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# On the Hausdorff Dimension of Bernoulli Convolutions\*

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## Abstract

We give an expression for the Garsia entropy of Bernoulli convolutions in terms of products of matrices. This gives an explicit rate of convergence of the Garsia entropy and shows that one can calculate the Hausdorff dimension of the Bernoulli convolution  $\nu_\beta$  to arbitrary given accuracy whenever  $\beta$  is algebraic. In particular, if the Garsia entropy  $H(\beta)$  is not equal to  $\log(\beta)$  then we have a finite time algorithm to determine whether or not  $\dim_{\mathbb{H}}(\nu_\beta) = 1$ .

## 1 Introduction

Bernoulli convolutions are a simple and interesting family of self-similar measures with overlaps. For  $\beta \in (1, 2)$ , the Bernoulli convolution  $\nu_\beta$  is defined as the weak-star limit of the family of measures  $\nu_\beta^{(n)}$  given by

$$\nu_\beta^{(n)} := \frac{1}{2^n} \sum_{a_1 \cdots a_n \in \{0,1\}^n} \delta_{\sum_{i=1}^n a_i \beta^{-i}}.$$

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The fundamental questions relating to the Bernoulli convolution  $\nu_\beta$  are whether it has Hausdorff dimension one, and if so, whether it is absolutely continuous.

Erdős proved that  $\nu_\beta$  is singular whenever  $\beta$  is a Pisot number [6], and it was later proved by Garsia that in fact  $\nu_\beta$  has Hausdorff dimension less than one whenever  $\beta$  is Pisot [14]. So far, Pisot numbers are the only class of  $\beta$  for which it is known that  $\nu_\beta$  is singular. Garsia gave a small explicit class of  $\beta$  for which  $\nu_\beta$  is absolutely continuous [13], until recently these were the only examples of Bernoulli convolutions for which it was known that the Hausdorff dimension is one. In [23] Solomyak proved that  $\nu_\beta$  is absolutely continuous for Lebesgue-almost every  $\beta \in (1, 2)$ .

A great deal of the recent progress on Bernoulli convolutions stems from Hochman's article [17], where it was proved that if  $\nu_\beta$  has Hausdorff dimension less than one then the sums in the definition of  $\nu_\beta^{(n)}$  must be superexponentially close. This can only happen on a set of  $\beta$  of Hausdorff dimension zero. Additionally, Hochman proved that if  $\beta$  is algebraic then  $\dim_{\mathbb{H}}(\nu_\beta)$  can be expressed in terms of the Garsia entropy of  $\beta$ , which will be defined in Section 1.1.

Further recent progress was made by Breuillard and Varjú [4], where it was proved that

$$H(\beta) \geq 0.44 \min\{\log 2, \log M_\beta\},$$

for any algebraic integer  $\beta \in (1, 2)$ , where  $H(\beta)$  is the Garsia entropy of  $\nu_\beta$  (see Section 1.1 for the definition) and  $M_\beta$  is the Mahler measure of  $\beta$  defined by  $M_\beta = \prod_{|\beta_i| > 1} |\beta_i|$ , where  $\beta_i$  are the algebraic conjugates (including  $\beta$  itself) of  $\beta$ . This implies that for an algebraic integer  $\beta \in (1, 2)$ ,  $\dim_{\mathbb{H}}(\nu_\beta) = 1$  if  $0.44 \min\{\log 2, \log M_\beta\} \geq \log \beta$  (see (1)).

In [5], Breuillard and Varjú showed, among other results, that

$$\{\beta \in (1, 2) : \dim_{\mathbb{H}}(\nu_\beta) < 1\} \subset \overline{\{\beta \in (1, 2) \cap \overline{\mathbb{Q}} : \dim_{\mathbb{H}}(\nu_\beta) < 1\}},$$

where  $\overline{\mathbb{Q}}$  is the set of algebraic numbers. This, together with Hochman's results, has sparked renewed interest in the study of Garsia entropy for algebraic parameters. If one were able to show that Pisot numbers are the only algebraic numbers corresponding to Bernoulli convolutions of dimension less than one, this would show that all non-Pisot  $\beta$  give rise to Bernoulli convolutions of dimension 1 (without the restriction that  $\beta$  should be algebraic).

There have also been recent results on the absolute continuity of Bernoulli convolutions. Shmerkin [22] proved further that  $\nu_\beta$  is absolutely continuous for all  $\beta \in (1, 2) \setminus \mathcal{E}$  where  $\mathcal{E}$  is a set of exceptions of Hausdorff dimension zero. In [24], Varjú gave new explicit examples of absolutely continuous Bernoulli convolutions. For a recent summary of progress on Bernoulli convolutions, see [25].

In this article we are interested in expressing the Garsia entropy and the dimension of Bernoulli convolutions  $\nu_\beta$  in terms of products of matrices. There is some precedent for this, see in particular [8, 10], but these previous ideas are based on tilings of the unit interval related to the construction of  $\nu_\beta$ , and cannot be generalised to the non-Pisot cases. We use a different approach to show that, for any algebraic integer  $\beta$ , one can construct matrices whose products encode information about the Garsia entropy. In particular, we give a sequence of lower bounds for the Garsia entropy which yield an explicit rate of convergence in the Garsia entropy formula.

## 1.1 Statement of Results

Let  $\Sigma := \{0, 1\}^{\mathbb{N}}$ . For  $p \in (0, 1)$ , let  $m_p$  denote the  $(p, 1-p)$ -Bernoulli product measure on  $\Sigma$  which gives weight  $p$  to digit 0 and weight  $1-p$  to digit 1. For  $\beta \in (1, 2)$ , the transformation  $\pi_\beta: \Sigma \rightarrow \mathbb{R}$  defined by

$$\pi_\beta: (a_i)_{i=1}^{\infty} \mapsto \sum_{i=1}^{\infty} a_i \beta^{-i},$$

maps the measure  $m_p$  to a measure  $\nu_{\beta,p}$  on  $\mathbb{R}$ . That is,  $\nu_{\beta,p} = m_p \circ \pi_\beta^{-1}$ . For  $p = \frac{1}{2}$ , we get the Bernoulli convolution  $\nu_\beta = \nu_{\beta, \frac{1}{2}}$ , which was defined in the previous section. For  $p \neq \frac{1}{2}$  we get a so-called biased Bernoulli convolution.

Given a word  $a_1 \cdots a_n \in \{0, 1\}^n$ , let the cylinder set  $[a_1 \cdots a_n]$  be defined by

$$[a_1 \cdots a_n] = \{\underline{b} = (b_i)_{i=1}^{\infty} \in \Sigma : b_1 \cdots b_n = a_1 \cdots a_n\}.$$

Given a sequence  $\underline{a} = (a_i)_{i=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ , let

$$\mathcal{N}_n(\underline{a}) = \mathcal{N}_n(a_1, \dots, a_n) = \left| \left\{ (b_1, \dots, b_n) \in \{0, 1\}^n : \sum_{i=1}^n b_i \beta^{-i} = \sum_{i=1}^n a_i \beta^{-i} \right\} \right|$$

and

$$\mathcal{M}_n(\underline{a}, p) = \sum_{\substack{b_1 \cdots b_n \in \{0,1\}^n \\ \sum_{i=1}^n b_i \beta^{-i} = \sum_{i=1}^n a_i \beta^{-i}}} m_p[b_1 \cdots b_n].$$

In what follows we write  $\mathcal{M}_n(\underline{a})$  or  $\mathcal{M}_n(a_1 \cdots a_n)$ , de-emphacising the dependence on  $p$  since we consider  $p$  to be fixed.

Let<sup>1</sup>

$$H_n(\beta, p) := - \sum_{a_1 \cdots a_n \in \{0,1\}^n} m_p[a_1 \cdots a_n] \log \mathcal{M}_n(a_1 \cdots a_n, p).$$

Finally we let

$$H(\beta, p) := \lim_{n \rightarrow \infty} \frac{1}{n} H_n(\beta, p).$$

$H(\beta, p)$  is called the Garsia entropy<sup>2</sup> of  $\nu_{\beta,p}$ . In particular, we write  $H_n(\beta) = H_n(\beta, 1/2)$  and  $H(\beta) = H(\beta, 1/2)$ .

Hochman [17] proved that if  $\beta \in (1, 2)$  is an algebraic number then the dimension of the Bernoulli convolution  $\nu_{\beta,p}$  is given by

$$\dim_{\mathbb{H}}(\nu_{\beta,p}) = \min \left\{ \frac{H(\beta, p)}{\log \beta}, 1 \right\}; \quad (1)$$

see also [4] for a more detailed explanation.

