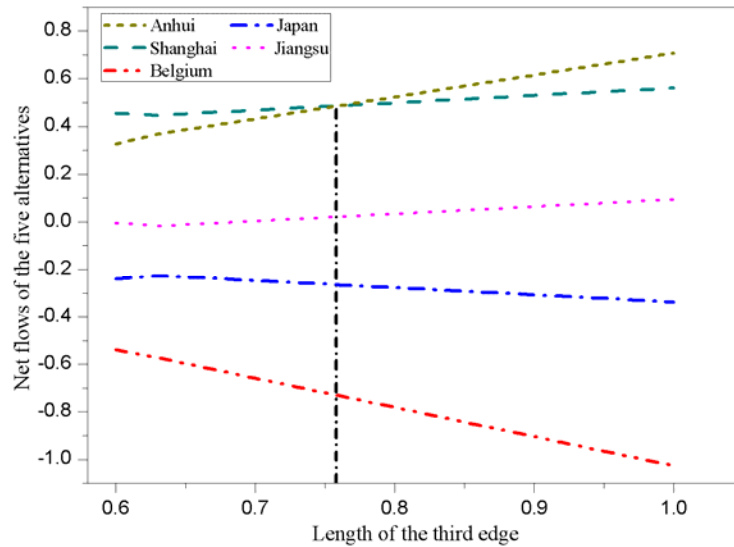

Fig. 6. Movement of net flows of the five alternatives when the length of the third edge changes from 0.6 to 1.

Altogether, the above discussion illustrates the significant influence of the preferences of the decision maker represented by triangles on solutions to the market selection problem. This also implies that the preferences of a decision maker significantly influence the generation of FPRs between nonadjacent alternatives from FPRs between adjacent alternatives when triangular bounded consistency is followed. Note that perfect rationality can be considered as a special case of bounded rationality. When all preferences of the decision maker satisfy additive or multiplicative consistency of FPRs, triangular bounded consistency will reduce to either additive or multiplicative consistency.

6.4 Comparative analysis

As presented in Section 3, triangular bounded consistency is inspired by additive and

multiplicative consistency. To highlight the difference between triangular bounded consistency and the two types of consistency, we conduct a comparative analysis based on the market selection problem described in Section 6.1.

By using additive and multiplicative consistency indicated in Definitions 2 and 3, respectively, we reimplement the process of generating the aggregated FPRs presented in Section 6.2, and the obtained results are presented in Tables 3 and 4.

Table 3

Aggregated FPRs between the five alternatives in accordance with additive consistency.

Areas	A_1	A_2	A_3	A_4	A_5
A_1	0.5	0.5155	0.464	0.5261	0.5377
A_2	0.4845	0.5	0.4485	0.5106	0.5222
A_3	0.536	0.5515	0.5	0.5621	0.5737
A_4	0.4739	0.4894	0.4379	0.5	0.5116
A_5	0.4623	0.4778	0.4263	0.4884	0.5

Table 4

Aggregated FPRs between the five alternatives in accordance with multiplicative consistency.

Areas	A_1	A_2	A_3	A_4	A_5
A_1	0.5	0.5155	0.4639	0.5262	0.5378
A_2	0.4845	0.5	0.4485	0.5107	0.5223
A_3	0.5361	0.5515	0.5	0.5621	0.5735
A_4	0.4738	0.4893	0.4379	0.5	0.5116
A_5	0.4622	0.4777	0.4265	0.4884	0.5

From the results in Tables 3 and 4, the net flows of alternative A_l (l = 1, …, 5) associated with additive and multiplicative consistency denoted by $\phi_A(A_l)$ and $\phi_M(A_l)$ are calculated as (0.0866, −0.0684, 0.4466, −0.1744, −0.2904) and (0.0868, −0.068, 0.4464, −0.1748, −0.2904), respectively. A common ranking order of the five alternatives is then generated, which is A_3 ꜰ A_1 ꜰ A_2 ꜰ A_4 ꜰ A_5. Shanghai (A_3) and Anhui (A_1) are still the most appropriate two areas, which is the same as the solution generated based on triangular bounded consistency. However, the net flow $\phi(A_l)$ is clearly different from $\phi_A(A_l)$ and $\phi_M(A_l)$, particularly for the net flows of alternatives A_1 and A_5. This difference is mainly because the preferences of the decision maker are bounded by triangular bounded consistency rather than additive or multiplicative consistency. Given a fixed set of FPRs of a decision maker, the solutions generated by adopting triangular bounded consistency may not be unique but within a range that reflects the bounded rationality of the decision maker. However, the solution is unique when additive or multiplicative consistency is adopted, implying that all decision makers must be strictly rational and make the same choice no matter who they are. In practice, different decision makers may make different decisions. From this perspective, triangular bounded consistency is more feasible and effective than additive or multiplicative consistency for generating solutions that are acceptable or satisfactory to a decision maker.

In Appendix D of the supplementary material, we further compare the relationship between the two types of consistency and triangular bounded consistency based on the market selection problem.

7. Conclusions

Consistency is an important issue concerning FPRs. There are two classical types of consistency

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in existing studies, namely additive and multiplicative consistency. Through a detailed analysis, it

is concluded that these two types of consistency cannot reflect the preferences of a decision maker

with bounded rationality. To address such an issue, a new consistency measure of an FPR matrix,

called triangular bounded consistency, is proposed. Using directed triangles to represent the

pairwise FPRs for any three alternatives, the RMMT of FPRs is geometrically represented by the

value range of the angles formed by two adjacent edges of triangles. This representation takes

different forms in the following two situations. The first is when the length of one of the two

adjacent edges is larger than 0.5, and the length of the other edge is smaller than 0.5. The second is

when the lengths of both the adjacent edges are either larger or smaller than 0.5. Given that

RMMT is satisfied, triangular bounded consistency is defined using the preference information of

a decision maker characterized by a set of directed triangles. The constructed triangular bounded

consistency is then applied to estimate the missing elements in an acceptable incomplete FPR

matrix. Finally, a problem of selecting areas to market the products of a company is investigated to

demonstrate the application of triangular bounded consistency of FPRs in MADA problems.

In the future, two significant issues concerning triangular bounded consistency must be

investigated. The first issue involves the set of triangles provided by a decision maker being

filtered to greatly reduce inconsistent triangles that satisfy RMMT but cannot represent the

bounded rationality of the decision maker. The other issue involves determining the importance

weight of the remaining triangles in determining the lower and upper bounds of triangular

bounded consistency of a decision maker. In addition, triangular bounded consistency will be used

to compare the consistency levels of different decision makers and check whether a decision

maker is more consistent in making some decisions than others. Furthermore, triangular bounded

consistency will be extended to other situations where preference information of a decision maker

is characterized by different types of FPRs, such as triangular FPRs (Meng and Chen, 2017).

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