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CICLO XXX

On the spectral stability of polyharmonic operators on singularly perturbed domains

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Riassunto

In questa tesi si studia la dipendenza degli autovalori di operatori differenziali ellittici di ordine superiore da perturbazioni singolari del dominio, con attenzione per gli operatori poliarmonici e per condizioni al bordo di tipo intermedio e Neumann. Si identificano opportune condizioni geometriche sul dominio iniziale, sui domini perturbati e sulla perturbazione al fine di assicurare la stabilità spettrale. Si caratterizzano i problemi differenziali limite, al variare dei parametri che regolano la deformazione del dominio iniziale. Si dimostra che, assumendo opportune ipotesi, gli autovalori e le proiezioni sugli autospazi associati al problema differenziale nel dominio perturbato convergono ai rispettivi autovalori e proiezioni associati al problema limite nel dominio iniziale. Inoltre si dimostra che i risolventi convergono compattamente al risolvente associato al problema limite.

In particolare, si analizza dapprima la convergenza spettrale di una famiglia di operatori autoaggiunti, ellittici, di ordine superiore, con condizioni al bordo di tipo intermedio, su domini perturbati definiti localmente dal sottografico di date funzioni. Si dimostra un teorema di stabilità spettrale assumendo che la convergenza delle funzioni che rappresentano localmente la frontiera convergano in modo sufficientemente regolare. Si utilizza poi tale risultato per studiare il comportamento spettrale di operatori poliarmonici con condizioni al bordo di tipo intermedio quando la frontiera del dominio è soggetta ad una oscillazione periodica e singolare, adattando delle tecniche utilizzate da J.M. Arrieta e P.D. Lamberti nel caso dell'operatore biarmonico. Si dimostra che il problema limite dipende dal rapporto tra l'ampiezza dell'oscillazione e il periodo di oscillazione. Infatti esiste un valore limite per questo rapporto al di sopra del quale si ha stabilità spettrale, cioè gli autovalori e le proiezioni sugli autospazi associati alla famiglia di domini perturbati convergono ai corrispondenti autovalori e proiezioni associati allo stesso operatore differenziale nel dominio limite; al di sotto di tale valore critico

invece l'operatore differenziale limite è differente, in quanto assume condizioni al bordo diverse sulla frontiera del dominio limite. Infine se il rapporto assume esattamente il valore critico, appare un 'termine strano' in una delle condizioni al bordo associate al problema limite, che è stato caratterizzato in funzione della soluzione di un dato problema al bordo ausiliario. In questo caso limite si sfruttano tecniche dimostrative tipiche dell'omogeneizzazione periodica, come il metodo di 'unfolding' e le decomposizioni micro-macroscopiche delle funzioni di Sobolev, presenti, ad esempio, in alcuni articoli di J. Casado-Diaz e collaboratori.
Nel piano euclideo si considerano inoltre l'operatore biarmonico e l'operatore associato al sistema di Reissner-Mindlin, con condizioni al bordo di tipo Neumann, su un dominio 'a bilanciere', che consiste di due domini regolari, limitati e disgiunti, collegati attraverso un canale sottile. Si analizza il comportamento limite dello spettro degli operatori e si caratterizza il limite degli autovalori e delle proiezioni sugli autospazi quando la larghezza del canale diminuisce fino ad annullarsi, adattando tecniche introdotte da J.M. Arrieta e collaboratori per l'operatore di Laplace con condizioni al bordo di tipo Neumann. Nelle applicazioni alla teoria dell'elasticità lineare, gli operatori in considerazione sono collegati alla deformazione di una piastra elastica, di materiale omogeneo e non vincolata, dovuta alla degenerazione di una delle sue dimensioni. In contrasto con il caso dell'operatore di Laplace, l'equazione limite risulta distorta da un coefficiente strano, che dipende dal coefficiente di Poisson della piastra modellizzata.

Abstract

In this thesis, we analyse the spectral convergence properties of higher order elliptic differential operators subject to singular domain perturbations and non-Dirichlet boundary conditions, with special attention to polyharmonic operators. We identify suitable conditions on the shape of the initial domain, on the shape of the perturbed domains, and on the geometry of the perturbation in order to assure the spectral stability. We find the limiting differential problem depending on the type of domain perturbation and the geometrical parameters governing the shape deformation of the initial domain. We prove that, under suitable conditions, the eigenvalues and the eigenprojections of the given differential operator in the perturbed domain converge to the eigenvalues and the eigenprojections of the limiting differential operator in the unperturbed domain. Finally, we prove convergence of the resolvent operators in the framework of the compact convergence of linear operators in Hilbert spaces.

More specifically, we first analyse the spectral convergence of a family of higher order self-adjoint elliptic operators subject to intermediate boundary conditions on perturbed domains defined locally by the hypographs of given functions. We prove a spectral stability theorem for this family of operators under the assumption that the convergence of the functions describing the boundary of the domain is sufficiently regular. Then we apply the theorem to study the spectral behaviour of polyharmonic operators with intermediate boundary conditions when the boundary of the domain undergoes a perturbation of oscillatory type, by adapting techniques introduced by J.M. Arrieta and P.D. Lamberti for the biharmonic operator. We prove that the limiting differential problem depends on the ratio between the amplitude and the period of the oscillation. Indeed there is a critical threshold above which there is spectral stability; that is, the eigenvalues and the eigenprojections of the perturbed problem converge to the corresponding eigenvalues and eigenprojections of the same differential problem in the limiting domain. Instead,

under that threshold there is a different behaviour depending on the order of the polyharmonic operator and on the type of intermediate boundary conditions imposed at the boundary. If the ratio assumes exactly the critical value, then the limiting differential problem exhibits a strange boundary condition, which is characterized in terms of an auxiliary function satisfying a suitable differential problem. In order to treat this critical case we use homogenization techniques and macroscopic-microscopic decompositions, inspired by arguments used by J. Casado-Diaz and collaborators.

Then we consider the biharmonic operator and the Reissner-Mindlin operator subject to homogeneous boundary conditions of Neumann type on a planar dumbbell domain which consists of two disjoint domains connected by a thin channel. We analyse the spectral behaviour of the operator, characterizing the limit of the eigenvalues and of the eigenprojections as the thickness of the channel goes to zero, in the spirit of the articles by J.M. Arrieta and collaborators for the Neumann Laplacian. In applications to linear elasticity, the operators under consideration are related to the deformation of a free elastic plate, a part of which shrinks to a segment. In contrast to the classical case of the Laplace operator, it turns out that the limiting equation is here distorted by a strange factor depending on a parameter which plays the role of the Poisson coefficient of the represented plate.

Introduction

The problem of studying the behaviour of eigenvalues and eigenfunctions of given differential operators when the domain is subject to perturbations has a long history. Rayleigh and Schröedinger can be considered the founders of this perturbation theory, which was initially studied in connection with the mechanics of vibrating systems and quantum theory (see [103, 105] and the introduction of [80]). Since those pioneering works the research in this field was first made mathematically rigorous and then considered as a topic of independent interest in the framework of Spectral Theory. Broadly speaking, the general problem consists in considering a differential operator H on a domain Ω and in studying the variation of the spectrum of H when Ω is subject to a perturbation. As an example, consider the Laplace operator $-\Delta$ subject to homogeneous boundary conditions (Dirichlet, Neumann, Robin, etc.) on a given bounded and connected open set Ω of $\mathbb{R}^N$, $N \geq 2$. We perturb the original domain Ω, to obtain a family of domains $(\Omega_\epsilon)_{\epsilon>0}$. The eigenvalue problem for $-\Delta$ on Ω_ϵ is defined by

$$\begin{cases} -\Delta u_\epsilon = \lambda[\Omega_\epsilon] u_\epsilon, & \text{in } \Omega_\epsilon, \\ \mathcal{B} u_\epsilon = 0, & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{0.0.1}$$

where $\mathcal{B}$ is an operator defining the boundary conditions. A natural problem is to find conditions on Ω, Ω_ϵ and the perturbation $\Omega \mapsto \Omega_\epsilon$, such that $\lambda_n[\Omega_\epsilon] \longrightarrow \lambda_n[\Omega]$ when $\epsilon \longrightarrow 0$, for all $n \in \mathbb{N}$. Perturbations $\Omega \mapsto \Omega_\epsilon$ such that both the eigenvalues and the eigenspaces of the problem in Ω_ϵ converge to the corresponding eigenvalues and eigenfunctions of the differential problem in Ω will be called here spectrally stable perturbations.

In domain perturbation problems it is essential to make a first distinction between regular and singular perturbations. We say that a perturbation $\Omega \mapsto \Omega_\epsilon$ is regular if there exists a family of diffeomorphisms $(\Phi_\epsilon)_{\epsilon>0}$ of class C^m such that Φ_ϵ maps Ω to Ω_ϵ and satisfies the following convergence condition

$$\|\Phi_\epsilon - \mathbb{I}\|_{C^m(\mathbb{R}^N)} \longrightarrow 0, \qquad \text{as } \epsilon \longrightarrow 0,$$

where the order of the considered operator is $2m$. Broadly speaking, we can say that regular perturbations guarantee the spectral convergence of any elliptic differential operator, independently of the boundary conditions imposed. Hence, the regularity condition via diffeomorphisms defined above is sufficient for the spectral stability. However, in general such a condition is far from being necessary.

On the other hand, if the perturbation is not regular, then spectral stability is intrinsically harder to achieve. Indeed, there exist many natural perturbations for which either it is impossible to construct diffeomorphisms between Ω and Ω_ϵ, or the family $(\Phi_\epsilon)_{\epsilon>0}$ of diffeomorphisms is not well-behaved with respect to ϵ. For example, let $\Omega = W \times (-1, 0)$, where W is a smooth bounded domain of $\mathbb{R}^{N-1}$ and let us consider the perturbed sets

$$\Omega_\epsilon = \{(\bar{x}, x_N) \in \mathbb{R}^N : \bar{x} \in W, a < x_N < g_\epsilon(\bar{x}) = \epsilon^\alpha b(\bar{x}/\epsilon)\} \tag{0.0.2}$$

for all $\epsilon > 0$, where b is a positive, smooth, non-constant periodic function of period $Y =]-1/2, 1/2[^{N-1}$. In this case it is possible to construct a family of diffeomorphisms $(\Phi_\epsilon)_{\epsilon>0}$ of class C^d mapping Ω_ϵ to Ω and defined by

$$\Phi_\epsilon(\bar{x}, x_N) = (\bar{x}, x_N - h_\epsilon(\bar{x}, x_N))$$

for all $\epsilon > 0$, where

$$h_\epsilon(\bar{x}, x_N) = \begin{cases} 0, & \text{if } -1 \leq x_N \leq -\epsilon, \\ g_\epsilon(\bar{x})\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^d, & \text{if } -\epsilon \leq x_N \leq g_\epsilon(\bar{x}), \end{cases}$$

for a suitable $d \in \mathbb{N}$, $d > 2$. On Ω_ϵ we consider the eigenvalue problem associated with $H_{\Omega_\epsilon} = \Delta^2 + \mathbb{I}$ subject to homogeneous boundary conditions of intermediate or Neumann type. Note that when $\alpha > 2$ then $\|\Phi_\epsilon - \mathbb{I}\|_{C^2(\mathbb{R}^N,\mathbb{R}^N)} < C$ for all $\epsilon > 0$, hence the perturbation is regular (and consequently spectrally stable). However, it is proved in [19] that the perturbation $\Omega \mapsto \Omega_\epsilon$ is spectrally stable for $\alpha > \bar{\alpha}$, where $\bar{\alpha} < 2$ depends on the boundary conditions. For example, $\bar{\alpha} = 3/2$ in the case of intermediate boundary conditions, see (0.0.5) below. Thus, this example shows that the family of spectrally stable perturbations comprehend both regular and singular perturbations.

It is then an interesting problem to characterise the spectrally stable perturbations which are not regular. Let us note that the family of admissible spectrally stable perturbations in the singular setting strongly depends on the differential operator and on the boundary conditions, as already mentioned in [57]. For example, if the boundary conditions are of Dirichlet type, then it is possible to choose more general perturbations than the ones allowed by other boundary conditions. Indeed, an old result in [22] states that for any elliptic differential operator of order $2m$ on Ω_ϵ with Dirichlet boundary conditions, the perturbation $(\Omega_\epsilon)_{\epsilon>0}$ is spectrally stable if the following conditions are satisfied:

(i) The initial domain Ω is such that $H_0^m(\Omega)$ coincides with the set of functions in $H^m(\mathbb{R}^N)$ which vanish in $\mathbb{R}^N \setminus \overline{\Omega}$;

(ii) For all compact set $K \subset \Omega$ there exists $\epsilon(K) > 0$ such that $K \subset \Omega_\epsilon$ for all $\epsilon \leq \epsilon(K)$;

(iii) For all open sets U such that $\overline{\Omega} \subset U$ there exists $\epsilon(U) > 0$ such that $\Omega_\epsilon \subset U$ for all $\epsilon \leq \epsilon(U)$.

We remark that (i) is a rather weak regularity condition on the set Ω, which is for example verified whenever the boundary $\partial\Omega$ is of class $C^{0,1}$. It is worth noticing that a similar result for non-Dirichlet boundary conditions is false in general. In [57] there is a classical counterexample for the Neumann Laplacian in the plane. It consists in a perturbation of the unit square $\Omega =]0,1[^2$ of $\mathbb{R}^2$ satisfying $(ii) - (iii)$ above, which is not spectrally stable, because $\lambda_2(\Omega) = \pi^2 \neq \lim_{\epsilon\to 0} \lambda_2(\Omega_\epsilon) = 0$. We refer to the books [28, 50, 59, 64, 77, 80, 98, 104] for further information on domain perturbation theory and spectral perturbation problems. We mention also the survey paper [73] and the papers [13, 23, 24, 36, 38, 39, 40, 55, 61, 84, 85, 86, 87, 88, 90, 114] where the authors tackled the problem of the spectral stability for elliptic differential operators from different points of view, either studying the dependence of the eigenvalues on additional parameters or by studying in full generality the properties of the map $\Omega \mapsto \lambda[\Omega]$, sometimes showing spectral stability estimates for differential operators subject to domain perturbation. In some cases singular domain perturbations can be studied via asymptotic analysis, see for example the recent article [56]. See also [46, 71] and the monographs [92, 93].

In the recent paper [19] the authors have tried to provide a unifying approach to the study of the spectral convergence properties of self-adjoint operators H_Q associated with quadratic forms Q given by

$$Q(u,v) = \sum_{|\alpha|=|\beta|=m} \int_\Omega A_{\alpha\beta} D^\alpha u D^\beta v dx + \int_\Omega uv dx, \tag{0.0.3}$$

for all $u, v \in V(\Omega)$. Here $V(\Omega) \subset H^m(\Omega)$ is a suitable function space containing $H_0^m(\Omega)$ and the coefficients $A_{\alpha\beta}$ are real valued functions satisfying $A_{\alpha\beta} = A_{\beta\alpha}$ for all $|\alpha| = |\beta| = m$ and the ellipticity condition

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq 0,$$

for all $x \in \mathbb{R}^N$, for all $\xi = (\xi_\alpha)_{|\alpha|=m} \in \mathbb{R}^M$ where M is the number of multiindices $\alpha \in \mathbb{N}^N$ with length $|\alpha| = m$. More precisely, recall that H_Q is uniquely defined by the relation

$$Q(u,v) =< H_Q^{1/2}u, H_Q^{1/2}v >_{L^2(\Omega)},$$

for all $u, v \in V(\Omega)$. In particular the domain of the square root $H_Q^{1/2}$ of H_Q is $V(\Omega)$ and a function u belongs to the domain of H_Q if and only if $u \in V(\Omega)$ and there exists $f \in L^2(\Omega)$ such that $Q(u, v) =< f, v >_{L^2(\Omega)}$ for all $v \in V(\Omega)$, in which case $H_Q u = f$. We refer to [60, Chp. 4] and Section 1.1 for a general introduction to the variational approach in the study of partial differential equations.
In [19] the authors give a condition, called condition (C) (see [19, Definition 3.1], and Definition 2.2.1), which implies the spectral convergence for the whole family of operators H_Q. Since this condition is rather general it is quite important to understand whether it has more geometrical equivalent characterisations, at least in some specific situations. It turns out that in the limiting cases $V(\Omega_\epsilon) = H^m(\Omega_\epsilon)$ and $V(\Omega_\epsilon) = H_0^m(\Omega_\epsilon)$ this is feasible. In the first case, which corresponds to Dirichlet boundary conditions, Condition (C) is equivalent to a Mosco-type convergence (see [94, 95, 96]) of the energy spaces $H_0^m(\Omega_\epsilon)$ to $H_0^m(\Omega)$. Mosco convergence is sometimes regarded as a particular type of Γ-convergence (see [28, 63]), which, in turn, is related to H-convergence (see [97], [108]). Let D be an bounded open set containing Ω and Ω_ϵ for all ϵ sufficiently small. We say that $H_0^m(\Omega_\epsilon)$ Mosco-converges to $H_0^m(\Omega)$ if the following statements hold:

(i) For all $u \in H_0^m(\Omega)$ there exists a sequence of functions $(u_\epsilon)_{\epsilon>0}$ such that $u_\epsilon \in H_0^m(\Omega_\epsilon)$ for all $\epsilon > 0$ and $u_\epsilon \to u$ in $H_0^m(D)$, as $\epsilon \to 0$.

(ii) For all sequences $(u_\epsilon)_{\epsilon>0}$ such that $u_\epsilon \in H_0^m(\Omega_\epsilon)$ and $\|u_\epsilon\|_{H^m(\Omega_\epsilon)} \leq C$ for all $\epsilon > 0$, there exist $\epsilon_k \to 0$ as $k \to \infty$, a subsequence $(u_{\epsilon_k})_{k\in\mathbb{N}}$ of $(u_\epsilon)_{\epsilon>0}$ and a function $u \in H_0^m(\Omega)$ such that $u_{\epsilon_k} \to u$ weakly in $H_0^m(D)$ as $k \to \infty$.

We remark that the critical condition here is the compactness condition (ii) which requires all the $H^m(D)$-weak cluster points of any sequence $(u_\epsilon)_\epsilon$ to lie in $H_0^m(\Omega)$. In the case of the Laplace operator $-\Delta$ with Dirichlet boundary conditions, it is possible to prove that if Ω_ϵ, Ω satisfy a uniform exterior cone condition (equivalently, Ω_ϵ, Ω are uniformly Lipschitz) and $\Omega_\epsilon \to \Omega$ in the Hausdorff complementary topology, then $H_0^1(\Omega_\epsilon)$ converges in the sense of Mosco to $H_0^1(\Omega)$. See the monographs [28, 64, 76] for more details on the relation between geometry of sets and spectral convergence. Note that very mild regularity assumptions on the boundaries of Ω_ϵ, Ω are assumed.

On the other hand, when $V(\Omega_\epsilon) = H^m(\Omega_\epsilon)$ for all $\epsilon > 0$, the situation is quite different. This case corresponds to Neumann boundary conditions, hence it is rather clear that more regularity at the boundary will be required to define in a proper way the differential problem. In this case Condition (C) turns out to be equivalent to Arrieta's condition (see e.g., [11, 18, 19]), which assumes the sets Ω_ϵ, Ω to be Lipschitz, and it requires the existence of Lipschitz open sets $K_\epsilon \subset \Omega_\epsilon \cap \Omega$ such that one of the two following equivalent conditions is satisfied:

1. If $u_\epsilon \in H^m(\Omega_\epsilon)$ and $\sup_{\epsilon>0}\|u_\epsilon\|_{H^m(\Omega_\epsilon)} < \infty$ then $\|u_\epsilon\|_{L^2(\Omega_\epsilon\setminus K_\epsilon)} \longrightarrow 0$ as $\epsilon \longrightarrow 0$;

2. $\lim_{\epsilon\to 0}\tau_\epsilon = \infty$, where

$$\tau_\epsilon = \inf_{\substack{\phi\in H^m(\Omega_\epsilon)\setminus\{0\}\\ \phi = 0 \text{ on } K_\epsilon}} \frac{Q_{\Omega_\epsilon}(\phi,\phi)}{\|\phi\|^2_{L^2(\Omega_\epsilon)}}.$$

When either (i) or (ii) (and hence both) are verified, then $|\Omega_\epsilon \setminus K_\epsilon| \longrightarrow 0$ as $\epsilon \longrightarrow 0$. We remark that to assure the spectral stability in this Neumann setting is crucial to avoid concentration of the L^2-norm of sequences u_ϵ in a neighbourhood of the boundary of Ω_ϵ. A classical situation in which Arrieta's condition is not verified is the case of a typical dumbbell perturbation (see [10, 11, 12, 15, 18, 78, 79]), which is analysed in this thesis in Chapter 5 and in Chapter 6 for the biharmonic operator and the Reisnner-Mindlin system, respectively. Typical dumbbell domains have the property that there exist K_ϵ as in Arrieta's condition and $|\Omega_\epsilon \setminus K_\epsilon| \longrightarrow 0$, but τ_ϵ remains bounded as $\epsilon \longrightarrow 0$.

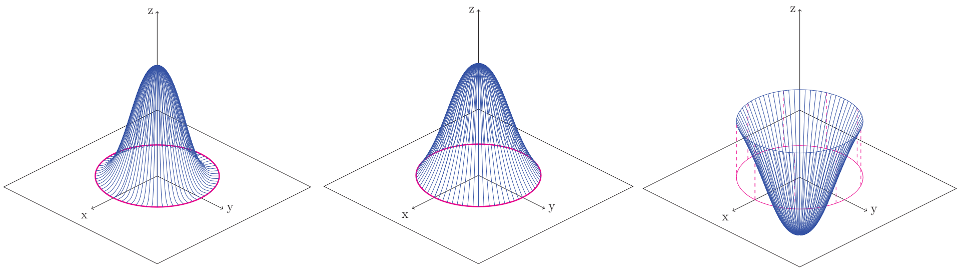
At this point two natural questions arise.

1. Is it possible to identify suitable geometrical conditions in order to guarantee that Condition (C) is verified, in the case of boundary conditions different from Dirichlet or Neumann?

2. What happens when condition (C) is not verified? In particular, what happens in the case of dumbbell perturbations?

In this thesis we try to give an answer to these questions, at least in some specific setting. The main part of the results will concern polyharmonic operators. Note that polyharmonic operators are a prototype for general higher order elliptic operators, at least in the study of eigenvalue problems. With regard to this, we mention that the study of polyharmonic operators, which began long time ago (see e.g., the pioneering articles by Almansi [5, 6, 7] and the book [99]) has recently attracted interest of many mathematicians. We refer to the extensive monograph [70] for more details on polyharmonic operators and we cite the articles [29, 30, 31, 33, 34, 48, 69, 101] where eigenvalue problems for polyharmonic operators have been considered.

Recall that the eigenvalues of the biharmonic operator in a domain $\Omega \subset \mathbb{R}^2$ have the physical meaning of representing the principal frequencies of a very thin three-dimensional vibrating plate of section Ω. More precisely, the eigenvalue problem for the biharmonic operator subject to Dirichlet boundary conditions is given by



(a) Clamped plate (b) Hinged plate (c) Free plate

Figure 1: Boundary conditions for a circular plate.

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{0.0.4}$$

Here $u(x)$ represents the vertical displacement of the plate at the point $x = (x_1, x_2) \in \Omega$, and n is the unit outer normal to $\partial\Omega$. In applications, Dirichlet boundary conditions are used to model a plate that is clamped at the boundary. In particular, the graph of u must touch tangentially the hyperplane $u = 0$ at the boundary of Ω, see Fig. 1(a). In a similar way, it is possible to consider the eigenvalue problem

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial^2 u}{\partial n^2} = 0, & \text{on } \partial\Omega. \end{cases} \tag{0.0.5}$$

which corresponds to the case of the hinged plate. Here the plate is not allowed to move vertically at the boundary of Ω but it may rotate around the line tangent to $\partial\Omega$ at $x \in \partial\Omega$, see Fig. 1(b). Finally, one can consider the eigenvalue problem

$$\begin{cases} \Delta^2 u + u = \lambda u, & \text{in } \Omega, \\ (1-\sigma)\frac{\partial^2 u}{\partial n^2} + \sigma\Delta u = 0, & \text{on } \partial\Omega \\ (1-\sigma)\,\mathrm{div}_{\partial\Omega}(D^2 u \cdot n)_{\partial\Omega} + \frac{\partial(\Delta u)}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{0.0.6}$$

which corresponds to the case of a free plate with Poisson's ratio $\sigma \in (-1, 1)$. Here $\mathrm{div}_{\partial\Omega}$ is the tangential divergence operator and $(F)_{\partial\Omega}$ stands for the projection of the vector field F on $\partial\Omega$. In this case the plate is free to move at the boundary, in particular it may present a non-trivial vertical displacement on $\partial\Omega$, see 1(c).

We note that this physical interpretation of Problems (0.0.4), (0.0.5) and (0.0.6) is deduced via the so-called Kirchhoff-Love model, which is valid for thin plates and for small oscillations. If instead we need to consider plates with a non-negligible thickness, then the Reissner-Mindlin system is more indicated. In the case of hard clamped boundary conditions (see [9] and Section 6.2.1) the eigenvalue problem for the Reissner-Mindlin system is defined by

$$\begin{cases} -\frac{\mu_1}{12}\Delta\beta - \frac{\mu_1+\mu_2}{12}\nabla(\operatorname{div}\beta) - \frac{\mu_1 k}{t^2}(\nabla w - \beta) = \lambda\frac{t^2}{12}\beta, & \text{in } \Omega, \\ -\frac{\mu_1 k}{t^2}(\Delta w - \operatorname{div}\beta) = \lambda w, & \text{in } \Omega, \\ \beta = w = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.0.7}$$

where $\Omega \subset \mathbb{R}^2$ represents the midplane of the plate, w is the transverse displacement of the midplane, $\beta = (\beta_1, \beta_2)$ is the fiber rotation and t is a non-negative real parameter representing the thickness of the plate. Moreover, μ_1 and μ_2 are coefficients related to the Lamé constants and $k > 0$ is a correction factor. We mention here the article [32], where the authors studied the dependence of the eigenvalues of (0.0.7) upon the shape of the domain Ω.

Problem (0.0.4) can be seen as a dimensional reduction of Problem (0.0.7) for $t \to 0$. Indeed, it is proved in [65] that the eigenvalues $\lambda_n(t)$ of Problem (0.0.7) converge to the eigenvalues of

$$\begin{cases} \frac{2\mu_1+\mu_2}{12}\Delta^2 w = \lambda w, & \text{in } \Omega, \\ w = \frac{\partial w}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.0.8}$$

as $t \to 0$. We mention that system (0.0.7) is used in numerical analysis as a second-order alternative to problem (0.0.4), see for example [26].

Inspired by the eigenvalue problems for the biharmonic operator, we consider more in general eigenvalue problems for polyharmonic operators, which can be written in the weak form as follows

$$\int_\Omega D^m u : D^m v \, dx = \lambda[\Omega] \int_\Omega uv \, dx, \tag{0.0.9}$$

for all $u, v \in V(\Omega)$, where

$$D^m u : D^m v = \sum_{i_1,\dots,i_m=1}^{N} \frac{\partial^m u}{\partial i_1 \dots \partial i_m} \frac{\partial^m v}{\partial i_1 \dots \partial i_m},$$

is the Frobenius product of the m-tensors $D^m u$ and $D^m v$, and $V(\Omega)$ is a subspace of $H^m(\Omega)$ satisfying $H^m_0(\Omega) \subset V(\Omega) \subset H^m(\Omega)$. We are particularly interested to polyharmonic operators with intermediate boundary conditions, corresponding to the case in which $V(\Omega) = H^m(\Omega) \cap H^k_0(\Omega)$, for $k \geq 1$, $k \leq m-1$. In the case when $k = m-1$ we say that $(-\Delta)^m$ satisfies *strong intermediate boundary conditions* and the corresponding eigenvalue problem reads

$$\begin{cases} (-\Delta)^m u = \lambda u, & \text{in } \Omega, \\ u = D^l u = 0, & \text{on } \partial\Omega, \text{ for all } 1 \leq l \leq m-2, \\ \frac{\partial^m u}{\partial n^m} = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.0.10}$$

where λ is the eigenvalue and u is the eigenfunction. It is important to underline the difference between problem (0.0.10) and the eigenvalue problem for $(-\Delta)^m$ with Dirichlet boundary conditions, which is given by

$$\begin{cases}(-\Delta)^m u = \lambda u, & \text{in } \Omega,\\ u = D^l u = 0, & \text{on } \partial\Omega, \text{ for all } 1 \le l \le m-2,\\ \frac{\partial^{m-1}u}{\partial n^{m-1}} = 0, & \text{on } \partial\Omega.\end{cases}$$

We refer to §4.1 in Chapter 4 for the mathematical deduction of these boundary value problems.

The main topic of this thesis is the analysis of the spectral convergence properties of higher order elliptic differential operators subject to singular perturbations and non-Dirichlet boundary conditions, with special attention to the eigenvalue problems mentioned above. In contrast to the theory of regular perturbations and the study of operators subject to Dirichlet boundary conditions, this field of research is rather recent. We mention that contributes in this direction can be found in topics as diverse as spectral theory on perturbed domains, see for example [30], shape optimization for eigenvalues see e.g., [31], [33], [106], study of the dynamics of suspension bridges, see e.g., [68], homogenization theory, see for example the monographs [92, 93].

The first part of the thesis (namely, Chapter 2, 3 and 4) can be seen as a continuation of the research begun in [19]. The main focus of attention throughout these chapters is the study of self-adjoint, higher order elliptic operators subject to intermediate boundary conditions on domains subject to boundary perturbations. In particular, we find conditions on the geometry of the domain and of the perturbation in order to guarantee the spectral stability. Given $\Omega \subset \mathbb{R}^N$ defined by

$$\Omega = \{(\bar{x}, x_N) \in W \times (a, c) : a < x_N < g(\bar{x})\},$$

where W is a $N-1$-dimensional open, connected and bounded set of class C^m, $g \in C^m(\overline{W})$, $a, c \in \mathbb{R}$, $a < \|g\|_{L^\infty(W)} < c$, we consider the family of perturbed domains

$$\Omega_\epsilon = \{(\bar{x}, x_N) \in W \times (a, c) : a < x_N < g_\epsilon(\bar{x})\}, \tag{0.0.11}$$

where $g_\epsilon \in C^m(\overline{W})$ for all $\epsilon > 0$. We note that the choice of these particular domains Ω, Ω_ϵ is not very restrictive. Indeed, it is always possible to assume that Ω and Ω_ϵ are locally in this form whenever they belong to suitable regular classes of domains such as those defined via a collection of local charts, say a fixed atlas. We refer to [37] for more information on the atlas class and its application to spectral stability problems. On the domain Ω we consider the higher order differential operator H_Ω associated with the quadratic form Q_Ω defined by (0.0.3) and we make the following assumptions:

(i) $V(\Omega) = W^{m,2}(\Omega) \cap W_0^{k,2}(\Omega)$ for some $1 \leq k < m$;

(ii) The coefficients $A_{\alpha\beta}$ are bounded measurable real-valued functions such that $A_{\alpha\beta} = A_{\beta\alpha}$ for all $|\alpha| = |\beta| = m$ and satisfy the uniform ellipticity condition

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq \theta \sum_{|\alpha|=m} |\xi_\alpha|^2,$$

for all $x \in \Omega$, for some $\theta > 0$.

Note that by Assumption (i) $V(\Omega)$ is compactly embedded in $L^2(\Omega)$, hence the differential operator H associated with the quadratic form Q has compact resolvent. Then we can prove that if for all $\epsilon > 0$ there exists $\kappa_\epsilon > 0$ such that

(i) $\kappa_\epsilon > \|g_\epsilon - g\|_\infty, \quad \forall \epsilon > 0$;

(ii) $\lim_{\epsilon\to 0} \kappa_\epsilon = 0$;

(iii) $\lim_{\epsilon\to 0} \frac{\|D^\beta(g_\epsilon - g)\|_\infty}{\kappa_\epsilon^{m-|\beta|-k+1/2}} = 0, \ \forall \beta \in \mathbb{N}^N$ with $|\beta| \leq m$,

then the perturbation $\Omega_\epsilon \mapsto \Omega$ is spectrally stable (see Lemma 2.2.2). Note that the case $k = 1$ was discussed in [19, Lemma 6.2]. The proof of this lemma is based on the construction of a family of diffeomorphisms Φ_ϵ from $\overline{\Omega}_\epsilon$ to $\overline{\Omega}$ preserving the boundary conditions of a generic $H^m \cap H_0^k$ function. Then $(\Phi_\epsilon)_{\epsilon>0}$ induces a pullback operator T_ϵ from $V(\Omega)$ to $V(\Omega_\epsilon)$ defined by

$$T_\epsilon\varphi = \varphi \circ \Phi_\epsilon,$$

for all $\varphi \in V(\Omega)$. In this way it is possible to prove that whenever hypothesis $(i) - (iii)$ are satisfied, Condition (C) is verified for the operator H_Ω on $V(\Omega)$. By [19, Theorem 3.5], Condition (C) implies that $H_{\Omega_\epsilon}^{-1}$ compactly converges to H_Ω^{-1} as $\epsilon \to 0$. The compact convergence of the operators is understood in the sense of [109, 110, 111, 112]. We mention that this notion of convergence allows to deduce the spectral convergence of family of operators $(H_\epsilon)_{\epsilon>0}$ defined in (possibly varying) Hilbert spaces from a suitable convergence of the associated Poisson problems with data $(f_\epsilon)_{\epsilon>0}$ lying in the Hilbert spaces in consideration. Since the compact convergence of $H_{\Omega_\epsilon}^{-1}$ implies the spectral stability, we deduce that the eigenvalues and eigenfunctions of H_{Ω_ϵ} converge to the eigenvalues and eigenfunctions of H_Ω as $\epsilon \to 0$.

In particular, Lemma 2.2.2 allows us to analyse in detail the spectral convergence of the polyharmonic operator $(-\Delta)^m$ on the domain Ω_ϵ defined by (0.0.2). To fix ideas, for each $\epsilon > 0$, let H_Ω, H_{Ω_ϵ} be the polyharmonic operator $-\Delta^m$ on Ω, Ω_ϵ respectively, with strong intermediate boundary conditions (which were defined in (0.0.10)). Moreover, define the operator $H_{\Omega,D}$ as the polyharmonic operator $(-\Delta)^m$ with Dirichlet boundary conditions on $W \times \{0\}$ and strong intermediate boundary conditions on $\partial\Omega \setminus (W \times \{0\})$. By applying Lemma 2.2.2 we prove that the family of operators H_{Ω_ϵ} spectrally converge to H_Ω whenever the exponent α is greater that $3/2$, for all $m \geq 2$. Moreover, the exponent $\alpha = 3/2$ is critical in the following sense: for $\alpha < \frac{3}{2}$, when the amplitude of the oscillation is slowly tending to zero, then H_{Ω_ϵ} spectrally converges to $H_{\Omega,D}$ as $\epsilon \longrightarrow 0$. In the critical case $\alpha = 3/2$, we are able to characterize the limit boundary conditions in terms of a function V satisfying a suitable differential boundary problem. This result is obtained by using the unfolding operator and macroscopic-miscroscopic analysis of the differential problem, which is common in homogenization theory (see e.g., [20, 21, 43, 44, 45, 46, 51, 53, 71, 108, 115]). More precisely, in Chapter 4, with the help of a polyharmonic Green formula (see Theorem 4.1.3) we are able to prove Theorem 4.2.1, which contains the fully description of the asymptotic spectral behaviour of the operator $(-\Delta)^m$ with strong intermediate boundary conditions. We remark that this result is interesting also from the point of view of the theory of function spaces, since Theorem 4.2.1 can be seen as the analysis of the convergence of the sequence of Sobolev spaces $V(\Omega_\epsilon) = H^m(\Omega_\epsilon) \cap H_0^{m-1}(\Omega_\epsilon)$ as $\epsilon \longrightarrow 0$. It is remarkable that in the critical case $\alpha = 3/2$ the limiting boundary conditions read

$$\begin{cases} u = 0, & \text{on } W \times \{0\}, \\ D^l u = 0, & \text{for all } l \leq m-2, \text{ on } W \times \{0\}, \\ \frac{\partial^m u}{\partial x_N^m} - K \frac{\partial^{m-1} u}{\partial x_N^{m-1}} = 0, & \text{on } W \times \{0\}. \end{cases}$$

where $K > 0$ is a positive real number (compare this problem to (0.0.10), for example). Here $-K\frac{\partial^{m-1} u}{\partial x_N^{m-1}}$ plays the role of the 'strange term', using the nomenclature introduced in the famous article [54].