In this article we are concerned with lower bounds for  $H(\beta, p)$ , and hence lower bounds for  $\dim_{\mathbb{H}}(\nu_{\beta,p})$ , when  $\beta$  is an algebraic number. If  $\beta$  is not an algebraic integer, i.e. not the root of a polynomial with integer coefficients where the leading coefficient is 1, then  $H(\beta, p) = \log 2$ . Thus we may restrict our interest to algebraic integers.

Given an algebraic integer  $\beta = \beta^{(1)}$  of degree  $d$ , let  $\beta^{(2)}, \dots, \beta^{(d)}$  denote its Galois conjugates, ordered by decreasing absolute value.

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<sup>1</sup>It would be more standard to write

$$H_n(\beta, p) = \sum_x \mathcal{M}_n^*(x, p) \log \mathcal{M}_n^*(x, p),$$

where the sum is over all  $x$  having a representation  $x = \sum_{i=1}^n a_i \beta^{-i}$  and  $\mathcal{M}_n^*(x, p)$  is just  $\mathcal{M}_n(\underline{a}, p)$  for any  $\underline{a}$  with  $x = \sum_{i=1}^n a_i \beta^{-i}$ . These expressions are clearly equivalent, we find ours more convenient since we work only with sequences and since the above makes the link with Lyapunov exponents of pairs of matrices more direct.

<sup>2</sup>Beware, there are two different conventions for the definition of Garsia entropy. Some authors divide by  $\log(\beta)$  in the definition.

**Theorem 1.1.** *Let  $\beta$  be an algebraic integer of degree  $d$  and let  $p \in (0, 1)$ . The Garsia entropy  $H(\beta, p)$  can be approximated with explicit error bounds. In particular,*

$$\frac{1}{n}H_n(\beta, p) - \frac{C + l \log(n+1)}{n} \leq H(\beta, p) \leq \frac{1}{n}H_n(\beta, p)$$

for all  $n \in \mathbb{N}$ , where

$$C = \log \left( 2^d \prod_{i:|\beta^{(i)}| \neq 1} \frac{1}{||\beta^{(i)}| - 1|} + 1 \right),$$

and  $l$  is the number of conjugates of  $\beta$  of absolute value 1.

Theorem 1.1 is proved by giving lower bounds for  $H(\beta, p)$  in terms of products of matrices.

**Theorem 1.2.** *There exists a pair of non-negative matrices  $M_0 = M_0(\beta, p)$  and  $M_1 = M_1(\beta, p)$ , with rows and columns indexed by a set  $\mathcal{A}$ , such that the sequence*

$$\frac{1}{n}L_n(\beta, p) := -\frac{1}{n} \sup_{i \in \mathcal{A}} \sum_{a_1 \cdots a_n \in \{0,1\}^n} m_p[a_1 \cdots a_n] \log \left( \sum_{j \in \mathcal{A}} (M_{a_1} \cdots M_{a_n})_{i,j} \right)$$

converges to  $H(\beta, p)$  from below as  $n \rightarrow \infty$ , and  $\frac{1}{n}L_n(\beta, p) \leq H(\beta, p)$ .

The set  $\mathcal{A}$  is finite (with size bounded by  $C(\beta)$  given by (2)) whenever  $\beta$  is hyperbolic, i.e. when it has no Galois conjugates of modulus one. In this case the matrices  $M_0, M_1$  are computable by a finite time algorithm. If  $\beta$  is not hyperbolic then  $\mathcal{A}$  might be countably infinite, but the matrices  $M_0, M_1$  have at most two non-zero terms in any row.

Theorems 1.1 and 1.2 are proved by bounding the difference between  $H_n(\beta, p)$  and  $L_n(\beta, p)$ . When  $\beta$  is hyperbolic, and so  $\mathcal{A}$  is finite, Theorem 1.2 can be expressed in the more familiar form of the Lyapunov exponent of the pair of matrices  $M_0, M_1$ .

**Theorem 1.3.** *When  $\beta$  is hyperbolic, the sequence*

$$\frac{1}{n}L'_n(\beta, p) := -\frac{1}{n} \sum_{a_1 \cdots a_n \in \{0,1\}^n} m_p[a_1 \cdots a_n] \log(\|M_{a_1} \cdots M_{a_n}\|)$$

converges to  $H(\beta, p)$  as  $n \rightarrow \infty$ , and  $\frac{1}{n}L'_n(\beta, p) \leq \frac{1}{n}L_n(\beta, p) \leq H(\beta, p)$ . Here, the norm that we use is the  $\ell^\infty$  operator norm given by  $\|B\| = \sup_{i \in \mathcal{A}} \sum_{j \in \mathcal{A}} |B_{ij}|$ .

An immediate corollary is that we can express the Garsia entropy as the Lyapunov exponent of the matrices  $M_0, M_1$  associated with the  $(p, 1 - p)$ -Bernoulli product measure.

**Corollary 1.4.** *If  $\beta$  is hyperbolic then the Garsia entropy  $H(\beta, p)$  is the limit of the sequence*

$$-\frac{1}{n} \log \|M_{a_1} \cdots M_{a_n}\|$$

for  $m_p$ -a.e.  $\underline{a} \in \{0, 1\}^{\mathbb{N}}$ .

That Corollary 1.4 follows from Theorem 1.3 is an immediate application of the Furstenberg–Kesten theorem on random matrices [12].

## 2 Preliminary Results

In this section we recall some standard algebraic lemmas as well as ideas about separation of polynomials originating in the work of Garsia [13].

Let  $\beta = \beta^{(1)} \in (1, 2)$  be an algebraic integer of degree  $d$ . Let  $\beta^{(2)}, \dots, \beta^{(r)}$  denote the algebraic conjugates of  $\beta$  of modulus strictly larger than one,  $\beta^{(r+1)}, \dots, \beta^{(r+l)}$  conjugates of modulus 1, and  $\beta^{(r+l+1)}, \dots, \beta^{(d)}$  conjugates of modulus less than one.

The following lemmas are standard.

**Lemma 2.1.** *If  $\sum_{i=1}^n \epsilon_i \beta^{-i} = 0$  for  $\epsilon_i \in \{-1, 0, 1\}$  then*

$$\sum_{i=1}^n \epsilon_i (\beta^{(j)})^{-i} = 0$$

for each  $j \in \{2, \dots, d\}$ .

**Lemma 2.2.** *Let  $P$  be a polynomial with integer coefficients. Then the product  $P(\beta)P(\beta^{(2)}) \cdots P(\beta^{(d)})$  is an integer.*

Note that this second lemma requires that  $\beta$  is an algebraic integer, i.e. the root of a polynomial with integer coefficients whose leading term is 1. It does not hold for all algebraic numbers.

For  $n \geq 0$ , define the set  $V_{\beta,n} \subset \left[ \frac{-1}{\beta-1}, \frac{1}{\beta-1} \right]$  by

$$V_{\beta,n} := \left\{ x = \sum_{i=0}^n \epsilon_i \beta^{n-i} : \epsilon_i \in \{-1, 0, 1\} \right. \\ \left. \text{and } \left| \sum_{i=0}^n \epsilon_i (\beta^{(j)})^{n-i} \right| \leq \frac{1}{|\beta^{(j)}| - 1} \text{ for all } j \in \{1, \dots, r\} \right\}.$$

Clearly,  $V_{\beta,n} \subset V_{\beta,n+1}$ . Let

$$V_{\beta} := \bigcup_{n=0}^{\infty} V_{\beta,n}.$$

**Lemma 2.3.** *Suppose that  $\sum_{i=0}^n \epsilon_i \beta^{n-i} = 0$  for some  $n \geq 1$ . Then*

$$\sum_{i=0}^m \epsilon_i \beta^{m-i} \in V_{\beta,m} \subset V_{\beta}$$

for each  $m \in \{0, \dots, n\}$ .

*Proof.* Suppose on the contrary that

$$\sum_{i=0}^m \epsilon_i \beta^{m-i} \notin V_{\beta} \quad \text{for some } m \in \{0, 1, \dots, n-1\}.$$

Then by definition, there exists  $j \in \{1, \dots, r\}$  such that

$$\left| \sum_{i=0}^m \epsilon_i (\beta^{(j)})^{m-i} \right| > \frac{1}{|\beta^{(j)}| - 1}.$$

Then

$$\begin{aligned} \left| \sum_{i=0}^{m+1} \epsilon_i (\beta^{(j)})^{m+1-i} \right| &= \left| \epsilon_{m+1} + \beta^{(j)} \sum_{i=0}^m \epsilon_i (\beta^{(j)})^{m-i} \right| \\ &\geq |\beta^{(j)}| \left| \sum_{i=0}^m \epsilon_i (\beta^{(j)})^{m-i} \right| - 1 \\ &> \frac{|\beta^{(j)}|}{|\beta^{(j)}| - 1} - 1 = \frac{1}{|\beta^{(j)}| - 1}. \end{aligned}$$



Iterating this argument gives that

$$\left| \sum_{i=0}^n \epsilon_i (\beta^{(j)})^{n-i} \right| > \frac{1}{|\beta^{(j)}| - 1}.$$

But by Lemma 2.1 the quantity on the left hand side is equal 0, since  $\sum_{i=0}^n \epsilon_i \beta^{n-i} = 0$ . This gives a contradiction.  $\square$

Let

$$C(\beta) := 2^d \prod_{j=1}^r \frac{1}{|\beta^{(j)}| - 1} \prod_{k=r+l+1}^d \frac{1}{1 - |\beta^{(k)}|} = 2^d \prod_{|\beta^{(j)}| \neq 1} \frac{1}{||\beta^{(j)}| - 1|}. \quad (2)$$

This is a product over all roots which do not have modulus one. The following lemma is essentially due to Garsia, see also [11, 2, 20].

