UNIVERSITÀ
DEGLI STUDI
DI PADOVA

DIPARTIMENTO
DI INGEGNERIA
DELL'INFORMAZIONE

On the Stability, Stabilizability and Control of Certain Classes of Positive Systems

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University of Padova
2017

Abstract

In this thesis stability, stabilizability and other control issues for certain classes of Positive Systems are investigated. In the first part, the focus is on Compartmental Systems: we start from Compartmental Switched Systems and show that, with respect to the general class of Positive Switched Systems, a much clearer picture of stability under arbitrary switching, stability under persistent switching, and stabilizability (where the control action may either pertain the switching function or involve the design of feedback controllers) can be drawn. Secondly, for the class of Compartmental Multi-Input Systems the problem of designing a state-feedback matrix that preserves the compartmental property of the resulting closed-loop system, meanwhile achieving asymptotic stability is addressed. Such an analysis finally leads to the development of an algorithm that allows to assess problem solvability and provides a possible solution whenever it exists.

The second part of the thesis is devoted to the Positive Consensus Problem: for a homogeneous Positive Multi-Agent System we investigate the problem of determining a state-feedback law that can be individually implemented by each agent, preserves the positivity of the overall system, and leads to the achievement of consensus.

Finally, for a particular class of Positive Bilinear Systems that arises in drugs concentration design for HIV treatment, we address the problem of determining an optimal constant input that stabilizes the system while maximizing its robustness against the presence of the external disturbance.

Acknowledgments

The three wonderful years spent as a PhD student represent a journey that enriched me a lot, contributing to my growth both as a researcher and as a person. While scientific results are extensively presented in the following part of the thesis, it is difficult to explain here what I was given in terms of personal growth. Conferences, PhD schools, and the period spent abroad gave me the possibility to get in touch with different cultures, to know people from different parts of the world, to visit faraway cities and countries. I have never had such opportunities before, and all these experiences gave me a lot.

I am indebted most of all to my advisor Professor Maria Elena Valcher. I will always be grateful to her for the time she spent with me, her patience, and her constant valuable guide. She taught me the love for research, and she showed me the perseverance, dedication and passion necessary to be a good researcher.

I sincerely thank Professor Anders Rantzer for his helpfulness and his guidance during the period I spent at the Department of Automatic Control at Lund University.

I am grateful to the reviewers of the thesis, Professor Yoshio Ebihara and Professor Oliver Mason, for their careful reading, their suggestions and their appreciation of the work presented in the thesis.

I would like to thank all the professors, postDocs and PhD students of the DEI Automatic group for making the Department such a pleasant working environment. A special thank goes to my officemates: the "old ones" Andrea, Diego, Giulia and Marco, who welcomed me as a new PhD student; my companions Chiara, Giacomo, Giulia, Michele, Nicoletta, Yutao; the many "young ones" to whom I wish wonderful doctoral years.

Finally, I sincerely thank my family for always supporting me, sharing with me joyful moments and giving me strength in the difficult ones. I will always be grateful for their essential invaluable support, and their constant unconditioned love.

Ringraziamenti

Il Dottorato di Ricerca costituisce un percorso che mi ha arricchita moltissimo, sia da un punto di vista scientifico sia da un punto di vista umano. I risultati scientifici, frutto di un lavoro tanto intenso quanto appassionante, sono raccolti in questa tesi e ampiamente presentati nel seguito. Più difficile è invece spiegare quanto ho ricevuto in termini umani. Conferenze, scuole di Dottorato, nove mesi trascorsi all'estero mi hanno dato la possibilità di entrare in contatto con culture diverse, conoscere persone provenienti da svariate parti del mondo, visitare città e paesi lontani. Sono opportunità che mai mi erano capitate prima, esperienze che mi hanno dato molto e che mi hanno fatta crescere come persona.

Ringrazio allora chi, più di ogni altro, ha permesso tutto ciò: il mio supervisore, la Professoressa Maria Elena Valcher. Non finirò mai di ringraziarla per il tempo che mi ha dedicato, la dedizione e la pazienza con cui mi ha costantemente seguita. Giorno dopo giorno mi ha trasmesso l'amore per la ricerca, mi ha mostrato la tenacia, la determinazione e la passione necessari per svolgere questo lavoro.

Ringrazio sinceramente il Professor Anders Ranzter per la sua disponibilità e la sua valida guida durante il periodo che ho svolto presso il Dipartimento di Automatica dell'Università di Lund.

La mia gratitudine va anche ai revisori della tesi, il Professor Yoshio Ebihara e il Professor Oliver Mason, per la loro attenta analisi, gli utili suggerimenti e l'apprezzamento espresso verso il lavoro presentato in questa tesi.

Desidero poi ringraziare professori, ricercatori, assegnisti e dottorandi del gruppo di Automatica che rendono il DEI un ambiente di lavoro piacevole. Un ringraziamento particolare va ai compagni di ufficio: i "dottorandi anziani" Andrea, Diego, Giulia e Marco, che mi hanno accolta da neodottoranda; i "compagni di avventura" Chiara, Giacomo, Giulia, Michele, Nicoletta, Yutao; i numerosi "giovani dottorandi" ai quali auguro un percorso altrettanto bello e costruttivo quanto il mio.

Infine, ringrazio di cuore la mia famiglia che mi è sempre stata vicina, gioendo con me nei momenti felici e incoraggiandomi in quelli difficili. Grazie per aver accettato, in molti casi capito, e sempre sostenuto con forza la mia scelta di essere fisicamente lontana in un

periodo difficile. Grazie perché in ventisette anni non mi è mai mancato quel sostegno fondamentale e insostituibile che solo la famiglia può dare.

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1

Introduction

In this chapter we first introduce the notation used throughout the thesis, and then provide formal definitions for those classes of continuous-time linear systems we will investigate in the sequel.

1.1 Notation

Scalars

$\mathbb{R}$ is the field of real numbers, and $\mathbb{R}_+$ is the semiring of nonnegative real numbers. $\mathbb{Z}$ and $\mathbb{N}$ denote the ring of integer numbers and the ring of natural numbers, respectively. Given $k, n \in \mathbb{Z}$, with $k \leq n$, the symbol $[k, n]$ denotes the integer set $\{k, k+1, \ldots, n\}$.

Matrices and vectors

We denote by $\mathbb{R}^{m \times n}$ and $\mathbb{R}^n$ the space of $m \times n$ matrices and the space of n-dimensional column vectors, respectively. We adopt notations $\mathbb{R}_+^{m \times n}$ and $\mathbb{R}_+^n$ for the set of $m \times n$ matrices with entries in $\mathbb{R}_+$, and the set of n-dimensional column vectors with entries in $\mathbb{R}_+$, respectively.

For a matrix $A \in \mathbb{R}^{m \times n}$ (a vector $\mathbf{v} \in \mathbb{R}^n$), notation $A^\top$ (respectively, $\mathbf{v}^\top$) denotes its transpose.

For a matrix A, its (i,j)th entry is denoted indifferently by $[A]_{ij}$ or a_{ij}. If A is block partitioned, we denote its (i,j)th block by $\text{block}_{ij}[A]$. In the special case of a vector $\mathbf{v}$, its ith entry is $[\mathbf{v}]_i$ or v_i, and its ith block is $\text{block}_i[\mathbf{v}]$.

Given a matrix $A \in \mathbb{R}^{m \times n}$, we denote by $\text{row}_i(A)$, $i \in [1,m]$, and $\text{col}_i(A)$, $i \in [1,n]$, the ith row of A and the ith column of A, respectively. We denote by $\text{tr}(A)$ and $\text{adj}(A)$ the trace of A and the adjugate matrix of A, respectively. The *nonzero pattern* $\overline{\text{ZP}}(A)$ is the set $\{(i,j) \in [1,m] \times [1,n] \colon [A]_{ij} \neq 0\}$. For a vector $\mathbf{v} \in \mathbb{R}^n$, the nonzero pattern is defined as $\overline{\text{ZP}}(\mathbf{v}) := \{i \in [1,n] \colon [\mathbf{v}]_i \neq 0\}$.

$\mathbf{1}_n$ is the n-dimensional vector with all entries equal to 1, and 0_n is the n-dimensional vector with all entries equal to 0. I_n and $0_{m \times n}$ denote the n-dimensional identity matrix and the $m \times n$ null matrix, respectively. When the dimension of $\mathbf{1}_n$, 0_n, I_n and $0_{m \times n}$ is clear from the context, it may be omitted.

The symbol $\mathbf{e}_i$ denotes the ith vector of the canonical basis in $\mathbb{R}^n$ (where n is always clear from the context), whose entries are all zero except for the ith one that is unitary. A vector $\mathbf{v}$ is called *ith monomial* (or, generically, monomial) if it is a positive multiple of the ith canonical vector, i.e., $\mathbf{v} = v\mathbf{e}_i$ for some positive scalar $v > 0$.

The *Kronecker product* of matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ is denoted by $A \otimes B$ and defined as

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{pm \times qn}.$$

For a vector $\mathbf{v} \in \mathbb{R}^n$, we define the ∞-*norm* and the 2-*norm* of $\mathbf{v}$ as $\|\mathbf{v}\|_\infty := \max_{i \in [1,n]} |v_i|$ and $\|\mathbf{v}\|_2 := \left(v_1^2 + \dots + v_n^2\right)^{1/2}$, respectively. Given a set $\mathcal{S} \subset \mathbb{R}^n$, the *distance of $\mathbf{v}$ from $\mathcal{S}$* is denoted by $\text{dist}\,(\mathbf{v}, \mathcal{S})$ and defined as $\text{dist}\,(\mathbf{v}, \mathcal{S}) := \inf_{\mathbf{z} \in \mathcal{S}} \|\mathbf{v} - \mathbf{z}\|_2$.

For a matrix $A \in \mathbb{R}^{n \times n}$, we denote by $\sigma(A)$ the spectrum of A. We adopt notation $\lambda_{\max}(A)$ for the *spectral abscissa* of A, defined as $\lambda_{\max}(A) := \max_{\lambda \in \sigma(A)} \Re(\lambda)$, and we say that A is *Hurwitz* if $\lambda_{\max}(A) < 0$, namely if all eigenvalues of A lie in the open left complex halfplane. We denote by $\bar{\sigma}(A)$ the largest singular value of A, defined as $\bar{\sigma}(A) := \max\left\{\sqrt{\lambda} \colon \lambda \in \sigma(A^\top A)\right\}$.

Special classes of matrices

A *permutation matrix* is a square matrix Π such that exactly one entry in each row and column is equal to 1, and all other entries are 0. Pre-multiplication (post-multiplication)

by Π effects a permutation of the rows (columns) of the matrix or vector Π is applied to.

We denote by $S_i \in \mathbb{R}^{(n-1)\times n}$ the *selection matrix* obtained by removing the ith row in the identity matrix I_n, namely

$$S_i = \left[\begin{array}{c|c|c} I_{i-1} & 0 & 0_{(i-1)\times(n-i)} \\ \hline 0_{(n-i)\times(i-1)} & 0 & I_{n-i} \end{array} \right].$$

The size n will always be clear from the context, namely from the size of the matrix or vector S_i is applied to. For any vector $\mathbf{v} \in \mathbb{R}^n$, $S_i\mathbf{v}$ is the vector obtained from $\mathbf{v}$ by removing the ith entry, while for any matrix $A \in \mathbb{R}^{n\times m}$, S_iA denotes the matrix obtained from A by removing the ith row.

Consistently with Frobenius (1912) (see also Schneider (1977)), we say that a nonzero matrix $A \in \mathbb{R}^{n\times n}$ is *reducible* "if we may partition $\{1,\ldots,n\}$ into two non-empty subsets $\mathcal{S}, \mathcal{T}$ such that $[A]_{ij} = 0$ if $i \in \mathcal{S}$, $j \in \mathcal{T}$." This is equivalent to saying that there exists a permutation matrix Π such that

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square (nonvacuous) matrices. A square matrix that is not reducible is said to be *irreducible*. It follows that 1×1 nonzero matrices are always irreducible.

A symmetric matrix $P = P^\top \in \mathbb{R}^{n\times n}$ is said to be *positive definite* ($P \succ 0$) if $\mathbf{x}^\top P \mathbf{x} > 0$ for every $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$, and *quadratic copositive* if $\mathbf{x}^\top P \mathbf{x} > 0$ for every $\mathbf{x} \in \mathbb{R}^n_+$, $\mathbf{x} \neq 0$. Clearly, a positive definite matrix is quadratic copositive, but the converse is not true.

A matrix $A \in \mathbb{R}^{m\times n}_+$ is a *nonnegative matrix* ($A \geq 0$); if $A \geq 0$ and at least one entry is positive, A is a *positive matrix* ($A > 0$); if all the entries of A are positive it is a *strictly positive matrix* ($A \gg 0$). The same notation is adopted for nonnegative, positive and strictly positive vectors. Notations and definitions straightforwardly extend to *nonpositive*, *negative* and *strictly negative* matrices and vectors. Given matrices A, $\bar{A} \in \mathbb{R}^{n\times n}$, notation $A \geq \bar{A}$ ($A > \bar{A}$) means $A - \bar{A} \geq 0$ ($A - \bar{A} > 0$).

A matrix $A \in \mathbb{R}^{n\times n}$ is a *Metzler matrix* if its off-diagonal entries are nonnegative. If, in addition, the entries of each of its columns sum up to a nonpositive number, i.e., $\mathbf{1}_n^\top A \leq 0^\top$, A is a *compartmental matrix*.

Functions

A function $V\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *copositive* if $V(0) = 0$ and $V(\mathbf{x}) > 0$ for every $\mathbf{x} > 0$. For a continuous and continuously differentiable function $V\colon \mathbb{R}^n \to \mathbb{R}$ we denote by ∇V its *gradient*, that is the vector-valued function $\nabla V\colon \mathbb{R}^n \to \mathbb{R}^{1 \times n}$ defined as

$$\nabla V := \begin{bmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix}.$$

Hence, for every $\mathbf{x}, \mathbf{v} \in \mathbb{R}^n$, $\nabla V(\mathbf{x})\mathbf{v}$ is the derivative of V in $\mathbf{x}$ along the direction $\mathbf{v}$.

Signals

Given a signal $\mathbf{u}\colon \mathbb{R}_+ \to \mathbb{R}^n$, the $\mathcal{L}_\infty$-norm of $\mathbf{u}$ is defined as $\|\mathbf{u}\|_\infty := \sup_{t \geq 0} \|\mathbf{u}(t)\|_\infty$. We denote by $\mathcal{L}_\infty^n[0, \infty)$ the space of signals $\mathbf{u}\colon \mathbb{R}_+ \to \mathbb{R}^n$ that have finite $\mathcal{L}_\infty$-norm. Similarly, we define the $\mathcal{L}_2$-norm of signal $\mathbf{u}\colon \mathbb{R}_+ \to \mathbb{R}^n$ as $\|\mathbf{u}\|_2 := \left(\int_0^{+\infty} \|\mathbf{u}(t)\|_2^2 \, dt \right)^{1/2}$, and we adopt the symbol $\mathcal{L}_2^n[0, \infty)$ to denote the space of signals $\mathbf{u}\colon \mathbb{R}_+ \to \mathbb{R}^n$ that have finite $\mathcal{L}_2$-norm.

Cones

A set $\mathcal{C} \subset \mathbb{R}^n$ is a *cone* if $\alpha\mathcal{C} \subseteq \mathcal{C}$ for all $\alpha \geq 0$. A cone $\mathcal{C}$ is said to be *polyhedral* if it can be expressed as the set of nonnegative linear combinations of a finite set of vectors, called *generating vectors*. When so, we denote by $\mathcal{C} = \mathrm{Cone}(\mathbf{v}_1, \ldots, \mathbf{v}_r)$, $r \geq 1$, the cone generated by $\mathbf{v}_i \in \mathbb{R}^n$, $i \in [1, r]$. We say that the set of vectors $\{\mathbf{v}_i\}_{i=1}^r$ is a family of *minimal generators* of $\mathcal{C}$ if no set with less than r vectors can be found generating the same cone. If the generating vectors are the columns of a matrix W, we adopt the notation $\mathcal{C} = \mathrm{Cone}(W)$. The *dual cone* of a cone $\mathcal{C} \subset \mathbb{R}^n$ is denoted by $\mathcal{C}^*$ and defined as

$$\mathcal{C}^* := \left\{ \mathbf{z} \in \mathbb{R}^n \colon \mathbf{v}^\top \mathbf{z} \geq 0, \forall \mathbf{v} \in \mathcal{C} \right\}.$$

We refer to Barker (1981) for more details on cones.

Graphs

A *graph* is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} := [1, N]$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs (or edges), and $\mathcal{A} \in \mathbb{R}_+^{N \times N}$ is the (nonnegative) *adjacency matrix*. There is an arc $(j, i) \in \mathcal{E}$ from j to i if and only if $[\mathcal{A}]_{ij} > 0$. When so, $[\mathcal{A}]_{ij}$ represents the *weight* of the arc (j, i). If all the nonzero weights equal 1, namely if $[\mathcal{A}]_{ij} > 0$ implies $[\mathcal{A}]_{ij} = 1$, we say that $\mathcal{G}$ is *unweighted*; otherwise $\mathcal{G}$ is *weighted*. The graph $\mathcal{G}$ is *undirected* if $\mathcal{A}$

is symmetric, i.e., $\mathcal{A} = \mathcal{A}^\top$; otherwise it is *directed*. We steadly assume that $\mathcal{G}$ has no self-loops, namely $[\mathcal{A}]_{ii} = 0$ for every index $i \in [1, N]$.

A sequence $j_1 \to j_2 \to \cdots \to j_k \to j_{k+1}$ is a *path* of length k from j_1 to j_{k+1} provided that $(j_1, j_2), \ldots, (j_k, j_{k+1})$ are elements of $\mathcal{E}$. The graph $\mathcal{G}$ is *connected* if for every $i, j \in \mathcal{V}$, $i \neq j$, there is a path from j to i, namely if there exists $k \in \mathbb{N}$ such that $[\mathcal{A}^k]_{ij} > 0$. The graph is *complete* if $[\mathcal{A}]_{ij} > 0$ for every $i, j \in \mathcal{V}$, $i \neq j$.

Given an undirected graph $\mathcal{G}$, we associate with it the *Laplacian matrix* $\mathcal{L} \in \mathbb{R}^{N \times N}$ defined as $\mathcal{L} := \mathcal{D} - \mathcal{A}$, where $\mathcal{D} \in \mathbb{R}_+^{N \times N}$ is a diagonal matrix whose ith diagonal entry is the *weighted degree* of vertex i, i.e., $[\mathcal{D}]_{ii} := \sum_{j=1}^N [\mathcal{A}]_{ij}$. Accordingly, the Laplacian matrix $\mathcal{L} = \mathcal{L}^\top$ takes the following form

$$
\mathcal{L} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ldots & \ell_{1N} \\ \ell_{12} & \ell_{22} & \ldots & \ell_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{1N} & \ell_{2N} & \ldots & \ell_{NN} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^N [\mathcal{A}]_{1j} & -[\mathcal{A}]_{12} & \ldots & -[\mathcal{A}]_{1N} \\ -[\mathcal{A}]_{12} & \sum_{j=1}^N [\mathcal{A}]_{2j} & \ldots & -[\mathcal{A}]_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -[\mathcal{A}]_{1N} & -[\mathcal{A}]_{2N} & \ldots & \sum_{j=1}^N [\mathcal{A}]_{Nj} \end{bmatrix}.
$$

For more details on graphs we refer to Mohar (1991); Merris (1994).

1.2   Positive Systems

In this section we introduce Positive Systems, a class of systems that encompasses the one of Compartmental Systems and that represents the starting point towards the introduction of Positive Switched Systems (and Compartmental Switched Systems). Generally speaking, a system of equations is said to be *positive* if, for every nonnegative initial condition and for every nonnegative input, the state and output variables remain nonnegative. Positivity constraint on system variables is oftentimes an intrinsic characteristic of the phenomena being described, a distinctive feature that any "good" model must be able to capture. Positivity is a pervasive trait as well: pressures, absolute temperatures, concentrations of substances, population levels and probabilities are all examples of variables that are confined to be positive or nonnegative.

An essential tool to provide a formal mathematical description of processes involving nonnegative variables is represented by nonnegative operators, that reduce to nonnegative matrices and Metzler matrices when restricting ourselves to linear finite-dimensional systems.

From a historical point of view, the interest in positive matrices dates back to the beginning of the XX century. Perron (1907) investigated spectral properties of strictly positive matrices and primitive matrices, and Frobenius (1912) extended these

results to the class of irreducible matrices. Since the appearance of their pioneering works, great impulse to the development of the newly initiated theory on nonnegative matrices came from its application to a variety of different disciplines and fields of research, e.g., economics Potron (1939); Leontief (1986), the theory of polynomials Wilf (1961), population dynamics Leslie (1945); Greiner (1984b), Markov chains Cohen (1979), Boltzmann equation Greiner (1984a). The interest in nonnegative operators and systems with positivity constraints persisted over decades, and the literature on the topic continued flourishing. On the one hand, from a purely algebraic point of view, the so called Perron-Frobenius theorems were extended to operators that leave a cone invariant in both finite-dimensional and infinite-dimensional spaces (a reference book is for instance Berman and Plemmons (1987)). On the other hand, positive linear systems, whose state-space representation involve positive matrices and Metzler matrices, were extensively studied (we refer to Farina and Rinaldi (2000) for a complete treatment on positive systems and their applications). One of the most attractive and challenging aspects in the investigation of positive systems is the fact that the positivity constraint on the state and input variables makes it possible to resort to new and ad hoc tools. Techniques based on cones and polytopes (that in the general, i.e., nonpositive, case cannot be employed) turned out to be the most appropriate ones to address reachability and controllability issues, Ohta, Maeda, and Kodama (1984); Commault and Alamir (2007); Valcher (2009). Diagonal Lyapunov functions and linear copositive Lyapunov functions, that in the general case allow to derive only conservative results, proved to be effective tools when dealing with stability and stabilizability problems, even in the presence of uncertainty or performance requirements, Ait Rami and Tadeo (2007); Ebihara, Peaucelle, and Arzelier (2011); Ait Rami (2011); Rantzer (2011); Tanaka and Langbort (2011); Briat (2013).

In the sequel we will consider (continuous-time linear) systems described by the following state-space model

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad t \in \mathbb{R}_+, \tag{1.1}$$

where $A \in \mathbb{R}^{n \times n}$ is the state-space matrix, $B \in \mathbb{R}^{n \times m}$ is the input-to-state matrix, $\mathbf{x}(t)$ is the n-dimensional state vector at time t, and $\mathbf{u}(t)$ is the m-dimensional input vector at time t. If B is the null matrix (or, equivalently, no input is acting on the system, i.e., $\mathbf{u}(t) \equiv 0$), system (1.1) is said to be *autonomous*.

When a scalar output measurement $y(t)$ is available, we assume its dynamics being described by

$$y(t) = C\mathbf{x}(t), \quad t \in \mathbb{R}_+, \tag{1.2}$$

where $C \in \mathbb{R}^{1 \times n}$ is the state-to-output matrix.

Definition 1.1. The continuous-time linear system (1.1)-(1.2) is *positive* if for every nonnegative initial condition and for every nonnegative input, the state and output variables remain nonnegative, namely if

$$\begin{cases} \mathbf{x}(0) \geq 0 \\ \mathbf{u}(t) \geq 0 \,\forall t \geq 0 \end{cases} \Rightarrow \begin{cases} \mathbf{x}(t) \geq 0 \,\forall t \geq 0 \\ y(t) \geq 0 \,\forall t \geq 0. \end{cases}$$

The following proposition states a well known necessary and sufficient condition for a system described as in (1.1)-(1.2) to be positive (see for instance Farina and Rinaldi (2000)).

Proposition 1.1. *The continuous-time linear system* (1.1)-(1.2) *is positive if and only if A is a Metzler matrix, $B \in \mathbb{R}_+^{n \times m}$ and $C \in \mathbb{R}_+^{1 \times n}$.*

1.2.1 Compartmental Systems

Compartmental Systems are a special class of positive state-space models that represent physical systems in which units, called compartments, exchange material and are subject to some conservation law (e.g., mass, energy, fluid). This is the case for instance when describing liquids flowing in a network of interconnected tanks, or the time evolution of temperatures in adjacent rooms. In all these cases, the ith entry of the state vector, $x_i(t)$, represents a quantity (the content of the ith compartment at time t), and hence it is intrinsically nonnegative. Moreover, when no external inflow is provided to the system (namely, when $\mathbf{u}(t) \equiv 0$), the total amount of material in the system $\sum_{i=1}^n x_i(t) = \mathbf{1}_n^\top \mathbf{x}(t)$ cannot increase with time. In mathematical terms, this means that $\mathbf{1}_n^\top \dot{\mathbf{x}}(t) = \mathbf{1}_n^\top A \mathbf{x}(t) \leq 0$ for every $\mathbf{x}(t) > 0$, and hence $\mathbf{1}_n^\top A \leq 0^\top$. This naturally leads to the following definition of Compartmental System.

Definition 1.2. The continuous-time linear system (1.1)-(1.2) is a *compartmental system* if it is positive and, in addition, its state-space matrix A is compartmental.

For a general introduction to compartmental systems we refer to Jacquez and Simon (1993).

1.3 Positive Switched Systems and Compartmental Switched Systems

Switched systems naturally arise as a way to describe the behaviour of systems undergoing different operating conditions. A switched system consists of a family of state-space models (the subsystems) and a switching function describing which of the subsystems is active at every time instant. In many cases, each of the configurations among which the system switches, namely every single operating mode, is modelled as a Positive System or a Compartmental System.

Definition 1.3. A *(Continuous-time Linear) Positive Switched System* is a system described by the following equation

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}u(t), \tag{1.3}$$

$$y(t) = C_{\sigma(t)}\mathbf{x}(t) \qquad\qquad t \in \mathbb{R}_+, \tag{1.4}$$

where $\sigma\colon \mathbb{R}_+ \to [1, M]$ is an arbitrary switching function, and, for each $i \in [1, M]$, $A_i \in \mathbb{R}^{n \times n}$ is a Metzler matrix, $B_i \in \mathbb{R}^n_+$ and $C_i \in \mathbb{R}^{1 \times n}_+$.

As an example, consider a population of n self replicating viral variants undergoing selection and mutation phenomena, and a set of M drugs (each one specific for certain viral variants) that can be administered to the patient. A Positive Switched System may be adopted to describe the behaviour of the virus population when switching among the available treatments, Hernandez-Vargas, Middleton, Colaneri, and Blanchini (2011).

Definition 1.4. A *(Continuous-time Linear) Compartmental Switched System* is a Positive Switched System (1.3)-(1.4) with compartmental subsystems, namely such that for each $i \in [1, M]$, $A_i \in \mathbb{R}^{n \times n}$ is a compartmental matrix.

Consider for instance a network of n interconnected tanks. A Compartmental Switched System describes fluid levels in the tanks when switching among different open/closed configurations of the pipes connecting the tanks, Blanchini, Colaneri, and Valcher (2015).

Remark 1.1. Before proceeding some important considerations pertaining the switching function σ are needed. First of all, we will tacitly assume that σ is piecewise constant, right continuous and has a finite number of discontinuities in every finite time interval. In this way we are ruling out the case of chattering, that is we are not considering the

case of an infinitely fast switching among the subsystems. On the other hand, it is well known (see, e.g., Hespanha, Liberzon, and Sontag (2002); Hespanha (2004); Bacciotti and Mazzi (2005)) that this set of switching functions does not exhibit sufficient regularity to address asymptotic stability of general switched systems. For this reason, in the following we will steadily assume that each switching function σ has a positive dwell-time (or, consistently with Bacciotti and Mazzi (2005), a "nonvanishing dwell-time"), by this meaning that for each σ a real number $\tau_\sigma > 0$ can be found such that for every pair of consecutive switching instants $t_{k+1} > t_k \geq 0$ it holds $t_{k+1} - t_k \geq \tau_\sigma$. We emphasize that the dwell-time τ_σ is not uniform over all switching functions but specific for the considered function σ.

While the interest in general (i.e., without any positivity assumption) switched systems dates back to more than fifty years ago and was originally motivated by their application to systems containing relays and/or hysteresis, the investigation of Positive Switched Systems was initiated only at the beginning of the past decade, both for theoretical reasons and for practical applications. On the one hand, the positivity constraints on the input and state variables offered the possibility to resort to new techniques and not yet settled approaches, thus attracting the attention of a significant number of researchers. Reachability, controllability, stability and stabilizability have been the object of an intense and fruitful study, see, e.g., Mason and Shorten (2003, 2004); Santesso and Valcher (2006); Gurvits, Shorten, and Mason (2007); Mason and Shorten (2007a,b); Valcher (2009); Fainshil, Margaliot, and Chigansky (2009); Fornasini and Valcher (2010); Valcher and Santesso (2010); Zappavigna, Colaneri, Geromel, and Shorten (2010); Blanchini, Colaneri, and Valcher (2012). On the other hand, additional motivations for investigating Positive Switched Systems came from the wide range of different application areas where they had been fruitfully employed. To mention the most significant ones: epidemiology, Moreno, Pastor-Satorras, and Vespignani (2002); Ait Rami, Bokharaie, Mason, and Wirth (2014); consensus and synchronization, Jadbabaie, Lin, and Morse (2003); congestion control, Shorten, Wirth, and Leith (2006); viral mitigation under drug treatment, Hernandez-Vargas et al. (2011); wireless power control, Zappavigna, Charalambous, and Knorn (2012).

As regards Compartmental Switched Systems, a systematic analysis was initiated only recently, Valcher and Zorzan (2015, 2016c, 2017a). Clearly, all results derived for the more general class of Positive Switched Systems hold true when assuming that every subsystem is a compartmental one. On the other hand, as we will see in Chapter 4 and Chapter 5, by exploiting spectral and structural properties of compartmental matrices we gain deeper insights and obtain much stronger characterizations of stability and

stabilizability.

2
Motivating examples

In this chapter we present some examples of practical relevance where Positive Systems and Compartmental Systems have been fruitfully employed. Our aim is to provide motivation for our subsequent analysis and, at the same time, to stimulate the reader's interest in the subject.

2.1 A compartmental model for cell proliferation

We start from a multicompartmental model, borrowed from Sundareshan and Fundakowski (1986) with slight modifications, describing the proliferation mechanism of a population of eukaryotic cells through the different phases of the cell cycle.

The cell cycle is a sequence of biochemical events by which a single cell matures, duplicates its DNA and divides to give rise to two daughter cells. In normal (i.e., healthy) tissues this proliferative activity is regulated by complex processes that aim at maintaining the population at a determined level. On the contrary, uncontrolled proliferation of abnormal cells results in unmanageable growth of the cell population that in the end may lead to the development of cancer. Mathematical modelling of the cell cycle and its analytical study are useful tools for investigating the proliferation mechanisms in both healthy and cancerous tissues, and addressing problems such as prognosis of cancer
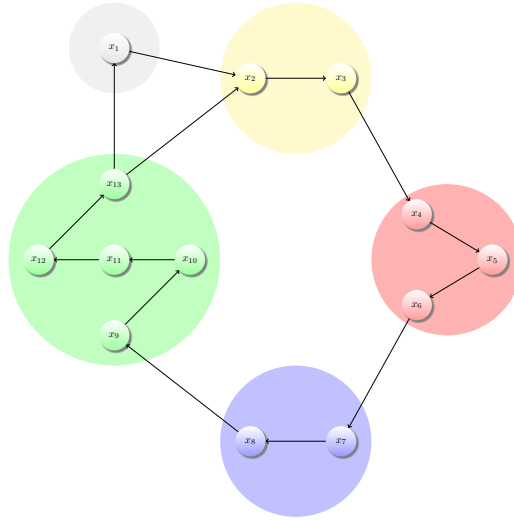
Figure 2.1: A multicompartmental model of cell proliferation through the cell cycle.

growth and regression, prediction of tumour response to chemotherapy, understanding the role of immunological systems.

The cell cycle consists of four distinct phases: G_1 (Gap 1) phase, during which the cell increases in size and synthesises RNA's and proteins; S (synthesis) phase, characterized by the duplication of the DNA in the cell's nucleus; G_2 (Gap 2) phase, during which the cell continues to grow, produces RNA's and proteins, and reorganizes its content in preparation for mitosis; M (mitosis) phase, during which the cell divides its duplicated DNA and cytoplasm to give rise to two new cells. Activation of each phase is dependent on the proper progression and completion of the previous one. After mitosis, the cell may continue in the aforementioned proliferative cycle entering again the G_1 phase, or it may temporarily stop dividing and enter a state of quiescence called G_0 phase.

Figure 2.1 shows the multicompartmental model adopted to describe the progression of the cell population through the cell cycle. The G_0 phase is represented by $g_0 = 1$ compartment, while the G_1, S, G_2 and M phases are represented by $g_1 = 2$, $s = 3$, $g_2 = 2$ and $m = 5$ compartments, respectively. So, the total number of compartments is $n = g_0 + g_1 + s + g_2 + m = 13$. We let $x_i(t)$, $i \in [1, 13]$, be the number of cells in the ith compartment at time t, where x_1 is the compartment associated with the G_0 phase and x_{13} is the last compartment of the M phase. We denote by λ_i, $i \in [1, 13]$, the transition rate of cells from the jth to the $(j + 1)$th compartment, $j \in [1, 12]$, and by μ_i, $i \in [1, 13]$, the loss rate due to cell death in the ith compartment. Consider the total number of cells that have completed mitosis and leave the 13th compartment: a fraction α of cells are

lost due to death at fission; a fraction β of them immediately begins another proliferative cycle by entering the G_1 phase; finally, a fraction $(1 - \alpha - \beta)$ enter the G_0 phase. Clearly, all the parameters λ_i, μ_i, $i \in [1, 13]$, α and β are nonnegative.

The dynamic evolution of cells in the ith compartment is governed by the mass conservation law and is described by the following equation:

$$
\begin{aligned}
\dot{x}_1(t) &= -(\lambda_1 + \mu_1)x_1(t) + (1 - \alpha - \beta)\lambda_{13}x_{13}(t) \\
\dot{x}_2(t) &= \lambda_1 x_1(t) - (\lambda_2 + \mu_2)x_2(t) + \beta\lambda_{13}x_{13}(t) \\
\dot{x}_i(t) &= \lambda_{i-1}x_{i-1}(t) - (\lambda_i + \mu_i)x_i(t), \qquad\qquad i \in [3, 13].
\end{aligned}
$$

Upon defining $\mathbf{x}(t) = \begin{bmatrix} x_1(t) & \ldots & x_{13}(t) \end{bmatrix}^\top$, if we put together the previous equations, the following Compartmental System describes the dynamics of the cell population:

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) \\
&= \begin{bmatrix}
-(\lambda_1 + \mu_1) & 0 & \ldots & \ldots & 0 & (1 - \alpha - \beta)\lambda_{13} \\
\lambda_1 & -(\lambda_2 + \mu_2) & \ldots & \ldots & 0 & \beta\lambda_{13} \\
0 & \lambda_2 & \ldots & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & & \vdots & \vdots \\
\vdots & \vdots & & \lambda_{11} & -(\lambda_{12} + \mu_{12}) & 0 \\
0 & 0 & \ldots & & \lambda_{12} & -(\lambda_{13} + \mu_{13})
\end{bmatrix} \mathbf{x}(t).
\end{aligned}
$$

Notice that the matrix $A \in \mathbb{R}^{13 \times 13}$ is compartmental and, since the associated graph is connected (see Figure 2.1), it is irreducible, too.

2.2 Room temperature regulation

Thermal systems represent an important application area of positive systems. Positivity is, indeed, an intrinsic characteristic of the system once the Kelvin scale is adopted as temperature measurement system. The following example, taken from Blanchini et al. (2015) with slight modifications, deals with temperature regulation in adjacent rooms.

Consider the thermal system of Figure 2.2: it consists of three rooms, two of them (room 2 and room 3) directly connected to the air-conditioning system. Let α, β and γ be the positive thermal transmission coefficients between the adjacent rooms $(1, 2)$, $(1, 3)$, and $(2, 3)$, respectively, when all doors are open. Denote by $x_i, i \in [1, 3]$, the (positive) difference between the temperature in the ith room and the desired temperature x_d. If we assume that the system is thermally isolated from the external environment, the time
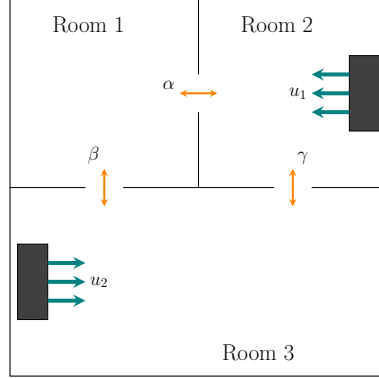
Figure 2.2: A three-room thermal system.

evolution of the temperatures in the three rooms is described by the following Multi-Input Compartmental System:

$$
\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)
$$

$$
= \begin{bmatrix} -(\alpha + \beta) & \alpha & \beta \\ \alpha & -(\alpha + \gamma) & \gamma \\ \beta & \gamma & -(\gamma + \beta) \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t).
$$

Notice that the compartmental matrix $A \in \mathbb{R}^{3 \times 3}$ is irreducible since each room can communicate with all the other rooms, and non-Hurwitz (indeed, $\mathbf{1}_3^\top A = 0^\top$) since the system is thermally isolated.

Now assume that the air-conditioning system has been switched off, and hence $\mathbf{u}(t) \equiv 0$: the resulting autonomous Compartmental System describes the unforced evolution of the temperatures in the rooms. If we also assume that doors among adjacent rooms $(1, 2)$ and $(1, 3)$ may be closed, different thermal transmission coefficients have to be considered. Specifically, the coefficients α and β take two possible values, namely $\alpha \in \{\alpha_{\min}, \alpha_{\max}\}$ and $\beta \in \{\beta_{\min}, \beta_{\max}\}$, with $\alpha_{\min} < \alpha_{\max}$ and $\beta_{\min} < \beta_{\max}$. So, there are four possible choices for the pair (α_i, β_i):

$$
\begin{aligned}
i = 1 : (\alpha_{\min}, \beta_{\min}); &\qquad i = 2 : (\alpha_{\max}, \beta_{\min}); \\
i = 3 : (\alpha_{\min}, \beta_{\max}); &\qquad i = 4 : (\alpha_{\max}, \beta_{\max}).
\end{aligned}
$$

Rooms temperatures evolution when switching among different open/closed configurations

of the two doors can be described by means of a Compartmental Switched System

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t),$$

where $\sigma\colon \mathbb{R}_+ \to [1, 4]$ is a switching function, and each matrix A_i, $i \in [1, 4]$, is characterized by the pair of parameters (α_i, β_i) described before.

2.3 Traffic congestion control

The example presented in this section, borrowed from Blanchini et al. (2015), deals with vehicular traffic in road junctions governed by traffic lights. It is worth noticing that similar models can be adopted to describe the behaviour of generic queueing systems with buffer variables, encountered for instance in many production processes or data processing tasks. These kinds of models are useful tools to improve traffic flowing over a network, test priority policies, address problems such as congestion avoidance and stabilization via periodic switching sequences.

Consider the three-way junction depicted in Figure 2.3 and composed of three road segments converging into a triangular interconnection. Buffer variables x_i, $i \in [1, 3]$, represent the number of vehicles queueing at the traffic lights inside the junction. We assume counterclockwise circulation inside the junction, and consider only those configurations with one red light. In addition, we make the following assumptions to model the behaviour of the ith, $i \in [1, 3]$, buffer variable[1]:

1) If x_i has green light and x_{i-1} has red light, x_i decays exponentially with rate $\alpha > 0$;

2) If x_i has red light and x_{i-1} has green light, x_i increases proportionally ($\beta > 0$) to x_{i-1};

3) If both x_i and x_{i-1} have green light, x_i remains approximately constant.

Consider the traffic lights configuration illustrated in Figure 2.3. With the previous modelling assumptions time evolution of the buffer state vector $\mathbf{x} := \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$ is described by

$$\dot{\mathbf{x}}(t) = A_3\mathbf{x}(t) = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \beta & 0 \end{bmatrix} \mathbf{x}(t).$$

---

[1]We have set $x_0 := x_3$.
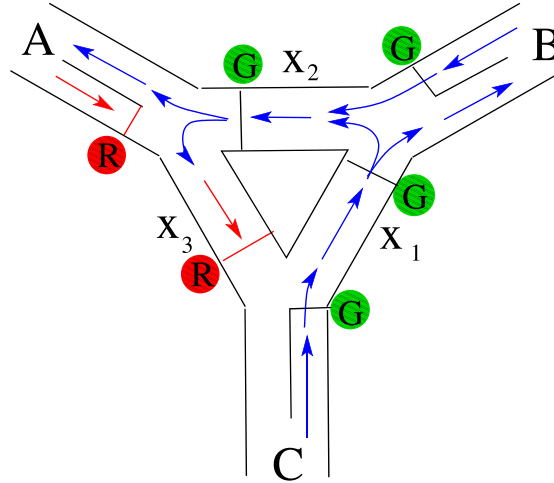
Figure 2.3: A three-way junction, Blanchini et al. (2015).

The other two configurations are obtained by a circular rotation of x_1, x_2 and x_3, and hence are characterized by the following state-space matrices

$$A_1 := \begin{bmatrix} 0 & 0 & \beta \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & -\alpha \end{bmatrix},$$

where A_i, $i \in [1, 2]$, represents the configuration with x_i having red light and the remaining variables having green light. When switching among the three possible configurations, the following Positive Switched System describes the dynamics of the buffer variables

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \tag{2.1}$$

where $\sigma \colon \mathbb{R}_+ \to [1, 3]$ is a control law implementing a certain switching policy.

2.4 A multicompartment lung model

The following example is taken from W.M. Haddad (2010) and concerns the mathematical modelling of the respiratory function. Due to their anatomical and functional heterogeneity, lungs can be regarded as an agglomeration of subunits (compartments) whose behaviour is significantly different in the inspiration and in the expiration phases. It is reasonable to assume that the bronchial tree has a dichotomy architecture, by this
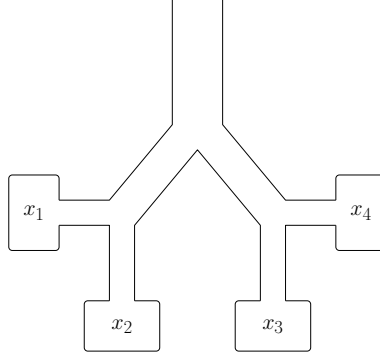
Figure 2.4: Four-compartment lung model

meaning that each airway unit progressively branches into two airway subunits of the subsequent generation. So, if there are ℓ generations of airway units, a compartmental model of size 2^ℓ, whose ith state variable x_i represents the volume of the ith compartment, can be adopted to describe the lung functioning.

Here we consider for simplicity the four-compartment lung model shown in Figure 2.4. We define the compliance matrix C as

$$C := \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{bmatrix},$$

where c_i, $i \in [1,4]$, denotes the compliance of the ith compartment. We introduce the resistance matrices R_{in} and R_{ex}, described as in equations (2.2) and (2.3) respectively, where $R_{j,i}^{\text{in}}$ ($R_{j,i}^{\text{ex}}$), $i \in [1, 2^j]$, $j \in [0,4]$, denotes the resistance to air flow of the ith airway in the jth generation during the inspiration (expiration) period, and $R_{0,1}^{\text{in}}$ ($R_{0,1}^{\text{ex}}$) denotes that of the inspiration (expiration) of the parent airway.

$$R_{\text{in}} := \begin{bmatrix} (R_{0,1}^{\text{in}}+R_{1,1}^{\text{in}}+R_{2,1}^{\text{in}}) & (R_{0,1}^{\text{in}}+R_{1,1}^{\text{in}}) & R_{0,1}^{\text{in}} & R_{0,1}^{\text{in}} \\ (R_{0,1}^{\text{in}}+R_{1,1}^{\text{in}}) & (R_{0,1}^{\text{in}}+R_{1,2}^{\text{in}}+R_{2,2}^{\text{in}}) & R_{0,1}^{\text{in}} & R_{0,1}^{\text{in}} \\ R_{0,1}^{\text{in}} & R_{0,1}^{\text{in}} & (R_{0,1}^{\text{in}}+R_{1,2}^{\text{in}}+R_{2,3}^{\text{in}}) & (R_{0,1}^{\text{in}}+R_{1,2}^{\text{in}}) \\ R_{0,1}^{\text{in}} & R_{0,1}^{\text{in}} & (R_{0,1}^{\text{in}}+R_{1,2}^{\text{in}}) & (R_{0,1}^{\text{in}}+R_{1,2}^{\text{in}}+R_{2,4}^{\text{in}}) \end{bmatrix} \tag{2.2}$$

$$R_{\text{ex}} := \begin{bmatrix} (R_{0,1}^{\text{ex}}+R_{1,1}^{\text{ex}}+R_{2,1}^{\text{ex}}) & (R_{0,1}^{\text{ex}}+R_{1,1}^{\text{ex}}) & R_{0,1}^{\text{ex}} & R_{0,1}^{\text{ex}} \\ (R_{0,1}^{\text{ex}}+R_{1,1}^{\text{ex}}) & (R_{0,1}^{\text{ex}}+R_{1,2}^{\text{ex}}+R_{2,2}^{\text{ex}}) & R_{0,1}^{\text{ex}} & R_{0,1}^{\text{ex}} \\ R_{0,1}^{\text{ex}} & R_{0,1}^{\text{ex}} & (R_{0,1}^{\text{ex}}+R_{1,2}^{\text{ex}}+R_{2,3}^{\text{ex}}) & (R_{0,1}^{\text{ex}}+R_{1,2}^{\text{ex}}) \\ R_{0,1}^{\text{ex}} & R_{0,1}^{\text{ex}} & (R_{0,1}^{\text{ex}}+R_{1,2}^{\text{ex}}) & (R_{0,1}^{\text{ex}}+R_{1,2}^{\text{ex}}+R_{2,4}^{\text{ex}}) \end{bmatrix} \tag{2.3}$$

We let $p_{\text{in}}(t)$ denote the pressure applied to the opening of the parent airway during inspiration and $p_{\text{ex}}(t)$ the external pressure (equal to the atmospheric pressure). Clearly,

inspiration and expiration periods alternate. So, if we let $\mathcal{T}_{\text{in}}$ ($\mathcal{T}_{\text{ex}}$) denote the set of all time instants at which inspiration (expiration) takes place, the lung dynamics can be described by the following Compartmental Switched System (see W.M. Haddad (2010) for an exhaustive derivation of the state equations):

$$\dot{\mathbf{x}}(t) = \begin{cases} A_{\text{in}}\mathbf{x}(t) + B_{\text{in}}p_{\text{in}}(t), & \text{if } t \in \mathcal{T}_{\text{in}}, \\ A_{\text{ex}}\mathbf{x}(t) + B_{\text{ex}}p_{\text{ex}}(t), & \text{if } t \in \mathcal{T}_{\text{ex}}, \end{cases}$$

where $A_{\text{in}} := -R_{\text{in}}^{-1}C$, $B_{\text{in}} := -R_{\text{in}}^{-1}\mathbf{1}_4$, $A_{\text{ex}} := -R_{\text{ex}}^{-1}C$ and $B_{\text{ex}} := -R_{\text{ex}}^{-1}\mathbf{1}_4$.

3

Preliminaries on Metzler matrices and compartmental matrices

In this chapter we introduce some properties of Metzler matrices and compartmental matrices that will be extensively exploited throughout the manuscript. Some of the results presented are basic facts known from the literature, some others are technical results that we will need in the following chapters.

3.1 Metzler matrices

The investigation of Positive Systems in the continuous-time setting is intimately related to the theory of Metzler matrices that characterize their state space representation. Since any Metzler matrix $A \in \mathbb{R}^{n \times n}$ can be written as $A = A_+ - aI_n$ for some nonnegative matrix $A_+ \in \mathbb{R}_+^{n \times n}$ and some positive scalar $a \in \mathbb{R}_+$, spectral properties of Metzler matrices are directly connected to those of nonnegative matrices.

Proposition 3.1 is the straightforward extension to the class of Metzler matrices of the Perron-Frobenius theorem as stated for instance in Berman and Plemmons (1987) for nonnegative matrices.

Proposition 3.1. *For a Metzler matrix $A \in \mathbb{R}^{n \times n}$ the following facts hold:*

   i) There exists in $\sigma(A)$ a real dominant eigenvalue (known as Frobenius eigenvalue*), i.e., there exists $\lambda_F(A) \in \sigma(A) \cap \mathbb{R}$ such that*

$$\lambda_F(A) > \Re(\lambda) \quad \forall \lambda \in \sigma(A), \lambda \neq \lambda_F(A).$$

   ii) There exists a positive eigenvector (known as Frobenius eigenvector*) corresponding to $\lambda_F(A)$, i.e., there exists $\mathbf{v}_F \in \mathbb{R}_+^n$ such that*

$$A\mathbf{v}_F = \lambda_F(A)\mathbf{v}_F.$$

   iii) If A is an irreducible matrix, then the Frobenius eigenvalue $\lambda_F(A)$ is necessarily simple (i.e., $\lambda_F(A)$ has algebraic multiplicity equal to 1), and the unique (up to scalar multiplication) Frobenius eigenvector $\mathbf{v}_F$ is strictly positive.

Notice that from Proposition 3.1, statement *i*), it follows that for a Metzler matrix A the spectral abscissa $\lambda_{\max}(A)$ is always a real eigenvalue and it coincides with the Frobenius eigenvalue $\lambda_F(A)$. When no confusion may arise, we will use λ_F instead of $\lambda_F(A)$ to denote the Frobenius eigenvalue of A.

A well known and useful property of Metzler matrices is the monotonicity of the spectral abscissa, Berman and Plemmons (1987); Son and Hinrichsen (1996).

Proposition 3.2. *Let $\bar{A}, A \in \mathbb{R}^{n \times n}$ be Metzler matrices such that $\bar{A} \geq A$. Then $\lambda_F(\bar{A}) \geq \lambda_F(A)$. If in addition $\bar{A}$ is irreducible, then $\bar{A} > A$ implies $\lambda_F(\bar{A}) > \lambda_F(A)$.*

Hurwitz stability of Metzler matrices has been extensively studied and many necessary and sufficient conditions for a Metzler matrix to be Hurwitz have been provided, Berman and Plemmons (1987).

Proposition 3.3. *For a Metzler matrix $A \in \mathbb{R}^{n \times n}$ the following facts are equivalent:*

   i) A is a Hurwitz matrix;

   ii) There exists a vector $\mathbf{v} \in \mathbb{R}_+^n$, $\mathbf{v} \gg 0$, such that $A\mathbf{v} \ll 0$;

   iii) A is non singular and $-A^{-1}$ is a positive matrix, i.e., $-A^{-1} > 0$;

   iv) The coefficients of the characteristic polynomial of A are all positive, namely if $\det(sI_n - A) := \alpha_0 + \alpha_1 z + \cdots + \alpha_{n-1}z^{n-1} + z^n$, then $\alpha_i > 0$ for every $i \in [0, n-1]$;

v) *There exists a positive diagonal matrix* $D \in \mathbb{R}^n_+$ *such that*

$$- \left(DA + A^\top D \right) \succ 0.$$

In addition, if A is a Metzler, irreducible and Hurwitz matrix, then $-A^{-1}$ is a strictly positive matrix, i.e., $-A^{-1} \gg 0$.

When dealing with Metzler matrices, it is oftentimes convenient to resort to row and column permutations to bring the matrix into some special form. Specifically, for a Metzler matrix $A \in \mathbb{R}^{n \times n}$, a permutation matrix Π can always be found such that

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ 0 & A_{22} & \dots & A_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & A_{ss} \end{bmatrix}, \tag{3.1}$$

where each diagonal block $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, s]$, is either scalar ($n_i = 1$) or irreducible. (3.1) is usually known as *Frobenius normal form* of A, Gantmacher (1960); Minc (1988).

Given a Metzler matrix $A \in \mathbb{R}^{n \times n}$, we associate with it (see Schneider (1986); Brualdi and Ryser (1991)) a directed graph (digraph) $\mathcal{D}(A) = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} := [1, n]$ and the adjacency matrix $\mathcal{A} \in \mathbb{R}^{n \times n}_+$ is defined as

$$[\mathcal{A}]_{ij} := \begin{cases} [A]_{ij}, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

Given two vertices $h, k \in \mathcal{V}$, $h \neq k$, we say that vertex h is *accessible* from vertex k if there exists a path in $\mathcal{D}(A)$ from k to h. Two distinct vertices h and k are said to *communicate* if each of them is accessible from the other. Each vertex is assumed to communicate with itself. The concept of communicating vertices allows to partition the set of vertices $\mathcal{V}$ into communicating classes. A *communication class* is a set of vertices such that each vertex within the class communicates with every other vertex in the class and with no other vertex. So, if we refer to the Frobenius normal from (3.1), each irreducible diagonal block A_{ii}, $i \in [1, s]$, is associated with a communication class, say $\mathcal{C}_i$, $i \in [1, s]$. We say that class $\mathcal{C}_j$ *accesses* class $\mathcal{C}_i$ if there is a path from some vertex $k \in \mathcal{C}_j$ to some vertex $h \in \mathcal{C}_i$[1]. Each class $\mathcal{C}_i$ has clearly access to itself. A class $\mathcal{C}_i$ is called *recurrent* if it has access to no other class except for itself, otherwise it is called

---

[1]Notice that the fact that class $\mathcal{C}_j$ accesses class $\mathcal{C}_i$ does not necessarily imply that block A_{ij} in (3.1) is nonzero.

transient, Kemeny and Snell (1960); Norris (1997). If $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ are the recurrent classes of $\mathcal{D}(A)$ and $\mathcal{C}_{\ell+1}, \ldots, \mathcal{C}_s$ its transient classes, then we can always assume without loss of generality that the Frobenius normal form (3.1) of A has the following structure

$$
\Pi^\top A \Pi = \begin{bmatrix}
A_{11} & \ldots & 0 & A_{1\ell+1} & \ldots & A_{1s} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & A_{\ell\ell} & A_{\ell\ell+1} & \ldots & A_{\ell s} \\
0 & \ldots & 0 & A_{\ell+1\ell+1} & \ldots & A_{\ell+1s} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & \ldots & A_{ss}
\end{bmatrix}. \tag{3.2}
$$

Notice that $\mathcal{D}(A)$ consists of a single communicating class, i.e., every pair of vertices $h, k \in \mathcal{V}$ communicate, if and only if A is irreducible. Hence, irreducibility of A and connectedness of the associated digraph $\mathcal{D}(A)$ are equivalent properties.