In the case of more general boundary conditions defined by the energy space $V(\Omega) = H^m(\Omega) \cap H_0^k(\Omega)$ with $1 \leq k < m-1$ we prove that, if $\alpha > m-k+1/2$, then $H_{\Omega_\epsilon}^{-1}$ compactly converges to H_Ω^{-1} as $\epsilon \longrightarrow 0$, see Theorem 2.2.4. Unfortunately we are not able to adapt the techniques of the case $k = m-1$ to study the spectral convergence of H_{Ω_ϵ} on $V(\Omega) = H^m(\Omega) \cap H_0^k(\Omega)$ with $1 \leq k < m-1$ for $\alpha \leq m-k+1/2$. In order to understand what happens in this case, in Chapter 3 we analyse the case of the triharmonic operator $-\Delta^3$ with intermediate boundary conditions. For the operator $-\Delta^3$ there are two natural choices of intermediate boundary conditions: either *strong intermediate boundary conditions*, associated with the energy space $V(\Omega) = H^3(\Omega) \cap H_0^2(\Omega)$, or *weak intermediate boundary conditions*, associated with the energy space $V(\Omega) = H^3(\Omega) \cap H_0^1(\Omega)$. In other words, the eigenvalue problem for $-\Delta^3$ is defined by

$$\begin{cases} -\Delta^3 u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \\ \frac{\partial^3 u}{\partial n^3} = 0, & \text{on } \partial\Omega, \end{cases}$$

in the case of strong intermediate boundary conditions, and by

$$\begin{cases} -\Delta^3 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \left((n^T D^3 u)_{\partial\Omega} : D_{\partial\Omega} n\right) - \frac{\partial^2(\Delta u)}{\partial n^2} - 2\,\mathrm{div}_{\partial\Omega}(D^3 u[n \otimes n])_{\partial\Omega} = 0, & \text{on } \partial\Omega, \\ \frac{\partial^3 u}{\partial n^3} = 0, & \text{on } \partial\Omega, \end{cases}$$

in the case of weak intermediate boundary conditions. This expression of the weak intermediate boundary conditions is deduced from a 'Triharmonic Green Formula', see Theorem 4.1.7. Moreover, we remark that $(\cdot)_{\partial\Omega}$ stands for the projection on the tangent hyperplane to $\partial\Omega$ and we refer to §1.4 for the definitions of the tangential operators $\mathrm{div}_{\partial\Omega}$ and $D_{\partial\Omega}$. By using a suitable degeneration lemma (see Lemma 3.1.2) we are able to prove that the asymptotic spectral behaviour of the triharmonic operator with weak intermediate boundary conditions is of the following type (see Theorem 3.3.1)

(i) If $\alpha > 5/2$, then the perturbation $\Omega_\epsilon \mapsto \Omega$ is spectrally stable;

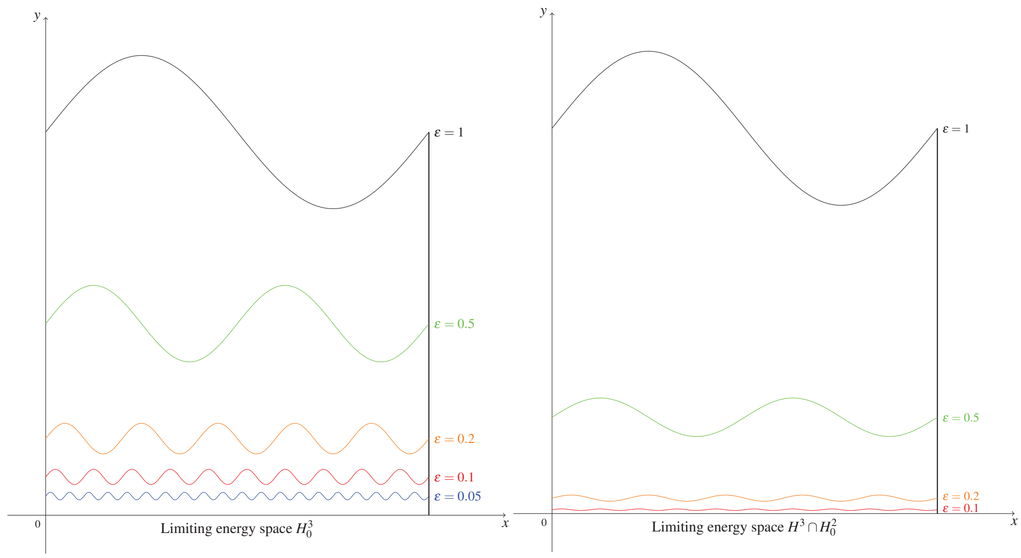
(ii) If $3/2 < \alpha < 5/2$, then the limiting differential problem in Ω exhibits strong intermediate boundary conditions on $W \times \{0\}$.

(iii) If $\alpha \leq 1$, then the limiting differential problem in Ω exhibits Dirichlet boundary conditions on $W \times \{0\}$.

See Fig. 2 for an example of how the exponent α changes the oscillation of the boundary of $\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 : x \in (0, 1), -1 < y < \epsilon^\alpha b(x/\epsilon)$ for the particular choice of $b(x) = 1 + 2\sin(2\pi x/5)$.

In the critical case $\alpha = 5/2$ we are able to give a full characterization of the limiting differential problem in terms of an auxiliary function, by using homogenization techniques similar to those used to treat the homogenization for the triharmonic operator with strong intermediate boundary conditions. Let us remark that it remains an open and quite difficult problem to understand the limiting spectral behaviour of the triharmonic operator with weak intermediate boundary conditions when $1 < \alpha \leq 3/2$.

The second part of the thesis (which consists of Chapter 5 and Chapter 6) is devoted to a spectral analysis of the biharmonic operator and of the Reissner-Mindlin system subject to Neumann boundary conditions on dumbbell domains.



(a) $\alpha = 1$ (b) $\alpha = 2$

Figure 2: Oscillations of the upper boundary of Ω_ϵ as $\epsilon \longrightarrow 0$, depending on α.

We consider planar dumbbell-shaped domains Ω_ϵ, with $\epsilon > 0$, described in Figure 3. Namely, given two bounded smooth domains Ω_L, Ω_R in $\mathbb{R}^2$ with the property that $\Omega_L \cap \Omega_R = \emptyset$, $(\Omega_R \cup \Omega_L) \cap ([0,1] \times [-1,1]) = \emptyset$, and such that $\partial\Omega_L \supset \{(0,y) \in \mathbb{R}^2 : -1 < y < 1\}$, $\partial\Omega_R \supset \{(1,y) \in \mathbb{R}^2 : -1 < y < 1\}$, we set

$$\Omega = \Omega_L \cup \Omega_R, \quad \text{and} \quad \Omega_\epsilon = \Omega \cup R_\epsilon \cup L_\epsilon\,,$$

for all $\epsilon > 0$ small enough. Here $R_\epsilon \cup L_\epsilon$ is a thin channel connecting Ω_L and Ω_R defined by

$$\begin{aligned} R_\epsilon &= \{(x,y) \in \mathbb{R}^2 : x \in (0,1), 0 < y < \epsilon g(x)\}, \\ L_\epsilon &= (\{0\} \times (0, \epsilon g(0)) \cup (\{1\} \times (0, \epsilon g(1)))), \end{aligned} \tag{0.0.12}$$

where $g \in C^2[0,1]$ is a positive real-valued function.

In Chapter 5 we study the following variant of Problem (0.0.6):

$$\begin{cases} \Delta^2 u - \tau \Delta u + u = \lambda\, u, & \text{in } \Omega_\epsilon, \\ (1-\sigma)\frac{\partial^2 u}{\partial n^2} + \sigma \Delta u = 0, & \text{on } \partial\Omega_\epsilon, \\ \tau \frac{\partial u}{\partial n} - (1-\sigma)\ \mathrm{div}_{\partial\Omega_\epsilon}(D^2 u \cdot n)_{\partial\Omega_\epsilon} - \frac{\partial(\Delta u)}{\partial n} = 0, & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{0.0.13}$$

where $\tau \in \mathbb{R}$, $\tau > 0$ is sometimes called lateral tension.

First of all, we prove that the eigenvalues of problem (0.0.13) can be asymptotically decomposed into two families of eigenvalues as

$$(\lambda_n(\Omega_\epsilon))_{n \geq 1} \approx (\omega_k)_{k \geq 1} \cup (\theta_l^\epsilon)_{l \geq 1}, \quad \text{as } \epsilon \longrightarrow 0, \tag{0.0.14}$$

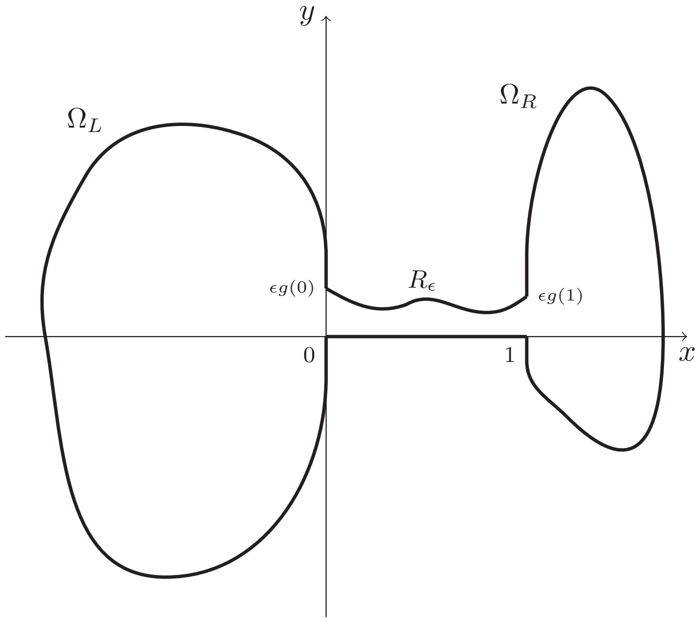


Figure 3: The dumbbell domain Ω_ϵ.

where $(\omega_k)_{k\geq 1}$ are the eigenvalues of problem

$$\begin{cases} \Delta^2 w - \tau\Delta w + w = \omega_k\, w, & \text{in } \Omega, \\ (1-\sigma)\frac{\partial^2 w}{\partial n^2} + \sigma\Delta w = 0, & \text{on } \partial\Omega, \\ \tau\frac{\partial w}{\partial n} - (1-\sigma)\,\mathrm{div}_{\partial\Omega}(D^2 w\cdot n)_{\partial\Omega} - \frac{\partial(\Delta w)}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.0.15}$$

and $(\theta_l^\epsilon)_{l\geq 1}$ are the eigenvalues of problem

$$\begin{cases} \Delta^2 v - \tau\Delta v + v = \theta_l^\epsilon\, v, & \text{in } R_\epsilon, \\ (1-\sigma)\frac{\partial^2 v}{\partial n^2} + \sigma\Delta v = 0, & \text{on } \Gamma_\epsilon, \\ \tau\frac{\partial v}{\partial n} - (1-\sigma)\,\mathrm{div}_{\Gamma_\epsilon}(D^2 v\cdot n)_{\Gamma_\epsilon} - \frac{\partial(\Delta v)}{\partial n} = 0, & \text{on } \Gamma_\epsilon, \\ v = 0 = \frac{\partial v}{\partial n}, & \text{on } L_\epsilon. \end{cases} \tag{0.0.16}$$

where $\Gamma_\epsilon = \partial R_\epsilon \setminus L_\epsilon$. The decomposition (0.0.14) is proved under the assumption that a certain condition on R_ϵ, called H-Condition (see Definition 5.2.7 in Chapter 5), is satisfied. We are able to prove that if we consider channels R_ϵ such that the profile function g has the following monotonicity property:

(MP): *there exists* $\delta \in]0, 1/2[$ *such that* g *is decreasing on* $[0, \delta)$ *and increasing on* $(1-\delta, 1]$.

then the H-Condition is verified. Hence, there exists a large class of dumbbell domains which satisfies the H-Condition.

According to the decomposition (0.0.14), in order to analyse the behaviour of $\lambda_n(\Omega_\epsilon)$ as $\epsilon \longrightarrow 0$, it suffices to study θ_l^ϵ as $\epsilon \longrightarrow 0$. We pass to the limit as $\epsilon \longrightarrow 0$ by using thin domain techniques (see e.g., [20, 21, 73, 74]) and we find the following limiting problem in the segment $(0, 1)$:

$$\begin{cases} \frac{1-\sigma^2}{g}(gh'')'' - \frac{\tau}{g}(gh')' + h = \theta h, & \text{in } (0,1), \\ h(0) = h(1) = 0, \\ h'(0) = h'(1) = 0. \end{cases} \tag{0.0.17}$$

Then, in Theorem 5.6.1, we establish the following alternative:

(A) either $\lambda_n(\Omega_\epsilon) \longrightarrow \omega_k$, for some $k \geq 1$ in which case the corresponding eigenfunctions converge in Ω to the eigenfunctions associated with ω_k.

(B) or $\lambda_n(\Omega_\epsilon) \longrightarrow \theta_l$ as $\epsilon \longrightarrow 0$ for some $l \in \mathbb{N}$ in which case the corresponding eigenfunctions behave in R_ϵ like the eigenfunctions associated with θ_l.

Moreover, all eigenvalues ω_k and θ_l are reached in the limit by the eigenvalues $\lambda_n(\Omega_\epsilon)$. We find it remarkable that for $\sigma \neq 0$ the limiting equation in (0.0.17) is distorted by the coefficient $1-\sigma^2 \neq 1$. This phenomenon shows that the dumbbell problem for our fourth order problem (0.0.13) with $\sigma \neq 0$ is significantly different from the second order problem considered in the literature (see e.g.,[11], [12], [78], [79] and references therein). We point out that, in contrast with the spectral problems for second order operators with either Neumann or Dirichlet boundary conditions on dumbbell domains, very little seems to be known about these problems for higher order operators. We refer to [114] for a recent analysis of the dumbbell problem in the case of elliptic systems subject to Dirichlet boundary conditions.

Finally, in Chapter 6 we prove spectral convergence results for the Reissner-Mindlin system on dumbbell domains $\Omega_\delta \subset \mathbb{R}^2$ subject to mixed free-clamped boundary conditions, which is defined by

$$\begin{cases} -\frac{\mu_1}{12}\Delta\beta - \frac{\mu_1+\mu_2}{12}\nabla(\operatorname{div}\beta) - \frac{\mu_1 k}{t^2}(\nabla w - \beta) = \lambda\frac{t^2}{12}\beta, & \text{in } \Omega_\delta, \\ -\frac{\mu_1 k}{t^2}(\Delta w - \operatorname{div}\beta) = \lambda w, & \text{in } \Omega_\delta, \\ \frac{\mu_1}{12}(\nabla\beta n + n^T\nabla\beta) + \frac{\mu_1+\mu_2}{12}(\operatorname{div}\beta)n = 0, & \text{on } \partial\Omega_\delta \setminus B, \\ s^T\epsilon(\beta)n + n^T\epsilon(\beta)s = 0, & \text{on } \partial\Omega_\delta \setminus B, \\ \frac{\mu_1 k}{t^2}(\nabla w - \beta)\cdot n = 0, & \text{on } \partial\Omega_\delta \setminus B, \\ \beta = w = 0, & \text{on } B, \end{cases} \tag{0.0.18}$$

where s is the unit vector tangent to $\partial\Omega_\delta$ obtained by rotating the unit normal vector n of $\pi/2$ anticlockwise and $B \subset \partial\Omega$ is an open subset of $\partial\Omega$. As we have mentioned before, the eigenvalues and eigenfunctions of this system are converging to the eigenvalues and the eigenfunctions of the biharmonic operator defined in (0.0.8) as $t \to 0$. Then it is an interesting question to understand whether the asymptotic spectral behaviour of the eigenvalues of (0.0.18) as $\delta \to 0$ is compatible with the asymptotic behaviour of the eigenvalues of the biharmonic operator investigated in Chapter 5. We are able to prove an asymptotic spectral decomposition result for the spectrum (0.0.18) by applying the results in Chapter 5. In particular we identify the limiting problem in the channel R_δ, which is given by

$$\begin{cases} -\frac{E}{12g}((h^1)'g)' - \frac{Ek}{2(1-\sigma)t^2}\left((h^2)' - h^1\right) = \frac{\theta t^2}{12}h^1, & \text{in } (0,1), \\ \frac{Ek}{2(1-\sigma)t^2}[(h^2)'' - (h^1)'] = \theta h^2, & \text{in } (0,1), \\ h(0) = h(1) = 0, \\ h'(0) = h'(1) = 0, \end{cases} \tag{0.0.19}$$

where θ is the eigenvalue and $h = (h^1, h^2)$ is the eigenfunction. We remark that

$$\frac{E}{12} = (1-\sigma^2)\frac{2\mu_1 + \mu_2}{12},$$

and by recalling (0.0.8), we note that the one dimensional Problem (6.5.1) is compatible with (5.1.9). Then we discuss briefly the behaviour of the spectrum of the Reissner-Mindlin system as $t \to 0$, by applying well-known techniques presented in the papers [9, 26, 65].

The thesis is organized as follows. Chapter 1 is dedicated to some preliminaries. In Chapter 2 we discuss the variational formulation of eigenvalue problems for higher order elliptic operators and we prove a spectral stability result for such operators on domains subject to boundary perturbations. In particular, we prove that polyharmonic operators with intermediate boundary conditions associated with the energy space $V(\Omega_\epsilon) = H^m(\Omega_\epsilon) \cap H^k_0(\Omega_\epsilon)$ are spectrally stable above the critical rate of oscillation $\alpha = m - k + 1/2$. In Chapter 3 we consider the spectral convergence for the triharmonic operator with intermediate boundary conditions on domains subject to boundary oscillations. We examine separately the case of strong intermediate boundary conditions and the case of weak intermediate boundary conditions and we prove spectral convergence theorems depending on the parameter α. In Chapter 4 we consider the spectral convergence for polyharmonic operators of any order with strong intermediate boundary conditions on domains subject to boundary oscillations. We also give a proof of a polyharmonic Green formula, which is of independent interest. Then we mainly consider the case in which there is no spectral stability (corresponding to $\alpha \le 3/2$). In the critical case $\alpha = 3/2$ we carry out the homogenization procedure needed to identify the limiting differential problem, independently of the order of the operator. In

Chapter 5 we find the limiting differential problem and the associated limiting spectrum for the biharmonic operator with Neumann boundary conditions on a planar dumbbell domain, as the width of the channel vanishes. Finally, Chapter 6 is devoted to the spectral analysis of the Reissner-Mindlin system with mixed free-clamped boundary conditions on dumbbell domains, with particular attention to the dependence of the eigenfunctions on the thickness parameter.

Part of these results have already been accepted for publication. The discussion about the triharmonic operator in Chapter 3 has been partially published in [17]. The spectral analysis for the biharmonic operator in Chapter 5 is contained in the article [16]. Parts of Chapter 2, of Chapter 3 and the material in Chapter 4 are contained in the article [67], in preparation.

Contents

Chapter 1

Notation and preliminary results

In this Chapter we recall for the convenience of the reader basic notation, results and definitions which will be used in the sequel.

1.1 Sobolev spaces and the variational method in Spectral Theory

Let $N, l \in \mathbb{N}$, $1 \le p \le \infty$ and let Ω be an open set in $\mathbb{R}^N$. We denote by $W^{l,p}(\Omega)$ the Sobolev space of real-valued functions in $L^p(\Omega)$ which admit weak derivatives up to order l in $L^p(\Omega)$. The space $W^{l,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{l,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{|\alpha|=l} \|D^\alpha u\|_{L^p(\Omega)}.$$

When $p = 2$ we set $W^{l,2}(\Omega) := H^l(\Omega)$. It is easy to check that $H^l(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{H^l(\Omega)} = \int_\Omega uv\, dx + \sum_{|\alpha|=l} \int_\Omega D^\alpha u\, D^\alpha v\, dx,$$

for all $u, v \in H^l(\Omega)$. If $p \ne \infty$, we denote by $W_0^{l,p}(\Omega)$ (resp. $H_0^l(\Omega)$) the closure in $W^{l,p}(\Omega)$ (resp. $H^l(\Omega)$) of the space $C_c^\infty(\Omega)$ of C^∞-functions with compact support in Ω.

In a similar way, for $m \in \mathbb{N}$, we define the Sobolev space $W^{l,p}(\Omega)^m$ of vector fields $u = (u_1, \dots, u_m)$ endowed with the norm

$$\|u\|_{W^{l,p}(\Omega)} = \sum_{k=1}^m \|u_k\|_{W^{l,p}(\Omega)}.$$

We refer to the monographs [25, 35, 91] for further information about Sobolev spaces and properties of $W^{l,p}$-functions.

Sobolev spaces are a fundamental tool in the analysis of partial differential equations because they are naturally associated with the variational formulation of many differential problems. In order to make this statement clearer we recall some well-known facts about the theory of operators in Hilbert spaces. First of all we restrict our analysis to the theory of self-adjoint operators, which is a natural choice when the aim is to address spectral problems. Let $\mathscr{H}$ be a given infinite dimensional separable Hilbert space (for example $L^2(\Omega)$). In applications to elliptic differential problems it is common to deal with problems in the form

$$Tu = \lambda u,$$

for all u in $\mathscr{D}(T)$, where T is a self-adjoint operator with domain $\mathscr{D}(T) \subset \mathscr{H}$ dense in $\mathscr{H}$, and compact resolvent. We recall the following

Theorem 1.1.1. *Let T be an unbounded self-adjoint operator densely defined on $\mathscr{H}$ with spectrum $\sigma \subseteq [0, +\infty[$. Then the following conditions are equivalent:*

(i) T has compact resolvent, i.e. $(T - \lambda\mathbb{I})^{-1}$ is a compact operator for all $\lambda \in \rho(T)$;

(ii) T has empty essential spectrum;

(iii) There exists a complete orthonormal set (ϕ_n) of eigenfunctions of T with corresponding eigenvalues $\lambda_n \geq 0$ such that $\lim_{n\to+\infty} \lambda_n = +\infty$.

There is an important correspondence between semibounded self-adjoint operators and semibounded quadratic forms. We first recall the following

Definition 1.1.2. A symmetric operator T with domain $\mathscr{D}$ is semibounded if there exists a constant $c \in \mathbb{R}$ s.t.

$$(Tu, u) \geq c\|u\|^2, \quad \forall u \in \mathscr{D}.$$

We also say that T is semibounded by the constant c. If $c = 0$ we say that T is non-negative. We write $T \geq c$ and $T \geq 0$, respectively.

Given a non-negative self-adjoint operator T on a Hilbert space $\mathscr{H}$ and $\alpha > 0$ it is possible to define in a rigorous way the non-negative self-adjoint operator T^α by functional calculus. In particular we have the following

Theorem 1.1.3. *T^α is canonically determined by the functional equality*

$$(T^\alpha + \mathbb{I})^{-1} = f(T),$$

where f is the continuous function on $\mathbb{R}$ defined by

$$f(x) = \frac{1}{|x|^\alpha + 1},$$

and $f(T)$ is defined via functional calculus (see e.g., [60, Chapter 2]). If $0 < \alpha < 1$, then $\mathscr{D}(T^\alpha)$ is related to $\mathscr{D}(T)$ in the following way. Given $f \in \mathscr{H}$, $f \in \mathscr{D}(T)$ if and only if $f \in \mathscr{D}(T^\alpha)$ and $T^\alpha f \in \mathscr{D}(T^{1-\alpha})$. Moreover, if this is verified, the following formula holds

$$Tf = T^{1-\alpha}(T^\alpha f).$$

Proof. We refer to [60, Theorems 4.3.3, 4.3.4]. □

Definition 1.1.4. We say that a quadratic form $Q : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ is

(a) densely defined if $\overline{\mathscr{D}(Q)} = \mathscr{H} \times \mathscr{H}$;

(b) symmetric if $Q(x, y) = Q(y, x)$;

(c) semibounded from below if there exists $c \in \mathbb{R}$ s.t. $Q(x, x) \geq -c\|x\|^2$ for all $x \in \mathscr{D}(Q)$. In particular if $c = 0$ we say that Q is non-negative;

(d) closed if Q is semibounded by a constant $c \in \mathbb{R}$ and $\mathscr{D}(Q)$ is complete with respect to the norm

$$\|x\|_Q \equiv \sqrt{Q(x, x) + (c + 1)\|x\|^2}.$$

(e) bounded if there exists $M > 0$ s.t.

$$|Q(x, y)| \leq M\|x\|\|y\|,$$

for all $x, y \in \mathscr{D}(Q)$.

Given a densely defined, non-negative self-adjoint operator T on $\mathscr{H}$ it is easy to verify that

$$Q_T(f, g) = (T^{1/2}f, T^{1/2}g) \quad \forall f, g \in \mathscr{D}(T^{1/2})$$

is a non-negative symmetric and densely defined quadratic form with domain $\mathscr{D}(T^{1/2})$. We shall call Q_T the quadratic form associated with T.

Lemma 1.1.5. *Let T be a non-negative self-adjoint operator densely defined on $\mathscr{H}$. Then $f \in \mathscr{H}$ lies in $\mathscr{D}(T)$ if and only if $f \in \mathscr{D}(T^{1/2})$ and also there exists $k \in \mathscr{H}$ such that*

$$Q_T(f, g) = (k, g)$$

for all $g \in \mathscr{D}(T^{1/2})$. In this case we have $Tf = k$.

Proof. Note that for every $f \in \mathscr{D}(T^{1/2})$ the condition

$$Q_T(f,g) = (T^{1/2}f, T^{1/2}g) = (k,g), \qquad \forall g \in \mathscr{D}(T^{1/2}),$$

is equivalent, by definition of adjoint operator, to the conditions that $T^{1/2}f \in \mathscr{D}((T^{1/2})^*)$ and $(T^{1/2})^*T^{1/2}f = k$. By the self-adjointness of T, $T^{1/2}$ is still self-adjoint. Thus by Proposition 1.1.3 in the case $\alpha = 1/2$ we conclude. □

Theorem 1.1.6. *The following conditions are equivalent:*

(i) Q is the quadratic form associated with a densely defined, non-negative self-adjoint operator T;

(ii) Q is a non-negative, symmetric and closed quadratic form on $\mathscr{H}$ with dense domain $\mathscr{D}(Q)$.

Proof. See [60, Theorem 4.4.2]. □

A very useful criterion to check if a given non-negative self-adjoint operator on a Hilbert space $\mathscr{H}$ has compact resolvent is given in the following

Theorem 1.1.7. *Let T be a non-negative self-adjoint operator on $\mathscr{H}$ and let Q be the quadratic form associated with T, defined on $\mathscr{D}(Q) = \mathscr{D}(T^{1/2})$. Then T has compact resolvent if and only if the embedding*

$$\iota : \mathscr{H}_Q \equiv (\mathscr{D}(Q), \|\cdot\|_Q) \to (\mathscr{H}, \|\cdot\|_{\mathscr{H}})$$

is a compact linear operator. In this case $\sigma(T)$ consists of an unbounded sequence of eigenvalues, whose eigenvectors form a complete orthonormal set in $\mathscr{H}$.

Proof. The last part of the statement is exactly Theorem 1.1.1. For simplicity we suppose that 0 is not in $\sigma(T)$, just to use directly the resolvent T^{-1} instead of $(T + \mathbb{I})^{-1}$. Note also that ι is continuous, since

$$\|f\|_{\mathscr{H}} \leq \|f\|_Q,$$

for all $f \in \mathscr{D}(Q)$. Next, we divide the proof in two steps.

Step 1. Our assert is that ι is compact if and only if $(T^{1/2})^{-1} \equiv T^{-1/2}$ is compact on $\mathscr{H}$. Since $0 \notin \sigma(T)$ by assumption, $T^{-1/2}$ is a bounded operator defined on the whole of $\mathscr{H}$.

Now, ι is compact if and only if given a sequence $(f_n)_n \subset \mathscr{D}(Q)$ with

$$\|T^{1/2}f_n\|_{\mathscr{H}} + \|f_n\|_{\mathscr{H}} \leq M, \quad \forall n \geq 1,$$

there exists a subsequence $(f_{n_k})_k$ of $(f_n)_n$ converging in $(\mathscr{H}, \|\cdot\|_{\mathscr{H}})$. We then set $T^{1/2} f_n = g_n$ for all n; since $T^{1/2}$ is invertible we deduce that ι is compact if and only if, given a sequence $(g_n)_n \subset \mathscr{H}$ with

$$\|g_n\|_{\mathscr{H}} + \|T^{-1/2} g_n\|_{\mathscr{H}} \leq M, \quad \forall n \geq 1,$$

there exists a subsequence $(g_{n_k})_k \subset (g_n)_n$ and $g \in \mathscr{H}$ such that

$$T^{-1/2} g_{n_k} \to g, \quad \text{in } \mathscr{H}, \text{ as } k \to \infty.$$

This proves the assert.

Step 2. We now prove that $T^{-1/2}$ is compact on $\mathscr{H}$ if and only if T^{-1} is compact on $\mathscr{H}$. The idea is to write

$$T^{-1} = T^{-1/2} \circ T^{-1/2}, \tag{1.1.1}$$

where the composition makes sense since $T^{-1/2}(\mathscr{D}(T^{1/2})) = \mathscr{D}(T)$, by Lemma [60, Lemma 4.4.1]. If $T^{-1/2}$ is compact on $\mathscr{H}$, by standard properties of compact operators we deduce that T^{-1} is compact.

Conversely, if T^{-1} is compact, by Theorem 1.1.1 there exists a complete orthonormal set $(u_n)_n$ for $\mathscr{H}$ of eigenvectors of T associated with the sequence of eigenvalues $(\lambda_n)_n$ such that $\lambda_n \to \infty$. This implies that there exists a complete orthonormal set $(u_n)_n$ for $\mathscr{H}$ of eigenvectors of $T^{1/2}$ associated with the sequence of eigenvalues $(\lambda_n^{1/2})_n$. Again by Theorem 1.1.1, this implies that $T^{-1/2}$ is compact.

By Steps 1 and 2 the compactness of ι is equivalent to the compactness of T^{-1}, and this concludes the proof. □

Moreover we can associate to each non-negative and symmetric operator T a canonical self-adjoint extension.

Theorem 1.1.8 (Friedrichs extension). *Let T be a non-negative, symmetric operator in a Hilbert space $\mathscr{H}$. Then there exists a self-adjoint extension T_F of T which is minimal in the following sense: if T' is a non-negative self-adjoint extension of T with associated quadratic form $Q_{T'}$, then $\mathscr{D}(Q_{T'}) \supset \mathscr{D}(Q_{T_F})$. This extension T_F is called the Friedrichs extension of T.*

Proof. We refer to [75, Theorem 4.4]. □

Let us introduce some notation. Let T be a non-negative self-adjoint operator on $\mathscr{H}$ and let L be any finite-dimensional subspace of $\mathscr{D}(T)$. We set

$$\lambda(L) \equiv \sup\{(Tf, f) : f \in L \text{ and } \|f\| = 1\}.$$

We then define a non-decreasing sequence of non-negative real numbers λ_n by

$$\lambda_n \equiv \inf\{\lambda(L) : L \subseteq \mathscr{D}(T) \text{ and } \dim(L) = n\}. \tag{1.1.2}$$

Let now S be any finite-dimensional subspace of $\mathscr{D}(T^{1/2})$. We define

$$\lambda'_n = \inf_{\substack{L \subset \mathscr{D}(T^{1/2}) \\ \dim(S)=n}} \sup_{\substack{f \in S \\ \|f\|=1}} Q_T(f, f).$$

In the case of non-negative self-adjoint operators with compact resolvent, the unbounded sequence of eigenvalues is fully determined by the min-max principle (1.1.2). More specifically, the following theorem holds.

Theorem 1.1.9. *Let T be a non-negative self-adjoint operator on $\mathscr{H}$ with compact resolvent. Then the sequence of real numbers $(\lambda_n)_{n\geq 1}$ defined by formula* (1.1.2) *coincides with the eigenvalues of T written in increasing order and repeated according to the multiplicity. Moreover, $\lambda_n = \lambda'_n$ for all $n \geq 1$.*

Proof. See for example [60, Theorem 4.5.3]. □

1.2 The unfolding method

The unfolding method is a powerful and versatile tool in the study of homogenization of partial differential equations introduced by D. Cioranescu, A. Damlamian and G. Griso, see e.g., [52, 53, 58]. Roughly speaking, this method is based on a change of variables which doubles the dimension of the space in order to take advantage of the periodicity involved in the setting of the problem. In particular, the so-called two-scale convergence of functions (which was the very first method used to study homogenization and multi-scale problems, see [4, 42]) turns out to be the weak convergence of the sequence of the unfolded functions. Moreover, the change of scale induces a macro-micro decomposition of functions which is especially suited for the weakly convergent sequences in Sobolev spaces. We are going to recall the main definitions and properties of the unfolding method.

Definition 1.2.1. Let Ω be an open set of $\mathbb{R}^N$ and let $Y = [-1/2, 1/2]^N$. For almost all $x \in \mathbb{R}^N$ we define $[x]_Y$ to be the unique vector in $\mathbb{Z}^N$, such that $x - [x]_Y$ lies in Y. Moreover we set $\{x\}_Y = x - [x]_Y \in Y$.

Then for almost all $x \in \mathbb{R}^N$ and every $\epsilon > 0$ we have the decomposition

$$x = \epsilon\left(\left[\frac{x}{\epsilon}\right]_Y + \left\{\frac{x}{\epsilon}\right\}\right).$$

We will use the following notation:

$$\begin{cases} \Xi_\epsilon = \{\xi \in \mathbb{Z}^N : \epsilon(\xi + Y) \subset \Omega\}, \\ \hat{\Omega}_\epsilon = \displaystyle\bigcup_{\xi \in \Xi_\epsilon} \epsilon(\xi + \overline{Y}), \\ \Lambda_\epsilon = \Omega \setminus \hat{\Omega}_\epsilon. \end{cases}$$

Then we have the following

Definition 1.2.2. The unfolding operator $\mathcal{T}_\epsilon$ is defined by

$$\mathcal{T}_\epsilon(\phi)(x,y) = \begin{cases} \phi\left(\epsilon\left[\frac{x}{\epsilon}\right] + \epsilon y\right), & \text{for a.a. } (x,y) \in \hat{\Omega}_\epsilon \times Y, \\ 0, & \text{for a.a. } (x,y) \in \Lambda_\epsilon \times Y. \end{cases}$$

for all Lebesgue measurable functions ϕ defined on Ω.

In the following we recall some basic properties of the unfolding operator $\mathcal{T}_\epsilon$.

Proposition 1.2.3. *Let $\epsilon > 0$ be fixed. The unfolding operator $\mathcal{T}_\epsilon$ maps Lebesgue-measurable functions on Ω to Lebesgue-measurable functions on $\Omega \times Y$. Moreover the following properties are satisfied:*

(i) For any Lebesgue-measurable functions v, w, $\mathcal{T}_\epsilon(vw) = \mathcal{T}_\epsilon(v)\mathcal{T}_\epsilon(w)$;

(ii) Let f be a Y-periodic Lebesgue-measurable function on $\mathbb{R}^N$. Let $f_\epsilon = f(x/\epsilon)$ for all $\epsilon > 0$. Then $\mathcal{T}_\epsilon(f_\epsilon|_\Omega) = f(y)\chi_{\hat{\Omega}_\epsilon \times Y}(x,y)$.

Then we have the following

Proposition 1.2.4. *Let $p \in [1, +\infty[$. Then the operator $\mathcal{T}_\epsilon$ is linear and continuous from $L^p(\Omega)$ to $L^p(\Omega \times Y)$ and*

$$\|\mathcal{T}_\epsilon(w)\|_{L^p(\Omega \times Y)} \leq \|w\|_{L^p(\Omega)},$$

for all $w \in L^p(\Omega)$. Moreover,

$$\int_{\Omega \times Y} \mathcal{T}_\epsilon(\phi)(x,y)dxdy = \int_{\hat{\Omega}_\epsilon} \phi(x)dx, \tag{1.2.1}$$

for all $\phi \in L^1(\Omega)$.

Proof. We give a proof of this proposition because property (1.2.1) will be used many times in Chapters 3 and 4. By definition of the set $\hat{\Omega}_\epsilon$ we have

$$\begin{aligned}\int_{\Omega\times Y}\mathcal{T}_\epsilon(\phi)(x,y)dxdy &= \int_{\hat{\Omega}_\epsilon\times Y}\mathcal{T}_\epsilon(\phi)(x,y)dxdy\\ &= \sum_{\xi\in\Xi_\epsilon}\int_{(\epsilon\xi+\epsilon Y)\times Y}\mathcal{T}_\epsilon(\phi)(x,y)dxdy.\end{aligned} \tag{1.2.2}$$

By definition of $\mathcal{T}_\epsilon$, for all $x\in(\epsilon\xi+\epsilon Y)$ we have that $\mathcal{T}_\epsilon(\phi)$ does not depend on x, for all $\xi\in\Xi_\epsilon$. As a consequence of this we can rewrite each integral appearing in the sum on the right-hand side of (1.2.2) as follows

$$\begin{aligned}\int_{(\epsilon\xi+\epsilon Y)\times Y}\mathcal{T}_\epsilon(\phi)(x,y)dxdy &= |\epsilon\xi+\epsilon Y|\int_Y\phi(\epsilon\xi+\epsilon y)dy\\ &= \int_{\epsilon\xi+\epsilon Y}\phi(x)dx,\end{aligned}$$

where in the last equality we used the change of variables $\epsilon\xi+\epsilon y = x$. Going back to (1.2.2) we then find that

$$\int_{\Omega\times Y}\mathcal{T}_\epsilon(\phi)(x,y)dxdy = \sum_{\xi\in\Xi_\epsilon}\int_{\epsilon\xi+\epsilon Y}\phi(x)dx = \int_{\hat{\Omega}_\epsilon}\phi(x)dx.$$

In a similar way one can prove that

$$\|\mathcal{T}_\epsilon(w)\|_{L^p(\Omega\times Y)} = \|w|_{\hat{\Omega}_\epsilon}\|_{L^p(\Omega)},$$

for all $w\in L^p(\Omega)$. □

Definition 1.2.5. Let $p\in[1,\infty]$. We define the operator $\mathcal{M}_Y$ from $L^p(\Omega\times Y)$ to $L^p(\Omega)$ by

$$\mathcal{M}_Y(\phi)(x) = \int_Y\phi(x,y)dy,$$

for almost all $x\in\Omega$, for all $\phi\in L^p(\Omega)$. Moreover, we define the operator $\mathcal{M}_\epsilon$ from $L^p(\Omega)$ to $L^p(\Omega)$ by

$$\mathcal{M}_\epsilon(\phi)(x) = \begin{cases}\frac{1}{\epsilon^N}\int_{\epsilon[\frac{x}{\epsilon}]+\epsilon Y}\phi(t)dt, & \text{if } x\in\hat{\Omega}_\epsilon,\\ 0, & \text{if } x\in\Lambda_\epsilon.\end{cases}$$

for almost all $x\in\Omega$, for all $\phi\in L^p(\Omega)$.