**Lemma 2.4.** *For  $n \geq 0$ , we have*

$$|V_{\beta,n}| \leq C(\beta)(n+1)^l + 1.$$

*In particular, if  $\beta$  is hyperbolic then  $V_{\beta}$  is finite.*

*Proof.* Fix  $n \geq 0$ . Let  $V'_{\beta,n} \subset \left[ \frac{-2}{\beta-1}, \frac{2}{\beta-1} \right]$  be given by

$$V'_{\beta,n} := \left\{ x = \sum_{i=0}^n \epsilon_i \beta^{n-i} : \epsilon_i \in \{-2, -1, 0, 1, 2\} \right. \\ \left. \text{and } \left| \sum_{i=0}^n \epsilon_i (\beta^{(j)})^{n-i} \right| \leq \frac{2}{|\beta^{(j)}| - 1} \text{ for all } j \in \{1, \dots, r\} \right\}.$$

For a non-zero  $x \in V'_{\beta,n}$ , given by

$$x = \sum_{i=0}^n \epsilon_i \beta^{n-i}$$

with  $\epsilon_i \in \{-2, -1, 0, 1, 2\}$ , write

$$x^{(j)} = \sum_{i=0}^n \epsilon_i (\beta^{(j)})^{n-i}, \quad j = 1, \dots, d.$$

Then

$$\prod_{j=1}^d |x^{(j)}| \geq 1, \quad (3)$$

since  $\prod_{j=1}^d x^{(j)}$  is an integer, which is non-zero as  $x \neq 0$ .

Now for  $j \in \{r+l+1, \dots, d\}$ ,

$$|x^{(j)}| = \left| \sum_{i=0}^n \epsilon_i (\beta^{(j)})^{n-i} \right| \leq \sum_{i=0}^n 2 |\beta^{(j)}|^{n-i} \leq \frac{2}{1 - |\beta^{(j)}|}.$$

Furthermore, for  $j \in \{2, \dots, r\}$ ,

$$|x^{(j)}| \leq \frac{2}{|\beta^{(j)}| - 1},$$

since  $x \in V'_{\beta, n}$ .

Finally, for  $j \in \{r+1, \dots, r+l\}$ ,

$$|x^{(j)}| = \left| \sum_{i=0}^n \epsilon_i (\beta^{(j)})^{n-i} \right| \leq \sum_{i=0}^n 2 \cdot 1^{n-i} \leq 2(n+1).$$

Then by (3),

$$|x| \geq \frac{1}{\prod_{i \in \{2, \dots, d\}} |x^{(i)}|} \geq C_0(n),$$

where

$$C_0(n) := 2^{-(d-1)} \left( \prod_{j \in \{2, \dots, r\}} (|\beta^{(j)}| - 1) \right) \frac{1}{(n+1)^l} \left( \prod_{j \in \{r+1, \dots, d\}} (1 - |\beta^{(j)}|) \right).$$

Hence, any  $x \in V'_{\beta, n} \setminus \{0\}$  has modulus at least  $C_0(n)$ .

Then for  $y, z \in V_{\beta, n}$  with  $y \neq z$ , we have

$$0 \neq y - z \in V'_{\beta, n}.$$

Hence  $|y - z| \geq C_0(n)$ . This shows that any two different elements of  $V_{\beta, n}$  are separated by at least  $C_0(n)$ . Therefore, since  $V_{\beta, n} \subset [-1/(\beta - 1), 1/(\beta - 1)]$ , this shows that  $V_{\beta, n}$  contains at most

$$\frac{2}{\beta - 1} \frac{1}{C_0(n)} + 1 = C(\beta)(n+1)^l + 1$$

elements. □

### 3 Matrices, Lyapunov Exponents, and Lower Bounds for Garsia Entropy

We now show how the sets  $V_{\beta,n}$  of the previous section have a natural graph structure, which allows one to compute lower bounds for Garsia entropy.

Start with the sets  $V_{\beta,0} = \{1, 0, -1\}$  and  $A_0 = \{1, 0, -1\}$ . At stage  $n \geq 1$  we let  $V_{\beta,n} = V_{\beta,n-1} \cup A_n$  where

$$A_n = \{\beta x - \epsilon_n : \epsilon_n \in \{-1, 0, 1\}, x \in A_{n-1}, \beta x - \epsilon_n \in V_{\beta}\}.$$

If  $\beta$  is hyperbolic, we stop the algorithm at the stage  $n$  for which  $V_{\beta,n} = V_{\beta,n-1}$ . Since in the hyperbolic case  $V_{\beta}$  is finite, the algorithm must stop in finite time with  $V_{\beta,n} = V_{\beta}$ . If  $\beta$  is not hyperbolic then  $V_{\beta}$  may be countably infinite, but  $V_{\beta,n}$  grows at most polynomially in  $n$ .

For each  $x, y \in V_{\beta}$ , draw a directed edge from  $x$  to  $y$ , labelled by  $\epsilon \in \{-1, 0, 1\}$ , whenever  $y = \beta x + \epsilon$ . Call the resulting graph  $\mathcal{G}$ .

There is a simple connection between the graph  $\mathcal{G}$  and the quantities  $\mathcal{N}_n(\underline{a})$ .

Suppose that for  $\underline{a} = (a_i)_{i=1}^{\infty}$  and  $\underline{b} = (b_i)_{i=1}^{\infty}$  we have

$$\sum_{i=1}^n a_i \beta^{-i} = \sum_{i=1}^n b_i \beta^{-i}.$$

Then, by the definition of  $V_{\beta}$  and Lemma 2.3, for each  $m \in \{1, \dots, n\}$  we have

$$d_m(\underline{a}, \underline{b}) := \beta^m \sum_{i=1}^m (a_i - b_i) \beta^{-i} \in V_{\beta}.$$

Then, letting  $d_0(\underline{a}, \underline{b}) := 0$ , we see that the word  $d_0(\underline{a}, \underline{b})d_1(\underline{a}, \underline{b}) \cdots d_n(\underline{a}, \underline{b})$  follows a path from 0 to 0 on the graph  $\mathcal{G}$ , following at each step  $i$  an edge labelled by  $(a_i - b_i) \in \{-1, 0, 1\}$ .

Given a word  $a_1 \cdots a_n \in \{0, 1\}^n$  and  $\epsilon_1 \cdots \epsilon_n \in \{-1, 0, 1\}^n$ , we write

$$\epsilon_1 \cdots \epsilon_n \sim a_1 \cdots a_n$$

if  $a_i - \epsilon_i \in \{0, 1\}$  for each  $i \in \{1, \dots, n\}$ . Then

$$\mathcal{N}_n(\underline{a}) = \left| \left\{ \epsilon_1 \cdots \epsilon_n \sim a_1 \cdots a_n \text{ such that there is a path from } 0 \text{ to } 0 \text{ in } \mathcal{G} \text{ obtained by following the edges } \epsilon_1 \cdots \epsilon_n \right\} \right|.$$

We can write down matrices which encode the choices of move  $\epsilon_i$  available given  $a_i$ .