We now introduce some nonzero pattern characterizations of the exponential of a Metzler matrix A. Lemma 3.4 is taken from Berman and Plemmons (1987), while Proposition 3.5 is borrowed from Santesso and Valcher (2007).

Lemma 3.4. *Let $A \in \mathbb{R}^{n \times n}$ be a Metzler matrix. Then, for every $t \geq 0$ the matrix e^{At} is nonnegative. If in addition A is irreducible, then for every $t > 0$ the matrix e^{At} is strictly positive.*

Proposition 3.5. *Let $A \in \mathbb{R}^{n \times n}$ be a Metzler matrix in Frobenius normal form* (3.1). *Then, at every time instant $t > 0$*

$$
\mathbb{A}(t) := e^{At} = \begin{bmatrix}
\mathbb{A}_{11}(t) & \mathbb{A}_{12}(t) & \ldots & \mathbb{A}_{1s}(t) \\
0 & \mathbb{A}_{22}(t) & \ldots & \mathbb{A}_{2s}(t) \\
\vdots & & \ddots & \vdots \\
0 & \ldots & & \mathbb{A}_{ss}(t)
\end{bmatrix},
$$

where $\mathbb{A}_{ii}(t) \gg 0$ for every $i \in [1, s]$, and $\mathbb{A}_{ij}(t)$, $i, j \in [1, s]$ and $i < j$, is either strictly positive or zero. Specifically, $\mathbb{A}_{ij}(t) \gg 0$ if class $\mathcal{C}_j$ has access to class $\mathcal{C}_i$, and is zero otherwise.

We conclude this section by introducing the notion of Metzler part of a matrix. Consistently with Hinrichsen and Plischke (2007), for a matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ we

define the *Metzler part* of A, and denote it by $\mathcal{M}(A)$, the matrix

$$[\mathcal{M}(A)]_{ij} := \begin{cases} a_{ii}, & \text{if } i = j; \\ |a_{ij}|, & \text{if } i \neq j. \end{cases}$$

Hinrichsen and Plischke (2007) proved the following monotonicity result for the spectral abscissa of the Metzler part of a matrix (see Fang, Loparo, and Feng (1993) for an alternative proof).

Lemma 3.6. *Let $\bar{A} \in \mathbb{R}^{n \times n}$ be a Metzler matrix. Then, for any $A \in \mathbb{R}^{n \times n}$ such that $\mathcal{M}(A) \leq \bar{A}$ the following inequalities hold:*

$$\lambda_{\max}(A) \leq \lambda_{\max}(\mathcal{M}(A)) \leq \lambda_{\max}(\bar{A}).$$

3.2 Compartmental matrices

We recall that a Metzler matrix $A \in \mathbb{R}^{n \times n}$ is compartmental if the entries of each of its columns sum up to a nonpositive number, i.e., $\mathbf{1}_n^\top A \leq 0^\top$. For any such matrix the following property holds, Farina and Rinaldi (2000).

Proposition 3.7. *For a compartmental matrix $A \in \mathbb{R}^{n \times n}$ the Frobenius eigenvalue is nonpositive, i.e., $\lambda_F \leq 0$. In addition, if $\lambda_F = 0$ then A is simply stable, by this meaning that it has the constant mode associated with $\lambda_F = 0$ but no unstable modes.*

Clearly, condition $\mathbf{1}_n^\top A \ll 0^\top$ ensures that A is Hurwitz (see Proposition 3.3). On the other hand, condition $\mathbf{1}_n^\top A = 0^\top$ means that $\mathbf{1}_n$ is a left eigenvector of A corresponding to $\lambda_F = 0$, and hence A is non-Hurwitz. The intermediate case when $\mathbf{1}_n^\top A < 0^\top$, but at least one of the entries of $\mathbf{1}_n^\top A$ is zero, does not allow to draw any conclusion on the Hurwitz property of A, unless A is irreducible. An equivalent formulation of the following result is credited to Taussky (1949) (see also Jacquez and Simon (1993)). We here provide a short proof for the sake of completeness.

Lemma 3.8. *Let $A \in \mathbb{R}^{n \times n}$ be a compartmental irreducible matrix. Then, A is Hurwitz if and only if $\mathbf{1}_n^\top A < 0^\top$.*

Proof. The necessity is always true, even for reducible matrices. Indeed, as A is compartmental, $\mathbf{1}_n^\top A \le 0^\top$. If it were $\mathbf{1}_n^\top A = 0^\top$, then, as previously noticed, the Frobenius eigenvalue of A would be zero, and hence A could not be Hurwitz.

Conversely, assume that A is irreducible and $\mathbf{1}_n^\top A < 0^\top$. By the Perron-Frobenius theorem, A exhibits a real dominant eigenvalue λ_F, and there exists a strictly positive eigenvector $\mathbf{v}_F \gg 0$ corresponding to λ_F, i.e., $A\mathbf{v}_F = \lambda_F \mathbf{v}_F$. So, by multiplying both sides of the previous inequality by $\mathbf{1}_n^\top$, one gets

$$\lambda_F(\mathbf{1}_n^\top \mathbf{v}_F) = \mathbf{1}_n^\top A\mathbf{v}_F < 0,$$

which implies $\lambda_F < 0$ and hence A is Hurwitz. ∎

The following result will be useful in Chapter 4.

Proposition 3.9. *Let $A \in \mathbb{R}^{n \times n}$ be a compartmental and Hurwitz matrix. Then, every principal submatrix of A is compartmental and Hurwitz.*

Proof. Let $A_{\mathcal{J}}$ denote the principal submatrix of A obtained by selecting rows and columns of A indexed by the set $\mathcal{J} = \{j_1, \dots, j_r\} \subseteq [1, n]$, $\mathcal{J} \ne \emptyset$. Since for every permutation matrix Π the matrix $\Pi^\top A \Pi$ is still compartmental and Hurwitz, it entails no loss of generality assuming that $\mathcal{J} = [1, r]$, $r \in [1, n]$, and hence the matrix A takes the following form

$$A = \begin{bmatrix} A_{\mathcal{J}} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Clearly, $A_{\mathcal{J}}$ is Metzler. Now notice that for every $j \in \mathcal{J}$ it holds

$$\mathbf{1}_r^\top \operatorname{col}_j(A_{\mathcal{J}}) \le \mathbf{1}_r^\top \operatorname{col}_j(A_{\mathcal{J}}) + \mathbf{1}_{n-r}^\top \operatorname{col}_j(A_{21}) = \mathbf{1}_n^\top \operatorname{col}_j(A) \le 0,$$

and hence $A_{\mathcal{J}}$ is compartmental. Moreover, let $[A]_{ii} =: d_i$, $i \in [r+1, n]$, and notice that the following relation holds

$$A \ge \bar{A} := \left[\begin{array}{c|ccc} A_{\mathcal{J}} & & 0_{1 \times (n-r)} & \\ \hline & d_{r+1} & \dots & 0 \\ 0_{n-r} & \vdots & \ddots & \vdots \\ & 0 & \dots & d_n \end{array} \right].$$

Then, by the monotonicity property of Metzler matrices and recalling that A is Hurwitz, one has

$$0 > \lambda_F(A) \geq \lambda_F(\bar{A}) = \max\left\{\lambda_F(A_{\mathcal{J}}), d_{r+1}, \ldots, d_n\right\},$$

and hence $A_{\mathcal{J}}$ is Hurwitz. ∎

Stability of Compartmental Switched Systems

In this chapter we consider Compartmental Switched Systems with autonomous subsystems, that is Compartmental Switched Systems described by

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \quad t \in \mathbb{R}_+, \tag{4.1}$$

where $\sigma \colon \mathbb{R}_+ \to [1, M]$ is an arbitrary switching function and A_i, $i \in [1, M]$, are compartmental matrices. For this class of systems, under the assumption that all subsystem matrices are Hurwitz, we first investigate stability under arbitrary switching. Secondly, we drop the Hurwitz stability assumption and provide classes of switching functions with special persistence properties that ensure the asymptotic convergence of the associated system trajectories, independently of the positive initial condition.

The analysis carried out throughout the chapter is mostly based on the results appeared in:

Valcher M. E. and Zorzan I. Stability and stabilizability of continuous-time compartmental switched systems. *IEEE Transactions on Automatic Control*, 61 (12): 3885–3897, 2016c.

4.1   Stability Under Arbitrary Switching

The definition of asymptotic stability under arbitrary switching for Compartmental Switched Systems is analogous to the one known from the literature for general (i.e., non-compartmental and not necessarily positive) switched systems. In addition, as for all linear switched systems, asymptotic stability is equivalent to exponential stability, Shorten, Wirth, Mason, Wulff, and King (2007).

Definition 4.1. The Compartmental Switched System (4.1) is *asymptotically stable under arbitrary switching* if for every positive initial condition $\mathbf{x}(0) > 0$ and every switching function $\sigma\colon \mathbb{R}_+ \to [1, M]$ the state trajectory $\mathbf{x}(t)$, $t \in \mathbb{R}_+$, converges to zero.

A standard tool for investigating stability under arbitrary switching of a switched system is represented by common Lyapunov functions. When dealing with Positive Switched Systems, we may loosen the constraints on the Lyapunov function and on its derivative, by allowing them to take positive and nonpositive values, respectively, only on $\mathbb{R}_+^n$, instead of on the whole space $\mathbb{R}^n$.

Definition 4.2. Let $V(\mathbf{x})\colon \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable copositive function.

- $V(\mathbf{x})$ is a *common weak copositive Lyapunov function* for the Compartmental Switched System (4.1) if for every $\mathbf{x} > 0$ and every $i \in [1, M]$ the derivative of V in $\mathbf{x}$ along the direction of the ith subsystem is nonpositive, namely if

$$\nabla V(\mathbf{x})A_i\mathbf{x} \le 0, \quad \forall \mathbf{x} > 0, \forall i \in [1, M].$$

- $V(\mathbf{x})$ is a *common copositive Lyapunov function*[1] for the Compartmental Switched System (4.1) if
$$\nabla V(\mathbf{x})A_i\mathbf{x} < 0, \quad \forall \mathbf{x} > 0, \forall i \in [1, M].$$

The following Proposition restates for the class of Compartmental Switched Systems and common (weak) copositive Lyapunov functions a well known result derived for switched systems without any positivity assumption[2], Liberzon (2003); Bacciotti and Mazzi (2005).

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[1]In the literature Lyapunov functions with negative derivative are also called *strict* Lyapunov functions (in contrast with *weak* Lyapunov functions whose derivative is nonpositive).

[2]Actually the result holds for the broader class of nonlinear switched systems.

Proposition 4.1. *i) If the Compartmental Switched System* (4.1) *admits a common weak copositive Lyapunov function, then it is stable under arbitrary switching.*[3]

ii) If the Compartmental Switched System (4.1) *admits a common copositive Lyapunov function, then it is asymptotically stable under arbitrary switching.*

When only common weak copositive Lyapunov functions are available, determining the asymptotic behaviour of system trajectories is still possible. To this aim, we need to introduce the concept of weakly invariant set, Bacciotti and Mazzi (2005); Liu, Liu, and Xie (2010). We say that a set $\mathcal{M}$ is *weakly invariant with respect to the ith mode of system* (4.1) if for every $\mathbf{x} \in \mathcal{M}$ there exists a real number $a > 0$ such that the solution of $\dot{\mathbf{x}}(t) = A_i \mathbf{x}(t)$ corresponding to the initial condition $\mathbf{x}(0) = \mathbf{x}$ is such that $\mathbf{x}(t) \in \mathcal{M}$ either for every $t \in [0, a]$ or for every $t \in [-a, 0]$. A compact set $\mathcal{M}$ is *weakly invariant with respect to system* (4.1) if for every $\mathbf{x} \in \mathcal{M}$ there exists an index $i \in [1, M]$ such that $\mathcal{M}$ is weakly invariant with respect to the ith mode of system (4.1).

Proposition 4.2. *Let $V(\mathbf{x})$ be a common weak copositive Lyapunov function for the Compartmental Switched System* (4.1). *Denote by $\mathcal{S}_\ell$, $\ell \in \mathbb{R}_+$ and $\ell > 0$, the level set $\mathcal{S}_\ell := \{\mathbf{x} \in \mathbb{R}_+^n : V(\mathbf{x}) < \ell\}$, and assume that $\mathcal{S}_\ell$ is bounded. Define the set $\mathcal{N}$ as*

$$\mathcal{N} := \{\mathbf{x} \in \mathbb{R}_+^n : \nabla V(\mathbf{x}) A_i \mathbf{x} = 0, \exists i \in [1, M]\}.$$

Finally, let $\mathcal{M}$ be the union of all the compact, weakly invariant sets which are contained in $\mathcal{S}_\ell \cap \mathcal{N}$. Then, for every positive initial condition $\mathbf{x}(0) \in \mathcal{S}_\ell$ and every switching function $\sigma : \mathbb{R}_+ \to [1, M]$ the state trajectory $\mathbf{x}(t)$, $t \in \mathbb{R}_+$, is attracted by $\mathcal{M}$, i.e.,

$$\mathbf{x}(0) \in \mathcal{S}_\ell \implies \lim_{t \to +\infty} \text{dist}(\mathbf{x}(t), \mathcal{M}) = 0.$$

In the following we will focus on two classes of common copositive Lyapunov functions: linear and quadratic positive definite Lyapunov functions. For these functions, the definition of Lyapunov function adjusts as follows.

Definition 4.3. A continuously differentiable copositive function $V(\mathbf{x})$ is a *Common Linear Copositive Lyapunov Function* for the Compartmental Switched System (4.1) if

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[3]We say that the Compartmental Switched System (4.1) is *stable under arbitrary switching* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\mathbf{x}(0)\|_2 < \delta \implies \|\mathbf{x}(t)\|_2 < \varepsilon, \quad \forall t \geq 0, \forall \sigma : \mathbb{R}_+ \to [1, M].$$

$V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$, for some strictly positive vector $\mathbf{v} \in \mathbb{R}^n_+$, and

$$\mathbf{v}^\top A_i \ll 0^\top, \quad \forall i \in [1, M].$$

Definition 4.4. A continuously differentiable copositive function $V(\mathbf{x})$ is a *Common Quadratic Positive Definite Lyapunov Function* for the Compartmental Switched System (4.1) if $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$, for some positive definite matrix $P = P^\top \in \mathbb{R}^{n \times n}$, and

$$\dot{V}_i(\mathbf{x}) = \mathbf{x}^\top \left[A_i^\top P + P A_i \right] \mathbf{x} < 0, \quad \forall \mathbf{x} > 0, \forall i \in [1, M],$$

namely the (symmetric) matrices $Q_i := - \left[A_i^\top P + P A_i \right]$, $i \in [1, M]$, are copositive.

Similarly, definitions of *Common Weak Linear Copositive Lyapunov Functions* and *Common Weak Quadratic Positive Definite Lyapunov Function* can be easily obtained by adjusting Definitions 4.3 and Definition 4.4, respectively.

Stability Under Arbitrary Switching of Positive Switched Systems

As already remarked, Compartmental Switched Systems inherit a number of results derived for the broader class of Positive Switched Systems. Asymptotic stability of this latter class of systems attracted a great deal of attention over the last ten years, and many efforts have been made to provide characterizations either in terms of convex combinations of the subsystem matrices or in terms of Lyapunov functions, see, e.g., Mason and Shorten (2006); Gurvits et al. (2007); Mason and Shorten (2007a); Fainshil et al. (2009); Knorn, Mason, and Shorten (2009); Fornasini and Valcher (2010); Zhao, Liu, and Li (2014). We refer to Blanchini et al. (2015) for an overview on the topic.

A necessary condition for stability under arbitrary switching of Positive Switched Systems is the fact that all convex combinations of the subsystem matrices are Hurwitz (actually, the result holds even for more general classes of switched systems, see for instance Liberzon (2003) where the result is stated for switched systems without any positivity assumption). Gurvits et al. (2007) proved that this condition is not sufficient for guaranteeing asymptotic stability of Positive Switched Systems (see also Fainshil et al. (2009), where a three-dimensional example can be found). Stability under arbitrary switching and Hurwitz stability of all convex combinations of the subsystem matrices are, indeed, equivalent conditions only for special classes of Positive Switched Systems, e.g., the case of second-order subsystems, Gurvits et al. (2007); and the case of Positive Switched Systems with rank 1 difference, Fornasini and Valcher (2014).

A necessary and sufficient condition for stability under arbitrary switching of Positive Switched Systems is the existence of an infinitely differentiable convex copositive Lyapunov function, Blanchini et al. (2015). Checking this condition without imposing any a priori structure on the Lyapunov function is computationally intractable. On the other hand, it has been proved that searching for linear copositive Lyapunov functions is, in general, too restrictive (see Fornasini and Valcher (2010) for an example). Whether asymptotically stable Positive Switched Systems necessarily admit a quadratic copositive Lyapunov function still represents, to the best of our knowledge, an unanswered question.

In the following, we show that for Compartmental Switched Systems a clear picture of stability under arbitrary switching can be obtained. In addition, we provide some characterizations in terms of Lyapunov functions that are much stronger than the ones available for the non-compartmental case.

4.1.1 Hurwitz property of the subsystem matrices

Clearly, a necessary condition for a linear switched system (and hence a Compartmental Switched System) to be stable under arbitrary switching is that all individual subsystems are asymptotically stable. On the other hand, in the general case neither the Hurwitz assumption on all the subsystem matrices, nor the stronger condition that every convex combination of the subsystem matrices is Hurwitz, is sufficient to guarantee stability under arbitrary switching, not even when we deal with continuous-time Positive Switched Systems and hence all matrices A_i, $i \in [1, M]$, are Metzler.

We now prove that for Compartmental Switched Systems, stability under arbitrary switching is equivalent to the fact that all the subsystem matrices are Hurwitz. To this aim, we need some preliminary technical results about Compartmental Systems.

Technical results about asymptotically stable Compartmental Systems

We consider Compartmental Systems described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \tag{4.2}$$

for some compartmental matrix $A \in \mathbb{R}^{n \times n}$, and prove that when A is a Hurwitz matrix, the weak linear copositive Lyapunov function $V(\mathbf{x}(t)) = \mathbf{1}_n^\top \mathbf{x}(t)$ is strictly decreasing along the system trajectories independently of the positive initial condition. We consider the irreducible case in Lemma 4.3 and then we remove the irreducibility hypothesis in Proposition 4.4.

Lemma 4.3. *Consider a Compartmental System described as in* (4.2), *and assume that the compartmental matrix $A \in \mathbb{R}^{n \times n}$ is irreducible and Hurwitz. Then, the weak linear copositive Lyapunov function $V(\mathbf{x}(t)) = \mathbf{1}_n^\top \mathbf{x}(t)$ is strictly decreasing along the system trajectories, independently of the positive initial condition, i.e.,*

$$\mathbf{1}_n^\top \mathbf{x}(t) < \mathbf{1}_n^\top \mathbf{x}(0), \quad \forall t > 0, \forall \mathbf{x}(0) > 0. \tag{4.3}$$

Proof. Since A is a compartmental irreducible and Hurwitz matrix, by Lemma 3.8, $\mathbf{1}_n^\top A < 0^\top$. Now, if $\mathbf{1}_n^\top A \ll 0^\top$, then $\dot{V}(\mathbf{x}(t)) = \mathbf{1}_n^\top A \mathbf{x}(t) < 0$ for every $t > 0$, independently of $\mathbf{x}(0) > 0$, and hence $V(\mathbf{x}(t))$ is strictly decreasing along the system trajectories, independently of the positive initial condition, namely (4.3) holds.

If $\mathbf{1}_n^\top A < 0^\top$, then, possibly by resorting to row and column permutations on A, we can assume without loss of generality that it takes the form $\mathbf{1}_n^\top A = \begin{bmatrix} 0_k^\top & -\mathbf{v}^\top \end{bmatrix}$, $\mathbf{v} \in \mathbb{R}_+^{n-k}$, $\mathbf{v} \gg 0$. Set

$$\mathcal{N} := \left\{ \mathbf{x} \in \mathbb{R}_+^n : \dot{V}(\mathbf{x}) = \mathbf{1}_n^\top A \mathbf{x} = 0 \right\} \tag{4.4}$$
$$= \mathrm{Cone}\left(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\right).$$

We want to show that $\mathcal{N}$ contains no system trajectory except for the zero trajectory, i.e., $\mathbf{x}(t) = 0$ for all $t \geq 0$. If $\mathbf{x}(0) \in \mathcal{N}$, $\mathbf{x}(0) \neq 0$, then $\mathbf{x}(0) = \begin{bmatrix} \mathbf{x}_{10}^\top & 0_{n-k}^\top \end{bmatrix}^\top$, $\mathbf{x}_{10} \in \mathbb{R}_+^k$, $\mathbf{x}_{10} > 0$. By the irreducibility assumption on A, $e^{At} \gg 0$ for all $t > 0$ (recall Lemma 3.4), and hence for every $t > 0$ $\mathbf{x}(t) = e^{At} \mathbf{x}(0) = \begin{bmatrix} \mathbf{x}_1(t)^\top & \mathbf{x}_2(t)^\top \end{bmatrix}^\top$, with $\mathbf{x}_2(t) \gg 0$. So, for every $t > 0$, $\mathbf{x}(t) \notin \mathcal{N}$, and therefore $V(\mathbf{x}(t))$ is strictly decreasing over any arbitrary small time interval $[0, t]$. ∎

Proposition 4.4. *Consider a Compartmental System described as in* (4.2), *and assume that the compartmental matrix $A \in \mathbb{R}^{n \times n}$ is Hurwitz. Then, the weak linear copositive Lyapunov function $V(\mathbf{x}(t)) = \mathbf{1}_n^\top \mathbf{x}(t)$ is strictly decreasing along the system trajectories independently of the positive initial condition, i.e.,* (4.3) *holds.*

Proof. The case when the compartmental matrix A is irreducible has been addressed in Lemma 4.3. So, we assume now that A is reducible. Consider the directed graph associated with A, and let $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ be its recurrent classes and $\mathcal{C}_{\ell+1}, \ldots, \mathcal{C}_s$ be its transient classes. It entails no loss of generality assuming that A is in Frobenius normal

form (3.2), namely

$$
A = \begin{bmatrix}
A_{11} & \dots & 0 & A_{1\ell+1} & \dots & A_{1s} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & A_{\ell\ell} & A_{\ell\ell+1} & \dots & A_{\ell s} \\
0 & \dots & 0 & A_{\ell+1\ell+1} & \dots & A_{\ell+1s} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & A_{ss}
\end{bmatrix},
$$

where $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, s]$, are irreducible matrices, since we can always reduce ourselves to this situation by resorting to a suitable permutation of the rows and columns of A that does not affect the compartmental property of A. Accordingly (see Proposition 3.5), e^{At} takes the following form

$$
\mathbb{A}(t) := e^{At} = \begin{bmatrix}
\mathbb{A}_{11}(t) & \dots & 0 & \mathbb{A}_{1\ell+1}(t) & \dots & \mathbb{A}_{1s}(t) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & \mathbb{A}_{\ell\ell}(t) & \mathbb{A}_{\ell\ell+1}(t) & \dots & \mathbb{A}_{\ell s}(t) \\
0 & \dots & 0 & \mathbb{A}_{\ell+1\ell+1}(t) & \dots & \mathbb{A}_{\ell+1s}(t) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & \mathbb{A}_{ss}(t)
\end{bmatrix},
$$

where, for every $t > 0$, the matrix $\mathbb{A}_{ij}(t) = \text{block}_{ij}\left[e^{At}\right] \in \mathbb{R}_+^{n_i \times n_j}$, $i \in [1, \ell]$, $j \in [\ell+1, s]$, is strictly positive if the class $\mathcal{C}_j$ has access to the class $\mathcal{C}_i$ and the null matrix otherwise. It follows from the definition of transient classes that, for every $j \in [\ell+1, s]$, there exists $i \in [1, \ell]$ such that $\mathbb{A}_{ij}(t) \in \mathbb{R}_+^{n_i \times n_j}$ is strictly positive. Moreover, since A_{ii}, $i \in [1, \ell]$, are irreducible matrices, the matrices $\mathbb{A}_{ii}(t) = e^{A_{ii}t}$, $i \in [1, \ell]$, are strictly positive for every $t > 0$. The state vector $\mathbf{x}(t)$ can be partitioned into s blocks, according to the partition of A. Now, define the set $\mathcal{N}$ as in (4.4), and let $\mathbf{x}(0) > 0$ be in $\mathcal{N}$. We first note that, by the irreducibility assumption on A_{ii}, $i \in [1, \ell]$, in every set of indices $\left\{\left(\sum_{k=0}^{i-1} n_k\right) + 1, \dots, \left(\sum_{k=0}^{i-1} n_k\right) + n_i\right\}$, $i \in [1, \ell]$, with $n_0 := 0$, there is at least one index j such that $\mathbf{1}_n^\top A \mathbf{e}_j < 0$. This implies that if $\mathbf{x}(t) > 0$ and $[\mathbf{x}(t)]_j > 0$ then $\mathbf{1}_n^\top A\mathbf{x}(t) < 0$, and hence $\mathbf{x}(t) \notin \mathcal{N}$. So, by the same reasoning adopted in Lemma 4.3, we can claim that every $\mathbf{x}(0) \in \mathcal{N}$, $\mathbf{x}(0) > 0$, whose nonzero entries belong only to the first ℓ blocks, necessarily generates a state trajectory that exits $\mathcal{N}$.

Assume, now, that there exists $k \in [\ell+1, s]$ such that $\text{block}_k[\mathbf{x}(0)] > 0$. Then, by the previous reasoning, there exists $i \in [1, \ell]$ such that $\text{block}_i\left[e^{At}\mathbf{x}(0)\right] \gg 0$ for every $t > 0$.

But this implies that $\mathbf{x}(t)$ cannot belong to $\mathcal{N}$. Thus, as $\dot{V}(\mathbf{x}(t)) = \mathbf{1}_n^\top A\mathbf{x}(t) \leq 0$ and $\mathcal{N}$ does not include system trajectories apart from the identically zero one, it follows that $V(\mathbf{x}(t))$ is strictly decreasing with t. ∎

To summarize, Proposition 4.4 proves that, if a Compartmental System is asymptotically stable, then the weak linear copositive Lyapunov function $V(\mathbf{x}(t)) = \mathbf{1}_n^\top \mathbf{x}(t)$ is strictly decreasing along the system trajectories (independently of the positive initial condition), even if its derivative may be zero at some point.

Remark 4.1. It is worth noticing that for an asymptotically stable Compartmental System described as in (4.2) it is not necessarily true that $\mathbf{1}_n^\top A \ll 0^\top$, which is the property a linear copositive Lyapunov function has to satisfy. However, it is clear that since A is a Metzler and Hurwitz matrix, there is always a vector $\mathbf{v} \gg 0$ such that $\mathbf{v}^\top A \ll 0^\top$. So, a linear copositive Lyapunov function always exists but it is not necessarily the one associated with the vector $\mathbf{1}_n$.

We are now in a position to prove the main result of the chapter.

Proposition 4.5. *Let $A_i \in \mathbb{R}^{n \times n}$, $i \in [1, M]$, be compartmental matrices. Then, the following facts are equivalent:*

i) The Compartmental Switched System (4.1) is asymptotically stable under arbitrary switching;

ii) For every choice of $\alpha_i \geq 0$, $i \in [1, M]$, with $\sum_{i=1}^M \alpha_i = 1$, the convex combination $\sum_{i=1}^M \alpha_i A_i$ is Hurwitz;

iii) A_i is Hurwitz for every $i \in [1, M]$.

Proof. i) $\Rightarrow$ ii) It is a well known result for continuous-time switched systems, Liberzon (2003).

ii) $\Rightarrow$ iii) It is obvious.

iii) $\Rightarrow$ i) The continuous and continuously differentiable copositive function $V(\mathbf{x}) = \mathbf{1}_n^\top \mathbf{x}$ is a Common Weak Linear Copositive Lyapunov Function for system (4.1), indeed

$$\nabla V(\mathbf{x}(t))\dot{\mathbf{x}}(t) = \mathbf{1}_n^\top A_i \mathbf{x}(t) \leq 0, \quad \forall \mathbf{x}(t) > 0, \forall i \in [1, M].$$

Proposition 4.1 ensures that the Compartmental Switched System is stable under arbitrary switching. However, we want to prove that the system is also asymptotically stable under

arbitrary switching. To this aim, define the set $\mathcal{N}$ as

$$\mathcal{N} := \left\{ \mathbf{x} \in \mathbb{R}_+^n \colon \nabla V(\mathbf{x}) A_i \mathbf{x} = \mathbf{1}_n^\top A_i \mathbf{x} = 0, \exists i \in [1, M] \right\}$$
$$= \bigcup_{i=1}^M \mathcal{N}_i,$$

where $\mathcal{N}_i := \left\{ \mathbf{x} \in \mathbb{R}_+^n \colon \mathbf{1}_n^\top A_i \mathbf{x} = 0 \right\}$. For every $\mathbf{x} \in \mathcal{N}$, $\mathbf{x} \neq 0$, define as well

$$\mathcal{I}_\mathbf{x} := \{ i \in [1, M] \colon \mathbf{x} \in \mathcal{N}_i \},$$
$$d_\mathbf{x} := \min_{i \notin \mathcal{I}_\mathbf{x}} \operatorname{dist}(\mathbf{x}, \mathcal{N}_i) > 0.$$

By the compartmental property of each subsystem (see the proof of Proposition 4.4), if $\mathbf{x}(0) \in \mathcal{N}_i$ for some $i \in [1, M]$, $\mathbf{x}(0) \neq 0$, then for every $\tau > 0$ sufficiently small $\mathbf{x}(\tau) \notin \mathcal{N}_i$. So, by choosing $\tau > 0$ sufficiently small we can ensure that $\mathbf{x}(\tau) \notin \mathcal{N}_i$ for every $i \in \mathcal{I}_{\mathbf{x}(0)}$. On the other hand, since the distance $d_{\mathbf{x}(0)}$ is finite, it is also true that if τ is sufficiently small $\mathbf{x}(\tau) \notin \mathcal{N}_i$ for every $i \notin \mathcal{I}_{\mathbf{x}(0)}$, and therefore $\mathbf{x}(\tau) \notin \mathcal{N}$. This ensures that the only compact, weakly invariant set contained in $\mathcal{N}$ is $\mathcal{M} = \{0\}$, and by Proposition 4.2 every state trajectory is attracted by $\mathcal{M}$, i.e., converges to the origin. ∎

4.1.2 Characterizations in terms of Lyapunov functions

First of all, we investigate the relationship between Common Linear Copositive and Common Quadratic Positive Definite Lyapunov functions. As a starting point, we have the following preliminary result, inherited from the general class of Positive Switched Systems, Blanchini et al. (2015).

Proposition 4.6. *Consider a Compartmental Switched System described as in* (4.1). *The following facts are equivalent:*

i) For every choice of M nonnegative diagonal matrices D_i, $i \in [1, M]$, with $\sum_{i=1}^M D_i = I_n$, the matrix $\sum_{i=1}^M A_i D_i$ is Metzler and Hurwitz;

ii) The Compartmental Switched System (4.1) *admits a Common Linear Copositive Lyapunov Function.*

If any of the previous equivalent conditions holds, then:

iii) The Compartmental Switched System (4.1) *admits a Common Quadratic Positive Definite Lyapunov Function.*

If iii) holds, then:

iv) *The Compartmental Switched System* (4.1) *is asymptotically stable under arbitrary switching.*

For general Positive Switched Systems, none of the latest implications $ii) \Rightarrow iii) \Rightarrow iv)$ can be reversed. Under the compartmental assumption, it is still true that $iii)$ does not imply $ii)$ (or any of the equivalent conditions), as the following example shows.

Example 4.1. Consider the Compartmental Switched System (4.1) with $M = 2$ and

$$A_1 = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}.$$

As $Q_i := - \left[A_i^\top + A_i \right] = -2A_i \succ 0$, $i = 1, 2$, condition $iii)$ holds for $P = I_2$. However, for every $\mathbf{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^\top \gg 0$, we have

$$\mathbf{v}^\top A_1 = \begin{bmatrix} v_2 - v_1 & \star \end{bmatrix} \ll 0^\top \quad \implies \quad v_2 < v_1$$
$$\mathbf{v}^\top A_2 = \begin{bmatrix} \star & v_1 - v_2 \end{bmatrix} \ll 0^\top \quad \implies \quad v_1 < v_2,$$

and hence the Compartmental Switched System does not admit a Common Linear Copositive Lyapunov Function. ♣

At the current stage of our research, it is not clear whether condition $iv)$ is equivalent to condition $iii)$ or not, since we have not been able to find either a counterexample or a proof. In the following we will show that under certain conditions on the matrices of the Compartmental Switched System, condition $iv)$ implies $iii)$, namely the Hurwitz property of all the compartmental matrices A_i, $i \in [1, M]$, guarantees the existence of a Common Quadratic Positive Definite Lyapunov Function for the associated Compartmental Switched System.

Converse results about Common Quadratic Positive Definite Lyapunov Functions

If the compartmental matrices A_i, $i \in [1, M]$, are Hurwitz, they all satisfy condition $\mathbf{1}_n^\top A_i < 0^\top$, and hence $V(\mathbf{x}) = \mathbf{x}^\top \mathbf{1}_n \mathbf{1}_n^\top \mathbf{x}$ represents a common weak copositive Lyapunov function[4] for the Compartmental Switched System. However, in general this is not a

---

[4]Note that $P = \mathbf{1}_n \mathbf{1}_n^\top$ is a copositive and positive semidefinite matrix, but it is not positive definite.

common copositive Lyapunov function, since there exist indices $i \in [1, M]$ and vectors $\mathbf{x} > 0$ such that $\dot{V}_i(\mathbf{x}) = \mathbf{x}^\top \left[A_i^\top P + P A_i \right] \mathbf{x} = 0$. In order to explore under what conditions the Hurwitz stability of the matrices A_i, $i \in [1, M]$, allows to construct a Common Quadratic Positive Definite Lyapunov Function, we focus our attention on the class of positive definite Lyapunov functions described as $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ with

$$P = P^\top = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon D \succ 0, \tag{4.5}$$

where $\varepsilon > 0$ and $D \in \mathbb{R}_+^{n \times n}$ is a diagonal matrix with positive diagonal entries.

Before proceeding let us state a preliminary lemma that will be used in the proofs of the following propositions.

Lemma 4.7. *Let $N = N^\top \in \mathbb{R}^{n \times n}$ have the following block structure:*

$$N = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix},$$

with $A = A^\top \in \mathbb{R}^{k \times k}$, $B \in \mathbb{R}^{k \times (n-k)}$, $C = C^\top \in \mathbb{R}^{(n-k) \times (n-k)}$. If $B \geq 0$ and both A and C are copositive, then N is copositive.

Proof. For every nonzero vector $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^\top & \mathbf{x}_2^\top \end{bmatrix}^\top$, $\mathbf{x}_1 \in \mathbb{R}_+^k$, $\mathbf{x}_2 \in \mathbb{R}_+^{n-k}$, one has

$$\begin{bmatrix} \mathbf{x}_1^\top & \mathbf{x}_2^\top \end{bmatrix} \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \mathbf{x}_1^\top A \mathbf{x}_1 + \mathbf{x}_2^\top B^\top \mathbf{x}_1 + \mathbf{x}_2^\top C \mathbf{x}_2 + \mathbf{x}_1^\top B \mathbf{x}_2$$

$$\geq \mathbf{x}_1^\top A \mathbf{x}_1 + \mathbf{x}_2^\top C \mathbf{x}_2 > 0,$$

where the last inequality follows from the properties of A and C, and fact that either $\mathbf{x}_1$ or $\mathbf{x}_2$ (or both) are nonzero. ∎

We can now provide a characterization of Compartmental Switched Systems admitting a Common Quadratic Positive Definite Lyapunov function of type (4.5). We first consider a single compartmental Hurwitz matrix in Lemma 4.8 and then we generalize this result to the case of a Compartmental Switched System described as in (4.1).

Lemma 4.8. *Let $A \in \mathbb{R}^{n \times n}$ be a compartmental and Hurwitz matrix such that*

$$\mathbf{1}_n^\top A = \mathbf{1}_n^\top \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0_k^\top & -\mathbf{v}^\top \end{bmatrix}, \quad \mathbf{v} \gg 0,$$

with $A_{11} \in \mathbb{R}^{k \times k}$ and $\mathbf{v} \in \mathbb{R}_+^{n-k}$. The following facts are equivalent:

  i) *There exists $\bar{\varepsilon} > 0$ such that, for every $0 < \varepsilon < \bar{\varepsilon}$, the matrix $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$ defines a Quadratic Positive Definite Lyapunov Function for A;*

 ii) *The matrix $-\left(A_{11} + A_{11}^\top\right)$ is copositive;*

iii) *The matrix $-\left(A_{11} + A_{11}^\top\right)$ is positive definite.*

Proof. Set $P := \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$, with $\varepsilon > 0$, and notice that the matrix $Q := -\left(A^\top P + PA\right)$ takes the following form

$$
\begin{aligned}
Q &= -\left(A^\top P + PA\right) \\
&= \begin{bmatrix} 0_{k \times k} & 0_{k \times (n-k)} \\ \mathbf{v}\mathbf{1}_k^\top & \mathbf{v}\mathbf{1}_{n-k}^\top \end{bmatrix} + \begin{bmatrix} 0_{k \times k} & \mathbf{1}_k \mathbf{v}^\top \\ 0_{(n-k) \times k} & \mathbf{1}_{n-k} \mathbf{v}^\top \end{bmatrix} - \varepsilon \begin{bmatrix} A_{11} + A_{11}^\top & A_{12} + A_{21}^\top \\ A_{12}^\top + A_{21} & A_{22} + A_{22}^\top \end{bmatrix} \\
&= \begin{bmatrix} -\varepsilon\left(A_{11} + A_{11}^\top\right) & \mathbf{1}_k \mathbf{v}^\top - \varepsilon\left(A_{12} + A_{21}^\top\right) \\ \mathbf{v}\mathbf{1}_k^\top - \varepsilon\left(A_{12}^\top + A_{21}\right) & \left(\mathbf{v}\mathbf{1}_{n-k}^\top + \mathbf{1}_{n-k}\mathbf{v}^\top\right) - \varepsilon\left(A_{22} + A_{22}^\top\right) \end{bmatrix}.
\end{aligned}
$$

  $i) \Rightarrow ii)$ If there exists $\varepsilon > 0$ such that Q is copositive, then for every positive vector $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^\top & 0_{n-k}^\top \end{bmatrix}^\top$ it holds

$$
0 < -\mathbf{x}^\top \left(A^\top P + PA\right) \mathbf{x} = -\varepsilon \mathbf{x}_1^\top \left(A_{11} + A_{11}^\top\right) \mathbf{x}_1, \quad \forall \mathbf{x}_1 > 0,
$$

and hence the matrix $-\left(A_{11} + A_{11}^\top\right)$ is copositive.

  $ii) \Rightarrow i)$ Assume that $-\left(A_{11} + A_{11}^\top\right)$ is copositive and notice that there always exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$

$$
\mathbf{1}_k \mathbf{v}^\top - \varepsilon\left(A_{12} + A_{21}^\top\right) \geq 0
$$
$$
\left(\mathbf{v}\mathbf{1}_{n-k}^\top + \mathbf{1}_{n-k}\mathbf{v}^\top\right) - \varepsilon\left(A_{22} + A_{22}^\top\right) \gg 0.
$$

Then, recalling that a symmetric strictly positive matrix is also a copositive matrix, by Lemma 4.7, for every such ε, matrix Q is copositive and hence $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$ defines a a Quadratic Positive Definite Lyapunov Function for A.

  $ii) \Leftrightarrow iii)$ The equivalence follows from Theorem 4 in Ping and Yu Yu (1993), stating that a symmetric matrix whose off-diagonal elements are all nonpositive is copositive if and only if it is positive definite. ■

Remark 4.2. If the compartmental matrix $A \in \mathbb{R}^{n \times n}$ is such that $\mathbf{1}_n^\top A \ll 0^\top$, then for every $\varepsilon > 0$ sufficiently small the matrix $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$ defines a Quadratic Positive Definite Lyapunov function for A since $Q = -\left(A^\top P + PA\right)$ is a strictly positive matrix. This immediately follows from the proof of the previous proposition in the case $k = 0$.

Proposition 4.9. *Consider a Compartmental Switched System described as in* (4.1), *and assume that $A_i \in \mathbb{R}^{n \times n}$, $i \in [1, M]$, are compartmental and Hurwitz matrices. Define the sets $\mathcal{J}_i$, $i \in [1, M]$, as*

$$\mathcal{J}_i := \left\{ j \in [1, n] \colon \mathbf{1}_n^\top \operatorname{col}_j(A_i) = 0 \right\}. \tag{4.6}$$

If $\mathcal{J}_i \neq \emptyset$ denote by $A_{\mathcal{J}_i}$ the submatrix of A_i obtained by selecting rows and columns of A_i indexed by $\mathcal{J}_i$. The following facts are equivalent:

i) There exists $\bar{\varepsilon} > 0$ such that, for all $0 < \varepsilon < \bar{\varepsilon}$, the matrix $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$ defines a Common Quadratic Positive Definite Lyapunov Function for the Compartmental Switched System (4.1);

ii) For every $i \in [1, M]$ with $\mathcal{J}_i \neq \emptyset$ the matrix $-\left(A_{\mathcal{J}_i}^\top + A_{\mathcal{J}_i}\right)$ is copositive;

iii) For every $i \in [1, M]$ with $\mathcal{J}_i \neq \emptyset$ the matrix $-\left(A_{\mathcal{J}_i}^\top + A_{\mathcal{J}_i}\right)$ is positive definite.

Proof. $i) \Leftrightarrow ii)$ If $i \in [1, M]$ is such that $\mathcal{J}_i = \emptyset$, then $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon I_n$ defines a Quadratic Positive Definite Lyapunov Function for A_i for every $\varepsilon > 0$ sufficiently small (see Remark 4.2). Otherwise, let Π_i be a permutation matrix such that:

$$\mathbf{1}_n^\top \underbrace{\Pi_i^\top A_i \Pi_i}_{\tilde{A}_i} = \mathbf{1}_n^\top \begin{bmatrix} A_{11}^{(i)} & A_{12}^{(i)} \\ A_{21}^{(i)} & A_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} 0_{k_i}^\top & -\mathbf{v}_i^\top \end{bmatrix},$$

with $\mathbf{v}_i \in \mathbb{R}_+^{n-k_i}$ and $\mathbf{v}_i \gg 0$, and notice that $A_{11}^{(i)} = A_{\mathcal{J}_i}$. Moreover, notice that $P = P^\top \succ 0$ defines a Quadratic Positive Definite Lyapunov Function for A_i, i.e., $-\left(A_i^\top P + PA_i\right)$ is copositive, if and only if the matrix

$$-\Pi_i^\top \left(A_i^\top P + PA_i\right) \Pi_i$$

is copositive. This, in turn amounts to saying that the matrix

$$-\left[\left(\Pi_i^\top A_i^\top \Pi_i\right)\left(\Pi_i^\top P \Pi_i\right) + \left(\Pi_i^\top P \Pi_i\right)\left(\Pi_i^\top A_i \Pi_i\right)\right] = -\left(\tilde{A}_i^\top P + P \tilde{A}_i\right)$$

is copositive, and by Lemma 4.8 such condition holds for every $\varepsilon > 0$ sufficiently small if and only if the matrix $-\left(A_{\mathcal{J}_i}^\top + A_{\mathcal{J}_i}\right)$ is copositive. Hence, for every $\varepsilon > 0$ sufficiently small the matrix $P = \mathbf{1}_n\mathbf{1}_n^\top + \varepsilon I_n$ defines a Common Quadratic Positive Definite Lyapunov Function for the Compartmental Switched System if and only if for every $i \in [1, M]$, with $\mathcal{J}_i \neq \emptyset$, the submatrix $-\left(A_{\mathcal{J}_i}^\top + A_{\mathcal{J}_i}\right)$ is copositive.

    $ii) \Leftrightarrow iii)$ Again, the equivalence follows from Theorem 4 in Ping and Yu Yu (1993).   ■

Corollary 4.10. *Consider a Compartmental Switched System described as in* (4.1), *and assume that $A_i \in \mathbb{R}^{n \times n}$, $i \in [1, M]$, are compartmental and Hurwitz matrices. If for every $i \in [1, M]$ the matrix A_i is such that the vector $\mathbf{1}_n^\top A_i < 0^\top$ has at most one entry equal to 0, then there exists $\varepsilon > 0$ such that $P = \mathbf{1}_n\mathbf{1}_n^\top + \varepsilon I_n$ defines a Common Quadratic Positive Definite Lyapunov Function for the Compartmental Switched System* (4.1).

Proof. If A_i is such that $\mathbf{1}_n^\top A_i \ll 0^\top$, then for every $\varepsilon > 0$ sufficiently small the matrix $P = \mathbf{1}_n\mathbf{1}_n^\top + \varepsilon I_n$ defines a Quadratic Positive Definite Lyapunov Function for A_i (see Remark 4.2). If the vector $\mathbf{1}_n^\top A_i < 0^\top$ has exactly one entry, say the jth one, equal to 0, then $A_{\mathcal{J}_i} = [A_i]_{jj}$. Since A_i is Metzler and Hurwitz, it must be $[A_i]_{jj} < 0$, but then $-\left(A_{\mathcal{J}_i}^\top + A_{\mathcal{J}_i}\right)$ is copositive and the thesis follows directly from Proposition 4.9.   ■

Remark 4.3. In the particular case where $A_i \in \mathbb{R}^{2 \times 2}$, $i \in [1, M]$, the previous corollary implies that there always exists $\varepsilon > 0$ such that $P = \mathbf{1}_2\mathbf{1}_2^\top + \varepsilon I_2$ defines a Common Quadratic Positive Definite Lyapunov Function for the Compartmental Switched System. Indeed, since each A_i is compartmental and Hurwitz, condition $\mathbf{1}_2^\top A_i \neq 0^\top$ ensures that $\mathbf{1}_2^\top A_i$ has either zero or one entry equal to 0. This proves that when dealing with two-dimensional Compartmental Switched Systems, conditions $iii)$ and $iv)$ in Proposition 4.6 are equivalent.

    We now explore a slightly larger class of Common Quadratic Positive Definite Lyapunov Functions with respect to those addressed in the previous results, since we replace the identity matrix in P with a diagonal matrix.

Proposition 4.11. *Consider a Compartmental Switched System described as in* (4.1), *and assume that $A_i \in \mathbb{R}^{n \times n}$, $i \in [1, M]$, are compartmental and Hurwitz matrices. Define the sets $\mathcal{J}_i$, $i \in [1, M]$, as in* (4.6). *If $\mathcal{J}_i \cap \mathcal{J}_j = \emptyset$ for every $i \neq j$, then there exist $\varepsilon > 0$ and a diagonal matrix D, with positive diagonal entries, such that the matrix $P := \mathbf{1}_n\mathbf{1}_n^\top + \varepsilon D$ defines a Common Quadratic Positive Definite Lyapunov Function for the Compartmental Switched System* (4.1).

Proof. Let $i \in [1, M]$ be such that $\mathcal{J}_i \neq \emptyset$. Define k_i as the cardinality of the set $\mathcal{J}_i$, $k_i := |\mathcal{J}_i|$. It entails no loss of generality assuming that $\mathcal{J}_i$ is an ordered k_i-tuple, with entries sorted in ascending order. Since by Proposition 3.9 the submatrix $A_{\mathcal{J}_i}$ is compartmental and Hurwitz, there always exists (see Proposition 3.3) a diagonal matrix $D_{\mathcal{J}_i} \in \mathbb{R}_+^{k_i \times k_i}$, with positive diagonal entries, such that

$$- \left(A_{\mathcal{J}_i}^\top D_{\mathcal{J}_i} + D_{\mathcal{J}_i} A_{\mathcal{J}_i} \right) \succ 0, \tag{4.7}$$

and hence, in particular, $- \left(A_{\mathcal{J}_i}^\top D_{\mathcal{J}_i} + D_{\mathcal{J}_i} A_{\mathcal{J}_i} \right)$ is copositive. Define the following positive diagonal matrix $D \in \mathbb{R}_+^{n \times n}$

$$[D]_{jj} = \begin{cases} 1, & \text{if } \nexists i \in [1, M] \text{ such that } j \in \mathcal{J}_i; \\ [D_{\mathcal{J}_i}]_{kk}, & \text{if } j \in \mathcal{J}_i \text{ and } j \text{ is the } k\text{th entry of } \mathcal{J}_i. \end{cases}$$

Now we show that there always exists $\varepsilon > 0$ such that $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon D$ defines a Common Quadratic Positive Definite Lyapunov Function for the Compartmental Switched System. Again, as in the proof of Proposition 3.9, we can assume without loss of generality that $\mathcal{J}_i = [1, k_i]$, $k_i \in [1, n]$, and hence

$$\mathbf{1}_n^\top A_i = \mathbf{1}_n^\top \begin{bmatrix} A_{\mathcal{J}_i} & A_{12}^{(i)} \\ A_{21}^{(i)} & A_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} 0_{k_i}^\top & -\mathbf{v}_i^\top \end{bmatrix},$$

for some $\mathbf{v}_i \in \mathbb{R}_+^{n-k_i}$, $\mathbf{v}_i \gg 0$, and

$$D = \begin{bmatrix} D_{\mathcal{J}_i} & 0 \\ 0 & \bar{D}_i \end{bmatrix}, \quad \bar{D}_i \in \mathbb{R}_+^{(n-k_i) \times (n-k_i)}.$$

The matrix $Q_i := - \left(A_i^\top P + P A_i \right)$ takes the form in (4.8).

Notice that there always exists $\bar{\varepsilon} > 0$ such that the following conditions are satisfied for every $\varepsilon \in (0, \bar{\varepsilon})$

$$-\varepsilon \left(\left(A_{21}^{(i)} \right)^\top \bar{D}_i + D_{\mathcal{J}_i} A_{12}^{(i)} \right) + \mathbf{1}_{k_i} \mathbf{v}_i^\top \geq 0$$

$$\mathbf{v}_i \mathbf{1}_{n-k_i}^\top + \mathbf{1}_{n-k_i} \mathbf{v}_i^\top - \varepsilon \left(\left(A_{22}^{(i)} \right)^\top \bar{D}_i + \bar{D}_i A_{22}^{(i)} \right) \gg 0.$$

Since, by assumption, condition (4.7) holds, by making use of Lemma 4.7 we can claim that for every $\varepsilon > 0$ sufficiently small the matrix Q_i is copositive, and hence $P = \mathbf{1}_n \mathbf{1}_n^\top + \varepsilon D$ defines a Common Quadratic Positive Definite Lyapunov Function for A_i for every

$$
\begin{aligned}
Q_i &= -\left(A_i^\top P + P A_i\right) \\
&= -\left(\begin{bmatrix} 0_{k_i\times k_i} & 0_{k_i\times(n-k_i)} \\ -\mathbf{v}_i\mathbf{1}_{k_i}^\top & -\mathbf{v}_i\mathbf{1}_{n-k_i}^\top \end{bmatrix} + \begin{bmatrix} 0_{k_i\times k_i} & -\mathbf{1}_{k_i}\mathbf{v}_i^\top \\ 0_{(n-k_i)\times k_i} & -\mathbf{1}_{n-k_i}\mathbf{v}_i^\top \end{bmatrix} + \right. \\
&\quad \left. + \varepsilon \begin{bmatrix} A_{\mathcal{J}_i}^\top D_{\mathcal{J}_i} & \left(A_{21}^{(i)}\right)^\top \bar{D}_i \\ \left(A_{12}^{(i)}\right)^\top D_{\mathcal{J}_i} & \left(A_{22}^{(i)}\right)^\top \bar{D}_i \end{bmatrix} + \begin{bmatrix} D_{\mathcal{J}_i} A_{\mathcal{J}_i} & D_{\mathcal{J}_i} A_{12}^{(i)} \\ \bar{D}_i A_{21}^{(i)} & \bar{D}_i A_{22}^{(i)} \end{bmatrix}\right) \\
&= \begin{bmatrix} -\varepsilon\left(A_{\mathcal{J}_i}^\top D_{\mathcal{J}_i} + D_{\mathcal{J}_i} A_{\mathcal{J}_i}\right) & -\varepsilon\left(\left(A_{21}^{(i)}\right)^\top \bar{D}_i + D_{\mathcal{J}_i} A_{12}^{(i)}\right) + \mathbf{1}_{k_i}\mathbf{v}_i^\top \\ \mathbf{v}_i\mathbf{1}_{k_i}^\top - \varepsilon\left(\left(A_{12}^{(i)}\right)^\top D_{\mathcal{J}_i} + \bar{D}_i A_{21}^{(i)}\right) & \mathbf{v}_i\mathbf{1}_{n-k_i}^\top + \mathbf{1}_{n-k_i}\mathbf{v}_i^\top - \varepsilon\left(\left(A_{22}^{(i)}\right)^\top \bar{D}_i + \bar{D}_i A_{22}^{(i)}\right) \end{bmatrix}
\end{aligned}
$$

$$(4.8)$$

$i \in [1, M]$ such that $\mathcal{J}_i \neq \emptyset$. On the other hand, if $i \in [1, M]$ is such that $\mathcal{J}_i = \emptyset$, then $k_i = 0$ and

$$
Q_i = \mathbf{v}_i\mathbf{1}_n^\top + \mathbf{1}_n\mathbf{v}_i^\top - \varepsilon\left(A_i^\top D + D A_i\right) \gg 0
$$

for sufficiently small $\varepsilon > 0$. So, the result is proved. ∎

Remark 4.4. From the proof of the previous proposition we can easily deduce that if there exists a positive diagonal matrix D such that for every $i \in [1, M]$ with $\mathcal{J}_i \neq \emptyset$ the matrix

$$
-\left(A_{\mathcal{J}_i}^\top D_{\mathcal{J}_i} + D_{\mathcal{J}_i} A_{\mathcal{J}_i}\right)
$$

is copositive (or, equivalently, positive definite), then there exists $\varepsilon > 0$ such that $P = \mathbf{1}_n\mathbf{1}_n^\top + \varepsilon D$ defines a Common Quadratic Positive Definite Lyapunov Function for the Compartmental Switched System.