Recall also the following classical definition.

Definition 1.2.6. Let $p \in [1,\infty[$ and let $(w_\epsilon)_{\epsilon>0}$ be a bounded sequence in $L^p(\Omega)$. We say that w_ϵ two-scale converges to a function $w \in L^p(\Omega \times Y)$ whenever

$$\int_\Omega w_\epsilon(x)\phi\Big(x, \frac{x}{\epsilon}\Big)dxdy \to \int_{\Omega\times Y} w(x,y)\phi(x,y)dxdy,$$

as $\epsilon \to 0$, for all bounded $\phi \in C^\infty(\Omega \times Y)$.

We then recall the principal convergence properties of the unfolding operator.

Proposition 1.2.7. *Let $p \in [1,\infty[$. The following statements hold true.*

(i) *Let $w \in L^p(\Omega)$. Then $\mathcal{T}_\epsilon(w) \to w$ in $L^p(\Omega \times Y)$, as $\epsilon \to 0$.*

(ii) *If $\mathcal{T}_\epsilon(w_\epsilon) \to w_0$ weakly in $L^p(\Omega \times Y)$ then $w_\epsilon \to \mathcal{M}_Y(w_0)$ weakly in $L^p(\Omega)$.*

(iii) *Let $(w_\epsilon)_\epsilon$ be a bounded sequence in $L^p(\Omega)$. Then $\mathcal{T}_\epsilon w_\epsilon \to w$ as $\epsilon \to 0$, weakly in $L^p(\Omega \times Y)$ if and only if w_ϵ two-scale converges to w.*

(iv) $\nabla_y(\mathcal{T}_\epsilon w) = \epsilon\mathcal{T}_\epsilon(\nabla w)$ *for all $w \in W^{1,p}(\Omega)$, hence $\mathcal{T}_\epsilon w \in L^p(\Omega; W^{1,p}(Y))$.*

(v) *Let $k \in \{1,\dots,N\}$, $p \in]1,\infty[$ and let $(w_\epsilon)_\epsilon$ be a bounded sequence in $L^p(\Omega)$ such that*

$$\epsilon\left\|\frac{\partial w_\epsilon}{\partial x_k}\right\| \leq C,$$

with the constant C not depending on ϵ. Then there exists a subsequence w_{ϵ_j} of w_ϵ (with $\epsilon_j \to 0$ as $j \to \infty$) and a function $w_0 \in L^p(\Omega \times Y)$ with $\frac{\partial w_0}{\partial y_k} \in L^p(\Omega \times Y)$ such that $\mathcal{T}_{\epsilon_j} w_{\epsilon_j} \to w_0$ weakly in $L^p(\Omega \times Y)$ and

$$\frac{\partial(\mathcal{T}_{\epsilon_j} w_{\epsilon_j})}{\partial y_k} \rightharpoonup \frac{\partial w_0}{\partial y_k}$$

weakly in $L^p(\Omega \times Y)$ as $j \to \infty$. Moreover w_0 is 1-periodic with respect to the variable y_k.

(vi) *Let $p \in]1,+\infty[$ and w_ϵ be a sequence in $W^{1,p}(\Omega)$ converging weakly to w in $W^{1,p}(\Omega)$ as $\epsilon \to 0$. Then $\mathcal{T}_\epsilon w_\epsilon \to w$ weakly in $L^p(\Omega; W^{1,p}(Y))$ as $\epsilon \to 0$ (that is, $\mathcal{T}_\epsilon w_\epsilon \to w$ weakly in $L^p(\Omega \times Y)$ and $\mathcal{T}_\epsilon(x,\cdot) \to w$ weakly in $W^{1,p}(Y)$ as $\epsilon \to 0$, for almost all $x \in \Omega$). Moreover if the convergence of w_ϵ to w is strong, then the convergence of $\mathcal{T}_\epsilon w_\epsilon$ to w is strong as well.*

Let us define $W^{1,p}_{per}(Y)$ to be the space of functions in $W^{1,p}_{loc}(\mathbb{R}^N)$ which are Y-periodic.

Theorem 1.2.8. *Let $p \in]1, \infty[$ and let w_ϵ be a sequence in $W^{1,p}(\Omega)$ converging weakly in $W^{1,p}(\Omega)$ to w. Then there exists w_0 in $L^p(\Omega; W^{1,p}_{per}(Y))$ and a subsequence of w_ϵ (which we still denote by w_ϵ) such that*

$$\mathcal{T}_\epsilon \nabla w_\epsilon \to \nabla w + \nabla_y w_0, \quad \text{weakly in } L^p(\Omega \times Y),$$

and

$$\frac{1}{\epsilon}(\mathcal{T}_\epsilon w_\epsilon - \mathcal{M}_\epsilon w_\epsilon) \to w_0 + y \cdot \nabla w, \quad \text{weakly in } L^p(\Omega; W^{1,p}(Y)).$$

Moreover $\mathcal{M}_Y w_0 = 0$.

When dealing with the boundary behaviour of Sobolev functions in multiscale problems it is often useful to define an anisotropic unfolding operator. Consider for example a domain $\Omega_\epsilon \subset \mathbb{R}^N$ defined by

$$\Omega_\epsilon = \{(\bar{x}, x_N) : \bar{x} \in W, -1 < x_N < \epsilon^\alpha b(x/\epsilon)\},$$

for a given smooth Y-periodic function b. Here, $Y =]-1/2, 1/2[^{N-1}$. We set the following notation. For any $k \in \mathbb{Z}^{N-1}$ and $\epsilon > 0$ we define

$$\begin{cases} C_\epsilon^k = \epsilon k + \epsilon Y, \\ I_{W,\epsilon} = \{k \in \mathbb{Z}^{N-1} : C_\epsilon^k \subset W\}, \\ \widehat{W}_\epsilon = \bigcup_{k \in I_{W,\epsilon}} C_\epsilon^k. \end{cases} \tag{1.2.3}$$

Then we give the following

Definition 1.2.9. Let u be a real-valued function defined in Ω. For any $\epsilon > 0$ sufficiently small the unfolding $\hat{u}$ of u is the real-valued function defined on $\hat{W}_\epsilon \times Y \times (-1/\epsilon, 0)$ by

$$\hat{u}(\bar{x}, \bar{y}, y_N) = u\left(\epsilon\left[\frac{\bar{x}}{\epsilon}\right] + \epsilon\bar{y}, \epsilon y_N\right),$$

for all $(\bar{x}, \bar{y}, y_N)) \in \hat{W}_\epsilon \times Y \times (-1/\epsilon, 0)$, where $\left[\frac{\bar{x}}{\epsilon}\right]$ denotes the integer part of the vector $\bar{x}\epsilon^{-1}$ with respect to Y, i.e., $[\bar{x}\epsilon^{-1}] = k$ if and only if $\bar{x} \in C_\epsilon^k$.

As a consequence of the anisotropy of the unfolding operator we end up with a weighted integration formula:

Lemma 1.2.10. *Let $a \in [-1, 0[$ be fixed. Then*

$$\int_{\widehat{W}_\epsilon \times (a,0)} u(x)dx = \epsilon \int_{\widehat{W}_\epsilon \times Y \times (a/\epsilon, 0)} \hat{u}(\bar{x}, y) d\bar{x} dy \tag{1.2.4}$$

for all $u \in L^1(\Omega)$ *and* $\epsilon > 0$ *sufficiently small. Moreover*

$$\int_{\widehat{W}_\epsilon \times (a,0)} \frac{\partial^l u(x)}{\partial x_{i_1} \cdots \partial x_{i_l}} dx = \epsilon \int_{\widehat{W}_\epsilon \times Y \times (a/\epsilon, 0)} \widehat{\frac{\partial^l u(x)}{\partial x_{i_1} \cdots \partial x_{i_l}}} d\bar{x} dy$$
$$= \epsilon^{1-l} \int_{\widehat{W}_\epsilon \times Y \times (a/\epsilon, 0)} \frac{\partial^l \hat{u}}{\partial y_{i_1} \cdots \partial y_{i_l}} (\bar{x}, y) d\bar{x} dy,$$

for all $l \leq m$*, for all* $u \in W^{m,1}(\Omega)$ *and* $\epsilon > 0$ *sufficiently small.*

1.3 Compact convergence results

In order to treat linear problems formulated on varying Banach spaces (as in Chapter 5) it is of main importance to have an abstract tool capable of relating these problems and their solutions in a suitable way. The theory of compact convergence of operators plays a central role in this case. According to [112], the very beginning of this theory goes back to Sobolev himself in the article [102]. This notion of convergence of linear operators was developed mainly by Vainikko in his articles (see e.g., [109], [110], [111], [112], [113], but see also [8], [49] and [107]) in connection with problems arising in numerical analysis, in particular in the convergence of approximation schemes. Indeed, consider for example a differential problem in the form $Tu = f$, with $u \in \mathcal{B}_1$, $f \in \mathcal{B}_2$, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are Banach spaces, and T is a linear and bounded operator from $\mathcal{B}_1$ to $\mathcal{B}_2$. A general strategy to find a solution to this problem is to consider approximating problems $T_n u_n = f_n$ cast on suitably chosen finite-dimensional subspaces $V_n \subset \mathcal{B}_1$ and to study if and how the sequence $(u_n)_n$ converges to u. The main problem here is that u_n and u (as well as f_n, f) lie in different spaces, hence it does not make sense in general to consider $\|u_n - u\|_{V_n}$. An interesting idea is then to consider a *connecting system* between $\mathcal{B}_1$ and V_n, i.e., a family of operators $(E_n)_n$ from $\mathcal{B}_1$ to V_n with the following properties

1. $\|E_n u\|_{V_n} \to \|u\|_{\mathcal{B}_1}$ as $n \to \infty$, for all $u \in \mathcal{B}_1$;
2. $\|E_n(\alpha u + \beta v) - \alpha E_n u - \beta E_n v\|_{V_n} \to 0$, as $n \to \infty$, for all $u, v \in \mathcal{B}_1$, $\alpha, \beta \in \mathbb{R}$.

It is often the case that the connecting system $(E_n)_n$ can be chosen linear and bounded, hence by the Banach-Steinhaus theorem it follows that $\|E_n\|_{\mathcal{L}(\mathcal{B}_1;V_n)} \leq C$ for all $n \in \mathbb{N}$. At this point it makes sense to consider $\|u_n - E_n u\|_{V_n}$ in order to study the convergence of u_n to u.

With this idea in mind we will now recall the main definition and results about the compact convergence on varying Hilbert spaces. Since we are interested mainly in the relation between compact convergence and spectral theory, we will stick to the case in which $\mathcal{B}_1 = \mathcal{B}_2 = \mathscr{H}_0$, with $\mathscr{H}_0$ a given separable Hilbert space. The following presentation is inspired by [15], [41].

Let $\mathscr{H}_\epsilon$, $\epsilon > 0$, be a family of separable Hilbert spaces. We assume the existence of a family of linear operators $\mathcal{E}_\epsilon \in \mathcal{L}(\mathscr{H}_0, \mathscr{H}_\epsilon)$, $\epsilon > 0$, such that

$$\|\mathcal{E}_\epsilon u_0\|_{\mathscr{H}_\epsilon} \longrightarrow \|u_0\|_{\mathscr{H}_0}, \quad \text{as } \epsilon \longrightarrow 0, \tag{1.3.1}$$

for all $u_0 \in \mathscr{H}_0$.

Definition 1.3.1. Let $\mathscr{H}_\epsilon$ and $\mathcal{E}_\epsilon$ be as above.

(i) Let $u_\epsilon \in \mathscr{H}_\epsilon$, $\epsilon > 0$. We say that u_ϵ $\mathcal{E}$-converges to u as $\epsilon \longrightarrow 0$ if $\|u_\epsilon - \mathcal{E}_\epsilon u\|_{\mathscr{H}_\epsilon} \longrightarrow 0$ as $\epsilon \longrightarrow 0$. We write $u_\epsilon \xrightarrow{E} u$.

(ii) Let $B_\epsilon \in \mathcal{L}(\mathscr{H}_\epsilon)$, $\epsilon > 0$. We say that B_ϵ $\mathcal{E}\mathcal{E}$-converges to a linear operator $B_0 \in \mathcal{L}(\mathscr{H}_0)$ if $B_\epsilon u_\epsilon \xrightarrow{E} B_0 u$ whenever $u_\epsilon \xrightarrow{E} u \in \mathscr{H}_0$. We write $B_\epsilon \xrightarrow{EE} B_0$.

(iii) Let $B_\epsilon \in \mathcal{L}(\mathscr{H}_\epsilon)$, $\epsilon > 0$. We say that B_ϵ compactly converges to $B_0 \in \mathcal{L}(\mathscr{H}_0)$ (and we write $B_\epsilon \xrightarrow{C} B_0$) if the following two conditions are satisfied:

   (a) $B_\epsilon \xrightarrow{EE} B_0$ as $\epsilon \longrightarrow 0$;
   (b) for any family $u_\epsilon \in \mathscr{H}_\epsilon$, $\epsilon > 0$, such that $\|u_\epsilon\|_{\mathscr{H}_\epsilon} = 1$ for all $\epsilon > 0$, there exists $\epsilon_k \longrightarrow 0$ as $k \longrightarrow \infty$, a subsequence $B_{\epsilon_k} u_{\epsilon_k}$ of $B_\epsilon u_\epsilon$ and $\bar{u} \in \mathscr{H}_0$ such that $B_{\epsilon_k} u_{\epsilon_k} \xrightarrow{E} \bar{u}$ as $k \longrightarrow \infty$.

Lemma 1.3.2. *Assume that a sequence of operators $B_\epsilon \in \mathcal{L}(\mathscr{H}_\epsilon)$ converges compactly to B_0 as $\epsilon \longrightarrow 0$. Then,*

1. *$\|B_\epsilon\|_{\mathcal{L}(\mathscr{H}_\epsilon)} \le C$ for some constant C not depending on ϵ.*
2. *If the kernel of $\mathbb{I} + B_0$ is $\{0\}$ then $\|(\mathbb{I} + B_\epsilon)^{-1}\|_{\mathcal{L}(\mathscr{H}_\epsilon)} \le M$ for small enough ϵ.*

Proof. See [15, Lemma 4.7]. □

For any $\epsilon \ge 0$, let A_ϵ be a (densely defined) closed, nonnegative differential operator on $\mathscr{H}_\epsilon$ with domain $\mathscr{D}(A_\epsilon) \subset \mathscr{H}_\epsilon$. We assume for simplicity that 0 does not belong to the spectrum of A_ϵ and that

$$\text{(H1): } A_\epsilon \text{ has compact resolvent } B_\epsilon := A_\epsilon^{-1} \text{ for any } \epsilon \in [0, 1),$$

and

$$\text{(H2): } B_\epsilon \xrightarrow{C} B_0, \text{ as } \epsilon \longrightarrow 0.$$

Let us also denote by $\sigma(A_\epsilon)$ the spectrum of the operator A_ϵ, and by $\rho(A_\epsilon)$ the resolvent set of A_ϵ, i.e., $\rho(A_\epsilon) = \mathbb{R} \setminus \sigma(A_\epsilon)$.

Lemma 1.3.3. *Let A_ϵ be as defined above and satisfying hypothesis $(H1)$ and $(H2)$. Then for any $\lambda \in \rho(A_0)$ there is an $\epsilon_\lambda > 0$ such that $\lambda \in \rho(A_\epsilon)$ for all $\epsilon \in [0, \epsilon_\lambda]$ and there is a constant $M_\lambda > 0$ such that*

$$\|(\lambda \mathbb{I} - A_\epsilon)^{-1}\|_{\mathcal{L}(\mathscr{H}_\epsilon)} \leq M,$$

for all $\epsilon \in [0, \epsilon_\lambda]$. Moreover, $(\lambda \mathbb{I} - A_\epsilon)^{-1}$ converges compactly to $(\lambda \mathbb{I} - A_0)^{-1}$ as $\epsilon \to 0$.

Proof. See [15, Lemma 4.8]. □

Lemma 1.3.4. *Let A_ϵ be as defined above and satisfying hypothesis $(H1)$ and $(H2)$. Let λ, δ be real numbers such that $S_\delta := \{\mu \in \mathbb{C} : |\mu - \lambda| = \delta\}$ satisfies $\sigma(A_0) \cap S_\delta = \emptyset$ then there exists $\epsilon_0 > 0$ depending only on S_δ such that $\sigma(A_\epsilon) \cap S_\delta = \emptyset$ for all $\epsilon \leq \epsilon_0$.*

Proof. See [15, Lemma 4.9]. □

Given an eigenvalue λ of A_0 we consider the generalized eigenspace $S(\lambda, A_0) := Q(\lambda, A_0)\mathscr{H}_0$, where Q is defined by

$$Q(\lambda, A_0) = \frac{1}{2\pi i} \int_{|\xi - \lambda| = \delta} (\xi \mathbb{I} - A_0)^{-1} \, d\xi$$

and $\delta > 0$ is such that the disk $\{\xi \in \mathbb{C} : |\xi - \lambda| \leq \delta\}$ does not contain any eigenvalue except for λ. In a similar way, if (H1),(H2) hold, then we can define

$$S(\lambda, A_\epsilon) := Q(\lambda, A_\epsilon) = \frac{1}{2\pi i} \int_{|\xi - \lambda| = \delta} (\xi \mathbb{I} - A_\epsilon)^{-1} \, d\xi.$$

This definition makes sense because of Lemma 1.3.4. Then the following theorem holds.

Theorem 1.3.5. *Let A_ϵ, A_0 be operators as above satisfying conditions (H1), (H2). Then the operators A_ϵ* are spectrally convergent *to A_0 as $\epsilon \to 0$, i.e., the following statements hold:*

(i) If λ_0 is an eigenvalue of A_0, then there exists a sequence $\epsilon_n \to 0$ and eigenvalues λ_n of A_{ϵ_n}, $n \in \mathbb{N}$, such that $\lambda_n \to \lambda_0$ as $n \to \infty$. Conversely, if for some sequence $\epsilon_n \to 0$, λ_n is an eigenvalue of A_{ϵ_n} for all $n \in \mathbb{N}$, and $\lambda_n \to \lambda_0$ as $n \to \infty$, then λ_0 is an eigenvalue of A_0.

(ii) There exists $\epsilon_0 > 0$ such that the dimension of the generalized eigenspace $S(\lambda_0, A_\epsilon)$ equals the dimension of $S(\lambda_0, A_0)$, for any eigenvalue λ_0 of A_0, for any $\epsilon \in [0, \epsilon_0)$.

(iii) *If* $\varphi_0 \in S(\lambda_0, A_0)$ *then for any* $\epsilon > 0$ *there exists* $\varphi_\epsilon \in S(\lambda_0, A_\epsilon)$ *such that* $\varphi_\epsilon \xrightarrow{E} \varphi_0$ *as* $\epsilon \to 0$.

(iv) *If* $\varphi_\epsilon \in S(\lambda_0, A_\epsilon)$ *satisfies* $\|\varphi_\epsilon\|_{\mathcal{H}_\epsilon} = 1$ *for all* $\epsilon > 0$, *then* φ_ϵ, $\epsilon > 0$, *has an* $\mathcal{E}$*-convergent subsequence whose limit is in* $S(\lambda_0, A_0)$.

Proof. See [15, Theorem 4.10]. □

1.4 Elements of tangential calculus

We recall here some basic definitions and results about the tangential calculus on the boundary of a regular open set of $\mathbb{R}^N$, which we shall use in particular in chapter 4. We refer to [64, Chapter 9] for details and further information.
Given $A \subset \mathbb{R}^N$ let d_A be the Euclidean distance function from A, defined by $d_A(x) = \inf_{y\in A}|x-y|$. We define the oriented distance function b_A from A by

$$b_A(x) = d_A(x) - d_{A^C}(x),$$

for all $x \in \mathbb{R}^N$. Here and in the sequel we denote with A^C the complementary of A in $\mathbb{R}^N$. Let Ω be an bounded open set of class C^2. In this case b_Ω coincides with the distance from $\partial\Omega$, with the convention that b_Ω is positive if x is in the interior of the Ω^C, and b_Ω is negative if x is in the interior of Ω.
Since Ω is of class C^2, it is well-known that there exists $h > 0$ and a tubular neighbourhood $S_{2h}(\partial\Omega)$ of radius h such that $b_\Omega \in C^2(S_{2h}(\partial\Omega))$.
We define the projection of a point x to $\partial\Omega$ by

$$p(x) = x - b_\Omega(x)\nabla b_\Omega(x), \tag{1.4.1}$$

for all $x \in S_{2h}(\partial\Omega)$. If $f \in C^0(\partial\Omega)$ we write

$$(f)_{\partial\Omega} = (f \circ p)|_{\partial\Omega}.$$

We define also the orthogonal projection operator P_Ω onto the tangent plane $T_{p(x)}\partial\Omega$ by

$$P_\Omega(x)[V] = (\mathbb{I} - \nabla b_\Omega(x) \otimes \nabla b_\Omega(x))V,$$

for all $V \in \mathbb{R}^N$. Note that P_Ω is the identity transformation on $T_{p(x)}\partial\Omega$. Indeed, it is possible to prove (see [64, §5 Chapter 9]) that $P_\Omega(x)$ coincides with the first fundamental form of $\partial\Omega$. Moreover it is immediate to see that $Dp|_{\partial\Omega} = P_\Omega$.
Now note that D^2b_Ω can be regarded as a linear transformation of $T_{p(x)}$ onto itself, because $(D^2b_\Omega(x))(n(x)) = D^2b_\Omega(x)\nabla b_\Omega(x) = 0$ (this is a consequence of the fact that $|\nabla b(x)| = 1$, for all $x \in S_{2h}(\partial\Omega)$). It is possible to prove that D^2b_Ω coincides with the second fundamental form associated with $\partial\Omega$ (see e.g., [64, §5.6 Chp.9] and Theorem 1.4.4 below).
We recall the following definition.

Definition 1.4.1. Let Ω be an bounded open set of class C^2 and let $h > 0$ be such that $b_\Omega \in C^2(S_{2h}(\partial\Omega))$. Let $f \in C^1(\partial\Omega)$ and let $F \in C^1(S_{2h}(\partial\Omega))$ be a C^1 extension of f to $S_{2h}(\partial\Omega)$ (that is, $F|_{\partial\Omega} = f$). We define the tangential gradient of f on $\partial\Omega$ by

$$\nabla_{\partial\Omega} f = \nabla F|_{\partial\Omega} - \frac{\partial F}{\partial n} n.$$

It is possible to prove (see [64, Theorem 5.1]) that this definition does not depend on the choice of the C^1 extension F of f. In particular, in this setting $\nabla_{\partial\Omega} f$ can be equivalently defined by

$$\nabla_{\partial\Omega} f = \nabla(f \circ p)|_{\partial\Omega} = (\nabla f)_{\partial\Omega},$$

where p is the projection on $\partial\Omega$ defined by (1.4.1). We can now define in a similar way other differential operators acting on the tangent space $T\partial\Omega$. We shall do this in the following.

Definition 1.4.2. Let $N \geq 1$, $v \in C^1(\partial\Omega)^N$. We define the tangential Jacobian matrix of v by

$$D_{\partial\Omega} v = D(v \circ p)|_{\partial\Omega},$$

and the tangential divergence of v by

$$\mathrm{div}_{\partial\Omega}(v \circ p)|_{\partial\Omega} = \mathrm{tr}(D_{\partial\Omega} v).$$

Assume now Ω is of class C^3 and $f \in C^2(\partial\Omega)$. We define the Laplace-Beltrami operator of f by

$$\Delta_{\partial\Omega} f = \Delta(f \circ p)|_{\partial\Omega} = \mathrm{div}_{\partial\Omega}(\nabla_{\partial\Omega} f),$$

and similarly we define the tangential Hessian matrix by

$$D^2_{\partial\Omega} f = D_{\partial\Omega}(\nabla_{\partial\Omega} f).$$

Remark 1.4.3. Let Ω be an bounded open set of class C^2 and let $h > 0$ be such that $b_\Omega \in C^2(S_{2h}(\partial\Omega))$. Let $\varphi \in C^1(S_{2h}(\partial\Omega))^N$. We can alternatively define the tangential Jacobian $D\varphi$ by

$$D\varphi|_{\partial\Omega} = D_{\partial\Omega}\varphi + (D\varphi n) \otimes n.$$

Then we have the following

Theorem 1.4.4. *Let Ω be of class C^2. Let $II_{\partial\Omega}$ be the second fundamental form associated with $\partial\Omega$. Then*

$$D_{\partial\Omega} n(x) = D^2 b_\Omega(x) = II_{\partial\Omega}(x),$$

for all $x \in \partial\Omega$.

Proof. We refer to [64, p.496]. □

As a consequence of Theorem (1.4.4), the curvature $\mathcal{H}$ of $\partial\Omega$, defined as the sum of the $N-1$ eigenvalues of the matrix $II_{\partial\Omega}$ is given by the formula

$$\mathrm{tr}\Big(D_{\partial\Omega} n\Big) = \mathcal{H} = \mathrm{tr}\Big(II_{\partial\Omega}\Big) = \Delta b|_{\partial\Omega}, \tag{1.4.2}$$

where $b := b_\Omega$ is the oriented distance function.

Remark 1.4.5. Note carefully that $D^2_{\partial\Omega} f$ does not coincide with $D^2(f \circ p)|_{\partial\Omega}$. Indeed, it is possible to prove that

$$D^2(f \circ p)|_{\partial\Omega} = D^2_{\partial\Omega} f - (D_{\partial\Omega} n \nabla_{\partial\Omega} f) \otimes n.$$

In particular, the projection of the Hessian matrix on a boundary with non-trivial curvature differs from the tangential Hessian by a lower-order factor, a fact which is well-known in Differential Geometry. In §4.1 we will show a possible block decomposition of the Hessian matrix of a function defined on $\partial\Omega$, see formula (4.1.13).

We conclude this section recalling the following important

Theorem 1.4.6 (Tangential Divergence Theorem)**.** *Let Ω be a bounded open set of class C^2 and let $v \in C^1(\partial\Omega)^N$. Then*

$$\int_{\partial\Omega} \mathrm{div}_{\partial\Omega} v \, dS = \int_{\partial\Omega} \mathcal{H}(v \cdot n)\, dS. \tag{1.4.3}$$

Let $f \in C^1(\partial\Omega)$. Then

$$\int_{\partial\Omega} (f \,\mathrm{div}_{\partial\Omega} v + \nabla_{\partial\Omega} f \cdot v)\, dS = \int_{\partial\Omega} \mathcal{H} f (v \cdot n)\, dS. \tag{1.4.4}$$

Proof. We refer to [64, §5.5 Chapter 9]. □

Formula (1.4.3) is the *tangential divergence formula*, which is a consequence of the standard Divergence Theorem. Identity (1.4.4) is sometimes called *tangential Green's formula*. We remark that when $v \in C^1(\partial\Omega)^N$ is a tangential vector field (i.e., $v \cdot n = 0$ on $\partial\Omega$), then by (1.4.4) we deduce that

$$\int_{\partial\Omega} f \,\mathrm{div}_{\partial\Omega} v \, dS = -\int_{\partial\Omega} \nabla_{\partial\Omega} f \cdot v \, dS.$$

Chapter 2

Higher order elliptic operators and their spectral convergence

The purpose of this chapter is the analysis of the eigenvalues and eigenfunctions of higher order elliptic operators defined on domains subject to boundary perturbations. We consider eigenvalue problems for such operators in variational form, by using the correspondence between non-negative self-adjoint operators and closed quadratic forms (see Theorem 1.1.6). As a matter of fact we shall identify the boundary conditions of the differential problem under consideration from the domain of the associated quadratic form (sometimes called 'energy space'). In Section 2.1 we describe this variational setting and we give some examples. Then, starting from Section 2.2, we consider higher order elliptic operators on domains subject to boundary perturbations. The focus is on the variation of the eigenvalues when the boundary conditions are of intermediate type. We shall prove a general lemma which relates the geometric parameters of the boundary perturbation to the spectral stability of the differential operators, see Lemma 2.2.2, by improving a former result presented in [19].

2.1 Variational formulation

Let M be the number of multiindices $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ with length $|\alpha| = \alpha_1 + \cdots + \alpha_N = m$. For all $\alpha, \beta \in \mathbb{N}^N$ such that $|\alpha| = |\beta| = m$, let $A_{\alpha\beta}$ be bounded measurable real-valued functions defined on $\mathbb{R}^N$ satisfying $A_{\alpha\beta} = A_{\beta\alpha}$ and the condition

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq 0, \tag{2.1.1}$$

for all $x \in \mathbb{R}^N$, $(\xi_\alpha)_{|\alpha|=m} \in \mathbb{R}^M$. For all open sets Ω of $\mathbb{R}^N$ we define

$$Q_\Omega(u,v) = \sum_{|\alpha|=|\beta|=m} \int_\Omega A_{\alpha\beta} D^\alpha u D^\beta v \, dx + \int_\Omega uv \, dx, \tag{2.1.2}$$

for all $u, v \in H^m(\Omega)$ and we set $Q_\Omega(u) = Q_\Omega(u,u)$. Note that by (2.1.1) Q_Ω is a positive quadratic form, densely defined in the Hilbert space $L^2(\Omega)$. Hence, $Q_\Omega(\cdot,\cdot)$ defines a scalar product in $H^m(\Omega)$.

Let $V(\Omega)$ be a linear subspace of $H^m(\Omega)$ containing $H^m_0(\Omega)$. By Theorem 1.1.6 we know that if the quadratic form Q_Ω is closed (equivalently, $V(\Omega)$ is complete with respect to $Q_\Omega^{1/2}$) then there exists a uniquely determined non-negative self-adjoint operator $H_{V(\Omega)}$ such that $\mathscr{D}(H^{1/2}_{V(\Omega)}) = V(\Omega)$ and

$$Q_\Omega(u,v) = \left(H^{1/2}_{V(\Omega)} u, H^{1/2}_{V(\Omega)} v\right)_{L^2(\Omega)}, \qquad \text{for all } u, v \in V(\Omega). \tag{2.1.3}$$

By Lemma 1.1.5 it is clear that the domain $\mathscr{D}(H_{V(\Omega)})$ of $H_{V(\Omega)}$ is the subset of $H^m(\Omega)$ containing all the functions $u \in V(\Omega)$ for which there exists $f \in L^2(\Omega)$ such that

$$Q_\Omega(u,v) = (f,v)_{L^2(\Omega)}, \quad \text{for all } v \in V(\Omega). \tag{2.1.4}$$

Moreover, $H_{V(\Omega)} u = f$. If u is a smooth function satisfying identity (2.1.4) and the coefficients $A_{\alpha\beta}$ are smooth, by integration by parts it is immediate to verify that (2.1.4) is the weak formulation of problem

$$Lu = f, \quad \text{in } \Omega,$$

where L is the operator defined by

$$Lu = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha (A_{\alpha\beta} D^\beta u) + u, \tag{2.1.5}$$

where the unknown u is (at least) $C^{2m}(\overline{\Omega})$ and it is subject to suitable boundary conditions depending on the choice of $V(\Omega)$. We shall provide some examples below.

By Theorem 1.1.7, if the embedding $V(\Omega) \subset L^2(\Omega)$ is compact, then the operator $H_{V(\Omega)}$ has compact resolvent. By Theorem 1.1.1, having compact resolvent is equivalent to having discrete spectrum, consisting of a sequence of isolated eigenvalues $\lambda_n[V(\Omega)]$ of finite multiplicity diverging to $+\infty$. By Theorem 1.1.9 the eigenvalues $\lambda_n[V(\Omega)]$ are determined by the following Min-Max principle:

$$\lambda_n[V(\Omega)] = \inf_{\substack{E \subset V(\Omega) \\ \dim E = n}} \sup_{\substack{u \in E \\ u \neq 0}} \frac{Q_\Omega(u)}{\|u\|^2_{L^2(\Omega)}},$$

for all $n \geq 1$. Furthermore, by Theorem 1.1.1 there exists an orthonormal basis in $L^2(\Omega)$ of eigenfunctions $\varphi_n[V(\Omega)]$ associated with the eigenvalues $\lambda_n[V(\Omega)]$.
We remark that since the coefficients $A_{\alpha\beta}$ are fixed and bounded, and $(V(\Omega), Q_\Omega^{1/2})$ is a complete Hilbert space, an application of the Open Mapping Theorem yields the existence of two positive constants $c, C \in \mathbb{R}$ independent of u such that

$$c\|u\|_{H^m(\Omega)} \leq Q_\Omega^{1/2}(u) \leq C\|u\|_{H^m(\Omega)}.$$

In other words, the two norms $Q_\Omega^{1/2}$ and $\|\|_{H^m(\Omega)}$ are equivalent on $V(\Omega)$. Note that in general the constant c may depend on Ω. However, if the coefficients $A_{\alpha\beta}$ satisfy the uniform ellipticity condition

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq \theta \sum_{|\alpha|=m} |\xi_\alpha|^2, \tag{2.1.6}$$

for all $x \in \mathbb{R}^N$, $(\xi_\alpha)_{|\alpha|=m} \in \mathbb{R}^M$ and for some $\theta > 0$, then c can be chosen independent of Ω.

Example: the biharmonic operator.
Let $m = 2$ and $A_{\alpha\beta} = \delta_{\alpha\beta}2/\alpha!$ for all $\alpha, \beta \in \mathbb{N}^N$ with $|\alpha| = |\beta| = 2$, where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$, and $\delta_{\alpha\beta} = 0$ otherwise. With this choice, condition (2.1.6) is satisfied. The quadratic form (2.1.2) in this case is given by

$$Q_\Omega(u, v) = \int_\Omega D^2u : D^2v\, dx + \int_\Omega uv\, dx, \tag{2.1.7}$$

for all $u, v \in V(\Omega) \subset H^2(\Omega)$. Here and in the sequel D^2u denotes the Hessian matrix of u and recall that we use the notation $D^2u : D^2v$ to denote the Frobenius product $\sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}$ of the two Hessian matrices.
Let $f \in L^2(\Omega)$ and consider

$$\int_\Omega D^2u : D^2v\, dx + \int_\Omega uv\, dx = \int_\Omega fv\, dx, \qquad \text{for all } v \in V(\Omega), \tag{2.1.8}$$

in the unknown $u \in V(\Omega)$. Let $k \in \{0, 1, 2\}$ and let $V(\Omega) = H^2(\Omega) \cap H_0^k(\Omega)$ (with the convention that $H_0^0(\Omega) = L^2(\Omega)$). If $V(\Omega) = H_0^2(\Omega)$ and Ω has finite Lebesgue measure, then the embedding $H_0^2(\Omega) \subset L^2(\Omega)$ is compact. If $V(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$ then, under very weak regularity assumptions on Ω (for example, Ω has a quasi-resolved boundary in the sense of Burenkov [35, §4.3]), $V(\Omega)$ is a closed subspace of $H^m(\Omega)$. If in addition Ω has finite Lebesgue measure then the embedding $V(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Finally, if $V(\Omega) = H^2(\Omega)$

and the open set Ω is bounded and of class C^0 (in the sense of Burenkov, [35]) then the embedding $V(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Then the self-adjoint operator $H^{1/2}_{V(\Omega)}$ associated with the quadratic form (2.1.7) has compact resolvent (see (2.1.3)).
The classical formulation of problem (2.1.8) depends on the choice of $V(\Omega)$. Assume here an in the sequel that Ω and f are regular enough to assure that the solution u is regular (say, of class $C^4(\overline{\Omega})$).
When $V(\Omega) = H^2_0(\Omega)$, a double integration by parts in the first integral in the left-hand side of (2.1.8) gives

$$\int_\Omega D^2u : D^2v\,dx = \int_\Omega \Delta^2 uv\,dx,$$

for all $v \in H^2_0(\Omega)$, hence we deduce that the classical formulation of (2.1.8) is given by

$$\begin{cases} \Delta^2 u + u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1.9}$$

This is the classical Dirichlet problem for the biharmonic operator. If $N = 2$ problem (2.1.9) is one of the existing models for the study of the bending of a clamped plate. We remark that by integration by parts it is easy to see that

$$\int_\Omega D^2u : D^2v\,dx = \int_\Omega \Delta u \Delta v\,dx,$$

for all $u, v \in H^2_0(\Omega)$. In particular, the norm defined by $(\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)})^{1/2}$ is equivalent to $\|u\|_{H^2(\Omega)}$ on $H^2_0(\Omega)$.
When $V(\Omega) = H^2(\Omega) \cap H^1_0(\Omega)$, by integation by parts we deduce that

$$\int_\Omega D^2u : D^2v\,dx = \int_\Omega \Delta^2 uv\,dx + \int_{\partial\Omega} \nabla\left(\frac{\partial u}{\partial n}\right) \cdot \nabla v\,dS,$$

for all $v \in H^2(\Omega) \cap H^1_0(\Omega)$. Since $v = 0$ on the boundary of Ω, $\nabla_{\partial\Omega} v = 0$ on $\partial\Omega$. Then, $\nabla v|_{\partial\Omega} = \frac{\partial v}{\partial n} n$ from which we deduce that

$$\int_\Omega D^2u : D^2v\,dx = \int_\Omega \Delta^2 uv\,dx + \int_{\partial\Omega} \frac{\partial^2 u}{\partial n^2}\frac{\partial v}{\partial n}\,dS,$$

for all $v \in H^2(\Omega) \cap H^1_0(\Omega)$. We refer to §1.4 for the definitions of the tangential differential operators. We deduce that the classical formulation of (2.1.8) is given by

$$\begin{cases} \Delta^2 u + u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \frac{\partial^2 u}{\partial n^2} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1.10}$$

This is the classical intermediate problem for the biharmonic operator. If $N = 2$ Problem (2.1.10) is used as a model for the study of the bending of hinged plates. We remark that the second boundary condition in (2.1.10) can be written as

$$\Delta u - \mathcal{H}\frac{\partial u}{\partial n} = 0,$$

where we recall that $\mathcal{H}$ is the curvature of the boundary, i.e., the sum of the principal curvatures.