Let  $x_1, x_2, \dots$  be some ordering of the elements of  $V_\beta$ , with  $x_1 = 0$ . Let  $\mathcal{A} = \{1, \dots, |V_\beta|\}$  if  $V_\beta$  is finite, and  $\mathbb{N}$  otherwise. We want to write down matrices  $M_0$  and  $M_1$  such that, for a word  $a_1 \cdots a_n$ ,

$$(M_{a_1} \cdots M_{a_n})_{i,j} = \sum_{\substack{b_1 \cdots b_n \in \{0,1\}^n \\ \beta^n x_i + \sum_{l=1}^n (a_l - b_l) \beta^{n-l} = x_j}} m_p[b_1 \cdots b_n]. \quad (4)$$

Let  $M_0$  be the  $|V_\beta| \times |V_\beta|$  matrix such that

$$(M_0)_{i,j} = \begin{cases} 1-p & \text{if } x_j = \beta x_i - 1 \\ p & \text{if } x_j = \beta x_i \\ 0 & \text{otherwise} \end{cases},$$

and let  $M_1$  be the  $|V_\beta| \times |V_\beta|$  matrix such that

$$(M_1)_{i,j} = \begin{cases} 1-p & \text{if } x_j = \beta x_i \\ p & \text{if } x_j = \beta x_i + 1 \\ 0 & \text{otherwise} \end{cases}.$$

**Lemma 3.1.** 1. For  $x_i, x_j \in V_\beta$ ,

$$(M_{a_1} \cdots M_{a_n})_{ij} = \sum_{\substack{b_1 \cdots b_n \in \{0,1\}^n \\ \beta^n x_i + \sum_{l=1}^n (a_l - b_l) \beta^{n-l} = x_j}} m_p[b_1 \cdots b_n].$$

2. If  $\sum_{i=1}^n a_i \beta^{-i} = \sum_{i=1}^n c_i \beta^{-i}$  for some  $a_1 \cdots a_n, c_1 \cdots c_n \in \{0,1\}^n$ , then

$$M_{a_1} \cdots M_{a_n} = M_{c_1} \cdots M_{c_n}.$$

*Proof.* Part 1 follows immediately from the definition of  $M_0$  and  $M_1$ , and part 2 follows directly from part 1.  $\square$

In particular,

$$\mathcal{M}_n(\underline{a}) = (M_{a_1} M_{a_2} \cdots M_{a_n})_{11},$$

and so we have immediately that

$$H_n(\beta, p) = - \sum_{a_1 \cdots a_n \in \{0,1\}^n} m_p[a_1 \cdots a_n] \log((M_{a_1} M_{a_2} \cdots M_{a_n})_{11}).$$

As  $H_n(\beta, p)$  is subadditive,  $\frac{1}{n} H_n(\beta, p)$  forms a sequence which converges to  $H(\beta, p)$ . We have  $H(\beta, p) \leq \frac{1}{n} H_n(\beta, p)$ .

### 3.1 Lower Bounds

For an algebraic integer  $\beta$ , we define

$$\begin{aligned} L_n(\beta, p) &:= - \sup_{i \in \mathcal{A}} \sum_{a_1 \cdots a_n \in \{0,1\}^n} m_p[a_1 \cdots a_n] \log \left( \sum_{j \in \mathcal{A}} (M_{a_1} \cdots M_{a_n})_{i,j} \right) \\ &= - \sup_{i \in \mathcal{A}} \int_{\underline{a} \in \{0,1\}^{\mathbb{N}}} \log \left( \sum_{j \in \mathcal{A}} (M_{a_1} \cdots M_{a_n})_{i,j} \right) dm(\underline{a}). \end{aligned}$$

Since

$$M_{11} \leq \sum_{j \in \mathcal{A}} M_{1j}$$

for any non-negative matrix  $M$ , we have, by choosing  $i = 1$  in the above definition, that  $L_n(\beta, p) \leq H_n(\beta, p)$ . Here, and in much of what follows, we note the minus in the definition of  $H_n(\beta, p)$  and  $L_n(\beta, p)$  which reverses a lot of inequalities.

**Lemma 3.2.**

$$L_{n+m}(\beta, p) \geq L_n(\beta, p) + L_m(\beta, p).$$

*Proof.* For  $i \in \mathcal{A}$ ,  $\underline{a} \in \Sigma$  we have

$$\begin{aligned} \sum_{j \in \mathcal{A}} (M_{a_1} \cdots M_{a_{n+m}})_{i,j} &= \sum_{j \in \mathcal{A}} \sum_{k \in \mathcal{A}} (M_{a_1} \cdots M_{a_n})_{i,k} (M_{a_{n+1}} \cdots M_{a_{n+m}})_{k,j} \\ &= \sum_{k \in \mathcal{A}} (M_{a_1} \cdots M_{a_n})_{i,k} \left( \sum_{j \in \mathcal{A}} (M_{a_{n+1}} \cdots M_{a_{n+m}})_{k,j} \right) \\ &\leq \sum_{k \in \mathcal{A}} (M_{a_1} \cdots M_{a_n})_{i,k} \left( \sup_{l \in \mathcal{A}} \sum_{j \in \mathcal{A}} (M_{a_{n+1}} \cdots M_{a_{n+m}})_{l,j} \right). \end{aligned}$$

It follows that

$$\begin{aligned} L_{n+m}(\beta, p) &= - \sup_{i \in \mathcal{A}} \int_{\underline{a} \in \{0,1\}^{\mathbb{N}}} \log \left( \sum_{j \in \mathcal{A}} (M_{a_1} \cdots M_{a_{n+m}})_{i,j} \right) dm(\underline{a}) \\ &\geq - \sup_{i \in \mathcal{A}} \sup_{l \in \mathcal{A}} \int_{\underline{a} \in \{0,1\}^{\mathbb{N}}} \int_{\underline{b} \in \{0,1\}^{\mathbb{N}}} \left( \log \left( \sum_{k \in \mathcal{A}} (M_{a_1} \cdots M_{a_n})_{i,k} \right) \right. \\ &\quad \left. + \log \left( \sum_{j \in \mathcal{A}} (M_{b_1} \cdots M_{b_m})_{l,j} \right) \right) dm(\underline{a}) dm(\underline{b}) \\ &= L_n(\beta, p) + L_m(\beta, p). \quad \square \end{aligned}$$

**Proposition 3.3.** *Let  $\beta \in (1, 2)$  be an algebraic integer. Then the sequence  $(\frac{1}{n}L_n(\beta, p))$  satisfies*

$$\frac{1}{n}H_n(\beta, p) - \frac{1}{n}\log(C(\beta)(n+1)^l + 1) \leq \frac{1}{n}L_n(\beta, p) \leq H(\beta, p) \leq \frac{1}{n}H_n(\beta, p).$$

*Proof.* We have proved that  $L_n(\beta, p)$  is superadditive. Since  $H_n(\beta, p)$  is subadditive,  $\frac{1}{n}H_n(\beta, p)$  converges to  $H(\beta, p)$  and  $\frac{1}{n}H_n(\beta, p) \geq \frac{1}{n}L_n(\beta, p)$ , we see that

$$H(\beta, p) \in \left( \frac{1}{n}L_n(\beta, p), \frac{1}{n}H_n(\beta, p) \right)$$

for all  $n \in \mathbb{N}$ . Hence we need only to prove the left hand inequality.

Let

$$X_n := \left\{ \sum_{i=1}^n a_i \beta^{-i} : a_i \in \{0, 1\} \right\}.$$

For  $x \in X_n$  let  $M_{x,n} := M_{a_1} \cdots M_{a_n}$  for any of the words  $a_1 \cdots a_n$  for which

$$x = \sum_{i=1}^n a_i \beta^{-i}.$$

This is well defined due to Lemma 3.1. Now

$$\begin{aligned} L_n(\beta, p) &:= - \sup_{i \in \mathcal{A}} \sum_{a_1 \cdots a_n \in \{0,1\}^n} m_p[a_1 \cdots a_n] \log \left( \sum_{j \in \mathcal{A}} (M_{a_1} \cdots M_{a_n})_{i,j} \right) \\ &= - \sup_{i \in \mathcal{A}} \sum_{x \in X_n} (M_{x,n})_{1,1} \log \left( \sum_{j \in \mathcal{A}} (M_{x,n})_{i,j} \right) \\ &= - \sum_{x \in X_n} (M_{x,n})_{1,1} \log((M_{x,n})_{1,1}) \\ &\quad - \sup_{i \in \mathcal{A}} \sum_{x \in X_n} (M_{x,n})_{1,1} \log \left( \frac{\sum_{j \in \mathcal{A}} (M_{x,n})_{i,j}}{(M_{x,n})_{1,1}} \right). \end{aligned}$$

The first term here is  $H_n(\beta, p)$ . Since  $\sum_{x \in X_n} (M_{x,n})_{1,1} = 1$ , we move this inside the log in the second term, and using the concavity of log we get

$$L_n(\beta, p) \geq H_n(\beta, p) - \sup_{i \in \mathcal{A}} \log \left( \sum_{x \in X_n} \sum_{j \in \mathcal{A}} (M_{x,n})_{i,j} \right).$$