4.2 Stability under persistent switching

We have seen in Section 4.1.1 that if all matrices A_i, $i \in [1, M]$, are compartmental and Hurwitz, then for every positive initial condition $\mathbf{x}(0) > 0$ and for every switching function $\sigma \colon \mathbb{R}_+ \to [1, M]$ the state trajectory of the Compartmental Switched System asymptotically converges to 0. In this section we relax the assumption that all the subsystem matrices are Hurwitz, and investigate which switching functions drive to zero the state trajectory independently of the positive initial condition.

To this purpose, we introduce the following ergodicity (or persistence) condition on the switching functions σ we will consider. Similar notions of ergodicity can be found in Cheng, Wang, and Hu (2008); Wang, Cheng, and Hu (2009); Liu et al. (2010); Zhao, Yin, Li, and Niu (2015).

Property 4.12. *There exists a subset $\mathcal{P}_\sigma \subseteq [1, M]$ such that for every $\bar{t} \geq 0$ and for every $p \in \mathcal{P}_\sigma$, consecutive switching instants $t_{k+1} > t_k \geq \bar{t}$ can be found such that $\sigma(t) = p$ for every $t \in [t_k, t_{k+1})$.*

Consistently with Definition 2.1 in Liu et al. (2010), for a switching function σ satisfying Property 4.12, we call any mode $p \in \mathcal{P}_\sigma$ a *persistent mode* of σ.

The following result, borrowed from Liu et al. (2010), will be useful in subsequent derivations.

Proposition 4.13. *Let $V(\mathbf{x})$ be a common weak Lyapunov function for the Compartmental Switched System (4.1). Let σ be a switching function satisfying Property 4.12, and let $p \in \mathcal{P}_\sigma$ be a persistent mode of σ. Define W_p^σ as the union of all the intervals $[t_k, t_{k+1})$ of length at least τ_σ and such that $\sigma(t) = p$ for every $t \in [t_k, t_{k+1})$. Assume that $V(\mathbf{x}(t))$ is nonincreasing on W_p^σ, namely for every pair of switching instants $t_j < t_k$ such that $\sigma(t_j) = \sigma(t_k) = p$ it holds*

$$V(\mathbf{x}(t_k)) \leq V(\mathbf{x}(t_j)).$$

Denote by $\mathcal{N}_p$ the set $\mathcal{N}_p := \left\{ \mathbf{x} \in \mathbb{R}_+^n \colon \mathbf{1}_n^\top A_p \mathbf{x} = 0 \right\}$. Finally, let $\mathcal{M}_p$ denote the largest weakly invariant set with respect to the pth mode of σ that is contained in $\mathcal{N}_p$. Then, the state trajectory $\mathbf{x}(t)$ weakly approaches $\mathcal{M}_p^*(c) := \mathcal{M}_p \cap V^{-1}(c)$, for some c, in the pth mode as $t \to +\infty$, meaning that

$$\lim_{\substack{t \to +\infty \\ t \in W_p^\sigma}} \mathrm{dist}\left(\mathbf{x}(t), \mathcal{M}_p^* \right) = 0.$$

We are now in a position to prove the following result.

Proposition 4.14. *Consider a Compartmental Switched System described as in (4.1). Define the sets $\mathcal{J}_i$, $i \in [1, M]$, as*

$$\mathcal{J}_i := \left\{ j \in [1, n] \colon \mathbf{1}_n^\top \mathrm{col}_j(A_i) = 0 \right\}.$$

Then, for every switching function σ satisfying Property 4.12 for a set $\mathcal{P}_\sigma$ such that $\cap_{i \in \mathcal{P}_\sigma} \mathcal{J}_i = \emptyset$, the state trajectory $\mathbf{x}(t)$, $t \in \mathbb{R}_+$, asymptotically converges to 0 for every positive initial condition $\mathbf{x}(0) > 0$.

Proof. As remarked in Proposition 4.5, $V(\mathbf{x}) = \mathbf{1}_n^\top \mathbf{x}$ is a Common Weak Linear Copositive Lyapunov Function for the Compartmental Switched System (4.1). Let σ be an arbitrary switching function satisfying Property 4.12 for a set $\mathcal{P}_\sigma$ such that $\cap_{i \in \mathcal{P}_\sigma} \mathcal{J}_i = \emptyset$ and let $p \in \mathcal{P}_\sigma$ be any (persistent) mode of σ. Denote by $\mathcal{M}_p$ the largest weakly invariant set with respect to the pth mode in $\mathcal{N}_p := \left\{ \mathbf{x} \in \mathbb{R}_+^n : \mathbf{1}_n^\top A_p \mathbf{x} = 0 \right\}$ and notice that $\mathcal{M}_p \subseteq \mathrm{Cone}(\mathbf{e}_j, j \in \mathcal{J}_p)$. Then, by Proposition 4.13, the state trajectory $\mathbf{x}(t)$ weakly approaches $\mathcal{M}_p^* := \mathcal{M}_p \cap V^{-1}(c)$, for some c, in the pth mode as $t \to +\infty$, i.e.,

$$\lim_{\substack{t \to +\infty \\ t \in W_p^\sigma}} \mathrm{dist}\left(\mathbf{x}(t), \mathcal{M}_p^* \right) = 0. \tag{4.9}$$

Now, if $n_p := |\mathcal{J}_p|$ and $n_p^c := n - n_p = |\mathcal{J}_p^c|$, $\mathcal{J}_p^c$ being the complementary set of $\mathcal{J}_p$ in $[1, n]$, denote by $\mathbf{x}_{\mathcal{J}_p} \in \mathbb{R}_+^{n_p}$ and $\mathbf{x}_{\mathcal{J}_p^c} \in \mathbb{R}_+^{n_p^c}$ the vectors formed by selecting the components of $\mathbf{x}$ indexed by the sets $\mathcal{J}_p$ and $\mathcal{J}_p^c$, respectively. By definition of $\mathcal{M}_p$, for every $\mathbf{x} \in \mathcal{M}_p^*$ we have $\mathbf{x}_{\mathcal{J}_p^c} = 0$, and hence equation (4.9) implies that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for every $t \geq t_\varepsilon$ with $\sigma(t) = p$ we have $\mathbf{1}_{n_p^c}^\top \mathbf{x}_{\mathcal{J}_p^c}(t) < \varepsilon$. But then, since for every $t \geq 0$ such that $\sigma(t) \neq p$ the function $V_p(\mathbf{x}(t)) = \mathbf{1}_{n_p^c}^\top \mathbf{x}_{\mathcal{J}_p^c}(t)$ is not increasing, it must be

$$\lim_{t \to +\infty} \mathbf{1}_{n_p^c}^\top \mathbf{x}_{\mathcal{J}_p^c}(t) = 0, \tag{4.10}$$

and hence $\mathbf{x}_{\mathcal{J}_p^c}(t) \to 0$ as $t \to +\infty$. Now, since this is true for every $p \in \mathcal{P}_\sigma$, the thesis follows from the fact that by hypothesis $\cap_{i \in \mathcal{P}_\sigma} \mathcal{J}_i = \emptyset$, i.e., $(\cap_{i \in \mathcal{P}_\sigma} \mathcal{J}_i)^c = [1, n]$, and hence by the De Morgan's law $\cup_{i \in \mathcal{P}_\sigma} \mathcal{J}_i^c = [1, n]$. ∎

Example 4.2. Consider the matrices:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

and note that $\mathcal{J}_1 = \{1\}$ and $\mathcal{J}_2 = \{2\}$. If $\bar{\sigma}$ is a switching function with a finite number of switchings, i.e., there exist $\bar{t} \geq 0$ and $p \in \{1, 2\}$ such that $\bar{\sigma}(t) = p$ for every $t \geq \bar{t}$, then Property 4.12 holds for $\mathcal{P}_\sigma = \{p\}$, but then $\cap_{i \in \mathcal{P}_\sigma} \mathcal{J}_i = \mathcal{J}_p = \{p\} \neq \emptyset$. Indeed, if we consider the state trajectory starting from any initial condition $\mathbf{x}(0) \gg 0$, clearly
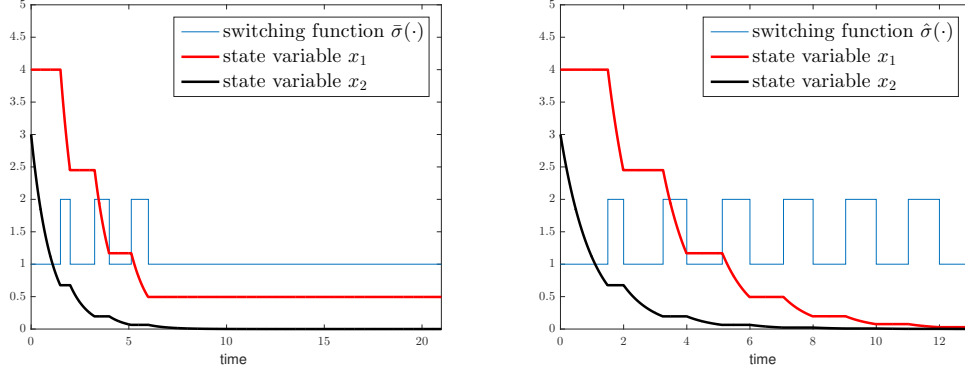
Figure 4.1: State trajectories corresponding to a switching function $\bar{\sigma}$ with a finite number of switchings (left) and to the switching function $\hat{\sigma}$ (right).

$[\mathbf{x}(t)]_p = [\mathbf{x}(\bar{t})]_p > 0$ for every $t \geq \bar{t}$, and hence the state trajectory cannot converge to zero, as shown in Figure 4.1, on the left. Now consider the switching function:

$$\hat{\sigma}(t) = \begin{cases} 1, & \text{if } t \in \left[2k, 2k+1+\frac{1}{2^{k+1}}\right), \ k = 0, 1, \dots \\ 2, & \text{elsewhere}, \end{cases}$$

and notice that in this case $\hat{\sigma}$ satisfies Property 4.12 for the set $\mathcal{P}_\sigma = \{1, 2\}$ and $\cap_{i \in \mathcal{P}_\sigma} \mathcal{J}_i = \emptyset$. Therefore, by Proposition 4.14, the state trajectory corresponding to any positive initial condition converges to the origin (see Figure 4.1, on the right). ♣

In the special case when the set

$$\mathcal{I}_{AS} := \{i \in [1, M] \colon A_i \text{ is Hurwitz}\}$$

is not empty, we can derive a similar result to the one given in Proposition 4.14, by considering all the switching functions σ for which the set $\mathcal{P}_\sigma$ intersects $\mathcal{I}_{AS}$.

Proposition 4.15. *Consider a Compartmental Switched System described as in* (4.1), *for which the index set $\mathcal{I}_{AS}$ is not empty. Then, for every switching function σ satisfying Property 4.12 for a set $\mathcal{P}_\sigma \subseteq [1, M]$ such that $\mathcal{P}_\sigma \cap \mathcal{I}_{AS} \neq \emptyset$, the state trajectory $\mathbf{x}(t)$, $t \in \mathbb{R}_+$, asymptotically converges to 0 for every positive initial condition $\mathbf{x}(0) > 0$.*

Proof. The proof proceeds as the proof of Proposition 4.14, but in this case we consider $p \in \mathcal{P}_\sigma \cap \mathcal{I}_{AS}$ as a persistent mode of σ and hence we can claim that $\mathcal{M}_p = \{0\}$. Then, by Proposition 4.13, the state trajectory $\mathbf{x}(t)$ weakly approaches $\mathcal{M}_p$ in the pth mode as

$t \to +\infty$, meaning that $\lim_{\substack{t \to +\infty \\ t \in W_p^\sigma}} \text{dist}\left(\mathbf{x}(t), \mathcal{M}_p\right) = 0$. This implies that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for every $t \geq t_\varepsilon$ for which $\sigma(t) = p$ we have $\mathbf{1}_n^\top \mathbf{x}(t) < \varepsilon$. But then, since for every $t \geq 0$ such that $\sigma(t) \neq p$ the function $V(\mathbf{x}(t)) = \mathbf{1}_n^\top \mathbf{x}(t)$ is not increasing, it must be $\lim_{t \to +\infty} \mathbf{1}_n^\top \mathbf{x}(t) = 0$, and hence $\mathbf{x}(t) \to 0$ as $t \to +\infty$. ∎

It is worth noticing that even in case the set $\mathcal{I}_{AS}$ is not empty, and we consider switching functions that take some value $p \in \mathcal{I}_{AS}$ an infinite number of times, the existence of a dwell-time $\tau_\sigma > 0$ such that $t_{k+1} - t_k \geq \tau_\sigma$ for every consecutive switching instants (recall Remark 1.1 in Chapter 1) is still fundamental. Indeed, requiring that for every $\bar{t} \geq 0$ the Lebesgue measure of the interval $\{t \geq \bar{t} : \sigma(t) = p\}$ is non-zero is not enough to guarantee asymptotic convergence to 0 of the state trajectories, as shown by the following example.

Example 4.3. Consider the compartmental matrices:

$$A_1 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},$$

and the state trajectory of the associated Compartmental Switched System corresponding to the initial condition $\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$ and the switching function

$$\sigma(t) = \begin{cases} 1, & \text{if } t \in \left[10k, 10k + \frac{1}{2^k}\right), \ k = 0, 1, \dots \\ 2, & \text{elsewhere.} \end{cases}$$

It is easy to see that the state trajectory $\mathbf{x}(t)$ converges to

$$\hat{\mathbf{x}} = \begin{bmatrix} e^{-\sum_{k=0}^{+\infty} \frac{1}{2^k}} & 0 \end{bmatrix}^\top = \begin{bmatrix} e^{-2} & 0 \end{bmatrix}^\top$$

when $t \to +\infty$. ♣

Now we consider switching functions having a common dwell-time $\tau^* > 0$, by this meaning that for every σ and every pair of consecutive switching instants t_k and t_{k+1} we have $t_{k+1} - t_k \geq \tau^*$ (this definition of dwell-time is consistent with the one introduced in Shorten et al. (2007); Zappavigna et al. (2010); Blanchini et al. (2015)).

Corollary 4.16 below follows directly from Proposition 4.15.

Corollary 4.16. *Consider a Compartmental Switched System described as in* (4.1) *and any set of switching functions* $\mathcal{S}_{dwell,AS}$ *satisfying the following two conditions:*

a) *There exists $\tau^* > 0$ such that all the switching functions have dwell-time τ^*;*

b) *For every $\sigma \in \mathcal{S}_{dwell,AS}$ and every $\bar{t} \geq 0$ the Lebesgue measure of the set $\{t \geq \bar{t} \colon \sigma(t) \in \mathcal{I}_{AS}\}$ is non-zero.*

Then, for every switching function $\sigma \in \mathcal{S}_{dwell,AS}$ and for every positive initial condition $\mathbf{x}(0) > 0$ the state trajectory $\mathbf{x}(t)$, $t \in \mathbb{R}_+$, converges to 0.

5

Stabilizability of Compartmental Switched Systems

This chapter is devoted to the study of stabilizability properties of Compartmental Switched Systems. First, we consider the case of autonomous compartmental subsystems and investigate under what conditions on the subsystem matrices there always exists a switching control law, possibly depending on the initial condition, that makes the state trajectory converge to zero. Secondly, we assume that all subsystems are compartmental single-input systems, and aim at determining a state-feedback control law (or, when a scalar output is available, an output-feedback control law) that preserves the compartmental property of the subsystems while ensuring asymptotic stability under arbitrary switching of the resulting closed-loop Compartmental Switched System.

It is worth noticing a change of perspective on the control action. Specifically, when dealing with autonomous subsystems the control action pertains the switching function σ, that is stabilization is achieved by properly selecting, at every time instant, the subsystem to be activated. On the contrary, when all the subsystems are endowed with a scalar control input, we aim at designing for each subsystem a stabilizing controller in such a way that the state trajectory converges to zero independently of the positive initial condition and of the switching function.

Stabilization of Compartmental Switched Systems with autonomous subsystems has

been addressed in:

Valcher M. E. and Zorzan I. On the stabilizability of continuous-time compartmental switched systems. In *Proceedings of the 54th IEEE Conference on Decision and Control*, pages 4246–4251, Osaka, Japan, 2015;

Valcher M. E. and Zorzan I. Stability and stabilizability of continuous-time compartmental switched systems. *IEEE Transactions on Automatic Control*, 61 (12): 3885–3897, 2016c.

The case with single-input compartmental subsystems is the subject of:

Valcher M. E. and Zorzan I. Continuous-time compartmental switched systems. In **F. Cacace R. S., L. Farina and Germani A.**, Editors, *Positive Systems. Lecture Notes in Control and Information Sciences*, Chapter 10, pages 123–138. Springer, 2017a.

5.1   Existence of stabilizing switching control functions

In this section we consider Compartmental Switched Systems with autonomous subsystems, that is Compartmental Switched Systems described by

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \quad t \in \mathbb{R}_+, \tag{5.1}$$

where $\sigma \colon \mathbb{R}_+ \to [1, M]$ is an arbitrary switching function and A_i, $i \in [1, M]$, are compartmental matrices. For this class of systems the definition of stabilizability is inherited from the broader class of Positive Switched Systems, Fornasini and Valcher (2012).

Definition 5.1. The Compartmental Switched System (5.1) is *stabilizable* if for every positive initial condition $\mathbf{x}(0) > 0$ there exists a switching function $\sigma \colon \mathbb{R}_+ \to [1, M]$ such that the corresponding state trajectory $\mathbf{x}(t)$, $t \in \mathbb{R}_+$, converges to zero.

Remark 5.1. Clearly, the stabilization problem is a non-trivial one only if all matrices A_i, $i \in [1, M]$, are not Hurwitz, and hence in the following we steadily make this assumption. On the other hand, if all matrices A_i, $i \in [1, M]$, would fulfill condition $\mathbf{1}_n^\top A_i = 0^\top$, stabilization would not be possible, since at every time $t \geq 0$ one would have $\mathbf{1}_n^\top \mathbf{x}(t) = \mathbf{1}_n^\top \mathbf{x}(0)$, and the state would never converge to zero. So, it must be $\mathbf{1}_n^\top A_i < 0^\top$ for at least one index $i \in [1, M]$.

A useful tool when addressing stabilization of Positive Switched Systems is represented by control Lyapunov functions (the counterpart of common Lyapunov functions introduced in Chapter 4 to investigate stability).

Definition 5.2. Let $V(\mathbf{x})\colon \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable copositive function. $V(\mathbf{x})$ is a *control Lyapunov function* for the Compartmental Switched System (5.1) if for every $\mathbf{x} > 0$ there exists $i = i(\mathbf{x}) \in [1, M]$ such that the derivative of V in $\mathbf{x}$ along the direction of the ith subsystem is negative, namely

$$\min_{i \in [1,M]} \nabla V(\mathbf{x}) A_i \mathbf{x} < 0, \quad \forall \mathbf{x} > 0.$$

In the following we will focus on two classes of control Lyapunov functions: linear and quadratic positive definite Lyapunov functions. For these functions, the definition of control Lyapunov function adjusts as follows.

Definition 5.3. A continuously differentiable copositive function $V(\mathbf{x})$ is a *Linear Copositive Control Lyapunov Function* for the Compartmental Switched System (5.1) if $V(\mathbf{x}) = \mathbf{v}^\top \mathbf{x}$, for some strictly positive vector $\mathbf{v} \in \mathbb{R}^n_+$, and

$$\min_{i \in [1,M]} \mathbf{v}^\top A_i \mathbf{x} < 0, \quad \forall \mathbf{x} > 0.$$

Definition 5.4. A continuously differentiable copositive function $V(\mathbf{x})$ is a *Quadratic Positive Definite Control Lyapunov Function* for the Compartmental Switched System (5.1) if $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$, for some $n \times n$ positive definite matrix $P = P^\top \succ 0$, and

$$\min_{i \in [1,M]} \mathbf{x}^\top \left[A_i^\top P + P A_i \right] \mathbf{x} < 0, \quad \forall \mathbf{x} > 0.$$

Stabilizability of Positive Switched Systems

It has already been pointed out that Compartmental Switched Systems, despite representing a marginally explored field of research in itself, inherit a number of results derived for the broader class of Positive Switched Systems. Stabilization of Positive Switched Systems has been extensively studied over the last ten years, see, e.g., Hernandez-Vargas et al. (2011); Fornasini and Valcher (2012); Blanchini, Colaneri, and Valcher (2011); Blanchini et al. (2012); Blanchini, Colaneri, and Valcher (2013). Blanchini et al. (2015) provides an overview on the topic.

Not surprisingly, techniques involving Hurwitz convex combinations of the subsystem matrices and copositive control Lyapunov functions proved to be the most effective tools to tackle stabilization problems. The existence of a Hurwitz convex combination of the subsystem matrices is a sufficient condition for stabilizability even for switched systems

without any positivity assumption, Wicks, Peleties, and De Carlo (1998). In recent times Blanchini et al. (2012) proved that, when dealing with two-dimensional Positive Switched Systems, the existence of a Hurwitz convex combination of the subsystem matrices is an equivalent condition to stabilizability, but this is no longer the case (see Blanchini et al. (2013)) when dealing with Positive Switched Systems of arbitrary dimension n. Indeed, a Hurwitz convex combination of the Metzler matrices $A_i, i \in [1, M]$, is equivalent to the existence of a positively homogeneous, smooth, copositive control Lyapunov function, but in the general case smoothness cannot be ensured for the copositive control Lyapunov functions that a stabilizable system admits.

In the following section we show that spectral and structural properties of compartmental matrices can be fruitfully exploited to derive much stronger characterizations and to gain remarkable insights into the nature of stabilizability of Compartmental Switched Systems.

5.1.1   Existence of a Hurwitz convex combination

As a first result we prove that for Compartmental Switched Systems stabilizability is equivalent to the existence of a Hurwitz convex combination of the matrices $A_i, i \in [1, M]$. To this end, we first consider, in Lemma 5.1, the case in which the matrix sum $\sum_{i=1}^{M} A_i$ is irreducible and then we remove this hypothesis in Lemma 5.2.

Lemma 5.1. *Consider a Compartmental Switched System described as in* (5.1)*, and assume that the matrix sum $A := \sum_{i=1}^{M} A_i$ of the compartmental matrices $A_i, i \in [1, M]$, is irreducible. If the Compartmental Switched System* (5.1) *is stabilizable, then every convex combination of $A_1, \ldots, A_M$ with positive coefficients is Hurwitz, i.e., for every $\alpha_i > 0, i \in [1, M]$, with $\sum_{i=1}^{M} \alpha_i = 1$, the matrix $\sum_{i=1}^{M} \alpha_i A_i$ is Hurwitz.*

Proof. Suppose by contradiction that there exist $\bar{\alpha}_i > 0, i \in [1, M]$, with $\sum_{i=1}^{M} \bar{\alpha}_i = 1$, such that the matrix $\bar{A} := \sum_{i=1}^{M} \bar{\alpha}_i A_i$ is not Hurwitz. By assumption, $\bar{A}$ is irreducible, too. By Lemma 3.8 this implies

$$0^\top = \mathbf{1}_n^\top \left(\sum_{i=1}^{M} \bar{\alpha}_i A_i \right) = \sum_{i=1}^{M} \bar{\alpha}_i \left(\mathbf{1}_n^\top A_i \right),$$

and hence $\mathbf{1}_n^\top A_i = 0^\top$ for all $i \in [1, M]$. But this contradicts (see Remark 5.1) the stabilizability assumption. ∎

Lemma 5.2. *If the Compartmental Switched System* (5.1) *is stabilizable, then there exists a Hurwitz convex combination of* $A_1, \ldots, A_M$.

Proof. If the matrix $\sum_{i=1}^{M} A_i$ is irreducible, then the statement follows from Lemma 5.1. If $\sum_{i=1}^{M} A_i$ is reducible, there exists a permutation matrix Π such that

$$
\begin{aligned}
\mathbb{A} :&= \Pi^\top \left(\sum_{i=1}^{M} A_i \right) \Pi = \sum_{i=1}^{M} \underbrace{\Pi^\top A_i \Pi}_{\mathbb{A}^{(i)}} \\
&= \begin{bmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} & \ldots & \mathbb{A}_{1s} \\ 0 & \mathbb{A}_{22} & \ldots & \mathbb{A}_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & \mathbb{A}_{ss} \end{bmatrix},
\end{aligned}
$$

where $\mathbb{A}_{jj} \in \mathbb{R}^{n_j \times n_j}$, $j \in [1, s]$, are irreducible matrices. Notice that, accordingly, each $\mathbb{A}^{(i)}$, $i \in [1, M]$, takes the following form:

$$
\mathbb{A}^{(i)} := \Pi^\top A_i \Pi = \begin{bmatrix} A_{11}^{(i)} & A_{12}^{(i)} & \ldots & A_{1s}^{(i)} \\ 0 & A_{22}^{(i)} & \ldots & A_{2s}^{(i)} \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & A_{ss}^{(i)} \end{bmatrix}.
$$

We want to prove that, under the stabilizability assumption, $\mathbb{A}$ is a Hurwitz matrix. To this aim, suppose by contradiction that $\mathbb{A}$ is not Hurwitz and let $k \in [1, s]$ be such that $\mathbb{A}_{kk}$ is compartmental (irreducible) and not Hurwitz. By Lemma 3.8 it must be $\mathbf{1}_{n_k}^\top \mathbb{A}_{kk} = 0^\top$ and hence $\sum_{i=1}^{M} \mathbf{1}_{n_k}^\top A_{kk}^{(i)} = 0^\top$. Since all matrices $A_{jj}^{(i)}$, $j \in [1, s]$, $i \in [1, M]$, are compartmental, the previous identity implies

$$
\mathbf{1}_{n_k}^\top A_{kk}^{(i)} = 0^\top, \quad \forall i \in [1, M]. \tag{5.2}
$$

Now consider the initial condition $\bar{\mathbf{x}}(0)$ (with respect to the new coordinate system, namely $\bar{\mathbf{x}}(0) = \Pi^\top \mathbf{x}(0)$) whose kth block is $\mathbf{1}_{n_k}$, while all the other blocks are zero, i.e.,

$$
\bar{\mathbf{x}}(0) = \begin{bmatrix} 0^\top & \ldots & \mathbf{1}_{n_k}^\top & \ldots & 0^\top \end{bmatrix}^\top.
$$

We want to show that, independently of the switching function σ, the corresponding state trajectory $\bar{\mathbf{x}}(t)$, $t \geq 0$, cannot converge to 0. Set $\bar{\mathbf{x}}_k(t) := \text{block}_k[\bar{\mathbf{x}}(t)]$, and notice

that its time evolution is described by the equation $\dot{\bar{\mathbf{x}}}_k(t) = A_{kk}^{\sigma(t)} \bar{\mathbf{x}}_k(t)$. By condition (5.2), for every $t > 0$ it holds

$$0 = \int_0^t \mathbf{1}_{n_k}^\top \dot{\bar{\mathbf{x}}}_k(\tau) d\tau = \mathbf{1}_{n_k}^\top \bar{\mathbf{x}}_k(t) - \mathbf{1}_{n_k}^\top \bar{\mathbf{x}}_k(0) = \mathbf{1}_{n_k}^\top \bar{\mathbf{x}}_k(t) - n_k, \tag{5.3}$$

where n_k is the dimension of $\mathbb{A}_{kk}$. Hence, for every switching function σ, we have $\mathbf{1}_{n_k}^\top \bar{\mathbf{x}}_k(t) = n_k$ for every $t \geq 0$, that contradicts stabilizability. Therefore, $\mathbb{A}$ must be Hurwitz and, by similarity, also $\sum_{i=1}^M A_i$ is Hurwitz. Hence, the positive convex combination $\sum_{i=1}^M \frac{1}{M} A_i$ is Hurwitz. ∎

The previous lemmas immediately lead to the following characterization of stabilizability.

Proposition 5.3. *Let $A_i \in \mathbb{R}^{n \times n}$, $i \in [1, M]$, be compartmental matrices. Then, the following facts are equivalent:*

i) The Compartmental Switched System (5.1) is stabilizable;

ii) There exists a Hurwitz convex combination of $A_1, \ldots, A_M$.

Proof. i) $\Rightarrow$ ii) follows from Lemma 5.2, while ii) $\Rightarrow$ i) is a well known result for switched systems, Wicks et al. (1998). ∎

5.1.2 Characterizations in terms of Lyapunov functions

In this subsection we aim to provide additional characterizations of stabilizability of Compartmental Switched Systems by making use of Copositive Control Lyapunov Functions.

In Blanchini et al. (2012) (see Theorem 3) the following result, that we restate here for Compartmental Switched Systems, was proved for the broader class of Positive Switched Systems.

Theorem 5.4. *Let $A_i \in \mathbb{R}^{n \times n}$, $i \in [1, M]$, be compartmental matrices. Then, the following facts are equivalent:*

i) There exists a Hurwitz convex combination of $A_1, \ldots, A_M$;

ii) The Compartmental Switched System (5.1) admits a Linear Copositive Control Lyapunov Function;

iii) The Compartmental Switched System (5.1) *admits a Quadratic Positive Definite Control Lyapunov Function.*

Thanks to Proposition 5.3, conditions *ii)* and *iii)* in Theorem 5.4 become equivalent characterizations of stabilizability for Compartmental Switched Systems. By making use of them, we can provide an additional characterization that allows to draw a very complete picture of the nature of stabilizability for Compartmental Switched Systems. To this goal we need a preliminary result.

Proposition 5.5. *Let* $A_i \in \mathbb{R}^{n \times n}$, $i \in [1, M]$, *be compartmental matrices. If there exist indices* $i_1, \ldots, i_n \in [1, M]$ *such that the matrix* $\tilde{A} := \begin{bmatrix} \text{col}_1(A_{i_1}) & \ldots & \text{col}_n(A_{i_n}) \end{bmatrix}$ *is Hurwitz, then there exist* $\alpha_i \geq 0$, $i \in [1, M]$, *with* $\sum_{i=1}^{M} \alpha_i = 1$, *such that the convex combination* $\sum_{i=1}^{M} \alpha_i A_i$ *is Hurwitz.*

Proof. Suppose, first, that $\tilde{A}$ is irreducible. Let $\mathcal{D}(\tilde{A}) := (\mathcal{V}, \mathcal{E}_{\tilde{A}})$ and $\mathcal{D}(A_{\Sigma}) := (\mathcal{V}, \mathcal{E}_{\Sigma})$ denote the digraphs associated with $\tilde{A}$ and $A_{\Sigma} := \sum_{i=1}^{M} A_i$ respectively, and notice that by construction $\mathcal{E}_{\tilde{A}} \subseteq \mathcal{E}_{\Sigma}$. Recalling that a matrix is irreducible if and only if its associated digraph is connected, the irreducibility assumption on $\tilde{A}$ guarantees that also A_{Σ} is irreducible. By hypothesis $\tilde{A}$ is Hurwitz, and hence there exists $k \in [1, n]$ such that $\mathbf{1}_n^{\top} \tilde{A} \mathbf{e}_k = \mathbf{1}_n^{\top} \text{col}_k(A_{i_k}) < 0$, but then $\mathbf{1}_n^{\top} \text{col}_k(A_{\Sigma}) = \sum_{i=1}^{M} \mathbf{1}_n^{\top} \text{col}_k(A_i) < 0$. This in turn implies, by Lemma 3.8, that A_{Σ} is Hurwitz, and hence also the convex combination $\sum_{i=1}^{M} \frac{1}{M} A_i$ is Hurwitz.

If $\tilde{A}$ is a reducible matrix, then the matrix $A_{\Sigma} := \sum_{i=1}^{M} A_i$ may either be irreducible or not and in the following the two cases will be considered separately. Assume first that A_{Σ} is irreducible and let Π be a permutation matrix that reduces $\tilde{A}$ to Frobenius normal form:

$$\Pi^{\top} \tilde{A} \Pi = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \ldots & \tilde{A}_{1s} \\ 0 & \tilde{A}_{22} & \ldots & \tilde{A}_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & \tilde{A}_{ss} \end{bmatrix},$$

where $\tilde{A}_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, s]$, are irreducible matrices. Accordingly, all matrices A_i are replaced by $\Pi^{\top} A_i \Pi$. Since, by hypothesis, $\tilde{A}_{11}$ is Hurwitz, there exists $k \in [1, n_1]$ such that $\mathbf{1}_{n_1}^{\top} \text{col}_k(\tilde{A}_{11}) < 0$, but then also $\mathbf{1}_n^{\top} \text{col}_k \left(\Pi^{\top} A_{\Sigma} \Pi \right) = \sum_{i=1}^{M} \mathbf{1}_n^{\top} \text{col}_k \left(\Pi^{\top} A_i \Pi \right) < 0$. This in turn implies, by Lemma 3.8, that A_{Σ} is Hurwitz, and hence also the convex combination $\sum_{i=1}^{M} \frac{1}{M} A_i$ is Hurwitz.

Consider now the case when A_Σ is reducible (this implies that every A_i, $i \in [1, M]$, is reducible). Let Π be a permutation matrix that reduces A_Σ to Frobenius normal form:

$$\hat{A}_\Sigma := \Pi^\top A_\Sigma \Pi = \sum_{i=1}^{M} \underbrace{\Pi^\top A_i \Pi}_{A_i^\Pi} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \ldots & \hat{A}_{1s} \\ 0 & \hat{A}_{22} & \ldots & \hat{A}_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & \hat{A}_{ss} \end{bmatrix},$$

where $\hat{A}_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, s]$, are compartmental and irreducible matrices. By hypothesis there exist indices $\hat{i}_1, \ldots, \hat{i}_n$ (related to $i_1, \ldots, i_n$ by the same permutation described by Π) such that the matrix $\tilde{\mathbb{A}} := \begin{bmatrix} \mathrm{col}_1\left(A_{\hat{i}_1}^\Pi\right) & \ldots & \mathrm{col}_n\left(A_{\hat{i}_n}^\Pi\right) \end{bmatrix}$ is Hurwitz and reducible. Moreover, $\tilde{\mathbb{A}}$ takes the following form:

$$\tilde{\mathbb{A}} = \begin{bmatrix} \tilde{\mathbb{A}}_{11} & \tilde{\mathbb{A}}_{12} & \ldots & \tilde{\mathbb{A}}_{1s} \\ 0 & \tilde{\mathbb{A}}_{22} & \ldots & \tilde{\mathbb{A}}_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & \tilde{\mathbb{A}}_{ss} \end{bmatrix},$$

with $\tilde{\mathbb{A}}_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, s]$. However, $\tilde{\mathbb{A}}$ might not be in Frobenius normal form and, if this is the case, there exists a permutation matrix $\tilde{\Pi}$ such that:

$$\bar{\mathbb{A}} := \tilde{\Pi}^\top \tilde{\mathbb{A}} \tilde{\Pi} = \begin{bmatrix} \bar{\mathbb{A}}_{11} & \bar{\mathbb{A}}_{12} & \ldots & \bar{\mathbb{A}}_{1s} \\ 0 & \bar{\mathbb{A}}_{22} & \ldots & \bar{\mathbb{A}}_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & \bar{\mathbb{A}}_{ss} \end{bmatrix},$$

where each diagonal block $\bar{\mathbb{A}}_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, s]$, has the following form:

$$\bar{\mathbb{A}}_{ii} = \begin{bmatrix} \bar{\mathbb{A}}_{11}^{(i)} & \bar{\mathbb{A}}_{12}^{(i)} & \ldots & \bar{\mathbb{A}}_{1\ell_i}^{(i)} \\ 0 & \bar{\mathbb{A}}_{22}^{(i)} & \ldots & \bar{\mathbb{A}}_{2\ell_i}^{(i)} \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & \bar{\mathbb{A}}_{\ell_i \ell_i}^{(i)} \end{bmatrix},$$

with $\bar{\mathbb{A}}_{jj}^{(i)} \in \mathbb{R}^{\bar{n}_j^{(i)} \times \bar{n}_j^{(i)}}$, $j \in [1, \ell_i]$, compartmental irreducible matrices and $\sum_{j=1}^{\ell_i} \bar{n}_j^{(i)} = n_i$. Now, notice that for every $j \in [1, \ell_i]$, $i \in [1, s]$, the matrix $\bar{\mathbb{A}}_{jj}^{(i)}$ is Hurwitz. In particular, $\bar{\mathbb{A}}_{11}^{(i)}$ is compartmental, irreducible and Hurwitz and therefore $\mathbf{1}_{\bar{n}_1^{(i)}}^\top \bar{\mathbb{A}}_{11}^{(i)} < 0^\top$, i.e., there exists $\bar{k} \in [1, \bar{n}_1^{(i)}]$ such that $\mathbf{1}_{\bar{n}_1^{(i)}}^\top \mathrm{col}_{\bar{k}}\left(\bar{\mathbb{A}}_{11}^{(i)}\right) < 0$. But then, there also exists $\tilde{k} \in [1, n_i]$,

with $\tilde{k}$ possibly different from $\bar{k}$, such that $\mathbf{1}_{n_i}^\top \mathrm{col}_{\tilde{k}}\left(\tilde{\mathbb{A}}_{ii}\right) < 0$. Hence, it is also true that $\mathbf{1}_{n_i}^\top \mathrm{col}_{\tilde{k}}\left(\hat{A}_{ii}\right) < 0$ and this in turn implies, by Lemma 3.8, that every diagonal block $\hat{A}_{ii}$, $i \in [1, s]$, is Hurwitz. So, finally, the matrix $\hat{A}_\Sigma$ and also the convex combination $\sum_{i=1}^M \frac{1}{M} A_i$ are Hurwitz. ∎

By putting together Proposition 5.3, Theorem 5.4 and Proposition 5.5, we finally derive the following set of necessary and sufficient conditions for stabilizability.

Theorem 5.6. *Let $A_i \in \mathbb{R}^{n \times n}$, $i \in [1, M]$, be compartmental matrices. Then, the following facts are equivalent:*

i) The Compartmental Switched System (5.1) is stabilizable;

ii) There exists a Hurwitz convex combination of $A_1, \ldots, A_M$;

iii) The Compartmental Switched System (5.1) admits a Linear Copositive Control Lyapunov Function;

iv) The Compartmental Switched System (5.1) admits a Quadratic Positive Definite Control Lyapunov Function;

v) There exist $i_1, \ldots, i_n \in [1, M]$ such that the matrix $\tilde{A} := \begin{bmatrix} \mathrm{col}_1(A_{i_1}) & \ldots & \mathrm{col}_n(A_{i_n}) \end{bmatrix}$ is Hurwitz;

vi) There exist M nonnegative diagonal matrices D_i, $i \in [1, M]$, with $\sum_{i=1}^M D_i = I_n$ such that the matrix $\sum_{i=1}^M A_i D_i$ is Hurwitz.

Proof. $i) \Leftrightarrow ii)$ It follows from Proposition 5.3.

$ii) \Leftrightarrow iii) \Leftrightarrow iv)$ It follows from Theorem 5.4.

$iii) \Rightarrow v)$ Assume that a vector $\mathbf{v} \gg 0$ can be found, such that for every $\mathbf{x} > 0$ there exists $i \in [1, M]$ such that $\mathbf{v}^\top A_i \mathbf{x} < 0$. Then, in particular, for every $j \in [1, n]$, there exists $i_j \in [1, M]$ such that $\mathbf{v}^\top A_{i_j} \mathbf{e}_j < 0$. So, the matrix $\tilde{A} := \begin{bmatrix} \mathrm{col}_1(A_{i_1}) & \ldots & \mathrm{col}_n(A_{i_n}) \end{bmatrix}$ satisfies $\mathbf{v}^\top \tilde{A} \ll 0^\top$, and this ensures that $\tilde{A}$ is Hurwitz.

$v) \Rightarrow ii)$ It follows from Proposition 5.5.

$v) \Rightarrow vi)$ For every $i \in [1, M]$ define the nonnegative diagonal matrix D_i as follows:

$$[D_i]_{kk} = \begin{cases} 1, & \text{if } i = i_k, \\ 0, & \text{otherwise,} \end{cases}$$

and notice that $\sum_{i=1}^{M} D_i = I_n$. Moreover, $\tilde{A} = \sum_{i=1}^{M} A_i D_i$ and hence by hypothesis it is Hurwitz.

$vi) \Rightarrow v)$ By hypothesis the matrix $\sum_{i=1}^{M} A_i D_i$ is Hurwitz, and hence there exists $\mathbf{v} \gg 0$ such that

$$\mathbf{z}^{\top} := \mathbf{v}^{\top} \left(\sum_{i=1}^{M} A_i D_i \right) \ll 0^{\top},$$

i.e., for every $k \in [1, n]$ it holds $[\mathbf{z}]_k = \sum_{i=1}^{M} \left[\mathbf{v}^{\top} A_i D_i \right]_k < 0$. This implies that for every $k \in [1, n]$ there is $i_k \in [1, M]$ such that $\left[\mathbf{v}^{\top} A_{i_k} D_{i_k} \right]_k < 0$. As D_{i_k} is a nonnegative diagonal matrix, the previous inequality implies that $\left[\mathbf{v}^{\top} A_{i_k} \right]_k = \mathbf{v}^{\top} \text{col}_k(A_{i_k}) < 0$. Hence, $\tilde{A} := \left[\text{col}_1(A_{i_1}) \quad \dots \quad \text{col}_n(A_{i_n}) \right]$ is such that $\mathbf{v}^{\top} \left[\text{col}_1(A_{i_1}) \quad \dots \quad \text{col}_n(A_{i_n}) \right] \ll 0^{\top}$, i.e., $\tilde{A}$ is Hurwitz. ∎

Remark 5.2. Notice that in order to prove $v) \Rightarrow vi)$ and $vi) \Rightarrow v)$ the compartmental assumption on the subsystem matrices is not required, and hence the equivalence between statements $v)$ and $vi)$ of Theorem 5.6 holds in the general (i.e., non-compartmental) case.

Remark 5.3. It is worth noticing an interesting consequence of the characterization provided in Theorem 5.6. Condition $v)$ involves up to n matrices, since the indices $i_1, i_2, \dots, i_n \in [1, M]$ are not necessarily distinct. If we look into the proof of Proposition 5.5, we easily realize that the existence of indices $i_1, i_2, \dots, i_n \in [1, M]$ such that $\left[\text{col}_1(A_{i_1}) \quad \dots \quad \text{col}_n(A_{i_n}) \right]$ is Hurwitz allows to say that the compartmental matrix $\sum_{k=1}^{n} \frac{1}{n} A_{i_k}$ is Hurwitz. Correspondingly, we can find $\mathbf{v} \gg 0$ such that for every $\mathbf{x} > 0$

$$\min_{k \in [1,n]} \mathbf{v}^{\top} A_{i_k} \mathbf{x} < 0,$$

and therefore the switching law

$$\sigma(t) = \arg\min_{k \in [1,n]} \mathbf{v}^{\top} A_{i_k} \mathbf{x}$$

is stabilizing. This shows that even when $M > n$, a stabilizable Compartmental Switched System can always be stabilized by switching among a number of subsystems not bigger than the system dimension n.

The following corollary provides a sufficient condition for the stabilizability of a Compartmental Switched System described as in (5.1).

Corollary 5.7. *Let $A_i \in \mathbb{R}^{n \times n}$, $i \in [1, M]$, be compartmental matrices. If for every $j \in [1, n]$ there exists $i_j \in [1, M]$ such that $\mathbf{1}_n^\top \mathrm{col}_j(A_{i_j}) < 0$, then the Compartmental Switched System* (5.1) *is stabilizable.*

Proof. By hypothesis $\tilde{A} := \begin{bmatrix} \mathrm{col}_1(A_{i_1}) & \ldots & \mathrm{col}_n(A_{i_n}) \end{bmatrix}$ satisfies $\mathbf{1}_n^\top \tilde{A} \ll 0^\top$, and hence is Hurwitz. So, by Theorem 5.6, the Compartmental Switched System (5.1) is stabilizable. $\blacksquare$

The sufficient condition stated in the previous Corollary 5.7 is not necessary, as shown by the following example.

Example 5.1. Consider the matrices:

$$A_1 = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix},$$

and notice that every convex combination of A_1 and A_2 is Hurwitz:

$$\alpha A_1 + (1 - \alpha) A_2 = \begin{bmatrix} -1 & 1 \\ 1 & \alpha - 3 \end{bmatrix}, \quad \alpha \in [0, 1].$$

Hence, by Proposition 5.3, the Compartmental Switched System (5.1) is stabilizable. However, the sufficient condition of Corollary 5.7 does not hold, since $\mathbf{1}_2^\top \mathrm{col}_1(A_1) = \mathbf{1}_2^\top \mathrm{col}_1(A_2) = 0$. ♣

5.2 Existence of stabilizing feedback control inputs

In this section we consider Compartmental Switched Systems whose subsystems are single-input compartmental systems, and hence are described by

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)} \mathbf{x}(t) + B_{\sigma(t)} u(t), \quad t \in \mathbb{R}_+, \tag{5.4}$$

where A_i, $i \in [1, M]$, are compartmental matrices, B_i, $i \in [1, M]$, are positive vectors (namely, $B_i > 0$ for every $i \in [1, M]$[1]), $\sigma \colon \mathbb{R}_+ \to [1, M]$ is an arbitrary switching function, and $u(t)$ is a scalar control input.

---

[1]As a matter of fact, this assumption is only for the sake of simplicity. Some of the B_i's could be zero, but in that case the corresponding matrices A_i's should be necessarily Hurwitz, in order for the stabilization problem we are going to address to be solvable.

The problem we want to address is the following one: assume that all the subsystem matrices A_i, $i \in [1, M]$, are non-Hurwitz and that the switching function σ is arbitrary but known at every time instant $t \geq 0$. We want to determine, if possible, a feedback control law, depending at each time $t \geq 0$ on the value of the switching function σ at t, such that the resulting feedback switched system is still compartmental and its state trajectories converge to zero for every positive initial condition $\mathbf{x}(0) > 0$ and every switching function σ, namely the resulting Compartmental Switched System is asymptotically stable under arbitrary switching.

We will investigate two kinds of feedback stabilization: state-feedback stabilization in Subsection 5.2.1 and output-feedback stabilization in Subsection 5.2.2.

5.2.1 State-feedback stabilization

The *state-feedback stabilization problem* for a Compartmental Switched System described as in (5.4) can be stated as follows: determine, if possible, a state-feedback control law

$$u(t) = K_{\sigma(t)}\mathbf{x}(t), \tag{5.5}$$

with $K_i \in \mathbb{R}^{1 \times n}$ for every $i \in [1, M]$, that makes the state trajectory converge to zero for every positive initial condition $\mathbf{x}(0) > 0$ and every switching function σ, while preserving the compartmental property of the resulting closed-loop switched system.

First of all, we observe that, under the state-feedback law (5.5), the resulting closed-loop switched system takes the following form

$$\dot{\mathbf{x}}(t) = \left(A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)}\right)\mathbf{x}(t), \tag{5.6}$$

and hence the control input (5.5) solves the state-feedback stabilization problem if and only if (5.6) is a Compartmental Switched System asymptotically stable under arbitrary switching. On the other hand, it has been proven in Chapter 4 (see Proposition 4.5) that for Compartmental Switched Systems stability under arbitrary switching is equivalent to the fact that all the subsystem matrices are Hurwitz. Hence, solving the state-feedback stabilization problem means determining state-feedback matrices K_i, $i \in [1, M]$, such that for every $i \in [1, M]$ the matrix $A_i + B_iK_i$ is compartmental and Hurwitz.

We first observe that if K_i is a positive matrix, then $A_i + B_iK_i > A_i$ and by the monotonicity of the Frobenius eigenvalue (see Proposition 3.2) we can claim that $\lambda_F(A_i + B_iK_i) \geq \lambda_F(A_i)$. So, as A_i is not Hurwitz, then $A_i + B_iK_i$ is not Hurwitz for every choice of $K_i > 0$. On the other hand, if there exists a matrix $K_i \in \mathbb{R}^{1 \times n}$, with both

positive and negative entries, that makes $A_i + B_i K_i$ compartmental and Hurwitz, we can always introduce a permutation matrix Π such that

$$K_i \Pi = \begin{bmatrix} K_{i+} & K_{i-} \end{bmatrix},$$

where $K_{i+} > 0$ and $K_{i-} \ll 0$. It is clearly seen that if $A_i + B_i K_i$ (and hence $\Pi^\top A_i \Pi + \Pi^\top B_i K_i \Pi$) is compartmental and Hurwitz, then also

$$\Pi^\top A_i \Pi + \Pi^\top B_i \begin{bmatrix} 0 & K_{i-} \end{bmatrix}$$

is compartmental and Hurwitz. So, we can always restrict our attention to matrices $K_i < 0$.

Now that we have focused our attention on negative state-feedback matrices, we can show that the solvability of the state-feedback stabilization problem only depends on the nonzero patterns of the pairs (A_i, B_i), $i \in [1, M]$.

Proposition 5.8. *Let $A_i \in \mathbb{R}^{n \times n}$, $i \in [1, M]$, be compartmental matrices. For every $i \in [1, M]$, let Π_i be a permutation matrix such that*

$$\Pi_i^\top A_i \Pi_i = \begin{bmatrix} A_{11}^{(i)} & A_{12}^{(i)} & \dots & A_{1s_i}^{(i)} \\ 0 & A_{22}^{(i)} & \dots & A_{2s_i}^{(i)} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & A_{s_i s_i}^{(i)} \end{bmatrix}, \quad \Pi_i^\top B_i = \begin{bmatrix} B_1^{(i)} \\ B_2^{(i)} \\ \vdots \\ B_{s_i}^{(i)} \end{bmatrix}, \tag{5.7}$$

where $A_{jj}^{(i)} \in \mathbb{R}^{n_j^{(i)} \times n_j^{(i)}}$, $j \in [1, s_i]$, are either scalar or irreducible matrices and $B_j^{(i)} \in \mathbb{R}_+^{n_j^{(i)}}$. For every $i \in [1, M]$, set $r_i := \max \left\{ j \in [1, s_i] \colon B_j^{(i)} \neq 0 \right\}$. The state-feedback stabilization problem is solvable if and only if for every $i \in [1, M]$ the following three conditions hold:

   a) $A_{jj}^{(i)}$ is (compartmental and) Hurwitz for every $j \neq r_i$;

   b) $B_j^{(i)} = 0$ for every $j \neq r_i$;

   c) There exists $\ell \in \left[1, n_{r_i}^{(i)} \right]$ such that $\overline{\mathrm{ZP}} \left(B_{r_i}^{(i)} \right) \setminus \{\ell\} \subseteq \overline{\mathrm{ZP}} \left(\mathrm{col}_\ell \left(A_{r_i r_i}^{(i)} \right) \right) \setminus \{\ell\}$.

Proof. We will prove that the existence of $K_i < 0$ such that $A_i + B_i K_i$ is compartmental and Hurwitz is equivalent to the fact that conditions *a)*, *b)* and *c)* hold for the pair (A_i, B_i). Since asymptotic stability under arbitrary switching of the Compartmental Switched System (5.6) is equivalent to the fact that all matrices $A_i + B_i K_i$, $i \in [1, M]$, are

$$A + BK =$$

$$= \begin{bmatrix} A_{11} + B_1 K_1 & \dots & A_{1r} + B_1 K_r & A_{1r+1} + B_1 K_{r+1} & \dots & A_{1s} + B_1 K_s \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_r K_1 & \dots & A_{rr} + B_r K_r & A_{rr+1} + B_r K_{r+1} & \dots & A_{rs} + B_r K_s \\ \hline & & & A_{r+1r+1} & \dots & A_{r+1s} \\ & & & \vdots & \ddots & \vdots \\ & & & 0 & \dots & A_{ss} \end{bmatrix} \quad (5.8)$$

compartmental and Hurwitz (see Chapter 4, Proposition 4.5), the result will immediately follow.

Consider the pair (A_i, B_i) for a specific value of the index $i \in [1, M]$. For the sake of simplicity, in the following we will drop the dependence on the index i, and hence refer to the pair as (A, B). Also, we will assume that the pair (A, B) is already in the form (5.7) (namely $\Pi = I_n$). This does not affect the substance of the proof, only the notation.

[Necessity] Let $K \in \mathbb{R}^{1 \times n}$, $K < 0$, be any state-feedback matrix such that $A + BK$ is compartmental and Hurwitz, and let us partition K in a way consistent with A and B, namely as

$$K = \begin{bmatrix} K_1 & K_2 & \dots & K_s \end{bmatrix},$$

with $K_j \in \mathbb{R}^{1 \times n_j}$, $K_j \leq 0$. We first prove necessity of condition *a)*. Notice that for every $K \in \mathbb{R}^{1 \times n}$ the matrix $A + BK$ takes the block-triangular form given in (5.8). As $K < 0$, if $r > 1$ a necessary condition for this matrix to be compartmental is that $B_r K_j = 0$ for every $j \in [1, r-1]$, and since $B_r > 0$, this means that $K_j = 0$ for every $j \in [1, r-1]$ (if $r = 1$ the result is trivially true). Consequently, the matrix $A + BK$ takes the same block triangular form as A, with each diagonal block $A_{jj} + B_j K_j$, $j \neq r$, coinciding with the corresponding diagonal block A_{jj} in A. So, in order for the matrix $A + BK$ to be Hurwitz, all the diagonal blocks A_{jj}, $j \neq r$, must be (compartmental and) Hurwitz.

To prove necessity of condition *b)* notice that since A_{rr} is compartmental, irreducible and non-Hurwitz, $\mathbf{1}_{n_r}^\top A_{rr} = 0^\top$, and hence, by the compartmental property of A, it must be $A_{jr} = 0$ for every $j < r$. But then, since $K_r < 0$, for every $j < r$ the matrix $A_{jr} + B_j K_r = B_j K_r \geq 0$ if and only if $B_j = 0$ for every $j < r$, namely condition *b)* holds.

To prove necessity of condition *c)* notice that for every $\ell \in [1, n_r]$, if there exists $h \in \overline{\text{ZP}}(B_r) \setminus \{\ell\}$ such that $h \notin \overline{\text{ZP}}(\text{col}_\ell(A_{rr})) \setminus \{\ell\}$, then $[A_{rr} + B_r K_r]_{h\ell} \geq 0$ if and only if $[A_{rr} + B_r K_r]_{h\ell} = 0$ namely if and only if $[K_r]_\ell = 0$. Hence, if there does not exist $\ell \in [1, n_r]$ such that $\overline{\text{ZP}}(B_r) \setminus \{\ell\} \subseteq \overline{\text{ZP}}(\text{col}_\ell(A_{rr})) \setminus \{\ell\}$, then $K_r = 0$ and the matrix

$A_{rr} + B_r K_r = A_{rr}$ cannot be Hurwitz.