Assume Ω of class C^2. Recall now the following 'Biharmonic Green Formula' (see [19, Lemma 8.56] and §4.1),

$$\int_\Omega \Delta^2 u\, v dx = \int_\Omega D^2 u : D^2 v\, dx \\ - \int_{\partial\Omega} \frac{\partial^2 u}{\partial n^2}\frac{\partial v}{\partial n}\, dS + \int_{\partial\Omega}\Big(\operatorname{div}_{\partial\Omega}((D^2 u \cdot n))_{\partial\Omega} + \frac{\partial \Delta u}{\partial n}\Big) v\, dS, \quad (2.1.11)$$

for all $v \in H^2(\Omega)$, where $u \in C^4(\overline{\Omega})$. Let $V(\Omega) = H^2(\Omega)$ in (2.1.7). By using formula (2.1.11) on the first integral in the left-hand side of (2.1.8) we deduce that

$$\int_\Omega \Delta^2 u\, v dx + \int_\Omega uv dx \\ + \int_{\partial\Omega} \frac{\partial^2 u}{\partial n^2}\frac{\partial v}{\partial n} dS - \int_{\partial\Omega}\Big(\operatorname{div}_{\partial\Omega}((D^2 u \cdot n))_{\partial\Omega} + \frac{\partial \Delta u}{\partial n}\Big) v dS = \int_\Omega f v dx,$$

for all $v \in H^2(\Omega)$, from which we deduce the classical Neumann problem for the biharmonic operator

$$\begin{cases} \Delta^2 u + u = f, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial n^2} = 0, & \text{on } \partial\Omega, \\ \operatorname{div}_{\partial\Omega}(D^2 u \cdot n)_{\partial\Omega} + \frac{\partial \Delta u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \quad (2.1.12)$$

where $\operatorname{div}_{\partial\Omega}$ is the tangential divergence operator defined in §1.4. We recall that if $N = 2$ the Neumann problem for the biharmonic operator is studied, for example, in connection with the study of the bending of free plates. In fact, problem (2.1.12) is a particular case of the family of problems (parametrized by $\sigma \in (-1, 1)$ and $\tau \geq 0$) defined by

$$\begin{cases} \Delta^2 u - \tau\Delta u + u = f, & \text{in } \Omega, \\ (1-\sigma)\frac{\partial^2 u}{\partial n^2} + \sigma\Delta u = 0, & \text{on } \partial\Omega, \\ \tau\frac{\partial u}{\partial n} - (1-\sigma)\ \operatorname{div}_{\partial\Omega}(D^2 u \cdot n)_{\partial\Omega} - \frac{\partial(\Delta u)}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

whose associated quadratic form is

$$\int_\Omega (1-\sigma)D^2u : D^2v + \sigma\Delta u\Delta v + \tau\nabla u\cdot\nabla v + uv\,dx = \int_\Omega fv\,dx,$$

for all $u, v \in H^2(\Omega)$. We will consider this differential problem in Chapter 5. Note that when $\sigma = 0$ and $\tau = 0$ we find again problem (2.1.12).

2.2 Higher order operators on domains with perturbed boundaries

Let $m \in \mathbb{N}$, $m \geq 2$ and let $\epsilon > 0$. Let $V(\Omega), V(\Omega_\epsilon)$ be subspaces of $H^m(\Omega)$ (respectively, $H^m(\Omega_\epsilon)$) containing $H^m_0(\Omega)$ (respectively, $H^m_0(\Omega_\epsilon)$). Moreover, let $H_{V(\Omega)}$, $H_{V(\Omega_\epsilon)}$, Q_Ω, Q_{Ω_ϵ} be as in (2.1.3). Then we recall the following

Definition 2.2.1. (see [19, Definition 3.1]). Given open sets Ω_ϵ, $\epsilon > 0$ and $\Omega \in \mathbb{R}^N$ with corresponding elliptic operators $H_{V(\Omega_\epsilon)}$, $H_{V(\Omega)}$ defined on Ω_ϵ, Ω respectively, we say that condition (C) is satisfied if there exists open sets $K_\epsilon \subset \Omega \cap \Omega_\epsilon$ such that

$$\lim_{\epsilon\to 0} |\Omega \setminus K_\epsilon| = 0, \tag{2.2.1}$$

and the following conditions are satisfied:
(C1) If $v_\epsilon \in V(\Omega_\epsilon)$ and $\sup_{\epsilon>0} Q_{\Omega_\epsilon}(v_\epsilon) < \infty$ then

$$\lim_{\epsilon\to 0} \|v_\epsilon\|_{L^2(\Omega_\epsilon\setminus K_\epsilon)} = 0; \tag{2.2.2}$$

(C2) For each $\epsilon > 0$ there exists an operator T_ϵ from $V(\Omega)$ to $V(\Omega_\epsilon)$ such that for all fixed $\varphi \in V(\Omega)$

(i) $\lim_{\epsilon\to 0} Q_{K_\epsilon}(T_\epsilon\varphi - \varphi) = 0$;

(ii) $\lim_{\epsilon\to 0} Q_{\Omega_\epsilon\setminus K_\epsilon}(T_\epsilon\varphi) = 0$;

(iii) $\lim_{\epsilon\to 0} \|T_\epsilon\varphi\|_{L^2(\Omega_\epsilon)} < \infty$.

(C3) For each $\epsilon > 0$ there exists an operator E_ϵ from $V(\Omega_\epsilon)$ to $H^m(\Omega)$ such that the set $E_\epsilon(V(\Omega_\epsilon))$ is compactly embedded in $L^2(\Omega)$ and such that

(i) If $v_\epsilon \in V(\Omega_\epsilon)$ is a sequence of functions such that $\sup_{\epsilon>0} Q_{V(\Omega_\epsilon)}(v_\epsilon) < \infty$, then $\lim_{\epsilon\to 0} Q_{K_\epsilon}(E_\epsilon v_\epsilon - v_\epsilon) = 0$;

(ii)

$$\sup_{\epsilon>0} \sup_{v\in V(\Omega_\epsilon)\setminus\{0\}} \frac{\|E_\epsilon v\|_{H^m(\Omega)}}{Q^{1/2}_{\Omega_\epsilon}(v)} < \infty;$$

(iii) If $v_\epsilon \in V(\Omega_\epsilon)$ is such that $\sup_{\epsilon>0} Q_{\Omega_\epsilon}(v_\epsilon) < \infty$ and there exists $v \in L^2(\Omega)$ such that, possibly passing to a subsequence, we have $\|E_\epsilon v_\epsilon - v\|_{L^2(\Omega)} \to 0$, then $v \in V(\Omega)$.

We recall moreover that the Condition (C) implies the compact convergence of the resolvent operators, hence the spectral convergence, see [19, Theorem 3.5].

2.2.1 A spectral convergence Lemma

Let $m \in \mathbb{N}$, $m \geq 2$, Ω, Ω_ϵ be bounded open connected open sets in $\mathbb{R}^N$ of class C^m, $g, g_\epsilon \in C^m(\overline{W})$, where $W \subset \mathbb{R}^{N-1}$ is an open, connected and bounded set of class C^m. We suppose that there exists a cuboid V of the form $W \times (a, b)$ such that

$$\Omega \cap V = \{(\bar{x}, x_N) \in W \times (a, b) : a < x_N < g(\bar{x})\} \tag{2.2.3}$$

and

$$\Omega_\epsilon \cap V = \{(\bar{x}, x_N) \in W \times (a, b) : a < x_N < g_\epsilon(\bar{x})\}. \tag{2.2.4}$$

We assume that the perturbation of the boundary is localized inside V, that is $\Omega \setminus V = \Omega_\epsilon \setminus V$. Let us consider quadratic forms on Ω, Ω_ϵ defined as in (2.1.2), where the coefficients $A_{\alpha\beta}$ satisfy the uniform ellipticity condition (2.1.6). Let us consider non-negative self-adjoint operators $H_{V(\Omega)}$ defined by (2.1.3) on the domain $V(\Omega) = W^{m,2}(\Omega) \cap W_0^{k,2}(\Omega)$ for some $1 \leq k < m$. Since Ω is of class C^m, $V(\Omega)$ is compactly embedded in $L^2(\Omega)$; hence, $H_{V(\Omega)}$ has compact resolvent.

We now state the main result of this chapter. Roughly speaking, we provide a criterion to verify whether the sequence of higher order operators $(H_{V(\Omega_\epsilon)})_{\epsilon>0}$ spectrally converges to $H_{V(\Omega)}$ as $\epsilon \to 0$ when $V(\Omega) = W^{m,2}(\Omega) \cap W_0^{k,2}(\Omega)$ for $k \in \{1, \ldots, m-1\}$. Such a criterion depends only on the convergence of the sequence of profile functions $(g_\epsilon)_{\epsilon>0}$ to g. The main point is that we do not require that g_ϵ converges to g in C^k, for any $k \geq 1$ (instead, the convergence in L^∞ is necessary). We note that this lemma is a generalization of [19, Lemma 6.2]. We refer to §1.3 for the definition and the properties of the compact convergence.

Lemma 2.2.2. *Suppose that* $V(\Omega) = W^{m,2}(\Omega) \cap W_0^{k,2}(\Omega)$ *for some* $1 \leq k < m$. *If for all* $\epsilon > 0$ *there exists* $\kappa_\epsilon > 0$ *such that*

(i) $\kappa_\epsilon > \|g_\epsilon - g\|_\infty, \quad \forall \epsilon > 0,$

(ii) $\lim_{\epsilon\to 0} \kappa_\epsilon = 0,$

(iii) $\lim_{\epsilon\to 0} \frac{\|D^\beta(g_\epsilon - g)\|_\infty}{\kappa_\epsilon^{m-|\beta|-k+1/2}} = 0,\ \forall \beta \in \mathbb{N}^N$ *with* $\beta \leq m$,

then $H^{-1}_{V(\Omega_\epsilon)} \xrightarrow{C} H^{-1}_{V(\Omega)}$.

Proof. The case $k = 1$ is proved in [19, Lemma 6.2]. Then, we suppose $k > 1$. Part of the proof is similar to the one of [19, Lemma 6.2]. However, for the sake of completeness we recall the basic notation and arguments. The idea of the proof is to construct suitable operators T_ϵ, E_ϵ such that Condition (C) is satisfied. It is possible to assume directly

$$\Omega = \{(\bar{x}, x_N) \in W \times (a, b) : a < x_N < g(\bar{x})\}$$

and

$$\Omega_\epsilon = \{(\bar{x}, x_N) \in W \times (a, b) : a < x_N < g_\epsilon(\bar{x})\}.$$

Define $k_\epsilon = M\kappa_\epsilon$ for a suitable constant $M > 2m$. Let $\tilde{g}_\epsilon = g_\epsilon - k_\epsilon$ and

$$K_\epsilon = \{(\bar{x}, x_N) \in W \times]a, b[: a < x_N < \tilde{g}_\epsilon(\bar{x})\}.$$

Note that with this definition of K_ϵ (2.2.1) is clearly satisfied. By the standard one dimensional estimate

$$\|f\|_{L^\infty(a,b)} \leq C\|f\|_{H^1(a,b)}, \tag{2.2.5}$$

and Tonelli Theorem it is easy to show that condition (C1) is satisfied.
We now construct a suitable family of diffeomorphisms mapping $\overline{\Omega}_\epsilon \to \overline{\Omega}$. Namely, we define $\Phi_\epsilon : \overline{\Omega}_\epsilon \to \overline{\Omega}$ by

$$\Phi_\epsilon(\bar{x}, x_N) = (\bar{x}, x_N - h_\epsilon(\bar{x}, x_N)),$$

for all $(\bar{x}, x_N) \in \overline{\Omega}_\epsilon$, where

$$h_\epsilon(\bar{x}, x_N) = \begin{cases} 0, & \text{if } a \leq x_N \leq \tilde{g}_\epsilon(\bar{x}), \\ (g_\epsilon(\bar{x}) - g(\bar{x}))\left(\frac{x_N - \tilde{g}_\epsilon(\bar{x})}{g_\epsilon(\bar{x}) - \tilde{g}_\epsilon(\bar{x})}\right)^{m+1} & \text{if } \tilde{g}_\epsilon(\bar{x}) < x_N \leq g_\epsilon(\bar{x}). \end{cases}$$

Then define T_ϵ to be the map from $V(\Omega)$ to $V(\Omega_\epsilon)$ defined by

$$T_\epsilon\varphi = \varphi \circ \Phi_\epsilon, \tag{2.2.6}$$

for all $\varphi \in V(\Omega)$. It is trivial to show that T_ϵ is well-defined and that condition (C2)(i) is satisfied. We now want to prove that condition (C2)(ii) is satisfied. We need to estimate the derivatives of $\varphi \circ \Phi_\epsilon$. Here we can improve the estimate given in [19, Lemma 6.2] by taking advantage of the decay of $D^\gamma\varphi$ in a neighbourhood of $\partial\Omega$, for $|\gamma| \leq k - 1$. We divide the proof in two steps.

Step 1. We aim to prove a decay inequality for the L^2-norm of the derivatives of φ near the boundary. First, note that

$$\Phi_\epsilon(\Omega_\epsilon \setminus K_\epsilon) = \Omega \setminus K_\epsilon = \{(\bar{x}, x_N) \in \Omega : \bar{x} \in W,\ g_\epsilon(\bar{x}) - k_\epsilon \leq x_N \leq g(\bar{x})\},$$

for any $\epsilon > 0$. Fix $x \in \Phi_\epsilon(\Omega_\epsilon \setminus K_\epsilon)$ and $\beta \in \mathbb{N}^N$, $|\beta| \le k-1$. Suppose for the moment $\varphi \in C^m(\overline{\Omega})$. By the Taylor's formula, we get that

$$\begin{aligned}
D^\beta\varphi(x) = & \sum_{l=0}^{k-1-|\beta|} \frac{1}{l!}\frac{\partial^l(D^\beta\varphi)}{\partial x_N^l}\bigg|_{x_N=g(\bar{x})}(x_N - g(\bar{x}))^l \\
&+ \frac{(x_N-g(\bar{x}))^{k-|\beta|}}{(k-|\beta|-1)!}\int_0^1 (1-t)^{k-1-|\beta|}\frac{\partial^{k-|\beta|}}{\partial x_N^{k-|\beta|}}D^\beta\varphi(\bar{x}, g(\bar{x}+t(x_N-g(\bar{x}))\,\mathrm{d}t \\
&= \sum_{l=0}^{k-1-|\beta|} \frac{1}{l!}\frac{\partial^l(D^\beta\varphi(\bar{x},g(\bar{x})))}{\partial x_N^l}(x_N-g(\bar{x}))^l + R(\beta,x),
\end{aligned}$$

where we have defined

$$R(\beta,x) := \frac{(x_N-g(\bar{x}))^{k-|\beta|}}{(k-|\beta|-1)!}\int_0^1 (1-t)^{k-1-|\beta|}\frac{\partial^{k-|\beta|}}{\partial x_N^{k-|\beta|}}D^\beta\varphi(\bar{x}, g(\bar{x}+t(x_N-g(\bar{x}))\,\mathrm{d}t$$

for $\beta \in \mathbb{N}^N$, $|\beta| \le k-1$ and for all $x = (\bar{x}, x_N) \in \Phi_\epsilon(\Omega_\epsilon \setminus K_\epsilon)$.

Note that $-2k_\epsilon \le g_\epsilon(\bar{x}) - g(\bar{x}) - k_\epsilon \le x_N - g(\bar{x}) \le 0$. By Jensen's inequality,

$$|R(\beta,x)|^2 \le 4k_\epsilon^{2(k-|\beta|)}\int_0^1 \left|\frac{\partial^{k-|\beta|}}{\partial x_N^{k-|\beta|}}D^\beta\varphi(\bar{x}, g(\bar{x})+t(x_N-g(\bar{x}))\right|^2 \mathrm{d}t. \tag{2.2.7}$$

An integration in the variable x_N in (2.2.7) and (2.2.5) yield

$$\begin{aligned}
&\int_{g_\epsilon-k_\epsilon}^{g(\bar{x})} |R(\beta,x)|^2\,\mathrm{d}x_N \\
&\le 4k_\epsilon^{2(k-|\beta|)}\int_0^1\int_{g_\epsilon-k_\epsilon}^{g(\bar{x})}\left|\frac{\partial^{k-|\beta|}}{\partial x_N^{k-|\beta|}}D^\beta\varphi(\bar{x}, g(\bar{x}+t(x_N-g(\bar{x})))\right|^2 \mathrm{d}x_N\,\mathrm{d}t \\
&\le Ck_\epsilon^{2(k-|\beta|)}\int_0^1\int_{g_\epsilon-k_\epsilon}^{g(\bar{x})}\left\|\frac{\partial^{k-|\beta|+1}}{\partial x_N^{k-|\beta|+1}}D^\beta\varphi(\bar{x},\cdot)\right\|^2_{W^{1,2}(a,g(\bar{x}))}\mathrm{d}x_N\,\mathrm{d}t \\
&\le Ck_\epsilon^{2(k-|\beta|)+1}\left\|\frac{\partial^{k-|\beta|+1}}{\partial x_N^{k-|\beta|+1}}D^\beta\varphi(\bar{x},\cdot)\right\|^2_{W^{1,2}(a,g(\bar{x}))}.
\end{aligned} \tag{2.2.8}$$

By integrating both sides of (2.2.8) with respect to $\bar{x} \in W$, we finally get

$$\int_{\Phi_\epsilon(\Omega_\epsilon\setminus K_\epsilon)} |R(\beta,x)|^2\,\mathrm{d}x \le Ck_\epsilon^{2(k-|\beta|)+1}\|\varphi\|^2_{W^{m,2}(\Omega)}, \tag{2.2.9}$$

for sufficiently small ϵ, for all $|\beta| \leq k-1$. Thus,

$$\int_{\Phi_\epsilon(\Omega_\epsilon \setminus K_\epsilon)} |D^\beta \varphi(x)|^2 \, dx \leq C k_\epsilon^{2(k-|\beta|)+1} \|\varphi\|^2_{W^{m,2}(\Omega)}, \\ + C \int_W \int_{g_\epsilon - k_\epsilon}^{g(\bar{x})} \left| \sum_{l=0}^{k-1-|\beta|} \frac{\partial^l (D^\beta \varphi(\bar{x}, g(\bar{x})))}{\partial x_N^l} \right|^2 |x_N - g(\bar{x})|^{2l} d\bar{x} \, dx_N, \tag{2.2.10}$$

for all $n \geq 1$, for all sufficiently small ϵ, and $|\beta| \leq k-1$. We now estimate the last integral in the right-hand side of (2.2.10) by

$$\begin{aligned} &\sum_{l=0}^{k-1-|\beta|} C \int_W \int_{g_\epsilon - k_\epsilon}^{g(\bar{x})} \left| \frac{\partial^l (D^\beta \varphi(\bar{x}, g(\bar{x})))}{\partial x_N^l} \right|^2 |x_N - g(\bar{x})|^{2l} d\bar{x} \, dx_N \\ &\leq \sum_{l=0}^{k-1-|\beta|} C k_\epsilon^{2l} \int_W \left| \frac{\partial^l (D^\beta \varphi(\bar{x}, g(\bar{x})))}{\partial x_N^l} \right|^2 d\bar{x} \\ &= \sum_{l=0}^{k-1-|\beta|} C k_\epsilon^{2l} \left\| \frac{\partial^l (D^\beta \varphi)}{\partial x_N^l} \right\|^2_{L^2(\partial\Omega)}. \end{aligned} \tag{2.2.11}$$

Thus, by (2.2.10), (2.2.11) we obtain

$$\int_{\Phi_\epsilon(\Omega_\epsilon \setminus K_\epsilon)} |D^\beta \varphi(x)|^2 \, dx \\ \leq \sum_{l=0}^{k-1-|\beta|} C k_\epsilon^{2l} \left\| \frac{\partial^l (D^\beta \varphi)}{\partial x_N^l} \right\|^2_{L^2(\partial\Omega)} + C k_\epsilon^{2(k-|\beta|)+1} \|\varphi\|^2_{W^{m,2}(\Omega)}. \tag{2.2.12}$$

Inequality (2.2.12) holds for smooth functions. If $\varphi \in H^m(\Omega) \cap H^k_0(\Omega)$, then we can choose a sequence $(\psi_n)_{n \geq 1} \subset C^\infty(\overline{\Omega})$ such that $\psi_n \to \varphi$ in $H^m(\Omega)$ (this is possible because $\partial\Omega$ is Lipschitz continuous). By using (2.2.12) on ψ_n we get

$$\int_{\Phi_\epsilon(\Omega_\epsilon \setminus K_\epsilon)} |D^\beta \psi_n(x)|^2 \, dx \\ \leq \sum_{l=0}^{k-1-|\beta|} C k_\epsilon^{2l} \left\| \frac{\partial^l (D^\beta \psi_n)}{\partial x_N^l} \right\|^2_{L^2(\partial\Omega)} + C k_\epsilon^{2(k-|\beta|)+1} \|\psi_n\|^2_{W^{m,2}(\Omega)}. \tag{2.2.13}$$

Now, by continuity of the trace operator

$$\left\| \frac{\partial^l (D^\beta \psi_n)}{\partial x_N^l} \right\|_{L^2(\partial\Omega)} \longrightarrow \left\| \mathrm{tr} \left[\frac{\partial^l (D^\beta \varphi)}{\partial x_N^l} \right] \right\|_{L^2(\partial\Omega)} = 0, \tag{2.2.14}$$

as $n \to \infty$, for all $0 \le l \le k-1-|\beta|$, $|\beta| \le k-1$. Since $\partial\Omega$ is Lipschitz, the convergence in $H^m(\Omega)$ implies the convergence of all the intermediate derivatives in $L^2(\Omega)$ (see e.g. [35, §4.4]). By using (2.2.14) we can pass to the limit as $n \to \infty$ in inequality (2.2.13) to get

$$\int_{\Phi_\epsilon(\Omega_\epsilon \setminus K_\epsilon)} |D^\beta \varphi(x)|^2 \, \mathrm{d}x \le C k_\epsilon^{2(k-|\beta|)+1} \|\varphi\|^2_{W^{m,2}(\Omega)}, \tag{2.2.15}$$

for all sufficiently small ϵ. Actually, inequality (2.2.15) holds also for $|\beta| = k$ (possibly modifying the constant in the right hand side). Indeed, $D^\beta\varphi \in W^{1,2}(\Omega)$, for any $|\beta| = k$, hence

$$\int_W \|D^\beta \varphi(\bar{x}, \cdot)\|_\infty \mathrm{d}\bar{x} \le C \|D^\beta \varphi\|_{W^{1,2}(\Omega)},$$

by standard boundedness of Sobolev functions on almost all vertical lines (see (2.2.5)). Thus,

$$\begin{aligned}\int_{\Phi_\epsilon(\Omega_\epsilon \setminus K_\epsilon)} |D^\beta \varphi(x)|^2 \, \mathrm{d}x &= \int_W \int_{g_\epsilon - k_\epsilon}^{g(\bar{x})} |D^\beta \varphi(x)|^2 \, \mathrm{d}x_N \, \mathrm{d}\bar{x} \\ &\le 2k_\epsilon \int_W \|D^\beta \varphi(\bar{x}, \cdot)\|^2_\infty \mathrm{d}\bar{x} \\ &\le 2Ck_\epsilon \|\varphi\|^2_{W^{m,2}(\Omega)}.\end{aligned}$$

This concludes Step 1.

Step 2. We are now ready to prove condition (C2)(ii). Let $\varphi \in V(\Omega)$ and let α be a fixed multiindex such that $|\alpha| = m$. We write

$$D^\alpha \varphi(\Phi_\epsilon(x)) = \sum_{1 \le |\beta| \le m} D^\beta \varphi(\Phi_\epsilon(x)) p^\alpha_{m,\beta}(\Phi_\epsilon)(x), \tag{2.2.16}$$

where $p^\alpha_{m,\beta}(\Phi_\epsilon)$ is a homogeneous polynomial of degree $|\beta|$ in derivatives of Φ_ϵ of order not exceeding $m - |\beta| + 1$. Note that the polynomial $p^\alpha_{m,\beta}(\Phi_\epsilon)$ appearing in (2.2.16) is the sum of several terms Θ in the following form

$$\Theta = D^{h_1}\left(\delta_{j_1,N} - \frac{\partial h_\epsilon}{\partial x_{j_1}}\right) \cdots D^{h_n}\left(\delta_{j_n,N} - \frac{\partial h_\epsilon}{\partial x_{j_n}}\right) \frac{\partial \Phi^{(j_{n+1})}}{\partial x_{i_{n+1}}} \cdots \frac{\partial \Phi^{(j_{|\beta|})}}{\partial x_{i_{|\beta|}}},$$

where[1] $1 \le n \le |\beta|$, $1 \le j_i \le N$ for all $i = 1, \dots, n$, $i_1, \dots, i_{|\beta|}$ are in $\{1, \dots, N-1\}$, and $h_1, \dots, h_n$ are multiindexes satisfying

$$|h_1| + \cdots + |h_n| = m - |\beta|. \tag{2.2.17}$$

[1]Here it is understood that for $|\beta| = 1$ the terms $\frac{\partial \Phi^{(j_{n+1})}}{\partial x_{i_{n+1}}} \cdots \frac{\partial \Phi^{(j_{|\beta|})}}{\partial x_{i_{|\beta|}}}$ are not present; recall that $m \ge 2$.

Moreover Θ is a sum of many summands Θ_r, and for each one there exist l multiindexes $L_1, ..., L_l$ with the following properties:

$$\Theta_r = D^{L_1} h_\epsilon \cdots D^{L_l} h_\epsilon, \qquad \text{for all } 1 \le l \le |\beta|, \tag{2.2.18}$$

and

$$|L_1| + \cdots + |L_l| = m - |\beta| + l. \tag{2.2.19}$$

This is easy to check if either all the h_i are different from $(0, \ldots, 0)^t$ or $j_i \neq N$ for all $1 \le i \le |\beta|$, which corresponds to the case $l = |\beta|$. For the other cases simply note that the number $\tilde{l} = |\beta| - l$ is related to the number of mutiindexes h_i which are identically zero.
Now by [19, Inequality (6.7)] and hypothesis (iii) we have

$$\begin{aligned}
\|D^{L_1} h_\epsilon \cdots D^{L_l} h_\epsilon\|_\infty &\le c\left(\sum_{|\gamma_1|\le|L_1|} \frac{\|D^{\gamma_1}(g_\epsilon - g)\|_\infty}{\kappa_\epsilon^{|L_1|-|\gamma_1|}}\right)\cdots\left(\sum_{|\gamma_l|\le|L_l|} \frac{\|D^{\gamma_l}(g_\epsilon - g)\|_\infty}{\kappa_\epsilon^{|L_l|-|\gamma_l|}}\right) \\
&\le Co(1)\left(\sum_{|\gamma_1|\le|L_1|} \frac{\kappa_\epsilon^{m-|\gamma_1|-k+1/2}}{\kappa_\epsilon^{|L_1|-|\gamma_1|}}\right)\cdots\left(\sum_{|\gamma_l|\le|L_l|} \frac{\kappa_\epsilon^{m-|\gamma_l|-k+1/2}}{\kappa_\epsilon^{|L_l|-|\gamma_l|}}\right) \\
&\le Co(1)\kappa_\epsilon^{l(m-k+1/2)-\sum_i |L_i|} \\
&= Co(1)\kappa_\epsilon^{l(m-k+1/2)-\sum_i |L_i| - |\beta| + k + 1/2} \cdot \kappa_\epsilon^{|\beta|-k-1/2} \\
&\le co(1)\kappa_\epsilon^{|\beta|-k-1/2}
\end{aligned}$$

where the last inequality holds provided that

$$l(m-k+1/2) - \sum_i |L_i| - |\beta| + k + 1/2 \ge 0.$$

By (2.2.19), we have to check that

$$l(m-k+1/2) - (m - |\beta| + l) - |\beta| + k + 1/2 \ge 0,$$

which is verified if and only if

$$l(m-k-1/2) \ge m - k - 1/2,$$

and this holds true because $m - k - 1/2 > 0$ and $l \ge 1$. Hence we have proved that

$$\|p^\alpha_{m,\beta}(\Phi_\epsilon)\|_\infty \le c\, o(1)\, \kappa_\epsilon^{|\beta|-k-1/2}, \tag{2.2.20}$$

for some constant $c > 0$.

By inequalities (2.2.15) and (2.2.20), we deduce that

$$
\begin{aligned}
Q_{\Omega_\epsilon\setminus K_\epsilon}(T_\epsilon\varphi) &\le \int_{\Omega_\epsilon\setminus K_\epsilon} |\varphi(\Phi_\epsilon)|^2\,\mathrm{d}x + C\sum_{|\alpha|=m}\int_{\Omega_\epsilon\setminus K_\epsilon} |D^\alpha\varphi(\Phi_\epsilon)|^2\,\mathrm{d}x \\
&\le C\int_{\Phi_\epsilon(\Omega_\epsilon\setminus K_\epsilon)} |\varphi|^2\,\mathrm{d}x + C\sum_{\substack{|\alpha|=m\\ 1\le|\beta|\le k}} \|p^\alpha_{m,\beta}(\Phi_\epsilon)\|^2_\infty \int_{\Omega_\epsilon\setminus K_\epsilon} |D^\beta\varphi(\Phi_\epsilon(x))|^2\,\mathrm{d}x \\
&+ C\sum_{\substack{|\alpha|=m\\ k<|\beta|\le m}} \|p^\alpha_{m,\beta}(\Phi_\epsilon)\|^2_\infty \int_{\Omega_\epsilon\setminus K_\epsilon} |D^\beta\varphi(\Phi_\epsilon(x))|^2\,\mathrm{d}x \\
&\le C\|\varphi\|^2_{L^2(\Omega\setminus K_\epsilon)} + Co(1)\kappa_\epsilon^{2(|\beta|-k-1/2)}\kappa_\epsilon^{2(k-|\beta|)+1} + CM\|\varphi\|^2_{W^{m,2}(\Omega\setminus K_\epsilon)},
\end{aligned}
\tag{2.2.21}
$$

where the constant M is such that

$$
\sum_{\substack{|\alpha|=m\\ k<|\beta|\le m}} \|p^\alpha_{m,\beta}(\Phi_\epsilon)\|^2_\infty \le C\sum_{\substack{|\alpha|=m\\ k<|\beta|\le m}} o(1)\kappa_\epsilon^{2(|\beta|-k)-1} \le M
$$

for all $\epsilon > 0$ sufficiently small. Since the right-hand side of (2.2.21) vanishes as $\epsilon \to 0$ we conclude that condition $(C2)(ii)$ is satisfied.
It remains to prove condition (C3). To prove that conditions (C3)(i), (C3)(ii) are satisfied it is sufficient to set $E_\epsilon u = (\mathrm{Ext}_{\Omega_\epsilon} u)|_\Omega$ for all $u \in V(\Omega_\epsilon)$, where $\mathrm{Ext}_{\Omega_\epsilon}$ is the standard Sobolev extension operator mapping $H^m(\Omega_\epsilon)$ to $H^m(\mathbb{R}^N)$. Finally, in order to prove condition (C3)(iii) it is sufficient to prove that the weak limit v of the uniformly bounded sequence v_ϵ lies in $H^k_0(\Omega)$. This is easily achieved by considering the extension-by-zero of the functions v_ϵ outside Ω_ϵ and passing to the limit, recalling that the limit set Ω has Lipschitz boundary. See [19, Lemma 6.2] for more details. □

Remark 2.2.3. Lemma (2.2.2) is a generalization of [19, Lemma 6.2] for all higher order elliptic operators and for all possible choices of intermediate boundary conditions. Clearly, in the case of fourth order elliptic operators, [19, Lemma 6.2] coincides with Lemma 2.2.2 because we have only one possible intermediate boundary condition (corresponding to the energy space $H^2(\Omega_\epsilon) \cap H^1_0(\Omega_\epsilon)$.

Let $\alpha \in \mathbb{R}$, $\alpha > 0$. Let $b \in C^\infty(\overline{W})$ a positive, non-constant periodic function, with periodicity cell given by $Y =]-1/2, 1/2[^{N-1}$. Let us set

$$
g_\epsilon(\bar{x}) = \epsilon^\alpha b\left(\frac{\bar{x}}{\epsilon}\right), \quad g(\bar{x}) = 0, \tag{2.2.22}
$$

for all $\bar{x} \in W$. Here, g_ϵ, g are the profile functions in the definitions (2.2.3),(2.2.4). Let $m \in \mathbb{N}$, $m \geq 2$. We define $H_{V(\Omega)}$ to be the self-adjoint operator associated with the quadratic form

$$Q_\Omega(u,v) = \int_\Omega D^m u : D^m v\, dx + \int_\Omega uv\, dx,$$

for all $u, v \in V(\Omega)$. Namely, $H_{V(\Omega)}$ is the polyharmonic operator $(-\Delta)^m$ on Ω. In a similar way we define $H_{V(\Omega_\epsilon)}$ to be the self-adjoint operators associated with Q_{Ω_ϵ}. Then we have the following

Theorem 2.2.4. *Let $m \in \mathbb{N}$, $m \geq 2$, $1 \leq k \leq m-1$. Let $V(\Omega) = W^{m,2}(\Omega) \cap W_0^{k,2}(\Omega)$. If $\alpha > m - k + \frac{1}{2}$, then $H^{-1}_{V(\Omega_\epsilon)} \xrightarrow{C} H^{-1}_{V(\Omega)}$ as $\epsilon \to 0$.*

Proof. We aim to apply Lemma 2.2.2. Let $0 < \epsilon \leq 1/2$ be fixed. In the statement of Lemma 2.2.2 we are allowed to choose $k = m-1$ and $\kappa_\epsilon = \epsilon^{\alpha\theta}\|b\|_\infty > \|g_\epsilon - g\|_\infty$, for some $\theta \in (0,1)$ to be specified. By the classical Gagliardo-Nirenberg interpolation inequality

$$\|D^\beta f\|_\infty \leq C\Big(\sum_{|\alpha|=m} \|D^\alpha f\|_\infty\Big)^{\frac{|\beta|}{m}} \|f\|_\infty^{1-\frac{|\beta|}{m}},$$

for all $f \in W^{m,\infty}(\Omega)$ (see e.g., [100, p.125]), in order to verify condition (iii) in Lemma 2.2.2 it is sufficient to verify it for $|\beta| = 0$ and $|\beta| = m$ (see also [19, Proposition 6.17]). When $|\beta| = 0$ we have

$$\lim_{\epsilon\to 0} \frac{\|g_\epsilon - g\|_\infty}{\kappa_\epsilon^{m-k+1/2}} = c \lim_{\epsilon\to 0} \frac{\epsilon^\alpha}{\epsilon^{\alpha\theta(m-k+1/2)}} = c \lim_{\epsilon\to 0} \epsilon^{\alpha(1-\theta(m-k-1/2)},$$

where c is a constant depending only on $\|b\|_\infty$. The right hand side clearly tends to 0 as soon as $\theta < \frac{1}{m-k+1/2}$.
When $|\beta| = m$, we must check that

$$\lim_{\epsilon\to 0} \frac{D^\beta g_\epsilon}{\kappa_\epsilon^{-k+1/2}} = 0.$$

Note that

$$\frac{D^\beta g_\epsilon}{\kappa_\epsilon^{-k+1/2}} = c\frac{\epsilon^{\alpha-m}}{\epsilon^{\alpha\theta(-k+1/2)}} = \epsilon^{\alpha(1-\theta(-k+1/2))-m},$$

and the right hand side tends to zero if and only if

$$\alpha\Big(1 + \theta\Big(k - \frac{1}{2}\Big)\Big) - m > 0. \tag{2.2.23}$$

By letting $\theta \to \frac{1}{m-k+1/2}$ in (2.2.23) we obtain that inequality (2.2.23) is satisfied when $\alpha > m-k+1/2$, true by assumption. By Lemma 2.2.2 we deduce the validity of Theorem 2.2.4. □

Remark 2.2.5. When $k = m-1$, Theorem 2.2.4 states that if $\alpha > \frac{3}{2}$, $H^{-1}_{V(\Omega_\epsilon)} \xrightarrow{C} H^{-1}_{V(\Omega)}$ as $\epsilon \to 0$, independently on $m \geq 2$. Actually, it is possible to prove that $\alpha = 3/2$ in this case is the critical exponent, in the sense that when $\alpha \leq 3/2$ $H^{-1}_{V(\Omega_\epsilon)}$ does not converge to $H^{-1}_{V(\Omega)}$ anymore. We refer to Theorem 4.2.1 for the complete discussion about the spectral convergence of $H_{V(\Omega_\epsilon)}$ depending on α.