Now recall that  $(M_{x,n})_{i,j}$  counts, for any  $a_1 \cdots a_n$  such that  $\sum_{l=1}^n a_l \beta^{-l} = x$ , the total measure of the words  $b_1 \cdots b_n$  for which

$$\beta^n x_i + \sum_{l=1}^n (a_l - b_l) \beta^{n-l} = x_j.$$

This can be rewritten as

$$\beta^n x_i + \beta^n x - \sum_{l=1}^n b_l \beta^{n-l} = x_j. \quad (5)$$

In order to sum this over all  $x \in X_n$  and  $j \in \mathcal{A}$ , we count for each  $b_1 \cdots b_n \in \{0,1\}^n$  the number of  $x \in X_n$  for which an equation of the form (5) is satisfied. This gives

$$\begin{aligned} \sum_{x \in X_n} \sum_{j \in \mathcal{A}} (M_{x,n})_{i,j} &= \sum_{x \in X_n} \sum_{j \in \mathcal{A}} \sum_{\substack{b_1 \cdots b_n \in \{0,1\}^n \\ (5) \text{ holds}}} m_p[b_1 \cdots b_n] \\ &= \sum_{b_1 \cdots b_n \in \{0,1\}^n} m_p[b_1 \cdots b_n] \cdot |X_n(i, b_1 \cdots b_n)|, \end{aligned}$$

where

$$X_n(i, b_1 \cdots b_n) = \left\{ x \in X_n : \beta^n x_i + \left( \beta^n x - \sum_{l=1}^n b_l \beta^{n-l} \right) \in V_\beta \right\}.$$

But now the separation arguments of Lemma 2.4 give that, for a fixed  $i$  and  $b_1 \cdots b_n$ , sums of the form  $\beta^n x - \sum_{l=1}^n b_l \beta^{n-l}$  are separated by at least  $C_0(n)$  unless they are equal. This bounds the number of elements of  $X_n(i, b_1 \cdots b_n)$ . Indeed, all possible values of

$$\beta^n x - \sum_{l=1}^n b_l \beta^{n-l}, \quad x \in X_n(i, b_1 \cdots b_n),$$

are contained in the interval  $[-\beta^n x_i - \frac{1}{\beta-1}, -\beta^n x_i + \frac{1}{\beta-1}]$  and they are separated by at least  $C_0(n)$ . Hence  $\beta^n x - \sum_{l=1}^n b_l \beta^{n-l}$  may attain at most

$$\frac{2}{\beta-1} \frac{1}{C_0(n)} + 1$$

different values, that is

$$|X_n(i, b_1 \cdots b_n)| \leq \frac{2}{\beta - 1} \frac{1}{C_0(n)} + 1 = C(\beta)(n + 1)^l + 1.$$

Thus

$$\begin{aligned} \sum_{x \in X_n} \sum_{j \in \mathcal{A}} (M_{x,n})_{i,j} &= \sum_{b_1 \cdots b_n \in \{0,1\}^n} m_p[b_1 \cdots b_n] |X_n(i, b_1 \cdots b_n)| \\ &\leq (C(\beta)(n + 1)^l + 1), \end{aligned} \tag{6}$$

and so

$$L_n(\beta, p) \geq H_n(\beta, p) - \log(C(\beta)(n + 1)^l + 1). \quad \square$$

This completes the proof of both Theorem 1.1 and 1.2.

As pointed out by one of the referees, Proposition 3.3 has the following interesting consequence.

**Corollary 3.4.** *Let  $\beta \in (1, 2)$  be an algebraic number. Then  $H(\beta, p)$  is continuous in  $p$ , and so is  $\dim_{\mathbb{H}} \nu_{\beta, p}$ .*

It arises a natural question whether  $\dim_{\mathbb{H}} \nu_{\beta, p}$  is continuous in  $p$  for every real  $\beta \in (1, 2)$ . To our best knowledge, the question is open. Nevertheless, it is known that  $\dim_{\mathbb{H}} \nu_{\beta, p}$  is always lower semi-continuous in  $(\beta, p)$  (see e.g. [18, Theorem 1.8]).

## 3.2 A matrix form for the hyperbolic case

We briefly comment on two alternative lower bounds which work for the hyperbolic case and are much easier to work with. Let

$$L'_n(\beta, p) := - \sum_{a_1 \cdots a_n \in \{0,1\}^n} m_p[a_1 \cdots a_n] \log(\|M_{a_1} \cdots M_{a_n}\|).$$

$L'_n(\beta, p)$  differs from  $L_n(\beta, p)$  in that the supremum over  $i \in \mathcal{A}$  happens inside the summation. Thus  $L'_n(\beta, p) \leq L_n(\beta, p)$ .

**Lemma 3.5.** *When the set  $V_\beta$  is finite, we have*

$$H(\beta, p) = \lim_{n \rightarrow \infty} \frac{1}{n} L'_n(\beta, p).$$



Proving this lemma completes the proof of Theorem 1.3.

*Proof.* Following the proof of Proposition 3.3 (or simply applying Lemma 9.9 in [26]) gives exactly

$$L'_n(\beta, p) \geq H_n(\beta, p) - \log\left(\sum_{x \in X_n} \|M_{x,n}\|\right).$$

But

$$\begin{aligned} -\log\left(\sum_{x \in X_n} \|M_{x,n}\|\right) &= -\log\left(\sum_{x \in X_n} \max_{i \in \mathcal{A}} \sum_{j \in \mathcal{A}} (M_{x,n})_{i,j}\right) \\ &\geq -\log\left(\sum_{x \in X_n} \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{A}} (M_{x,n})_{i,j}\right) \\ &\geq -\log(|\mathcal{A}|(C(\beta) + 1)), \end{aligned}$$

where the last line uses inequality (6) (with  $l = 0$ ), summing both sides over  $i \in \mathcal{A}$ . Then

$$\frac{1}{n} L'_n(\beta, p) \geq \frac{1}{n} H_n(\beta, p) - \frac{1}{n} \log(|\mathcal{A}|(C(\beta) + 1)),$$

giving that  $\frac{1}{n} L'_n(\beta, p)$  converges to  $H(\beta, p)$  as required.  $\square$

Norms of random products of matrices are extremely well studied, and so putting our lower bound for  $H(\beta, p)$  in the above form may yield useful computations.

We now describe another bound from below on  $H(\beta, p)$ , which is computationally very simple, and which is sometimes sufficient to conclude that the Hausdorff dimension of  $\nu_{\beta,p}$  is 1.

**Proposition 3.6.** *Suppose that  $V_\beta$  is finite. Let  $\lambda$  be the largest eigenvalue of the matrix  $((1-p)M_0 + pM_1)$ . Then*

$$-\log \lambda \leq H(\beta, p),$$

and hence  $\dim_{\text{H}} \nu_\beta \geq \min\left\{1, \frac{-\log(\lambda)}{\log(\beta)}\right\}$ .

*Proof.* We use the norm  $\|M\|_1 = \sum_{i,j} |M_{i,j}|$ . For non-negative matrices  $A$  and  $B$ , we have  $\|A\|_1 + \|B\|_1 = \|A + B\|_1$ .

We have

$$\begin{aligned}
\frac{1}{n} L'_n(\beta, p) &= -\frac{1}{n} \sum_{a_1 \cdots a_n} m_p[a_1 \cdots a_n] \log \left( \|M_{a_1} \cdots M_{a_n}\| \right) \\
&\geq -\frac{1}{n} \sum_{a_1 \cdots a_n} m_p[a_1 \cdots a_n] \log \left( \|M_{a_1} \cdots M_{a_n}\|_1 \right) \\
&\geq -\frac{1}{n} \log \left( \sum_{a_1 \cdots a_n} m_p[a_1 \cdots a_n] \|M_{a_1} \cdots M_{a_n}\|_1 \right) \\
&= -\frac{1}{n} \log \left( \left\| \sum_{a_1 \cdots a_n} m_p(a_1 \cdots a_n) M_{a_1} \cdots M_{a_n} \right\|_1 \right) \\
&= -\frac{1}{n} \log \left\| \left( (1-p)M_0 + pM_1 \right)^n \right\|_1.
\end{aligned}$$

By Proposition 3.3,  $\frac{1}{n} L'_n(\beta, p)$  is a lower bound on  $H(\beta, p)$  and since

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left\| \left( (1-p)M_0 + pM_1 \right)^n \right\|_1 = \log \lambda,$$

we have  $H(\beta, p) \geq -\log \lambda$ . □

Since computing eigenvalues is extremely rapid, this approach is the one that we use in practice for proving that  $\dim_{\mathbb{H}}(\nu_{\beta,p}) = 1$  for a variety of examples.

### 3.3 Computational ideas and examples

In this section we describe how to use Proposition 3.3 to get explicit bounds on  $H(\beta) = H(\beta, 1/2)$  and hence on  $\dim_{\mathbb{H}} \nu_{\beta}$  for specific examples. For the remainder of the article we concern ourselves only with the case of unbiased Bernoulli convolutions, and no longer include  $p$  as a variable.