[Sufficiency] We now prove that when conditions *a)*, *b)* and *c)* hold, a matrix $K < 0$ such that $A + BK$ is compartmental and Hurwitz exists. Let $\ell \in [1, n_r]$ be such that $\overline{\text{ZP}}(B_r) \setminus \{\ell\} \subseteq \overline{\text{ZP}}(\text{col}_\ell(A_{rr})) \setminus \{\ell\}$, and set

$$k_\ell^* := \min_{\substack{j \in [1, n_r] \\ j \neq \ell}} \frac{[A_{rr}]_{j\ell}}{[B_r]_j}.$$

Then, for every k_ℓ with $-k_\ell^* \leq k_\ell < 0$, the matrix

$$\bar{K}_r = \begin{bmatrix} 0 & \dots & k_\ell & \dots & 0 \end{bmatrix} = k_\ell \mathbf{e}_\ell^\top$$

is such that $A_{rr} + B_r \bar{K}_r < A_{rr}$ is still compartmental. Moreover, recalling that A_{rr} is irreducible, by the monotonicity property of the spectral abscissa (see Proposition 3.2), $\lambda_F(A_{rr} + B_r \bar{K}_r) < \lambda_F(A_{rr}) = 0$. Finally, set $K = \begin{bmatrix} 0 & \dots & 0 & \bar{K}_r & 0 & \dots & 0 \end{bmatrix}$. Condition *b)* ensures that $A + BK$ is still compartmental, while condition *a)* ensures that $A + BK$ is also Hurwitz. ∎

Remark 5.4. Proposition 5.8 extends to the class of Compartmental Switched Systems the results about stabilization of Positive Systems obtained by de Leenheer and Aeyels (2001). Indeed, in Theorem 1 and Theorem 2 of de Leenheer and Aeyels (2001), for a single pair (A_i, B_i), where A_i is a Metzler matrix and B_i is a positive vector, necessary and sufficient conditions for the existence of a static controller K_i that stabilizes the system while preserving the Metzler property of the resulting matrix $A_i + B_i K_i$, have been provided. The compartmental assumption on both A_i and the closed loop matrix $A_i + B_i K_i$ had two consequences: on the one hand, it allowed us to derive the previous characterization without introducing restrictive assumptions as in de Leenheer and Aeyels (2001) (see, in particular, the hypothesis that there exists $\bar{\mathbf{x}} \gg 0$ such that $A_i \bar{\mathbf{x}} = 0$). On the other hand, it led to a set of conditions that are slightly more restrictive than those derived in Theorem 2 of de Leenheer and Aeyels (2001).

Remark 5.5. As an immediate consequence of the previous proof (see necessity of condition *b)*), it follows that if A is a compartmental and reducible matrix in Frobenius normal form (3.1), A is non-Hurwitz if and only if there exists $i \in [1, s]$ such that A_{ii} is compartmental, irreducible (or scalar) and non-Hurwitz. For every such block A_{ii}, it must be $A_{ji} = 0$ for every $j \in [1, i-1]$. Hence, in the general case, a compartmental reducible matrix is non-Hurwitz if and only if a permutation matrix Π can be found such that $\Pi^\top A \Pi$ has

the following structure

$$
\Pi^\top A \Pi =
\begin{bmatrix}
A_{11} & 0 & \dots & 0 & A_{1r+1} & \dots & A_{1s} \\
0 & A_{22} & \dots & 0 & A_{2r+1} & \dots & A_{2s} \\
\vdots & & \ddots & \vdots & \vdots & & \vdots \\
0 & \dots & \dots & A_{rr} & A_{rr+1} & \dots & A_{rs} \\
0 & \dots & \dots & 0 & A_{r+1r+1} & \dots & A_{r+1s} \\
\vdots & & & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & \dots & 0 & \dots & \dots & A_{ss}
\end{bmatrix},
\quad
\begin{aligned}
& \mathbf{1}_{n_j}^\top A_{jj} = 0, \forall\, j \in [1, r], \\
& A_{jj} \text{ Hurwitz}, \forall\, j \in [r+1, s].
\end{aligned}
$$

$$(5.9)$$

5.2.2   Output-feedback stabilization

In this subsection we assume that for the Compartmental Switched System (5.4) a scalar output measurement $y(t) = C_{\sigma(t)}\mathbf{x}(t)$ is available, namely we consider Compartmental Switched Systems taking the following form:

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}u(t), \tag{5.10}$$

$$y(t) = C_{\sigma(t)}\mathbf{x}(t), \qquad\qquad t \in \mathbb{R}_+, \tag{5.11}$$

where for every $i \in [1, M]$ the matrices $A_i \in \mathbb{R}^{n \times n}$ are compartmental, and the vectors $B_i \in \mathbb{R}_+^n$ and $C_i \in \mathbb{R}_+^{1 \times n}$ are positive. In this new set-up, we consider a problem similar to the one considered in the previous subsection, but the control input is now an output-feedback control input, i.e., $u(t) = k_{\sigma(t)}y(t) = k_{\sigma(t)}C_{\sigma(t)}\mathbf{x}(t)$, where we can constrain our attention (by the same reasoning we adopted before) to the case $k_i < 0$ for every $i \in [1, M]$. Specifically, the *output-feedback stabilization problem* can be stated as follows: determine, if possible, an output-feedback control input

$$u(t) = k_{\sigma(t)}y(t), \tag{5.12}$$

with $k_i < 0$ for every $i \in [1, M]$, that makes the state trajectory converge to zero for every positive initial condition $\mathbf{x}(0) > 0$ and every switching function σ, while preserving the compartmental property of the resulting closed-loop switched system.

Notice that, when the control input (5.12) is applied to system (5.10), the resulting closed-loop switched system is given by

$$\dot{\mathbf{x}}(t) = \left(A_{\sigma(t)} + k_{\sigma(t)}B_{\sigma(t)}C_{\sigma(t)} \right) \mathbf{x}(t). \tag{5.13}$$

By following the same reasoning as before, we can claim that the control input (5.12) solves the output-feedback stabilization problem if and only if the matrix $A_i + k_i B_i C_i$ is compartmental and Hurwitz for every $i \in [1, M]$. Of course, solving the output-feedback stabilization problem means solving the state-feedback stabilization problem with the additional constraint that every feedback matrix $K_i := k_i C_i$ is a scaled version of the output matrix C_i. It is then clear that conditions *a)*, *b)* and *c)* of Proposition 5.8 are necessary conditions also for the solvability of the output-feedback stabilization problem (however, they are not sufficient). To investigate the solvability of the output-feedback stabilization problem, let us assume that, when the pair (A_i, B_i) is described as in (5.7), also the output matrix C_i is partitioned in a way consistent with A_i and B_i, namely as

$$C_i = \begin{bmatrix} C_1^{(i)} & C_2^{(i)} & \dots & C_{s_i}^{(i)} \end{bmatrix}, \tag{5.14}$$

with $C_j^{(i)} \in \mathbb{R}_+^{1 \times n_j^{(i)}}$, $j \in [1, s_i]$. Again, it turns out that the solvability of the output-feedback stabilization problem only depends on the nonzero pattern of all triples (A_i, B_i, C_i), $i \in [1, M]$.

Proposition 5.9. *Let $A_i \in \mathbb{R}^{n \times n}$, $i \in [1, M]$, be compartmental matrices. For every $i \in [1, M]$, let Π_i be a permutation matrix such that the pair $\left(\Pi_i^\top A_i \Pi_i, \Pi_i^\top B_i \right)$ and the output matrix $C_i \Pi_i$ are described as in (5.7) and (5.14), respectively. Assume that for every $i \in [1, M]$ conditions a) and b) of Proposition 5.8 are satisfied, where r_i is the index of the unique nonzero block in $\Pi_i^\top B_i$. For every $i \in [1, M]$, set*

$$t_i := \min \left\{ j \in [1, s_i] \colon C_j^{(i)} \neq 0 \right\}.$$

The output-feedback stabilization problem is solvable if and only if for every $i \in [1, M]$ the following three conditions hold:

c1) The first nonzero block in $C_i \Pi_i$ corresponds to the unique nonzero block in $\Pi_i^\top B_i$, namely $t_i = r_i$;

c2) For every $j \in [r_i + 1, s_i]$ such that $C_j^{(i)} \neq 0$ the following property holds:

$$\overline{\mathrm{ZP}} \left(B_{r_i}^{(i)} \right) \times \overline{\mathrm{ZP}} \left(C_j^{(i)} \right) \subseteq \overline{\mathrm{ZP}} \left(A_{r_i j}^{(i)} \right); \tag{5.15}$$

c3) $\left(\overline{\mathrm{ZP}} \left(B_{r_i}^{(i)} \right) \times \overline{\mathrm{ZP}} \left(C_{r_i}^{(i)} \right) \right) \backslash \left\{ (\ell, \ell) : \ell \in \left[1, n_{r_i}^{(i)} \right] \right\} \subseteq \overline{\mathrm{ZP}} \left(A_{r_i r_i}^{(i)} \right) \backslash \left\{ (\ell, \ell) : \ell \in \left[1, n_{r_i}^{(i)} \right] \right\}.$

Proof. As in the proof of Proposition 5.8, we will show that there exists $k_i < 0$ such that $A_i + k_i B_i C_i$ is compartmental and Hurwitz if and only if conditions *c1)*, *c2)* and *c3)* hold for the triple (A_i, B_i, C_i). Since asymptotic stability under arbitrary switching of system (5.13) is equivalent to the fact that all matrices $A_i + k_i B_i C_i$, $i \in [1, M]$, are compartmental and Hurwitz, the result follows.

Consider the triple (A_i, B_i, C_i) for a specific value of the index $i \in [1, M]$. For the sake of simplicity, similarly to what we did in the proof of the previous Proposition 5.8, we drop the dependence on the index i, and hence refer to the triple as (A, B, C), and we assume that the triple is already in the desired block form (i.e., $\Pi = I_n$).

As condition *b)* of Proposition 5.8 holds, namely $B_j = 0$ for every $j \neq r$, then for every scalar k the matrix $A + kBC$ takes the following form

$$A + kBC = \left[\begin{array}{ccc|cccc} A_{11} & \dots & A_{1r} & A_{1r+1} & \dots & A_{1s} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ kB_rC_1 & \dots & A_{rr} + kB_rC_r & A_{rr+1} + kB_rC_{r+1} & \dots & A_{rs} + kB_rC_s \\ \hline & & & A_{r+1r+1} & \dots & A_{r+1s} \\ & & & \vdots & \ddots & \vdots \\ & & & 0 & \dots & A_{ss} \end{array} \right].$$

Recalling that $B_r \neq 0$, a negative scalar k is such that $A + kBC$ is compartmental if and only if the following conditions hold:

1) $kB_rC_j = 0$ for every $j \in [1, r-1]$, namely $C_j = 0$ for every $j \in [1, r-1]$ (i.e., condition *c1)* holds);

2) $A_{rr} + kB_rC_r$ is compartmental;

3) $A_{rj} + kB_rC_j$ is a nonnegative matrix for every $j \in [r+1, s]$.

It is easy to verify that there exists $k < 0$ satisfying condition 3) if and only if for every $j \in [r+1, s]$ such that $C_j \neq 0$ condition (5.15) holds. On the other hand, since by hypothesis condition *a)* of Proposition 5.8 holds true, $A + kBC$ is compartmental and Hurwitz if and only if the matrix $A_{rr} + kB_rC_r$ is compartmental and Hurwitz. Recalling that A_{rr} is compartmental, irreducible and non-Hurwitz, by the monotonicity property of the spectral abscissa it follows that there exists $k < 0$ such that $A_{rr} + kB_rC_r$ is compartmental and Hurwitz if and only if also condition *c3)* holds true. ∎

5.3 Affine Compartmental Switched Systems

An *Affine Compartmental Switched System* is a Compartmental Switched System of type (5.4) and for which the input function takes a constant value, i.e., $u(t) = \bar{u}$ for all $t \geq 0$. An Affine Compartmental Switched System is thus described by

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + b_{\sigma(t)}, \quad t \in \mathbb{R}_+, \tag{5.16}$$

where $b_{\sigma(t)} := B_{\sigma(t)}\bar{u}$. Clearly, for every $i \in [1, M]$, A_i is a compartmental matrix and b_i is a positive vector. We assume that all pairs (A_i, b_i), $i \in [1, M]$, are stabilizable, in the standard sense of linear systems. This amounts to saying that when the matrix A_i is not Hurwitz, and hence $\lambda_F(A_i) = 0$, then the Hautus test matrix evaluated at zero, $\left[sI_n - A_i \mid b_i \right]\big|_{s=0}$, has rank n, namely b_i cannot be expressed as a linear combination of the columns of A_i. In particular, $b_i \neq 0$.

We say that a state $\bar{\mathbf{x}} > 0$ is a *switched equilibrium point* of (5.16) if the origin is included in the convex hull of the vectors $A_i\bar{\mathbf{x}} + b_i$, $i \in [1, M]$ (see Aubin and Cellina (1984) for details on discontinuous differential equations). Upon denoting by $\mathcal{A}_M$ the set of vectors $\boldsymbol{\alpha} \in \mathbb{R}_+^M$ such that $\mathbf{1}_M^\top \boldsymbol{\alpha} = 1$, $\bar{\mathbf{x}} > 0$ is a switched equilibrium point if there exists $\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 & \dots & \alpha_M \end{bmatrix}^\top \in \mathcal{A}_M$ such that

$$0 = \sum_{i=1}^{M} \alpha_i (A_i\bar{\mathbf{x}} + b_i) = A(\boldsymbol{\alpha})\bar{\mathbf{x}} + b(\boldsymbol{\alpha}), \tag{5.17}$$

where $A(\boldsymbol{\alpha}) := \sum_{i=1}^{M} \alpha_i A_i$, and $b(\boldsymbol{\alpha}) := \sum_{i=1}^{M} \alpha_i b_i$. Notice that, in general, $\bar{\mathbf{x}}$ is not an equilibrium point of any of the affine subsystems $\dot{\mathbf{x}}(t) = A_i\mathbf{x}(t) + b_i$, $i \in [1, M]$.

By exploiting Theorem 5.6, we want to provide a characterization of all the switched equilibria that can be "reached" under some stabilizing switching function σ (see also Bolzern, Colaneri, and Nicolao (2004)), by this meaning that for every $\epsilon > 0$ we can ensure that there exists $\bar{t} > 0$ such that for every $t \geq \bar{t}$ the distance between the state trajectory and the switched equilibrium point is smaller than ϵ. To this end we need a preliminary lemma.

Lemma 5.10. *Let $A \in \mathbb{R}^{n \times n}$ be a compartmental, reducible and non-Hurwitz matrix in Frobenius normal form:*

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ 0 & A_{22} & \dots & A_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & A_{ss} \end{bmatrix}, \tag{5.18}$$

where the diagonal blocks $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, s]$, are either scalar ($n_i = 1$) or irreducible matrices. Let $\mathcal{C}_i$ denotes the communication class in $\mathcal{D}(A)$ associated with the block A_{ii}. The matrix A admits as left Frobenius eigenvector $\mathbf{v}_F$, corresponding to $\lambda_F = 0$, a positive vector that can be partitioned according to the block-partition of A, namely as

$$\mathbf{v}_F = \begin{bmatrix} \mathbf{v}_1^\top & \mathbf{v}_2^\top & \dots & \mathbf{v}_s^\top \end{bmatrix}^\top,$$

and whose blocks $\mathbf{v}_i \in \mathbb{R}_+^{n_i}$, $i \in [1, s]$, satisfy the following conditions:

1) *If $\mathcal{C}_i$ is a* conservative class, *namely it is associated with a non-Hurwitz block A_{ii}, then $\mathbf{v}_i = \mathbf{1}_{n_i}$;*

2) *If $\mathcal{C}_i$ is associated with a Hurwitz block A_{ii}, and $\mathcal{C}_i$ has not access to any conservative class, then $\mathbf{v}_i = 0$;*

3) *If $\mathcal{C}_i$ is associated with a Hurwitz block A_{ii}, and $\mathcal{C}_i$ has access to some conservative class, then $\mathbf{v}_i \gg 0$.*

Proof. We first note that, by the assumptions on A (see Remark 5.5), a permutation matrix Π can be found such that $\Pi^\top A \Pi$ is described as in (5.9), where the first r (irreducible or scalar) diagonal blocks are singular and satisfy $\mathbf{1}_{n_i}^\top A_{ii} = 0^\top$, while the remaining $s - r$ diagonal blocks are Hurwitz. It entails no loss of generality assuming that A has the structure given in (5.9) (namely $\Pi = I_n$), since this can be achieved by simply permuting the blocks of A and hence those of $\mathbf{v}_F$.

If we denote by $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ the set of indices of the classes in *1)*, *2)* and *3)*, respectively, then clearly $\mathcal{I}_1 = [1, r]$, while $\mathcal{I}_2 \cup \mathcal{I}_3 = [r + 1, s]$. Moreover, no class $\mathcal{C}_i, i \in \mathcal{I}_2$, has access to any class $\mathcal{C}_j, j \in \mathcal{I}_1 \cup \mathcal{I}_3$.

1) The conservative classes are those corresponding to the first r diagonal blocks, and we have already pointed out that $\mathbf{1}_{n_i}^\top A_{ii} = 0^\top$ for every $i \in \mathcal{I}_1 = [1, r]$. So, the (essential) uniqueness of the left Frobenius eigenvector of an irreducible matrix, ensures that $\mathbf{v}_i = \mathbf{1}_{n_i}$ for every $i \in \mathcal{I}_1$.

2) We prove this result by induction. Let $i \in [r + 1, s]$ be the smallest index in $\mathcal{I}_2$. Then $\mathcal{C}_i$ is a communication class associated with a Hurwitz block and it has access to no other class, namely $A_{ji} = 0$ for every $j < i$. So, condition

$$\mathbf{v}_1^\top A_{1i} + \mathbf{v}_2^\top A_{2i} + \cdots + \mathbf{v}_j^\top A_{ji} + \cdots + \mathbf{v}_i^\top A_{ii} = 0, \tag{5.19}$$

becomes $\mathbf{v}_i^\top A_{ii} = 0^\top$, and since A_{ii} is nonsingular, then $\mathbf{v}_i = 0$.

Suppose, now, that $i \in [r+2, s]$, $i \in \mathcal{I}_2$, and we have shown that for every $j \in \mathcal{I}_2$, $j < i$, condition *2)* holds. Then, for every $j < i$, if $j \in \mathcal{I}_1 \cup \mathcal{I}_3$ then $A_{ji} = 0$, if $j \in \mathcal{I}_2$ then $\mathbf{v}_j = 0$. Consequently, (5.19) becomes, again, $\mathbf{v}_i^\top A_{ii} = 0^\top$, and since A_{ii} is nonsingular, $\mathbf{v}_i = 0$.

3) We prove also this fact by induction. Let $i \in [r+1, s]$ be the smallest index in $\mathcal{I}_3$. Then, $\mathcal{C}_i$ is a communication class associated with a Hurwitz block and it has direct access to some conservative class $\mathcal{C}_j$, $j \in \mathcal{I}_1 = [1, r]$, by this meaning that there is an arc from some vertex in $\mathcal{C}_i$ to some vertex in $\mathcal{C}_j$. This amounts to saying that $A_{ji} > 0$ for some $j \in \mathcal{I}_1$. On the other hand, for every $k < i$, $k \in \mathcal{I}_2$, if any, we have already proved that $\mathbf{v}_k = 0$. So, condition (5.19) implies

$$
\begin{aligned}
\mathbf{v}_i^\top &= \left[\mathbf{v}_1^\top A_{1i} + \mathbf{v}_2^\top A_{2i} + \cdots + \mathbf{v}_j^\top A_{ji} + \cdots + \mathbf{v}_{i-1}^\top A_{i-1i} \right] (-A_{ii})^{-1} \\
&\geq \left[\mathbf{v}_j^\top A_{ji} \right] (-A_{ii})^{-1},
\end{aligned}
$$

where we used the fact that A_{ii} is Hurwitz and irreducible (or scalar), and hence the matrix $(-A_{ii})^{-1}$ is strictly positive (recall Proposition 3.3). On the other hand, $\mathbf{v}_j = \mathbf{1}_{n_j} \gg 0$ and $A_{ji} > 0$. This ensures that $\left[\mathbf{v}_j^\top A_{ji} \right] (-A_{ii})^{-1} \gg 0$, and hence $\mathbf{v}_i \gg 0$.

Suppose, now, that $i \in [r+2, s]$, $i \in \mathcal{I}_3$, and we have shown that for every $j \in \mathcal{I}_3$, $j < i$, condition *3)* holds. Then, for every $j < i$, if $j \in \mathcal{I}_1 \cup \mathcal{I}_3$ then $\mathbf{v}_j \gg 0$ (and there exists $j \in \mathcal{I}_1 \cup \mathcal{I}_3$ such that $A_{ji} > 0$), if $j \in \mathcal{I}_2$ then $\mathbf{v}_j = 0$. Consequently, (5.19) leads, again, to $\mathbf{v}_i^\top \geq \left[\mathbf{v}_j^\top A_{ji} \right] (-A_{ii})^{-1} \gg 0$. This completes the proof. ∎

We are now in a position to introduce the main result of this section, that adapts to the class of Affine Compartmental Switched Systems the characterization first given in Blanchini et al. (2012).

Theorem 5.11. *Consider an Affine Compartmental Switched System described as in* (5.16), *and suppose that when all the b_i's are set to zero the system is stabilizable in the sense of Definition 5.1. Also, assume that each pair (A_i, b_i), $i \in [1, M]$, is stabilizable. Then, the set of all switched equilibrium points that can be reached by resorting to some switching control law σ is given by*

$$
\mathcal{E} := \left\{ \bar{\mathbf{x}} > 0 \colon \bar{\mathbf{x}} = -A(\boldsymbol{\alpha})^{-1} b(\boldsymbol{\alpha}), \exists \boldsymbol{\alpha} \in \mathcal{A}_M^H \right\},
$$

where $\mathcal{A}_M^H := \{ \boldsymbol{\alpha} \in \mathcal{A}_M \colon A(\boldsymbol{\alpha})$ is Hurwitz $\}$.

Proof. We preliminary notice that the stabilizability assumption on the Compartmental Switched System with autonomous subsystems ensures, by Theorem 5.6 of the previous section, that the set $\mathcal{A}_M^H$ is not empty, and hence $\mathcal{E} \neq \emptyset$. Clearly, all elements of $\mathcal{E}$ are switched equilibrium points, since they satisfy equation (5.17). We now prove the converse, namely that all equilibria belong to $\mathcal{E}$. This amounts to saying that if $\bar{\mathbf{x}} > 0$ satisfies $A(\boldsymbol{\alpha})\bar{\mathbf{x}} + b(\boldsymbol{\alpha}) = 0$ for some $\boldsymbol{\alpha} \in \mathcal{A}_M$, then $A(\boldsymbol{\alpha})$ is Hurwitz.

Suppose, by contradiction, it is not. Then, being the convex combination of compartmental matrices, it will be compartmental with $\lambda_F(A(\boldsymbol{\alpha})) = 0$. If $A(\boldsymbol{\alpha})$ is irreducible, then $\mathbf{1}_n^\top A(\boldsymbol{\alpha}) = 0^\top$. Consequently, for every $i \in [1, M]$ such that $\alpha_i > 0$, one has $\mathbf{1}_n^\top A_i = 0^\top$, thus implying that A_i is not Hurwitz. On the other hand, $\mathbf{1}_n^\top b(\boldsymbol{\alpha}) = \mathbf{1}_n^\top (A(\boldsymbol{\alpha})\bar{\mathbf{x}} + b(\boldsymbol{\alpha})) = 0$, and this implies that for every $i \in [1, M]$ such that $\alpha_i > 0$, one has $b_i = 0$. This contradicts the stabilizability assumption on the pairs (A_i, b_i), $i \in [1, M]$, such that $\alpha_i > 0$.

Suppose, now, that $A(\boldsymbol{\alpha})$ is reducible. It entails no loss of generality assuming that $A(\boldsymbol{\alpha})$ is in Frobenius normal form (5.18) and $b(\boldsymbol{\alpha})$ is accordingly partitioned as

$$b(\boldsymbol{\alpha}) = \begin{bmatrix} B_1(\boldsymbol{\alpha}) \\ B_2(\boldsymbol{\alpha}) \\ \vdots \\ B_s(\boldsymbol{\alpha}) \end{bmatrix},$$

where $A_{ii}(\boldsymbol{\alpha}) \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, s]$, are either scalar ($n_i = 1$) or irreducible matrices, and $B_i(\boldsymbol{\alpha}) \in \mathbb{R}_+^{n_i}$. This is a not restrictive assumption, since we can always reduce ourselves to this situation by resorting to a suitable permutation matrix Π, and hence moving from the pair $(A(\boldsymbol{\alpha}), b(\boldsymbol{\alpha}))$ to the pair $\left(\Pi^\top A(\boldsymbol{\alpha})\Pi, \Pi^\top b(\boldsymbol{\alpha}) \right)$. Now consider the left Frobenius eigenvector of $A(\boldsymbol{\alpha})$, $\mathbf{v}_F$, corresponding to $\lambda_F(A(\boldsymbol{\alpha})) = 0$ and given in Lemma 5.10, partitioned accordingly to the block-partition of $A(\boldsymbol{\alpha})$ and $b(\boldsymbol{\alpha})$ as

$$\mathbf{v}_F = \begin{bmatrix} \mathbf{v}_1^\top & \mathbf{v}_2^\top & \dots & \mathbf{v}_s^\top \end{bmatrix}^\top, \quad \mathbf{v}_i \in \mathbb{R}_+^{n_i}, i \in [1, s].$$

By the previous lemma, we know that $\mathbf{v}_i \neq 0$ if and only if the class $\mathcal{C}_i$ is either conservative ($A_{ii}(\boldsymbol{\alpha})$ is singular) or it has access to some conservative class, and if $\mathbf{v}_i \neq 0$ then $\mathbf{v}_i \gg 0$. We denote by $\mathcal{I}$ the set of indices $i \in [1, s]$ such that $\mathbf{v}_i \gg 0$. According to the notation used within the proof of Lemma 5.10, $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_3$. So, condition

$$\mathbf{v}_F^\top (A(\boldsymbol{\alpha})\bar{\mathbf{x}} + b(\boldsymbol{\alpha})) = 0^\top$$

implies $\mathbf{v}_F^\top b(\boldsymbol{\alpha}) = 0$, and hence $B_i(\boldsymbol{\alpha}) = 0$ for every $i \in \mathcal{I}$. This allows to say that a

(new) permutation matrix $\tilde{\Pi}$ can be found such that

$$\tilde{\Pi}^\top A(\boldsymbol{\alpha})\tilde{\Pi} = \begin{bmatrix} D_{11}(\boldsymbol{\alpha}) & 0 & D_{13}(\boldsymbol{\alpha}) \\ 0 & D_{22}(\boldsymbol{\alpha}) & D_{23}(\boldsymbol{\alpha}) \\ 0 & 0 & D_{33}(\boldsymbol{\alpha}) \end{bmatrix}, \quad \tilde{\Pi}^\top B(\boldsymbol{\alpha}) = \begin{bmatrix} 0 \\ E_2(\boldsymbol{\alpha}) \\ 0 \end{bmatrix},$$

where $D_{11}(\boldsymbol{\alpha})$ is a block diagonal matrix that groups together all the diagonal blocks $A_{ii}(\boldsymbol{\alpha})$ in $A(\boldsymbol{\alpha})$ that are irreducible and conservative, $D_{22}(\boldsymbol{\alpha})$ is a block triangular matrix that groups together all the diagonal blocks $A_{ii}(\boldsymbol{\alpha})$ in $A(\boldsymbol{\alpha})$ that are irreducible, Hurwitz and correspond to classes that have no access to conservative classes, and finally $D_{33}(\boldsymbol{\alpha})$ is a block triangular matrix that groups together all the diagonal blocks $A_{ii}(\boldsymbol{\alpha})$ in $A(\boldsymbol{\alpha})$ that are irreducible, Hurwitz and correspond to classes that have access to some conservative class. Also, $D_{13}(\boldsymbol{\alpha}) > 0$, $D_{23}(\boldsymbol{\alpha}) \geq 0$ and $E_2(\boldsymbol{\alpha}) > 0$. It is easily seen that for every $j \in [1, M]$ such that $\alpha_j > 0$ one has

$$\tilde{\Pi}^\top A_j \tilde{\Pi} = \begin{bmatrix} D_{11}^{(j)} & 0 & D_{13}^{(j)} \\ 0 & D_{22}^{(j)} & D_{23}^{(j)} \\ 0 & 0 & D_{33}^{(j)} \end{bmatrix}, \quad \tilde{\Pi}^\top b_j = \begin{bmatrix} 0 \\ E_2^{(j)} \\ 0 \end{bmatrix},$$

and that $D_{11}^{(j)}$ is a block diagonal matrix whose diagonal blocks are conservative and hence singular. It is also clear that for every $K_j \tilde{\Pi} = \begin{bmatrix} K_1^{(j)} & K_2^{(j)} & K_3^{(j)} \end{bmatrix}$ one has

$$\begin{aligned} \tilde{\Pi}^\top \left(A_j + b_j K_j \right) \tilde{\Pi} &= \tilde{\Pi}^\top A_j \tilde{\Pi} + \tilde{\Pi}^\top b_j K_j \tilde{\Pi} \\ &= \begin{bmatrix} D_{11}^{(j)} & 0 & D_{13}^{(j)} \\ E_2^{(j)} K_1^{(j)} & D_{22}^{(j)} + E_2^{(j)} K_2^{(j)} & D_{23}^{(j)} + E_2^{(j)} K_3^{(j)} \\ 0 & 0 & D_{33}^{(j)} \end{bmatrix}, \end{aligned}$$

and hence 0 is an eigenvalue of $A_j + b_j K_j$ for every K_j, thus contradicting the stabilizability assumption on the pair (A_j, b_j). Therefore, $A(\boldsymbol{\alpha})$ must be Hurwitz and hence $\bar{\mathbf{x}}$ belongs to $\mathcal{E}$.

The second part of the proof proceeds like the one in Blanchini et al. (2015) and we report it here only for the sake of completeness. We now want to prove that all points in $\mathcal{E}$ are equilibria achievable by means of some stabilizing switching control function. Let $A(\boldsymbol{\alpha})$, $\boldsymbol{\alpha} \in \mathcal{A}_M^H$, be a Hurwitz matrix and let $P = P^\top$ be a positive definite matrix such that $A^\top(\boldsymbol{\alpha})P + PA(\boldsymbol{\alpha})$ is negative definite. Let $\bar{\mathbf{x}}$ be the element of $\mathcal{E}$ corresponding to $A(\boldsymbol{\alpha})$, and consider the Quadratic Positive Definite Control Lyapunov

Function $V(\mathbf{x} - \bar{\mathbf{x}}) := (\mathbf{x} - \bar{\mathbf{x}})^\top P(\mathbf{x} - \bar{\mathbf{x}})$ and the control strategy $\sigma(t) = u(\mathbf{x}(t))$, where

$$
\begin{aligned}
u(\mathbf{x}) :&= \underset{i \in [1,M]}{\arg\min} \left\{ (A_i \mathbf{x} + b_i)^\top P(\mathbf{x} - \bar{\mathbf{x}}) + (\mathbf{x} - \bar{\mathbf{x}})^\top P(A_i \mathbf{x} + b_i) \right\} \\
&= \underset{i \in [1,M]}{\arg\min} \left\{ 2(\mathbf{x} - \bar{\mathbf{x}})^\top P(A_i \mathbf{x} + b_i) \right\}.
\end{aligned}
\tag{5.20}
$$

Keeping in mind that $A(\boldsymbol{\alpha})\bar{\mathbf{x}} = -b(\boldsymbol{\alpha})$, we have for $\mathbf{x} \neq \bar{\mathbf{x}}$

$$
\begin{aligned}
(\mathbf{x} - \bar{\mathbf{x}})^\top P(A_i \, \mathbf{x} + b_i) = \underbrace{(\mathbf{x} - \bar{\mathbf{x}})^\top P A(\boldsymbol{\alpha})(\mathbf{x} - \bar{\mathbf{x}})}_{<0} \\
+ (\mathbf{x} - \bar{\mathbf{x}})^\top P \left[(A_i \mathbf{x} + b_i) - (A(\boldsymbol{\alpha})\mathbf{x} + b(\boldsymbol{\alpha})) \right].
\end{aligned}
$$

The first term on the right hand side is negative. On the other hand, by construction, the vector $A(\boldsymbol{\alpha})\mathbf{x} + b(\boldsymbol{\alpha})$ belongs to the convex hull of the vectors $A_i \mathbf{x} + b_i$, and hence

$$
\min_{i \in [1,M]} \left\{ (\mathbf{x} - \bar{\mathbf{x}})^\top P \left[(A_i \mathbf{x} + b_i) - (A(\boldsymbol{\alpha})\mathbf{x} + b(\boldsymbol{\alpha})) \right] \right\} \leq 0.
$$

This ensures that $\min_{i \in [1,M]} \left\{ 2(\mathbf{x} - \bar{\mathbf{x}})^\top P(A_i \mathbf{x} + b_i) \right\}$ is negative, and hence we have a stabilizing switching law that leads the system evolution to $\bar{\mathbf{x}}$. ∎

5.3.1 Characterization of the set $\mathcal{A}_M^H$

In this subsection we want to understand under which conditions the convex combination $\sum_{i=1}^M \alpha_i A_i$, where $\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 & \dots & \alpha_M \end{bmatrix}^\top \in \mathcal{A}_M$, is (compartmental and) Hurwitz. In order to provide a more complete characterization of the set $\mathcal{A}_M^H$, let us remove the assumption that for every $i \in [1, M]$ the matrix A_i is non-Hurwitz and hence assume that there may exist $i \in [1, M]$ such that A_i is Hurwitz.

Lemma 5.12. *Let $A_i \in \mathbb{R}^{n \times n}$, $i \in [1, M]$, be compartmental matrices, and define the set $\mathcal{A}_M^H$ as in Theorem 5.11, i.e., $\mathcal{A}_M^H := \{ \boldsymbol{\alpha} \in \mathcal{A}_M : A(\boldsymbol{\alpha})$ is Hurwitz $\}$. Then,*

 i) If $\boldsymbol{\alpha} \in \mathcal{A}_M^H$, then for every $\boldsymbol{\beta} \in \mathcal{A}_M$ with $\overline{\mathrm{ZP}}(\boldsymbol{\alpha}) \subseteq \overline{\mathrm{ZP}}(\boldsymbol{\beta})$, $\boldsymbol{\beta} \in \mathcal{A}_M^H$.

As a consequence

 ii) If $\mathcal{A}_M^H \neq \emptyset$, then all vectors $\boldsymbol{\alpha} \in \mathcal{A}_M$ with $\boldsymbol{\alpha} \gg 0$ belong to $\mathcal{A}_M^H$.

Proof. i) Consider first the case when $A(\boldsymbol{\beta})$ is irreducible. Since $A(\boldsymbol{\alpha})$ is Hurwitz, then $\mathbf{1}_n^\top A(\boldsymbol{\alpha}) < 0^\top$, which implies that there exist indices $\ell \in [1, n]$ and $i \in [1, M]$, such that $\alpha_i \neq 0$ (and hence $\beta_i \neq 0$), and $\mathbf{1}_n^\top \mathrm{col}_\ell(A_i) < 0$. Consequently, $\mathbf{1}_n^\top A(\boldsymbol{\beta}) < 0^\top$, and the irreducibility of $A(\boldsymbol{\beta})$ ensures that $A(\boldsymbol{\beta})$ is Hurwitz and hence $\boldsymbol{\beta} \in \mathcal{A}_M^H$.

Consider now the case when $A(\boldsymbol{\beta})$ is reducible, and let Π be a permutation matrix such that $\Pi^\top A(\boldsymbol{\beta})\Pi$ is in Frobenius normal form (5.9). Suppose by contradiction that $A(\boldsymbol{\beta})$ is non-Hurwitz (namely, in (5.9) $r \geq 1$) and hence the block $A_{11}(\boldsymbol{\beta})$ is irreducibile and non-Hurwitz. This implies that for every $\ell \in [1, n_1]$ and for every $i \in [1, M]$ such that $\beta_i > 0$, $\mathbf{1}_{n_1}^\top \mathrm{col}_\ell(A_{11}^{(i)}) = 0$. Now observe that the matrix $\Pi^\top A(\boldsymbol{\alpha})\Pi$ has the same upper triangular structure as $\Pi^\top A(\boldsymbol{\beta})\Pi$ and it is still true that for every $\ell \in [1, n_1]$ and for every $i \in [1, M]$ such that $\alpha_i > 0$, $\mathbf{1}_{n_1}^\top \mathrm{col}_\ell(A_{11}^{(i)}) = 0$. But then $A_{11}(\boldsymbol{\alpha})$, and hence $A(\boldsymbol{\alpha})$, is not Hurwitz, and this contradicts the assumption that $\boldsymbol{\alpha} \in \mathcal{A}_M^H$.

ii) Follows from i). $\blacksquare$

Remark 5.6. It is worth noticing that the case may occur that $\mathcal{A}_M^H$ coincides with the set of strictly positive vectors in $\mathcal{A}_M$, but no vector with a smaller nonzero pattern can be found in $\mathcal{A}_M^H$. Consider for instance the compartmental non-Hurwitz matrices

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It is clear that $\tilde{\boldsymbol{\alpha}} = \frac{1}{3}\mathbf{1}_3 \in \mathcal{A}_M^H$, but every $\boldsymbol{\alpha} \in \mathcal{A}_M$ with $|\overline{\mathrm{ZP}}(\boldsymbol{\alpha})| < 3$ does not belong to $\mathcal{A}_M^H$.

Stabilization of Multi-Input Compartmental Systems

In this chapter we address state-feedback stabilization of Multi-Input Compartmental Systems described as

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{6.1}$$

where $A \in \mathbb{R}^{n \times n}$ is a compartmental matrix and $B \in \mathbb{R}_+^{n \times m}$, $m > 1$, is a positive matrix devoid of zero columns. For the Compartmental System (6.1), under the assumption that the matrix A is non-Hurwitz, solving the *state-feedback stabilization problem* means determining a controller $K \in \mathbb{R}^{m \times n}$ such that the state-feedback control law $\mathbf{u}(t) = K\mathbf{x}(t)$ makes the closed-loop system asymptotically stable while preserving the compartmental property, namely such that $A + BK$ is compartmental and Hurwitz.

It is worth saying that extending to Multi-Input Compartmental Systems the characterization of stabilizability derived in the previous chapter for Single-Input Compartmental Systems is not straightforward. While stabilizability of the latter relies only on the nonzero patterns of the matrices involved in the system description (see Chapter 5, Section 5.2, as well as de Leenheer and Aeyels (2001)), for the former this is no more the case since stabilizability depends also on the specific entries of the matrices.

The content of the present chapter is largely based on:

Valcher M. E. and Zorzan I. On the state-feedback stabilisation of compartmental systems. In *Proceedings of 56th IEEE Conference on Decision and Control*, pages 6543–6548, Melbourne, Australia, 2017c;

Valcher M. E. and Zorzan I. State-feedback stabilization of multi-input compartmental systems. *Systems & Control Letters*, 2017e. (submitted).

Feedback stabilization of Positive Systems

The stabilization of Positive Systems and the dual problem of positive observer design have been the object of an intense study, even in the multi-input case, see, e.g., van den Hof (1998); Dautrebande and Bastin (1999); de Leenheer and Aeyels (2001); Gao, Lam, Wang, and Xu (2005); Ait Rami and Tadeo (2007); Back and Astolfi (2008); Roszak and Davison (2009); Briat (2013); Ebihara, Peaucelle, and Arzelier (2013). Most of the literature focused on the broader class of Positive Systems and translated the positive stabilization problem either into a Linear Matrix Inequality (LMI) or into a Linear Programming (LP) problem. A solution in terms of LMI has been provided in Gao et al. (2005) by making use of the fact that the Metzler matrix (positive matrix for discrete-time systems) obtained by means of a state-feedback is Hurwitz (Schur, respectively) if and only if it admits a positive diagonal Lyapunov function. In a similar way, Ait Rami and Tadeo (2007) resorted to linear copositive Lyapunov functions, and restated the positive stabilization problem as an LP problem. The solution in terms of LP, even if equivalent from a theoretical viewpoint, is preferable due to its lower computational complexity. Even more, it is prone to be easily extended to cope with robust stabilization in the presence of polytopic uncertainties, stabilization with restricted sign controls and stabilization with bounded controls.

Alternative approaches to the positive stabilization problem have been proposed. The characterization derived by Roszak and Davison (2009) is based on the construction of certain polytopes and on verifying whether a selection of their vertices can be used to construct a stabilizing state-feedback matrix. In Briat (2013) the problem of achieving by means of a state-feedback not only positivity and stability, but also certain $\mathcal{L}_1$ and $\mathcal{L}_\infty$ performances has been investigated. Also in this case, necessary and sufficient conditions for the problem solution have been expressed as LP problems.

Differently from the general class of Positive Systems, a systematic stabilizability analysis for the specific case of Multi-Input Compartmental Systems has been initiated only recently, Valcher and Zorzan (2017c,e). Interestingly enough, taking into account the additional properties compartmental matrices are endowed with allows to considerably

simplify results available for Positive Systems.

6.1   Problem solution when A is irreducible

We start from the case when the compartmental matrix A in (6.1) is irreducible. We show that, when so, the state-feedback stabilization problem is solvable if and only if it can be solved by resorting to a state-feedback that depends on a single compartment.

Recall that the selection matrix $S_i \in \mathbb{R}^{(n-1)\times n}$ is the matrix obtained by removing the ith row in the identity matrix I_n, and hence a matrix $A \in \mathbb{R}^{n\times n}$ is Metzler if and only if all vectors $S_h A\mathbf{e}_h$, $h \in [1,n]$, are nonnegative. We now introduce a technical lemma that will be exploited in subsequent analysis.

Lemma 6.1. *Consider a Multi-Input Compartmental System described as in* (6.1), *and assume that A is irreducible. If there exist $\mathbf{v} \in \mathbb{R}^m$ and $h \in [1,n]$ such that*

$$S_h(A\mathbf{e}_h + B\mathbf{v}) \geq 0, \tag{6.2a}$$

$$\mathbf{1}_n^\top B\mathbf{v} < 0, \tag{6.2b}$$

then for every $\varepsilon \in (0,1)$ the feedback matrix $K = \varepsilon \mathbf{v}\mathbf{e}_h^\top \in \mathbb{R}^{m\times n}$ makes $A + BK$ compartmental, irreducible, and Hurwitz.

Proof. Preliminarily notice that the unique nonzero column of the feedback matrix K is the hth one, and hence all columns of $A + BK$ are exactly the same as those of A except for the hth one. Condition (6.2a) ensures that for every $\varepsilon \in (0,1)$ the following two conditions hold

$$S_h A\mathbf{e}_h + \varepsilon S_h B\mathbf{v} \geq 0,$$

$$\overline{\mathrm{ZP}}(S_h A\mathbf{e}_h + \varepsilon S_h B\mathbf{v}) \supseteq \overline{\mathrm{ZP}}(S_h A\mathbf{e}_h).$$

This implies that for every $\varepsilon \in (0,1)$ the matrix $A + \varepsilon B\mathbf{v}\mathbf{e}_h^\top$ is Metzler and irreducible. If (6.2b) holds, then it is also true that for every $\varepsilon \in (0,1)$ one has $\mathbf{1}_n^\top(A\mathbf{e}_h + \varepsilon B\mathbf{v}) < 0$, and hence $\mathbf{1}_n^\top(A + \varepsilon B\mathbf{v}\mathbf{e}_h^\top) < 0^\top$. This ensures that $A + BK$ is compartmental (irreducible) and also Hurwitz (see Lemma 3.8). ∎

Proposition 6.2 below states that when A is irreducible and the state-feedback stabilization problem is solvable, there always exists a solution $K \in \mathbb{R}^{m\times n}$ with a unique nonzero column.

Proposition 6.2. *Consider a Multi-Input Compartmental System described as in* (6.1), *and assume that A is irreducible and non-Hurwitz. If the state-feedback stabilization problem is solvable, then there exist $\mathbf{v} \in \mathbb{R}^m$ and $h \in [1, n]$ such that for every $\varepsilon \in (0, 1)$ the matrix $K := \varepsilon \mathbf{v} \mathbf{e}_h^\top \in \mathbb{R}^{m \times n}$ is a possible solution.*

Proof. Let $\bar{K} \in \mathbb{R}^{m \times n}$ be a solution of the state-feedback stabilization problem. Let $h \in [1, n]$ be such that $0 > \mathbf{1}_n^\top (A + B\bar{K})\mathbf{e}_h = \mathbf{1}_n^\top A\mathbf{e}_h + \mathbf{1}_n^\top B\bar{K}\mathbf{e}_h = \mathbf{1}_n^\top B\bar{K}\mathbf{e}_h$ (such an index exists, otherwise $\lambda_F(A + B\bar{K}) = 0$), and set $\mathbf{v} := \bar{K}\mathbf{e}_h$. Clearly, $\mathbf{1}_n^\top B\mathbf{v} < 0$, and $S_h(A\mathbf{e}_h + B\mathbf{v}) = S_h(A + B\bar{K})\mathbf{e}_h \geq 0$. So, by Lemma 6.1, for every $\varepsilon \in (0, 1)$ the feedback matrix $K = \varepsilon \mathbf{v} \mathbf{e}_h^\top \in \mathbb{R}^{m \times n}$ makes $A + BK$ compartmental, irreducible and Hurwitz. ∎

Example 6.1. Consider the Multi-Input Compartmental System

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -5 & 0 \\ 0 & 5 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{u}(t).$$

Notice that A is an irreducible and non-Hurwitz matrix. It is easy to verify that any feedback matrix of the form

$$K = \begin{bmatrix} k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix}, \quad k_1, k_2 \in \mathbb{R},$$

does not solve the state-feedback stabilization problem. However, for every $\varepsilon \in (0, 1)$ the two feedback matrices

$$K = \varepsilon \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}, \quad K = \varepsilon \begin{bmatrix} 0 & 0 & -0.5 \\ 0 & 0 & 0.25 \end{bmatrix},$$

make $A + BK$ compartmental and Hurwitz.   ♣

   The previous example shows that, even if a solution $K \in \mathbb{R}^{m \times n}$ with a unique nonzero column can be found, not all columns (i.e., not all indices $h \in [1, n]$) play an equivalent role. The following proposition provides, for a fixed $h \in [1, n]$, equivalent conditions for the existence of a vector $\mathbf{k} \in \mathbb{R}^m$ such that $K := \mathbf{k} \mathbf{e}_h^\top$ is a possible solution.

Proposition 6.3. *Consider a Multi-Input Compartmental System described as in* (6.1), *and assume that A is irreducible and non-Hurwitz. Let h be a fixed index in $[1, n]$ and*

introduce the set $H := \{j \in [1, n], j \neq h \colon [A]_{jh} = 0\}$. Let B_H be the matrix obtained by selecting only the rows of B indexed in H, and $\mathcal{C}_H$ the dual cone of the polyhedral cone generated by[1] $(B_H)^\top$, namely $\mathcal{C}_H := \left(\text{Cone}(B_H^\top)\right)^*$. Let $\{\mathbf{w}_i\}_{i=1}^N$ be a family of minimal generators of $\mathcal{C}_H$, namely $\mathcal{C}_H = \text{Cone}(W)$ where $W = \begin{bmatrix} \mathbf{w}_1 & \dots & \mathbf{w}_N \end{bmatrix} \in \mathbb{R}^{m \times N}$. The following facts are equivalent:

    i) There exists $\mathbf{v} \in \mathbb{R}^m$ such that conditions (6.2) hold;

    ii) The N-dimensional row vector $\mathbf{1}_n^\top BW$ has at least one negative entry;

    iii) There exists $i \in [1, N]$ such that the vector $\mathbf{w}_i = W\mathbf{e}_i$ satisfies

$$\mathbf{1}_n^\top B\mathbf{w}_i < 0 \tag{6.3a}$$

$$[B\mathbf{w}_i]_j \geq 0, \forall j \in H; \tag{6.3b}$$

    iv) There exists $\mathbf{k} \in \mathbb{R}^m$ such that $K = \mathbf{k}\mathbf{e}_h^\top \in \mathbb{R}^{m \times n}$ is a solution to the state-feedback stabilization problem.

Proof. Let d be the cardinality of the index set H, $d := |H|$. By definition of dual cone, $\mathcal{C}_H$ can be expressed as

$$\begin{aligned}
\mathcal{C}_H &= \left\{ \mathbf{x} \in \mathbb{R}^m \colon \mathbf{y}^\top \mathbf{x} \geq 0, \forall \mathbf{y} \in \text{Cone}(B_H^\top) \right\} \\
&= \left\{ \mathbf{x} \in \mathbb{R}^m \colon \mathbf{u}^\top B_H \mathbf{x} \geq 0, \forall \mathbf{u} \in \mathbb{R}_+^d \right\} \\
&= \left\{ \mathbf{x} \in \mathbb{R}^m \colon B_H \mathbf{x} \geq 0 \right\}.
\end{aligned}$$

i) $\Rightarrow$ *ii)* Clearly, any vector $\mathbf{v} \in \mathbb{R}^m$ satisfying condition (6.2a) is such that $B_H \mathbf{v} \geq 0$. This means that $\mathbf{v} \in \mathcal{C}_H = \text{Cone}(W)$, and hence there exists $\mathbf{u} \in \mathbb{R}_+^N$, such that $\mathbf{v} = W\mathbf{u}$. Then, condition (6.2b) can be rewritten as $0 > \mathbf{1}_n^\top B\mathbf{v} = \mathbf{1}_n^\top BW\mathbf{u}$, and this implies that the vector $\mathbf{1}_n^\top BW$ has at least one negative entry, namely *ii)* holds.

ii) $\Rightarrow$ *iii)* Let $i \in [1, N]$ be such that $0 > \left[\mathbf{1}_n^\top BW\right]_i = \mathbf{1}_n^\top BW\mathbf{e}_i = \mathbf{1}_n^\top B\mathbf{w}_i$. Clearly, $\mathbf{w}_i$ satisfies condition (6.3a). Moreover, since $\mathbf{w}_i \in \text{Cone}(W) = \mathcal{C}_H$, we have $B_H \mathbf{w}_i \geq 0$, namely $[B\mathbf{w}_i]_j \geq 0$ for every $j \in H$.

iii) $\Rightarrow$ *i)* We want to prove that if *iii)* holds, there exists $\varepsilon > 0$ such that $\mathbf{v} := \varepsilon \mathbf{w}_i$ satisfies *i)*. Clearly, $\mathbf{1}_n^\top B\mathbf{v} = \varepsilon \mathbf{1}_n^\top B\mathbf{w}_i < 0$ for every $\varepsilon > 0$, namely $\mathbf{v}$ satisfies (6.2b) for every $\varepsilon > 0$. Moreover, since $\mathbf{v} \in \mathcal{C}_H$, we have $B_H \mathbf{v} \geq 0$ for every $\varepsilon > 0$, namely

---

[1] In the following, to keep the notation simple, we will use $B_H^\top$ instead of $(B_H)^\top$. Note that without this clarification the simplified notation would have been ambiguous, since $(B_H)^\top \neq (B^\top)_H$.

$[B\mathbf{v}]_j \geq 0$ for every $j \in H$, $\varepsilon > 0$. On the other hand, for every $j \notin H$, $j \neq h$, we have $[A]_{jh} > 0$, and hence there always exists $\varepsilon_j > 0$ such that $[A]_{jh} + \varepsilon_j \mathbf{e}_j^\top B\mathbf{w}_i \geq 0$. So, if we choose $\varepsilon := \min_{j \notin (H \cup \{h\})} \varepsilon_j$, the vector $\mathbf{v}$ satisfies conditions (6.2).

$i) \Rightarrow iv)$ The implication follows from Lemma 6.1, by assuming $\mathbf{k} = \varepsilon \mathbf{v}$, with ε arbitrary in $(0, 1)$.

$iv) \Rightarrow i)$ Set $\mathbf{v} := \mathbf{k}$. Since $A + BK = A + B\mathbf{v}\mathbf{e}_h^\top$ is compartmental and Hurwitz, the first property ensures that (6.2a) holds. Meanwhile the Hurwitz property of the compartmental matrix $A + B\mathbf{v}\mathbf{e}_h^\top$ implies that $\mathbf{1}_n^\top(A + B\mathbf{v}\mathbf{e}_h^\top) < 0^\top$ and since $\mathbf{1}_n^\top A = 0^\top$ this implies that (6.2b) holds.   ■

Example 6.2. Consider the Multi-Input Compartmental System

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) = \begin{bmatrix} -1-\alpha & 1 & 0 & 0 \\ 1 & -1-\alpha & 0 & \alpha \\ 0 & 0 & -\alpha & \alpha \\ \alpha & \alpha & \alpha & -2\alpha \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \mathbf{u}(t),$$

where $\alpha > 0$ is an arbitrary positive scalar. Notice that A is irreducible and non-Hurwitz (as $\mathbf{1}_4^\top A = 0^\top$). It is easy to verify that for every $h \in [1, 4]$ a vector $\mathbf{v} \in \mathbb{R}^2$ satisfying conditions (6.2) does not exist. By Proposition 6.3 this means that for every $h \in [1, 4]$ there does not exist $K \in \mathbb{R}^{2 \times 4}$ whose unique nonzero column is the hth one, and that makes $A + BK$ compartmental and Hurwitz. It then follows from Proposition 6.2 that the state-feedback stabilization problem for the pair (A, B) is not solvable.   ♣

Proposition 6.3 provides, under the assumption that A is a compartmental irreducible and non-Hurwitz matrix, necessary and sufficient conditions for the existence of a solution taking the form $K = \mathbf{k}\mathbf{e}_h^\top$, for some $\mathbf{k} \in \mathbb{R}^m$, where h is a fixed index in $[1, n]$. On the other hand, by Proposition 6.2, if the state-feedback stabilization problem is solvable, then there always exists a solution K taking that form for some index $h \in [1, n]$. This immediately leads to the following necessary and sufficient condition for state-feedback stabilization.

Theorem 6.4. *Consider a Multi-Input Compartmental System described as in* (6.1)*, and assume that A is irreducible and non-Hurwitz. The state-feedback stabilization problem is solvable if and only if there exist a vector $\mathbf{v} \in \mathbb{R}^m$ and an index $h \in [1, n]$ such that conditions* (6.2) *(or any of the equivalent conditions of Proposition 6.3) hold.*
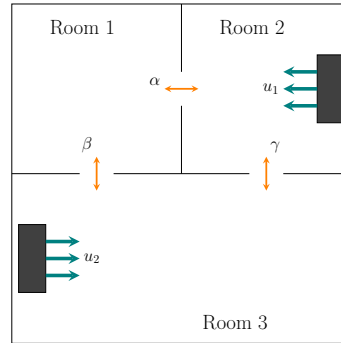
Figure 6.1: A simple three-room thermal system.