Chapter 3

Triharmonic operator on singularly perturbed domains

In this chapter we discuss eigenvalue problems for the triharmonic operator $-\Delta^3$ on a domain $\Omega \subset \mathbb{R}^N$ subject to singular boundary perturbations. Let us describe the appropriate context. We consider the quadratic form defined by

$$Q_\Omega(u,v) = \int_\Omega D^3u : D^3v \, dx + \int_\Omega uv \, dx, \tag{3.0.1}$$

for all $u, v \in V(\Omega)$, where $V(\Omega) \subset H^3(\Omega)$ and $H^3_0(\Omega) \subset V(\Omega)$. Note that (3.0.1) corresponds to (2.1.2) with the choice of $A_{\alpha\beta} = \delta_{\alpha\beta}6/\alpha!$ for all $\alpha, \beta \in \mathbb{N}^N$. In particular, (3.0.1) satisfies condition (2.1.6), hence there exists a densely defined, non-negative and self-adjoint operator $H_{V(\Omega)}$ with domain $\mathscr{D}(H_{V(\Omega)}) \subset H^3(\Omega)$ such that

$$Q_\Omega(u,v) = (H^{1/2}_{V(\Omega)}u, \, H^{1/2}_{V(\Omega)}v),$$

for all $u, v \in V(\Omega)$. Assume that the embedding of $V(\Omega)$ in $L^2(\Omega)$ is compact (this is certainly true when Ω is bounded and $\partial\Omega$ has C^0 boundary in the sense of Burenkov, see [35]). Then, by Theorem 1.1.7, $H_{V(\Omega)}$ has compact resolvent. Let us consider the eigenvalue problem

$$\int_\Omega D^3u : D^3v \, dx + \int_\Omega uv \, dx = \lambda \int_\Omega uv \, dx, \tag{3.0.2}$$

in the unknowns λ (the eigenvalue) and $u \in V(\Omega)$ (the eigenfunction), for all $v \in V(\Omega)$.

Let $k \in \mathbb{N}$, $0 \le k \le 3$ and let us set $V(\Omega) = H^3(\Omega) \cap H^k_0(\Omega)$. If $k = 3$ then $V(\Omega) = H^3_0(\Omega)$. By integration by parts we realise that

$$\int_\Omega D^3u : D^3v \, dx = \int_\Omega \nabla(\Delta u) \cdot \nabla(\Delta v) \, dx,$$

for all $u, v \in H^3_0(\Omega)$. The classical operator L associated with this quadratic form is $Lu = -\Delta^3 u + u$, subject to Dirichlet boundary conditions on $\partial\Omega$. Namely, (3.0.2) corresponds to the eigenvalue problem

$$\begin{cases} -\Delta^3 u + u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = \frac{\partial^2 u}{\partial n^2} = 0, & \text{on } \partial\Omega. \end{cases}$$

Let now $k \leq 2$. In order to identify the boundary conditions satisfied by the classic operator L associated with the quadratic form (3.0.1) we need the following Triharmonic Green Formula

$$\begin{aligned} \int_\Omega D^3 u : D^3 v \, dx = &- \int_\Omega \Delta^3 u \, v \, dx + \int_{\partial\Omega} \frac{\partial^3 u}{\partial n^3} \frac{\partial^2 v}{\partial n^2} \, dS \\ &+ \int_{\partial\Omega} \left(\left((n^T D^3 u)_{\partial\Omega} : D_{\partial\Omega} n \right) - \frac{\partial^2(\Delta u)}{\partial n^2} - 2 \operatorname{div}_{\partial\Omega} (D^3 u[n \otimes n])_{\partial\Omega} \right) \frac{\partial v}{\partial n} \, dS \\ &+ \int_{\partial\Omega} \Bigg(\operatorname{div}^2_{\partial\Omega} \left((n^T D^3 u)_{\partial\Omega} \right) + \operatorname{div}_{\partial\Omega} \left(D_{\partial\Omega} n (D^3 u[n \otimes n])_{\partial\Omega} \right) \\ &\qquad + \frac{\partial(\Delta^2 u)}{\partial n} + \operatorname{div}_{\partial\Omega} \left(n^T D^2(\Delta u) \right)_{\partial\Omega} \Bigg) v \, dS. \end{aligned} \tag{3.0.3}$$

which is proved in Theorem 4.1.7. Moreover we refer to §1.4 for the definition of the tangential operators appearing in (3.0.3).

When $k = 2$, $V(\Omega) = H^3(\Omega) \cap H^2_0(\Omega)$. Assume henceforth that $u \in H^6(\Omega)$. By (3.0.3) we deduce that

$$\int_\Omega D^3 u : D^3 v dx = - \int_\Omega \Delta^3 u \, v dx + \int_{\partial\Omega} \frac{\partial^3 u}{\partial n^3} \frac{\partial^2 v}{\partial n^2} dS,$$

for all $v \in V(\Omega)$. Then we deduce that the classical eigenvalue problem associated with (3.0.2) on $V(\Omega)$ is defined by

$$\begin{cases} -\Delta^3 u + u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \\ \frac{\partial^3 u}{\partial n^3} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.0.4}$$

In this case we say that the classical operator $Lu = -\Delta^3 u + u$ associated with problem (3.0.4) satisfies *strong intermediate boundary conditions* on $\partial\Omega$.

When $k = 1$, $V(\Omega) = H^3(\Omega) \cap H^1_0(\Omega)$. By (3.0.3) we deduce that

$$\int_\Omega D^3u : D^3v dx = -\int_\Omega \Delta^3 u\, v\, dx + \int_{\partial\Omega} \frac{\partial^3 u}{\partial n^3}\frac{\partial^2 v}{\partial n^2}\, dS + \int_{\partial\Omega} \left(\left((n^T D^3 u)_{\partial\Omega} : D_{\partial\Omega} n\right) - \frac{\partial^2(\Delta u)}{\partial n^2} - 2\,\mathrm{div}_{\partial\Omega}(D^3u[n\otimes n])_{\partial\Omega} \right) \frac{\partial v}{\partial n}\, dS,$$

for all $v \in V(\Omega)$. Thus, the classical eigenvalue problem associated with (3.0.2) on $V(\Omega)$ is defined by

$$\begin{cases} -\Delta^3 u + u = \lambda u, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega,\\ \left((n^T D^3u)_{\partial\Omega} : D_{\partial\Omega} n\right) - \frac{\partial^2(\Delta u)}{\partial n^2} - 2\,\mathrm{div}_{\partial\Omega}(D^3u[n\otimes n])_{\partial\Omega} = 0, & \text{on } \partial\Omega,\\ \frac{\partial^3 u}{\partial n^3} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.0.5}$$

In this case, we say that the classical operator $Lu = -\Delta^3 u + u$ associated with problem (3.0.4) satisfies *weak intermediate boundary conditions* on $\partial\Omega$. Note that the geometry of $\partial\Omega$ (in particular, the curvature tensor $D_{\partial\Omega} n$) appears non-trivially in the second boundary condition.

Finally, when $k = 0$, $V(\Omega) = H^3(\Omega)$, hence by definition we get the Neumann problem for $H_{V(\Omega)}$. By (3.0.3) we deduce that the classical Neumann eigenvalue problem associated with (3.0.2) on $V(\Omega)$ is defined by

$$\begin{cases} -\Delta^3 u + u = \lambda u, & \text{in } \Omega,\\ \mathrm{div}^2_{\partial\Omega}\left((n^T D^3u)_{\partial\Omega}\right) + \mathrm{div}_{\partial\Omega}\left(D_{\partial\Omega} n (D^3u[n\otimes n])_{\partial\Omega}\right) & \\ \qquad + \frac{\partial(\Delta^2 u)}{\partial n} + \mathrm{div}_{\partial\Omega}\left(n^T D^2(\Delta u)\right)_{\partial\Omega} = 0, & \text{on } \partial\Omega,\\ \left((n^T D^3u)_{\partial\Omega} : D_{\partial\Omega} n\right) - \frac{\partial^2(\Delta u)}{\partial n^2} - 2\,\mathrm{div}_{\partial\Omega}(D^3u[n\otimes n])_{\partial\Omega} = 0, & \text{on } \partial\Omega,\\ \frac{\partial^3 u}{\partial n^3} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.0.6}$$

The main focus of attention throughout this chapter is on the triharmonic operator $H_{V(\Omega)}$ subject to intermediate boundary conditions, either of strong type (see (3.0.4)) or of weak type (see (3.0.5)). Namely, we set $V(\Omega) = H^3(\Omega) \cap H^k_0(\Omega)$, with either $k = 2$ or $k = 1$. Let $\epsilon > 0$ and let $\Omega_\epsilon \subset \mathbb{R}^N$ be defined by

$$\Omega_\epsilon = \{(\bar{x}, x_N) : \mathbb{R}^N : \bar{x} \in W, -1 < x_N < g_\epsilon(\bar{x})\},$$

where $W \subset \mathbb{R}^{N-1}$ is a bounded domain of class C^3 and $g_\epsilon(\bar{x})$ is defined in (2.2.22). Moreover, let $\Omega = W \times (-1, 0)$. We are interested in the spectral convergence of $H_{V(\Omega_\epsilon)}$ as $\epsilon \to 0$, depending on the value of the parameter $\alpha > 0$ appearing in (2.2.22). By Lemma 2.2.2 (see also Theorem 2.2.4) there exists an exponent $\bar{\alpha} > 1$

depending on the choice of $V(\Omega)$ such that $H^{-1}_{V(\Omega_\epsilon)} \overset{C}{\to} H^{-1}_{V(\Omega)}$ for all $\alpha > \bar{\alpha}$. Hence, a large part of the chapter is devoted to the study of the spectral convergence of $H_{V(\Omega_\epsilon)}$ when $\alpha \le \bar{\alpha}$. In particular, we analyse the behaviour of the energy spaces $V(\Omega_\epsilon)$ as $\epsilon \to 0$, by proving suitable degeneration results (see Section 3.1 below).

3.1 A general degeneration Lemma

The aim of this section is to prove a general Lemma concerning the limiting boundary behaviour of sequences $(u_\epsilon)_\epsilon$ such that $u_\epsilon \in H^3(\Omega_\epsilon) \cap H^k_0(\Omega_\epsilon)$ and $\|u_\epsilon\|_{H^3(\Omega_\epsilon)} < \infty$, for all $\epsilon > 0$, for $k = 1, 2$. More precisely we will prove the following results.

Lemma 3.1.1. *Let $Y = [-1/2, 1/2]^{N-1}$, $\alpha \in \mathbb{R}$, $\alpha > 0$. Let $\Omega = W \times (-1, 0)$, where $W \subset \mathbb{R}^{N-1}$ is a C^3 bounded domain. Let*

$$\Omega_\epsilon = \{(\bar{x}, x_N) \in \mathbb{R}^N : \bar{x} \in W, -1 < x_N < g_\epsilon(\bar{x}) := \epsilon^\alpha b(\bar{x}/\epsilon)\}, \tag{3.1.1}$$

where $b \in C^3(W)$ is a positive, non-constant Y-periodic function. Let $(u_\epsilon)_{\epsilon>0}$ be such that $u_\epsilon \in H^3(\Omega_\epsilon) \cap H^2_0(\Omega_\epsilon)$ for all $\epsilon > 0$ and $u_\epsilon|_\Omega \to u$ weakly in $H^3(\Omega)$. Let also $\hat{u} \in L^2(W, H^3(Y \times (-1, 0)))$ be defined by (3.1.17). *Then:*

(i) If $\alpha > 3/2$ then $u \in H^3(\Omega) \cap H^2_0(\Omega)$;

(ii) If $\alpha = 3/2$ then $u \in H^3(\Omega) \cap H^2_0(\Omega)$ and

$$\frac{\partial^2 \hat{u}}{\partial y_N \partial y_j}(\bar{x}, \bar{y}, 0) = -\frac{\partial^2 u}{\partial x_N^2}(\bar{x}, \bar{y}, 0)\frac{\partial b(\bar{y})}{\partial y_j}, \qquad \text{for all } j \in \{1, \dots, N-1\}.$$

(iii) If $0 < \alpha < 3/2$ then

$$u \in \left\{\phi \in H^3(\Omega) \cap H^2_0(\Omega) : \frac{\partial^2 \phi}{\partial x_N^2}(\bar{x}, 0) = 0 \text{ for a.a. } \bar{x} \in W\right\}.$$

Proof. We postpone this proof, since it follows easily from the proof of Lemma 3.1.2 below. □

Lemma 3.1.2. *Let $Y = [-1/2, 1/2]^{N-1}$, $\alpha \in \mathbb{R}$, $\alpha > 0$. Let $\Omega = W \times (-1, 0)$, where $W \subset \mathbb{R}^{N-1}$ is bounded domain of class C^3. Let Ω_ϵ be as in* (3.1.1)*. Let $(u_\epsilon)_{\epsilon>0}$ be such that $H^3(\Omega_\epsilon) \cap H^1_0(\Omega_\epsilon)$ for all $\epsilon > 0$ and $u_\epsilon|_\Omega \to u$ weakly in $H^3(\Omega)$. Let also $\hat{u} \in L^2(W, H^3(Y \times (-1, 0)))$ be defined by* (3.1.17). *Then:*

(i) If $\alpha > 5/2$ then $u \in H^3(\Omega) \cap H^1_0(\Omega)$;

(ii) *If* $\alpha = 5/2$ *then* $u \in H^3(\Omega) \cap H^1_0(\Omega)$ *and*

$$\frac{\partial^2 \hat{u}}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, 0) = -\frac{\partial u}{\partial x_N}(\bar{x}, \bar{y}, 0)\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j}, \qquad \text{for all } i, j \in \{1, \dots, N-1\}.$$

(iii) *If* $3/2 < \alpha < 5/2$ *then*

$$u \in \left\{\phi \in H^3(\Omega) \cap H^1_0(\Omega) : \frac{\partial \phi}{\partial x_N}(\bar{x}, 0) = 0 \text{ for a.a. } \bar{x} \in W\right\};$$

(iv) *If* $\alpha = 3/2$ *then there exists* $\epsilon_0 > 0$ *such the sequence* $\left(\frac{1}{\epsilon^2}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, 0)\right)_{\epsilon \in [0, \epsilon_0]}$ *is uniformly bounded in* $L^2(W \times Y)$*; up to a subsequence there exists* $f \in L^2(W)$ *such that*

$$\frac{1}{\epsilon^2}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, 0) \rightharpoonup f(\bar{x}),$$

in $L^2(W \times Y)$ *and*

$$\frac{\partial^2 \hat{u}}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, 0) + f(\bar{x})\frac{\partial b(\bar{y})}{\partial y_i \partial y_j} = 0, \qquad \text{for all } i, j \in \{1, \dots, N-1\};$$

(v) *If* $0 < \alpha \leq 1$ *then*

$$u \in \left\{\phi \in H^3(\Omega) \cap H^1_0(\Omega) : \frac{\partial \phi}{\partial x_N}(\bar{x}, 0) = \frac{\partial^2 \phi}{\partial x_N^2}(\bar{x}, 0) = 0 \text{ for a.a. } \bar{x} \in W\right\}.$$

Proof of Lemma 3.1.2. Fix $0 < \epsilon < 1$. We find convenient to treat first the case $\alpha \geq 3/2$. Since $u_\epsilon \in H^1_0(\Omega_\epsilon)$

$$u_\epsilon(\bar{x}, g_\epsilon(\bar{x})) = 0, \quad \text{for a.e. } \bar{x} \in W. \tag{3.1.2}$$

Note that the function $u_\epsilon(\cdot, g_\epsilon(\cdot)) \in H^{5/2}(W) \subset H^2(W)$. We differentiate (3.1.2) with respect to x_i, $1 \leq i \leq N-1$ and we get

$$\frac{\partial u_\epsilon}{\partial x_i}(\bar{x}, g_\epsilon(\bar{x})) + \frac{\partial u_\epsilon}{\partial x_N}(\bar{x}, g_\epsilon(\bar{x}))\frac{\partial g_\epsilon(\bar{x})}{\partial x_i} = 0,$$

for a.e. $\bar{x} \in W$. We then differentiate with respect to x_j, $1 \leq j \leq N-1$ in order to get

$$\begin{aligned}
&\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_j}(\bar{x}, g_\epsilon(\bar{x})) + \frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}(\bar{x}, g_\epsilon(\bar{x}))\frac{\partial g_\epsilon(\bar{x})}{\partial x_j} + \frac{\partial^2 u_\epsilon}{\partial x_j \partial x_N}(\bar{x}, g_\epsilon(\bar{x}))\frac{\partial g_\epsilon(\bar{x})}{\partial x_i} \\
&+ \frac{\partial^2 u_\epsilon}{\partial x_N^2}(\bar{x}, g_\epsilon(\bar{x}))\frac{\partial g_\epsilon(\bar{x})}{\partial x_i}\frac{\partial g_\epsilon(\bar{x})}{\partial x_j} + \frac{\partial u_\epsilon}{\partial x_N}(\bar{x}, g_\epsilon(\bar{x}))\frac{\partial^2 g_\epsilon(\bar{x})}{\partial x_i \partial x_j} = 0,
\end{aligned} \tag{3.1.3}$$

for a.e. $\bar{x} \in W$. Let now

$$\hat{v}(\bar{x}, y) = v\Big(\epsilon\Big[\frac{\bar{x}}{\epsilon}\Big] + \epsilon\bar{y}, \epsilon y_N\Big),$$

for all $v \in H^1(\Omega_\epsilon)$, for all $\bar{x} \in \widehat{W}_\epsilon$, $\bar{y} \in Y$, $y_N \in (-1/\epsilon, \epsilon^{\alpha-1} b(\bar{y}))$. It is understood that $\hat{v}$ is set to be zero for all $\bar{x} \in W \setminus \widehat{W}_\epsilon$. Set for simplicity $\epsilon^{\alpha-1} b(\bar{y}) = y_\epsilon$, and note that by periodicity of b, $b(\bar{y}) = b([\bar{x}/\epsilon] + \bar{y}) = \epsilon^{-\alpha}\widehat{g_\epsilon}(\bar{x}, \bar{y})$ for all $(\bar{x}, \bar{y}) \in C^k_\epsilon \times Y$. We unfold equality (3.1.3) and we use property (iv) in Proposition 1.2.7 in order to obtain

$$\begin{aligned}
&\frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, y_\epsilon) + \frac{\epsilon^{\alpha-1}}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, y_\epsilon)\frac{\partial b(\bar{y})}{\partial y_j} + \frac{\epsilon^{\alpha-1}}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x}, \bar{y}, y_\epsilon)\frac{\partial b(\bar{y})}{\partial y_i}\\
&+ \frac{\epsilon^{2\alpha-2}}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{y}, y_\epsilon)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} + \frac{\epsilon^{\alpha-2}}{\epsilon}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, y_\epsilon)\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0,
\end{aligned}$$

for a.e. $\bar{x} \in W$, for a.e. $\bar{y} \in Y$.

We now define

$$\begin{aligned}
\hat{\Psi}_\epsilon(\bar{x}, y) =& \frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x}, y) + \frac{\epsilon^{\alpha-1}}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, y)\frac{\partial b(\bar{y})}{\partial y_j} + \frac{\epsilon^{\alpha-1}}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x}, y)\frac{\partial b(\bar{y})}{\partial y_i}\\
&+ \frac{\epsilon^{2\alpha-2}}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, y)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} + \frac{\epsilon^{\alpha-2}}{\epsilon}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, y)\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j},
\end{aligned}$$

and if we set $\hat{Y} = \{y \in \mathbb{R}^N : \bar{y} \in Y, -1 < y_N < \epsilon^{\alpha-1} b(\bar{y})\}$, then $\hat{\Psi}_\epsilon \in L^2(W, H^1(\hat{Y}))$. Since $\hat{\Psi}_\epsilon(\bar{x}, y, y_\epsilon) = 0$ we have that

$$|\hat{\Psi}_\epsilon(\bar{x}, \bar{y}, 0)| \leq \int_0^{y_\epsilon} \left|\frac{\partial \hat{\Psi}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, t)\right| dt, \qquad \text{for a.e. } \bar{x} \in W, \bar{y} \in Y.$$

By definition of $\hat{\Psi}_\epsilon$ and Hölder inequality we now deduce that

$$\begin{aligned}
|\hat{\Psi}_\epsilon(\bar{x}, \bar{y}, 0)| \leq& \left(\epsilon^{\alpha-1}\|b\|_\infty\right)^{1/2}\Bigg[\frac{1}{\epsilon^2}\left\|\frac{\partial^3 \hat{u}_\epsilon}{\partial y_i \partial y_j \partial y_N}(\bar{x}, \bar{y}, \cdot)\right\|_{L^2(0,y_\epsilon)}\\
&+ \frac{\epsilon^{\alpha-1}}{\epsilon^2}\|\nabla b\|_\infty\left\|\frac{\partial^3 \hat{u}_\epsilon}{\partial y_i \partial y_N^2}(\bar{x}, \bar{y}, \cdot)\right\|_{L^2(0,y_\epsilon)} + \frac{\epsilon^{\alpha-1}}{\epsilon^2}\|\nabla b\|_\infty\left\|\frac{\partial^3 \hat{u}_\epsilon}{\partial y_j \partial y_N^2}(\bar{x}, \bar{y}, \cdot)\right\|_{L^2(0,y_\epsilon)}\\
&+ \frac{\epsilon^{2\alpha-2}}{\epsilon^2}\|\nabla b\|^2_\infty\left\|\frac{\partial^3 \hat{u}_\epsilon}{\partial y_N^3}(\bar{x}, \bar{y}, \cdot)\right\|_{L^2(0,y_\epsilon)} + \frac{\epsilon^{\alpha-2}}{\epsilon}\|D^2 b\|_\infty\left\|\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{y}, \cdot)\right\|_{L^2(0,y_\epsilon)}\Bigg],
\end{aligned} \tag{3.1.4}$$

Let us define $\hat{Y}_{>0} := \hat{Y} \cap \{y_N \in \mathbb{R} : y_N > 0\}$. We square both hand sides of (3.1.4) and integrate over $W \times Y$ to get

$$\int_W \int_Y |\hat{\Psi}_\epsilon(\bar{x}, \bar{y}, 0)|^2 d\bar{y}d\bar{x} \le C(\|b\|^2_{C^2(Y)} + \|\nabla b\|^4_\infty)\epsilon^{\alpha-1}\Bigg[\frac{1}{\epsilon^4}\|D^3_y \hat{u}_\epsilon\|^2_{L^2(W\times\hat{Y}_{>0})} + \frac{\epsilon^{2\alpha-2}}{\epsilon^4}\|D^3_y \hat{u}_\epsilon\|^2_{L^2(W\times\hat{Y}_{>0})} + \frac{\epsilon^{4\alpha-4}}{\epsilon^4}\|D^3_y \hat{u}_\epsilon\|^2_{L^2(W\times\hat{Y}_{>0})} + \frac{\epsilon^{2\alpha-4}}{\epsilon^2}\left\|\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}\right\|^2_{L^2(W\times\hat{Y}_{>0})}\Bigg], \tag{3.1.5}$$

We now take advantage of the exact integration formula (see (1.2.4)). We deduce that

$$\begin{aligned}\int_W \int_Y |\hat{\Psi}_\epsilon(\bar{x}, \bar{y}, 0)|^2 d\bar{y}d\bar{x} &\le C\epsilon^{\alpha-1}\Bigg[\frac{\epsilon^5}{\epsilon^4}\|D^3 u_\epsilon\|^2_{L^2(\Omega_\epsilon\setminus\Omega)} + \frac{\epsilon^{2\alpha-2}}{\epsilon^4}\epsilon^5\|D^3 u_\epsilon\|^2_{L^2(\Omega_\epsilon\setminus\Omega)} \\ &\quad + \frac{\epsilon^{4\alpha-4}}{\epsilon^4}\epsilon^5\|D^3 u_\epsilon\|^2_{L^2(\Omega_\epsilon\setminus\Omega)} + \frac{\epsilon^{2\alpha-4}}{\epsilon^2}\epsilon^3\left\|\frac{\partial^2 u_\epsilon}{\partial x_N^2}\right\|^2_{L^2(\Omega_\epsilon\setminus\Omega)}\Bigg] \\ &\le C\|D^3 u_\epsilon\|^2_{L^2(\Omega_\epsilon)}(\epsilon^\alpha + \epsilon^{3\alpha-2} + \epsilon^{5\alpha-4}) + C\epsilon^{4\alpha-4}\left\|\frac{\partial^2 u_\epsilon}{\partial x_N^2}\right\|^2_{W^{1,2}(\Omega_\epsilon)} \\ &\le C(\epsilon^\alpha + \epsilon^{4\alpha-4}) + o(\epsilon^\alpha)\end{aligned} \tag{3.1.6}$$

Note that since $\alpha \ge 3/2 > 4/3$ then $\epsilon^{4\alpha-4} = o(\epsilon^\alpha)$. We deduce that

$$\int_W \int_Y |\hat{\Psi}_\epsilon(\bar{x}, \bar{y}, 0)|^2 d\bar{y}d\bar{x} = O(\epsilon^\alpha), \qquad \int_W \int_Y \left|\int_Y \hat{\Psi}_\epsilon(\bar{x}, \bar{z}, 0)d\bar{z}\right|^2 d\bar{y}d\bar{x} = O(\epsilon^\alpha),$$

as $\epsilon \to 0$. Since $\alpha > 1$ we deduce that

$$\int_W \int_Y \epsilon^{-1}\left|\hat{\Psi}_\epsilon(\bar{x}, \bar{y}, 0) - \int_Y \hat{\Psi}_\epsilon(\bar{x}, \bar{z}, 0)d\bar{z}\right|^2 d\bar{y}d\bar{x} = O(\epsilon^{\alpha-1}) \to 0, \tag{3.1.7}$$

as $\epsilon \to 0$. The left-hand side of (3.1.7) is the integral of the square of the sum of five terms, that we are going to denote by $T_1, \dots, T_5$. Hence we can rewrite (3.1.7) as

$$\int_W \int_Y \left|T_1 + \cdots + T_5\right|^2 d\bar{y}d\bar{x} \to 0, \quad \text{as } \epsilon \to 0, \tag{3.1.8}$$

where

$$T_1 = \frac{1}{\epsilon^{5/2}}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x},\bar{y},0) - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x},\bar{z},0)d\bar{z}\right);$$

$$T_2 = \frac{\epsilon^{\alpha-1}}{\epsilon^{5/2}}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x},\bar{y},0)\frac{\partial b(\bar{y})}{\partial y_j} - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x},\bar{z},0)\frac{\partial b(\bar{z})}{\partial y_j}d\bar{z}\right);$$

$$T_3 = \frac{\epsilon^{\alpha-1}}{\epsilon^{5/2}}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x},\bar{y},0)\frac{\partial b(\bar{y})}{\partial y_i} - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x},\bar{z},0)\frac{\partial b(\bar{z})}{\partial y_i}d\bar{z}\right);$$

$$T_4 = \frac{\epsilon^{2\alpha-2}}{\epsilon^{5/2}}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x},\bar{y},0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x},\bar{z},0)\frac{\partial b(\bar{z})}{\partial y_i}\frac{\partial b(\bar{z})}{\partial y_j}d\bar{z}\right);$$

$$T_5 = \frac{\epsilon^{\alpha-2}}{\epsilon^{3/2}}\left(\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},0)\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} - \int_Y \frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{z},0)\frac{\partial^2 b(\bar{z})}{\partial y_i \partial y_j}d\bar{z}\right).$$

Recall that the function U_ϵ defined by

$$U_\epsilon(\bar{x},y) = \hat{u}_\epsilon(\bar{x},y) - \int_Y \left(\hat{u}_\epsilon(\bar{x},\bar{\zeta},0) - \sum_{|\eta|=2}\int_Y D_y^\eta \hat{u}_\epsilon(\bar{x},\bar{\zeta},0)\,\mathrm{d}\bar{\zeta}\right)\frac{\bar{\zeta}^\eta}{\eta!}\,\mathrm{d}\bar{\zeta}$$
$$- \int_Y \nabla_y \hat{u}_\epsilon(\bar{x},\bar{\zeta},0)\,\mathrm{d}\bar{\zeta}\cdot y - \sum_{|\eta|=2}\int_Y D_y^\eta \hat{u}_\epsilon(\bar{x},\bar{\zeta},0)\,\mathrm{d}\bar{\zeta}\,\frac{y^\eta}{\eta!},$$

is such that the sequence $(\epsilon^{-5/2}U_\epsilon)$ is uniformly bounded in $L^2(W,H^3(Y\times(d,0))$, for any $d<0$, see for example Lemma 3.2.3. Note also that $D_y^\eta U_\epsilon = D_y^\eta \hat{u}_\epsilon - \int_Y D_y^\eta \hat{u}_\epsilon(\cdot,\bar{z},\cdot)d\bar{z}$ for any $|\eta|=2$. Using these facts we deduce that

$$\int_W\int_Y |T_1|^2 d\bar{y}d\bar{x} = \int_W\int_Y \left|\epsilon^{-5/2}\frac{\partial^2 U_\epsilon}{\partial y_i\partial y_j}(\bar{x},\bar{y},0)\right|^2 d\bar{y}d\bar{x}$$
$$\leq C\left\|\epsilon^{-5/2}\frac{\partial^2 U_\epsilon}{\partial y_i \partial y_j}\right\|^2_{L^2(W,H^1(Y\times(-1,0))}$$
$$\leq C\|\epsilon^{-5/2}D_y^3\hat{u}_\epsilon\|^2_{L^2(W\times Y\times(-1,0))} \leq C\|D^3 u_\epsilon\|^2_{L^2(\Omega)},$$

where we have used a trace inequality, the Poincaré-Wirtinger inequality, property (iv) in Proposition 1.2.7 and the exact integration formula (1.2.4). Hence T_1 is bounded in $L^2(W\times Y)$.

We proceed to estimate the other terms T_i. Note that the function $\frac{\partial b}{\partial y_j}$ has null average over Y because of periodicity. Hence,

$$\int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x},\bar{z},0)\frac{\partial b(\bar{z})}{\partial y_j}d\bar{z} = \int_Y \frac{\partial b(\bar{z})}{\partial y_j}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x},\bar{z},0) - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x},\bar{t},0)d\bar{t}\right)d\bar{z}$$

and

$$\int_W \int_Y \epsilon^{2\alpha-2-5} \left| \int_Y \frac{\partial b(\bar{z})}{\partial y_j} \left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{z}, 0) - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{t}, 0) d\bar{t} \right) d\bar{z} \right|^2 d\bar{y} d\bar{x}$$
$$\leq C \int_W \int_Y \epsilon^{2\alpha-2} \int_Y \left| \epsilon^{-5/2} \left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{z}, 0) - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{t}, 0) d\bar{t} \right) \right|^2 d\bar{z} d\bar{y} d\bar{x}$$
$$\leq C\epsilon^{2\alpha-2} \|\epsilon^{-5/2} \partial^2_{y_i y_N} U_\epsilon(\cdot, \cdot, 0)\|^2_{L^2(W\times Y)} \leq C\epsilon^{2\alpha-2} \|\epsilon^{-5/2} D^3_y \hat{u}_\epsilon\|^2_{L^2(W\times Y\times(-1,0))} \longrightarrow 0, \tag{3.1.9}$$

as $\epsilon \longrightarrow 0$, for all $\alpha > 1$. We deduce that

$$\int_W \int_Y |T_2|^2 d\bar{y} d\bar{x} \leq C \int_W \int_Y \left| \frac{\epsilon^{\alpha-1}}{\epsilon^{5/2}} \left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, 0) \frac{\partial b(\bar{y})}{\partial y_j} \right) \right|^2 d\bar{y} d\bar{x}$$
$$+ C \int_W \int_Y \left| \int_Y \epsilon^{\alpha-1-5/2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{z}, 0) \frac{\partial b(\bar{z})}{\partial y_j} d\bar{z} \right|^2 d\bar{y} d\bar{x}$$
$$\leq C \int_W \int_Y \left| \frac{\epsilon^{\alpha}}{\epsilon^{3/2}} \left(\frac{1}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, 0) \frac{\partial b(\bar{y})}{\partial y_j} \right) \right|^2 d\bar{y} d\bar{x} + o(1), \tag{3.1.10}$$

as $\epsilon \longrightarrow 0$. We claim that

$$\frac{1}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, 0) \frac{\partial b(\bar{y})}{\partial y_j} \longrightarrow \frac{\partial^2 u}{\partial x_i \partial x_N}(\bar{x}, 0) \frac{\partial b(\bar{y})}{\partial y_j}, \tag{3.1.11}$$

in $L^2(W \times Y)$ as $\epsilon \longrightarrow 0$. Since $u_\epsilon \longrightarrow u$ weakly in $H^3(\Omega)$, by the compactness of the trace operator we have that

$$\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}(\bar{x}, 0) \longrightarrow \frac{\partial^2 u}{\partial x_i \partial x_N}(\bar{x}, 0), \tag{3.1.12}$$

in $L^2(W)$, as $\epsilon \longrightarrow 0$. Now define

$$\overline{\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}}(\bar{x}) := \frac{1}{\epsilon^{N-1}} \int_{C_\epsilon(\bar{x})} \frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}(\bar{t}, 0)\, d\bar{t},$$

where $C_\epsilon(\bar{x})$ is as in (1.2.3). Note that, by a change of variable,

$$\overline{\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}}(\bar{x}) = \int_Y \widehat{\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}}(\bar{x}, \bar{z}, 0) d\bar{z} = \frac{1}{\epsilon^2} \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{z}, 0)\, d\bar{z}.$$

By (3.1.12) we deduce that

$$\overline{\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}} \to \frac{\partial^2 u}{\partial x_i \partial x_N}(\cdot, 0),$$

strongly in $L^2(W)$ as $\epsilon \to 0$ (here we use the fact that if a sequence of functions v_ϵ converges strongly in L^2 to v then $\overline{v_\epsilon}$ converges strongly in L^2 to v. We give a proof of this fact in Lemma 3.1.4 below). Moreover we know that

$$\frac{1}{\epsilon^2}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\cdot, \cdot, 0) - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\cdot, \bar{z}, 0) d\bar{z}\right) \to 0,$$

in $L^2(W \times Y)$ as $\epsilon \to 0$ (since $\epsilon^{-5/2}\partial_{y_i y_N} U_\epsilon$ is uniformly bounded in $L^2(W \times Y)$, for all $\epsilon > 0$). Hence

$$\frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, 0) \to \frac{\partial^2 u}{\partial x_i \partial x_N}(\bar{x}, 0),$$

in $L^2(W \times Y)$ as $\epsilon \to 0$, which proves the claim. Since $\alpha > 3/2$, by recalling (3.1.10) we then deduce that T_2 vanishes in $L^2(W \times Y)$ as $\epsilon \to 0$.

We consider T_3. Since T_3 is exactly T_2 with swapped indexes i and j we deduce that T_3 vanishes in $L^2(W \times Y)$ as $\epsilon \to 0$.

We then consider T_4. By arguing as in (3.1.11) we deduce that

$$\frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{y}, 0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} \to \frac{\partial^2 u}{\partial y_N^2}(\bar{x}, 0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j},$$

in $L^2(W \times Y)$ as $\epsilon \to 0$. This implies that

$$\int_Y \frac{1}{\epsilon^2}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{z}, 0)\frac{\partial b(\bar{z})}{\partial y_i}\frac{\partial b(\bar{z})}{\partial y_j}\right) d\bar{z} \to$$
$$\int_Y \frac{\partial^2 u}{\partial y_N^2}(\bar{x}, 0)\frac{\partial b(\bar{z})}{\partial y_i}\frac{\partial b(\bar{z})}{\partial y_j} d\bar{z} = \frac{\partial^2 u}{\partial y_N^2}(\bar{x}, 0)\int_Y \frac{\partial b(\bar{z})}{\partial y_i}\frac{\partial b(\bar{z})}{\partial y_j} d\bar{z},$$

in $L^2(W \times Y)$, as $\epsilon \to 0$. It is then clear that

$$T_4 = \frac{\epsilon^{2\alpha}}{\epsilon^{5/2}}\left(\frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{y}, 0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} - \int_Y \frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{z}, 0)\frac{\partial b(\bar{z})}{\partial y_i}\frac{\partial b(\bar{z})}{\partial y_j} d\bar{z}\right) \to 0, \tag{3.1.13}$$

in $L^2(W \times Y)$ as $\epsilon \to 0$ for all $\alpha > 5/4$, hence in particular for any $\alpha \geq 3/2$.
Finally we consider T_5. Arguing as in the proof of Claim (3.1.11) we can prove that

$$\frac{1}{\epsilon}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, 0)\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} \to \frac{\partial u}{\partial y_N}(\bar{x}, 0)\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j}, \tag{3.1.14}$$

in $L^2(W \times Y)$ as $\epsilon \to 0$ and

$$\int_Y \frac{1}{\epsilon}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{z}, 0)\frac{\partial^2 b(\bar{z})}{\partial y_i \partial y_j} d\bar{z} \to \frac{\partial u}{\partial y_N}(\bar{x}, 0)\int_Y \frac{\partial^2 b(\bar{z})}{\partial y_i \partial y_j} d\bar{z} = 0, \tag{3.1.15}$$

in $L^2(W \times Y)$ as $\epsilon \to 0$, where the right-hand side of (3.1.15) is zero due to periodicity of b.

We now consider different cases according to the value of α.