Suppose  $\beta$  is hyperbolic. Then one easily writes a computer program which finds the (finite) graph  $\mathcal{G}$  and the matrices  $M_0$  and  $M_1$ . By Proposition 3.3, we have

$$\frac{1}{n} L_n(\beta) \leq H(\beta) \leq \frac{1}{n} H_n(\beta).$$

Expressed as a bound on  $\dim_{\mathbb{H}} \nu_{\beta}$ , it says

$$\min \left\{ 1, \frac{1}{n} \frac{L_n(\beta)}{\log \beta} \right\} \leq \dim_{\mathbb{H}} \nu_{\beta} \leq \min \left\{ 1, \frac{1}{n} \frac{H_n(\beta)}{\log \beta} \right\}.$$

Given an  $n$  one can, with a computer, calculate numerically the above lower and upper bounds on  $H(\beta)$  and  $\dim_{\mathbb{H}} \nu_{\beta}$ . Unfortunately, the convergence is quite slow, and the computational complexity is high, since evaluating  $L_n(\beta)$  and  $H_n(\beta)$  involves summing over  $2^n$  different sequences.

There is a way to somewhat improve the convergence by pruning the graph  $\mathcal{G}$ . Call a vertex  $x$  redundant if there is no path to  $\{0\}$  from  $x$  along edges in the graph. Clearly, a vertex is redundant, if and only if all edges from the vertex lead to redundant vertices. We remove all redundant vertices from  $\mathcal{G}$  and get a new graph which we denote by  $\mathcal{G}'$ . Using instead this pruned graph to define  $\tilde{L}_n(\beta)$  in the same way as the definition of  $L'_n(\beta)$ , the above bounds on  $H(\beta)$  and  $\dim_{\mathbb{H}} \nu_{\beta}$  hold with  $L'_n(\beta)$  replaced by  $\tilde{L}_n(\beta)$ .

**Example 3.7.** To illustrate the above, we let  $\beta$  be the largest root of the equation

$$\beta^4 - \beta^3 - \beta^2 + \beta - 1 = 0.$$

Here  $\beta \approx 1.5129$  has one other conjugate  $\beta^{(2)}$  larger than one in modulus,  $\beta^{(2)} \approx -1.1787$ .  $\beta$  also has two conjugates less than one in modulus, both of which are complex.  $\mathcal{G}$  consists of 71 vertices and  $\mathcal{G}'$  consists of 21 vertices.

Using the graph  $\mathcal{G}$  and  $n = 9$ , we find that

$$\frac{1}{n} \frac{L'_n(\beta)}{\log \beta} = 0.77199 \leq \frac{H(\beta)}{\log \beta} \leq 1.5763 = \frac{1}{n} \frac{H_n(\beta)}{\log \beta}.$$

Using instead the pruned graph  $\mathcal{G}'$  and  $n = 9$ , we find that

$$\frac{1}{n} \frac{\tilde{L}_n(\beta)}{\log \beta} = 1.0006 \leq \frac{H(\beta)}{\log \beta}.$$

We conclude that  $\dim_{\mathbb{H}}(\nu_{\beta}) = 1$ . We remark that this result does not follow from the aforementioned work of Breuillard and Varjú [4], since in this example

$$\frac{0.44 \min\{\log 2, \log M_{\beta}\}}{\log \beta} \approx 0.6146 < 1.$$

As is illustrated in the above example, even if the upper and lower bounds are far apart, they can still be useful to prove that the Hausdorff dimension is 1. For the number  $\beta$  in the example it is sufficient to take  $n = 9$  in order to prove that the dimension is 1. For some other numbers, one needs to take larger values of  $n$ , resulting in very long computation times.

### 3.3.1 Using Proposition 3.6

We now give some examples to show the advantage of using Proposition 3.6. The key advantage of this proposition lies in the fact that eigenvalues are numerically quick to compute.

**Example 3.8.** We take  $\beta$  as in Example 3.7. Proposition 3.6 gives

$$\frac{H(\beta)}{\log \beta} \geq 1.3867$$

and hence  $\dim_{\mathbb{H}} \nu_{\beta} = 1$ .

The lower bound for  $H(\beta, p)$  given in Proposition 3.6 is not tight. By looking at Bernoulli convolutions associated with Pisot numbers one can see how far off the true value it is for some examples.

**Example 3.9.** Let  $\beta$  be the Golden ratio. Alexander and Zagier showed that that  $\dim_{\mathbb{H}} \nu_{\beta} = 0.995570\dots$  [3]. Proposition 3.6 gives

$$\frac{H(\beta)}{\log \beta} \geq 0.9924$$

and hence  $\dim_{\mathbb{H}} \nu_{\beta} \geq 0.9924$ .

The fact that the dimension of the Bernoulli convolution in the previous example is known so accurately is due to special properties of the Golden ratio. Outside of a special class of Pisot numbers known as multinacci numbers, there are not many examples of Pisot numbers for which the Hausdorff dimension of  $\nu_{\beta}$  was known to three decimal places before the present work, see [16]. See also [8], in which Feng calculated the Hausdorff dimension with high precision for multinacci numbers.

Below, we give an example of what information can be obtained from Proposition 3.6 for Pisot numbers that are not multinacci numbers.

**Example 3.10.** Let  $\beta$  be the Pisot number given by  $\beta^3 - \beta - 1 = 0$ . Since  $\beta$  is a Pisot number, we have  $\dim_{\mathbb{H}} \nu_{\beta} < 1$ . Proposition 3.6 gives

$$\frac{H(\beta)}{\log \beta} \geq 0.99999.$$

Hence  $0.99999 \leq \dim_{\mathbb{H}} \nu_{\beta} < 1$  and we have obtained the Hausdorff dimension of  $\nu_{\beta}$  to five decimal places.

In fact, the numbers in Examples 3.9 and 3.10 already appeared in a work of Lau [19], but not in connection with Hausdorff dimension. The link between the lower bound for  $\dim_{\mathbb{H}}(\nu_{\beta})$  of our Proposition 3.6 and Lau's Mean Quadratic Variation Dimension are explained in Section 4. We will then see that the lower bound obtained in Example 3.10 follows from Lau's work together with that of Ngai [21].

Finally, we apply our methods to the study of hyperbolic  $\beta$  of degree 4 and 5.

**Example 3.11.** Let  $\beta$  satisfy

$$a_5\beta^5 + a_4\beta^4 + \cdots + a_0 = 0$$

with each  $a_i \in \{-1, 0, 1\}$ . Suppose that  $\beta$  is hyperbolic. Then either  $\beta$  is Pisot and  $\nu_{\beta}$  has Hausdorff dimension less than one, or  $\beta$  is not Pisot and  $\nu_{\beta}$  has dimension one. The computations are given in Tables 1 and 2, which shows all hyperbolic  $\beta$  that are roots of a  $\{-1, 0, 1\}$ -polynomial of degree 2, 3, 4 or 5. Paul Mercat has also done these computations and confirms our results (private communication).

## 4 Connections with mean quadratic variation dimension

In this section, we give an interesting relation between the lower bound  $\frac{-\log \lambda}{\log \beta}$  which follows from Proposition 3.6, and another quantity (the so-called mean quadratic variation dimension or  $L^2$ -spectrum) of Bernoulli convolutions. To begin with, let  $\mu$  be a Borel probability measure in  $\mathbb{R}^d$  with compact support.

Polynomial	$\beta$	type	lower bound ( $\frac{-\log \lambda}{\log \beta}$ )	size of $\mathcal{G}'$
$x^2 - x - 1$	1.6180	Pisot	0.99240	5
$x^3 - x^2 - x - 1$	1.8393	Pisot	0.96422	7
$x^3 - x^2 - 1$	1.4656	Pisot	0.99912	49
$x^3 - x - 1$	1.3247	Pisot	0.99999	179
$x^4 - x^3 - x^2 - x - 1$	1.9276	Pisot	0.97333	9
$x^4 - x^3 - x^2 + x - 1$	1.5129	not Pisot	1.38670	21
$x^4 - x^3 - 1$	1.3803	Pisot	0.99999	1253
$x^4 - x^3 + x^2 - x - 1$	1.2906	not Pisot	2.50349	9
$x^4 - x^2 - 1$	1.2720	not Pisot	1.98480	25
$x^4 - x - 1$	1.2207	not Pisot	1.61576	1253
$x^4 + x^3 - x^2 - x - 1$	1.1787	not Pisot	3.49147	21

Table 1: Lower bounds on  $\dim_{\mathbb{H}} \nu_{\beta}$  for all hyperbolic  $\beta$  of degree 2, 3 and 4. If the lower bound is larger than one, then  $\dim_{\mathbb{H}} \nu_{\beta} = 1$ .