Remark 6.1. In Theorem 6.4 (and Proposition 6.3) necessary and sufficient conditions for the state-feedback stabilization problem to be solvable are given in the form of LP problems. Since these conditions involve a single column, they are simpler than the general ones obtained for Multi-Input Positive Systems.

Example 6.3 (Room temperature regulation)**.** Consider the thermal system of Figure 6.1: it consists of three rooms, two of them (room 2 and room 3) directly connected to the air-conditioning system. Let $\alpha = \beta = 0.7$ and $\gamma = 1$ be the thermal transmission coefficients between the adjacent rooms $(1,2)$, $(1,3)$, and $(2,3)$, respectively. Denote by $x_i, i \in [1,3]$, the (positive) difference between the temperature in the ith room and the desired temperature x_d. If we assume that the system is thermally isolated from the external environment, the time evolution of the temperatures in the three rooms is described by the following compartmental model:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) = \begin{bmatrix} -(\alpha + \beta) & \alpha & \beta \\ \alpha & -(\alpha + \gamma) & \gamma \\ \beta & \gamma & -(\gamma + \beta) \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t).$$

Notice that A is irreducible and non-Hurwitz (as $\mathbf{1}_3^\top A = 0^\top$). We want to determine, if possible, a state-feedback control law for the air-conditioning system that allows to regulate all temperatures by making use only of the temperature of room 1. It is easy to verify that for the fixed index $h = 1$, the vector $\mathbf{v} = \begin{bmatrix} -0.5 & -0.5 \end{bmatrix}^\top$ satisfies conditions (6.2). Then, it follows from Lemma 6.1 that for every $\varepsilon \in (0,1)$ the matrix $K = \varepsilon \mathbf{v} \mathbf{e}_1^\top$ solves the state-feedback stabilization problem. ♣

6.2 The case when A is reducible

In this section we consider the case when the state-space matrix A involved in the Compartmental System description (6.1) is a (compartmental and) reducible matrix. Extending the characterization provided in the previous section for the irreducible case is not straightforward, and the main difficulty in doing so comes from the fact that for a compartmental and reducible matrix A condition $\mathbf{1}_n^\top A < 0^\top$ is a necessary (but not sufficient) condition for A to be Hurwitz. For this reason, in order to provide sufficient conditions for the solvability of the the state-feedback stabilization problem we will exploit the notion of directed graph associated with a compartmental matrix, and interpret Hurwitz stability in terms of connection properties among the vertices of the associated graph.

6.2.1 Preliminaries: graphical interpretation of Hurwitz stability

It is well known that when dealing with compartmental matrices Hurwitz stability can be assessed by just examining the associated directed graph. Fife (1972) proved that a compartmental system is singular if and only if it contains a *trap*, i.e., a set of compartments from which there are no flows to the environments and no flows to compartments that are not in the trap. An equivalent formulation of this result that will be extensively exploited in the following, must be credited to Eisenfeld (1982) (but see also Jacquez and Simon (1993); Bastin and Guffens (2006)).

Consistently with Bastin and Guffens (2006), we say that: 1) the ith compartment has *outflow to the environment* if $\mathbf{1}_n^\top A \mathbf{e}_i < 0$, namely if there is a direct flow from the ith compartment to the environment; 2) the ith compartment is *outflow connected* if there is a path from that compartment to some jth compartment (i.e., from vertex i to vertex j in $\mathcal{D}(A)$), that has outflow to the environment. These concepts allow us to introduce the following criterion for Hurwitz stability of a compartmental matrix.

Lemma 6.5. *A compartmental matrix $A \in \mathbb{R}^{n \times n}$ is Hurwitz if and only if all its compartments are outflow connected.*

Remark 6.2. Notice that when A is a compartmental and irreducible matrix, Lemma 6.5 reduces to Lemma 3.8. Indeed, recalling that the irreducibility of A and the connectedness of the associated directed graph $\mathcal{D}(A)$ are equivalent properties, it is easy to see that A is Hurwitz if and only if one at least of its compartments has outflow to the environment, i.e., $\mathbf{1}_n^\top A < 0^\top$.

In the following, under the assumption that $A \in \mathbb{R}^{n \times n}$ is a compartmental reducible and non-Hurwitz matrix, we will refer to the following Frobenius normal form (see Chapter 5, Remark 5.5), with either scalar or irreducible diagonal blocks $A_{ii}, i \in [1, s]$, and $r \geq 1$ such that:

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & 0 & \dots & 0 & A_{1r+1} & \dots & A_{1s} \\ 0 & A_{22} & \dots & 0 & A_{2r+1} & \dots & A_{2s} \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & \dots & A_{rr} & A_{rr+1} & \dots & A_{rs} \\ 0 & \dots & \dots & 0 & A_{r+1r+1} & \dots & A_{r+1s} \\ \vdots & & & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & \dots & \dots & A_{ss} \end{bmatrix}, \quad \begin{aligned} \lambda_F(A_{jj}) &= 0, \forall\, j \in [1, r], \\ \lambda_F(A_{jj}) &< 0, \forall\, j \in [r+1, s]. \end{aligned}$$

(6.4)

Consequently, $\mathbf{1}_{n_j}^\top A_{jj} = 0^\top$ for every $j \in [1, r]$, and $\mathbf{1}_{n_j}^\top A_{jj} < 0^\top$ for all $j \in [r+1, s]$.

When referring to the Frobenius form (6.4) we will denote by Ω_i, $i \in [1, s]$, the set of all row and column indices corresponding to the ith diagonal block A_{ii}, i.e.,

$$\Omega_i := \left[\sum_{k=1}^{i-1} n_k + 1, \sum_{k=1}^{i} n_k \right].$$

(6.5)

6.2.2 Sufficient conditions for the problem solution

The following sufficient condition for the solvability of the state-feedback stabilization problem can be seen as an extension to the case when the compartmental matrix A is a reducible matrix of the sufficient condition stated in Lemma 6.1 for the irreducible case.

Proposition 6.6. *Consider a Multi-Input Compartmental System described as in* (6.1), *and assume that A is non-Hurwitz. Assume without loss of generality that A is in the Frobenius normal form* (6.4), *with scalar or irreducible diagonal blocks $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, and the index $r \in [1, s]$, $r \geq 1$, is such that $\lambda_F(A_{ii}) = 0$ for every $i \in [1, r]$, and $\lambda_F(A_{ii}) < 0$ for every $i \in [r+1, s]$. Also, let the sets Ω_i, $i \in [1, s]$, be defined as in* (6.5). *If for every $i \in [1, r]$ there exist $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \Omega_i$ such that*

$$S_{\ell_i}(A\mathbf{e}_{\ell_i} + B\mathbf{v}_i) \geq 0, \tag{6.6a}$$

$$\mathbf{1}_n^\top B\mathbf{v}_i < 0, \tag{6.6b}$$

then for every $\varepsilon \in (0,1)$ the feedback matrix

$$K := \varepsilon \sum_{i=1}^{r} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top \tag{6.7}$$

solves the state-feedback stabilization problem.

Proof. We want to prove that for every $\varepsilon \in (0,1)$ the matrix K given in (6.7) makes $A + BK$ compartmental and Hurwitz. To this end, let us first partition the input-to-state matrix B and the state-feedback matrix K in a way consistent with the block-partition (6.4) of A, namely as

$$B = \begin{bmatrix} B_1^\top & B_2^\top & \dots & B_s^\top \end{bmatrix}^\top, \quad K = \begin{bmatrix} K_1 & K_2 & \dots & K_s \end{bmatrix}, \tag{6.8}$$

where $B_i \in \mathbb{R}_+^{n_i \times m}$ and $K_i \in \mathbb{R}^{m \times n_i}$, for every $i \in [1, s]$. By assumption, for every $i \in [1, r]$, there exist $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \Omega_i$ such that (6.6) hold. Consequently, for every $\varepsilon \in (0, 1)$ the following three conditions hold true:

$$S_{\ell_i} A \mathbf{e}_{\ell_i} + \varepsilon S_{\ell_i} B \mathbf{v}_i \geq 0, \tag{6.9a}$$

$$\mathbf{1}_n^\top (A \mathbf{e}_{\ell_i} + \varepsilon B \mathbf{v}_i) < 0, \tag{6.9b}$$

$$\overline{\mathrm{ZP}}(S_{h_i} A_{ii} \mathbf{e}_{h_i} + \varepsilon S_{h_i} B_i \mathbf{v}_i) \supseteq \overline{\mathrm{ZP}}(S_{h_i} A_{ii} \mathbf{e}_{h_i}), \tag{6.9c}$$

where $h_i := \ell_i - \sum_{k=1}^{i-1} n_k \in [1, n_i]$. We preliminarily notice that every hth column of $A + BK$, $h \in [1, n] \setminus \{\ell_1, \dots, \ell_r\}$, coincides with the hth column of A and hence $S_h(A+BK)\mathbf{e}_h \geq 0$ and $\mathbf{1}_n^\top(A+BK)\mathbf{e}_h \leq 0$. On the other hand, for every $h \in \{\ell_1, \dots, \ell_r\}$ condition (6.9a) ensures that $S_h(A + BK)\mathbf{e}_h \geq 0$, while condition (6.9b) guarantees that $\mathbf{1}_n^\top(A + BK)\mathbf{e}_h < 0$. Consequently, $A + BK$ is Metzler and compartmental. Finally, (6.9c) implies that for every $\varepsilon \in (0,1)$ the matrix $K_i = \varepsilon \mathbf{v}_i \mathbf{e}_{h_i}^\top \in \mathbb{R}^{m \times n_i}$ (see (6.8)) makes the compartmental matrix $A_{ii} + B_i K_i$ irreducible (and Hurwitz).

To prove that $A + BK$ is Hurwitz, we will prove that in the closed-loop system every compartment is outflow connected (i.e., for every vertex p of the directed graph $\mathcal{D}(A+BK)$ there exists a vertex q, possibly coinciding with p, such that there is a path from p to q and $\mathbf{1}_n^\top(A+BK)\mathbf{e}_q < 0$). By Lemma 6.5 this fact ensures that $A+BK$ is Hurwitz. In order to prove the outflow connectedness of all compartments of the closed-loop system, we need to partition its compartments in a convenient way. Specifically, we group its compartments according to the communication classes introduced in (6.4) for the open-loop system, namely the sets Ω_i, $i \in [1, s]$, defined in (6.5). Notice that these are not necessarily the

communication classes of $\mathcal{D}(A + BK)$. Indeed, the matrices A and $A + BK$ in general have different nonzero patterns (the off-diagonal blocks $B_j K_i$, $j \neq i$, are nonnegative, and not necessarily zero), namely the interconnection topology among compartments in the closed-loop system is potentially different from the one characterizing the open-loop system. However, this is not a problem, as our goal is to prove that all the compartments of $A + BK$ are outflow connected, and not to determine the communication classes of $\mathcal{D}(A + BK)$. To this aim, we make the following considerations:

1) Every Ω_i, with $i \in [1, r]$, is such that (a) there is a path from any compartment of Ω_i to any other compartment of Ω_i (since $A_{ii} + B_i K_i$ is irreducible); and (b) there is a compartment in Ω_i (the ℓ_ith one) that has outflow to the environment (since $\mathbf{1}_n^\top (A + BK) \mathbf{e}_{\ell_i} < 0$).

2) Every Ω_i, with $i \in [r + 1, s]$, still exhibits property (a) since the ith diagonal block of $A + BK$ coincides with the irreducible matrix A_{ii}. We now prove by induction that either Ω_i satisfies property (b), or it has access to Ω_j, $j < i$, for which (b) holds. Start from $i = r + 1$, namely from Ω_{r+1}, and note that each $(j, r + 1)$th block of $A + BK$ coincides with the original block in A, namely A_{jr+1}. Two cases may occur: either there exists $\ell_{r+1} \in \Omega_{r+1}$ such that

$$\mathbf{1}_n^\top (A + BK) \mathbf{e}_{\ell_{r+1}} = \sum_{k=1}^{r+1} \mathbf{1}_{n_k}^\top A_{kr+1} \mathbf{e}_{\ell_{r+1}} < 0,$$

and hence Ω_{r+1} exhibits property (b); or $\sum_{k=1}^{r+1} \mathbf{1}_{n_k}^\top A_{kr+1} = 0^\top$, and, if this is the case, the Hurwitz property of A_{r+1r+1} implies that there exists $j \in [1, r]$ such that $A_{jr+1} > 0$, namely Ω_{r+1} has access to some Ω_j, $j \in [1, r]$, that, by part 1), satisfies property (b).
Now, let $\bar{k} \in [r + 1, s - 1]$ and suppose that, for every $k \in [r + 1, \bar{k}]$, Ω_k either satisfies property (b) or it has access to Ω_j, $j < k$, that satisfies property (b). By applying to $\Omega_{\bar{k}+1}$ the same reasoning adopted for Ω_{r+1} we can claim that either $\Omega_{\bar{k}+1}$ exhibits property (b) or it has access to Ω_j, $j < \bar{k} + 1$, for which (b) holds. So, every compartment in every Ω_i, with $i \in [r + 1, s]$, is outflow connected.

To conclude, every compartment of the closed-loop system $A + BK$ is outflow connected and hence $A + BK$ is also Hurwitz. This means that the feedback matrix K defined in (6.7) solves the state-feedback stabilization problem for every $\varepsilon \in (0, 1)$. ∎

Example 6.4. Consider the following Multi-Input Compartmental System

$$
\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix} \mathbf{u}(t) = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathbf{u}(t).
$$

Clearly, A_{11} is irreducible and non-Hurwitz, while A_{22} is irreducible and Hurwitz. So, $r = 1$ and $s = 2$. It is easy to verify that the vector $\mathbf{v}_1 = \begin{bmatrix} 1 & -2 \end{bmatrix}^\top$ satisfies conditions (6.6) for $\ell_1 = 1$. This means that the sufficient condition of Proposition 6.6 is satisfied. It is a matter of simple computation to see that, for every $\varepsilon \in (0, 1)$, the feedback matrix

$$
K = \varepsilon \mathbf{v}_1 \mathbf{e}_1^\top = \varepsilon \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}
$$

makes $A + BK$ compartmental and Hurwitz. ♣

The previous stabilizability condition requires the existence of r vectors $\mathbf{v}_i$, $i \in [1, r]$, satisfying conditions (6.6). If such vectors cannot be found, the stabilization problem may still be solvable as described by the following result.

Proposition 6.7. *Consider a Multi-Input Compartmental System described as in* (6.1), *and suppose that A is non-Hurwitz. Assume without loss of generality that A is in the Frobenius normal form* (6.4), *with scalar or irreducible diagonal blocks $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, and the index $r \in [1, s]$, $r \geq 1$, is such that $\lambda_F(A_{ii}) = 0$ for every $i \in [1, r]$, and $\lambda_F(A_{ii}) < 0$ for every $i \in [r + 1, s]$. Also, let the sets Ω_i, $i \in [1, s]$, be defined as in* (6.5) *and assume that B is block-partitioned as in* (6.8). *Introduce the following sets:*

$$
E_0 := \left\{ i \in [1, s] : \exists \ell_i \in \Omega_i \ s.t. \ \mathbf{1}_n^\top A \mathbf{e}_{\ell_i} < 0 \right\} \subseteq [r + 1, s]
$$

$$
E_1 := \left\{ i \in ([1, s] \setminus E_0) : \exists \mathbf{v}_i \in \mathbb{R}^m, \ \ell_i \in \Omega_i \ s.t. \ S_{\ell_i}(A \mathbf{e}_{\ell_i} + B \mathbf{v}_i) \geq 0 \ and \ \mathbf{1}_n^\top B \mathbf{v}_i < 0 \right\}.
$$

If for every $i \in E_2 := ([1, r] \setminus E_1)$ there exist $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \Omega_i$ such that[2]

$$
S_{\ell_i}(A \mathbf{e}_{\ell_i} + B \mathbf{v}_i) \geq 0, \tag{6.10a}
$$

$$
\mathbf{1}_n^\top B \mathbf{v}_i = 0, \tag{6.10b}
$$

$$
B_d \mathbf{v}_i > 0, \quad \exists d \in (E_0 \cup E_1), \tag{6.10c}
$$

---

[2]Note that if $E_0 \cup E_1 = \emptyset$, condition (6.10c) does not hold, and hence this sufficient condition cannot be applied. As we will see in the following, if $E_0 \cup E_1 = \emptyset$ the stabilization problem cannot be solved.

then, for every $\varepsilon \in (0,1)$, the feedback matrix

$$K := \varepsilon \sum_{i \in (E_1 \cup E_2)} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top = \begin{bmatrix} K_1 & K_2 & \dots & K_s \end{bmatrix} \tag{6.11}$$

solves the state-feedback stabilization problem.

Proof. We want to prove that for every $\varepsilon \in (0,1)$ the matrix K given in (6.11) makes $A + BK$ compartmental and Hurwitz. To this end, preliminarily notice that: (1) $K_i = 0$ for every $i \notin (E_1 \cup E_2)$; (2) for every $h \in [1, n] \setminus \{\ell_i, i \in E_1 \cup E_2\}$ the hth column of $A + BK$ coincides with the hth column of A. Consequently,

$$S_h(A + BK)\mathbf{e}_h \geq 0, \qquad \text{and} \qquad \mathbf{1}_n^\top(A + BK)\mathbf{e}_h \leq 0. \tag{6.12}$$

To prove that $A + BK$ is compartmental, we need to prove that inequalities (6.12) hold true also for every $h \in \{\ell_i, i \in E_1 \cup E_2\}$.

Consider first the indices belonging to E_1. For every $i \in E_1$, there exist $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \Omega_i$ such that conditions (6.6) hold. Define $h_i := \ell_i - \sum_{k=1}^{i-1} n_k \in [1, n_i]$. By proceeding as in the proof of Proposition 6.6, we can claim that for every $\varepsilon \in (0,1)$ and every $h \in \{\ell_i, i \in E_1\}$ conditions (6.12) hold true and, in addition, the matrix $K_i = \varepsilon \mathbf{v}_i \mathbf{e}_{h_i}^\top \in \mathbb{R}^{m \times n_i}$ makes $A_{ii} + B_i K_i$ (compartmental and) irreducible.

Consider now the set $E_2 \subseteq [1, r]$. For every $i \in E_2$ there exist $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \Omega_i$ such that conditions (6.10) hold. Define also in this case $h_i := \ell_i - \sum_{k=1}^{i-1} n_k$. We observe that for every $\varepsilon \in (0,1)$ the following three conditions hold true:

$$S_{\ell_i} A \mathbf{e}_{\ell_i} + \varepsilon S_{\ell_i} B \mathbf{v}_i \geq 0, \tag{6.13a}$$

$$\mathbf{1}_n^\top (A \mathbf{e}_{\ell_i} + \varepsilon B \mathbf{v}_i) = 0, \tag{6.13b}$$

$$\overline{\text{ZP}}(S_{h_i} A_{ii} \mathbf{e}_{h_i} + \varepsilon S_{h_i} B_i \mathbf{v}_i) \supseteq \overline{\text{ZP}}(S_{h_i} A_{ii} \mathbf{e}_{h_i}). \tag{6.13c}$$

By following the same reasoning as before, we can claim that for every $\varepsilon \in (0,1)$ and every $h \in \{\ell_i, i \in E_2\}$ the inequalities (6.12) hold (and therefore, at this stage, we can claim that $A + BK$ is compartmental), and the matrix $K_i = \varepsilon \mathbf{v}_i \mathbf{e}_{h_i}^\top \in \mathbb{R}^{m \times n_i}$ makes $A_{ii} + B_i K_i$ (compartmental and) irreducible. Moreover, there exists $d \in (E_0 \cup E_1)$ such that $A_{di} + B_d K_i = 0 + B_d K_i > 0$, where we used the fact that $i \in E_2 \subseteq [1, r]$ implies $A_{di} = 0$.

To prove that $A + BK$ is Hurwitz, we will prove that in the closed-loop system every compartment is outflow connected. To this aim, we partition (as we did in the proof of Proposition 6.6) the compartments of the closed-loop system $A + BK$ according to the

partition into communication classes Ω_i, $i \in [1, s]$, of the open-loop system. Again, this is only a convenient partition of the compartments of the closed-loop system that does not necessarily coincide with the partition into communication classes.

We want to show that, for every $i \in [1, s]$, every compartment in Ω_i is outflow connected, considering all possible cases:

- $i \in E_0$: Clearly, $i \in E_0$ if and only if there exists in Ω_i a compartment, the ℓ_ith one, that has outflow to the environment in the open-loop system. On the other hand, condition $K_i = 0$ ensures that $A_{ii} + B_i K_i = A_{ii}$ is irreducible, and $A_{ji} + B_j K_i = A_{ji}$ for every $j \in [1, s], j \neq i$. Therefore every compartment in Ω_i is outflow connected also in the closed-loop system $A + BK$.

- $i \in E_1$: Every Ω_i, $i \in E_1$, is such that: (a) there is a path from any compartment of Ω_i to any other compartment of Ω_i (since the diagonal blocks $A_{ii} + B_i K_i$ are irreducible); and (b) there is a compartment, the ℓ_ith one, that has outflow to the environment (see (6.9b)).

- $i \in E_2$: Every Ω_i with $i \in E_2$ is such that: (a) there is a path from any compartment of Ω_i to any other compartment of Ω_i (again because the blocks $A_{ii} + B_i K_i$ are irreducible); and (b) there is a path from a compartment in Ω_i to a compartment in Ω_d with $d \in (E_0 \cup E_1)$, namely Ω_i has access to Ω_d (since $B_d K_i > 0$). So, even the compartments in Ω_i with $i \in E_2$ are outflow connected.

- $i \notin \cup_{i=0}^{2} E_i$: We first note that $[1, r] \subseteq E_1 \cup E_2$. Therefore any $i \notin \cup_{i=0}^{2} E_i$ necessarily belongs to $[r + 1, s]$ and satisfies the following conditions: $A_{ii} + B_i K_i = A_{ii}$ is a compartmental, irreducible, Hurwitz matrix and $\sum_{k=1}^{i} \mathbf{1}_{n_k}^{\top} A_{ki} = 0^{\top}$. This implies that there exists $j \in [1, i - 1]$ such that $A_{ji} > 0$, namely Ω_i has access to Ω_j, $j \in [1, i - 1]$. Set $\bar{k} := \min \{i \notin (E_0 \cup E_1 \cup E_2)\}$. By proceeding as in the proof of Proposition 6.6, we can show that (a) there is a path from any compartment of $\Omega_{\bar{k}}$ to any other compartment of $\Omega_{\bar{k}}$, (b) there is a path from a compartment in $\Omega_{\bar{k}}$ to a compartment in Ω_i with $i \in (E_0 \cup E_1 \cup E_2)$. Consequently, all the compartments in $\Omega_{\bar{k}}$ are outflow connected. By proceeding recursively, conditions (a) and (b) prove to be true for all the remaining $i \notin (E_0 \cup E_1 \cup E_2)$.

To conclude, every compartment of the closed-loop system $A + BK$ is outflow connected, and hence $A + BK$ is also Hurwitz. This means that the feedback matrix K defined in (6.11) solves the state-feedback stabilization problem. ∎

Remark 6.3. Note that a solution K with exactly r non-zero columns (where r is the number of conservative, i.e., non-Hurwitz, communication classes of A) might not exist even if the state-feedback stabilization problem is solvable. Indeed, the minimum number of non-zero columns of any solution K is at least r but it might be greater, as shown in the following example.

Example 6.5. Consider the following Multi-Input Compartmental System

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{u}(t) = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathbf{u}(t).$$

Clearly, $s = 2$, A_{11} is irreducible and non-Hurwitz, while A_{22} is irreducible and Hurwitz. So, $r = 1$. In this case we have: $E_0 = \emptyset$; $E_1 = \{2\}$ and the vector $\mathbf{v}_2 = \begin{bmatrix} 0 & -1/2 \end{bmatrix}^\top$ satisfies conditions (6.6) for $\ell_2 = 4$; $E_2 = \{1\}$ and there exists $\mathbf{v}_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ that satisfies conditions (6.10) for $\ell_1 = 1$. This means that the sufficient condition of Proposition 6.7 is satisfied. Then, it is easy to verify that for every $\varepsilon \in (0, 1)$ the feedback matrix

$$K = \varepsilon(\mathbf{v}_1 \mathbf{e}_1^\top + \mathbf{v}_2 \mathbf{e}_4^\top) = \varepsilon \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1/2 \end{bmatrix}$$

makes $A + BK$ compartmental and Hurwitz. Notice that in this case the sufficient condition of Proposition 6.6 does not hold. ♣

6.3 Necessary and sufficient conditions for the problem solution: a general algorithm

The reasoning behind the proofs of Proposition 6.6 and Proposition 6.7 can be further exploited to determine a necessary and sufficient condition for the solvability of the state-feedback stabilization problem. To this aim, we need to define the concept of *distance from the environment* of the sets Ω_i, $i \in [1, s]$, which provide a partition of the n compartments, namely of the set $[1, n]$, and for which the following property holds: for every $p, q \in \Omega_i$ there is a path $p \to q_1 \to \cdots \to q_k \to q$ with $q_d \in \Omega_i$, $d \in [1, k]$.

We say that Ω_i has *direct outflow to the environment*, or *distance $\delta = 0$ from the environment*, if there exists a compartment in Ω_i with outflow to the environment. This

ensures that, given any compartment p of Ω_i, either it has outflow to the environment, or there is a path $p \to q_1 \to \cdots \to q_k \to q_{k+1}$ with $q_d \in \Omega_i$, $d \in [1, k+1]$, to a compartment q_{k+1} that has outflow to the environment.

If Ω_i has no direct outflow to the environment, we say that it has *distance* $\delta \geq 1$ *from the environment* if there exists an arc from a compartment p of Ω_i to some compartment $q \in \Omega_j$, where Ω_j is a set having distance $\delta - 1$ from the environment, and no $\delta' < \delta$ can be found for which the previous property holds.

We say that Ω_i has *infinite distance from the environment* if none of the compartments of Ω_i is outflow connected.

The concept of distance from the environment allows us to restate the state-feedback stabilization problem in slightly different terms. Specifically, we can claim that the state-feedback stabilization problem is solvable if and only if there exists $K \in \mathbb{R}^{m \times n}$ such that in the closed-loop system (namely, for the closed-loop matrix $A + BK$) each set Ω_i, $i \in [1, s]$, has finite distance from the environment[3].

We want now to comment on the sufficient conditions provided in Propositions 6.6 and Proposition 6.7, in terms of the previously defined concepts. In Proposition 6.6 we require that for each conservative communication class of the original matrix A, namely for each Ω_i, $i \in [1, r]$, a state feedback can be found such that the set Ω_i has direct outflow to the environment in the resulting feedback system. If this is the case, then also the other sets Ω_i, $i \in [r+1, s]$, will have finite distance from the environment, either because they have, in turn, direct outflow to the environment or because they have access (through a path of finite length) to other sets Ω_j, $j < i$, that have direct outflow to the environment.

In Proposition 6.7 we move a step further. We denote by E_0 the set of indices of the classes Ω_i with distance $\delta = 0$ from the environment for the original matrix A. The set E_1, on the other hand, includes the indices of all sets Ω_i that are not directly connected to the environment but can acquire this property as a result of a state feedback acting on one of its compartments. So, if for each conservative class of the original system, Ω_i, $i \in [1, r]$, whose index i does not belong to E_1, we can find a state feedback that connects Ω_i with some class Ω_j, $j \in E_0 \cup E_1$, then each such class will have distance $\delta = 1$ from the environment, and all the remaining classes Ω_j, $j \in [r+1, s] \setminus (E_0 \cup E_1)$, will have finite distance from the environment in turn, thus ensuring that $A + BK$ is stable.

The following algorithm extends the idea of Proposition 6.7 as follows. At the initial step, it considers all the sets Ω_i that either have direct outflow to the environment

---

[3]Note that we find it convenient to refer to the sets Ω_i, $i \in [1, s]$, but the result would hold true for any other partition of the set of compartments $[1, n]$, provided that all the compartments in each Ω_i communicate with each other.

($i \in E_0$) or can gain direct outflow to the environment by means of a state-feedback ($i \in E_1$). If in this way we have considered all the sets ($E_0 \cup E_1 = [1, s]$), then a solution is immediately provided[4]. If there are sets whose indices do not belong to $E_0 \cup E_1$, namely the set of indices of the "remaining sets" $R^{(0)}$ is not empty, then we consider first the sets Ω_i, $i \in R^{(0)}$, that have access to some set Ω_j, $j \in E_0 \cup E_1$, (and hence distance $\delta = 1$ from the environment). We let $N^{(1)}$ denote the set of indices of these sets, and consider now the sets Ω_i, $i \in R^{(0)} \setminus N^{(1)}$, for which the access to some set Ω_j, $j \in E_0 \cup E_1$, can be obtained by means of a state-feedback. We let $E_2^{(1)}$ denote the set of such indices. The union set $N^{(1)} \cup E_2^{(1)}$ represents the set of indices of all sets Ω_i that have distance $\delta = 1$ from the environment either because of the structure of A or as a result of a state-feedback. Subsequently we update the distance δ to the value 2, update the index set of the remaining Ω_i's (now $R^{(1)}$) by subtracting $N^{(1)}$ and $E_2^{(1)}$ from $R^{(0)}$, and determine the sets $N^{(2)}$ and $E_2^{(2)}$. If at some step δ we have emptied the index set $R^{(\delta)}$ of the remaining sets, then the stabilization problem is solvable and a solution is explicitly proposed. If, on the other hand, at some step we have not decreased the cardinality of $R^{(\delta)}$ then the problem is not solvable.

Before stating the algorithm, we recall the assumptions under which we are operating. We assume that $A \in \mathbb{R}^{n \times n}$ is a non-Hurwitz compartmental matrix in Frobenius normal form (6.4), with scalar or irreducible diagonal blocks $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $\lambda_F(A_{ii}) = 0$ for every $i \in [1, r]$, $r \geq 1$, and $\lambda_F(A_{ii}) < 0$ for every $i \in [r+1, s]$. The sets Ω_i, $i \in [1, s]$, are defined as in (6.5) and B is block-partitioned as in (6.8).

Algorithm 1:

A0. Define the sets E_0 and E_1 as in Proposition 6.7. To every $i \in E_1$ we associate the pair $(\mathbf{v}_i, \ell_i)$, where $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \Omega_i$ are such that $S_{\ell_i} A \mathbf{e}_{\ell_i} + S_{\ell_i} B \mathbf{v}_i \geq 0$ and $\mathbf{1}_n^\top B \mathbf{v}_i < 0$.

Initialize $\delta = 0$, $\tilde{E}_2^{(\delta)} = E_2^{(\delta)} = \emptyset$ and $R^{(\delta)} = [1, s] \setminus \left(E_0 \cup E_1 \cup \tilde{E}_2^{(\delta)} \right)$.

A1. If $R^{(\delta)} = \emptyset$, STOP. Then the stabilization problem is solvable and for every $\varepsilon \in (0, 1)$ the feedback matrix

$$K := \varepsilon \sum_{i \in E_1 \cup E_2^{(0)} \cup \cdots \cup E_2^{(\delta)}} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top$$

is a solution.

If $R^{(\delta)} \neq \emptyset$, go to A2.

---

[4]This represents a special case of Proposition 6.7: the case when, by means of a state-feedback, we can ensure that all sets Ω_i, $i \in [1, s]$, have direct outflow to the environment, i.e., distance $\delta = 0$.

A2. Define $N^{(\delta+1)} = \left\{ i \in R^{(\delta)} \colon A_{d_i i} \neq 0, \exists d_i \in E_0 \cup E_1 \cup \tilde{E}_2^{(0)} \cup \cdots \cup \tilde{E}_2^{(\delta)} \right\} \subseteq [r+1, s]$.

Set $E_2^{(\delta+1)} = \left\{ i \in R^{(\delta)} \setminus N^{(\delta+1)} \colon \exists \mathbf{v}_i \in \mathbb{R}^m, \ell_i \in \Omega_i \text{ s.t. conditions (6.14) hold} \right\}$,

$$S_{\ell_i} A \mathbf{e}_{\ell_i} + S_{\ell_i} B \mathbf{v}_i \geq 0, \tag{6.14a}$$

$$\mathbf{1}_n^\top B \mathbf{v}_i = 0, \tag{6.14b}$$

$$B_{d_i} \mathbf{v}_i > 0, \quad \exists d_i \in E_0 \cup E_1 \cup \tilde{E}_2^{(0)} \cup \cdots \cup \tilde{E}_2^{(\delta)}. \tag{6.14c}$$

To every $i \in E_2^{(\delta+1)}$ we associate the pair $(\mathbf{v}_i, \ell_i)$, where $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \Omega_i$ satisfy equations (6.14). Set $\tilde{E}_2^{(\delta+1)} = E_2^{(\delta+1)} \cup N^{(\delta+1)}$.

A3. Set $R^{(\delta+1)} = R^{(\delta)} \setminus \tilde{E}_2^{(\delta+1)}$. If $R^{(\delta+1)} = R^{(\delta)}$, STOP. The system is not stabilizable.

Otherwise, update $\delta = \delta + 1$ and repeat from A1.

Remark 6.4. The algorithm could be made more efficient if at the step A2, when determining $E_2^{(\delta+1)}$, we would consider among the set of classes for which we have already guaranteed a finite distance from the environment also the classes indexed in $N^{(\delta+1)}$, namely if we would check condition (6.14c) for $d_i \in E_0 \cup E_1 \cup \tilde{E}_2^{(0)} \cup \cdots \cup \tilde{E}_2^{(\delta)} \cup N^{(\delta+1)}$. However, in doing so, δ would no longer represent the distance of the class from the environment.

It is clear, by the way the algorithm has been conceived, that it always comes to an end in no more than $s - 1$ steps (namely, δ cannot be greater than $s - 1$). To prove that the stabilization problem is solvable if and only if the algorithm ends with $R^{(\delta)} = \emptyset$ for some $\delta \in \mathbb{N}$, we need the following result.

Lemma 6.8. *Consider a Multi-Input Compartmental System described as in* (6.1)*, and suppose that A is non-Hurwitz. Assume without loss of generality that A is in the Frobenius normal form* (6.4)*, with scalar or irreducible diagonal blocks $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, and the index $r \in [1, s]$, $r \geq 1$, is such that $\lambda_F(A_{ii}) = 0$ for every $i \in [1, r]$, and $\lambda_F(A_{ii}) < 0$ for every $i \in [r + 1, s]$. Also, let the sets Ω_i, $i \in [1, s]$, be defined as in* (6.5)*, and assume that B is block-partitioned as in* (6.8)*. If the state-feedback stabilization problem is solvable, then there exist $\mathbf{v}_i \in \mathbb{R}^m$ and $\ell_i \in \Omega_i$, $i \in [1, s]$, such that for every $\varepsilon \in (0, 1)$ the matrix*

$$K^* = \varepsilon \sum_{i=1}^{s} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top = \begin{bmatrix} K_1^* & \ldots & K_s^* \end{bmatrix} \tag{6.15}$$

makes $A + BK^*$ compartmental and Hurwitz, and for every $i \in [1, s]$ the diagonal blocks $A_{ii} + B_i K_i^*$ are irreducible.

Proof. Let $K \in \mathbb{R}^{m \times n}$ be an arbitrary solution to the state-feedback stabilization problem. Define the matrix $K' = \begin{bmatrix} K'_1 & \ldots & K'_s \end{bmatrix} \in \mathbb{R}^{m \times n}$, column by column, as follows:

$$K'\mathbf{e}_h := \begin{cases} 0, & \text{if } \mathbf{1}_n^\top A\mathbf{e}_h < 0, \\ \varepsilon K\mathbf{e}_h, & \text{if } \mathbf{1}_n^\top A\mathbf{e}_h = 0, \end{cases}$$

where ε is arbitrary in $(0,1)$. We first observe that for every $h \in [1,n]$ (and for every $\varepsilon \in (0,1)$) $S_h(A + BK')\mathbf{e}_h \geq 0$ and

$$\overline{\mathrm{ZP}}(S_h(A + BK')\mathbf{e}_h) \supseteq \overline{\mathrm{ZP}}(S_h A\mathbf{e}_h). \tag{6.16}$$

This implies, in particular, that for every $i \in [1,s]$ and every $h_i \in [1,n_i]$

$$\overline{\mathrm{ZP}}(S_{h_i}(A_{ii} + B_i K'_i)\mathbf{e}_{h_i}) \supseteq \overline{\mathrm{ZP}}(S_{h_i} A_{ii}\mathbf{e}_{h_i}).$$

This ensures that for every $\varepsilon \in (0,1)$ the matrix K' makes $A + BK'$ Metzler with all diagonal blocks $A_{ii} + B_i K'_i$ irreducible. To prove that $A + BK'$ is still compartmental and Hurwitz, it is sufficient to note that:

1) For every $h \in [1,n]$

$$\mathbf{1}_n^\top (A + BK')\mathbf{e}_h = \begin{cases} < 0, & \text{if } \mathbf{1}_n^\top A\mathbf{e}_h < 0, \\ \leq 0, & \text{if } \mathbf{1}_n^\top A\mathbf{e}_h = 0. \end{cases}$$

   This ensures that $A + BK'$ is compartmental and the set of compartments of $A + BK'$ with direct outflow to the environment includes the set of compartments of $A + BK$ with direct outflow to the environment.

2) By the way K' has been defined, if the hth compartment of $A + BK'$, $h \in [1,n]$, does not have direct outflow to the environment then h is such that $\mathbf{1}_n^\top A\mathbf{e}_h = 0$ and hence $K'\mathbf{e}_h = \varepsilon K\mathbf{e}_h$. Therefore $(A + BK')\mathbf{e}_h = (A + \varepsilon BK)\mathbf{e}_h$, and condition $\varepsilon \in (0,1)$ ensures that all the arcs from the hth compartment that appear in the digraph associated with $A + BK$ also appear in the digraph associated with $A + BK'$.

So, the Hurwitz property of $A + BK$ ensures that all the compartments in $A + BK$ are outflow connected, and this property is preserved in $A + BK'$, thus ensuring that $A + BK'$ is Hurwitz, too.

We initialize the matrix K^* by assuming $K^* := K'$. We then proceed as follows:

- Set $D_0 := \left\{ i \in [1,s] \colon \mathbf{1}_n^\top (A + BK')\mathbf{e}_{\ell_i} < 0, \exists \ell_i \in \Omega_i \right\}$. This is the set of indices of the classes in $A + BK^* = A + BK'$ that have direct outflow to the environment. For every $i \in D_0$, we select one such ℓ_i and impose that for every $h \in \Omega_i$, $h \neq \ell_i$, $K^* \mathbf{e}_h = 0$. Note that after this change D_0 still represents the set of indices of the classes in $A + BK^*$, for the updated K^*, that have direct outflow to the environment.

- Set $D_1 := \{i \in [1,s] \setminus D_0 \colon A_{ji} + B_j K'_i \neq 0, \exists j \in D_0\}$. If $i \in D_1$, this means that there exist $j \in D_0$, $\ell_i \in \Omega_i$ and $q \in \Omega_j$ such that $[A + BK']_{q\ell_i} \neq 0$. For every $i \in D_1$, we set $K^* \mathbf{e}_h = 0$ for every $h \in \Omega_i$, $h \neq \ell_i$. Also in this case D_1 represents the set of indices of classes in $A + BK^*$ that have distance one from the environment, both before and after having updated K^*.

- By proceeding in this way, we determine all the sets D_δ and we set to zero the columns of K^* accordingly. Since $A + BK'$ is Hurwitz all the sets Ω_i have finite distance from the environment and this property is preserved in all the subsequent modifications that lead to the final K^*. By the way we have proceeded, K^* has the structure (6.15), and the nonzero vectors $\mathbf{v}_i$ coincide with $K\mathbf{e}_{\ell_i}$.

$\blacksquare$

Theorem 6.9. *Consider a Multi-Input Compartmental System described as in* (6.1), *and suppose that A is non-Hurwitz. Assume without loss of generality that A is in the Frobenius normal form* (6.4), *with scalar or irreducible diagonal blocks $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, and the index $r \in [1,s]$, $r \geq 1$, is such that $\lambda_F(A_{ii}) = 0$ for every $i \in [1,r]$, and $\lambda_F(A_{ii}) < 0$ for every $i \in [r+1, s]$. Also, let the sets Ω_i, $i \in [1,s]$, be defined as in* (6.5), *and assume that B is block-partitioned as in* (6.8). *The state-feedback stabilization problem is solvable if and only if Algorithm 1 applied to the pair (A, B) ends with $R^{(\delta)} = \emptyset$ for some $\delta > 0$. When so, the feedback matrix K generated by the algorithm (see step A1) represents a solution.*

Proof. [Sufficiency] The proof trivially follows from the fact that if the Algorithm ends with $R^{(\delta)} = \emptyset$, then the final closed-loop matrix $A + BK$ is compartmental, and for every compartment $h \in [1,n]$ of the matrix $A + BK$ we have been able to either guarantee that it belongs to a set that has direct outflow to the environment (this is the case if $h \in \Omega_i$, for some $i \in E_0 \cup E_1$) or to ensure that it belongs to a set Ω_i that has finite distance from the environment (this is the case if $i \in \tilde{E}_2^{(\delta)}$ for some $\delta \geq 1$). Therefore in

the closed-loop system, all the compartments are outflow connected, and this ensures the Hurwitz property.

[Necessity] By Lemma 6.8 if the state-feedback stabilization problem is solvable, then there exists a solution taking the structure (6.15). The Algorithm constructs, among all possible solutions (6.15), one in which each set Ω_i, $i \in [1, s]$, has in the resulting closed-loop system $A + BK$ the minimum possible distance from the environment. So, if the Algorithm does not end with a solution, a solution (6.15) does not exist. ∎

Example 6.6. Consider the following Multi-Input Compartmental System

$$
\dot{\mathbf{x}}(t) = \left[\begin{array}{ccc|cc|c}
-2 & 1 & 1 & 1 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -2 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & -2
\end{array}\right] \mathbf{x}(t) + \left[\begin{array}{cc}
0 & 1 \\
0 & 0 \\
0 & 0 \\
1 & 1 \\
0 & 0 \\
1 & 0
\end{array}\right] \mathbf{u}(t)
$$

$$
= \left[\begin{array}{ccc}
A_{11} & A_{12} & A_{13} \\
0 & A_{22} & A_{23} \\
0 & 0 & A_{33}
\end{array}\right] \mathbf{x}(t) + \left[\begin{array}{c}
B_1 \\
B_2 \\
B_3
\end{array}\right] \mathbf{u}(t).
$$

Notice that A is in Frobenius normal form (5.9), with diagonal block A_{11} irreducible and non-Hurwitz, and diagonal blocks A_{22} and A_{33} irreducible and Hurwitz. So, $s = 3$ and $r = 1$. We apply Algorithm 1 to the pair (A, B) and determine, if possible, a solution to the stabilization problem:

$\delta = 0$: We set $E_0 = \emptyset$, $E_1 = \{2\}$, and we associate the pair $(\mathbf{v}_2, \ell_2) := \left(\begin{bmatrix} 1/4 & -1/2 \end{bmatrix}^\top, 4\right)$ to $i = 2$. We initialize $\tilde{E}_2^{(0)} = E_2^{(0)} = \emptyset$ and $R^{(0)} = \{1, 3\}$.

We define $N^{(1)} = \{3\}$, $E_2^{(1)} = \emptyset$, and $\tilde{E}_2^{(1)} = E_2^{(1)} \cup N^{(1)} = \{3\}$.

We set $R^{(1)} = R^{(0)} \setminus \tilde{E}_2^{(1)} = \{1\}$, and since $R^{(1)} \subset R^{(0)}$ we update δ.

$\delta = 1$: We define $N^{(2)} = \emptyset$ and $E_2^{(2)} = \{1\}$. We associate the pair $(\mathbf{v}_1, \ell_1) := \left(\begin{bmatrix} 1 & -1 \end{bmatrix}^\top, 1\right)$ to $i = 1$. We set $\tilde{E}_2^{(2)} = E_2^{(2)} \cup N^{(2)} = \{1\}$.

We set $R^{(2)} = R^{(1)} \setminus \tilde{E}_2^{(2)} = \emptyset$. Since $R^{(2)} = \emptyset$ the stabilization problem is solvable and for every $\varepsilon \in (0, 1)$ the feedback matrix

$$
K := \varepsilon \sum_{i \in E_1 \cup E_2^{(1)} \cup E_2^{(2)}} \mathbf{v}_i \mathbf{e}_{\ell_i}^\top = \begin{bmatrix}
1 & 0 & 0 & \frac{1}{4} & 0 & 0 \\
-1 & 0 & 0 & -\frac{1}{2} & 0 & 0
\end{bmatrix}
$$

is a possible solution. ♣

Example 6.7. Consider the following Multi-Input Compartmental System

$$
\dot{\mathbf{x}}(t) = \left[\begin{array}{cc|cc|cc|c}
-1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -3 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -2
\end{array}\right] \mathbf{x}(t) + \left[\begin{array}{cc}
0 & 2 \\
0 & 0 \\
\hline
1 & 0 \\
0 & 0 \\
\hline
1 & 1 \\
0 & 0 \\
\hline
1 & 0
\end{array}\right] \mathbf{u}(t)
$$

$$
= \begin{bmatrix}
A_{11} & 0 & A_{13} & A_{14} \\
0 & A_{22} & A_{23} & 0 \\
0 & 0 & A_{33} & A_{34} \\
0 & 0 & 0 & A_{44}
\end{bmatrix} \mathbf{x}(t) + \begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4
\end{bmatrix} \mathbf{u}(t).
$$

Notice that A is in Frobenius normal form (6.4), with diagonal blocks A_{11} and A_{22} irreducible and non-Hurwitz, and diagonal blocks A_{33} and A_{44} irreducible and Hurwitz. So, $s = 4$ and $r = 2$. We apply Algorithm 1 to the pair (A, B) and determine, if possible, a solution to the stabilization problem:

$\delta = 0$: We set $E_0 = \emptyset$, $E_1 = \{3\}$, and we associate the pair $(\mathbf{v}_3, \ell_3) := \left(\begin{bmatrix} 1/4 & -1/2 \end{bmatrix}^\top, 5\right)$ to $i = 3$. We initialize $\tilde{E}_2^{(0)} = E_2^{(0)} = \emptyset$ and $R^{(0)} = \{1, 2, 4\}$.

We define $N^{(1)} = \{4\}$, $E_2^{(1)} = \emptyset$, and $\tilde{E}_2^{(1)} = E_2^{(1)} \cup N^{(1)} = \{4\}$.

We set $R^{(1)} = R^{(0)} \setminus \tilde{E}_2^{(1)} = \{1, 2\}$, and since $R^{(1)} \subset R^{(0)}$ we update δ.

$\delta = 1$: We define $N^{(2)} = \emptyset$ and $E_2^{(2)} = \{1\}$. We associate the pair $(\mathbf{v}_1, \ell_1) := \left(\begin{bmatrix} 1 & -1 \end{bmatrix}^\top, 1\right)$ to $i = 1$. We set $\tilde{E}_2^{(2)} = E_2^{(2)} \cup N^{(2)} = \{1\}$.

We set $R^{(2)} = R^{(1)} \setminus \tilde{E}_2^{(2)} = \{2\}$, and since $R^{(2)} \subset R^{(1)}$ we update δ.

$\delta = 2$: We define $N^{(3)} = \emptyset$, $E_2^{(3)} = \emptyset$ and $\tilde{E}_2^{(3)} = E_2^{(3)} \cup N^{(3)} = \emptyset$.

We set $R^{(3)} = R^{(2)} \setminus \tilde{E}_2^{(3)} = \{2\}$. Since $R^{(3)} = R^{(2)}$ we stop and conclude that the stabilization problem is not solvable. ♣

7

Control of Positive Multi-Agent Systems: the Positive Consensus Problem

In this chapter we consider Positive Multi-Agent Systems, i.e., systems composed of multiple interacting agents, each one modelled by a Positive System. Under the assumptions that the agents are homogeneous and affected by a single (scalar) input, and that the mutual interactions among them are cooperative, we aim at determining a state-feedback law that each agent can individually implement in a distributed form, in such a way that consensus is achieved and the positivity of the overall state trajectories is preserved.

The content of the present chapter is largely based on:

Valcher M. E. and Zorzan I. On the consensus problem with positivity constraints. In *Proceedings of the 2016 American Control Conference*, pages 2846–2851, Boston, MA, 2016b;

Valcher M. E. and Zorzan I. New results on the solution of the positive consensus problem. In *Proceedings of the 55th IEEE Conference on Decision and Control*, pages 5251–5256, Las Vegas, Nevada, USA, 2016a;

Valcher M. E. and Zorzan I. On the consensus of homogeneous multi-agent systems with positivity constraints. *IEEE Transactions on Automatic Control*, 62 (10): 5096–5110, 2017b;

Valcher M. E. and Zorzan I. Positive consensus problem: the case of complete

communication. In **F. Cacace R. S., L. Farina and Germani A.**, Editors, *Positive Systems. Lecture Notes in Control and Information Sciences*, Chapter 19, pages 239–252. Springer, 2017d.

From consensus for general multi-agent systems to Positive Consensus

Multi-agent systems and consensus problems have been very lively research topics in the last decade. Early contributions on these subjects date back to the seventies, DeGroot (1974), followed by a few additional contributions in the eighties and nineties, Tsitsiklis (1984); Smith (1995), but it was only ten years ago that milestone papers like Lin, Morse, and Anderson (2003); Fax and Murray (2004); Ren and Beard (2005); Olfati-Saber, Fax, and Murray (2007), stimulated a wide interest in these problems within the Systems and Control community. The reason for such an impressive success must be credited to the number of diverse and challenging practical problems that can be formalized as consensus problems among agents: flocking and swarming in animal groups, dynamics of opinion forming, coordination in sensor networks, clock synchronization, distributed tasks among mobile robots/vehicles. In all these contexts, the common ingredient is the existence of different individuals/units (the agents), each of them performing a task and communicating with its neighbours to achieve a common goal. Consensus algorithms propose distributed control strategies that each agent implements, based on the received information, to reach a common target, to converge to some common value or set of values (see, e.g., Ren and Beard (2005); Hui and Haddad (2008); Scardovi and Sepulchre (2009)).

In a number of these contexts, the information that the agents acquire and update, based on the communication with the other agents, and on which they search for consensus, is the value of variables that are intrinsically nonnegative. This is the case, for instance, when dealing with wireless sensor networks in greenhouses, Chaudhary, Nayse, and Waghmare (2011), since the parameters that the sensors measure and exchange are light intensity, humidity, and CO_2 concentration, and the sensors must converge to some common values for these parameters, based on which shading or artificial lights will be controlled, watering/heating systems will be activated, CO_2 will be injected, and so on.

Another interesting problem, formalized as a consensus problem with positivity constraint, is the emissions control for a fleet of Plugin Hybrid Vehicles, Knorn, Corless, and Shorten (2011). Each vehicle has a parallel power-train configuration that allows for any arbitrary combination of the power generated by the combustion engine and the electric motor. Moreover, the vehicles can communicate. Under these assumptions, an algorithm was proposed in Knorn et al. (2011) to regulate in a cooperative way the CO_2

emissions, so that no vehicle has a higher emission level than the others.

Finally, the distributed multi-vehicle coordination problem through local information exchange investigated in Ren (2007); Ren and Atkins (2007) is an example of consensus problem among agents (the vehicles) whose dynamics is described by a linear positive state-space model.

In addition to the previous contributions, specifically addressing the positive consensus problem, there have been a number of contributions dealing with properties and performances (stability, stabilization, L_1-gain, optimal and distributed control) of Positive Multi-Agent Systems and more generally of interconnected Positive Systems, Rantzer (2011); Ebihara, Peaucelle, and Arzelier (2012a); Ebihara et al. (2013); Ebihara, Peaucelle, and Arzelier (2014).

Stimulated by this stream of research and by the aforementioned application problems, we have investigated the consensus problem for homogeneous multi-agent systems, whose agents are described by a continuous-time, single-input, positive state-space model. The agents are supposed to be cooperative and the communication graph describing the agents' mutual interactions is an undirected, weighted and connected graph. Also, agents adopt a distributed state-feedback control strategy, based on the information available on the states of their neighbouring agents. As the agents' states are intrinsically nonnegative, a natural requirement to introduce, in addition to consensus, is the positivity of the overall controlled multi-agent system. This amounts to saying that the state feedback law adopted to achieve consensus must constrain the state trajectories to remain in the positive orthant. The consensus problem under positivity constraint was first addressed in Valcher and Misra (2013), under quite different assumptions on the communication structure, the feedback control law and the final goal. Indeed, first of all no communication graph was introduced and an n-dimensional supervisory output-feedback controller, instead of a distributed state-feedback control law, was adopted. This implies, in particular, that each agent was assumed to interact with all the other agents. Finally, consensus was imposed (for all the nonnegative initial conditions) to a common constant value, a requirement that constrains the spectrum of the state matrix of the agents' state-space description and has here been removed.

7.1   Positive consensus: problem statement

Consider a group of N identical agents, each of them described by the same n-dimensional continuous-time Single-Input Positive System:

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t), \quad t \in \mathbb{R}_+, \tag{7.1}$$

where $\mathbf{x}_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$ are the state vector and the input of the ith agent, respectively, $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is a non-Hurwitz Metzler matrix, and $B = [b_i] \in \mathbb{R}_+^n$ is a positive vector. The communication among the N agents is described by an undirected, weighted and connected communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, with $\mathcal{V} = [1, N]$ and $\mathcal{A} = \mathcal{A}^\top \in \mathbb{R}_+^{N \times N}$. The positivity assumption on the adjacency matrix corresponds to assuming that the interactions between pairs of agents are cooperative.