Case $3/2 < \alpha < 5/2$**.** In this case, by summarising the previous results we have that T_1 is bounded in $L^2(W \times Y)$ whereas T_2, T_3, T_4 tend to zero in $L^2(W \times Y)$ as $\epsilon \to 0$. This implies that

$$\left(\int_W \int_Y |T_5|^2 d\bar{y} d\bar{x}\right)^{1/2} \le \left(\int_W \int_Y |T_1 + T_2 + T_3 + T_4|^2 d\bar{y} d\bar{x}\right)^{1/2} + o(1) \le M,$$

for a big enough positive constant M independent of ϵ, for all small enough $\epsilon > 0$. This implies that

$$\left\| \frac{1}{\epsilon}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, 0)\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} - \int_Y \frac{1}{\epsilon}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{z}, 0)\frac{\partial^2 b(\bar{z})}{\partial y_i \partial y_j} d\bar{z} \right\|_{L^2(W\times Y)} = O(\epsilon^{5/2-\alpha}),$$

as $\epsilon \to 0$. By letting $\epsilon \to 0$ and recalling (3.1.14) and (3.1.15) we deduce that

$$\frac{\partial u}{\partial y_N}(\bar{x}, 0)\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0,$$

for a.e. $\bar{x} \in W$, for a.e. $\bar{y} \in Y$, and since b is not affine we deduce that

$$\frac{\partial u}{\partial x_N}(\bar{x}, 0) = 0, \tag{3.1.16}$$

for a.e. $\bar{x} \in W$. We conclude that

$$u \in \left\{\phi \in H^3(\Omega) \cap H^1_0(\Omega) : \frac{\partial \phi}{\partial x_N}(\bar{x}, 0) = 0 \text{ for a.a. } \bar{x} \in W\right\}.$$

Case $\alpha = 5/2$**.** We have the estimate

$$\left(\int_W \int_Y |T_1 + T_5|^2 d\bar{y} d\bar{x}\right)^{1/2} \le \left(\int_W \int_Y |T_2 + T_3 + T_4|^2 d\bar{y} d\bar{x}\right)^{1/2} + o(1) = o(1),$$

as $\epsilon \longrightarrow 0$. Thus,

$$\frac{1}{\epsilon^{5/2}}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x},\bar{y},0) - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x},\bar{z},0)d\bar{z}\right) + \frac{1}{\epsilon}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},0)\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} \longrightarrow 0,$$

as $\epsilon \longrightarrow 0$. Now since $(\epsilon^{-5/2}U_\epsilon)$ is uniformly bounded in $L^2(W \times Y \times (d,0))$, there exists a subsequence of $(\epsilon^{-5/2}U_\epsilon)$ and a function $\hat{u} \in L^2(W, H^3(Y \times (d,0)))$ such that

$$\epsilon^{-5/2}U_\epsilon \rightharpoonup \hat{u}, \tag{3.1.17}$$

in $L^2(W, H^3(Y \times (d,0)))$. This implies that

$$\frac{1}{\epsilon^{5/2}}\left(\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x},\bar{y},0) - \int_Y \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x},\bar{z},0)d\bar{z}\right) \longrightarrow \frac{\partial^2 \hat{u}}{\partial y_i \partial y_j}(\bar{x},\bar{y},0),$$

strongly in $L^2(W \times Y)$ as $\epsilon \longrightarrow 0$. Moreover, according to (3.1.14) we deduce that

$$\frac{\partial^2 \hat{u}}{\partial y_i \partial y_j}(\bar{x},\bar{y},0) = -\frac{\partial u}{\partial y_N}(\bar{x},0)\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j}, \tag{3.1.18}$$

for a.e. $\bar{x} \in W$, a.e. $\bar{y} \in Y$.

Case $\alpha = 3/2$**.** In this case the first three terms T_1, T_2, T_3 are uniformly bounded in ϵ. Moreover we have the following convergences

$$T_1 \longrightarrow \frac{\partial^2 \hat{u}}{\partial y_i \partial y_j}(\bar{x},\bar{y},0), \tag{3.1.19}$$

$$T_2 \longrightarrow \frac{\partial^2 u}{\partial x_i x_N}(\bar{x},0)\frac{\partial b}{\partial y_j}(\bar{y}), \tag{3.1.20}$$

$$T_3 \longrightarrow \frac{\partial^2 u}{\partial x_j x_N}(\bar{x},0)\frac{\partial b}{\partial y_i}(\bar{y}), \tag{3.1.21}$$

where all the limits are in $L^2(W \times Y)$, as $\epsilon \longrightarrow 0$. Note that since $\alpha < 2$ we know, by Proposition 3.1.3 below, that $\frac{\partial u}{\partial x_N}(\bar{x},0) = 0$ for almost all $\bar{x}$, hence

$$\frac{\partial^2 u}{\partial x_i x_N}(\bar{x},0) = 0, \qquad \frac{\partial^2 u}{\partial x_j x_N}(\bar{x},0) = 0, \tag{3.1.22}$$

for almost all $\bar{x} \in W$. Thus, T_2, T_3 are vanishing as $\epsilon \longrightarrow 0$. Moreover, we have already proved in (3.1.13) that T_4 vanishes as $\epsilon \longrightarrow 0$. We then deduce that

$$\|T_5\|^2_{L^2(W\times Y)} \le \sum_{i=1}^{4}\|T_i\|^2_{L^2(W\times Y)} \le C,$$

where the constant C does not depend on ϵ. Hence, up to a subsequence there exists a function $f \in L^2(W)$ such that

$$\frac{1}{\epsilon^2}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, 0) \rightharpoonup f(\bar{x}),$$

in $L^2(W \times Y)$ as $\epsilon \to 0$. The fact that f does not depend on $\bar{y}$ can be shown as follows. Let $\varphi \in C_c^\infty(W \times Y)$ and let $i \in \{1, \dots, N-1\}$. Then

$$\int_{W\times Y} \frac{1}{\epsilon^2}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, 0)\frac{\partial \varphi}{\partial y_i}(\bar{x}, \bar{y})\, d\bar{x}d\bar{y} = -\int_{W\times Y} \frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N \partial y_i}(\bar{x}, \bar{y}, 0)\varphi(\bar{x}, \bar{y})\, d\bar{x}d\bar{y} \to 0,$$

as $\epsilon \to 0$, since

$$\frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N \partial y_i}(\bar{x}, \bar{y}, 0) \to \frac{\partial^2 u}{\partial x_N \partial x_i}(\bar{x}, 0) = 0,$$

strongly in $L^2(W \times Y)$ as $\epsilon \to 0$, for all $i = 1, \dots, N-1$. We then deduce that up to a subsequence

$$\frac{1}{\epsilon^2}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, 0)\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} \to f(\bar{x})\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j},$$

weakly in $L^2(W \times Y)$, as $\epsilon \to 0$.

Case $\alpha \leq 1$. In this case we give a more direct proof based on a different definition of the unfolding operator. We define

$$\hat{Y} = \{(\bar{y}, y_N) : \bar{y} \in Y, -1 < y_N < b(\bar{y})\}, \tag{3.1.23}$$

and

$$\hat{u}_\epsilon(\bar{x}, \bar{y}, y_N) := u_\epsilon\left(\epsilon\left[\frac{\bar{x}}{\epsilon}\right] + \epsilon\bar{y}, \epsilon^\alpha y_N\right), \tag{3.1.24}$$

for all $(\bar{x}, y) \in W \times \hat{Y}$, for all $u_\epsilon \in H^3(\Omega_\epsilon)$. Then starting from the identity (3.1.2) we deduce the analogous of (3.1.3), which namely reads

$$\begin{aligned}
&\frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, b(\bar{y})) + \frac{\epsilon^{\alpha-1}}{\epsilon^{\alpha+1}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_j}\\
&+ \frac{\epsilon^{\alpha-1}}{\epsilon^{\alpha+1}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_i} + \frac{\epsilon^{2\alpha-2}}{\epsilon^{2\alpha}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j}\\
&+ \frac{\epsilon^{\alpha-2}}{\epsilon^\alpha}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0.
\end{aligned} \tag{3.1.25}$$

If $\alpha = 1$, by arguing as in (3.1.34) below, we have

$$\frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, b(\bar{y})) \longrightarrow \frac{\partial^2 u}{\partial x_i \partial x_j}(\bar{x}, 0),$$

$$\frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_j} \longrightarrow \frac{\partial^2 u}{\partial x_i \partial x_N}(\bar{x}, 0)\frac{\partial b(\bar{y})}{\partial y_j},$$

$$\frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_i} \longrightarrow \frac{\partial^2 u}{\partial x_j \partial x_N}(\bar{x}, 0)\frac{\partial b(\bar{y})}{\partial y_i},$$

$$\frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} \longrightarrow \frac{\partial^2 u}{\partial x_N^2}(\bar{x}, 0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j},$$

as $\epsilon \longrightarrow 0$, where the limits are taken in $L^2(W \times Y)$. According to (3.1.25), we immediately discover that

$$\left\| \frac{1}{\epsilon}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, b(\bar{y})) \right\|_{L^2(W\times\hat{Y})} \leq C\epsilon, \tag{3.1.26}$$

for all $\epsilon > 0$. By (3.1.26) we deduce that

$$\frac{\partial u}{\partial x_N}(\bar{x}, 0) = 0, \tag{3.1.27}$$

and that there exists a function $\zeta \in L^2(W)$ such that, up to a subsequence,

$$\frac{1}{\epsilon^2}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, b(\bar{y})) \rightharpoonup \zeta(\bar{x}),$$

in $L^2(W\times Y)$ as $\epsilon \longrightarrow 0$. The fact that ζ does not depend on $\bar{y}$ is an easy consequence of the following argument. Let $\varphi \in C_c^\infty(W \times Y)$. Then

$$\int_{W\times Y} \frac{1}{\epsilon^2}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial \varphi}{\partial y_i}\, d\bar{x}d\bar{y} = -\int_{W\times Y} \frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N \partial y_i}(\bar{x}, \bar{y}, b(\bar{y}))\varphi\, d\bar{x}d\bar{y},$$

and passing to the limit as $\epsilon \longrightarrow 0$ we deduce that

$$\int_{W\times Y} \zeta \frac{\partial \varphi}{\partial y_i}\, d\bar{x}d\bar{y} = -\int_{W\times Y} \frac{\partial^2 u}{\partial x_N \partial x_i}(\bar{x}, 0)\varphi\, d\bar{x}d\bar{y} = 0, \tag{3.1.28}$$

where we have used that $\frac{\partial^2 u}{\partial x_N \partial x_i}(\bar{x}, 0) = 0$ because of (3.1.27). Since equality (3.1.28) holds for all the functions $\varphi \in C_c^\infty(W \times Y)$ we deduce that ζ has a weak derivative in y_i and that $\frac{\partial \zeta}{\partial y_i} = 0$.

Then we can pass to the limit - strongly $L^2(W \times Y)$ - as $\epsilon \to 0$ in (3.1.25) in order to obtain

$$\begin{aligned} &\frac{\partial^2 u}{\partial x_i \partial x_j}(\bar{x},0) + \frac{\partial^2 u}{\partial x_i \partial x_N}(\bar{x},0)\frac{\partial b(\bar{y})}{\partial y_j} + \frac{\partial^2 u}{\partial x_j \partial x_N}(\bar{x},0)\frac{\partial b(\bar{y})}{\partial y_i} \\ &+ \frac{\partial^2 u}{\partial x_N^2}(\bar{x},0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} + \zeta(\bar{x})\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0. \end{aligned} \tag{3.1.29}$$

Note that the first three summands in (3.1.29) are zero because of the vanishing of u and ∇u on $W \times \{0\}$ (this follows from (3.1.27)). Hence we are left with the equality

$$\frac{\partial^2 u}{\partial x_N^2}(\bar{x},0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} + \zeta(\bar{x})\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0.$$

Recall now that since b is Y-periodic, its derivatives are periodic and with null average on Y. Hence an integration in Y yields

$$\frac{\partial^2 u}{\partial x_N^2}(\bar{x},0)\int_Y \frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j}d\bar{y} = 0,$$

for almost all $\bar{x} \in W$. Since this holds for all $i, j = 1, \ldots, N-1$ we can in particular choose $i = j$. This implies

$$\frac{\partial^2 u}{\partial x_N^2}(\bar{x},0)\int_Y |\nabla b|^2 d\bar{y} = 0,$$

and since b is non constant it must be $\frac{\partial^2 u}{\partial x_N^2}(\bar{x},0) = 0$ for almost all $\bar{x} \in W$.

If $\alpha < 1$ we can argue in a similar way. Namely, we multiply each side of (3.1.25) by $\epsilon^{2-2\alpha}$ in order to obtain

$$\begin{aligned} &\frac{1}{\epsilon^{2\alpha}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x},\bar{y},b(\bar{y})) + \frac{\epsilon^{1-\alpha}}{\epsilon^{\alpha+1}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_j} \\ &+ \frac{\epsilon^{1-\alpha}}{\epsilon^{\alpha+1}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_i} + \frac{1}{\epsilon^{2\alpha}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} \\ &+ \frac{1}{\epsilon^{2\alpha}}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0. \end{aligned} \tag{3.1.30}$$

Now note that the first three summands in (3.1.30) are vanishing as $\epsilon \to 0$ (here we use the fact that $\alpha < 1$ and $u(\bar{x},0) = 0$ for almost all $x \in W$). Then we deduce that

$$\frac{\partial^2 u}{\partial x_N^2}(\bar{x},0)\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} + \lim_{\epsilon \to 0}\frac{1}{\epsilon^{2\alpha}}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x},\bar{y},b(\bar{y}))\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0.$$

This first implies that

$$\left\| \frac{1}{\epsilon^\alpha} \frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, b(\bar{y})) \frac{\partial^2 b}{\partial y_i \partial y_j} \right\|_{L^2(W\times Y)} \leq C\epsilon^\alpha,$$

hence $\frac{\partial u}{\partial x_N}(\bar{x}, 0) = 0$. Moreover, we deduce that up to a subsequence there exists $\zeta \in L^2(W)$ such that

$$\frac{1}{\epsilon^{2\alpha}} \frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, b(\bar{y})) \rightharpoonup \zeta(\bar{x}),$$

in $L^2(W \times Y)$ as $\epsilon \to 0$. Then arguing as in the case $\alpha = 1$ we deduce that $\frac{\partial^2 u}{\partial x_N^2}(\bar{x}, 0) = 0$. □

Proof of Lemma 3.1.1. The proof uses arguments similar to the one of Lemma 3.1.2. However, since $u_\epsilon \in H^3(\Omega_\epsilon) \cap H^2_0(\Omega_\epsilon)$, the calculations are easier. Indeed, suppose $u_\epsilon|_\Omega \to u$ weakly in $H^3(\Omega)$. Then, since $u_\epsilon = 0 = \frac{\partial u_\epsilon}{\partial n}$ on $\Gamma_\epsilon = \{(\bar{x}, x_N) \in \mathbb{R}^N : \bar{x} \in W, x_N = g_\epsilon(\bar{x})\}$ we deduce that $\nabla u_\epsilon = 0$ on Γ_ϵ. Hence, by differentiating with respect to x_j the identity $\frac{\partial u_\epsilon}{\partial x_i}(\bar{x}, g_\epsilon(\bar{x})) = 0$ and by unfolding we obtain

$$\frac{1}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, y_\epsilon) + \frac{\epsilon^{\alpha-1}}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, y_\epsilon) \frac{\partial b(\bar{y})}{\partial y_j} = 0.$$

Note that here $i \in \{1, \ldots, N\}$, whereas $j \in \{1, \ldots, N-1\}$. Hence, if

$$\hat{\Phi}_\epsilon(\bar{x}, \bar{y}, y_N) := \frac{1}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, y_N) + \frac{\epsilon^{\alpha-1}}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, y_N) \frac{\partial b(\bar{y})}{\partial y_j},$$

arguing as in (3.1.6) we deduce that

$$\int_W \int_Y \epsilon^{-1} \left| \hat{\Phi}_\epsilon(\bar{x}, \bar{y}, 0) - \int_Y \hat{\Phi}_\epsilon(\bar{x}, \bar{z}, 0) d\bar{z} \right|^2 d\bar{y} d\bar{x} \leq C \|u_\epsilon\|^2_{H^3(\Omega_\epsilon)} \epsilon^{\alpha-1} \to 0,$$

as $\epsilon \to 0$, and exactly as in the discussion for $u_\epsilon \in H^3(\Omega_\epsilon) \cap H^1_0(\Omega_\epsilon)$ we deduce that if $\alpha = 3/2$ then

$$\frac{\partial^2 u}{\partial x_i \partial x_N}(\bar{x}, 0) \frac{\partial b(\bar{y})}{\partial y_j} = -\frac{\partial^2 \hat{u}}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, 0),$$

while, if $\alpha < 3/2$, then

$$\frac{\partial^2 u}{\partial x_i \partial x_N}(\bar{x}, 0) = 0,$$

for almost all $\bar{x} \in W$, for all $i \in \{1, \ldots, N\}$. In particular, $\frac{\partial^2 u}{\partial x_N^2}(\bar{x}, 0) = 0$, concluding the proof. □

Proposition 3.1.3. *Let $u \in H^3(\Omega) \cap H^1_0(\Omega)$ be the function defined in the statement of Lemma (3.1.2). If $1 < \alpha < 2$ then $\frac{\partial u}{\partial x_N}(\bar{x}, 0) = 0$ for almost all $\bar{x} \in W$.*

Proof. In this proof we use the definition of $\hat{Y}$ and $\hat{u}$ introduced in (3.1.23) and (3.1.24). Note that

$$\epsilon^\alpha \int_W \int_{\hat{Y}} \frac{1}{\epsilon^2} |\nabla_{\bar{y}} \hat{u}_\epsilon|^2 + \frac{1}{\epsilon^{2\alpha}} \left| \frac{\partial \hat{u}_\epsilon}{\partial y_N} \right|^2 d\bar{x} dy = \int_{\Omega_\epsilon} |\nabla u_\epsilon|^2 \, dx, \tag{3.1.31}$$

where we have used formula (1.2.4). Since $\alpha < 2$ we deduce that $\nabla_y \hat{u}_\epsilon \to 0$ in $L^2(W \times \hat{Y})^N$. In a similar way one proves that $D^\beta_y \hat{u}_\epsilon \to 0$ for all the multiindexes β such that $1 \le |\beta| \le 3$. Now note that

$$\int_W \int_Y |\hat{u}_\epsilon(\bar{x}, \bar{y}, 0)|^2 d\bar{y} d\bar{x} = \int_W |u_\epsilon(\bar{x}, 0)|^2 d\bar{x} \le C,$$

uniformly in $\epsilon > 0$. Thus,

$$\begin{aligned}
&\int_W \int_{\hat{Y}} |\hat{u}_\epsilon(\bar{x}, y)|^2 dy d\bar{x} \\
&\le 2 \int_W \int_{\hat{Y}} |\hat{u}_\epsilon(\bar{x}, y) - \hat{u}_\epsilon(\bar{x}, \bar{y}, 0)|^2 dy d\bar{x} + 2(b(\bar{y}) + 1) \int_W \int_Y |\hat{u}_\epsilon(\bar{x}, \bar{y}, 0)|^2 d\bar{y} d\bar{x} \\
&\le 2 \int_W \int_{\hat{Y}} \int_0^{y_N} \left| \frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, t) \right|^2 dt |y_N| dy d\bar{x} + C \le C',
\end{aligned}$$

hence $\hat{u}_\epsilon$ is uniformly bounded in $L^2(W, H^3(\hat{Y}))$ and up to a subsequence

$$\hat{u}_\epsilon \rightharpoonup \hat{u} \qquad \text{in } L^2(W, H^3(\hat{Y})),$$

for some function $\hat{u} \in L^2(W, H^3(\hat{Y}))$. Actually $\hat{u}$ does not depend on y; indeed $\nabla_y \hat{u}_\epsilon \to 0$ in $L^2(W \times \hat{Y})^N$ implies that $\nabla_y \hat{u} = 0$.
Since $u_\epsilon \to u$ weakly in $H^3(\Omega)$, by the Trace Theorem, $u_\epsilon(\bar{x}, 0) \to u(\bar{x}, 0)$ strongly in $L^2(W)$. By Lemma 3.1.4 below, we deduce that

$$\overline{u_\epsilon}(\bar{x}) = \frac{1}{\epsilon^{N-1}} \int_{C_\epsilon(\bar{x})} u_\epsilon(\bar{t}, 0) d\bar{t} \to u(\bar{x}, 0), \tag{3.1.32}$$

strongly in $L^2(W)$ as $\epsilon \to 0$. By a change of variable it is easy to see that

$$\overline{u_\epsilon}(\bar{x}) = \int_Y \hat{u}_\epsilon(\bar{x}, \bar{z}, 0) \, d\bar{z},$$

for almost all $\bar{x} \in W$. By Poincaré inequality it is also easy to prove that

$$\left\| \hat{u}_\epsilon - \int_Y \hat{u}_\epsilon(\cdot, \bar{z}, 0) d\bar{z} \right\|_{L^2(W \times \hat{Y})} \le C \|\nabla_{\bar{y}} \hat{u}_\epsilon\|_{L^2(W \times \hat{Y})} \to 0, \tag{3.1.33}$$

as $\epsilon \longrightarrow 0$, according to (3.1.31). Then, by (3.1.32) and (3.1.33) we have

$$\|\hat{u}_\epsilon - u(\bar{x},0)\|_{L^2(W\times\hat{Y})} \leq \left\|\hat{u}_\epsilon - \int_Y \hat{u}_\epsilon(\cdot,\bar{z},0)d\bar{z}\right\|_{L^2(W\times\hat{Y})} + \left\|\int_Y \hat{u}_\epsilon(\cdot,\bar{z},0)d\bar{z} - u(\bar{x},0)\right\|_{L^2(W\times\hat{Y})} \longrightarrow 0, \tag{3.1.34}$$

as $\epsilon \longrightarrow 0$, which implies that $\hat{u}(\bar{x}) = u(\bar{x},0)$ for almost all $\bar{x} \in W$. Now we unfold the following identity

$$\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_j}(\bar{x}, g_\epsilon(\bar{x})) + \frac{\partial^2 u_\epsilon}{\partial x_i \partial x_N}(\bar{x}, g_\epsilon(\bar{x}))\frac{\partial g_\epsilon(\bar{x})}{\partial x_j} + \frac{\partial^2 u_\epsilon}{\partial x_j \partial x_N}(\bar{x}, g_\epsilon(\bar{x}))\frac{\partial g_\epsilon(\bar{x})}{\partial x_i}$$
$$+ \frac{\partial^2 u_\epsilon}{\partial x_N^2}(\bar{x}, g_\epsilon(\bar{x}))\frac{\partial g_\epsilon(\bar{x})}{\partial x_i}\frac{\partial g_\epsilon(\bar{x})}{\partial x_j} + \frac{\partial u_\epsilon}{\partial x_N}(\bar{x}, g_\epsilon(\bar{x}))\frac{\partial^2 g_\epsilon(\bar{x})}{\partial x_i \partial x_j} = 0,$$

in order to obtain

$$\frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, b(\bar{y})) + \frac{\epsilon^{\alpha-1}}{\epsilon^{\alpha+1}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_j} + \frac{\epsilon^{\alpha-1}}{\epsilon^{\alpha+1}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_j \partial y_N}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_i}$$
$$+ \frac{\epsilon^{2\alpha-2}}{\epsilon^{2\alpha}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial b(\bar{y})}{\partial y_i}\frac{\partial b(\bar{y})}{\partial y_j} + \frac{\epsilon^{\alpha-2}}{\epsilon^{\alpha}}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0. \tag{3.1.35}$$

Note that

$$\frac{1}{\epsilon^2}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, b(\bar{y})) \longrightarrow \frac{\partial^2 u}{\partial x_i \partial x_j}(\bar{x}, 0) = 0,$$
$$\frac{1}{\epsilon^{\alpha+1}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, b(\bar{y})) \longrightarrow \frac{\partial^2 u}{\partial x_i \partial x_N}(\bar{x}, 0),$$
$$\frac{1}{\epsilon^{2\alpha}}\frac{\partial^2 \hat{u}_\epsilon}{\partial y_N^2}(\bar{x}, \bar{y}, b(\bar{y})) \longrightarrow \frac{\partial^2 u}{\partial x_N^2}(\bar{x}, 0),$$

as $\epsilon \longrightarrow 0$, where the limits are in $L^2(W \times Y)$. Hence, if $1 < \alpha < 2$ we deduce that all the summands in (3.1.35) are vanishing in $L^2(W \times Y)$ with the possible exception of

$$\frac{\epsilon^{\alpha-2}}{\epsilon^{\alpha}}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j}.$$

Since equality (3.1.35) must hold, this implies that also this last summand is bounded; hence,

$$\frac{1}{\epsilon^{\alpha}}\frac{\partial \hat{u}_\epsilon}{\partial y_N}(\bar{x}, \bar{y}, b(\bar{y}))\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} \longrightarrow 0,$$

in $L^2(W \times Y)$ as $\epsilon \longrightarrow 0$, and consequently,

$$\frac{\partial u}{\partial x_N}(\bar{x}, 0)\frac{\partial^2 b(\bar{y})}{\partial y_i \partial y_j} = 0,$$

for almost all $(\bar{x}, \bar{y}) \in (W \times Y)$. Then $\frac{\partial u}{\partial x_N}(\bar{x}, 0) = 0$ for almost all $\bar{x} \in W$, concluding the proof. □

Lemma 3.1.4. *Let $(v_\epsilon)_\epsilon$ be a sequence of functions in $L^2(\Theta)$, for a given bounded open set $\Theta \subset \mathbb{R}^N$. Let $v \in L^2(\Theta)$, and assume that $v_\epsilon \longrightarrow v$ in $L^2(\Theta)$. For all $\epsilon > 0$ let $C_\epsilon(x) = \{y \in \mathbb{R}^N : |x - y| < \epsilon\}$ and we define*

$$\overline{v_\epsilon}(x) = \frac{1}{\epsilon^N}\int_{C_\epsilon(x)} v_\epsilon(y)\, dy,$$

for almost all $x \in \Theta$. Then $\overline{v_\epsilon} \longrightarrow v$ in $L^2(\Theta)$ as $\epsilon \longrightarrow 0$.

Proof. We claim that

$$\overline{v}(x) := \frac{1}{\epsilon^N}\int_{C_\epsilon(x)} v(y)dy \longrightarrow v(x), \tag{3.1.36}$$

strongly in $L^2(\Theta)$ as $\epsilon \longrightarrow 0$. Let $\delta > 0$ be fixed and let $w \in C^1(\Theta) \cap L^2(\Theta)$ such that $\|v - w\|_{L^2(\Theta)} \leq \delta$. Then

$$\begin{aligned}\overline{v}(x) - v(x) &= \frac{1}{\epsilon^N}\int_{C_\epsilon(x)} (v(y) - v(x))\, dy \\ &= \frac{1}{\epsilon^N}\int_{C_\epsilon(x)} (v(y) - w(y))\, dy + (w(x) - v(x)) \\ &+ \frac{1}{\epsilon^N}\int_{C_\epsilon(x)} (w(y) - w(x))\, dy.\end{aligned}$$

Let us define $\Theta^\epsilon = \{x \in \Theta : \mathrm{dist}(x, \partial\Theta) > \epsilon\}$. Note that

$$\begin{aligned}\int_{\Theta^\epsilon}\left|\frac{1}{\epsilon^N}\int_{C_\epsilon(x)}(v(y) - w(y))dy\right|^2 dx &\leq \int_{\Theta^\epsilon}\frac{1}{\epsilon^N}\int_{C_\epsilon(x)} |v(y) - w(y)|^2\, dydx \\ &= \int_{\Theta^\epsilon} |v(y) - w(y)|^2\left(\frac{1}{\epsilon^N}\int_{C_\epsilon(y)} dx\right) dy \leq C\delta^2\end{aligned}$$

where we have used Jensen's inequality and Tonelli Theorem. Moreover, it is clear that

$$\left\|\frac{1}{\epsilon^N}\int_{C_\epsilon(x)}(w(y) - w(x))dy\right\|_{L^2(\Theta)} \leq C\epsilon.$$

Hence,

$$\|\overline{v} - v\|_{L^2(\Theta^\epsilon)} \leq C(\delta + \epsilon) \leq C'\delta.$$

The arbitrariness of δ proves the validity of claim (3.1.36). Now note that

$$\|\overline{v_\epsilon} - \overline{v}\|_{L^2(\Theta^\epsilon)} \leq \left(\int_{\Theta^\epsilon} \left(\frac{1}{\epsilon^N} \int_{C_\epsilon(x)} |v_\epsilon(y) - v(y)|^2 dy \right) dx \right)^{1/2}.$$

By Tonelli Theorem we can exchange the order of the integrals in order to obtain

$$\int_{\Theta^\epsilon} \left(\frac{1}{\epsilon^N} \int_{C_\epsilon(x)} |v_\epsilon(y) - v(y)|^2 dy \right) dx \leq \|v_\epsilon - v\|^2_{L^2(\Theta)} \frac{1}{\epsilon^N} \int_{C_\epsilon(y)} dx = \|v_\epsilon - v\|^2_{L^2(\Theta)}.$$

Hence

$$\|\overline{v_\epsilon} - \overline{v}\|_{L^2(\Theta^\epsilon)} \leq \|v_\epsilon - v\|_{L^2(\Theta)},$$

and this implies that

$$\|\overline{v_\epsilon} - v\|_{L^2(\Theta^\epsilon)} \leq \|\overline{v_\epsilon} - \overline{v}\|_{L^2(\Theta^\epsilon)} + \|\overline{v} - v\|_{L^2(\Theta^\epsilon)} \leq \|v_\epsilon - v\|_{L^2(\Theta)} + \|\overline{v} - v\|_{L^2(\Theta^\epsilon)},$$

and the right-hand side tends to zero as $\epsilon \to 0$. □

3.2 The triharmonic operator with strong intermediate boundary conditions

We begin by recalling the notation. We consider $\Omega = W \times (-1, 0)$, where W is a bounded C^3-domain in $\mathbb{R}^{N-1}$. Recall that $g_\epsilon(\bar{x}) = \epsilon^\alpha b(\bar{x}/\epsilon)$ for all $\bar{x} \in W$, and that b is a positive, non-constant Y-periodic function of class C^3. Let $\epsilon \in \mathbb{R}$, $\epsilon > 0$ be fixed. We define the perturbed sets

$$\Omega_\epsilon = \{(\bar{x}, x_N) : \bar{x} \in W, -1 < x_N < g_\epsilon(\bar{x})\}. \tag{3.2.1}$$

We shall consider the family of operators

$$H_{\Omega_\epsilon,S} = -\Delta^3 + \mathbb{I},$$

with domain Ω_ϵ, associated with the quadratic form (3.0.1) for all $u, v \in V(\Omega_\epsilon) \equiv W^{3,2}(\Omega_\epsilon) \cap W^{2,2}_0(\Omega_\epsilon)$. Note that the operators $H_{\Omega_\epsilon,S}$ are satisfying strong intermediate boundary conditions on $\partial\Omega_\epsilon$ (see Problem (3.0.4)). Let us set $W = W \times \{0\}$. We denote by $H_{\Omega,D}$ the operator $\Delta^3 + \mathbb{I}$ subject to Dirichlet boundary conditions on W and strong intermediate boundary conditions on $\partial\Omega \setminus W$.
The following theorem shows that there are three possible cases for the spectral convergence of $H_{\Omega_\epsilon,S}$, depending on α. This result is the triharmonic analogous of [19, Theorem 7.3].

Theorem 3.2.1 (Spectral convergence). *With the notation above, the following statements hold true.*

(i) [Spectral stability] *If* $\alpha > 3/2$, *then* $H^{-1}_{\Omega_\epsilon,S} \xrightarrow{C} H^{-1}_{\Omega,S}$, *as* $\epsilon \to 0$.

(ii) [Instability] *If* $\alpha < 3/2$, *then* $H^{-1}_{\Omega_\epsilon,S} \xrightarrow{C} H^{-1}_{\Omega,D}$ *as* $\epsilon \to 0$.

(iii) [Strange term] *If* $\alpha = 3/2$, *then* $H^{-1}_{\Omega_\epsilon,S} \xrightarrow{C} \hat{H}^{-1}_{\Omega}$ *as* $\epsilon \to 0$, *where* $\hat{H}_\Omega$ *is the operator* $\Delta^3 + \mathbb{I}$ *with intermediate boundary conditions on* $\partial\Omega \setminus W$ *and the following boundary conditions on* W: $u = 0$, $\nabla u = 0$, $\frac{\partial^3 u}{\partial x_N^3} + K\frac{\partial^2 u}{\partial x_N^2} = 0$, *where the factor* K *is given by*

$$K = -\int_Y \left(\Delta\left(\frac{\partial^2 V}{\partial y_N^2}\right) + 2\Delta_{N-1}\left(\frac{\partial^2 V}{\partial y_N^2}\right)\right) b(\bar{y})\mathrm{d}\bar{y} = \int_{Y\times(-\infty,0)} |D^3 V|^2 \,\mathrm{d}y,$$

and the function V *is* Y*-periodic in* $\bar{y}$ *and satisfies the following microscopic problem*

$$\begin{cases} \Delta^3 V = 0, & \text{in } Y \times (-\infty, 0), \\ V(\bar{y}, 0) = 0, & \text{on } Y, \\ \frac{\partial V}{\partial y_N}(\bar{y}, 0) = b(\bar{y}), & \text{on } Y, \\ \frac{\partial^3 V}{\partial y_N^3}(\bar{y}, 0) = 0, & \text{on } Y. \end{cases}$$

Proof. (i) has been proved in Theorem (2.2.4). The proof of (ii) is analogous to the proof of [19, Theorem 7.3]. Namely, it is sufficient to show that Condition (C) (see Definition 2.2.1) holds with $V(\Omega_\epsilon) = H^3(\Omega_\epsilon) \cap H^2_0(\Omega_\epsilon)$ and $V(\Omega) = H^3_{0,W}(\Omega)$, where $H^3_{0,W}(\Omega)$ is defined as the set of functions $u \in H^3(\Omega) \cap H^2_0(\Omega)$ such that $\frac{\partial^2 u}{\partial x_N^2} = 0$ on $W \times \{0\}$. By following the proof of [19, Theorem 7.3] it is not difficult to see that condition (2.2.1), condition $(C1)$, $(C2)$ and $(C3)(i), (ii)$ are satisfied. Finally, condition $(C3)(iii)$ follows from Lemma 3.1.1. We shall discuss (iii) in the following sections, see Section 3.2.1 and Section 3.2.2. □

3.2.1 Critical case - Macroscopic problem.

Following the approach in [19], we will use the unfolding method from homogenization theory in order to pass to the limit as $\epsilon \to 0$. Let us define

$$\Phi_\epsilon(\bar{x}, x_N) = (\bar{x}, x_N - h_\epsilon(\bar{x}, x_N)), \quad \text{for all } x = (\bar{x}, x_N) \in \Omega_\epsilon,$$

where h_ϵ is defined by

$$h_\epsilon(\bar{x}, x_N) = \begin{cases} 0, & \text{if } -1 \le x_N \le -\epsilon, \\ g_\epsilon(\bar{x})\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4, & \text{if } -\epsilon \le x_N \le g_\epsilon(\bar{x}). \end{cases}$$

With this definition Φ_ϵ is a diffeomorphism of class C^3, even though the L^∞ norms of the highest order derivatives may blow up, as $\epsilon \longrightarrow 0$. Moreover, one can prove the following

Lemma 3.2.2. *The map Φ_ϵ is a diffeomorphism of class C^3 and there exists a constant $c > 0$ independent of ϵ such that*

$$|h_\epsilon| \le c\epsilon^\alpha, \quad \left|\frac{\partial h_\epsilon}{\partial x_i}\right| \le c\epsilon^{\alpha-1}, \quad \left|\frac{\partial^2 h_\epsilon}{\partial x_i \partial x_j}\right| \le c\epsilon^{\alpha-2}, \quad \left|\frac{\partial^3 h_\epsilon}{\partial x_i \partial x_j \partial x_k}\right| \le c\epsilon^{\alpha-3},$$

for all $\epsilon > 0$ sufficiently small.

Proof. Follows directly from the definition of h_ϵ. □

As in [19, Section 8.1], we introduce the pullback operator T_ϵ from $L^2(\Omega)$ to $L^2(\Omega_\epsilon)$ given by

$$T_\epsilon u = u \circ \Phi_\epsilon,$$

for all $u \in L^2(\Omega)$. See [19, Section 8.1] for the properties of this operator, with the trivial replacement of $W^{2,2}(\Omega)$ with $W^{3,2}(\Omega)$.

Let us recall that the basic setting and definitions in the unfolding method theory have already been introduced in §1.2. We shall use in particular the notation introduced in (1.2.3), in Definition 1.2.9 and Lemma 1.2.10.

Let $W^{3,2}_{\mathrm{Per}_Y,\mathrm{loc}}(Y\times(-\infty,0))$ be the space of functions in $W^{3,2}_{\mathrm{loc}}(\mathbb{R}^N\times(-\infty,0))$ which are Y-periodic in the first $(N-1)$ variables $\bar{y}$. Then we define $W^{3,2}_{\mathrm{loc}}(Y\times(-\infty,0))$ to be the space of functions in $W^{3,2}_{\mathrm{Per}_Y,\mathrm{loc}}(Y\times(-\infty,0))$ restricted to $Y\times(-\infty,0)$. Finally we set

$$w^{3,2}_{\mathrm{Per}_Y}(Y\times(-\infty,0)) := \big\{u \in W^{3,2}_{\mathrm{Per}_Y,\mathrm{loc}}(Y\times(-\infty,0)) : \\ \|D^\gamma u\|_{L^2(Y\times(-\infty,0))} < \infty, \forall |\gamma| = 3\big\}.$$

Lemma 3.2.3. *The following statements hold:*

(i) Let $v_\epsilon \in W^{3,2}(\Omega)$ with $\|v_\epsilon\|_{W^{3,2}(\Omega)} \le M$, for all $\epsilon > 0$. Let V_ϵ be defined by

$$V_\epsilon(\bar{x}, y) = \hat{v}_\epsilon(\bar{x}, y) - \int_Y \hat{v}_\epsilon(\bar{x}, \bar{y}, 0)\, d\bar{y} \\ - \int_Y \nabla_y \hat{v}_\epsilon(\bar{x}, \bar{y}, 0)\, d\bar{y} \cdot y - \sum_{|\alpha|=2} \int_Y D^\alpha_y \hat{v}_\epsilon(\bar{x}, \bar{y}, 0)\, d\bar{y}\, \frac{y^\alpha}{\alpha!},$$

for $(\bar{x}, y) \in \widehat{W_\epsilon} \times Y \times (-1/\epsilon, 0)$. *Then there exists a limit function* $\hat{v} \in L^2(W, w^{3,2}_{\mathrm{Per}_Y}(Y \times (-\infty, 0)))$ *such that*

(a) $\frac{D^\gamma_y V_\epsilon}{\epsilon^{5/2}} \rightharpoonup D^\gamma_y \hat{v}$ *in* $L^2(W \times Y \times (d, 0))$ *as* $\epsilon \to 0$, *for any* $d < 0$, *for any* $\gamma \in \mathbb{N}^N$, $|\gamma| \leq 2$.