For  $x \in \mathbb{R}^d$  and  $r > 0$ , let  $B_r(x)$  denote the closed ball centered at  $x$  of radius  $r$ . For  $q \in \mathbb{R}$ , the  $L^q$ -spectrum of  $\mu$  is defined as

$$\tau(\mu, q) = \liminf_{r \rightarrow 0} \frac{\log \Theta_{\mu}(q; r)}{\log r},$$

where

$$\Theta_{\mu}(q; r) = \sup \sum_i \mu(B_r(x_i))^q, \quad r > 0, q \in \mathbb{R}, \quad (7)$$

and the supremum is taken over all families of disjoint balls  $\{B_r(x_i)\}_i$  with  $x_i \in \text{supp}(\mu)$ . In the case  $q = 2$ ,  $\tau_{\mu}(2)$  is also called the *mean quadratic variation dimension* of  $\mu$  (see [19]).

By definition,  $\tau(\mu, 1) = 0$  and  $\tau(\mu, q)$  is a concave function of  $q$  over  $\mathbb{R}$ . It follows that  $\frac{\tau(\mu, q)}{q-1}$  is monotone decreasing over  $(1, \infty)$ , and so  $\tau'(\mu, 1+) \geq \frac{\tau(\mu, q)}{q-1}$  for each  $q > 1$ . In [21], Ngai proved that  $\dim_{\mathbb{H}}(\mu) \geq \tau'(\mu, 1+)$ . As a consequence,  $\dim_{\mathbb{H}}(\mu) \geq \frac{\tau(\mu, q)}{q-1}$  for every  $q > 1$ ; moreover, the equality  $\dim_{\mathbb{H}} \mu = \frac{\tau(\mu, q)}{q-1}$ , if it holds for some  $q > 1$ , will imply that  $\tau(\mu, \cdot)$  is affine over  $[1, q]$ .

The concept of  $L^q$ -spectrum plays a fundamental role in multifractal analysis. In [19], Lau computed the  $L^2$ -spectrum of the Bernoulli convolution  $\nu_{\beta}$  when  $\beta$  is a Pisot number. As pointed out by one of the referees, in the Pisot case

Polynomial	$\beta$	type	lower bound ( $\frac{-\log \lambda}{\log \beta}$ )	size of $\mathcal{G}'$
$x^5 - x^4 - x^3 - x^2 - x - 1$	1.9659	Pisot	0.98357	11
$x^5 - x^4 - x^3 - x^2 - 1$	1.8885	Pisot	0.98227	739
$x^5 - x^4 - x^3 - x^2 + 1$	1.7785	Pisot	0.99576	947
$x^5 - x^4 - x^3 - x^2 + x - 1$	1.7924	not Pisot	1.12741	13
$x^5 - x^4 - x^3 - x - 1$	1.8124	Pisot	0.98243	349
$x^5 - x^4 - x^3 - x + 1$	1.6804	not Pisot	1.17467	139
$x^5 - x^4 - x^3 - 1$	1.7049	Pisot	0.99304	339
$x^5 - x^4 - x^3 + x - 1$	1.5499	not Pisot	1.1971	1133
$x^5 - x^4 - x^3 + x^2 - 1$	1.4432	Pisot	0.99998	5385
$x^5 - x^4 - x^2 - x - 1$	1.6851	not Pisot	1.1072	1265
$x^5 - x^4 - x^2 - x + 1$	1.5262	not Pisot	1.4420	139
$x^5 - x^4 - x^2 - 1$	1.5702	Pisot	0.99986	841
$x^5 - x^4 - x^2 + x - 1$	1.4036	not Pisot	1.3664	1239
$x^5 - x^4 - x - 1$	1.4971	not Pisot	1.4216	57
$x^5 - x^4 + x^2 - x - 1$	1.2628	not Pisot	2.4946	131
$x^5 - x^4 + x^3 - x^2 - x - 1$	1.4076	not Pisot	1.9447	11
$x^5 - x^4 + x^3 - x^2 - 1$	1.2499	not Pisot	1.8291	1617
$x^5 - x^4 + x^3 - x - 1$	1.2083	not Pisot	3.3882	11
$x^5 - x^3 - x^2 - x - 1$	1.5342	Pisot	0.99983	2635
$x^5 - x^3 - x^2 - x + 1$	1.3690	not Pisot	1.8252	947
$x^5 - x^3 - x^2 - 1$	1.4291	not Pisot	1.6106	43
$x^5 - x^3 - x^2 + x - 1$	1.2828	not Pisot	2.5554	13
$x^5 - x^3 - 1$	1.2365	not Pisot	1.4629	11173
$x^5 - x^3 + x^2 - x - 1$	1.2000	not Pisot	3.1452	947
$x^5 - x^2 - 1$	1.1939	not Pisot	1.9356	3387
$x^5 - x - 1$	1.1673	not Pisot	2.2193	2807
$x^5 + x^3 - x^2 - x - 1$	1.1436	not Pisot	4.3718	97
$x^5 + x^4 - x^3 - x^2 - 1$	1.1595	not Pisot	4.3005	13
$x^5 + x^4 - x^3 - x - 1$	1.1408	not Pisot	4.6298	139
$x^5 + x^4 - x^2 - x - 1$	1.1237	not Pisot	2.5707	6485

Table 2: Lower bounds on  $\dim_{\mathbb{H}} \nu_{\beta}$  for all hyperbolic  $\beta$  of degree 5. If the lower bound is larger than one, then  $\dim_{\mathbb{H}} \nu_{\beta} = 1$ .

there is certain coincidence between the numerical estimates of our bound  $\frac{-\log \lambda}{\log \beta}$  in Tables 1 and 2 and that of the  $L^2$ -spectrum of  $\nu_\beta$  given by Lau [19]. The following result justifies this connection.

**Proposition 4.1.** *Suppose that  $\beta \in (1, 2)$  is a Pisot number. Let  $\lambda$  be the largest eigenvalue of the matrix  $((1-p)M_0 + pM_1)$ . Then*

$$\tau(\nu_{\beta,p}, 2) = -\frac{\log \lambda}{\log \beta}.$$

Moreover,  $\dim_{\mathbb{H}}(\nu_{\beta,p}) > \frac{\tau(\nu_{\beta,p}, q)}{q-1}$  for any  $q > 1$ .

*Proof.* Let  $\beta$  be a Pisot number in  $(1, 2)$ . The second part of the proposition follows from the fact that  $\tau(\nu_{\beta,p}, q)$  is strictly concave over the region  $(0, \infty)$ . This fact comes from some results obtained in [7, 9]. Indeed, by Theorem 5.2 in [7], there exist  $\ell, r \in \mathbb{N}$  and a tuple of non-negative  $r \times r$  matrices  $(A_1, \dots, A_\ell)$  such that  $\sum_{i=1}^{\ell} A_i$  is irreducible and for any  $q > 0$ ,

$$\tau(\nu_{\beta,p}, q) = -\frac{P(q)}{\log \beta},$$

where  $P(q) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i_1 \dots i_n \in \{1, \dots, \ell\}^n} (\|A_{i_1} \cdots A_{i_n}\|_1)^q$ . Suppose on the contrary that  $\tau(\nu_{\beta,p}, \cdot)$  is not strictly concave on  $(0, \infty)$ . Then  $\tau(\nu_{\beta,p}, \cdot)$  is affine on some interval  $(a, b)$  with  $0 < a < b$ , and so is the function  $P(\cdot)$ . This implies that (see the proof of Theorem 1.8 in [9])  $(A_1, \dots, A_\ell)$  has uniform exponent modulo 0 in the sense that there exist  $c, t > 0$  such that for each  $n \in \mathbb{N}$  and any  $i_1 \cdots i_n \in \{1, \dots, \ell\}^n$ ,

$$\text{either } A_{i_1} \dots A_{i_n} = 0 \quad \text{or} \quad c^{-1}t^n \leq \|A_{i_1} \cdots A_{i_n}\|_1 \leq ct^n.$$

Furthermore, this implies that  $\nu_{\beta,p}$  is absolutely continuous (see Theorem 6.2 of [9]), which gives a contradiction since  $\nu_{\beta,p}$  is singular ([6]).