Consider the state-feedback control law[1]:

$$u_i(t) = K \sum_{j=1}^{N} [\mathcal{A}]_{ij} \left[\mathbf{x}_j(t) - \mathbf{x}_i(t) \right],$$

where $K \in \mathbb{R}^{1 \times n}$ is a feedback matrix to be designed. If we denote by $\mathbf{x}(t) \in \mathbb{R}^{Nn}$ and $\mathbf{u}(t) \in \mathbb{R}^N$ the state vector and the input vector of the multi-agent system, respectively, i.e.,

$$\mathbf{x}(t) := \begin{bmatrix} \mathbf{x}_1^\top(t) & \dots & \mathbf{x}_N^\top(t) \end{bmatrix}^\top$$
$$\mathbf{u}(t) := \begin{bmatrix} u_1(t) & \dots & u_N(t) \end{bmatrix}^\top,$$

the overall dynamics is described by:

$$\dot{\mathbf{x}}(t) = (I_N \otimes A) \, \mathbf{x}(t) + (I_N \otimes B) \, \mathbf{u}(t)$$
$$\mathbf{u}(t) = - (\mathcal{L} \otimes K) \, \mathbf{x}(t),$$

or equivalently by:

$$\dot{\mathbf{x}}(t) = [(I_N \otimes A) - (I_N \otimes B)(\mathcal{L} \otimes K)] \, \mathbf{x}(t), \tag{7.2}$$

where $\mathcal{L} \in \mathbb{R}^{N \times N}$ is the Laplacian matrix associated with the communication graph $\mathcal{G}$. The consensus problem with positivity constraints, or *positive consensus problem*, can be stated as follows: find a feedback matrix $K \in \mathbb{R}^{1 \times n}$ such that:

I) The overall system (7.2) is positive, i.e., $\mathbb{A} := (I_N \otimes A) - (I_N \otimes B)(\mathcal{L} \otimes K)$ is a

---

[1]This state-feedback control law is known in the literature as De Groot's type law, since the first formal study of consensus is credited to DeGroot (1974). This kind of state-feedback law has been assumed in the literature as the standard consensus algorithm since the early contributions on the subject (see Olfati-Saber et al. (2007); Ren, Beard, and Atkins (2007) and references therein). Even nowadays, possibly with modifications that account for the existence of time-varying communications among the agents or for the fact that relationships among agents may be cooperative or competitive, this is the most common consensus protocol, Wieland, Kim, and Allgöwer (2011); Su and Huang (2012); Xia, Cao, and Johansson (2016).

Metzler matrix;

II) The overall system (7.2) reaches consensus, i.e., $\lim_{t \to +\infty} \mathbf{x}_i(t) - \mathbf{x}_j(t) = 0$ for every $i, j \in [1, N]$, for almost all positive initial conditions.[2]

The first step to determine under what conditions on the agents' representation (7.1) and on the communication graph $\mathcal{G}$, positive consensus can be achieved is the reformulation of the problem in algebraic terms. Before proceeding, we recall some properties of the Laplacian matrix associated with undirected, weighted graphs.

Basic facts on undirected, weighted graphs and their Laplacian matrices

First of all, notice that for any undirected graph $\mathcal{G}$, all rows of the associated Laplacian matrix $\mathcal{L}$ sum up to 0, and hence $\mathbf{1}_N$ is always a right eigenvector of $\mathcal{L}$ corresponding to the zero eigenvalue. Secondly, recalling that a Metzler matrix is irreducible if and only if the associated graph is connected (see Chapter 3, Section 3.1), it follows that for a connected graph $\mathcal{G}$ both the adjacency matrix $\mathcal{A}$ and the Laplacian $\mathcal{L}$ are irreducible. The connectedness assumption on $\mathcal{G}$ also implies that the weighted degree of each of its vertices is greater than zero, i.e., $\ell_{ii} > 0$ for every $i \in [1, N]$.

We denote by λ_i, $i \in [1, N]$, the eigenvalues of $\mathcal{L}$ sorted in non-decreasing order, namely as $\lambda_i \leq \lambda_{i+1}$ for every $i \in [1, N-1]$. The following result provides a useful characterization of the spectrum of $\mathcal{L}$, Ren and Beard (2005); Wieland et al. (2011).

Lemma 7.1. *Let $\mathcal{G}$ be an undirected, weighted graph with N vertices. If $\mathcal{G}$ is connected, then the Laplacian $\mathcal{L}$ is a symmetric positive semidefinite matrix, and 0 is a simple eigenvalue of $\mathcal{L}$. As a consequence, all eigenvalues of $\mathcal{L}$ are real and*

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N. \tag{7.3}$$

We now characterize the relationship between the maximal weighted degree $\ell^* := \max_{i \in [1,N]} \ell_{ii}$ and the Laplacian eigenvalues. Lemma 7.2 is a straightforward application of the results provided by Fu and Markham (1993) for Hermitian matrices.

---

[2] Requiring that consensus is achieved for almost all initial conditions is a standard set-up, see, e.g., Proskurnikov, Matveev, and Cao (2016); Xia et al. (2016), to the point that often it is not even mentioned. In the special case of the positive consensus problem, we restrict our attention to positive initial conditions, and initial conditions for which consensus may not be achieved necessarily belong to the boundary of the positive orthant. The case when all nonnegative initial conditions lead to consensus would require to introduce the irreducibility assumption on both the matrix A and the matrix $\mathbb{A}$, see Valcher and Misra (2013), constraints that seem unnecessary.

Lemma 7.2. *Let $\mathcal{G}$ be an undirected, weighted graph with N vertices and positive adjacency matrix $\mathcal{A} \in \mathbb{R}_+^{N \times N}$. Then, the following inequalities hold: $0 < \ell^* < \lambda_N$.*

As we will see, the order relation between ℓ^* and λ_2 plays a crucial role when looking for a solution to the positive consensus problem. Such a relationship is characterized in Lemma 7.3 below (see Fiedler (1973); Mohar (1991); Goldberg (2006) for statement *i*); and Pejic (2008) for statement *ii*)).

Lemma 7.3. *i) Let $\mathcal{G}$ be an undirected, unweighted graph with N vertices. Then, $\ell^* < \lambda_2$ if and only if $\mathcal{G}$ is complete, i.e., $[\mathcal{A}]_{ij} = 1$ for every $i, j \in \mathcal{V}$, $i \neq j$. When so, $\ell^* = N - 1$ and $\lambda_i = N$ for every $i \in [2, N]$.*
ii) Let $\mathcal{G}$ be an undirected, weighted graph with N vertices. If $\ell^* < \lambda_2$, then $\mathcal{G}$ is complete.

It is worth noticing that completeness of an undirected, weighted graph does not imply $\ell^* < \lambda_2$. Consider for instance the complete graph corresponding to the Laplacian matrix

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & 3 \end{bmatrix}.$$

The maximal weighted degree is $\ell^* = 3$, the smallest nonzero eigenvalue is $\lambda_2 = 3$, and hence $\ell^* = \lambda_2$.

Positive consensus: a simultaneous constrained stabilization problem

The positive consensus problem can be restated in algebraic terms, as proved in the following proposition.

Proposition 7.4. *Define the matrix $K^* = [k_i^*] \in \mathbb{R}_+^{1 \times n}$ as[3]:*

$$k_i^* := \begin{cases} \min\limits_{\substack{j \in [1,n] \\ j \neq i}} \dfrac{a_{ji}}{b_j} \dfrac{1}{\ell^*}, & \text{if } \exists j \neq i \text{ s.t. } b_j \neq 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, the positive consensus problem is solvable if and only if there exists a matrix $K \in \mathbb{R}_+^{1 \times n}$ such that $0 \leq K \leq K^*$ and all matrices $A - \lambda_i B K$, $i \in [2, N]$, are Hurwitz.

---

[3]Note that the only situation when K^* is not a finite row vector is when $B = b\mathbf{e}_i$ for some $i \in [1, n]$, and if so the only infinite entry is $k_i^* = +\infty$.

Proof. The positive consensus problem is solvable if and only if conditions I) and II) hold. As far as requirement I) is concerned, we notice that $\mathbb{A}$ takes the following form:

$$\mathbb{A} = \begin{bmatrix} A - \ell_{11}BK & -\ell_{12}BK & \dots & -\ell_{1N}BK \\ -\ell_{12}BK & A - \ell_{22}BK & \dots & -\ell_{2N}BK \\ \vdots & \vdots & \ddots & \vdots \\ -\ell_{1N}BK & -\ell_{2N}BK & \dots & A - \ell_{NN}BK \end{bmatrix}$$

and hence $\mathbb{A}$ is Metzler if and only if (a) all blocks $A - \ell_{ii}BK$, $i \in [1, N]$, are Metzler and (b) all blocks $-\ell_{ij}BK$, $i, j \in [1, N]$, $i \neq j$, are nonnegative. Since $\ell_{ij} \leq 0$ for every $i, j \in [1, N]$, $i \neq j$, and they cannot be all zero (if so $\mathcal{A}$ would be the null matrix), condition (b) holds if and only if $BK \geq 0$, but since B is a positive column vector, this amounts to saying that $K \geq 0$.

On the other hand, by the definition of K^*, condition (a) holds if and only if $K \leq K^*$. Therefore, K makes $\mathbb{A}$ Metzler, namely condition I) holds, if and only if $0 \leq K \leq K^*$.

Finally, for requirement II) we can rely on a classical result about consensus, Fax and Murray (2004); Wieland et al. (2011): a necessary and sufficient condition for the agents to achieve consensus is that all matrices $A - \lambda_i BK$, $i \in [2, N]$, are Hurwitz. This completes the proof. ∎

Notice that Hurwitz stability of all matrices $A - \lambda_i BK$, $i \in [2, N]$, requires the pair (A, B) to be stabilizable, and hence in the following we will always make this assumption. Note, also, that this is a special case of simultaneous stabilization problem, since we need to simultaneoulsy stabilize all the pairs $(A, \lambda_i B)$, $i \in [2, N]$, by resorting to state feedback matrices that belong to the hypercube of vertices 0 and K^*.

7.2   Preliminary analysis: a necessary condition

As a first step, we want to understand under what conditions on the structure of the matrices A and B the positive consensus problem is solvable. To this end, we preliminarily assume that the Metzler matrix A is in Frobenius normal form (3.1) and the positive vector B is partitioned consistently with A, namely

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1s} \\ 0 & A_{22} & \dots & A_{2s} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & A_{ss} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_s \end{bmatrix}, \tag{7.4}$$

$$A - \ell^* B K^* =$$

$$= \left[\begin{array}{cccc|ccc}
A_{11}-\ell^* B_1 K_1^* & A_{12}-\ell^* B_1 K_2^* & \ldots & A_{1r}-\ell^* B_1 K_r^* & A_{1r+1}-\ell^* B_1 K_{r+1}^* & \ldots & A_{1s}-\ell^* B_1 K_s^* \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-\ell^* B_r K_1^* & -\ell^* B_r K_2^* & \ldots & A_{rr}-\ell^* B_r K_r^* & A_{rr+1}-\ell^* B_r K_{r+1}^* & \ldots & A_{rs}-\ell^* B_r K_s^* \\
\hline
 & & & & A_{r+1r+1} & \ldots & A_{r+1s} \\
 & & & & \vdots & \ddots & \vdots \\
 & & & & 0 & \ldots & A_{ss}
\end{array} \right] \quad (7.5)$$

where $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, s]$, are either scalar ($n_i = 1$) or irreducible matrices, and $B_i \in \mathbb{R}_+^{n_i}$. Again, this is a not restrictive assumption, since we can always reduce ourselves to this situation by resorting to a suitable permutation matrix Π, and hence moving from the pair (A, B) to the pair $\left(\Pi^\top A \Pi, \Pi^\top B \right)$. It turns out that, in order for the positive consensus problem to be solvable, only one of the irreducible diagonal blocks A_{ii} can be non-Hurwitz. Specifically, we have the following result.

Proposition 7.5. *Assume without loss of generality that the Metzler matrix A and the positive vector B are described as in (7.4), where $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, s]$, are either scalar or irreducible matrices, and $B_i \in \mathbb{R}_+^{n_i}$. Set $r := \max \{i \in [1, s] \colon B_i \neq 0\}$. If the positive consensus problem is solvable, then A_{ii} is (Metzler and) Hurwitz for every $i \neq r$.*

Proof. Any matrix $K \in \mathbb{R}_+^{1 \times n}$, with $0 \leq K \leq K^*$, can be partitioned in a way consistent with A and B, namely as $K = \begin{bmatrix} K_1 & K_2 & \ldots & K_s \end{bmatrix}$, with $K_j \in \mathbb{R}_+^{1 \times n_j}$. By the definition of K^*, $A - \ell^* B K^*$ is necessarily Metzler and takes the block-triangular form given in (7.5). If $r > 1$ the only way for this matrix to be Metzler is that $-\ell^* B_r K_j^* = 0$ for every $j \in [1, r-1]$, and since $\ell^* > 0$ and $B_r > 0$, this means that $K_j^* = 0$ for every $j \in [1, r-1]$ (if $r = 1$ the result is trivially true). So, if $K \in \mathbb{R}_+^{1 \times n}$, $0 \leq K \leq K^*$, is any solution to the positive consensus problem, then all its blocks K_j, $j \in [1, r-1]$, must be zero. Consequently, each matrix $A - \lambda_i B K$, $i \in [2, N]$, takes the same block triangular form as A, with each diagonal block $A_{jj} - \lambda_i B_j K_j$, $j \neq r$, coinciding with the corresponding diagonal block A_{jj} in A. So, the Hurwitz property of $A - \lambda_i B K$, $i \in [2, N]$, implies that all the diagonal blocks A_{jj}, $j \neq r$, are (Metzler and) Hurwitz. ∎

The following corollary immediately follows from the previous Proposition 7.4 and Proposition 7.5.

Corollary 7.6. *Assume without loss of generality that the Metzler matrix A and the positive vector B are described as in (7.4), where $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i \in [1, s]$, are either scalar or irreducible matrices, and $B_i \in \mathbb{R}_+^{n_i}$. Define the matrix $K^* \in \mathbb{R}_+^{1 \times n}$ as in Proposition 7.4, and partition it accordingly to the partition of A and B. Set $r := \max\{i \in [1, s]: B_i \neq 0\}$ and let $K_r^* \in \mathbb{R}_+^{1 \times n_r}$ be the rth block of K^*. The positive consensus problem is solvable if and only if the following conditions hold:*

  i) A_{ii} is (Metzler and) Hurwitz for every $i \neq r$;

  ii) There exists a matrix $K_r \in \mathbb{R}_+^{1 \times n_r}$, $0 \leq K_r \leq K_r^*$, that makes the matrices $A_{rr} - \lambda_i B_r K_r$, $i \in [2, N]$, Hurwitz.

If the previous conditions hold, then the row matrix $K \in \mathbb{R}_+^{1 \times n}$, having K_r as rth block and all remaining blocks equal to zero, is a solution.

Remark 7.1. The previous corollary entails far rich consequences, since it tells us that once the non-Hurwitz Metzler matrix A is brought to Frobenius normal form (7.4), then the solvability of the positive consensus problem requires to first check that all the diagonal blocks of $\Pi^\top A \Pi$ are (Metzler and) Hurwitz, except for the (scalar or irreducible) diagonal block A_{rr}, and then to investigate the positive consensus problem for the pair (A_{rr}, B_r), by assuming as upper bound on the vector $K_r \in \mathbb{R}_+^{1 \times n_r}$, the largest positive vector $K_r^* \in \mathbb{R}_+^{1 \times n_r}$ such that

$$\begin{cases} A_{rr} - \ell^* B_r K_r & \text{is Metzler} \\ A_{jr} - \ell^* B_j K_r \geq 0, & \forall j \in [1, r-1]. \end{cases}$$

Example 7.1. Consider the following single-input positive state-space model for the generic agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t)$$

$$= \begin{bmatrix} -1 & 0 & 1 & 4 & 1 & 1 \\ 0 & -2 & 1 & 4 & 1 & 1 \\ 0 & 1 & -3 & 2 & \frac{1}{2} & 2 \\ 0 & 0 & 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 3 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} u_i(t).$$

The pair (A, B) is stabilizable. A is in Frobenius normal form (7.4), with $s = 4$, $n_1 = n_4 = 1$, $n_2 = n_3 = 2$, and the only non-Hurwitz diagonal block is the one associated

with the last nonzero block in B. Specifically, $r = 3$ and

$$A_{33} = \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore, condition i) in Corollary 7.6 is satisfied. Assume that there are $N = 3$ agents and that the Laplacian matrix of the communication graph is the following one:

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

In this case $\ell^* = 2$, the eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0 < \lambda_2 = 1 < \lambda_3 = 3$ and $K^* = \begin{bmatrix} 0 & | & 0 & 0 & | & 1 & \frac{1}{4} & | & \frac{1}{4} \end{bmatrix}$. We note, however, that

$$A_{33} - \ell^* B_3 K_3^* = \begin{bmatrix} -3 & 1/2 \\ 1 & -3/2 \end{bmatrix},$$

so it is true that $A_{33} - \ell^* B_3 K_3^*$ is Metzler, but there exist matrices $K_3 > K_3^*$ such that $A_{33} - \ell^* B_3 K_3$ is Metzler, too. It is easy to see that the matrices $A_{rr} - \lambda_i B_r K_r^*$, $i \in [2,3]$, are Hurwitz, and hence condition ii) in Corollary 7.6 holds. Therefore, the positive consensus problem is solvable. ♣

Corollary 7.6 immediately leads to the complete solution of the case when the diagonal block A_{rr} in (7.4) is scalar.

Corollary 7.7. *Assume that A, B and r are as in Corollary 7.6, and $n_r = 1$, namely A_{rr} and B_r are scalar. The positive consensus problem is solvable if and only if the following conditions hold:*

  a) A_{rr} is the only non-Hurwitz diagonal block of A;

  b) The scalar matrix $A_{rr} - \lambda_2 B_r k_r^*$ is negative (and hence Hurwitz).

If so, a possible solution is given by the matrix $\bar{K} = \begin{bmatrix} 0_{n_1}^\top & \dots & 0_{n_{r-1}}^\top & k_r^* & 0_{n_{r+1}}^\top & \dots & 0_{n_s}^\top \end{bmatrix}$.

Proof. For $n_r = 1$ Corollary 7.6 reads as follows: the positive consensus problem is solvable if and only if A_{jj} is Hurwitz for every $j \neq r$ (namely condition a) holds), and there exists a real number k_r, with $0 \leq k_r \leq k_r^*$, such that

$$A_{rr} - \lambda_i B_r k_r < 0, \quad \forall i \in [2, N]. \tag{7.6}$$

But inequality (7.6) holds for some k_r, with $0 \leq k_r \leq k_r^*$, and every $i \in [2, N]$ if and only if it holds for $k_r = k_r^*$ and $i = 2$, which amounts to saying that condition b) holds. This also shows that $\bar{K}$ solves the positive consensus problem. ∎

As already mentioned in Remark 7.1 (see also Example 7.1), the matrix K_r^* is not determined only by the constraint of keeping $A_{rr} - \ell^* B_r K_r^*$ Metzler and with off-diagonal entries as small as possible, but also by the additional constraints $A_{jr} - \ell^* B_j K_r^* \geq 0$, $j \in [1, r-1]$. Consequently, there might exist $K_r \in \mathbb{R}_+^{1 \times n_r}$, $K_r > K_r^*$, such that $A_{rr} - \ell^* B_r K_r$ is still Metzler. Before proceeding it is thus convenient to define $\bar{K}_r^* = \begin{bmatrix} \bar{k}_i^* \end{bmatrix} \in \mathbb{R}_+^{1 \times n_r}$, $\bar{K}_r^* \geq K_r^*$, as

$$\bar{k}_i^* := \begin{cases} \min_{\substack{j \in [1, n_r] \\ j \neq i}} \dfrac{[A_{rr}]_{ji}}{[B_r]_j} \dfrac{1}{\ell^*}, & \text{if } \exists j \neq i \text{ s.t. } [B_r]_j \neq 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

Analogously, there might exist $\ell > \ell^*$ such that $A_{rr} - \ell B_r K_r^*$ is still Metzler, and hence we define

$$\bar{\ell}^* := \max \left\{ \lambda \in \mathbb{R}_+ : A_{rr} - \lambda B_r K_r^* \text{ is Metzler} \right\}$$
$$= \min_{\substack{i, j \in [1, n_r] \\ j \neq i, [B_r]_i k_j^* \neq 0}} \frac{[A_{rr}]_{ij}}{[B_r]_i k_j^*} \geq \ell^*.$$

Notice that if $\bar{\ell}^* = \ell^*$ then there exists $j \in [1, n_r]$ such that $\bar{k}_j^* = k_j^*$, while if $\bar{\ell}^* > \ell^*$ then $\bar{K}_r^* \gg K_r^*$. Also, it is always true that $\ell^* \bar{K}_r^* \geq \bar{\ell}^* K_r^*$.

Example 7.2. Consider the multi-agent system consisting of $N = 3$ agents and described in Example 7.1. We have already seen that $\ell^* = 2$, $r = 3$ and $K_3^* = \begin{bmatrix} 1 & \frac{1}{4} \end{bmatrix}$. It is easy to see that $\bar{\ell}^* = 3 > 2 = \ell^*$, and hence $\bar{K}_3^* \gg K_3^*$, specifically $\bar{K}_3^* = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \end{bmatrix} \gg \begin{bmatrix} 1 & \frac{1}{4} \end{bmatrix} = K_3^*$. ♣

In the following, to simplify the notation, we will drop the subscript r and hence refer to $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}_+^n$ and $K \in \mathbb{R}_+^{1 \times n}$, with $0 \leq K \leq K^*$, under the steady assumption that $n \geq 2$, A is irreducible and the matrix K^* is assigned and satisfies the constraint $K^* \leq \bar{K}^*$, where $\bar{K}^*$ is the largest of the positive row vectors K such that $A - \ell^* BK$ is Metzler. Clearly, we rule out the trivial case when $K^* = 0$ and hence the problem has no solution. Under the previous assumptions, we denote by $\mathcal{K}^H$ the set of solutions of the

positive consensus problem, i.e.,

$$\mathcal{K}^H := \{K : 0 \le K \le K^*, A - \lambda_i BK \text{ Hurwitz}, i \in [2, N]\}.$$

7.3 The case $\lambda_{\max}(A) = 0$

The case when the Metzler matrix A is irreducible and its spectral abscissa, i.e., its Frobenius eigenvalue, is 0 deserves an independent analysis that easily leads to the conclusion that under these conditions the positive consensus problem is always solvable.

Proposition 7.8. *Assume that A is an $n \times n$, $n \ge 2$, Metzler, irreducible matrix with $\lambda_{\max}(A) = 0$, and $B \in \mathbb{R}_+^n$ is a positive vector. Then, the positive consensus problem is always solvable and every K such that $0 < K \ll K^*$ is a solution, i.e.,*

$$\mathcal{K}^H \supseteq \{K : 0 < K \ll K^*\}.$$

In addition, if $\bar{\ell}^* \ge \lambda_N$, then $\mathcal{K}^H = \{K : 0 < K \le K^*\}$.

Proof. Let $\mathbf{v}_F \gg 0$ be the Frobenius eigenvector of the Metzler irreducible matrix A. We first note that for every matrix K, the strictly positive vector $(\mathbf{1}_N \otimes \mathbf{v}_F)$ is an eigenvector of $\mathbb{A} = (I_N \otimes A) - (I_N \otimes B)(\mathcal{L} \otimes K)$ corresponding to the zero eigenvalue. For every K, with $0 < K \le K^*$, the matrix $\mathbb{A}$ is Metzler. On the other hand, if $0 < K \ll K^*$ the matrices $A - \ell_{ii}BK$, $i \in [1, N]$, have exactly the same nonzero pattern as the matrix A, and hence are irreducible. This implies (the proof is a minor modification of the proof of Lemma 2 in Valcher and Misra (2013)) that $\mathbb{A}$ is irreducible. Therefore, for every $0 < K \ll K^*$, $\mathbb{A}$ is a Metzler, irreducible matrix, having the strictly positive vector $(\mathbf{1}_N \otimes \mathbf{v}_F)$ as eigenvector corresponding to the zero eigenvalue. This ensures (recall Proposition 3.1) that $\lambda_{\max}(\mathbb{A}) = 0$ and all the other eigenvalues have negative real part. Being $\sigma(\mathbb{A}) = \sigma(A) \cup \sigma(A - \lambda_2 BK) \cup \cdots \cup \sigma(A - \lambda_N BK)$ (see Wieland, Kim, Scheu, and Allgöwer (2008); Wieland et al. (2011)), it follows that all matrices $A - \lambda_i BK$, $i \in [2, N]$, are Hurwitz, and hence consensus is achieved. On the other hand, if we assume that

$\bar{\ell}^* \geq \lambda_N$, then for every K, with $0 < K \leq K^*$, we have

$$A > A - \lambda_i BK$$
$$\geq A - \lambda_i B \left(\frac{\bar{\ell}^*}{\lambda_N} K^* \right)$$
$$= A - \bar{\ell}^* B \left(\frac{\lambda_i}{\lambda_N} K^* \right)$$
$$\geq A - \bar{\ell}^* BK^*.$$

As the matrices $A - \lambda_i BK$, $i \in [2, N]$, are lower bounded by a Metzler matrix, they are Metzler, too. On the other hand, by the irreducibility assumption on A and the monotonicity of the spectral abscissa, for every $i \in [2, N]$ we have

$$0 = \lambda_{\max}(A) > \lambda_{\max}(A - \lambda_i BK), \tag{7.7}$$

i.e., K solves the positive consensus problem. ∎

Remark 7.2. 1) The reasoning adopted within the first part of the previous proof does not extend to the general case of arbitrary $\lambda_{\max}(A) \geq 0$, since the fact that $\mathbb{A}$ is an irreducible matrix having $\lambda_{\max}(A)$ as Frobenius eigenvalue does not ensure that the matrices $A - \lambda_i BK$, $i \in [2, N]$, are Hurwitz. 2) Condition $0 < K \ll K^*$ ensures the irreducibility of $\mathbb{A}$. When some of the entries of K coincide with their upper bound, it is possible that one or more of the diagonal blocks $A - \ell_{ii} BK$, $i \in [1, N]$, are not irreducible and hence $\mathbb{A}$ is not necessarily irreducible. Consensus may still be possible (in particular, as enlightened in the second part of the statement, if $\bar{\ell}^* \geq \lambda_N$) but it cannot be deduced through this path.

Example 7.3. Consider the positive single-input agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_i(t).$$

A is a Metzler, irreducible, non-Hurwitz matrix with $\lambda_{\max}(A) = 0$. The pair (A, B) is

stabilizable. Assume that there are $N = 8$ agents and that the Laplacian matrix is:

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 3 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$

The eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = \lambda_4 = 2$, $\lambda_5 = \lambda_6 = \lambda_7 = 4$ and $\lambda_8 = 6$, while $\ell^* = 3$. We assume $K^* = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} \end{bmatrix} < \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \end{bmatrix} = \bar{K}^*$, and hence $\bar{\ell}^* = \ell^*$. It is easy to see that for every $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ with $0 < K \le K^*$ the matrices

$$A - \lambda_i BK = \begin{bmatrix} -1 - \lambda_i k_1 & 1 - \lambda_i k_2 \\ 1 - 2\lambda_i k_1 & -1 - 2\lambda_i k_2 \end{bmatrix}, \quad i \in [2, 8],$$

have negative traces and positive determinants and hence are Hurwitz. So, the positive consensus problem is solvable. ♣

As the case $\lambda_{\max}(A) = 0$ has been solved, in the following we will steadily assume that $\lambda_{\max}(A) > 0$.

7.4 Some technical results: the sets $\mathcal{K}_2^{MH}$ and $\mathcal{K}_N^{MH}$

In order to investigate the positive consensus problem solvability, it is convenient to introduce the following sets for $j = 2, N$:

$$\mathcal{K}_j^M := \{K \colon 0 \le K \le K^*, A - \lambda_j BK \text{ Metzler}\},$$
$$\mathcal{K}_j^H := \{K \colon 0 \le K \le K^*, A - \lambda_j BK \text{ Hurwitz}\}.$$

We also define the sets $\mathcal{K}_j^{MH}$, $j = 2, N$, as the sets of matrices satisfying the usual bounding conditions $0 \le K \le K^*$, and making $A - \lambda_j BK$ Metzler and Hurwitz, i.e,

$$\mathcal{K}_j^{MH} := \{K \colon 0 \le K \le K^*, A - \lambda_j BK \text{ Metzler and Hurwitz}\} = \mathcal{K}_j^M \cap \mathcal{K}_j^H. \tag{7.8}$$

By resorting to the definition of $\bar{K}^*$, it is easily seen that

$$\mathcal{K}_j^M = \left\{ K : 0 \le K \le K^* \right\} \cap \left\{ K : 0 \le K \le \frac{\ell^*}{\lambda_j} \bar{K}^* \right\},$$

and hence if we define the matrices $\hat{K}_j$, $j = 2, N$, as follows:

$$\left[\hat{K}_j \right]_i := \min \left\{ k_i^*, \frac{\ell^*}{\lambda_j} \bar{k}_i^* \right\}, \quad i \in [1, n],$$

then $\mathcal{K}_j^M = \left\{ K : 0 \le K \le \hat{K}_j \right\}$, $j = 2, N$.

In general, providing a complete description for the sets $\mathcal{K}_j^H$, $j = 2, N$, is not an easy task. On the contrary, by exploiting the monotonicity property of Metzler matrices, a clear picture of the sets $\mathcal{K}_j^{MH}$, $j = 2, N$, can be drawn. For simplicity, we focus on $\mathcal{K}_2^{MH}$, but the same analysis trivially extends to $\mathcal{K}_N^{MH}$.

The following lemma provides a necessary and sufficient condition for the set $\mathcal{K}_2^{MH}$ to be non-empty.

Lemma 7.9. *Assume that A is an $n \times n$, $n \ge 2$, Metzler, irreducible, non-Hurwitz matrix, and $B \in \mathbb{R}_+^n$ is a positive vector. Consider the set $\mathcal{K}_2^{MH}$ defined in* (7.8). *Then, $\mathcal{K}_2^{MH}$ is non-empty if and only if $A - \lambda_2 B \hat{K}_2$ is (Metzler and) Hurwitz.*

Proof. Sufficiency is obvious. To prove necessity assume, by contradiction, that $A - \lambda_2 B \hat{K}_2$ is non-Hurwitz, namely $\lambda_{\max}(A - \lambda_2 B \hat{K}_2) \ge 0$. Then, for every $K \in \mathcal{K}_2^M$ we have $A - \lambda_2 B K \ge A - \lambda_2 B \hat{K}_2$, and, by the monotonicity property of the spectral abscissa, it follows that $\lambda_{\max}(A - \lambda_2 B K) \ge \lambda_{\max}(A - \lambda_2 B \hat{K}_2) \ge 0$, and hence $\mathcal{K}_2^{MH} = \emptyset$. ∎

In order to investigate the structure of the set $\mathcal{K}_2^{MH}$ we need the following preliminary lemma.

Lemma 7.10. *Assume that A is an $n \times n$, $n \ge 2$, Metzler, irreducible, non-Hurwitz matrix, and $B \in \mathbb{R}_+^n$ is a positive vector. If there exist $\lambda \in \mathbb{R}_+$, $\lambda > 0$, and $K \in \mathbb{R}_+^{1 \times n}$ such that $A - \lambda B K$ is Metzler and Hurwitz, then:*

i) $\lambda_{\max}(A)$ *is a simple eigenvalue, and it is the only nonnegative real eigenvalue of $\sigma(A)$;*

ii) If A is non-singular, then $A^{-1}B$ is a positive vector and $K A^{-1} B > \frac{1}{\lambda} > 0$.

Proof. i) The fact that $\lambda_{\max}(A)$ is a simple nonnegative eigenvalue follows from the irreducibility of A. The proof of the fact that there are no other nonnegative real eigenvalues relies on some results about the positive observer problem reported in Back and Astolfi (2008). Let $\lambda \in \mathbb{R}_+$, $\lambda > 0$, and $K \in \mathbb{R}_+^{1 \times n}$ be such that $(A - \lambda BK)^\top = A^\top - (\lambda K^\top)B^\top$ is Metzler Hurwitz. By Lemma 4.6 and Theorem 4.7 in Back and Astolfi (2008), this implies that the positive system

$$\dot{\mathbf{z}}(t) = A^\top \mathbf{z}(t),$$
$$\mathbf{y}(t) = B^\top \mathbf{z}(t),$$

admits a positive observer and hence the number of nonnegative real eigenvalues of $A^\top$, counting the multiplicity, is at most 1. As $\sigma(A) = \sigma(A^\top)$ and A is non-Hurwitz, then condition *i*) holds.

ii) As $A - \lambda BK$ is Metzler and Hurwitz, its inverse exists and it is a negative matrix (see Proposition 3.3, statement *iii*)). On the other hand, if A is nonsingular, recalling that $B > 0$, we have

$$0 > (A - \lambda BK)^{-1} B$$
$$= \left[I_n - \lambda A^{-1} BK \right]^{-1} \left(A^{-1} B \right)$$
$$= A^{-1} B \left(1 - \lambda K A^{-1} B \right)^{-1}.$$

Since $\left(1 - \lambda K A^{-1} B \right)$ is a scalar, all the nonzero entries of the vector $A^{-1}B$ must have the same sign. Suppose by contradiction that $A^{-1}B$ is a negative vector, namely $A^{-1}B = -\mathbf{v}$, for some vector $\mathbf{v} \in \mathbb{R}_+^n$. Let $\mathbf{w} \in \mathbb{R}_+^n$, $\mathbf{w} > 0$, be the left Frobenius eigenvector of A, so that $\mathbf{w}^\top A = \lambda_{\max}(A)\mathbf{w}^\top$. From $B = -A\mathbf{v}$, upon multiplying by $\mathbf{w}^\top$ on both sides, we get $0 \le \mathbf{w}^\top B = -\mathbf{w}^\top A\mathbf{v} = -\lambda_{\max}(A)\mathbf{w}^\top \mathbf{v} \le 0$, which implies $\mathbf{w}^\top B = 0$. But if this were the case, then $\mathbf{w}^\top (A - \lambda BK) = \lambda_{\max}(A)\mathbf{w}^\top$, namely $\mathbf{w}^\top$ would be a left eigenvector of $A - \lambda BK$ corresponding to the positive eigenvalue $\lambda_{\max}(A)$, thus contradicting the Hurwitz assumption on $A - \lambda BK$. Hence, $A^{-1}B$ must be a nonnegative vector. Since $A^{-1}B \left(1 - \lambda K A^{-1} B \right)^{-1} < 0$, this also means that $1 - \lambda K A^{-1} B < 0$, namely $1 < \lambda K A^{-1} B$, and since $\lambda > 0$ the inequalities $K A^{-1} B > \frac{1}{\lambda} > 0$ follow. ∎

We can now provide a complete description of the set $\mathcal{K}_2^{MH}$.

Proposition 7.11. *Assume that A is an $n \times n$, $n \ge 2$, Metzler, irreducible, non-singular matrix with $\lambda_{\max}(A) > 0$, and $B \in \mathbb{R}_+^n$ is a positive vector. Assume also that $A - \lambda_2 B\hat{K}_2$*

is Metzler and Hurwitz. Then

$$\mathcal{K}_2^{MH} = \left\{ K \colon 0 \leq K \leq \hat{K}_2 \text{ and } KA^{-1}B > \frac{1}{\lambda_2} \right\}.$$

Proof. To prove the previous identity, we make the following observations:

1) The set $\mathcal{K}_2^{MH}$ is convex. Indeed, let $K_1, K_2 \in \mathcal{K}_2^{MH}$ and for every $j \in [1, 2]$ and $i \in [2, N]$ define the polynomials

$$p_{j,i}(s) := \det(sI_n - A + \lambda_i BK_j) = d(s) + \lambda_i n_j(s),$$

where we have set $d(s) := \det(sI_n - A)$ and $n_j(s) := K_j \operatorname{adj}(sI_n - A)B$. Since $A - \lambda_i BK_j$ is Metzler and Hurwitz for every $j \in [1, 2]$ and $i \in [2, N]$, it follows that for every $j \in [1, 2]$ and $i \in [2, N]$ the polynomials $p_{j,i}(s)$ have positive coefficients. Now consider the feedback matrix $K_\alpha := \alpha K_1 + (1 - \alpha)K_2$, with $\alpha \in [0, 1]$, and notice that $0 \leq K_\alpha \leq K^*$ and

$$A - \lambda_i BK_\alpha = \alpha \left(A - \lambda_i BK_1 \right) + (1 - \alpha) \left(A - \lambda_i BK_2 \right)$$

is still Metzler for every $i \in [2, N]$. Moreover, the characteristic polynomial of $A - \lambda_i BK_\alpha$ can be written as

$$
\begin{aligned}
p_{\alpha,i}(s) :&= \det(sI_n - A + \lambda_i BK_\alpha) \\
&= d(s) + \lambda_i \left[K_\alpha \operatorname{adj}(sI_n - A)B \right] \\
&= d(s) + \lambda_i \left[\alpha n_1(s) + (1 - \alpha)n_2(s) \right] \\
&= \alpha p_{1,i}(s) + (1 - \alpha)p_{2,i}(s),
\end{aligned}
$$

and hence it has positive coefficients. By Proposition 3.3, statement *iv)*, this implies that the Metzler matrix $A - \lambda_i BK_\alpha$ is Hurwitz for every $i \in [2, N]$. Hence, $K_\alpha \in \mathcal{K}_2^{MH}$ and the convexity of set $\mathcal{K}_2^{MH}$ is proved.

2) For every K with $0 \leq K \leq \hat{K}_2$ the matrix $A - \lambda_2 BK$ is Metzler, so we need to understand for which K it is Hurwitz and for which K it is not. Clearly, $0 \notin \mathcal{K}_2^{MH}$ and hence $\mathcal{K}_2^{MH} \subsetneq \left\{ K \colon 0 \leq K \leq \hat{K}_2 \right\}$;

3) By assumption $\hat{K}_2 \in \mathcal{K}_2^{MH}$ and hence there exists $\varepsilon > 0$ such that for every $K \in B(\hat{K}_2, \varepsilon)$, the ball of center $\hat{K}_2$ and radius ε, $A - \lambda_2 BK$ is Hurwitz. This

ensures, in particular, that the set $\mathcal{K}_2^{MH}$ intersects (possibly includes) the n faces $\mathcal{F}$ of the hypercube $\left\{K\colon 0 \leq K \leq \hat{K}_2\right\}$ having one vertex in $\hat{K}_2$.

In order to complete the description of $\mathcal{K}_2^{MH}$, we only need to determine which matrices K in the interior of the hybercube $\left\{K\colon 0 \leq K \leq \hat{K}_2\right\}$ belong to the boundary of $\mathcal{K}_2^{MH}$. Clearly, such matrices K leave $A - \lambda_2 BK$ Metzler and irreducible, and hence they necessarily correspond to the case when $A - \lambda_2 BK$ loses the Hurwitz property by becoming singular (with all the remaining eigenvalues in the open left complex half-plane)[4]. This amounts to saying that $\det(A - \lambda_2 BK) = 0$, and since A is non-singular this means that $\det A \cdot \det \left(I_n - \lambda_2 A^{-1}BK\right) = 0$, and hence $1 - \lambda_2 KA^{-1}B = 0$, which means that $KA^{-1}B = 1/\lambda_2$. So, to conclude the interior of the convex set $\mathcal{K}_2^{MH}$ consists of all the matrices in the interior of the hypercube $\left\{K\colon 0 \leq K \leq \hat{K}_2\right\}$ that are strictly greater than some matrix $\bar{K}$ belonging to the hyperplane $\bar{K}A^{-1}B = 1/\lambda_2$. By Lemma 7.10, part *ii*), the vector $A^{-1}B$ is positive and hence a matrix K belonging to the interior of the hypercube $\left\{K\colon 0 \leq K \leq \hat{K}_2\right\}$ satisfies $K \gg \bar{K}$ for some vector $\bar{K}$ with $\bar{K}A^{-1}B = 1/\lambda_2$ if and only if $KA^{-1}B > 1/\lambda_2$. ∎

The analysis carried on for the set $\mathcal{K}_2^{MH}$ can be easily extended to the set $\mathcal{K}_N^{MH}$, thus leading to the following corollary.

Corollary 7.12. *Assume that A is an $n \times n$, $n \geq 2$, Metzler, irreducible, non-Hurwitz matrix, and $B \in \mathbb{R}_+^n$ is a positive vector. Consider the set $\mathcal{K}_N^{MH}$ defined in (7.8). Then, $\mathcal{K}_N^{MH}$ is non-empty if and only if $A - \lambda_N B\hat{K}_N$ is (Metzler and) Hurwitz. When so*

$$\mathcal{K}_N^{MH} = \left\{K\colon 0 \leq K \leq \hat{K}_N \ and \ KA^{-1}B > \frac{1}{\lambda_N}\right\}.$$

7.5 Sufficient conditions for the problem solvability: the set $\mathcal{K}^{MH}$

In this section we provide a set of sufficient conditions for the solvability of the positive consensus problem that introduce additional constraints on the matrices $A - \lambda_i BK$, $i \in [2, N]$, with respect to that of being Hurwitz: we investigate the case when one at least of the solutions K, with $0 \leq K \leq K^*$, of the positive consensus problem make the resulting matrices $A - \lambda_i BK$, $i \in [2, N]$, not only Hurwitz but also Metzler.

---

[4]Note that this also means that this "lower boundary" of $\mathcal{K}_2^{MH}$ belongs to the closure of $\mathcal{K}_2^{MH}$, but not to $\mathcal{K}_2^{MH}$ itself.

Define the set

$$\mathcal{K}^{MH} := \left\{ K \colon 0 \le K \le K^*, A - \lambda_i BK \text{ Metzler and Hurwitz}, i \in [2, N] \right\},$$

and notice that the following relationships hold: $\left(\mathcal{K}_2^H \cap \mathcal{K}_N^H \right) \supseteq \mathcal{K}^H \supseteq \mathcal{K}^{MH}$.

Proposition 7.13 below provides necessary and sufficient conditions ensuring that the set $\mathcal{K}^{MH}$ is not empty.

Proposition 7.13. *Assume that A is an $n \times n$, $n \ge 2$, Metzler, irreducible, non-Hurwitz matrix, and $B \in \mathbb{R}_+^n$ is a positive vector. Then $\mathcal{K}^{MH} = \mathcal{K}_2^H \cap \mathcal{K}_N^M$. In addition, the following facts are equivalent:*

i) $\mathcal{K}^{MH} \ne \emptyset$;

ii) $A - \lambda_2 B\hat{K}_N$ *is Metzler and Hurwitz;*

iii) *The set* $\left\{ \alpha \in (0, 1] \colon A - \alpha \lambda_N B\hat{K}_N \text{ Hurwitz} \right\}$ *is not empty and*

$$\hat{\alpha} := \inf \left\{ \alpha \in (0, 1] \colon A - \alpha \lambda_N B\hat{K}_N \text{ Hurwitz} \right\}$$

satisfies $\hat{\alpha} < \frac{\lambda_2}{\lambda_N}$.

Proof. If $K \in \mathcal{K}^{MH}$, then K satisfies $0 \le K \le K^*$, and makes all matrices $A - \lambda_i BK$, $i \in [2, N]$, Metzler and Hurwitz. This ensures that $A - \lambda_2 BK$ is Hurwitz (and hence $K \in \mathcal{K}_2^H$) and $A - \lambda_N BK$ is Metzler (and hence $K \in \mathcal{K}_N^M$). Conversely, let K be a matrix satisfying $0 \le K \le K^*$, and such that $A - \lambda_2 BK$ is Hurwitz and $A - \lambda_N BK$ is Metzler. As $A - \lambda_2 BK \ge A - \lambda_i BK \ge A - \lambda_N BK$ for every $i \in [2, N]$, the fact that the lower bound is Metzler ensures that all matrices are Metzler. The fact that the upper bound of this set of Metzler matrices is Hurwitz ensures that all the matrices are Hurwitz. Consequently, $K \in \mathcal{K}^{MH}$. So, we have shown that $\mathcal{K}^{MH} = \mathcal{K}_2^H \cap \mathcal{K}_N^M$.

i) $\Rightarrow$ *ii)* Clearly, since $\lambda_2 \le \lambda_N$, the matrix $A - \lambda_2 B\hat{K}_N \ge A - \lambda_N B\hat{K}_N$, being lower bounded by a Metzler matrix, is Metzler, in turn. Now suppose that $\mathcal{K}^{MH} \ne \emptyset$ and let $K \in \mathcal{K}^{MH}$. Since $K \in \mathcal{K}_N^M$, it holds $K \le \hat{K}_N$ and hence $A - \lambda_2 BK \ge A - \lambda_2 B\hat{K}_N$. On the other hand, $A - \lambda_2 BK$ is Metzler and Hurwitz and hence, by the monotonicity property of the spectral abscissa, also the Metzler matrix $A - \lambda_2 B\hat{K}_N$ is Hurwitz.

ii) $\Rightarrow$ *iii)* If $A - \lambda_2 B\hat{K}_N$ is Metzler and Hurwitz, then even the Metzler matrix $A - \lambda_N B\hat{K}_N \le A - \lambda_2 B\hat{K}_N$ is Hurwitz, and hence $\frac{\lambda_2}{\lambda_N} \in \left\{ \alpha \in (0, 1] \colon A - \alpha \lambda_N B\hat{K}_N \text{ Hurwitz} \right\}$. This also implies that $\hat{\alpha} < \frac{\lambda_2}{\lambda_N}$.

iii) $\Rightarrow$ *i)* Observe that for every $k \in [2, N]$ the matrix $A - \lambda_k B\hat{K}_N$ satisfies $A - \lambda_k B\hat{K}_N \geq A - \lambda_N B\hat{K}_N$, and hence it is Metzler. On the other hand, if the set $\left\{\alpha \in (0, 1]: A - \alpha\lambda_N B\hat{K}_N \text{ Hurwitz}\right\}$ is not empty and the infimum value of the set $\hat{\alpha}$ is such that $\hat{\alpha} < \frac{\lambda_2}{\lambda_N}$, then all matrices $A - \lambda_k B\hat{K}_N = A - \frac{\lambda_k}{\lambda_N}\lambda_N B\hat{K}_N$, $k \in [2, N]$, are (Metzler and) Hurwitz. This proves that $\hat{K}_N \in \mathcal{K}^{MH}$, and hence $\mathcal{K}^{MH} \neq \emptyset$.

■

Remark 7.3. Since $A - \lambda_N B\hat{K}_N$ is Metzler, the set $\left\{\alpha \in (0, 1]: A - \alpha\lambda_N B\hat{K}_N \text{ Hurwitz}\right\}$ coincides with the set $\left\{\alpha \in (0, 1]: A - \alpha\lambda_N B\hat{K}_N \text{ Metzler and Hurwitz}\right\}$. Moreover, if the set is not empty, then the Metzler matrix $A - \alpha\lambda_N B\hat{K}_N$ satisfies $A - \lambda_N B\hat{K}_N \leq A - \hat{\alpha}\lambda_N B\hat{K}_N$ and hence it is necessarily Hurwitz. So, Proposition 7.13 above essentially states that the set $\mathcal{K}^{MH}$ is not empty, namely there exists a state feedback matrix K, satisfying the usual bounding conditions, that makes all matrices $A - \lambda_i BK$, $i \in [2, N]$, Metzler and Hurwitz, if and only if such a solution can be found in the set of matrices $\left\{\alpha\hat{K}_N: \alpha \in (0, 1]\right\}$. Note that not only the set $\left\{\alpha \in (0, 1]: A - \alpha\lambda_N B\hat{K}_N \text{ Hurwitz}\right\}$ must be not empty, and hence the parameter $\hat{\alpha}$ well defined, but the interval $(\hat{\alpha}, 1]$ must be sufficiently "large" to include the interval $\left[\frac{\lambda_2}{\lambda_N}, 1\right]$. Only in this way we can determine a matrix of the form $K = \alpha\hat{K}_N$ that makes $A - \lambda_i BK$ Metzler and Hurwitz for every $i \in [2, N]$.

Note that, by putting together Proposition 7.13, and the descriptions of the sets $\mathcal{K}_N^M$ and $\mathcal{K}_2^{MH}$, we obtain

$$\mathcal{K}^{MH} = \left\{K: 0 \leq K \leq \hat{K}_N \text{ and } KA^{-1}B > \frac{1}{\lambda_2}\right\}.$$

Example 7.4. Consider the positive single-input agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_i(t).$$

A is a Metzler, irreducible, non-Hurwitz matrix with $\lambda_{\max}(A) > 0$. The pair (A, B) is stabilizable. If we assume that the Laplacian is the same as in Example 7.3, then we easily find that $\bar{K}^*$ remains the same. We assume that also K^* is the same. We observe that $\frac{\ell^*}{\lambda_N}\bar{K}^* = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \end{bmatrix} < \begin{bmatrix} \frac{1}{6} & \frac{1}{6} \end{bmatrix} = K^*$, and hence $\hat{K}_N = \frac{\ell^*}{\lambda_N}\bar{K}^*$. It is easy to check that

$A - \lambda_2 B \hat{K}_N$ is (Metzler and) Hurwitz, and indeed

$$\mathcal{K}^{MH} = \left\{ K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} : 0 \le k_1 \le 1/12, 0 \le k_2 \le 1/6 \text{ and } 5k_1 + k_2 > 1/2 \right\}.$$

♣

The fact that $\mathcal{K}^{MH} = \emptyset$, however, does not preclude the solvability of the positive consensus problem, as clarified in the following example.

Example 7.5. Consider the positive single-input agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & -2 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} u_i(t).$$

Notice that A is a Metzler, irreducible, non-singular and non-Hurwitz matrix and that the pair (A, B) is stabilizable. Assume that there are $N = 3$ agents and that the Laplacian matrix of the communication graph is the following one:

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

In this case $\ell^* = 2$ and the eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0 < \lambda_2 = 1 < \lambda_3 = 3$. We assume that $K^* = \begin{bmatrix} 0 & 1 & \frac{1}{4} \end{bmatrix} = \bar{K}^*$, and therefore $\hat{K}_N = \frac{\ell^*}{\lambda_N} \bar{K}^* = \frac{2}{3} K^*$ and $A - \lambda_2 B \hat{K}_N = A - \frac{2}{3} B K^*$ is not Hurwitz. This implies that $\mathcal{K}^{MH} = \emptyset$. However, it is easy to verify that $\bar{K} = \begin{bmatrix} 0 & 1 & \frac{1}{6} \end{bmatrix} \le K^*$ solves the positive consensus problem, since $A - \lambda_i B \bar{K}, i \in [2, 3]$, are Hurwitz matrices, but

$$A - \lambda_3 B \bar{K} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -5 & 0 \\ 1 & -1 & -2.5 \end{bmatrix}$$

is not Metzler.　　　　　　　　　　　　　　　　　　　　　　　　　　　　　　♣

Remark 7.4. It is worth mentioning that Ait Rami and Tadeo (2007) provided a complete parametrization, expressed as the set of solutions of an LP problem, of all the matrices K that make $A + BK$ Metzler and Hurwitz, where A is a Metzler matrix and B a positive matrix. Such a parametrization could be adapted to this specific case, keeping in mind

that we have not a single pair (A, B), but $N - 1$ pairs $(A, \lambda_i B)$, $i \in [2, N]$, and that we must take into account the additional constraint $0 \le K \le K^*$. We will make use of the aforementioned parametrization later in Section 7.8.1.

Remark 7.5. To conclude, we would like to remark another reason why the case when the set $\mathcal{K}^{MH}$ is non-empty is of particular interest. If $K \in \mathcal{K}^{MH}$, all matrices $A - \lambda_i BK$, $i \in [2, N]$, are not only Metzler and Hurwitz, but they also admit a Common Linear Copositive Lyapunov Function in the sense of Definition 4.3 (Chapter 4, Section 4.1). Indeed, for every choice of n indices $i_1, i_2, \ldots, i_n \in [2, N]$, the Metzler matrix $\begin{bmatrix} \mathrm{col}_1(A - \lambda_{i_1} BK) & \mathrm{col}_2(A - \lambda_{i_2} BK) & \ldots & \mathrm{col}_n(A - \lambda_{i_n} BK) \end{bmatrix}$ is upper bounded by the Metzler Hurwitz matrix $A - \lambda_2 BK$, and hence is Hurwitz, in turn. This ensures, see Knorn et al. (2009); Fornasini and Valcher (2010), the existence of a Common Linear Copositive Lyapunov function. Even more, every Linear Copositive Lyapunov function for the matrix $A - \lambda_2 BK$ is necessarily a Linear Copositive Lyapunov function for all the matrices $A - \lambda_i BK$, $i \in [2, N]$.

7.6 The case of non-complete communication graph

In this section we consider the case when the communication graph describing the interconnection topology among the agents is not complete, namely it is not true that each agent communicates with all the other agents, a quite realistic scenario for N sufficiently large. This assumption ensures (recall Lemma 7.3) that the smallest nonzero eigenvalue of the Laplaian $\mathcal{L}$ is not greater than the maximal weighted degree ℓ^*, i.e., $\lambda_2 \le \ell^*$. This implies that $\hat{K}_2 = K^*$, and hence $\mathcal{K}_2^M = \{K \colon 0 \le K \le K^*\}$ and $\mathcal{K}_2^H = \mathcal{K}_2^{MH}$. Such a condition allows to simplify the analysis and will be extensively exploited throughout this section [5].

As a first result, the following proposition provides some necessary conditions for the positive consensus problem solvability.

Proposition 7.14. *Assume that A is an $n \times n$, $n \ge 2$, Metzler, irreducible, non-Hurwitz matrix, $B \in \mathbb{R}_+^n$ is a positive vector and $K^* \in \mathbb{R}_+^{1 \times n}$ is assigned. If the positive consensus problem is solvable, then:*

---

[5]Actually, the results derived in the present section hold their validity whenever condition $\lambda_2 \le \ell^*$ is satisfied. As a consequence, they can be applied when the undirected, weighted communication graph is complete, provided that $\lambda_2 \le \ell^*$.

  i) $A - \lambda_2 BK^*$ *is a (Metzler and) Hurwitz matrix*[6]*;*

  ii) $\lambda_{\max}(A)$ *is a simple eigenvalue, and it is the only nonnegative real eigenvalue of* $\sigma(A)$.

  iii) *If A is non-singular, then $A^{-1}B$ is a positive vector and $K^*A^{-1}B > \frac{1}{\lambda_2} > 0$.*

Proof. *i*) We have already remarked that, under the assumption that the communication graph $\mathcal{G}$ is undirected, connected but not complete, $\lambda_2 \leq \ell^*$, and hence $\mathcal{K}_2^H = \mathcal{K}_2^{MH}$. Since $\mathcal{K}_2^H \supseteq \mathcal{K}^H$, it follows from Lemma 7.9 that a necessary condition for the positive consensus problem to be solvable is that $A - \lambda_2 B \hat{K}_2 = A - \lambda_2 BK^*$ is (Metzler and) Hurwitz, namely *i*) holds.

  ii), *iii*) These conditions directly follow from condition *i*) and Lemma 7.10. ■

  By Proposition 7.14 a necessary condition for the positive consensus problem to be solvable is that $\lambda_{\max}(A)$ is the only nonnegative real eigenvalue of A. Since the case $\lambda_{\max}(A) = 0$ has already been solved in Section 7.3 (see Proposition 7.8), in the remaining part of this section we will assume that A is non-singular.