(b) $\frac{D^\gamma_y V_\epsilon}{\epsilon^{5/2}} = \frac{D^\gamma_y \hat{v}_\epsilon}{\epsilon^{5/2}} \rightharpoonup D^\gamma_y \hat{v}$ *in* $L^2(W \times Y \times (-\infty, 0))$ *as* $\epsilon \to 0$, *for any* $\gamma \in \mathbb{N}^N$, $|\gamma| = 3$,

where it is understood that functions $V_\epsilon, D^\alpha_y V_\epsilon, D^\gamma_y V_\epsilon$ *are extended by zero in the whole of* $W \times Y \times (-\infty, 0)$ *outside their natural domain of definition* $\widehat{W_\epsilon} \times Y \times (-1/\epsilon, 0)$.

(ii) *Let* $\psi \in W^{1,2}(\Omega)$. *Then*

$$\widehat{(T_\epsilon \psi)_{|\Omega}} \overset{\epsilon \to 0}{\to} \psi(\bar{x}, 0), \quad \text{in } L^2(W \times Y \times (-1, 0)).$$

Proof. The proof is similar to the one in [19, Lemma 8.9]. The main idea is to note that $D^\gamma_y V_\epsilon = D^\gamma_y \hat{v}_\epsilon$ for any $|\gamma| = 3$ and that

$$\left\| \frac{D^\gamma_y V_\epsilon}{\epsilon^{5/2}} \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1/\epsilon, 0))} \leq \|D^\gamma v_\epsilon\|_{L^2(\Omega)} \leq C,$$

for all $\epsilon > 0$, according to the exact integration formula (see Lemma 1.2.10) and the Poincaré inequality. To prove the periodicity it is sufficient to use an argument similar to the one contained in Lemma 4.3 in [43] to $D^2 V_\epsilon$ to obtain that $\nabla_y \hat{v}$ is periodic. Then we find out that $\hat{v}$ is also periodic because

$$\int_Y \nabla_y \hat{v}(\bar{x}, \bar{y}, 0) \mathrm{d}\bar{y} = 0.$$

Indeed, all the functions V_ϵ have this property, and the weak limit preserves the integral mean. □

Let $f_\epsilon \in L^2(\Omega_\epsilon)$ and $f \in L^2(\Omega)$ be such that $f_\epsilon \rightharpoonup f$ in $L^2(\mathbb{R}^N)$ as $\epsilon \to 0$, with the understanding that the functions are extended by zero outside their natural domains. Let $v_\epsilon \in V(\Omega_\epsilon) = W^{3,2}(\Omega_\epsilon) \cap W^{2,2}_0(\Omega_\epsilon)$ be such that

$$H_{\Omega_\epsilon, I} v_\epsilon = f_\epsilon, \tag{3.2.2}$$

for all $\epsilon > 0$ small enough. Then $\|v_\epsilon\|_{W^{3,2}(\Omega_\epsilon)} \leq M$ for all $\epsilon > 0$ sufficiently small, hence, possibly passing to a subsequence there exists $v \in W^{3,2}(\Omega) \cap W^{2,2}_0(\Omega)$ such that $v_\epsilon \rightharpoonup v$ in $W^{3,2}(\Omega)$ and $v_\epsilon \to v$ in $L^2(\Omega)$.

Let $\varphi \in V(\Omega) = W^{3,2}(\Omega) \cap W_0^{2,2}(\Omega)$ be a fixed test function. Since $T_\epsilon\varphi \in V(\Omega_\epsilon)$, by (3.2.2) we get

$$\int_{\Omega_\epsilon} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x + \int_{\Omega_\epsilon} v_\epsilon T_\epsilon \varphi \, \mathrm{d}x = \int_{\Omega_\epsilon} f_\epsilon T_\epsilon \varphi \, \mathrm{d}x, \tag{3.2.3}$$

and passing to the limit as $\epsilon \to 0$ we have that

$$\int_{\Omega_\epsilon} v_\epsilon T_\epsilon \varphi \, \mathrm{d}x \to \int_{\Omega} v\varphi \, \mathrm{d}x, \quad \int_{\Omega_\epsilon} f_\epsilon T_\epsilon \varphi \, \mathrm{d}x \to \int_{\Omega} f\varphi \, \mathrm{d}x. \tag{3.2.4}$$

Now consider the first integral in the right hand-side of (3.2.3). Set $K_\epsilon = W \times (-1, -\epsilon)$. By splitting the integral in three terms corresponding to $\Omega_\epsilon \setminus \Omega$, $\Omega \setminus K_\epsilon$ and K_ϵ and by arguing as in [19, Section 8.3] one can show that

$$\int_{K_\epsilon} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x \to \int_{\Omega} D^3 v : D^3 \varphi \, \mathrm{d}x, \quad \int_{\Omega_\epsilon \setminus \Omega} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x \to 0,$$

as $\epsilon \to 0$. We mention here that to prove that $\int_{\Omega_\epsilon \setminus \Omega} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x \to 0$ it is needed to fully exploit the fact that $\varphi \in H^3(\Omega_\epsilon) \cap H_0^2(\Omega_\epsilon)$; more precisely we can estimate the higher order derivatives of φ by using the Poincaré inequality in the x_N direction and ultimately the boundedness of Sobolev functions along a.a. parallel lines. For example, we can estimate the derivatives of $\varphi \circ \Phi_\epsilon$ in the following way

$$\begin{aligned} \int_{\Omega_\epsilon \setminus \Omega} \left| \frac{\partial^3 v_\epsilon}{\partial x_j \partial x_k \partial x_l} \frac{\partial \varphi}{\partial x_i}(\widehat{\Phi}_\epsilon) \frac{\partial^3 \Phi_\epsilon^i}{\partial x_j \partial x_k \partial x_l} \right|^2 dx &\leq C\epsilon^{-3} \int_{\Omega_\epsilon \setminus \Omega} \left| \frac{\partial \varphi(\Phi_\epsilon)}{\partial x_i} \right|^2 dx \\ &\leq C\epsilon^{-3}\epsilon^3 \int_{\Phi_\epsilon(\Omega_\epsilon \setminus \Omega)} \left| \frac{\partial^2 \varphi}{\partial x_i \partial x_N} \right|^2 dx \\ &\leq C\epsilon^{3/2} \|\varphi\|^2_{H^3(\Omega)} \to 0, \end{aligned}$$

as $\epsilon \to 0$, where in the second inequality we have used the one-dimensional Poincaré inequality, while in the third we have used the boundedness of $\frac{\partial^2 \varphi}{\partial x_i \partial x_N} \in H^1(\Omega)$ in the x_N direction and the fact that $|\Phi_\epsilon(\Omega_\epsilon \setminus \Omega)| \leq C\epsilon^{3/2}$. Let us define

$$Q_\epsilon = \widehat{W_\epsilon} \times (-\epsilon, 0). \tag{3.2.5}$$

We split again the remaining integral in two summands, as follows

$$\int_{\Omega_\epsilon \setminus K_\epsilon} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x = \int_{\Omega_\epsilon \setminus (K_\epsilon \cup Q_\epsilon)} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x + \int_{Q_\epsilon} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x. \tag{3.2.6}$$

It is possible to prove (see [19, Section 8.3]) that

$$\int_{\Omega_\epsilon\setminus(K_\epsilon\cup Q_\epsilon)} D^3v_\epsilon : D^3T_\epsilon\varphi\,\mathrm{d}x \longrightarrow 0,$$

as $\epsilon \longrightarrow 0$. It remains to analyse the limit as $\epsilon \longrightarrow 0$ of the last summand in the right hand-side of (3.2.6). Before doing this we state the following:

Lemma 3.2.4. *For all $y \in Y \times (-1,0)$ and $i,j,k = 1,\dots,N$ the functions $\hat{h}_\epsilon(\bar{x},y)$, $\widehat{\frac{\partial h_\epsilon}{\partial x_i}}(\bar{x},y)$, $\widehat{\frac{\partial^2 h_\epsilon}{\partial x_i\partial x_j}}(\bar{x},y)$ and $\widehat{\frac{\partial^3 h_\epsilon}{\partial x_i\partial x_j\partial x_k}}(\bar{x},y)$ are independent of $\bar{x}$. Moreover, $\hat{h}_\epsilon(\bar{x},y) = O(\epsilon^{3/2})$, $\widehat{\frac{\partial h_\epsilon}{\partial x_i}}(\bar{x},y) = O(\epsilon^{1/2})$ as $\epsilon \longrightarrow 0$,*

$$\epsilon^{1/2}\widehat{\frac{\partial^2 h_\epsilon}{\partial x_i\partial x_j}}(\bar{x},y) \to \frac{\partial^2(b(\bar{y})(y_N+1)^4)}{\partial y_i\partial y_j}, \tag{3.2.7}$$

as $\epsilon \longrightarrow 0$, for all $i,j = 1,\dots,N$, uniformly in $y \in Y \times (-1,0)$, and

$$\epsilon^{3/2}\widehat{\frac{\partial^3 h_\epsilon}{\partial x_i\partial x_j\partial x_k}}(\bar{x},y) \to \frac{\partial^3(b(\bar{y})(y_N+1)^4)}{\partial y_i\partial y_j\partial y_k}, \tag{3.2.8}$$

as $\epsilon \longrightarrow 0$, for all $i,j,k = 1,\dots,N$, uniformly in $y \in Y \times (-1,0)$.

Proof. The proof is just a matter of calculations, which can be carried out as in [19, Lemma 8.27]. For the convenience of reader we write down the more important steps in the computations. First of all note that the first part of the statement involving the asymptotic behaviour of $\widehat{h_\epsilon}$ as $\epsilon \longrightarrow 0$ follows directly from Lemma 3.2.2, because the unfolding does not change the rate of decay in ϵ. Then note that

$$\begin{aligned}\widehat{\frac{\partial^2 h_\epsilon}{\partial x_i\partial x_j}}(\bar{x},y) &= \epsilon^{\alpha-2}\frac{\partial^2 b(\bar{y})}{\partial y_i\partial y_j}(\bar{y})\left(\widehat{\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}}\right)^4 + \epsilon^{\alpha-1}\frac{\partial b(\bar{y})}{\partial y_i}\widehat{\frac{\partial}{\partial x_j}}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4 \\ &\quad + \epsilon^{\alpha-1}\frac{\partial b(\bar{y})}{\partial y_j}\widehat{\frac{\partial}{\partial x_i}}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4 + \epsilon^{\alpha}b(\bar{y})\widehat{\frac{\partial^2}{\partial x_i\partial x_j}}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4,\end{aligned} \tag{3.2.9}$$

$$\begin{aligned}&\widehat{\frac{\partial^3 h_\epsilon}{\partial x_i\partial x_j\partial x_l}}(\bar{x},y) = \epsilon^{\alpha-3}\frac{\partial^3 b(\bar{y})}{\partial y_i\partial y_j\partial y_l}(\bar{y})\left(\widehat{\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}}\right)^4 \\ &+ \epsilon^{\alpha-2}\left[\frac{\partial^2 b(\bar{y})}{\partial y_i\partial y_j}\widehat{\frac{\partial}{\partial x_l}}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4 + \frac{\partial^2 b(\bar{y})}{\partial y_i\partial y_l}\widehat{\frac{\partial}{\partial x_j}}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4 + \frac{\partial^2 b(\bar{y})}{\partial y_j\partial y_l}\widehat{\frac{\partial}{\partial x_i}}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4\right] \\ &+ \epsilon^{\alpha-1}\left[\frac{\partial b(\bar{y})}{\partial y_i}\widehat{\frac{\partial^2}{\partial x_j\partial x_l}}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4 + \frac{\partial b(\bar{y})}{\partial y_j}\widehat{\frac{\partial^2}{\partial x_i\partial x_l}}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4 + \frac{\partial b(\bar{y})}{\partial y_l}\widehat{\frac{\partial^2}{\partial x_i\partial x_j}}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4\right] \\ &+ \epsilon^{\alpha}b(\bar{y})\widehat{\frac{\partial^3}{\partial x_i\partial x_j\partial x_l}}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4.\end{aligned} \tag{3.2.10}$$

Moreover we have the following formulae for the derivatives of $\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4$:

$$\widehat{\frac{\partial}{\partial x_i}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4} = \frac{4}{\epsilon}\frac{(y_N+1)^3}{(\epsilon^{\alpha-1}b(\bar{y})+1)^4}\delta iN - 4\epsilon^{\alpha-2}\frac{\partial b(\bar{y})}{\partial y_i}\frac{(y_N+1)^4}{(\epsilon^{\alpha-1}b(\bar{y})+1)^5}, \quad (3.2.11)$$

$$\widehat{\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4} = \frac{12}{\epsilon^3}\delta_{iN}\delta_{jN}\frac{(y_N+1)^2}{(\epsilon^{\alpha-1}b(\bar{y})+1)^4} - 4\epsilon^{\alpha-3}\frac{\partial^2}{\partial y_i\partial y_j}\frac{(y_N+1)^4}{(\epsilon^{\alpha-1}b(\bar{y})+1)^5}$$
$$- 20\epsilon^{2\alpha-4}\frac{\partial b}{\partial y_i}\frac{\partial b}{\partial y_j}\frac{(y_N+1)^4}{(\epsilon^{\alpha-1}b(\bar{y})+1)^6} - 16\epsilon^{\alpha-3}\frac{(y_N+1)^3}{(\epsilon^{\alpha-1}b(\bar{y})+1)^5}\left(\delta_{iN}\frac{\partial b}{\partial y_j} + \delta_{jN}\frac{\partial b}{\partial y_i}\right), \quad (3.2.12)$$

and

$$\widehat{\frac{\partial^3}{\partial x_i\partial x_j\partial x_l}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^4} = \frac{24}{\epsilon^3}\delta_{iN}\delta_{jN}\delta_{lN}\frac{(1+y_N)}{(\epsilon^{\alpha-1}b(\bar{y})+1)^4}$$
$$- 48\epsilon^{\alpha-4}\frac{(1+y_N)^2}{(\epsilon^{\alpha-1}b(\bar{y})+1)^5}\left(\frac{\partial b}{\partial y_i}\delta_{jN}\delta_{lN} + \frac{\partial b}{\partial y_j}\delta_{iN}\delta_{lN} + \frac{\partial b}{\partial y_l}\delta_{iN}\delta_{jN}\right)$$
$$+ 80\epsilon^{2\alpha-5}\frac{(1+y_N)^3}{(\epsilon^{\alpha-1}b(\bar{y})+1)^6}\left(\frac{\partial b}{\partial y_i}\frac{\partial b}{\partial y_j}\delta_{lN} + \frac{\partial b}{\partial y_j}\frac{\partial b}{\partial y_l}\delta_{iN} + \frac{\partial b}{\partial y_i}\frac{\partial b}{\partial y_l}\delta_{jN}\right)$$
$$- 16\epsilon^{\alpha-4}\frac{(1+y_N)^3}{(\epsilon^{\alpha-1}b(\bar{y})+1)^5}\left(\frac{\partial^2 b}{\partial y_i\partial y_j}\delta_{lN} + \frac{\partial^2 b}{\partial y_j\partial y_l}\delta_{iN} + \frac{\partial^2 b}{\partial y_i\partial y_l}\delta_{jN}\right)$$
$$- 4\epsilon^{\alpha-4}\frac{(1+y_N)^4}{(\epsilon^{\alpha-1}b(\bar{y})+1)^5}\frac{\partial^3 b}{\partial y_i\partial y_j\partial y_l} + 120\epsilon^{3\alpha-6}\frac{(1+y_N)^4}{(\epsilon^{\alpha-1}b(\bar{y})+1)^7}\frac{\partial b}{\partial y_i}\frac{\partial b}{\partial y_j}\frac{\partial b}{\partial y_l}$$
$$+ 20\epsilon^{2\alpha-5}\frac{(1+y_N)^4}{(\epsilon^{\alpha-1}b(\bar{y})+1)^6}\left(\frac{\partial^2 b}{\partial y_i\partial y_j}\frac{\partial b}{\partial y_l} + \frac{\partial^2 b}{\partial y_j\partial y_l}\frac{\partial b}{\partial y_i} + \frac{\partial^2 b}{\partial y_i\partial y_l}\frac{\partial b}{\partial y_j}\right), \quad (3.2.13)$$

By using (3.2.11), (3.2.12) and (3.2.13) in (3.2.9), (3.2.10) and passing to the limit as $\epsilon \longrightarrow 0$ we deduce (3.2.7) and (3.2.8). □

Now we have the tools to prove the following

Lemma 3.2.5. *Let $v_\epsilon \in V(\Omega_\epsilon) = W^{3,2}(\Omega_\epsilon) \cap W_0^{2,2}(\Omega_\epsilon)$ be such that $\|v_\epsilon\|_{W^{3,2}(\Omega_\epsilon)} \leq M$ for all $\epsilon > 0$. Let $v \in W^{3,2}(\Omega) \cap W_0^{2,2}(\Omega)$ the weak limit of $(v_\epsilon)_\epsilon$ in $W^{3,2}(\Omega_\epsilon)$. Let φ be a fixed function in $V(\Omega)$. Let $\hat{v} \in L^2(W, w^{3,2}_{\mathrm{Per}_Y}(Y \times (-\infty, 0)))$ be as in*

Lemma 3.2.3. Then

$$\begin{aligned}
&\int_{Q_\epsilon} D^3 v_\epsilon : D^3(T_\epsilon \varphi)\,\mathrm{d}x \longrightarrow \\
&-3 \int_W \int_{Y\times(-1,0)} D^2_y\Big(\frac{\partial \hat{v}}{\partial y_N}\Big) : D^2_y(b(\bar{y})(1+y_N)^4)\,\mathrm{d}y\,\frac{\partial^2 \varphi}{\partial x_N^2}(\bar{x},0)\mathrm{d}\bar{x} \\
&-\int_W \int_{Y\times(-1,0)} y_N(D^3_y(\hat{v}) : D^3(b(\bar{y})(1+y_N)^4)\,\mathrm{d}y\,\frac{\partial^2 \varphi}{\partial x_N^2}(\bar{x},0)\mathrm{d}\bar{x}.
\end{aligned} \tag{3.2.14}$$

Proof. In this proof we use the index notation and we drop the summation symbols. A direct calculation shows that

$$\begin{aligned}
&\int_{Q_\epsilon} D^3 v_\epsilon : D^3(T_\epsilon \varphi)\,\mathrm{d}x = \int_{Q_\epsilon} \frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h}\frac{\partial^3 (\varphi\circ\Phi_\epsilon)}{\partial x_i \partial x_j \partial x_h}\,\mathrm{d}x \\
&= \int_{Q_\epsilon} \frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h}\frac{\partial^3 \varphi}{\partial x_k \partial x_l \partial x_m}(\Phi_\epsilon(x))\,\frac{\partial \Phi_\epsilon^{(k)}}{\partial x_i}\frac{\partial \Phi_\epsilon^{(l)}}{\partial x_j}\frac{\partial \Phi_\epsilon^{(m)}}{\partial x_h}\,\mathrm{d}x \\
&+ \int_{Q_\epsilon} \frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h}\frac{\partial^2 \varphi}{\partial x_k \partial x_l}(\Phi_\epsilon(x))\left[\frac{\partial \Phi_\epsilon^{(k)}}{\partial x_i}\frac{\partial^2 \Phi_\epsilon^{(l)}}{\partial x_j \partial x_h} + \frac{\partial \Phi_\epsilon^{(k)}}{\partial x_j}\frac{\partial^2 \Phi_\epsilon^{(l)}}{\partial x_i \partial x_h} + \frac{\partial \Phi_\epsilon^{(k)}}{\partial x_h}\frac{\partial^2 \Phi_\epsilon^{(l)}}{\partial x_i \partial x_j}\right]\mathrm{d}x \\
&+ \int_{Q_\epsilon} \frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h}\frac{\partial \varphi}{\partial x_k}(\Phi_\epsilon(x))\,\frac{\partial^3 \Phi_\epsilon^{(k)}}{\partial x_i \partial x_j \partial x_h}\,\mathrm{d}x.
\end{aligned} \tag{3.2.15}$$

We consider separately the three integrals in the right hand side of (3.2.15). Let us remark for future use that

$$\frac{\partial \Phi_\epsilon^{(k)}}{\partial x_i} = \begin{cases} \delta_{ki}, & \text{if } k \neq N, \\ \delta_{Ni} - \frac{\partial h_\epsilon}{\partial x_i}, & \text{if } k = N, \end{cases}$$

whence

$$\frac{\partial^2 \Phi_\epsilon^{(k)}}{\partial x_i \partial x_j} = \begin{cases} 0, & \text{if } k \neq N, \\ -\frac{\partial^2 h_\epsilon}{\partial x_i \partial x_j}, & \text{if } k = N, \end{cases} \qquad \frac{\partial^3 \Phi_\epsilon^{(k)}}{\partial x_i \partial x_j \partial x_h} = \begin{cases} 0, & \text{if } k \neq N, \\ -\frac{\partial^3 h_\epsilon}{\partial x_i \partial x_j \partial x_h}, & \text{if } k = N. \end{cases}$$

Consider now the first integral in the right-hand side of (3.2.15). An application of the exact integration formula (1.2.4) yields

$$\begin{aligned}
&\left|\epsilon\int_W\int_{Y\times(-1,0)}\widehat{\frac{\partial^3 v_\epsilon}{\partial x_i\partial x_j\partial x_h}}\frac{\partial^3\varphi}{\partial x_k\partial x_l\partial x_m}(\widehat{\Phi_\epsilon(x)})\,\widehat{\frac{\partial\Phi_\epsilon^{(k)}}{\partial x_i}}\,\widehat{\frac{\partial\Phi_\epsilon^{(l)}}{\partial x_j}}\,\widehat{\frac{\partial\Phi_\epsilon^{(m)}}{\partial x_h}}\,\mathrm{d}x\right|\\
&=\epsilon^{-2}\left|\int_W\int_{Y\times(-1,0)}\frac{\partial^3\widehat{v_\epsilon}}{\partial y_i\partial y_j\partial y_h}\frac{\partial^3\varphi}{\partial x_k\partial x_l\partial x_m}(\widehat{\Phi_\epsilon(x)})\,\widehat{\frac{\partial\Phi_\epsilon^{(k)}}{\partial x_i}}\,\widehat{\frac{\partial\Phi_\epsilon^{(l)}}{\partial x_j}}\,\widehat{\frac{\partial\Phi_\epsilon^{(m)}}{\partial x_h}}\,\mathrm{d}x\right|\\
&\le c\epsilon^{-2}\epsilon^{5/2}\int_W\int_{Y\times(-1,0)}\left|\epsilon^{-5/2}\frac{\partial^3\widehat{v_\epsilon}}{\partial x_i\partial x_j\partial x_h}\right|\left|\frac{\partial^3\varphi}{\partial x_k\partial x_l\partial x_m}(\widehat{\Phi_\epsilon(x)})\right|\mathrm{d}x\\
&\le c\left\|\epsilon^{-5/2}\frac{\partial^3\widehat{v_\epsilon}}{\partial x_i\partial x_j\partial x_h}\right\|_{L^2(\widehat{W_\epsilon}\times Y\times(-1,0))}\left\|\epsilon^{1/2}\frac{\partial^3\varphi}{\partial x_k\partial x_l\partial x_m}(\widehat{\Phi_\epsilon(x)})\right\|_{L^2(\widehat{W_\epsilon}\times Y\times(-1,0))}\\
&\le c\left\|\frac{\partial^3\varphi}{\partial x_k\partial x_l\partial x_m}(\Phi_\epsilon(x))\right\|_{L^2(Q_\epsilon)}\le c\left\|\frac{\partial^3\varphi}{\partial x_k\partial x_l\partial x_m}\right\|_{L^2(\Phi_\epsilon(Q_\epsilon))},
\end{aligned}$$

which vanishes as $\epsilon\longrightarrow 0$.
Consider now the second integral in the right hand side of (3.2.15). Note that all the terms with $l\neq N$ vanish. Thus, without loss of generality we set $l=N$. With respect to the index k, we prefer to consider separately two cases. If $k\neq N$, then

$$\begin{aligned}
&\left|\int_{Q_\epsilon}\frac{\partial^3 v_\epsilon}{\partial x_i\partial x_j\partial x_h}\frac{\partial^2\varphi}{\partial x_k\partial x_N}(\Phi_\epsilon(x))\,\delta_{ki}\frac{\partial^2\Phi_\epsilon^{(N)}}{\partial x_j\partial x_h}\,\mathrm{d}x\right|\\
&\qquad\le C\epsilon^{-1/2}\left|\int_{Q_\epsilon}\frac{\partial^3 v_\epsilon}{\partial x_i\partial x_j\partial x_h}\frac{\partial^2\varphi}{\partial x_i\partial x_N}(\Phi_\epsilon(x))\,\mathrm{d}x\right|\\
&\qquad\le C\epsilon^{-1/2}\|v_\epsilon\|_{W^{3,2}(Q_\epsilon)}\left(\int_{Q_\epsilon}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_N}(\Phi_\epsilon(x))\right|^2\mathrm{d}x\right)^{1/2}\\
&\qquad\le C\epsilon^{-1/2}\|v_\epsilon\|_{W^{3,2}(Q_\epsilon)}\left(\int_{W\times(-\epsilon,0)}\left|\frac{\partial^2\varphi}{\partial x_i\partial x_N}(x)\right|^2\mathrm{d}x\right)^{1/2}\\
&\qquad\le C\epsilon^{-1/2}\|v_\epsilon\|_{W^{3,2}(Q_\epsilon)}\epsilon\|\varphi\|_{W^{3,2}(\Omega)},
\end{aligned}$$

where the last inequality follows by the one-dimensional Poincaré inequality (note that $\frac{\partial\varphi}{\partial x_N}(\bar{x},0)=0$ because $\varphi\in W_0^{2,2}(\Omega)$, hence $\frac{\partial^2\varphi}{\partial x_k\partial x_N}(\bar{x},0)=0$ for almost all $\bar{x}\in W$). Since the right hand side vanishes as $\epsilon\longrightarrow 0$, we deduce that all the integrals with $k\neq N$ vanish as well.

If instead $k = N$, then the unfolding method yields

$$\begin{aligned}
&\int_{Q_\epsilon} \frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h} \frac{\partial^2 \varphi}{\partial x_N^2}(\Phi_\epsilon(x)) \left[\frac{\partial \Phi_\epsilon^{(N)}}{\partial x_i} \frac{\partial^2 \Phi_\epsilon^{(N)}}{\partial x_j \partial x_h} + \frac{\partial \Phi_\epsilon^{(N)}}{\partial x_j} \frac{\partial^2 \Phi_\epsilon^{(N)}}{\partial x_i \partial x_h} + \frac{\partial \Phi_\epsilon^{(N)}}{\partial x_h} \frac{\partial^2 \Phi_\epsilon^{(N)}}{\partial x_i \partial x_j} \right] \mathrm{d}x \\
&= \epsilon^{-5} \int_{\widehat{W_\epsilon} \times Y \times (-1,0)} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 \varphi}{\partial x_N^2}(\widehat{\Phi_\epsilon}(y)) \left[\frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_j \partial y_h} \right. \\
&\qquad\qquad \left. + \frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_j} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i \partial y_h} + \frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_h} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i \partial y_j} \right] \mathrm{d}\bar{x} \mathrm{d}y,
\end{aligned}$$

and since we are summing on the indexes $i, j, h \in 1, \ldots, N$, the right-hand side equals

$$3\epsilon^{-5} \int_{\widehat{W_\epsilon} \times Y \times (-1,0)} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 (\varphi(\widehat{\Phi_\epsilon}(y)))}{\partial x_N^2} \frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_j \partial y_h} \mathrm{d}\bar{x} \mathrm{d}y.$$

Note now that

$$\frac{\partial \widehat{\Phi_\epsilon}^{(k)}}{\partial y_i} = \begin{cases} \epsilon \delta_{ki}, & \text{if } k \neq N, \\ \epsilon \delta_{Ni} - \epsilon \widehat{\frac{\partial h_\epsilon}{\partial x_i}}, & \text{if } k = N, \end{cases} \qquad \frac{\partial^2 \widehat{\Phi_\epsilon}^{(k)}}{\partial y_i \partial y_j} = \begin{cases} 0, & \text{if } k \neq N, \\ -\epsilon^2 \widehat{\frac{\partial^2 h_\epsilon}{\partial x_i \partial x_j}}, & \text{if } k = N. \end{cases}$$

Thus, we have

$$\begin{aligned}
&3\epsilon^{-5} \int_{\widehat{W_\epsilon} \times Y \times (-1,0)} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 (\varphi(\widehat{\Phi_\epsilon(y)}))}{\partial x_N^2} \frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_j \partial y_h} \mathrm{d}\bar{x} \mathrm{d}y \\
&= -3\epsilon^{-2} \int_{\widehat{W_\epsilon} \times Y \times (-1,0)} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 (\varphi(\widehat{\Phi_\epsilon}(y)))}{\partial x_N^2} \left(\delta_{Ni} - \widehat{\frac{\partial h_\epsilon}{\partial x_i}} \right) \widehat{\frac{\partial^2 h_\epsilon}{\partial x_j \partial x_h}} \mathrm{d}\bar{x} \mathrm{d}y \\
&= -3 \int_{\widehat{W_\epsilon} \times Y \times (-1,0)} \epsilon^{-5/2} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_N \partial y_j \partial y_h} \overline{T_\epsilon \left(\frac{\partial^2 \varphi}{\partial x_N^2} \right)} \epsilon^{1/2} \overline{\frac{\partial^2 h_\epsilon}{\partial x_j \partial x_h}} \mathrm{d}\bar{x} \mathrm{d}y \\
&+ 3 \int_{\widehat{W_\epsilon} \times Y \times (-1,0)} \epsilon^{-5/2} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 (\varphi(\widehat{\Phi_\epsilon}(y)))}{\partial x_N^2} \widehat{\frac{\partial h_\epsilon}{\partial x_i}} \epsilon^{1/2} \overline{\frac{\partial^2 h_\epsilon}{\partial x_j \partial x_h}} \mathrm{d}\bar{x} \mathrm{d}y.
\end{aligned} \tag{3.2.16}$$

The last summand in the right-hand side of (3.2.16) vanishes as $\epsilon \longrightarrow 0$. Indeed,

$$\begin{aligned}
&\left| \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \epsilon^{-5/2} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 \varphi}{\partial x_N^2}(\widehat{\Phi_\epsilon}(y)) \widehat{\frac{\partial h_\epsilon}{\partial x_i}} \epsilon^{1/2} \widehat{\frac{\partial^2 h_\epsilon}{\partial x_j \partial x_h}} \, \mathrm{d}\bar{x}\mathrm{d}y \right| \\
&\leq C\epsilon^{1/2} \left| \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \epsilon^{-5/2} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 \varphi}{\partial x_N^2}(\widehat{\Phi_\epsilon}(y)) \, \mathrm{d}\bar{x}\mathrm{d}y \right| \\
&\leq C\epsilon^{1/2} \left\| \epsilon^{-5/2} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \right\|_{L^2(\widehat{W_\epsilon}\times Y\times(-1,0))} \left\| \frac{\partial^2 \varphi}{\partial x_N^2}(\widehat{\Phi_\epsilon}(y)) \right\|_{L^2(\widehat{W_\epsilon}\times Y\times(-1,0))} \\
&\leq C'\epsilon^{1/2}\epsilon^{-1/2} \left\| \frac{\partial^2 \varphi}{\partial x_N^2}(\Phi_\epsilon(x)) \right\|_{L^2(Q_\epsilon)} \leq C'\epsilon^{1/2} \|\varphi\|_{W^{3,2}(\Omega)} \longrightarrow 0,
\end{aligned}$$

as $\epsilon \longrightarrow 0$. The only remaining term in (3.2.16) is

$$-3 \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \epsilon^{-5/2} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_N \partial y_j \partial y_h} \widehat{T_\epsilon\left(\frac{\partial^2 \varphi}{\partial x_N^2}\right)} \epsilon^{1/2} \widehat{\frac{\partial^2 h_\epsilon}{\partial x_j \partial x_h}} \, \mathrm{d}\bar{x}\mathrm{d}y,$$

which tends to

$$-3 \int_W \int_{Y\times(-1,0)} D_y^2\Big(\frac{\partial \hat{v}}{\partial y_N}\Big) : D_y^2(b(\bar{y})(1+y_N)^4) \, \mathrm{d}y \, \frac{\partial^2 \varphi}{\partial x_N^2}(\bar{x},0)\mathrm{d}\bar{x},$$

as $\epsilon \longrightarrow 0$, by Lemma 3.2.3 (i) and (ii), and by Lemma 3.2.4.
It remains to treat only the third integral in the right-hand side of (3.2.15). We apply the unfolding method in order to obtain

$$\begin{aligned}
&\epsilon \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \widehat{\frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h}} \frac{\partial \varphi}{\partial x_N}(\widehat{\Phi_\epsilon}(y)) \widehat{\frac{\partial^3 \Phi_\epsilon^{(N)}}{\partial x_i \partial x_j \partial x_h}} \, \mathrm{d}\bar{x}\mathrm{d}y \\
&= \epsilon^{-2} \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \frac{\partial^3 \widehat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h} \frac{\partial \varphi}{\partial x_N}(\widehat{\Phi_\epsilon}(y)) \widehat{\frac{\partial^3 \Phi_\epsilon^{(N)}}{\partial x_i \partial x_j \partial x_h}} \, \mathrm{d}\bar{x}\mathrm{d}y \\
&= \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \left[\epsilon^{-5/2} \frac{\partial^3 \widehat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h}\right] \left[\epsilon^{-1} \frac{\partial \varphi}{\partial x_N}(\widehat{\Phi_\epsilon}(y))\right] \left[\epsilon^{3/2} \widehat{\frac{\partial^3 \Phi_\epsilon^{(N)}}{\partial x_i \partial x_j \partial x_h}}\right] \mathrm{d}\bar{x}\mathrm{d}y.
\end{aligned}$$

By former results it is clear that

$$\epsilon^{-5/2} \frac{\partial^3 \widehat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h} \longrightarrow \frac{\partial^3 \hat{v}}{\partial y_i \partial y_j \partial y_h}, \quad \epsilon^{3/2} \widehat{\frac{\partial^3 \Phi_\epsilon^{(N)}}{\partial x_i \partial x_j \partial x_h}} \longrightarrow -\frac{\partial^3 (b(\bar{y})(1+y_N)^4)}{\partial y_i \partial y_j \partial y_h},$$

as $\epsilon \longrightarrow 0$, where the limits are in $L^2(W \times Y)$ and in $L^\infty(W \times Y)$, respectively. To conclude the proof is then sufficient to show that

$$\epsilon^{-1}\frac{\partial \varphi}{\partial x_N}(\widehat{\Phi_\epsilon(y)}) \longrightarrow y_N \frac{\partial^2 \varphi}{\partial x_N^2}(\bar{x}, 0),$$

strongly in $L^2(\widehat{W_\epsilon} \times Y \times (-1,0))$, as $\epsilon \longrightarrow 0$.
To see this, set first $\psi = \frac{\partial \varphi}{\partial x_N}$ and note that $\psi \in H^2 \cap H_0^1(\Omega)$. Assume for the moment $\psi \in H^2 \cap H_0^1(\Omega) \cap C^\infty(\Omega)$. Now we estimate

$$\begin{aligned}
&\int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \left| \frac{\psi(\widehat{\Phi_\epsilon})}{\epsilon} - \frac{\partial \psi}{\partial x_N}(\bar{x},0) y_N \right|^2 \mathrm{d}\bar{x}\, \mathrm{d}y \\
&= \sum_{k\in I_{W,\epsilon}} \int_{-1}^0 \int_{C_\epsilon^k \times Y} \left| \frac{\psi(\epsilon\left[\frac{\bar{x}}{\epsilon}\right] + \epsilon\bar{y}, \epsilon y_N - h_\epsilon(\epsilon\left[\frac{\bar{x}}{\epsilon}\right] + \epsilon\bar{y}, \epsilon y_N))}{\epsilon} - \frac{\partial \psi}{\partial x_N}(\bar{x},0) y_N \right|^2 \mathrm{d}\bar{y}\mathrm{d}\bar{x}\mathrm{d}y_N \\
&= \int_{-1}^0 \sum_{k\in I_{W,\epsilon}} \int_{C_\epsilon^k} \int_{C_\epsilon^k} \left| \frac{\psi(\bar{z}, \epsilon y_N - h_\epsilon(\bar{z}, \epsilon y_N))}{\epsilon} - \frac{\partial \psi}{\partial x_N}(\bar{x},0) y_N \right|^2 \frac{\mathrm{d}\bar{z}}{\epsilon^{N-1}}\, \mathrm{d}\bar{x}\, \mathrm{d}y_N. \qquad (*)
\end{aligned}$$