Next we prove the first part of the proposition, i.e.  $\tau(\nu_{\beta,p}, 2) = -\frac{\log \lambda}{\log \beta}$ . For this purpose, we first claim that for each  $q > 0$ , there exists a constant  $C = C(\beta, p, q) > 0$  such that

$$C^{-1}u_n(q) \leq \Theta_{\nu_{\beta,p}}(q; \beta^{-n}) \leq Cu_n(q), \tag{8}$$

where  $\Theta$  is defined as in (7) and

$$u_n(q) := \sum_{a_1 \cdots a_n \in \{0,1\}^n} m_p[a_1 \cdots a_n] ((M_{a_1} \cdots M_{a_n})_{1,1})^{q-1}.$$



To prove (8), for each  $n \in \mathbb{N}$ , we define an equivalence relation  $\sim_n$  on  $\{1, \dots, \ell\}^n$  by

$$a_1 \cdots a_n \sim_n b_1 \cdots b_n \quad \text{if} \quad \sum_{i=1}^n a_i \beta^{-i} = \sum_{i=1}^n b_i \beta^{-i},$$

and let  $\Omega_n := \{1, \dots, \ell\}^n / \sim_n$  be the corresponding quotient set. Define two maps  $S_0, S_1: \mathbb{R} \rightarrow \mathbb{R}$  by  $S_i(x) = (x + i)/\beta$ ,  $i = 0, 1$ . Write for brevity  $S_{a_1 \cdots a_n} = S_{a_1} \circ \cdots \circ S_{a_n}$ . It is clear that  $S_{a_1 \cdots a_n} = S_{b_1 \cdots b_n}$  if and only if  $a_1 \cdots a_n \sim_n b_1 \cdots b_n$ . Hence by the self-similarity of  $\nu_{\beta,p}$ ,

$$\nu_{\beta,p} = \sum_{a_1 \cdots a_n \in \{0,1\}^n} m_p[a_1 \cdots a_n] \nu_{\beta,p} \circ S_{a_1 \cdots a_n}^{-1} = \sum_{[a_1 \cdots a_n] \in \Omega_n} (M_{a_1} \cdots M_{a_n})_{1,1} \nu_{\beta,p} \circ S_{a_1 \cdots a_n}^{-1}, \quad (9)$$

recalling that  $\sum_{b_1 \cdots b_n \in \{0,1\}^n: b_1 \cdots b_n \sim_n a_1 \cdots a_n} m_p[b_1 \cdots b_n] = (M_{a_1} \cdots M_{a_n})_{1,1}$ . Since  $\beta$  is a Pisot number, there exists an integer  $N$  such that for any  $n \in \mathbb{N}$  and any interval  $\Delta \subset I := \text{supp}(\nu_{\beta,p}) = [0, 1/(\beta - 1)]$  of length  $\beta^{-n}$ , there are at most  $N$  many elements  $[a_1 \cdots a_n] \in \Omega_n$  such that  $S_{a_1 \cdots a_n}(I) \cap \Delta \neq \emptyset$  (see e.g. [19, 7]). Applying this property and (9) one can show (with some additional standard arguments) that  $\Theta_{\nu_{\beta,p}}(q; \beta^{-n})$  is comparable to  $\sum_{[a_1 \cdots a_n] \in \Omega_n} ((M_{a_1} \cdots M_{a_n})_{1,1})^q$  (i.e. the ratios between them are uniformly bounded away from 0). We leave the details to the reader. Now it is easy to see that  $\sum_{[a_1 \cdots a_n] \in \Omega_n} ((M_{a_1} \cdots M_{a_n})_{1,1})^q = u_n(q)$ , and so (8) holds.

In the end, we show that  $u_n(2)$  is comparable to  $v_n(2)$ , where

$$v_n(2) := \sum_{a_1 \cdots a_n \in \{0,1\}^n} m_p[a_1 \cdots a_n] \|M_{a_1} \cdots M_{a_n}\|_1.$$

This will conclude our desired result since we have already proved that  $\lim_{n \rightarrow \infty} (\log v_n(2))/n = \log \lambda$  in the proof of Proposition 3.6.

Clearly  $v_n(2) \geq u_n(2)$ , and

$$\begin{aligned} v_n(2) &= \sum_{[a_1 \cdots a_n] \in \Omega_n} (M_{a_1} \cdots M_{a_n})_{1,1} \|M_{a_1} \cdots M_{a_n}\|_1 \\ &\leq \sum_{[a_1 \cdots a_n] \in \Omega_n} (\|M_{a_1} \cdots M_{a_n}\|_1)^2 =: w_n(2). \end{aligned}$$

For two elements  $[b_1 \cdots b_n], [a_1 \cdots a_n] \in \Omega_n$ , we say that  $[b_1 \cdots b_n]$  is a neighbour of  $[a_1 \cdots a_n]$  if there exist  $i, j \in \mathcal{A}$  such that  $\beta^n x_i + \sum_{l=1}^n (a_l - b_l) \beta^{n-l} =$

$x_j$ . Assume that  $n$  is large enough so that  $\beta^n x_i - x_j \neq \beta^n x_{i'} - x_{j'}$  when  $(i, j) \neq (i', j')$ . Then by the definition (4), we have

$$\|M_{a_1} \cdots M_{a_n}\|_1 = \sum (M_{b_1} \cdots M_{b_n})_{1,1},$$

where the sum is over all the neighbours  $[b_1 \cdots b_n]$  of  $[a_1 \cdots a_n]$  in  $\Omega_n$ . Since any element in  $\Omega_n$  has at most  $|\mathcal{A}|^2$  many neighbours, and conversely each element in  $\Omega_n$  is the neighbour of at most  $|\mathcal{A}|^2$  many elements in  $\Omega_n$ . It follows that

$$\begin{aligned} w_n(2) &= \sum_{[a_1 \cdots a_n] \in \Omega_n} (\|M_{a_1} \cdots M_{a_n}\|_1)^2 \\ &= \sum_{[a_1 \cdots a_n] \in \Omega_n} \left( \sum_{\substack{[b_1 \cdots b_n] \in \Omega_n \\ [b_1 \cdots b_n] \text{ is a neighbour of } [a_1 \cdots a_n]}} (M_{b_1} \cdots M_{b_n})_{1,1} \right)^2 \\ &\leq |\mathcal{A}|^4 \sum_{[a_1 \cdots a_n] \in \Omega_n} \left( \sum_{\substack{[b_1 \cdots b_n] \in \Omega_n \\ [b_1 \cdots b_n] \text{ is a neighbour of } [a_1 \cdots a_n]}} ((M_{b_1} \cdots M_{b_n})_{1,1})^2 \right) \\ &\leq |\mathcal{A}|^6 \sum_{[b_1 \cdots b_n] \in \Omega_n} ((M_{b_1} \cdots M_{b_n})_{1,1})^2 = |\mathcal{A}|^6 u_n(2). \end{aligned}$$

Hence we obtain that  $u_n(2) \leq v_n(2) \leq w_n(2) \leq |\mathcal{A}|^6 u_n(2)$ . That is,  $u_n(2)$  and  $v_n(2)$  are comparable. It completes the proof of the proposition.  $\square$

**Remark.** When  $\beta \in (1, 2)$  is a hyperbolic algebraic integer, the above proof can be adapted to show that  $\tau(\nu_{\beta,p}, 2) \leq -\frac{\log \lambda}{\log \beta}$ .

## 5 Further Comments

1. As can be seen from Table 2, when  $\beta$  has Galois conjugates close to 1 in modulus the graph  $\mathcal{G}$  can be very large. In these cases, calculating the graph  $\mathcal{G}$  may not be the most efficient way of proving that  $\nu_\beta$  has dimension 1. In a follow up article we show how one can perform counting estimates broadly similar to those of [15, 16] on a higher

dimensional self-affine set with contraction ratios equal to the Galois conjugates of  $\beta$ . These estimates often yield that  $\dim_{\mathbb{H}}(\nu_{\beta}) = 1$ , and work even in the case of non-hyperbolic  $\beta$ .

2. A short argument of Mercat (personal communication) shows that  $H(\beta) \leq \log(\beta)$  whenever  $\beta$  is a Salem number. Therefore it will hold that  $L_n(\beta) < \log(\beta)$  for all  $n \in \mathbb{N}$  and so our finite time approximation methods will not be able to show that  $\dim_{\mathbb{H}}(\nu_{\beta}) = 1$  for  $\beta$  Salem.
3. According to Lemma 2.4,  $V_{\beta}$  is finite if  $\beta$  is hyperbolic. Is it true that if  $\beta$  is a non-hyperbolic algebraic integer, then  $V_{\beta}$  is infinite?

If we in the definition of  $V_{\beta}$  only consider  $\beta$  and not its conjugates, and define

$$\tilde{V}_{\beta,n} := \left\{ x = \sum_{i=0}^n \epsilon_i \beta^{n-i} : \epsilon_i \in \{-1, 0, 1\} \right. \\ \left. \text{and } \left| \sum_{i=0}^n \epsilon_i \beta^{n-i} \right| \leq \frac{1}{\beta - 1} \right\}, \\ \tilde{V}_{\beta} := \bigcup_{n=0}^{\infty} \tilde{V}_{\beta,n},$$

then, according to a result by Akiyama and Komornik [1],  $\tilde{V}_{\beta}$  is finite if and only if  $\beta$  is a Pisot number.

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