7.6.1 The case when $\mathcal{K}^H = \mathcal{K}^{MH}$

We now investigate the case when all the solutions K of the positive consensus problem make the resulting matrices $A - \lambda_i BK$, $i \in [2, N]$, not only Hurwitz but also Metzler. In other terms, we want to determine necessary and sufficient conditions ensuring that $\mathcal{K}^H = \mathcal{K}^{MH}$. To this end we need this preliminary lemma.

Lemma 7.15. *Assume that A is an $n \times n$, $n \geq 2$, Metzler, irreducible, non-singular matrix with $\lambda_{\max}(A) > 0$, and $B \in \mathbb{R}^n_+$ is a positive vector. The following facts are equivalent:*

  i) $\bar{\ell}^* \geq \lambda_N$;

  ii) $\bar{K}^* \frac{\ell^*}{\lambda_N} \geq K^*$;

  iii) $A - \lambda_i BK^*$ *is Metzler for every* $i \in [2, N]$.

---

[6]In the special case when B is an ith monomial vector and hence the ith entry of K^* is $+\infty$, in order to define $A - \lambda_2 BK^*$ we assume that k_i^* is arbitrarily large but otherwise finite. This will be a steady assumption also in the following.

Proof. $i) \Rightarrow ii)$ By the way $\bar{\ell}^*$ has been defined, we have that $\bar{\ell}^* \leq \frac{a_{ij}}{b_i k_j^*}$, for every $i, j \in [1, n]$, $i \neq j$, and hence

$$k_j^* \leq \frac{a_{ij}}{b_i \bar{\ell}^*}, \quad \forall i, j \in [1, n], i \neq j.$$

If $i)$ holds, then $k_j^* \leq \frac{a_{ij}}{b_i \lambda_N} = \frac{\ell^*}{\lambda_N} \frac{a_{ij}}{b_i \ell^*}$, for every $i, j \in [1, n]$, $i \neq j$, and hence

$$k_j^* \leq \frac{\ell^*}{\lambda_N} \min_{\substack{i \in [1, n] \\ j \neq i, b_i \neq 0}} \frac{a_{ij}}{b_i \ell^*} = \frac{\ell^*}{\lambda_N} \bar{k}_j^*, \quad \forall j \in [1, n],$$

thus proving that $ii)$ holds.

$ii) \Rightarrow iii)$ If $\bar{K}^* \frac{\ell^*}{\lambda_N} \geq K^*$, then for every $i \in [2, N]$

$$A - \lambda_i B K^* \geq A - \lambda_i B \left(\frac{\ell^*}{\lambda_N} \bar{K}^* \right)$$
$$= A - \ell^* B \left(\frac{\lambda_i}{\lambda_N} \bar{K}^* \right)$$
$$\geq A - \ell^* B \bar{K}^*.$$

Since all the matrices $A - \lambda_i B K^*$ are lower bounded by the Metzler matrix $A - \ell^* B \bar{K}^*$, they are Metzler, too, and hence condition $iii)$ holds.

$iii) \Rightarrow i)$ By definition, $\bar{\ell}^* := \max \{\lambda \in \mathbb{R}_+ : A - \lambda B K^* \text{ is Metzler}\}$, so it is immediately seen that if $iii)$ holds then $\lambda_i \leq \bar{\ell}^*$ for every $i \in [2, N]$, and hence $i)$ holds. ∎

We can now make use of the previous lemma to derive the following result.

Proposition 7.16. *Assume that A is an $n \times n$, $n \geq 2$, Metzler, irreducible, non-singular matrix with $\lambda_{\max}(A) > 0$, and $B \in \mathbb{R}_+^n$ is a positive vector. Assume also that $A - \lambda_2 B K^*$ is a (Metzler and) Hurwitz matrix. The following facts are equivalent:*

i) $\bar{\ell}^* \geq \lambda_N$;

ii) $\bar{K}^* \frac{\ell^*}{\lambda_N} \geq K^*$;

iii) $A - \lambda_i B K^*$ is Metzler and Hurwitz for every $i \in [2, N]$;

iv) $\mathcal{K}^H = \mathcal{K}^{MH}$.

If any of the previous equivalent conditions hold, then $\mathcal{K}^H = \mathcal{K}^{MH}$ coincides with $\mathcal{K}_2^H$.
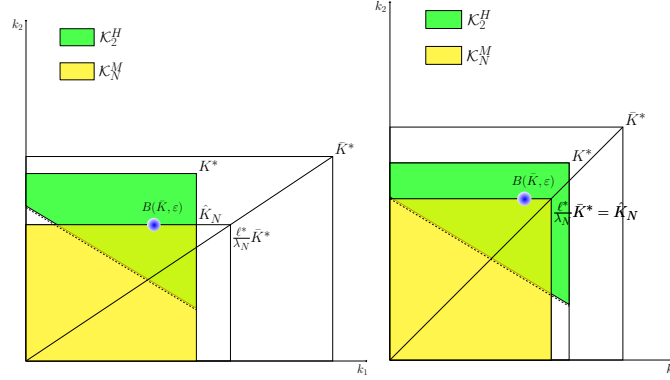
Figure 7.1: Sets $\mathcal{K}_2^H$ and $\mathcal{K}_N^M$ for the case $n = 2$. On the left, we consider the case when $\bar{K}^* \frac{\ell^*}{\lambda_N}$ is neither greater nor smaller than K^*. On the right, the case $\bar{K}^* \frac{\ell^*}{\lambda_N} < K^*$.

Proof. The equivalence of i), ii) and iii), under the assumption that $A - \lambda_2 BK^*$ is Metzler and Hurwitz, is an immediate consequence of Lemma 7.15. To prove that ii) and iv) are equivalent, observe that, in general, $\mathcal{K}_2^H \supseteq \mathcal{K}^H \supseteq \mathcal{K}^{MH}$. On the other hand, by Proposition 7.13,

$$\mathcal{K}^{MH} = \mathcal{K}_2^H \cap \mathcal{K}_N^M = \mathcal{K}_2^H \cap \left\{ K : 0 \leq K \leq \hat{K}_N \right\}.$$

So, if ii) holds, then $\hat{K}_N = K^*$ and this implies that $\mathcal{K}_2^H \cap \mathcal{K}_N^M = \mathcal{K}_2^H$. Consequently, $\mathcal{K}^{MH} = \mathcal{K}^H = \mathcal{K}_2^H$, namely iv) holds. This also proves the final statement of the proposition.

Conversely, assume that condition ii) does not hold. Consequently, $\mathcal{K}^{MH} = \mathcal{K}_2^H \cap \mathcal{K}_N^M = \mathcal{K}_2^H \cap \left\{ K : 0 \leq K \leq \hat{K}_N \right\} \subsetneq \mathcal{K}_2^H$. In particular, by the structure of the sets $\mathcal{K}_2^H$ and $\left\{ K : 0 \leq K \leq \hat{K}_N \right\}$, there exists $\bar{K}$ that satisfies two requirements (see Figure 7.1 for the case $n = 2$): (1) $\bar{K}$ belongs to the interior of $\mathcal{K}_2^H$; and (2) $\bar{K}$ belongs to the boundary of $\mathcal{K}^{MH}$, and specifically to some face $\mathcal{F}$ of $\left\{ K : 0 \leq K \leq \hat{K}_N \right\}$ having $\hat{K}_N$ as one of its vertices.

Clearly, $A - \lambda_i B\bar{K}$ is Metzler and Hurwitz for every $i \in [2, N]$. On the other hand, an $\varepsilon > 0$ can be found such that $B(\bar{K}, \varepsilon) \subset \mathcal{K}_2^H$ and for every $\tilde{K} \in B(\bar{K}, \varepsilon)$ the matrices $A - \lambda_i B\tilde{K}$, $i \in [2, N]$, are Hurwitz. This implies that there exists $\tilde{K} \in \mathcal{K}^H \setminus \mathcal{K}^{MH}$, thus contradicting iv). ∎

7.6.2 Sufficient conditions for the problem solvability: the set $\tilde{\mathcal{K}}$

In this subsection we exploit the notion of Metzler part of a matrix to determine a subset of the solution set that is potentially larger than $\mathcal{K}^{MH}$. To this aim, define the matrix

$\tilde{K} := \left[\tilde{k}_j\right]$ as follows:

$$\tilde{k}_j := \min\left\{k_j^*, \frac{2\ell^*}{\lambda_2 + \lambda_N}\bar{k}_j^*\right\}, \quad j \in [1, n],$$

and introduce the set $\tilde{\mathcal{K}} := \left\{K \colon 0 \leq K \leq \tilde{K}\right\} \cap \mathcal{K}_2^H$. We have the following result.

Proposition 7.17. *Assume that A is an $n \times n$, $n \geq 2$, Metzler, irreducible, non-singular matrix with $\lambda_{\max}(A) > 0$, and $B \in \mathbb{R}_+^n$ is a positive vector. Assume also that $A - \lambda_2 BK^*$ is a (Metzler and) Hurwitz matrix. Then, the following relationships hold:*

$$\mathcal{K}^H \supseteq \tilde{\mathcal{K}} \supseteq \mathcal{K}^{MH}.$$

Proof. Since $\lambda_2 \leq \lambda_N$, we have $\frac{\ell^*}{\lambda_N} \leq \frac{2\ell^*}{\lambda_2 + \lambda_N}$, and hence for every $j \in [1, n]$ it holds

$$\left[\hat{K}_N\right]_j = \min\left\{k_j^*, \frac{\ell^*}{\lambda_N}\bar{k}_j^*\right\} \leq \min\left\{k_j^*, \frac{2\ell^*}{\lambda_2 + \lambda_N}\bar{k}_j^*\right\} = \left[\tilde{K}\right]_j,$$

namely $\hat{K}_N \leq \tilde{K}$. Recalling that $\mathcal{K}^{MH}$ is the set of all matrices K that belong to the intersection of the hypercube $\left\{K \colon 0 \leq K \leq \hat{K}_N\right\}$ with the set $\mathcal{K}_2^H$, this implies that $\mathcal{K}^{MH}$ is a subset of $\tilde{\mathcal{K}}$, namely $\tilde{\mathcal{K}} \supseteq \mathcal{K}^{MH}$. We now prove that $\mathcal{K}^H \supseteq \tilde{\mathcal{K}}$. Let $K \in \tilde{\mathcal{K}}$ and notice that this implies that $K \in \mathcal{K}_2^H$, namely $A - \lambda_2 BK$ is Metzler and Hurwtiz. For $k \in [3, N]$, consider the matrix $A_k := A - \lambda_k BK$. Its Metzler part is equivalently defined as

$$[\mathcal{M}(A_k)]_{ij} = \begin{cases} \lambda_k b_i k_j - a_{ij}, & \text{if } i \neq j \text{ and } a_{ij} - \lambda_k b_i k_j < 0; \\ a_{ij} - \lambda_k b_i k_j, & \text{otherwise.} \end{cases}$$

Note that condition $a_{ij} - \lambda_k b_i k_j < 0$ necessarily implies $b_i > 0$. We want to show that $\mathcal{M}(A_k) \leq A - \lambda_2 BK$. Clearly, for all pairs (i, j) such that $[\mathcal{M}(A_k)]_{ij} = [A_k]_{ij}$ this is true because $a_{ij} - \lambda_k b_i k_j \leq a_{ij} - \lambda_2 b_i k_j$. On the other hand, since for every $j \in [1, n]$

$$k_j \leq \tilde{k}_j \leq \frac{2\ell^*}{\lambda_2 + \lambda_N}\bar{k}_j^* = \frac{2\ell^*}{\lambda_2 + \lambda_N}\min_{\substack{i \in [1, n] \\ i \neq j}}\frac{a_{ij}}{b_i}\frac{1}{\ell^*},$$

then for every pair of indices $i, j \in [1, n]$, with $i \neq j$,

$$k_j \leq \frac{2\ell^*}{\lambda_2 + \lambda_N}\frac{a_{ij}}{b_i}\frac{1}{\ell^*} \leq \frac{2}{\lambda_2 + \lambda_k}\frac{a_{ij}}{b_i},$$
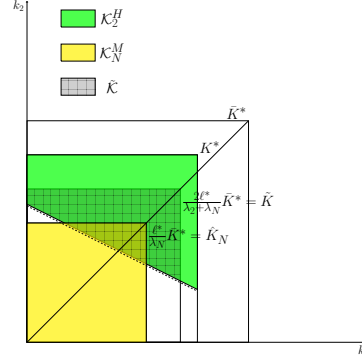
Figure 7.2: Sets $\mathcal{K}_2^H$, $\mathcal{K}_N^M$ and $\tilde{\mathcal{K}}$ for the case $n = 2$ and $\hat{K}_N < \tilde{K} \in \mathcal{K}_2^H$.

and this implies that

$$\lambda_k b_i k_j - a_{ij} \leq a_{ij} - \lambda_2 b_i k_j.$$

So, condition $\mathcal{M}(A_k) \leq A - \lambda_2 BK$ and the Hurwitz property of $A - \lambda_2 BK$ ensure that $\lambda_{\max}(A_k) < 0$, and hence all matrices $A - \lambda_k BK$, $k \in [2, N]$, are Hurwitz, namely $K \in \mathcal{K}^H$. $\blacksquare$

We now provide a necessary and sufficient condition for the set $\tilde{\mathcal{K}}$ to be non-empty.

Proposition 7.18. *Assume that A is an $n \times n$, $n \geq 2$, Metzler, irreducible, non-singular matrix with $\lambda_{\max}(A) > 0$, and $B \in \mathbb{R}_+^n$ is a positive vector. Assume also that $A - \lambda_2 BK^*$ is a (Metzler and) Hurwitz matrix. Then, $\tilde{\mathcal{K}} \neq \emptyset$ if and only if $A - \lambda_2 B\tilde{K}$ is (Metzler and) Hurwitz.*

Proof. By the way $\tilde{K}$ has been defined, it holds $\tilde{K} \leq K^*$, and hence $\tilde{\mathcal{K}}$ is the set of all matrices K that belong to the hypercube $\left\{K : 0 \leq K \leq \tilde{K}\right\}$ and satisfy $KA^{-1}B > 1/\lambda_2$. Recalling that $A^{-1}B$ is a positive vector, either $\tilde{K}$ satisfies $\tilde{K}A^{-1}B > 1/\lambda_2$, namely it belongs to $\mathcal{K}_2^H$ and hence it makes $A - \lambda_2 B\tilde{K}$ (Metzler and) Hurwitz, or $\tilde{\mathcal{K}} = \emptyset$. $\blacksquare$

Remark 7.6. Notice that $\mathcal{K}^{MH}$ and $\tilde{\mathcal{K}}$ can be described as the intersection of $\mathcal{K}_2^H$ with the hypercubes $\left\{K : 0 \leq K \leq \hat{K}_N\right\}$ and $\left\{K : 0 \leq K \leq \tilde{K}\right\}$, respectively. So, if $\hat{K}_N < \tilde{K}$ and $\tilde{K} \in \mathcal{K}_2^H$, then the set $\mathcal{K}^{MH}$ is a proper subset of $\tilde{\mathcal{K}}$, i.e, $\tilde{\mathcal{K}} \supsetneq \mathcal{K}^{MH}$ (see Figure 7.2 for the case $n = 2$).

7.6.3   Communication graphs whose Laplacian matrices satisfy special conditions

When the agents' state-space representation and the communication graph are such that $\lambda_2 + \lambda_N \leq 2\bar{\ell}^*$, the fact that the set $\tilde{\mathcal{K}}$ is not empty is a necessary and sufficient condition for the positive consensus problem solvability, as stated in the following proposition.

Proposition 7.19. *Assume that A is an $n \times n$, $n \geq 2$, Metzler, irreducible, non-singular matrix with $\lambda_{\max}(A) > 0$, and $B \in \mathbb{R}_+^n$ is a positive vector. Assume also that $A - \lambda_2 B K^*$ is a (Metzler and) Hurwitz matrix and that $\lambda_2 + \lambda_N \leq 2\bar{\ell}^*$. Then, the positive consensus problem is solvable if and only if $A - \lambda_2 B\tilde{K}$ is a (Metzler and) Hurwitz matrix.*

Proof. [Sufficiency] If $A - \lambda_2 B\tilde{K}$ is Metzler and Hurwitz, by Proposition 7.18 the set $\tilde{\mathcal{K}}$ is not empty, and hence by Proposition 7.17 the positive consensus problem is solvable.

[Necessity] Suppose that $A - \lambda_2 B\tilde{K}$ is non-Hurwitz, and hence $\tilde{K} \notin \mathcal{K}_2^H$. We want to prove that, if this is the case, the positive consensus problem is not solvable. To this aim, observe that, since for every $j \in [1, n]$ either $\tilde{k}_j = k_j^*$ or $\tilde{k}_j = \frac{2\ell^*}{\lambda_2 + \lambda_N} \bar{k}_j^*$, it entails no loss of generality assuming that

$$\tilde{k}_j = \begin{cases} k_j^*, & j \in [1, r] \\ \frac{2\ell^*}{\lambda_2 + \lambda_N} \bar{k}_j^*, & j \in [r+1, n], \end{cases} \tag{7.9}$$

for some $r \in [0, n]$. So, if we partition K^* and $\bar{K}^*$ as

$$K^* = \begin{bmatrix} \mathbf{k}_1^* & \mathbf{k}_2^* \end{bmatrix},$$
$$\bar{K}^* = \begin{bmatrix} \bar{\mathbf{k}}_1^* & \bar{\mathbf{k}}_2^* \end{bmatrix},$$

where $\mathbf{k}_1^*, \bar{\mathbf{k}}_1^* \in \mathbb{R}_+^{1 \times r}$ and $\mathbf{k}_2^*, \bar{\mathbf{k}}_2^* \in \mathbb{R}_+^{1 \times (n-r)}$, the matrix $\tilde{K}$ can be rewritten as

$$\tilde{K} = \begin{bmatrix} \mathbf{k}_1^* & \frac{2\ell^*}{\lambda_2 + \lambda_N} \bar{\mathbf{k}}_2^* \end{bmatrix}.$$

If we partition the positive vector $A^{-1}B$ in a way consistent with $\tilde{K}$, namely

$$A^{-1}B = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix},$$

where $\mathbf{v}_1 \in \mathbb{R}^r_+$, $\mathbf{v}_2 \in \mathbb{R}^{n-r}_+$, since by assumption $\tilde{K} \notin \mathcal{K}^H_2$, it holds

$$\tilde{K}A^{-1}B = \mathbf{k}^*_1\mathbf{v}_1 + \frac{2\ell^*}{\lambda_2 + \lambda_N}\bar{\mathbf{k}}^*_2\mathbf{v}_2 \leq \frac{1}{\lambda_2}.$$

Recalling that $\ell^*\bar{K}^* \geq \bar{\ell}^*K^*$, if condition $\lambda_2 + \lambda_N \leq 2\bar{\ell}^*$ holds, then

$$\frac{2\ell^*}{\lambda_2 + \lambda_N}\bar{\mathbf{k}}^*_2 \geq \frac{2\bar{\ell}^*}{\lambda_2 + \lambda_N}\mathbf{k}^*_2 \geq \mathbf{k}^*_2.$$

This implies that

$$K^*A^{-1}B = \mathbf{k}^*_1\mathbf{v}_1 + \mathbf{k}^*_2\mathbf{v}_2 \leq \mathbf{k}^*_1\mathbf{v}_1 + \frac{2\ell^*}{\lambda_2 + \lambda_N}\bar{\mathbf{k}}^*_2\mathbf{v}_2 = \tilde{K}A^{-1}B \leq \frac{1}{\lambda_2},$$

namely $K^* \notin \mathcal{K}^H_2$. Since the fact that $K^* \in \mathcal{K}^H_2$ is a necessary condition for the problem solvability (see Proposition 7.14), we can conclude that the positive consensus problem is not solvable. ∎

The previous proposition allows us to identify a class of communication graphs for which problem solvability can be completely characterized and the solution set $\mathcal{K}^H$ coincides with the whole set $\mathcal{K}^H_2$.

Corollary 7.20. *Assume that A is an $n \times n$, $n \geq 2$, Metzler, irreducible, non-singular matrix with $\lambda_{\max}(A) > 0$, and $B \in \mathbb{R}^n_+$ is a positive vector. Assume also that the eigenvalues of the Laplacian matrix are such that $\lambda_2 + \lambda_N \leq 2\ell^*$. Then, the positive consensus problem is solvable if and only if $A - \lambda_2 BK^*$ is a (Metzler and) Hurwitz matrix. When so, $\mathcal{K}^H = \mathcal{K}^H_2$.*

Proof. We already know by Proposition 7.14 that the fact that $A - \lambda_2 BK^*$ is a (Metzler and) Hurwitz matrix is a necessary condition for the problem solvability. On the other hand, in general, the following relationship holds: $\mathcal{K}^H_2 \supseteq \mathcal{K}^H \supseteq \tilde{\mathcal{K}}$. If condition $\lambda_2 + \lambda_N \leq 2\ell^*$ holds, then $\tilde{K} = K^*$ and hence $\tilde{\mathcal{K}} = \{K : 0 \leq K \leq K^*\} \cap \mathcal{K}^H_2 = \mathcal{K}^H_2$ and this implies $\mathcal{K}^H = \mathcal{K}^H_2$. So, since $K^* \in \mathcal{K}^H_2$, K^* also belongs to $\mathcal{K}^H$ and hence the positive consensus problem is solvable. ∎

Condition $\lambda_2 + \lambda_N \leq 2\ell^*$ in Corollary 7.20 only depends on the interconnection topology among the agents. The interest in this condition comes from the fact that several meaningful unweighted graphs satisfy it. Among them it is worth mentioning complete bipartite graphs $K_{p,q}$ with $p \geq 2q$ (see Theorem 2.21 in Merris (1994)), any

tree with a unique vertex of degree ℓ^* (see Theorem 8 in Bapat (1996), Theorem 2.1 in ch. Das (2003)). As a further example, consider the case of $N \geq 3$ agents whose interconnection topology is described by a star graph, by this meaning that there is an internal node (say node 1) communicating with the remaining $N - 1$ nodes, and there is no other interaction among the agents. The Laplacian matrix is given by:

$$
\mathcal{L} = \begin{bmatrix} N-1 & -1 & \ldots & -1 \\ -1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \ldots & 1. \end{bmatrix}
$$

In this case $\ell^* = N - 1$, and since $\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \ell_{ii} = 2N - 2$, it follows that $\lambda_2 + \lambda_N \leq 2(N-1) = 2\ell^*$. In addition, if we replace any ray of the star graph with a complete (sub)graph of arbitrary dimension, the algebraic condition $\lambda_2 + \lambda_N \leq 2\ell^*$ still holds (see Theorem 4.4 in Bapat, Lal, and Pati (2012)). Notice that this communication topology describes quite a realistic situation: the existence of an agent (playing the role of a coordinator) that communicates with all the other agents, and the partition of the remaining agents of the network into groups (not necessarily of the same dimension) such that every agent communicates with all (and only) the agents belonging to its own group.

On the other hand, consider the m-dimensional hypercube defined in Harary, Hayes, and Wu (1988); Saad and Schultz (1988), namely the graph whose vertex set $\mathcal{V}$ consists of the $N := 2^m$ m-tuples with binary coordinates 0 or 1 and where two vertices are adjacent whenever their corresponding vectors differ in exactly one entry. In this case $\ell^* = \ell_{ii} = m$ for every $i \in [1, N]$ and the Laplacian matrix has (distinct) eigenvalues $\tilde{\lambda}_k = 2k$ with multiplicity $\binom{m}{k}$ for $k \in [0, m]$. Consequently, it is always true that $\lambda_2 + \lambda_N = 2 + 2m > 2m = 2\ell^*$.

7.6.4   The case when B is a canonical vector

We consider the case when B is a monomial vector, namely $B = b\mathbf{e}_i$, for some $b > 0$ and $i \in [1, n]$. The interest in this case comes from the fact that a good number of physical systems that can be modelled through positive or compartmental state-space models have an input-to-state matrix B which is canonical. This happens every time the control input directly affects only one of the state variables (e.g., the gene expression model, some thermal or fluid network models, some chemical reaction networks). As an additional example, in the vehicle model used in Ren (2007); Ren and Atkins (2007), to investigate the distributed multi-vehicle coordination problem, the matrix A is Metzler

and unstable, while the matrix B is a canonical vector. In this situation, it entails no loss of generality assuming $B = \mathbf{e}_1$, since we can always reduce ourselves to this situation by resorting to a suitable permutation and to a scaling factor that modify the numeric values of the possible solutions, but do not affect the problem solvability. Accordingly, we can express the matrix A as

$$A = \begin{bmatrix} a_{11} & \mathbf{r}^\top \\ \mathbf{c} & A_{22} \end{bmatrix}, \tag{7.10}$$

where $a_{11} \in \mathbb{R}$, $\mathbf{r}, \mathbf{c} \in \mathbb{R}_+^{n-1}$ are nonnegative vectors, and $A_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ is a Metzler matrix. Therefore, for every $K = [k_j] \in \mathbb{R}^{1\times n}$ and λ_i we have

$$A - \lambda_i B K = \begin{bmatrix} a_{11} - \lambda_i k_1 & a_{12} - \lambda_i k_2 & \dots & a_{1n} - \lambda_i k_n \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - \lambda_i k_1 & \mathbf{r}^\top - \lambda_i \begin{bmatrix} k_2 & \dots & k_n \end{bmatrix} \\ \mathbf{c} & A_{22} \end{bmatrix}. \tag{7.11}$$

As previously remarked, there is no upper bound on the first entry of the matrix $\bar{K}^*$, and hence $\bar{K}^* = \begin{bmatrix} \bar{k}_1^* & \bar{k}_2^* & \dots & \bar{k}_n^* \end{bmatrix} = \begin{bmatrix} +\infty & \frac{a_{12}}{\ell^*} & \dots & \frac{a_{1n}}{\ell^*} \end{bmatrix} = \begin{bmatrix} +\infty & \frac{1}{\ell^*}\mathbf{r}^\top \end{bmatrix}$. We consider first the case when also the first entry of K^* is $+\infty$. Note that if we consider the block form of the pair (A, B) (see (7.4)), this situation arises only if $B_i = 0$ for every $i \in [1, r-1]$, and hence B_r is the only nonzero block in B.

Proposition 7.21. *Assume that A is an $n \times n$, $n \geq 2$, Metzler, irreducible, non-singular matrix with $\lambda_{\max}(A) > 0$ and described as in (7.10). Assume also that $B = \mathbf{e}_1$ and $K^* = \begin{bmatrix} +\infty & k_2^* & \dots & k_n^* \end{bmatrix}$. Then, the positive consensus problem is solvable if and only if A_{22} is (Metzler and) Hurwitz.*

Proof. [Sufficiency] Let $\alpha_0, \dots, \alpha_{n-2}$ and $\alpha_{n-1} = 1$ be the coefficients of the characteristic polynomial of A_{22}, namely $\det(sI_{n-1} - A_{22}) = s^{n-1} + \alpha_{n-2}s^{n-2} + \dots + \alpha_1 s + \alpha_0$, and notice that if A_{22} is (Metzler and) Hurwitz, then $\alpha_i > 0$ for every $i \in [0, n-1]$. We now prove that, if this is the case, there always exists a feedback matrix $K = \begin{bmatrix} k_1 & 0 & \dots & 0 \end{bmatrix}$, with $0 < K \leq K^*$ (in practice, $k_1 > 0$, since there are no bounds on k_1^*), that solves the positive consensus problem. To this aim, notice that for this choice of K the Metzler

matrix $A - \lambda_2 BK$ takes the form:

$$A - \lambda_2 BK = \begin{bmatrix} a_{11} - \lambda_2 k_1 & \mathbf{r}^\top \\ \mathbf{c} & A_{22} \end{bmatrix}.$$

So, if we define $\bar{a} := -a_{11} + \lambda_2 k_1$, and set $\mathbf{r}^\top \operatorname{adj}(sI_{n-1} - A_{22})\mathbf{c} =: \beta_{n-2}s^{n-2} + \cdots + \beta_1 s + \beta_0$, the characteristic polynomial of $A - \lambda_2 BK$ can be expressed as:

$$\begin{aligned}
\det(sI_n - A + \lambda_2 BK) &= \\
&= \det(sI_{n-1} - A_{22})(s + \bar{a}) - \mathbf{r}^\top \operatorname{adj}(sI - A_{22})\mathbf{c} \\
&= (s^{n-1} + \alpha_{n-2}s^{n-2} + \cdots + \alpha_1 s + \alpha_0)(s + \bar{a}) - (\beta_{n-2}s^{n-2} + \cdots + \beta_1 s + \beta_0) \\
&= s^n + (\alpha_{n-2} + \bar{a})s^{n-1} + (\alpha_{n-3} + \bar{a}\alpha_{n-2} - \beta_{n-2})s^{n-2} + \\
&\quad + \cdots + (\alpha_0 + \bar{a}\alpha_1 - \beta_1)s + (\bar{a}\alpha_0 - \beta_0).
\end{aligned}$$

Therefore, if we take $k_1 > 0$ large enough so that $\bar{a} > \max_{i \in [0, n-1]} \frac{\beta_i - \alpha_{i-1}}{\alpha_i}$, where we set $\alpha_{-1} = \beta_{n-1} = 0$, then the Metzler matrix $A - \lambda_2 BK$ is Hurwitz since all the coefficients of its characteristic polynomial are positive (see Proposition 3.3, statement iv)). Moreover, for every $i \in [3, N]$, the Metzler matrix $A - \lambda_i BK$ is such that $A - \lambda_i BK \leq A - \lambda_2 BK$ and, by the monotonicity property of the spectral abscissa, it follows that K solves the positive consensus problem.

[Necessity] If the consensus problem is solvable, then $\hat{A} := A - \lambda_2 BK^*$ is Metzler and Hurwitz, and a necessary condition for this to happen is that its principal submatrix $\hat{A}_{22}$, obtained by deleting the first row and the first column in $\hat{A}$, is (Metzler and) Hurwitz. As $\hat{A}_{22} = A_{22}$, the result follows. ∎

Remark 7.7. If we refer to the partition of A given in (7.10) and partition its inverse A^{-1} in a consistent way as

$$A^{-1} = \begin{bmatrix} x & \mathbf{y}^\top \\ \mathbf{z} & W \end{bmatrix},$$

where $x \in \mathbb{R}$, $\mathbf{y}, \mathbf{z} \in \mathbb{R}^{n-1}$ and $W \in \mathbb{R}^{(n-1) \times (n-1)}$, it is easy to verify that x, $\mathbf{y}$, $\mathbf{z}$ and W

take the following expressions

$$x = \left(a_{11} - \mathbf{r}^\top A_{22}^{-1} \mathbf{c} \right)^{-1},$$

$$\mathbf{y}^\top = -\frac{1}{a_{11}} \mathbf{r}^\top \left(A_{22} - \frac{\mathbf{c} \mathbf{r}^\top}{a_{11}} \right)^{-1},$$

$$\mathbf{z} = -A_{22}^{-1} \mathbf{c} \left(a_{11} - \mathbf{r}^\top A_{22}^{-1} \mathbf{c} \right)^{-1},$$

$$W = \left(A_{22} - \frac{\mathbf{c} \mathbf{r}^\top}{a_{11}} \right)^{-1}.$$

Then, from the description of $\mathcal{K}_2^H$ given in Lemma 7.11, a matrix K of the form $K = \begin{bmatrix} k_1 & 0 & \dots & 0 \end{bmatrix}$ belongs to $\mathcal{K}_2^H$ if and only if

$$KA^{-1}B = k_1 x = k_1 \left(a_{11} - \mathbf{r}^\top A_{22}^{-1} \mathbf{c} \right)^{-1} > \frac{1}{\lambda_2}.$$

Recalling that if A_{22} is Metzler and Hurwitz then the matrix $-A_{22}^{-1}$ is a positive matrix, it follows that $K = \begin{bmatrix} k_1 & 0 & \dots & 0 \end{bmatrix}$ belongs to $\mathcal{K}_2^H$ for every $k_1 > \lambda_2^{-1} \left(a_{11} - \mathbf{r}^\top A_{22}^{-1} \mathbf{c} \right)$.

Example 7.6. Consider the positive single-input agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_i(t).$$

Notice that A is a Metzler, irreducible and non-Hurwitz matrix and that the pair (A, B) is stabilizable. Consider $N = 3$ agents and the same adjacency matrix as in Example 7.1. The eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0$ and $\lambda_2 = 1 < \lambda_3 = 3$. The matrix A_{11}, obtained by deleting the third row and the third column of A, is Hurwitz, and indeed by choosing $K = \begin{bmatrix} 0 & 0 & 7 \end{bmatrix}$ we get that both matrices $A - \lambda_2 BK$ and $A - \lambda_3 BK$ are Metzler and Hurwitz.

On the other hand, if we assume as vector B the canonical vector $B = \mathbf{e}_2$, it is easily seen that the positive consensus problem is not solvable. ♣

It is worth underlying that the necessary condition given in Proposition 7.21 is independent of the fact that the first entry of K^* is infinite or finite. Indeed, when $B = \mathbf{e}_1$, a necessary condition for the solvability of the positive consensus problem is that A_{22} is Metzler and Hurwitz. However, when $k_1^* < +\infty$, this is no longer sufficient. Indeed, the possibility of resorting to a feedback matrix K whose unique nonzero entry

is the first one works if and only if

$$k_1^* \geq k_1 > \lambda_2^{-1} \left(a_{11} + \max_{i \in [0, n-1]} \frac{\beta_i - \alpha_{i-1}}{\alpha_i} \right),$$

where, as in the proof of Proposition 7.21, α_i, $i \in [0, n-2]$, and $\alpha_{n-1} = 1$ are such that $\det(sI_{n-1} - A_{22}) = s^{n-1} + \alpha_{n-2}s^{n-2} + \cdots + \alpha_1 s + \alpha_0$; β_i, $i \in [0, n-2]$, are such that $\mathbf{r}^\top \operatorname{adj}(sI_{n-1} - A_{22})\mathbf{c} = \beta_{n-2}s^{n-2} + \cdots + \beta_1 s + \beta_0$; and $\alpha_{-1} = \beta_{n-1} = 0$. Differently, the characteristic polynomial of the matrices $A - \lambda_i BK$, $i \in [2, N]$, would not have positive coefficients, thus ruling out the Hurwitz property of these matrices. Therefore the study of the conditions that ensure the positive consensus when $k_1^* < +\infty$ requires, in the general case, a completely different analysis, that keeps into account the specific values taken by the matrix K^*, and is still an open problem.

7.6.5   Second-order agents

Consensus among agents described by second-order models has been the subject of a good number of papers (see, e.g., Zhang and Liu (2013), and references therein). In this subsection we investigate the case when each agent is modelled by a second-order positive linear system, i.e.,

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_i(t), \tag{7.12}$$

with a_{12}, a_{21}, b_1 and b_2 nonnegative real numbers. Note that we steadily assume that A is Metzler, non-Hurwitz and non-singular.

Recalling that any matrix $M \in \mathbb{R}^{2 \times 2}$ is Hurwitz if and only if $\operatorname{tr}(M) < 0$ and $\det(M) > 0$, after elementary manipulations it can be seen that for every $A \in \mathbb{R}^{2 \times 2}$, $B \in \mathbb{R}^2$ and $K \in \mathbb{R}^{1 \times 2}$, the matrix $M := A - \lambda BK$ is Hurwitz if and only if

$$\begin{cases} \lambda KB & > & \operatorname{tr}(A); \\ \lambda K \operatorname{adj}(A)B & < & \det(A). \end{cases} \tag{7.13}$$

This simple observation allows to prove the following Lemma.

Lemma 7.22. *Given $A \in \mathbb{R}^{2 \times 2}$ and non-singular, $B \in \mathbb{R}^2$ and $K \in \mathbb{R}^{1 \times 2}$, for every choice of the $N - 1$ positive real numbers $0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$, the following facts are equivalent:*

i) $A - \lambda BK$ is Hurwitz for every λ such that $\lambda_2 \leq \lambda \leq \lambda_N$;

ii) $A - \lambda_i BK$ *is Hurwitz for every* $i \in [2, N]$;

iii) $A - \lambda_i BK$ *is Hurwitz for* $i = 2, N$.

Proof. $i) \Rightarrow ii) \Rightarrow iii)$ are obvious. Assume that $iii)$ holds. If $A - \lambda_2 BK$ and $A - \lambda_N BK$ are Hurwitz matrices, inequalities (7.13) hold true for $\lambda = \lambda_2$ and $\lambda = \lambda_N$, but then such inequalities are obviously verified also for every λ with $\lambda_2 < \lambda < \lambda_N$. This ensures that $A - \lambda BK$ is Hurwitz for every $\lambda_2 \leq \lambda \leq \lambda_N$, namely $i)$ holds. ∎

From the previous lemma it follows that, when dealing with second-order agents, checking whether a candidate feedback matrix $K \in \mathbb{R}_+^{1 \times 2}$, $0 \leq K \leq K^*$, solves the positive consensus problem amounts to checking whether $A - \lambda_2 BK$ and $A - \lambda_N BK$ are both Hurwitz.

The following proposition provides necessary and sufficient conditions for the solvability of the positive consensus problem when dealing with two-dimensional agents.

Proposition 7.23. *Assume that each agent is described by a second-order positive state-space model* (7.12)*, with* A *Metzler, non-singular and non-Hurwitz. Then the positive consensus problem is solvable if and only if:*

i) $A - \lambda_2 BK^*$ *is a (Metzler and) Hurwitz matrix;*

ii) $\sigma(A) = \{\lambda_{\max}(A), \mu\}$, *with* $\lambda_{\max}(A) > 0$ *and* $\mu < 0$.

Moreover, when conditions $i)$ *and* $ii)$ *hold, then* $K^* \in \mathcal{K}^H$ *and*

$$\mathcal{K}^H = \left\{ K \in \mathbb{R}_+^{1 \times 2} \colon 0 \leq K \leq K^*, KA^{-1}B > \frac{1}{\lambda_2} \right\} = \mathcal{K}_2^H. \tag{7.14}$$

Proof. [Necessity] If the positive consensus problem is solvable, then $i)$ and $ii)$ follow immediately from Proposition 7.14.

[Sufficiency] We first show that, under assumption $ii)$, if inequalities (7.13) hold for $i = 2$ and some $K \geq 0$, then they hold also for $i = N$ and the same $K \geq 0$. As $KB \geq 0$, $\lambda_N KB \geq \lambda_2 KB > \text{tr}(A)$. On the other hand, condition $ii)$ implies that $\det(A) < 0$ and therefore $\lambda_2 K \text{adj}(A)B < \det(A)$ implies $K \text{adj}(A)B < 0$. This also ensures that $\lambda_N K \text{adj}(A)B < \lambda_2 K \text{adj}(A)B < \det(A)$. By Lemma 7.22, we have shown that $K \in \mathcal{K}^H$ if and only if K satisfies inequalities (7.13) for $i = 2$, namely $A - \lambda_2 BK$ is Hurwitz, which amounts to saying that $K \in \mathcal{K}_2^H$. As a result, condition $i)$ ensures that $K^* \in \mathcal{K}^H$, and $\mathcal{K}^H = \mathcal{K}_2^H$ can be described as in (7.14). ∎

Example 7.7. Consider the positive single-input agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_i(t).$$

Assume that there are $N = 3$ agents and that the Laplacian matrix of the communication graph is as in Example 7.1, so that $\ell^* = 2$ and the eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0 < \lambda_2 = 1 < \lambda_3 = 3$. Clearly the spectrum of A satisfies condition $ii)$ of Proposition 7.23. Also, we assume $K^* = \begin{bmatrix} 3/2 & 1/2 \end{bmatrix}$. Therefore $A - BK^*$ is Metzler and Hurwitz. So, the positive consensus problem is solvable and K^* is a solution (note, however, that $A - \lambda_3 BK^*$ is Hurwitz but not Metzler). ♣

7.7   The case when $\mathcal{G}$ is complete and $\bar{\ell}^* < \lambda_2$

In this section we address the case when the communication graph $\mathcal{G}$ is complete and the maximal weighted degree ℓ^* satisfies $\ell^* \leq \bar{\ell}^* < \lambda_2$. In this scenario, if we sort the eigenvalues of $\mathcal{L}$ as in (7.3), then $0 = \lambda_1 < \ell^* \leq \bar{\ell}^* < \lambda_2 \leq \cdots \leq \lambda_N$. As we will see, the apparently more restrictive situation that $\ell^* \leq \bar{\ell}^* < \lambda_2$ makes the investigation of the problem solution more difficult. Indeed, a major consequence of the fact that $\ell^* \leq \bar{\ell}^* < \lambda_2$ is that $\{K : 0 \leq K \leq K^*\} \supsetneq \mathcal{K}_2^M$. As a consequence, one of the main necessary conditions for the positive consensus problem solvability we exploited in the previous section, namely the fact that the matrix $A - \lambda_2 BK^*$ is Metzler and Hurwitz, does not hold anymore. As $\bar{\ell}^*$ is smaller than λ_2, by the way K^* is defined the matrix $A - \lambda_2 BK^*$ (and hence all matrices $A - \lambda_i BK^*$, $i \in [2, N]$) is not Metzler, and the case may occur that $A - \lambda_2 BK$ is Hurwitz even if $A - \lambda_2 BK^*$ is not.

Some necessary conditions for the problem solvability, however, can be determined, as they are independent of the relationship between $\bar{\ell}^*$ and λ_2.

Proposition 7.24. *Assume that A is an $n \times n$, $n \geq 2$, Metzler, irreducible, non-Hurwitz matrix with $\lambda_{\max}(A) > 0$, and $B \in \mathbb{R}_+^n$ is a positive vector. If the positive consensus problem is solvable, then $K^*B > \operatorname{tr}(A)/\lambda_2$.*

Proof. As the trace of a matrix equals the sum of its eigenvalues, a necessary condition for the matrices $A - \lambda_i BK$, $i \in [2, N]$, to be Hurwitz is that their traces are negative, i.e., $\operatorname{tr}(A - \lambda_i BK) = \operatorname{tr}(A) - \lambda_i KB < 0$ for every $i \in [2, N]$. However, since both B and K are positive vectors, if there exists a matrix K such that $0 \leq K \leq K^*$ and

$A - \lambda_i BK$ is Hurwitz, then $K^* B \geq KB > \frac{\mathrm{tr}(A)}{\lambda_i}$ for every $i \in [2, N]$. Finally, note that if $\mathrm{tr}(A) < 0$ the previous condition is trivial. If $\mathrm{tr}(A) \geq 0$ then $\frac{\mathrm{tr}(A)}{\lambda_2} \geq \frac{\mathrm{tr}(A)}{\lambda_i}$ for every $i \in [2, N]$. So, in both cases, condition $K^* B > \frac{\mathrm{tr}(A)}{\lambda_i}$ holds for every $i \in [2, N]$ if and only if $K^* B > \frac{\mathrm{tr}(A)}{\lambda_2}$. ∎

Remark 7.8. By looking into the proof of Proposition 7.24 we easily realize that the above condition represents a necessary condition for the problem solvability even when the communication graph is not complete. On the other hand, condition $K^* B > \mathrm{tr}(A)/\lambda_2$ trivially follows from statement i) of Proposition 7.14, and hence Proposition 7.14 provides, for the case when the communication graph is not complete, a much stronger condition with respect to that of Proposition 7.24.

Remark 7.9. In the following the undirected, unweighted, complete graph with N vertices will be denoted by $\mathcal{G}_N$. In this case $\ell^* = \ell_{ii} = N - 1$ for every $i \in [1, N]$, the associated Laplacian matrix can be expressed as $\mathcal{L} = N I_N - \mathbf{1}_N \mathbf{1}_N^\top$, and, by Lemma 7.3, the nonzero eigenvalues of $\mathcal{L}$ are $\lambda_2 = \cdots = \lambda_N = N$.

The necessary condition stated in Proposition 7.14 is not sufficient to guarantee solvability of the positive consensus problem, not even when dealing with $N = 2$ agents described by a two-dimensional ($n = 2$) model, as the following elementary example shows.

Example 7.8. Consider the positive single-input agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_i(t).$$

A is a Metzler, irreducible, non-Hurwitz matrix and the pair (A, B) is stabilizable. Assume that there are $N = 2$ agents and assume that the interconnection topology is described by the undirected, unweighted and complete graph $\mathcal{G}_2$, namely

$$\mathcal{L} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Then (see Lemma 7.3) $0 = \lambda_1 < \ell^* = 1 < \lambda_2 = 2$. Assume that $\bar{\ell}^* = \ell^* = 1$ and that the matrix K^* is given by $K^* = \bar{K}^* = \begin{bmatrix} 1 & 1 \end{bmatrix}$, and hence condition $2 = K^* B > \mathrm{tr}(A)/\lambda_2 = 1$ holds. Yet, for every $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, with $0 \leq k_i \leq 1$, $i \in [1, 2]$, $A - \lambda_2 BK$ is not Hurwitz. So, the positive consensus problem is not solvable. ♣

Example 7.9. Consider the positive single-input agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 6 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u_i(t).$$

Notice that A is a Metzler, irreducible, non-Hurwitz matrix and that the pair (A, B) is stabilizable. Consider $N = 3$ agents and assume that the interconnection topology is described by the undirected, unweighted and complete graph $\mathcal{G}_3$. In this case (see Lemma 7.3) $\ell^* = 2$ and the eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = 3$. Assume that $\bar{\ell}^* = \ell^* = 2$ and that the matrix K^* is given by $K^* = \bar{K}^* = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$. As $K^*B = 1 < \frac{4}{3} = \frac{\mathrm{tr}(A)}{\lambda_2}$, we conclude that the positive consensus problem is not solvable. ♣

7.7.1 The case when B is a canonical vector

We consider now the case when B is a monomial vector. As already remarked, we can assume without loss of generality that $B = \mathbf{e}_1$. The following result provides, in the case $\ell^* \le \bar{\ell}^* < \lambda_2$, an analysis that parallels the one carried on in subsection 7.6.4.

Proposition 7.25. *Assume that A is an $n \times n$, $n \ge 2$, Metzler, irreducible, non-Hurwitz matrix described as in (7.10). Assume also that $B = \mathbf{e}_1$ and $K^* = \begin{bmatrix} +\infty & k_2^* & \ldots & k_n^* \end{bmatrix}$.*
i) If the positive consensus problem is solvable, then every eigenvalue of A_{22} with nonnegative real part has geometric multiplicity equal to 1.
ii) If A_{22} is Hurwitz, then the positive consensus problem is solvable.

Proof. *i*) Assume that the positive consensus problem is solvable and suppose by contradiction that there exists $\mu \in \sigma(A_{22})$ with $\Re\{\mu\} \ge 0$ and geometric multiplicity $d > 1$. Partition the feedback matrix $K \in \mathbb{R}_+^{1 \times n}$, $0 \le K \le K^*$, in a consistent way wtih A, namely as $K = \begin{bmatrix} k_1 & \mathbf{k}_2 \end{bmatrix}$, where $\mathbf{k}_2 \in \mathbb{R}_+^{1 \times (n-1)}$. Now, notice that for every $i \in [2, N]$ the characteristic polynomial of $A - \lambda_i BK$ can be written as

$\det(sI_n - A + \lambda_i BK) =$

$\quad = \det(sI_n - A) + \lambda_i K \operatorname{adj}(sI_n - A)B$

$\quad = \det(sI_{n-1} - A_{22}) \left[s - a_{11} - \mathbf{r}^\top (sI_{n-1} - A_{22})^{-1}\mathbf{c} \right] + \lambda_i \begin{bmatrix} k_1 & \mathbf{k}_2 \end{bmatrix} \begin{bmatrix} \det(sI_{n-1} - A_{22}) \\ \operatorname{adj}(sI_{n-1} - A_{22})\mathbf{c} \end{bmatrix}$

$\quad = (s - a_{11} + \lambda_i k_1) \det(sI_{n-1} - A_{22}) + \left(\lambda_i \mathbf{k}_2 - \mathbf{r}^\top \right) \operatorname{adj}(sI_{n-1} - A_{22})\mathbf{c}.$

If $\mu \in \sigma(A_{22})$, then $\det(\mu I_{n-1} - A_{22}) = 0$ and, since the geometric multiplicity of μ as an eigenvalue of A_{22} is $d > 1$, it also holds that $\text{adj}(\mu I_{n-1} - A_{22}) = 0$, and hence $\det(\mu I_n - A + \lambda_i BK) = 0$ for every $K \in \mathbb{R}_+^{1 \times n}$, which contradicts the assumption of the solvability of the positive consensus problem.

ii) It is the same as the proof of the sufficiency part of Proposition 7.21. ∎

Differently from the case $\lambda_2 \leq \ell^*$, the Hurwitz condition on the submatrix A_{22} is sufficient for the problem solvability, but it is not necessary, as shown in Example 7.10 below.

Example 7.10. Consider the positive single-input agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & -1 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_i(t).$$

Notice that A is a Metzler, irreducible and non-Hurwitz matrix and that the pair (A, B) is stabilizable. Consider $N = 3$ agents and the same adjacency matrix as in Example 7.9, so that $\ell^* = 2$ and $\lambda_2 = \lambda_3 = 3$. Assume that $\bar{\ell}^* = \ell^* = 2$ and that $K^* = \bar{K}^* = \begin{bmatrix} 1 & 1 & +\infty \end{bmatrix}$. $B = \mathbf{e}_3$ and the matrix A_{11}, obtained from A by deleting the third row and the third column, is non-Hurwitz, however this does not preclude the problem solvability. It is easy to verify that the positive consensus problem is solvable since for $K = \begin{bmatrix} \frac{2}{3} & 1 & 0 \end{bmatrix} \in \mathbb{R}_+^{1 \times 3}$, with $0 \leq K \leq K^*$, we get

$$A - \lambda_2 BK = A - \lambda_3 BK = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

which is Hurwitz. ♣

7.7.2 Second-order agents

We investigate now the case when the agents are modelled by a second-order positive linear system, namely as in (7.12). As a straightforward consequence of Lemma 7.22 and of the fact that $KB \geq 0$ (and hence $\lambda_N KB \geq \lambda_2 KB$), it follows that for 2-dimensional agents the set of feedback matrices that solve the positive consensus problem is the set

of matrices $K \in \mathbb{R}_+^{1 \times 2}$ that satisfy the following LMIs:

$$K^* \geq K \geq 0;$$

$$\lambda_2 KB > \text{tr}(A);$$

$$\det(A) > \lambda_i K \, \text{adj}(A)B, \quad i = 2, N.$$

This ensures that also in the case $\ell^* \leq \bar{\ell}^* < \lambda_2$ the set of solutions is convex.

When the agents are described by second-order state-space models the case of B monomial and $K^* = \begin{bmatrix} +\infty & k_2^* \end{bmatrix}$ can be completely solved. To this aim recall that from Proposition 7.25, statement ii), it follows that condition $a_{22} < 0$ ensures the solvability of the positive consensus problem, but as we have shown this is not a necessary condition. So, in the following we assume $a_{22} \geq 0$ and investigate under which additional conditions on the matrix A and on the interconnection topology the positive consensus problem is solvable.

Proposition 7.26. *Assume that each agent is described by a second-order positive state-space model* (7.12), *with A Metzler, non-singular and non-Hurwitz. Assume also that $B = \mathbf{e}_1$, $A_{22} = a_{22} \geq 0$ and $K^* = \begin{bmatrix} +\infty & k_2^* \end{bmatrix}$. Then, the positive consensus problem is solvable if and only if $a_{21} > 0$ and the following condition holds:*

$$\max\left\{0, \frac{\text{tr}(A)a_{22}}{\lambda_2}\right\} < \frac{a_{12}a_{21}}{\ell^*} + \min\left\{\frac{\det(A)}{\lambda_2}, \frac{\det(A)}{\lambda_N}\right\}. \tag{7.15}$$

When so, there is always a solution of the form $K = \begin{bmatrix} \max\left\{0, \frac{\text{tr}(A)a_{22}}{\lambda_2}\right\} + \epsilon & \frac{a_{12}}{\ell^*} \end{bmatrix}$, with $\epsilon > 0$ and arbitrarily small.

Proof. Note first that as $B = \mathbf{e}_1$ and $a_{22} \geq 0$, if the positive consensus problem is solvable, then a_{21} must be positive, otherwise a_{22} would be an eigenvalue of every matrix $A - \lambda_i BK$, $i \in [2, N]$. Conversely, it is easy to see that condition (7.15) implies $a_{21} > 0$. So, in the following we will assume $a_{21} > 0$. Set $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. Then $KB = k_1$, $K^* = \begin{bmatrix} +\infty & \frac{a_{12}}{\ell^*} \end{bmatrix}$,

and the previous LMIs become

$$k_1 \geq 0,$$

$$k_1 > \frac{\text{tr}(A)}{\lambda_2},$$

$$\frac{a_{12}}{\ell^*} \geq k_2 \geq 0,$$

$$\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} a_{22} \\ -a_{21} \end{bmatrix} < \min \left\{ \frac{\det(A)}{\lambda_2}, \frac{\det(A)}{\lambda_N} \right\}. \tag{7.16}$$

It is clear that, as $a_{21} > 0$, inequality (7.16) holds if and only if it holds for $k_2 = k_2^* = \frac{a_{12}}{\ell^*}$. So, inequality (7.16) becomes

$$k_1 a_{22} < \frac{a_{12} a_{21}}{\ell^*} + \min \left\{ \frac{\det(A)}{\lambda_2}, \frac{\det(A)}{\lambda_N} \right\}. \tag{7.17}$$

If $\text{tr}(A) < 0$, then the only constraint on k_1 is the nonnegativity and condition (7.17) holds if and only if it holds for $k_1 = 0$. And if this is the case it also holds for $k_1 = \varepsilon$, with $\epsilon > 0$ and arbitrarily small. On the other hand, if $\text{tr}(A) \geq 0$, then the problem is solvable if and only if it is solvable by assuming $k_1 = \frac{\text{tr}(A)}{\lambda_2} + \epsilon$, with $\epsilon > 0$ arbitrarily small, and this happens if and only if

$$\frac{\text{tr}(A)}{\lambda_2} a_{22} < \frac{a_{12} a_{21}}{\ell^*} + \min \left\{ \frac{\det(A)}{\lambda_2}, \frac{\det(A)}{\lambda_N} \right\}.$$

$\blacksquare$

When the N agents are described by a second-order state-space model, $B = \mathbf{e}_1$, $A_{22} = a_{22} > 0$ and the communication among them is described by $\mathcal{G}_N$, Proposition 7.26 allows us to draw the following conclusion concerning the number of agents.

Corollary 7.27. *Assume that each agent is described by a second-order positive state-space model (7.12), with A Metzler, non-singular and non-Hurwitz. Assume also that $B = \mathbf{e}_1$, $A_{22} = a_{22} > 0$, $K^* = \begin{bmatrix} +\infty & k_2^* \end{bmatrix}$ and the communication graph is described by the undirected, unweighted and complete graph $\mathcal{G}_N$. Then, there exists $\bar{N}$ such that for every $N \geq \bar{N}$ positive consensus cannot be reached.*

Proof. The Laplacian of $\mathcal{G}_N$ has $\ell^* = N - 1$ and eigenvalues $\lambda_2 = \cdots = \lambda_N = N$. So,

condition (7.15) becomes

$$\max\left\{0, \frac{\text{tr}(A)a_{22}}{N}\right\} < \frac{a_{12}a_{21}}{N-1} + \frac{\det(A)}{N}$$

and it implies $a_{22}^2 < \frac{1}{N-1}a_{12}a_{21}$. Clearly, the term on the right goes to 0 as N tends to $+\infty$, while $a_{22}^2 > 0$. So, there exists $\bar{N}$ such for every $N \geq \bar{N}$ the previous inequality and hence condition (7.15) do not hold, i.e., positive consensus cannot be reached. ∎

Example 7.11. Consider the positive single-input agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}\mathbf{x}_i(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u_i(t).$$

Assume that the interconnection topology among the agents is described by $\mathcal{G}_N$: it follows from (7.15) that for every $N \geq \bar{N} = 4$ the positive consensus problem is not solvable.

7.8 Sufficient conditions for problem solvability

In this section we present some sufficient conditions for the problem solvability that rely on the theory of robust stability of positive systems and on the theory of robust stability of polynomials, and in general lead to matrices $A - \lambda_i BK$ that are Hurwitz but not necessarily Metzler. The key idea is to start from some $K \in \mathcal{K}_2^{MH}$, if any, and to determine sufficient conditions that make such a solution "robust", in the sense that it does not hold only for $\lambda = \lambda_2$ but for every $\lambda \in [\lambda_2, \lambda_N]$.

7.8.1 Sufficient conditions from robust stability of positive systems

In this subsection some results on the robust stability of positive linear systems are exploited to derive sufficient conditions for the solvability of the positive consensus problem. In particular, Proposition 7.28 below provides a sufficient condition for a matrix $K \in \mathbb{R}_+^{1 \times n}$ to solve the positive consensus problem expressed as an LMI.

Proposition 7.28. *Given a state feedback matrix $K \in \mathcal{K}_2^{MH}$, if any of the following two equivalent conditions holds:*

i) $|K(A - \lambda_2 BK)^{-1}B| < (\lambda_N - \lambda_2)^{-1}$;

ii) There exists $\mathbf{p} \in \mathbb{R}_+^n$, $\mathbf{p} \gg 0$, *that solves the LP*

$$\begin{cases} \mathbf{p}^\top (A - \lambda_2 BK) + K \ll 0 \\ \mathbf{p}^\top B < \frac{1}{\lambda_N - \lambda_2} \end{cases} \tag{7.18}$$

then $K \in \mathcal{K}^H$. *This implies that*

$$\mathcal{K}^H \supseteq \left\{ K \in \mathcal{K}_2^{MH} \colon |K (A - \lambda_2 BK)^{-1} B| < (\lambda_N - \lambda_2)^{-1} \right\} =: \mathcal{K}_{LP}. \tag{7.19}$$

Proof. We first prove the equivalence between conditions *i)* and *ii)*, which relies on some L_1-gain characterization for positive systems reported in Briat (2013). Consider the positive system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t) \tag{7.20}$$

$$y(t) = K\mathbf{x}(t) \tag{7.21}$$

and assume that the input u obeys the output feedback control law $u(t) = -\lambda_2 y(t) + v(t)$, where $v(t)$ is an independent input. The resulting closed-loop system is described by

$$\dot{\mathbf{x}}(t) = (A - \lambda_2 BK)\mathbf{x}(t) + Bv(t) \tag{7.22}$$

$$y(t) = K\mathbf{x}(t) \tag{7.23}$$

and its transfer function is given by $W(s) = K(sI - A + \lambda_2 BK)^{-1}B$. Since $K \in \mathcal{K}_2^{MH}$, the positive system (7.22) is asymptotically stable, and by Proposition 2 in Briat (2013) its L_1-gain g can be expressed in terms of its transfer function as $g = W(0) = -K (A - \lambda_2 BK)^{-1} B = |K(A - \lambda_2 BK)^{-1}B|$.

Condition *i)* amounts to saying that the L_1-gain of the positive system (7.22) is smaller than $(\lambda_N - \lambda_2)^{-1}$, and by Lemma 1 in Briat (2013) this is true if and only if the LP (7.18) is feasible.

To prove that condition *i)* ensures that K solves the positive consensus problem, namely $K \in \mathcal{K}^H$, we make use of a result on robust stability of positive systems reported in Son and Hinrichsen (1996). Set $\bar{A} := A - \lambda_2 BK$, $\Delta := \lambda_2 - \lambda$ and notice that for every $\lambda \in [\lambda_2, \lambda_N]$

$$A - \lambda BK = \bar{A} + \Delta BK, \tag{7.24}$$

where $\bar{A}$ is Metzler and Hurwitz and ΔBK is a perturbation matrix. Specifically, the matrix BK gives the structure of the perturbation, while Δ can be regarded as an

unknown scalar disturbance that gives the size of the perturbation.

By Theorem 5 in Son and Hinrichsen (1996), the stability radius of the positive system $\dot{\mathbf{x}}(t) = \bar{A}\mathbf{x}(t)$ with respect to perturbations described as in (7.24), namely $r(\bar{A}; B, K) := \inf\left\{|\Delta|\colon \lambda_{\max}(\bar{A} + \Delta BK) \geq 0\right\}$, can be computed as $|K(A - \lambda_2 BK)^{-1}B|^{-1}$. So, if condition $i)$ holds, then

$$r(\bar{A}; B, K) = \frac{1}{|K(A - \lambda_2 BK)^{-1}B|} > \lambda_N - \lambda_2,$$

and this ensures that $A - \lambda BK$ is Hurwitz for every $\lambda \in [\lambda_2, \lambda_N]$. Therefore, K solves the positive consensus problem and the final statement follows. ∎

Remark 7.10. The parametrization provided in Ait Rami and Tadeo (2007) can be used to provide an alternative statement of the previous conditions $i)$ and $ii)$. Indeed, we first note that condition $K \in \mathcal{K}_2^{MH}$ is equivalent to saying that

$$K = \begin{bmatrix} z_1 & z_2 & \ldots & z_n \end{bmatrix} \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}^{-1} =: \mathbf{z}^\top D^{-1},$$

where the n-dimensional vectors $\mathbf{z} > 0$, $\mathbf{d} := D\mathbf{1}_n \gg 0$, satisfy

$$A\mathbf{d} + B\mathbf{1}_n^\top \mathbf{z} < 0,$$
$$a_{ij}d_j + b_i z_j \geq 0,$$
$$\mathbf{z}^\top \leq K^* D.$$

By keeping in mind that $|K(A - \lambda_2 BK)^{-1}B| = -K(A - \lambda_2 BK)^{-1}B$ (see the proof of Proposition 7.28), we can rewrite condition $i)$ in Proposition 7.28 as:

$$-\mathbf{z}^\top \left(AD - \lambda_2 B\mathbf{z}^\top\right)^{-1} B < \frac{1}{\lambda_N - \lambda_2}$$

and condition $ii)$ as the existence of a vector $\mathbf{p} \in \mathbb{R}_+^n$, $\mathbf{p} \gg 0$, that solves the LP

$$\begin{cases} \mathbf{p}^\top \left(AD - \lambda_2 B\mathbf{z}^\top\right) + \mathbf{z}^\top \ll 0 \\ \mathbf{p}^\top B < \frac{1}{\lambda_N - \lambda_2} \end{cases}$$

The LMI formulation (7.18) of Proposition 7.28 is useful when some matrix $K \in \mathcal{K}_2^{MH}$, is already available, and we want to check whether $A - \lambda BK$ is Hurwitz for every

$\lambda \in [\lambda_2, \lambda_N]$. On the other hand, if we are looking for a feedback matrix $K \in \mathcal{K}_2^{MH}$, such that the sufficient condition of Proposition 7.28 holds, we should first try with the matrix $\hat{K}_2$, as shown in Proposition 7.29 below.

Proposition 7.29. *Assume that A is an $n \times n$, $n \geq 2$, Metzler, irreducible and non-Hurwitz matrix, and $B \in \mathbb{R}_+^n$ is a positive vector. Assume also that $A - \lambda_2 B \hat{K}_2$ is a (Metzler and) Hurwitz matrix. If the sufficient condition given in Proposition 7.28 holds for some $K \in \mathcal{K}_2^{MH}$, namely if $\mathcal{K}_{LP}$ is not empty, then:*

i) The Laplacian eigenvalues are such that $\lambda_2 > \frac{\lambda_N}{2}$;

ii) $\hat{K}_2 \in \mathcal{K}_{LP}$.

In addition, the set $\mathcal{K}_{LP}$ can be described as

$$\mathcal{K}_{LP} = \left\{ K : 0 \leq K \leq \hat{K}_2 \text{ and } KA^{-1}B > (2\lambda_2 - \lambda_N)^{-1} \right\}.$$

Proof. Notice that $K(A - \lambda_2 BK)^{-1} B$ can be rewritten in the following way

$$\begin{aligned} K(A - \lambda_2 BK)^{-1} B &= K \left(I_n - \lambda_2 A^{-1} BK \right)^{-1} A^{-1} B \\ &= \left(KA^{-1}B \right) \left(1 - \lambda_2 KA^{-1}B \right)^{-1}. \end{aligned}$$

Then, any feedback matrix $K \in \mathcal{K}_2^{MH}$ satisfies condition $i)$ of Proposition 7.28 if and only if

$$\frac{KA^{-1}B}{\lambda_2(KA^{-1}B) - 1} < (\lambda_N - \lambda_2)^{-1},$$

i.e., if and only if

$$(\lambda_N - 2\lambda_2)(KA^{-1}B) < -1. \tag{7.25}$$

$i)$ Suppose by contradiction that $\lambda_2 < \frac{\lambda_N}{2}$. If this were the case, since by Lemma 7.10 the vector $A^{-1}B$ is positive, there could not exist $K \in \mathcal{K}_2^{MH}$ satisfying condition (7.25), and hence $\mathcal{K}_{LP} = \emptyset$.

$ii)$ Clearly, since $A^{-1}B$ is a positive vector, either $\hat{K}_2$ satisfies condition (7.25) or any matrix $K \in \mathcal{K}_2^{MH}$ does not satisfy condition (7.25). ∎

Statement $i)$ of the above proposition provides a necessary condition for the set $\mathcal{K}_{LP}$ to be non-empty that involves only the eigenvalues of the Laplacian matrix. Interestingly enough, such a condition holds true for every undirected, weighted and complete graph with $\ell^* < \lambda_2$, as shown in the following lemma.

Lemma 7.30. *Let $\mathcal{G}$ be an undirected, weighted and complete graph with N vertices. If $\ell^* < \lambda_2$, then $\lambda_2 > \frac{\lambda_N}{2}$.*

Proof. Consider the Metzler matrix $\bar{\mathcal{L}} = \lambda_2 I_N - \mathcal{L}$ and notice that: (1) all the diagonal entries of $\bar{\mathcal{L}}$ are positive since $\ell^* < \lambda_2$; (2) for every $i, j \in [1, N]$, $i \neq j$, it holds $\left[\bar{\mathcal{L}}\right]_{ij} > 0$ since $\mathcal{G}$ is complete. This means that $\bar{\mathcal{L}}$ is a strictly positive, and hence irreducible, matrix. Therefore, its Frobenius eigenvalue is necessarily simple. But then, as the spectrum of $\bar{\mathcal{L}}$ is given by

$$\sigma(\bar{\mathcal{L}}) = \{\lambda_2, 0, \lambda_2 - \lambda_3, \ldots, \lambda_2 - \lambda_N\},$$

it follows that $\lambda_2 > |\lambda_2 - \lambda_N|$, i.e., $\lambda_2 > \frac{\lambda_N}{2}$. ∎

When looking for feedback matrices $K \in \mathcal{K}_2^{MH}$ such that the sufficient condition of Proposition 7.28 holds, a change of variable is needed in order to restate the problem in the form of a set of LMIs. This is shown in Proposition 7.31 below.

Proposition 7.31. *Assume there exist $\mathbf{v} = [v_i] \in \mathbb{R}_+^n$, $\mathbf{v} \gg 0$, and $\mu_i \in \mathbb{R}_+$, $i \in [1, n]$, such that the LP*

$$A\mathbf{v} - \lambda_2 B \sum_{i=1}^{n} \mu_i + B \ll 0 \tag{7.26a}$$

$$\sum_{i=1}^{n} \mu_i < \frac{1}{\lambda_N - \lambda_2} \tag{7.26b}$$

$$0 \leq \mu_i \leq k_i^* v_i \tag{7.26c}$$

is feasible. Then, the positive consensus problem is solvable and a possible solution is given by $K = [k_i] = \left[\frac{\mu_i}{v_i}\right]$.

Proof. Consider again the positive system (7.20) with control input $u(t) = -\lambda_2 y(t) + v(t)$. By Lemma 3 in Briat (2013) there exists an output feedback matrix $K \in \mathbb{R}^{1 \times n}$ such that the resulting closed-loop system (7.22) is asymptotically stable and exhibits an L_1-gain smaller than $(\lambda_N - \lambda_2)^{-1}$ (which amounts to saying that the sufficient condition of Proposition 7.28 holds) if and only if the LP (7.26a)-(7.26b) is feasible. Then, the additional inequality (7.26c) ensures that $0 \leq K \leq K^*$ and the thesis directly follows from Proposition 7.28. ∎

7.8.2 A sufficient condition from the theory of robust stability for polynomials

Another sufficient condition for a matrix $K \in \mathbb{R}_+^{1 \times n}$ to solve the positive consensus problem can be derived from a criterion of robust stability of polynomials, i.e., stability of polynomials with uncertain coefficients. Before proceeding, consistently with Barmish (1994), we introduce the notion of *Hurwitz matrix* associated with a given polynomial.

Definition 7.1. Consider the polynomial $d(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$ of degree n $(a_n \neq 0)$. The *Hurwitz matrix* associated with $d(s)$ is the $n \times n$ real matrix

$$
H_{d,n} := \begin{bmatrix}
a_{n-1} & a_{n-3} & a_{n-5} & \ldots & 0 \\
a_n & a_{n-2} & a_{n-4} & \ldots & 0 \\
0 & a_{n-1} & a_{n-3} & \ldots & 0 \\
0 & a_n & a_{n-2} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & a_0
\end{bmatrix}
$$

For the sake of clarity, the matrix $H_{d,n}$ has the following structure: the first and the second rows coincide with the second and the first row of the Routh table, respectively, completed with zeros; every couple of consecutive rows is obtained by the previous couple of rows by means of a one-step shift to the right (and the insertion of a 0 on the left). Note that if $a_0 \neq 0$, then $H_{d,n}$ is non-singular, Barmish (1994).

Proposition 7.32. *Denote by $d(s)$ the characteristic polynomial of $A \in \mathbb{R}^{n \times n}$, i.e., $d(s) := \det(sI_n - A)$, and let $\bar{\lambda} \in [\lambda_2, \lambda_N]$ and $K \in \mathbb{R}_+^{1 \times n}$, $0 \leq K \leq K^*$, be such that $A - \bar{\lambda}BK$ is Hurwitz. Define the polynomial, of degree (at most) $n - 1$, $q(s) := K \operatorname{adj}(sI_n - A)B$, and denote by $H_{q,n} \in \mathbb{R}^{n \times n}$ the Hurwitz matrix associated with $q(s)$ regarded as a polynomial of degree n. Define also the matrix pencil*

$$
H_{p,n}(\lambda) := H_{d,n} + \lambda H_{q,n}, \quad \lambda \in \mathbb{R}, \tag{7.27}
$$

and denote by $0 \leq \mu_1 < \cdots < \mu_k$ the nonnegative, real, distinct eigenvalues of $H_{p,n}(\lambda)$, namely the nonnegative, real, distinct values for which the matrix pencil $H_{p,n}(\lambda)$ becomes a singular matrix. If there exists $j \in [1, k]$ such that $[\lambda_2, \lambda_N] \subset (\mu_j, \mu_{j+1})$, where $\mu_{k+1} = +\infty$, then for every $i \in [2, N]$ the matrix $A - \lambda_i BK$ is Hurwitz, i.e., K solves the positive consensus problem.

Before proceeding with the proof of Proposition 7.32 we need to state the following lemma on the robust stability of polynomials.

Lemma 7.33. *Let $d(s)$ and $q(s)$ be two polynomials such that $\deg d(s) = n > \deg q(s)$. Consider the family of polynomials parametrized by λ, $p(s, \lambda) = d(s) + \lambda q(s)$, where $\lambda \in [\lambda_-, \lambda_+]$ and $\lambda_- > 0$. Assume that there exists $\bar{\lambda} \in [\lambda_-, \lambda_+]$ such that $p(s, \bar{\lambda})$ is Hurwitz. Then, the polynomial $p(s, \lambda)$ is Hurwitz for every $\lambda \in [\lambda_-, \lambda_+]$ if and only if the Hurwitz matrix associated with $p(s, \lambda)$, namely $H_{p,n}(\lambda)$, is nonsingular for every $\lambda \in [\lambda_-, \lambda_+]$.*

The proof of Lemma 7.33 above follows from a simple application of Lemma 4.8.3 in Barmish (1994) once we notice that for every $\lambda \in [\lambda_-, \lambda_+]$ the polynomial $p(s, \lambda)$ can be written as $p(s, \lambda) = p(s, \bar{\lambda}) + (\lambda - \bar{\lambda})q(s)$. For the sake of brevity, the proof is omitted.

We can now prove Proposition 7.32.

Proof. (Proposition 7.32) Introduce the characteristic polynomial $p(s, \lambda)$ of the matrix $A - \lambda BK$

$$p(s, \lambda) := \det(sI_n - A + \lambda BK) = d(s) + \lambda q(s),$$

and note that the Hurwitz matrix associated with $p(s, \lambda)$ is the matrix pencil $H_{p,n}(\lambda)$ defined in (7.27). Since by hypothesis $p(s, \bar{\lambda})$ is Hurwitz and there exists $j \in [1, k]$ such that $H_{p,n}(\lambda)$ is nonsingular for every $\lambda \in (\mu_j, \mu_{j+1}) \supset [\lambda_2, \lambda_N]$, then it follows from Lemma 7.33 that $p(s, \lambda)$ is Hurwitz for every $\lambda \in [\lambda_2, \lambda_N]$, and hence K solves the positive consensus problem. ∎

Example 7.12. Consider the positive single-input agent:

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0.5 & 2 & -3 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} u_i(t)$$

Assume that there are $N = 5$ agents and that the Laplacian matrix of the communication graph is the following one:

$$\mathcal{L} = \begin{bmatrix} 1.05 & -0.8 & -0.25 & 0 & 0 \\ -0.8 & 1.05 & 0 & -0.25 & 0 \\ -0.25 & 0 & 1 & -0.25 & -0.5 \\ 0 & -0.25 & -0.25 & 1 & -0.5 \\ 0 & 0 & -0.5 & -0.5 & 1 \end{bmatrix}.$$

In this case $\ell^* = 1.05$, $\lambda_2 = 0.3876$, $\lambda_5 = 1.9405$ and $\bar{K}^* = \begin{bmatrix} 0 & 1.9048 & 0.4762 \end{bmatrix}$. We assume $K^* = \bar{K}^*$. Notice that the necessary condition of Proposition 7.14 is satisfied since $\lambda_{\max}(A) = 1.4901$ is the only nonnegative real eigenvalue of $\sigma(A)$ and $A - \lambda_2 B K^*$ is a (Metzler and) Hurwitz matrix. Now, since $k_1^* = 0$, for every $K \in \mathbb{R}_+^{1 \times 3}$, $0 \leq K \leq K^*$, the matrix $H_{q,n}$ takes the following form:

$$H_{q,n} = \begin{bmatrix} 2k_2 + k_3 & 7k_2 + 4k_3 & 0 \\ 0 & 9k_2 + 4k_3 & 0 \\ 0 & 2k_2 + k_3 & 7k_2 + 4k_3 \end{bmatrix},$$

and the matrix pencil defined in (7.27) results

$$H_{p,n}(\lambda) = \begin{bmatrix} 3 & -5.5 & 0 \\ 1 & -3 & 0 \\ 0 & 3 & -5.5 \end{bmatrix}$$
$$+ \lambda \begin{bmatrix} 2k_2 + k_3 & 7k_2 + 4k_3 & 0 \\ 0 & 9k_2 + 4k_3 & 0 \\ 0 & 2k_2 + k_3 & 7k_2 + 4k_3 \end{bmatrix}$$

Then, it is easy to verify that the eigenvalues of the matrix pencil are $\tilde{\mu}_1 = \frac{5.5}{7k_2 + 4k_3}$ and the zeros of the polynomial $g(\tilde{\mu}) = \tilde{\mu}^2 (2k_2 + k_3)(9k_2 + 4k_3) + \tilde{\mu}(14k_2 + 5k_3) - 3.5$. If we choose $K = K^*$ we have $\mu_1 = 0.0951 < \mu_2 = 0.3609$, and hence $[\lambda_2, \lambda_5] \subset (\mu_2, +\infty)$ and by Proposition 7.32 K^* solves the positive consensus problem. ♣

Optimal Control of Positive Bilinear Systems

In this chapter we address $\mathcal{L}_1$-optimal and $\mathcal{H}_\infty$-optimal control problems for a particular class of Positive Bilinear Systems that arise in drug dosage design for HIV treatment.

In the following we develop an approach and propose a solution by referring to this specific practical application of the considered class of models. However, the extensions of such approach and solution to the whole class of Positive Bilinear Systems is quite immediate.

The results of this chapter are somewhat preliminary, and this problem is still under investigation. The content of the present chapter is the subject of:

Zorzan I. and Rantzer A. $\mathcal{L}_1$ and $\mathcal{H}_\infty$ optimal control of positive bilinear systems. In *Proceedings of the 56th IEEE Conference on Decision and Control*, pages 727–732, Melbourne, Australia, 2017.

Input-output gain characterization of Positive Systems

Robustness of a dynamical system against external disturbances and parametric uncertainties is intimately related to its input-output gain. It is then reasonable to consider as performance index of the system the induced norm of the system transfer matrix, defined as the maximal amplification (in terms of some signal norm) from input to output. In recent times, input-output gain analysis of positive systems has been extensively studied,

and many efforts have been made to translate the stabilization problem with performance requirements either into an LMI or an LP. Tanaka and Langbort (2011) exploited the fact that for internally positive systems the storage function matrix appearing in the "Bounded Real Lemma" can be chosen to be diagonal without loss of generality, and proved that, when dealing with internally positive systems, the design of structured $\mathcal{H}_\infty$ static state feedback controllers achieving stability and internal positivity in closed loop can be formulated as a convex and tractable problem. Ebihara et al. (2011) pointed out that L_1 gain[1] with weightings on the input and output signals plays an essential role in the stability analysis of interconnected positive systems. L_1 optimal feedback controller synthesis for positive systems with given weighting vectors has been addressed in Ebihara, Peaucelle, and Arzelier (2012b): it is proved that an optimal state-feedback gain, which is designed for a fixed positive system and a fixed pair of weighing vectors, is robustly optimal against variations of the input matrix, the direct feedthrough matrix of the controlled positive system as well as variations of the weighting vector for the disturbance input signal. Briat (2013) employed linear copositive Lyapunov functions and dissipativity theory with linear supply-rates to address robust stability and stabilization of uncertain linear positive systems, and provided new characterizations of $\mathcal{L}_1$ and $\mathcal{H}_\infty$ gains. Rantzer (2015) developed methods for the synthesis of distributed controllers based on linear Lyapunov functions and storage functions, thus proving that $\mathcal{H}_\infty$ analysis can be extended to large-scale positive systems.

In the following, focusing on a particular class of Positive Bilinear Systems, we address both $\mathcal{L}_1$-optimal and $\mathcal{H}_\infty$-optimal control problems. Specifically, we exploit existent characterizations of the $\mathcal{L}_1$-norm for Positive Systems to provide an equivalent convex formulation of the $\mathcal{L}_1$-optimal control problem. As for the $\mathcal{H}_\infty$-case, we propose an algorithm based on the iterative solution of a convex feasibility problem, that approximates an $\mathcal{H}_\infty$-optimal controller with arbitrary accuracy.

The considered class of Positive Bilinear Systems has been employed in a number of applications, e.g., in network theory and in this context optimal control issues concern the proper selection of leaders in a directed graph, Dhingra, Colombino, and Jovanovic (2016). Another interesting application comes from the biological area, specifically from the so called quasispecies model. This model has been often adopted to describe the behaviour of a group of self replicating genotypes undergoing selection and mutation phenomena. This is the case, for instance, when modeling the evolutionary dynamics either of heterogeneous tumor cells, Giordano, Rantzer, and Jonsson (2016), or of a

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[1]Differently from the notion of $\mathcal{L}_1$ gain adopted in the present chapter, the one introduced in Ebihara et al. (2011) involves weighting vectors (possibly different from the vector of all ones) on the input and output signals of the system.

population of HIV viral variants, Jonsson, Matni, and Murray (2013); Jonsson, Rantzer, and Murray (2014); Rantzer and Bernhardsson (2014). It is worth saying that, in the context of HIV infection treatment, a certain number of recent contributions have been devoted to the problem of optimal drug scheduling. For instance, Hernandez-Vargas et al. (2011); Colaneri, Middleton, Chen, Caporale, and Blanchini (2014) addressed the $\mathcal{L}_1$-optimal control problem over finite horizon; while Dhingra et al. (2016) established convexity with respect to the control inputs of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm. Finally, Colombino, Dhingra, Jovanovic, and Smith (2016) considered the case when uncertainty is introduced into the Positive Bilinear System: convexity of the structured singular value with respect to the control parameters still holds. However, minimization of the structured singular value cannot be formulated as a conic program that can be efficiently solved with standard solvers, and, for this reason, an expression of its subgradient is derived.

8.1 Robustness measures and their characterizations

Consider a (linear) Multi-Input Multi-Output Positive System described by the following equations

$$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{w}(t) \\ \mathbf{y}(t) = C\mathbf{x}(t), \qquad\qquad t \in \mathbb{R}_+, \end{cases} \tag{8.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the n-dimensional state vector at time t, $\mathbf{w}(t) \in \mathbb{R}^m$ is the (arbitrary) m-dimensional external disturbance at time t, $\mathbf{y}(t) \in \mathbb{R}^p$ is the p-dimensional output vector, $A \in \mathbb{R}^{n \times n}$ is a Metzler matrix, and $B \in \mathbb{R}_+^{n \times m}$, $C \in \mathbb{R}_+^{p \times n}$ are positive matrices. We denote by $G(s)$ the transfer matrix of system (8.1) from disturbance $\mathbf{w}$ to output $\mathbf{y}$, defined as

$$G(s) := C(sI - A)^{-1}B.$$

In order to quantify robustness of system (8.1) against the presence of the external disturbance $\mathbf{w}$, the following two metrics will be considered.

Definition 8.1. The $\mathcal{L}_1$-norm of system (8.1) is the maximal amplification in terms of $\mathcal{L}_\infty$-norm, from disturbance $\mathbf{w}$ to output $\mathbf{y}$, i.e.,

$$\|G\|_{\mathcal{L}_1} := \sup_{\mathbf{w} \in \mathcal{L}_\infty^m[0,\infty)} \frac{\|\mathbf{y}\|_\infty}{\|\mathbf{w}\|_\infty}.$$

Definition 8.2. The $\mathcal{H}_\infty$-norm of system (8.1) is the maximal amplification in terms of

$\mathcal{L}_2$-norm, from disturbance $\mathbf{w}$ to output $\mathbf{y}$, i.e.,

$$\|G\|_{\mathcal{H}_\infty} := \sup_{\mathbf{w} \in \mathcal{L}_2^m[0,\infty)} \frac{\|\mathbf{y}\|_2}{\|\mathbf{w}\|_2}.$$

When dealing with Positive Systems we can assess whether the $\mathcal{L}_1$-norm of the system is less than γ, with $\gamma > 0$ fixed, by solving an LP. This is shown in Lemma 8.1 below, Briat (2013); Rantzer (2015).

Lemma 8.1. *Consider a Positive System described as in* (8.1). *The following statements are equivalent:*

i) The Metzler matrix A is Hurwitz and $\|G\|_{\mathcal{L}_1} < \gamma$;

ii) There exists $\boldsymbol{\lambda} \in \mathbb{R}_+^n$, $\boldsymbol{\lambda} \gg 0$, such that

$$A\boldsymbol{\lambda} + B\mathbf{1}_m < 0 \tag{8.2}$$

$$C\boldsymbol{\lambda} - \gamma\mathbf{1}_p < 0. \tag{8.3}$$

A remarkable feature of Positive Systems is that the $\mathcal{H}_\infty$-norm is intimately related to the static gain, as stated in the following lemma, Rantzer (2011).

Lemma 8.2. *Consider a Positive System described as in* (8.1), *and assume that $A \in \mathbb{R}^{n \times n}$ is a (Metzler and) and Hurwitz matrix. Then, the $\mathcal{H}_\infty$-norm of the system equals the largest singular value of the transfer matrix $G(s)$ evaluated at $s = 0$, namely*

$$\|G\|_{\mathcal{H}_\infty} = \bar{\sigma}(G(0)).$$

8.2 Problem statement: optimal control of Positive Bilinear Systems with application to HIV treatment

A well-known model describing the evolutionary dynamics of a population of HIV viral variants in the presence of a combination of drug is the so called quasispecies model, Jonsson et al. (2013, 2014); Colombino et al. (2016); Dhingra et al. (2016)). The model is described by the following equations

$$\begin{cases} \dot{\mathbf{x}}(t) = \left(A + \sum_{i=1}^M u_i D_i\right)\mathbf{x}(t) + \mathbf{w}(t) \\ \mathbf{y}(t) = C\mathbf{x}(t), \qquad\qquad\qquad t \in \mathbb{R}_+, \end{cases} \tag{8.4}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector whose ith component $x_i(t)$ represents the population of the ith HIV viral variant at time $t \geq 0$, $\mathbf{w}(t) \in \mathbb{R}^n$ is an arbitrary positive disturbance, and $\mathbf{y}(t) \in \mathbb{R}^p$ is the output. $C \in \mathbb{R}_+^{p \times n}$ is a positive matrix. It is common in this context to consider as output either the total viral population, i.e., $C = \mathbf{1}_n^\top$, or the concentration of a particularly dangerous viral variant, i.e., $C = \mathbf{e}_j^\top$ for some $j \in [1, n]$. The matrix $A \in \mathbb{R}^{n \times n}$ is a Metzler and non-Hurwitz matrix describing the evolutionary dynamics of the entire viral population (composed of n mutants) in the absence of any drug. Specifically, each diagonal entry a_{ii}, $i \in [1, n]$, represents the replication rate of mutant i, and each off-diagonal entry a_{ij}, $i \neq j$, $i, j \in [1, n]$, represents the mutation rate from viral variant j to viral variant i. We assume that M different drugs can be simultaneously administered, and their effects on the n variants are described by the diagonal matrices D_i, $i \in [1, M]$. The control inputs u_i, $i \in [1, M]$, represent the dosage of each drug.

The control objective is to design a constant drug concentration profile, say a controller $\mathbf{u} := \begin{bmatrix} u_1 & \ldots & u_M \end{bmatrix}^\top$, $\mathbf{u} > 0$, that makes the resulting closed-loop system asymptotically stable, maximizes its robustness against the influence of the external disturbance $\mathbf{w}$ (with respect to either the $\mathcal{L}_1$-norm or the $\mathcal{H}_\infty$-norm), and satisfies the constraint $\mathbf{1}_M^\top \mathbf{u} \leq 1$. The requirement $\mathbf{1}_M^\top \mathbf{u} \leq 1$ is justified by the practical need of keeping the total drug concentration small. Indeed, due to the side effects that these drugs inevitably have, it is reasonable to assume that the total amount of drug tolerated by the patients is limited.

We can now state in a formal way the robust control problems we are going to solve in the following sections.

$\mathcal{L}_1$-*optimal robust control problem for HIV treatment:* determine, if possible, a static controller $\mathbf{u} \in \mathbb{R}_+^M$, with $\mathbf{u} > 0$ and $\mathbf{1}_M^\top \mathbf{u} \leq 1$, that stabilizes the HIV model (8.4) and minimizes its $\mathcal{L}_1$-norm.

$\mathcal{H}_\infty$-*optimal robust control problem for HIV treatment:* determine, if possible, a static controller $\mathbf{u} \in \mathbb{R}_+^M$, with $\mathbf{u} > 0$ and $\mathbf{1}_M^\top \mathbf{u} \leq 1$, that stabilizes the HIV model (8.4) and minimizes its $\mathcal{H}_\infty$-norm.

8.3 $\mathcal{L}_1$-optimal drugs concentration design

In this section we consider the problem of designing a drug concentration profile that is optimal with respect to the $\mathcal{L}_1$-norm of the system. We will show that this $\mathcal{L}_1$-optimal robust control problem can be equivalently formulated as a convex optimization problem. As a consequence, a solution, if any, can always be computed with standard optimization software.

By applying Lemma 8.1 to the HIV model (8.4), it follows that a stabilizing controller $\mathbf{u} \in \mathbb{R}_+^M$ satisfying the constraint $\mathbf{1}_M^\top \mathbf{u} \leq 1$ exists if and only if scalar $\gamma > 0$ and vectors $\mathbf{u} \in \mathbb{R}_+^M$, $\boldsymbol{\lambda} \in \mathbb{R}_+^n$, $\boldsymbol{\lambda} \gg 0$, can be found such that

$$\left(A + \sum_{i=1}^M u_i D_i \right) \boldsymbol{\lambda} + \mathbf{1}_n \leq 0$$

$$C\boldsymbol{\lambda} - \gamma \mathbf{1}_p \leq 0$$

$$\mathbf{1}_M^\top \mathbf{u} \leq 1.$$

Hence, solving the $\mathcal{L}_1$-optimal robust control problem means solving, over variables $\mathbf{u} \in \mathbb{R}_+^M$, $\boldsymbol{\lambda} \in \mathbb{R}_+^n$, $\gamma \in \mathbb{R}_+$, the following minimization problem

$$\underset{\substack{\mathbf{u}>0, \boldsymbol{\lambda} \gg 0, \\ \gamma > 0}}{\text{minimize}} \quad \gamma \tag{8.5a}$$

subject to

$$\left(A + \sum_{i=1}^M u_i D_i \right) \boldsymbol{\lambda} + \mathbf{1}_n \leq 0 \tag{8.5b}$$

$$C\boldsymbol{\lambda} - \gamma \mathbf{1}_p \leq 0 \tag{8.5c}$$

$$\mathbf{1}_M^\top \mathbf{u} \leq 1. \tag{8.5d}$$

Clearly, the product $\sum_{i=1}^M u_i D_i \boldsymbol{\lambda}$ in the inequality constraint (8.5b) makes the minimization problem (8.5) non-convex. We now provide an equivalent convex formulation of problem (8.5). In the following, and for the rest of the chapter, given a vector $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{z} := \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}^\top$, we denote by $e^{\mathbf{z}} \in \mathbb{R}^n$ the vector $e^{\mathbf{z}} := \begin{bmatrix} e^{z_1} & \dots & e^{z_n} \end{bmatrix}^\top$. Similarly, for a scalar $z \in \mathbb{R}$, the symbol $e^{\mathbf{z}-z}$ denotes the vector $e^{\mathbf{z}-z} := \begin{bmatrix} e^{z_1-z} & \dots & e^{z_n-z} \end{bmatrix}^\top$.

Proposition 8.3. *The non-convex optimization problem* (8.5) *is equivalent to the fol-*

lowing convex problem

$$\underset{\substack{\mathbf{u}\in\mathbb{R}_+^M, \mathbf{z}\in\mathbb{R}^n, \\ \gamma\in\mathbb{R}_+}}{\text{minimize}} \quad \gamma \tag{8.6a}$$

subject to

$$\text{row}_j(A)e^{\mathbf{z}-z_j} + \sum_{i=1}^M u_i\left[D_i\right]_j + e^{-z_j} \leq 0, \quad j \in [1,n] \tag{8.6b}$$

$$Ce^{\mathbf{z}} - \gamma\mathbf{1}_p \leq 0 \tag{8.6c}$$

$$\mathbf{1}_M^\top\mathbf{u} \leq 1. \tag{8.6d}$$

Proof. Consider the non-convex problem (8.5) and notice that the jth row, $j \in [1,n]$, of the vector inequality (8.5b) can be rewritten as

$$\text{row}_j(A)\boldsymbol{\lambda} + \sum_{i=1}^M u_i\left[D_i\right]_j \lambda_j + 1 \leq 0, \tag{8.7}$$

where we exploited the fact that the D_i, $i \in [1,M]$, are diagonal matrices. Since $\boldsymbol{\lambda} \gg 0$, i.e., $\lambda_j > 0$ for every $j \in [1,n]$, we can divide both sides of inequality (8.7) by λ_j, thus obtaining

$$\text{row}_j(A)\frac{\boldsymbol{\lambda}}{\lambda_j} + \sum_{i=1}^M u_i\left[D_i\right]_j + \frac{1}{\lambda_j} \leq 0. \tag{8.8}$$

Now, define the new variables $z_j := \ln\lambda_j$, $j \in [1,n]$ (again, this is always possible since $\boldsymbol{\lambda}$ is a strictly positive vector). In the new variables, inequality (8.8) becomes

$$\text{row}_j(A)e^{\mathbf{z}-z_j} + \sum_{i=1}^M u_i\left[D_i\right]_j + e^{-z_j} \leq 0,$$

where we have set $\mathbf{z} := \begin{bmatrix} z_1 & \ldots & z_n \end{bmatrix}^\top$, and

$$e^{\mathbf{z}-z_j} := \begin{bmatrix} e^{z_1-z_j} & \ldots & e^{z_n-z_j} \end{bmatrix}^\top.$$

Finally, notice that with this change of variables inequality (8.5c) becomes

$$Ce^{\mathbf{z}} - \gamma\mathbf{1}_p \leq 0,$$

and thus equivalence between minimization problems (8.5) and (8.6) is proven. ∎

The convex minimization problem (8.6) allows to efficiently solve the $\mathcal{L}_1$-optimal robust control problem, as summarized in the following theorem.

Theorem 8.4. *The $\mathcal{L}_1$-optimal robust control problem for HIV treatment is solvable if and only if the convex minimization problem (8.6) is solvable. When so, the $\mathcal{L}_1$-optimal controller $\mathbf{u}^*_{\mathcal{L}_1}$ is given by the solution $\mathbf{u}^*$ of problem (8.6).*

8.4 $\mathcal{H}_\infty$-optimal drugs concentration design

This section tackles the problem of determining a static controller $\mathbf{u} \in \mathbb{R}^M_+$, with $\mathbf{1}^\top_M \mathbf{u} \leq 1$, that stabilizes the HIV model (8.4) and minimizes its $\mathcal{H}_\infty$-norm. Differently from the $\mathcal{L}_1$-case, there is no way, to the authors' knowledge, to provide an equivalent convex formulation of the $\mathcal{H}_\infty$-optimal control problem. However, in the following we will propose a procedure to determine a controller whose performance is arbitrarily close to the one obtained by the $\mathcal{H}_\infty$-optimal controller. To this aim, we need an additional assumption on the HIV model (8.4), specifically the fact that the state space matrix A is an irreducible matrix, a straightforward assumption from now on.

The following proposition plays for the $\mathcal{H}_\infty$-case and under the irreducibility assumption on the matrix A, a similar role to the one played by Lemma 8.1 for the $\mathcal{L}_1$-case.

Proposition 8.5. *Consider a Positive System described as in (8.1), and assume that the Metzler matrix $A \in \mathbb{R}^{n \times n}$ is irreducible. The following statements are equivalent:*

i) The matrix A is Hurwitz and $\|G\|_{\mathcal{H}_\infty} = \gamma$;

ii) There exist $\boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathbb{R}^n_+$, $\boldsymbol{\eta}, \boldsymbol{\zeta} \gg 0$, such that

$$A\boldsymbol{\eta} + \boldsymbol{\zeta} = 0 \tag{8.9}$$

$$C^\top C\boldsymbol{\eta} + \gamma^2 A^\top \boldsymbol{\zeta} = 0. \tag{8.10}$$

Proof. i) $\Rightarrow$ ii) Notice that system (8.1) is positive and asymptotically stable, and hence, by Lemma 8.2, $\|G\|_\infty = \bar{\sigma}(G(0))$. By explicitly writing the expression of $G(0)$ and recalling the definition of singular value, we have

$$\begin{aligned} \gamma^2 &= \lambda_{\max}\left(G^\top(0)G(0)\right) \\ &= \lambda_{\max}\left(-A^{-\top}C^\top\left(-CA^{-1}\right)\right). \end{aligned} \tag{8.11}$$

Since A is Metzler, irreducible and Hurwitz, the matrix $-A^{-1}$ is strictly positive, and hence the matrix $M_+ := -A^{-\top}C^\top\left(-CA^{-1}\right)$ is positive and irreducible, too. Then, from the Perron-Frobenius theorem, (8.11) holds if and only if γ^2 is the Perron-Frobenius eigenvalue of M_+ and there exists a strictly positive eigenvector (the Perron-Frobenius eigenvector) $\boldsymbol{\zeta} \in \mathbb{R}_+^n$, $\boldsymbol{\zeta} \gg 0$, corresponding to γ^2. This means that

$$
\begin{aligned}
\gamma^2\boldsymbol{\zeta} &= M_+\boldsymbol{\zeta} \\
&= -A^{-\top}C^\top\left(-CA^{-1}\right)\boldsymbol{\zeta} \\
&= -A^{-\top}C^\top C\boldsymbol{\eta},
\end{aligned}
$$

where we have set $\boldsymbol{\eta} := -A^{-1}\boldsymbol{\zeta}$. Clearly, $\boldsymbol{\eta} \gg 0$ since $-A^{-1} \gg 0$ and $\boldsymbol{\zeta} \gg 0$. So, strictly positive vectors $\boldsymbol{\eta}$, $\boldsymbol{\zeta} \in \mathbb{R}_+^n$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta} \gg 0$, exist such that

$$
\begin{aligned}
-A^{-1}\boldsymbol{\zeta} &= \boldsymbol{\eta} \\
-A^{-\top}C^\top C\boldsymbol{\eta} &= \gamma^2\boldsymbol{\zeta},
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
A\boldsymbol{\eta} + \boldsymbol{\zeta} &= 0 \\
C^\top C\boldsymbol{\eta} + \gamma^2 A^\top\boldsymbol{\zeta} &= 0,
\end{aligned}
$$

namely statement ii) holds.

$ii) \Rightarrow i$) The fact that A is Hurwitz directly follows from equation (8.9) (recall Lemma 3.3, statement ii)). By substituting $\boldsymbol{\eta} = -A^{-1}\boldsymbol{\zeta}$ into equation (8.10) we get

$$
C^\top C\left(-A^{-1}\boldsymbol{\zeta}\right) + \gamma^2 A^\top\boldsymbol{\zeta} = 0, \tag{8.12}
$$

namely

$$
\begin{aligned}
\gamma^2\boldsymbol{\zeta} &= -A^{-\top}C^\top C\left(-A^{-1}\boldsymbol{\zeta}\right) \\
&= \left(A^{-\top}C^\top CA^{-1}\right)\boldsymbol{\zeta} \\
&= G^\top(0)G(0)\boldsymbol{\zeta}.
\end{aligned}
$$

Hence, by recalling that $\boldsymbol{\zeta} \gg 0$, γ^2 is the Perron-Frobenius eigenvalue of $G^\top(0)G(0)$, i.e., $\gamma^2 = \lambda_{\max}\left(G^\top(0)G(0)\right)$ and $\|G\|_{\mathcal{H}_\infty} = \gamma$. ∎

By applying Proposition 8.5 to the HIV model (8.4), the $\mathcal{H}_\infty$-optimal robust control

problem can be formulated as a minimization problem over variables $\mathbf{u} \in \mathbb{R}_+^M$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta} \in \mathbb{R}_+^n$ and $\gamma \in \mathbb{R}_+$, namely as

$$\begin{array}{c} \text{minimize} \\ \mathbf{u}>0, \boldsymbol{\eta}\gg 0, \\ \boldsymbol{\zeta}\gg 0, \gamma>0 \end{array} \quad \gamma \tag{8.13a}$$

subject to

$$\left(A + \sum_{i=1}^{M} u_i D_i \right) \boldsymbol{\eta} + \boldsymbol{\zeta} \leq 0 \tag{8.13b}$$

$$C^\top C \boldsymbol{\eta} + \gamma^2 \left(A + \sum_{i=1}^{M} u_i D_i \right)^\top \boldsymbol{\zeta} \leq 0 \tag{8.13c}$$

$$\mathbf{1}_M^\top \mathbf{u} \leq 1. \tag{8.13d}$$

The minimization problem (8.13) is clearly non-convex, and two reasons for non-convexity can, indeed, be identified: 1) the product $\sum_{i=1}^{M} u_i D_i \boldsymbol{\eta}$ in inequality constraint (8.13b); and 2) the product $\gamma^2 \left(A + \sum_{i=1}^{M} u_i D_i \right)^\top \boldsymbol{\zeta}$ in inequality constraint (8.13c). While the first non-convexity can be solved in a similar way to what done for the $\mathcal{L}_1$-optimal robust control problem, there is no way, to the authors' knowledge, to solve the second non-convexity. However, we will be able to compute a controller $\mathbf{u}_{\mathcal{H}_\infty}^*$ arbitrarily close to the $\mathcal{H}_\infty$-optimal controller. To this aim, we first formulate a convex feasibility problem to determine whether, with $\gamma > 0$ fixed, a controller $\mathbf{u} > 0$, with $\mathbf{1}_M^\top \mathbf{u} \leq 1$, that makes $\|G\|_{\mathcal{H}_\infty} = \gamma$ exists. Then, a simple bisection algorithm over γ allows to determine the controller $\mathbf{u}_{\mathcal{H}_\infty}^*$.

Proposition 8.6. *Consider the HIV model* (8.4) *and assume that the Metzler matrix A is irreducible. Let $\gamma > 0$ be fixed. The following statements are equivalent:*

i) There exists a controller $\mathbf{u} \in \mathbb{R}_+^M$, with $\mathbf{u} > 0$ and $\mathbf{1}_M^\top \mathbf{u} \leq 1$, that makes the system (8.4) *asymptotically stable and such that $\|G\|_{\mathcal{H}_\infty} = \gamma$;*

ii) There exist vectors $\mathbf{u} \in \mathbb{R}_+^M$, $\mathbf{v}, \mathbf{z} \in \mathbb{R}^n$ such that

$$\text{row}_j(A)e^{\mathbf{v}-v_j} + \sum_{i=1}^{M} u_i [D_i]_j + e^{z_j - v_j} \leq 0, \quad j \in [1, n] \tag{8.14a}$$

$$\gamma^2 \left(\text{col}_j^\top(A)e^{\mathbf{z}-z_j} + \sum_{i=1}^{M} u_i [D_i]_j \right) + \text{row}_j \left(C^\top C \right) e^{\mathbf{v}-z_j} \leq 0, \quad j \in [1, n] \tag{8.14b}$$

$$\mathbf{1}_M^\top \mathbf{u} \leq 1. \tag{8.14c}$$

Proof. By applying Proposition 8.5 we can claim that a controller $\mathbf{u} \in \mathbb{R}_+^M$, with $\mathbf{u} > 0$ and $\mathbf{1}_M^\top \mathbf{u} \leq 1$, that stabilizes the positive system (8.4) and makes $\|G\|_{\mathcal{H}_\infty} = \gamma$, exists if and only if vectors $\mathbf{u} \in \mathbb{R}_+^M$, $\mathbf{u} > 0$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta} \in \mathbb{R}_+^n$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta} \gg 0$, can be found that satisfy equations (8.13b), (8.13c), and (8.13d). Recalling that D_i, $i \in [1, M]$, are diagonal matrices, inequalities (8.13b) and (8.13c) can be rewritten as

$$\text{row}_j(A)\boldsymbol{\eta} + \sum_{i=1}^M u_i \left[D_i \right]_j \eta_j + \zeta_j \leq 0, \quad j \in [1, n], \tag{8.15}$$

and

$$\text{row}_j \left(C^\top C \right) \boldsymbol{\eta} + \gamma^2 \left(\text{col}_j^\top (A)\boldsymbol{\zeta} + \sum_{i=1}^M u_i \left[D_i \right]_j \zeta_j \right) \leq 0, \quad j \in [1, n], \tag{8.16}$$

respectively. Since $\boldsymbol{\eta}$, $\boldsymbol{\zeta} \gg 0$, dividing inequalities (8.15) and (8.16) by η_j and ζ_j, respectively, yields

$$\text{row}_j(A)\frac{\boldsymbol{\eta}}{\eta_j} + \sum_{i=1}^M u_i \left[D_i \right]_j + \frac{\zeta_j}{\eta_j} \leq 0, \quad j \in [1, n], \tag{8.17}$$

$$\text{row}_j \left(C^\top C \right) \frac{\boldsymbol{\eta}}{\zeta_j} + \gamma^2 \left(\text{col}_j^\top (A)\frac{\boldsymbol{\zeta}}{\zeta_j} + \sum_{i=1}^M u_i \left[D_i \right]_j \right) \leq 0, \quad j \in [1, n]. \tag{8.18}$$

Now, introduce the variables $v_j := \ln \eta_j$, $j \in [1, n]$, and $z_j := \ln \zeta_j$, $j \in [1, n]$. Upon defining $\mathbf{v} := \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}^\top$ and $\mathbf{z} := \begin{bmatrix} z_1 & \ldots & z_n \end{bmatrix}^\top$, it is easy to see that in the new variables inequalities (8.17) and (8.18) become (8.14a) and (8.14b), respectively, and the proof is concluded. ∎

We are now in a position to provide an algorithm to determine a controller $\mathbf{u}_{\mathcal{H}_\infty}^*$ arbitrary close to the solution of the $\mathcal{H}_\infty$-optimal control problem for HIV treatment. To keep notation simple, by solving the problem $\mathcal{F}_\gamma(\mathbf{u}, \mathbf{v}, \mathbf{z})$ we mean solving the feasibility problem (8.14) over variables $\mathbf{u}$, $\mathbf{v}$, $\mathbf{z}$ and with $\gamma > 0$ fixed. We will denote by $(\mathbf{u}^*, \mathbf{v}^*, \mathbf{z}^*)$ its solution.

Algorithm:

A0. Select a lower bound γ_l and an upper bound γ_u such that $0 < \gamma_l \leq \gamma^* \leq \gamma_u$, where γ^* is the best $\mathcal{H}_\infty$ performance that can be achieved. Initialise $\gamma_{\mathcal{H}_\infty}^* = 0$ and $\mathbf{u}_{\mathcal{H}_\infty}^* = 0$.

A1. If $(\gamma_u - \gamma_l)/\gamma_l \leq \varepsilon$, where ε is a fixed value specifying how close γ^* and $\gamma_{\mathcal{H}_\infty}^*$ will be, stop. Otherwise, go to step A2.

A2. Set $\gamma = (\gamma_l + \gamma_u)/2$, and solve the convex feasibility problem $\mathcal{F}_\gamma(\mathbf{u}, \mathbf{v}, \mathbf{z})$.

A3. If $\mathcal{F}_\gamma(\mathbf{u}, \mathbf{v}, \mathbf{z})$ is feasible, set $\gamma_u = \gamma$, $\gamma^*_{\mathcal{H}_\infty} = \gamma$, and $\mathbf{u}^*_{\mathcal{H}_\infty} = \mathbf{u}^*$. Otherwise, set $\gamma_l = \gamma$. Repeat from step A1.

8.5   Numerical example

To conclude, we illustrate the derived results with a numerical example.

Consider the HIV model (8.4) with $n = 5$ viral variants, and assume that the Metzler matrix $A \in \mathbb{R}^{5 \times 5}$ describing the evolutionary dynamics of the viral population in the absence of any drug is given by

$$A = \begin{bmatrix} 0.6 & 0 & 0.9 & 0 & 0 \\ 0.2 & 0.2 & 0 & 0.9 & 0 \\ 0 & 0.2 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.9 \\ 0 & 0 & 0.2 & 0 & 0.5 \end{bmatrix}.$$

Notice that the Metzler matrix A is irreducible and non-Hurwitz. Consider as output the concentration of the first viral variant, namely $C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$. Assume that $M = 3$ drugs are simultaneously administered, whose effects are described by the diagonal matrices

$$D_1 = \begin{bmatrix} -7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}.$$

Solving the convex minimization problem (8.6) yields the $\mathcal{L}_1$-optimal controller $\mathbf{u}^*_{\mathcal{L}_1} = \begin{bmatrix} 0.3673 & 0.3233 & 0.3094 \end{bmatrix}^\top$. Notice that $\mathbf{1}_5^\top \mathbf{u}^*_{\mathcal{L}_1} = 1$, the closed-loop system is asymptotically stable and has $\mathcal{L}_1$-norm equals to $\gamma^*_{\mathcal{L}_1} = 1.7561$.

Running the proposed algorithm for the solution of the $\mathcal{H}_\infty$-optimal control problem we obtain $\mathbf{u}^*_{\mathcal{H}_\infty} = \begin{bmatrix} 0.3795 & 0.3145 & 0.3060 \end{bmatrix}^\top$. It is easy to verify that $\mathbf{1}_5^\top \mathbf{u}^*_{\mathcal{H}_\infty} = 1$, the closed-loop system is asymptotically stable and its $\mathcal{H}_\infty$-norm is $\gamma^*_{\mathcal{H}_\infty} = 0.8613$.

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