By the Lagrange Theorem there exists $\xi \in (0, \epsilon y_N - h_\epsilon(\bar{z}, \epsilon y_N))$ such that

$$\begin{aligned}
\psi(\bar{z}, \epsilon y_N - h_\epsilon(\bar{z}, \epsilon y_N)) &= \psi(\bar{z}, \epsilon y_N - h_\epsilon(\bar{z}, \epsilon y_N)) - \psi(\bar{z}, 0) \\
&= \frac{\partial \psi}{\partial x_N}(\bar{z}, \xi)(\epsilon y_N - h_\epsilon(\bar{z}, \epsilon y_N)).
\end{aligned}$$

Thus, $(*)$ equals

$$\begin{aligned}
&\int_{-1}^0 \sum_{k\in I_{W,\epsilon}} \int_{C_\epsilon^k} \int_{C_\epsilon^k} \left| \frac{\partial \psi}{\partial x_N}(\bar{z}, \xi) \frac{(\epsilon y_N - h_\epsilon(\bar{z}, \epsilon y_N))}{\epsilon} - \frac{\partial \psi}{\partial x_N}(\bar{x},0) y_N \right|^2 \frac{\mathrm{d}\bar{z}}{\epsilon^{N-1}}\, \mathrm{d}\bar{x}\, \mathrm{d}y_N \\
&\le C \int_{-1}^0 \sum_{k\in I_{W,\epsilon}} \int_{C_\epsilon^k} \int_{C_\epsilon^k} \left| \left(\frac{\partial \psi}{\partial x_N}(\bar{z}, \xi) - \frac{\partial \psi}{\partial x_N}(\bar{x},0) \right) y_N \right|^2 \frac{\mathrm{d}\bar{z}}{\epsilon^{N-1}}\, \mathrm{d}\bar{x}\, \mathrm{d}y_N \\
&+ C \int_{-1}^0 \sum_{k\in I_{W,\epsilon}} \int_{C_\epsilon^k} \int_{C_\epsilon^k} \left| \frac{\partial \psi}{\partial x_N}(\bar{z}, \xi) \frac{h_\epsilon(\bar{z}, \epsilon y_N))}{\epsilon} \right|^2 \frac{\mathrm{d}\bar{z}}{\epsilon^{N-1}}\, \mathrm{d}\bar{x}\, \mathrm{d}y_N,
\end{aligned} \qquad (3.2.17)$$

and the right-hand side of (3.2.17) is estimated from above by

$$
\begin{aligned}
&C\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial\psi}{\partial x_N}(\bar z,\xi)-\frac{\partial\psi}{\partial x_N}(\bar z,0)\right|^2\frac{\mathrm{d}\bar z}{\epsilon^{N-1}}\,\mathrm{d}\bar x\,\mathrm{d}y_N\\
&+C\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial\psi}{\partial x_N}(\bar z,0)-\frac{\partial\psi}{\partial x_N}(\bar x,0)\right|^2\frac{\mathrm{d}\bar z}{\epsilon^{N-1}}\,\mathrm{d}\bar x\,\mathrm{d}y_N \qquad (3.2.18)\\
&+C\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial\psi}{\partial x_N}(\bar z,\xi)\frac{h_\epsilon(\bar z,\epsilon y_N))}{\epsilon}\right|^2\frac{\mathrm{d}\bar z}{\epsilon^{N-1}}\,\mathrm{d}\bar x\,\mathrm{d}y_N.
\end{aligned}
$$

We estimate separately the three integrals appearing in (3.2.18). By the Fundamental Theorem of Calculus we can estimate the first integral as follows

$$
\begin{aligned}
&\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\int_0^{\xi}\frac{\partial^2\psi}{\partial x_N^2}(\bar z,t)dt\right|^2\frac{\mathrm{d}\bar z}{\epsilon^{N-1}}\,\mathrm{d}\bar x\,\mathrm{d}y_N\\
&\le C\epsilon\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\int_{-C\epsilon}^{0}\left|\frac{\partial^2\psi}{\partial x_N^2}(\bar z,t)\right|^2 dt\frac{\mathrm{d}\bar z}{\epsilon^{N-1}}\,\mathrm{d}\bar x\le C\epsilon\left\|\frac{\partial^2\psi}{\partial x_N^2}(\bar z,t)\right\|^2_{L^2(W\times(-C\epsilon,0))}.
\end{aligned}
\qquad (3.2.19)
$$

Then the second integral in (3.2.18) can be rewritten as

$$
\begin{aligned}
&\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\frac{\partial\psi}{\partial x_N}(\bar z,0)-\frac{\partial\psi}{\partial x_N}(\bar x,0)}{|\bar z-\bar x|^{N/2}}\right|^2|\bar z-\bar x|^N\frac{\mathrm{d}\bar z}{\epsilon^{N-1}}\,\mathrm{d}\bar x\,\mathrm{d}y_N\\
&\qquad\le C\epsilon\left\|\frac{\partial\psi}{\partial x_N}(\cdot,0)\right\|^2_{B_2^{1/2}(W)}\le C\epsilon\left\|\frac{\partial\psi}{\partial x_N}\right\|^2_{H^1(\Omega)},
\end{aligned}
\qquad (3.2.20)
$$

where $B_2^{1/2}(W)$ is the Besov space of parameters 2, 1/2, which is estimated from above by the H^1 norm in Ω according to the Trace Theorem. Finally we estimate the third integral in the right-hand side of (3.2.18) as follows:

$$
\begin{aligned}
&\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial\psi}{\partial x_N}(\bar z,\xi)\frac{h_\epsilon(\bar z,\epsilon y_N))}{\epsilon}\frac{\mathrm{d}\bar z}{\epsilon^{N-1}}\right|\mathrm{d}\bar x\,\mathrm{d}y_N\\
&\qquad\le C\epsilon\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\sup_{\xi\in(-C\epsilon,0)}\left|\frac{\partial\psi}{\partial x_N}(\bar z,\xi)\right|^2\frac{\mathrm{d}\bar z}{\epsilon^{N-1}}\,\mathrm{d}\bar x\,\mathrm{d}y_N\\
&\qquad\le C\epsilon\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\left\|\frac{\partial\psi}{\partial x_N}(\bar z,\cdot)\right\|^2_{L^\infty(-1,0)}\mathrm{d}\bar z\le C\epsilon\left\|\frac{\partial\psi}{\partial x_N}\right\|^2_{H^1(\Omega)},
\end{aligned}
$$

where in the last inequality we have used the boundedness of Sobolev functions on almost all vertical lines. We finally deduce that

$$\int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \left|\frac{\psi(\widehat{\Phi_\epsilon})}{\epsilon} - \frac{\partial\psi}{\partial x_N}(\bar{x},0)y_N\right|^2 \mathrm{d}\bar{x}\,\mathrm{d}y \leq C\epsilon \left\|\frac{\partial\psi}{\partial x_N}\right\|^2_{H^1(\Omega)},$$

and the right-hand side tends to zero as $\epsilon \to 0$. This concludes the proof for smooth functions. If ψ is a general $H^2(\Omega)\cap H^1_0(\Omega)$ function, according to [35, Theorem 9, p.77], we can approximate it by a sequence of functions $(\psi_n)_n$ in $H^2(\Omega)\cap C^\infty(\Omega)$ such that $\psi_n \to \psi$ in $H^2(\Omega)$ and $\mathrm{Tr}_{\partial\Omega}D^\gamma\psi_n = \mathrm{Tr}_{\partial\Omega}D^\gamma\psi$ for all $n\in\mathbb{N}$, for all $|\gamma|\leq 1$. Then

$$\begin{aligned}
\left\|\frac{\psi(\widehat{\Phi}_\epsilon(y))}{\epsilon} - y_N\frac{\partial\psi}{\partial x_N}(\bar{x},0)\right\|_{L^2(\widehat{W}_\epsilon\times Y\times(-1,0))} &\leq \left\|\frac{\psi(\widehat{\Phi}_\epsilon(y))}{\epsilon} - \frac{\psi_n(\widehat{\Phi}_\epsilon(y))}{\epsilon}\right\|_{L^2(\widehat{W}_\epsilon\times Y\times(-1,0))} \\
+\left\|\frac{\psi_n(\widehat{\Phi}_\epsilon(y))}{\epsilon} - y_N\frac{\partial\psi_n}{\partial x_N}(\bar{x},0)\right\|_{L^2(\widehat{W}_\epsilon\times Y\times(-1,0))} &+ \left\|\frac{\partial(\psi_n-\psi)}{\partial x_N}(\bar{x},0)y_N\right\|_{L^2(\widehat{W}_\epsilon\times Y\times(-1,0))}.
\end{aligned} \tag{3.2.21}$$

We first consider

$$\left\|\frac{\psi(\widehat{\Phi}_\epsilon(y))}{\epsilon} - \frac{\psi_n(\widehat{\Phi}_\epsilon(y))}{\epsilon}\right\|_{L^2(\widehat{W}_\epsilon\times Y\times(-1,0))}.$$

By the exact integration formula we can directly consider

$$\begin{aligned}
&\frac{1}{\epsilon^2}\frac{1}{\epsilon}\int_W\int_{-\epsilon}^0 |\psi(\Phi_\epsilon(x)) - \psi_n(\Phi_\epsilon(x))|^2\, d\bar{x}dx_N \\
&\leq \frac{C}{\epsilon^3}\int_W\int_{-\epsilon}^{-h_\epsilon(\bar{x},0)} |\psi(x)-\psi_n(x)|^2\, d\bar{x}dx_N \\
&\leq \frac{C}{\epsilon^3}\int_W\int_{-\epsilon}^0 |\psi(x)-\psi_n(x)|^2\, d\bar{x}dx_N.
\end{aligned} \tag{3.2.22}$$

Now by the Poincaré inequality and the fact that $D^\gamma\psi(\bar{x},0) - D^\gamma\psi_n(\bar{x},0) = 0$ for almost all $\bar{x}\in W$, for all $|\gamma|\leq 1$ (which is a consequence of the choice of ψ_n), we deduce that the right-hand side of (3.2.22) is estimated from above as follows

$$\frac{C\epsilon^4}{\epsilon^3}\int_W\int_{-\epsilon}^0 \left|\frac{\partial^2\psi}{\partial x_N^2}(\bar{x},x_N) - \frac{\partial^2\psi_n}{\partial x_N^2}(\bar{x},x_N)\right|^2 d\bar{x}dx_N \leq C\epsilon\|\psi-\psi_n\|_{H^2(\Omega)}.$$

Going back to (3.2.21) we have

$$\begin{aligned}
&\left\| \frac{\psi(\widehat{\Phi}_\epsilon(y))}{\epsilon} - y_N \frac{\partial \psi}{\partial x_N}(\bar{x}, 0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \\
&\leq C\|\psi - \psi_n\|_{H^2(\Omega)} + \left\| \frac{\psi_n(\widehat{\Phi}_\epsilon(y))}{\epsilon} - y_N \frac{\partial \psi_n}{\partial x_N}(\bar{x}, 0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \\
&+ \left\| y_N \frac{\partial \psi_n}{\partial x_N}(\bar{x}, 0) - y_N \frac{\partial \psi}{\partial x_N}(\bar{x}, 0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))}.
\end{aligned} \tag{3.2.23}$$

Fix $\delta > 0$ arbitrarily small. Choose n big enough so that $\|\psi - \psi_n\|_{H^2(\Omega)} \leq \delta$ and

$$\left\| y_N \frac{\partial \psi_n}{\partial x_N}(\bar{x}, 0) - y_N \frac{\partial \psi}{\partial x_N}(\bar{x}, 0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \leq \delta.$$

Note that this is possible by the Trace Theorem and the convergence of ψ_n to ψ in H^2. Now, with the choice of n and δ above take $\epsilon > 0$ small enough in such a way that

$$\left\| \frac{\psi_n(\widehat{\Phi}_\epsilon(y))}{\epsilon} - y_N \frac{\partial \psi_n}{\partial x_N}(\bar{x}, 0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \leq \delta.$$

This is possible by the previous discussion on the behaviour of smooth functions. Finally we deduce that for every δ there exists $\epsilon > 0$ such that

$$\left\| \frac{\psi(\widehat{\Phi}_\epsilon(y))}{\epsilon} - y_N \frac{\partial \psi}{\partial x_N}(\bar{x}, 0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \leq 3\delta. \tag{3.2.24}$$

By the arbitrariness of δ in (3.2.24) we deduce that

$$\lim_{\epsilon \to 0} \left\| \frac{\psi(\widehat{\Phi}_\epsilon(y))}{\epsilon} - y_N \frac{\partial \psi}{\partial x_N}(\bar{x}, 0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} = 0,$$

concluding the proof. □

We summarize the previous discussion in the following

Theorem 3.2.6. *Let $f_\epsilon \in L^2(\Omega_\epsilon)$ and $f \in L^2(\Omega)$ be such that $f_\epsilon \rightharpoonup f$ in $L^2(\Omega)$. Let $v_\epsilon \in H^3(\Omega_\epsilon) \cap H^2_0(\Omega_\epsilon)$ be the solutions to $H_{\Omega_\epsilon} v_\epsilon = f_\epsilon$. Then, possibly passing to a subsequence, there exists $v \in H^3(\Omega) \cap H^2_0(\Omega)$ and $\hat{v} \in L^2(W, w^{3,2}_{Per_Y}(Y \times (\infty, 0)))$*

such that $v_\epsilon \rightharpoonup v$ *in* $H^3(\Omega)$, $v_\epsilon \to v$ *in* $L^2(\Omega)$ *and such that statements* (a) *and* (b) *in Lemma 3.2.3 hold. Moreover,*

$$-3\int_W\int_{Y\times(-1,0)} D_y^2\Big(\frac{\partial \hat{v}}{\partial y_N}\Big) : D_y^2(b(\bar{y})(1+y_N)^4)\,\mathrm{d}y\,\frac{\partial^2\varphi}{\partial x_N^2}(\bar{x},0)\mathrm{d}\bar{x}$$
$$-\int_W\int_{Y\times(-1,0)} y_N(D_y^3(\hat{v}) : D^3(b(\bar{y})(1+y_N)^4)\,\mathrm{d}y\,\frac{\partial^2\varphi}{\partial x_N^2}(\bar{x},0)\mathrm{d}\bar{x}$$
$$+\int_\Omega D^3 v : D^3\varphi + u\varphi\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x.$$

for all $\varphi \in H^3(\Omega)\cap H_0^2(\Omega)$.

Notation. Let $g(y) = b(\bar{y})(1+y_N)^4$ for all $y \in Y\times(-1,0)$. In the sequel we will refer to

$$-\int_W\int_{Y\times(-1,0)}\left(3D_y^2\Big(\frac{\partial \hat{v}}{\partial y_N}\Big) : D_y^2 g(y) + y_N(D_y^3(\hat{v}) : D^3 g(y))\right)\mathrm{d}y\,\frac{\partial^2\varphi}{\partial x_N^2}(\bar{x},0)\mathrm{d}\bar{x}, \quad (3.2.25)$$

as the *strange term* appearing in the homogenization.

3.2.2 Critical case - Microscopic problem.

Let $\psi \in C^\infty(\overline{W}\times\overline{Y}\times]-\infty,0])$ be such that $\mathrm{supp}\,\psi \subset C\times\overline{Y}\times[d,0]$ for some compact set $C\subset W$ and $d\in]-\infty,0[$, and $\psi(\bar{x},\bar{y},0) = \nabla\psi(\bar{x},\bar{y},0) = 0$ for all $(\bar{x},\bar{y})\in W\times Y$. Assume also ψ to be Y-periodic in the variable $\bar{y}$. We set

$$\psi_\epsilon(x) = \epsilon^{\frac{5}{2}}\psi\Big(\bar{x},\frac{\bar{x}}{\epsilon},\frac{x_N}{\epsilon}\Big),$$

for all $\epsilon > 0$, $x\in W\times]-\infty,0]$. Then $T_\epsilon\psi_\epsilon \in V(\Omega_\epsilon)$ for sufficiently small ϵ, hence we can use it in the weak formulation of the problem in Ω_ϵ, getting

$$\int_{\Omega_\epsilon} D^3 v_\epsilon : D^3 T_\epsilon\psi_\epsilon\,\mathrm{d}x + \int_{\Omega_\epsilon} v_\epsilon T_\epsilon\psi_\epsilon\,\mathrm{d}x = \int_{\Omega_\epsilon} f_\epsilon T_\epsilon\psi_\epsilon\,\mathrm{d}x.$$

As in [19], it is possible to prove that

$$\int_{\Omega_\epsilon} v_\epsilon T_\epsilon\psi_\epsilon\,\mathrm{d}x \to 0, \qquad \int_{\Omega_\epsilon} f_\epsilon T_\epsilon\psi_\epsilon\,\mathrm{d}x \to 0, \quad (3.2.26)$$

as $\epsilon\to 0$, and

$$\int_{\Omega_\epsilon\setminus\Omega} D^3 v_\epsilon : D^3 T_\epsilon\psi_\epsilon\, dx \to 0, \quad (3.2.27)$$

as $\epsilon \longrightarrow 0$. Moreover, a slight modification of [19, Lemma 8.47] yields

$$\int_\Omega D^3 v_\epsilon : D^3 T_\epsilon \psi_\epsilon \, dx \to \int_{W\times Y\times(-\infty,0)} D^3_y \hat{v}(\bar{x}, y) : D^3_y \psi(\bar{x}, y) \, d\bar{x}dy. \tag{3.2.28}$$

Now we have the following

Theorem 3.2.7. *Let $\hat{v} \in L^2(W, w^{3,2}_{Per_Y}(Y \times (-\infty, 0)))$ be the function from Theorem 3.2.6. Then*

$$\int_{W\times Y\times(-\infty,0)} D^3_y \hat{v}(\bar{x}, y) : D^3_y \psi(\bar{x}, y) \, d\bar{x}dy = 0,$$

for all $\psi \in L^2(W, w^{3,2}_{Per_Y}(Y \times (-\infty, 0)))$ such that $\psi(\bar{x}, \bar{y}, 0) = \nabla\psi(\bar{x}, \bar{y}, 0) = 0$ on $W \times Y$. Moreover, for any $i = 1, \dots, N-1$, we have

$$\frac{\partial^2 \hat{v}}{\partial y_i \partial y_N}(\bar{x}, \bar{y}, 0) = -\frac{\partial b}{\partial y_i}(\bar{y})\frac{\partial^2 v}{\partial x_N^2}(\bar{x}, 0) \qquad \text{on } W \times Y, \tag{3.2.29}$$

and

$$\frac{\partial^2 \hat{v}}{\partial y_i \partial y_j}(\bar{x}, \bar{y}, 0) = 0, \qquad \text{on } W \times Y, \tag{3.2.30}$$

for all $j = 1, \dots, N-1$.

Proof. The first part of the statement follows from (3.2.26), (3.2.27), (3.2.28) (see also [19, Theorem 8.53]). In order to prove formulas (3.2.29) and (3.2.30) it is sufficient to apply Lemma (3.1.1), case $\alpha = 3/2$, on the sequence $(v_\epsilon)_{\epsilon>0}$, since $v_\epsilon \in H^3(\Omega_\epsilon) \cap H^2_0(\Omega_\epsilon)$ and $\|v_\epsilon\|_{H^3(\Omega_\epsilon)} \leq C$, for all $\epsilon > 0$. □

Now we have the following

Lemma 3.2.8. *There exists $V \in w^{3,2}_{Per_Y}(Y \times (-\infty, 0))$ satisfying the equation*

$$\int_{Y\times(-\infty,0)} D^3 V : D^3 \psi \, dy = 0, \tag{3.2.31}$$

for all $\psi \in w^{3,2}_{Per_Y}(Y \times (-\infty, 0))$ such that $\psi(\bar{y}, 0) = 0 = \nabla\psi(\bar{y}, 0)$ on Y, and the boundary conditions

$$\begin{cases} V(\bar{y}, 0) = 0, & \text{on } Y, \\ \frac{\partial V}{\partial y_N}(\bar{y}, 0) = b(\bar{y}), & \text{on } Y. \end{cases}$$

Function V is unique up to a sum of a monomial in y_N of the form ay_N^2. Moreover $V \in W^{6,2}_{Per_Y}(Y \times (d, 0))$ for any $d < 0$ and it satisfies the equation

$$\Delta^3 V = 0, \qquad \text{in } Y \times (d, 0),$$

subject to the boundary condition

$$\frac{\partial^3 V}{\partial y_N^3}(\bar{y}, 0) = 0, \qquad \text{on } Y.$$

Proof. Similar to the proof of [19, Lemma 8.60]. By a standard minimizing procedure (see [19, Lemma 8.60]) it is easy to prove that there exists a function $V \in w^{3,2}_{Per_Y}(Y\times(-\infty, 0))$ satisfying (3.2.31). With regard to the uniqueness, note that if there exist two functions V_1 and V_2 satisfying (3.2.31) then by using $\tilde{V} = V_1 - V_2$ as test function in (3.2.31) we have

$$\int_{Y\times(-\infty,0)} D^3 V_1 : D^3 \tilde{V}\, \mathrm{d}y = 0 = \int_{Y\times(-\infty,0)} D^3 V_2 : D^3 \tilde{V}\, \mathrm{d}y,$$

hence,

$$\int_{Y\times(-\infty,0)} |D^3 \tilde{V}|^2\, \mathrm{d}y = 0.$$

From the periodicity in $\bar{y}$ and the fact that $\frac{\partial \tilde{V}}{\partial y_N}(\bar{y}, 0) = 0$ for almost all $\bar{y} \in Y$ we easily deduce that $V_1(y) - V_2(y) = a y_N^2$, for some $a \in \mathbb{R}$.

Regularity of V is classical. Finally, by using in the weak formulation (3.2.31) test functions ψ with bounded support in the y_N direction and by formula (4.1.12) we deduce that

$$\int_{Y\times(-\infty,0)} D^3 V : D^3 \psi\, \mathrm{d}y = -\int_{Y\times(-\infty,0)} \Delta^3 V \psi\, \mathrm{d}y + \int_Y \frac{\partial^3 V}{\partial y_N^3} \frac{\partial^2 \psi}{\partial y_N^2}\, d\bar{y}.$$

By the arbitrariness of ψ we then deduce that V is triharmonic and satisfies the boundary conditions in the statement of the Lemma. □

Theorem 3.2.9. *Let V be the function defined in Lemma 3.2.8. Let $v, \hat{v}$ be as in Theorem 3.2.6. Then*

$$\hat{v}(\bar{x}, y) = -V(y) \frac{\partial^2 v}{\partial x_N^2}(\bar{x}, 0) + a(\bar{x}) y_N^2.$$

In particular, the strange term (3.2.25) *is given by*

$$-\int_Y \left(\Delta\left(\frac{\partial^2 V}{\partial y_N^2} \right) + 2\Delta_{N-1} \left(\frac{\partial^2 V}{\partial y_N^2} \right) \right) b(\bar{y}) \mathrm{d}\bar{y} \int_W \frac{\partial^2 v}{\partial x_N^2}(\bar{x}, 0) \frac{\partial^2 \varphi}{\partial x_N^2}(\bar{x}, 0) \mathrm{d}\bar{x}.$$

Proof. First note that the function

$$\hat{v}(\bar{x}, y) = -V(y) \frac{\partial^2 v}{\partial x_N^2}(\bar{x}, 0), \tag{3.2.32}$$

satisfies the conditions found in Theorem 3.2.7, and it is well defined because the function V is unique up to the sum of a monomial of order 2 in y_N, which does not modifies the boundary conditions. In order to simplify the notation we set

$$g(y) = b(\bar{y})(1 + y_N)^4, \qquad \text{for all } y \in Y \times (-1, 0).$$

By using (3.2.32) in (3.2.25) we deduce that the strange term equals

$$\int_{Y\times(-1,0)} 3D^2_y\Big(\frac{\partial V}{\partial y_N}\Big) : D^2_y(g(y)) + y_N(D^3_y V : D^3(g(y))\,\mathrm{d}y \int_W \frac{\partial^2 v}{\partial x_N^2}(\bar{x}, 0)\frac{\partial^2 \varphi}{\partial x_N^2}(\bar{x}, 0)\mathrm{d}\bar{x}.$$

Thus it is sufficient to check that

$$\int_{Y\times(-1,0)} 3D^2_y\Big(\frac{\partial V}{\partial y_N}\Big) : D^2_y(g(y)) + y_N(D^3_y V : D^3(g(y))\,\mathrm{d}y =$$
$$- \int_Y \left(\Delta\left(\frac{\partial^2 V}{\partial y_N^2}\right) + 2\Delta_{N-1}\left(\frac{\partial^2 V}{\partial y_N^2}\right)\right) b(\bar{y})\mathrm{d}\bar{y}.$$

We set

$$(\gamma) := \int_{Y\times(-1,0)} D^2_y\Big(\frac{\partial V}{\partial y_N}\Big) : D^2_y(g(y))\mathrm{d}y,$$

and we proceed integrating by parts the other integral

$$\int_{Y\times(-1,0)} y_N(D^3_y V : D^3(g(y))\,\mathrm{d}y. \tag{3.2.33}$$

Before proceeding we remark that the boundary terms coming from the integration by parts formula will be identically zero, due to the periodicity of the integrands and their vanishing when $y_N = 0$ or $y_N = -1$. We rewrite the integral in (3.2.33) with index notation, dropping the summation symbols. In this way,

$$\begin{aligned}
&\int_{Y\times(-1,0)} y_N\left(\frac{\partial^3 V}{\partial y_i \partial y_j \partial y_h}\frac{\partial^3 g(y)}{\partial y_i \partial y_j \partial y_h}\right)\mathrm{d}y \\
&= -\int_{Y\times(-1,0)} y_N\left(\frac{\partial^4 V}{\partial y_i^2 \partial y_j \partial y_h}\frac{\partial^2 g(y)}{\partial y_j \partial y_h}\right)\mathrm{d}y - \int_{Y\times(-1,0)} \frac{\partial^3 V}{\partial y_N \partial y_j \partial y_h}\frac{\partial^2 g(y)}{\partial y_j \partial y_h}\,\mathrm{d}y \\
&= -\int_{Y\times(-1,0)} y_N\left(\frac{\partial^4 V}{\partial y_i^2 \partial y_j \partial y_h}\frac{\partial^2 g(y)}{\partial y_j \partial y_h}\right)\mathrm{d}y - (\gamma).
\end{aligned} \tag{3.2.34}$$

A further integration by parts on the first integral in the right-hand side of (3.2.34) yields

$$\int_{Y\times(-1,0)} y_N\left(\frac{\partial^5 V}{\partial y_i^2 \partial y_j^2 \partial y_h}\frac{\partial g(y)}{\partial y_h}\right)\mathrm{d}y + \int_{Y\times(-1,0)} \frac{\partial^4 V}{\partial y_i^2 \partial y_N \partial y_h}\frac{\partial g(y)}{\partial y_h}\,\mathrm{d}y. \tag{3.2.35}$$

Recall that $\Delta^3 V \equiv 0$ on $Y \times (-1, 0)$ by Lemma 3.2.8; hence, an integration by parts on the first integral in the right hand side of (3.2.35) yields

$$\begin{aligned} &- \int_{Y\times(-1,0)} y_N \Delta^3 V g(y)\,\mathrm{d}y - \int_{Y\times(-1,0)} \frac{\partial^5 V}{\partial y_i^2 \partial y_j^2 \partial y_N} g(y)\,\mathrm{d}y \\ &= - \int_{Y\times(-1,0)} \frac{\partial^5 V}{\partial y_i^2 \partial y_j^2 \partial y_N} g(y)\,\mathrm{d}y. \end{aligned} \tag{3.2.36}$$

Thus,

$$\begin{aligned} &\int_{Y\times(-1,0)} y_N (D_y^3 V : D^3(b(\bar{y})(1+y_N)^4)\,\mathrm{d}y = \\ &-(\gamma) + \int_{Y\times(-1,0)} \frac{\partial^4 V}{\partial y_i^2 \partial y_N \partial y_h} \frac{\partial g(y)}{\partial y_h}\,\mathrm{d}y - \int_{Y\times(-1,0)} \frac{\partial^5 V}{\partial y_i^2 \partial y_j^2 \partial y_N} g(y)\,\mathrm{d}y. \end{aligned} \tag{3.2.37}$$

We consider now the last integral in the right hand side of (3.2.37) and we integrate twice by parts in y_i and y_j. A calculation yields

$$\begin{aligned} &\int_{Y\times(-1,0)} \frac{\partial^5 V}{\partial y_i^2 \partial y_j^2 \partial y_N} g(y)\,\mathrm{d}y = \\ &\quad (\gamma) + \int_Y \frac{\partial^4 V}{\partial y_j^2 \partial y_N^2}(\bar{y},0) b(\bar{y})\,\mathrm{d}\bar{y} - \int_Y \frac{\partial^3 V}{\partial y_i \partial y_N^2}(\bar{y},0) \frac{\partial g(\bar{y},0)}{\partial y_i}\,\mathrm{d}\bar{y}. \end{aligned} \tag{3.2.38}$$

We rewrite equality (3.2.38) in compact form, getting

$$\begin{aligned} &\int_{Y\times(-1,0)} \frac{\partial^5 V}{\partial y_i^2 \partial y_j^2 \partial y_N} g(y)\,\mathrm{d}y = \\ &\quad (\gamma) + \int_Y \Delta\left(\frac{\partial^2 V(\bar{y},0)}{\partial y_N^2}\right) b(\bar{y})\,\mathrm{d}\bar{y} - \int_Y \nabla\left(\frac{\partial^2 V(\bar{y},0)}{\partial y_N^2}\right) \cdot \nabla(g(\bar{y},0))\,\mathrm{d}\bar{y}. \end{aligned} \tag{3.2.39}$$

In the last integral in the right hand side of (3.2.39) we split the gradient in the normal and tangential part, and we apply the Divergence Theorem in order to get

$$\begin{aligned} -\int_Y \nabla\left(\frac{\partial^2 V(\bar{y},0)}{\partial y_N^2}\right) \cdot \nabla(g(\bar{y},0))\,\mathrm{d}\bar{y} = &- \int_Y \frac{\partial^3 V(\bar{y},0)}{\partial y_N^3} \frac{\partial(g(\bar{y},0))}{\partial y_N}\,\mathrm{d}\bar{y} \\ &+ \int_Y \Delta_{N-1}\left(\frac{\partial^2 V(\bar{y},0)}{\partial y_N^2}\right) b(\bar{y})\,\mathrm{d}\bar{y}, \end{aligned}$$

and since $\frac{\partial^3 V(\bar{y},0)}{\partial y_N^3} = 0$ by Lemma 3.2.8, we have

$$\int_{Y\times(-1,0)} \frac{\partial^5 V}{\partial y_i^2 \partial y_j^2 \partial y_N} g(y)\,\mathrm{d}y = (\gamma) + \int_Y \Delta\left(\frac{\partial^2 V(\bar{y},0)}{\partial y_N^2}\right) b(\bar{y})\,\mathrm{d}\bar{y} + \int_Y \Delta_{N-1}\left(\frac{\partial^2 V(\bar{y},0)}{\partial y_N^2}\right) b(\bar{y})\,\mathrm{d}\bar{y}. \quad (3.2.40)$$

Going back to (3.2.37), with similar arguments (integration by parts and the Divergence Theorem) we obtain that

$$\int_{Y\times(-1,0)} \frac{\partial^4 V}{\partial y_i^2 \partial y_N \partial y_h} \frac{\partial g(y)}{\partial y_h}\,\mathrm{d}y = -(\gamma) - \int_Y \Delta_{N-1}\left(\frac{\partial^2 V(\bar{y},0)}{\partial y_N^2}\right) b(\bar{y})\,\mathrm{d}\bar{y}. \quad (3.2.41)$$

By collecting (3.2.25), (3.2.37), (3.2.40), and (3.2.41), we deduce that

$$\begin{aligned}&\int_{Y\times(-1,0)} 3D_y^2\left(\frac{\partial V}{\partial y_N}\right) : D_y^2(g(y)) + y_N(D_y^3 V : D^3(g(y))\,\mathrm{d}y = \\ &3(\gamma) - (\gamma) + \int_{Y\times(-1,0)} \frac{\partial^4 V}{\partial y_i^2 \partial y_N \partial y_h} \frac{\partial g(y)}{\partial y_h}\,\mathrm{d}y - \int_{Y\times(-1,0)} \frac{\partial^5 V}{\partial y_i^2 \partial y_j^2 \partial y_N} g(y)\,\mathrm{d}y \\ &= -2\int_Y \Delta_{N-1}\left(\frac{\partial^2 V(\bar{y},0)}{\partial y_N^2}\right) b(\bar{y})\,\mathrm{d}\bar{y} - \int_Y \Delta\left(\frac{\partial^2 V(\bar{y},0)}{\partial y_N^2}\right) b(\bar{y})\,\mathrm{d}\bar{y}.\end{aligned}$$

This concludes the proof. □

Theorem 3.2.10. *Let V be as in Lemma 3.2.8. Then*

$$\int_{Y\times(-1,0)} \left(3D_y^2\left(\frac{\partial V}{\partial y_N}\right) : D_y^2(b(\bar{y})(1+y_N)^4) + y_N(D_y^3 V : D^3(b(\bar{y})(1+y_N)^4)\right)\mathrm{d}y = \int_{Y\times(-\infty,0)} |D^3 V|^2\,\mathrm{d}y. \quad (3.2.42)$$

Proof. Let ϕ be the real-valued function defined on $Y\times]-\infty,0]$ by

$$\phi(y) = \begin{cases} y_N b(\bar{y})(1+y_N)^4, & \text{if } -1 \le y_N \le 0, \\ 0, & \text{if } y_N < -1. \end{cases}$$

Then $\phi \in W^{3,2}(Y\times(-\infty,0))$, $\phi(\bar{y},0) = 0$ and

$$\nabla\phi(\bar{y},0) = (0,0,\ldots,0,b(\bar{y})).$$

Now note that the function $\psi = V - \phi$ is a suitable test-function in equation (3.2.31); by plugging it in we get

$$\int_{Y\times(-\infty,0)} |D^3V|^2\,\mathrm{d}y = \int_{Y\times(-1,0)} D^3V : D^3\phi\,\mathrm{d}y.$$

By the Leibnitz rule it is easy to check that

$$\int_{Y\times(-1,0)} D^3V : D^3\phi\,\mathrm{d}y = 3\int_{Y\times(-1,0)} D^2_y\Big(\frac{\partial V}{\partial y_N}\Big) : D^2_y(b(\bar y)(1+y_N)^4)\mathrm{d}y$$
$$+ \int_{Y\times(-1,0)} y_N(D^3_yV : D^3(b(\bar y)(1+y_N)^4)\,\mathrm{d}y.$$

□

We are now ready to conclude the proof of (iii) of Theorem 3.2.1.

Proof of Theorem 3.2.1(iii). Let us set $g(y) = b(\bar y)(1+y_N)^4$ for all $y \in Y\times(-1,0)$. The function v in Theorem 3.2.6 satisfies

$$\int_W \int_{Y\times(-1,0)} \left(3D^2_y\Big(\frac{\partial V}{\partial y_N}\Big) : D^2_y g(y) + y_N(D^3_yV : D^3g(y))\right)\mathrm{d}y\,\frac{\partial^2 v}{\partial x_N^2}\frac{\partial^2\varphi}{\partial x_N^2}(\bar x,0)\mathrm{d}\bar x$$
$$+\int_\Omega D^3v : D^3\varphi + u\varphi\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x, \tag{3.2.43}$$

for all $\varphi \in H^3(\Omega)\cap H^2_0(\Omega)$, where V is the function from Lemma 3.2.8. By Theorem 3.2.9 and Theorem 3.2.10 we can rewrite the first integral on the left-hand side of (3.2.43) as

$$\int_W\left(\int_{Y\times(-\infty,0)}|D^3V|^2\,\mathrm{d}y\right)\frac{\partial^2 v}{\partial x_N^2}\frac{\partial^2\varphi}{\partial x_N^2}(\bar x,0)\mathrm{d}\bar x,$$

and by the Triharmonic Green Formula (see formula (4.1.17)) we deduce that

$$\int_\Omega D^3v : D^3\varphi\,dx = -\int_\Omega \Delta^3 v\varphi + \int_{\partial\Omega}\frac{\partial^3 v}{\partial n^3}\frac{\partial^2\varphi}{\partial n^2}\,dS, \tag{3.2.44}$$

for all $\varphi \in H^3(\Omega)\cap H^2_0(\Omega)$. Hence, on $W\times\{0\}$ we find the following boundary integral

$$\int_W\left(\frac{\partial^3 v}{\partial x_N^3}(\bar x,0) + \left(\int_{Y\times(-\infty,0)}|D^3V|^2\,dy\right)\frac{\partial^2 v}{\partial x_N^2}(\bar x,0)\right)\frac{\partial^2\varphi}{\partial x_N^2}(\bar x,0)\,\mathrm{d}\bar x, \tag{3.2.45}$$

for all $\varphi \in H^3(\Omega)\cap H^2_0(\Omega)$. By (3.2.43), (3.2.44), (3.2.45) and the arbitrariness of φ we deduce the statement of Theorem 3.2.1, part (iii). □

3.3 The triharmonic operator with weak intermediate boundary conditions

Let Ω, Ω_ϵ be defined as in (3.2.1). We shall consider the operators

$$H_{\Omega_\epsilon,I} = -\Delta^3 + \mathbb{I},$$

on the open sets Ω_ϵ, associated with weak intermediate boundary conditions on $\partial\Omega_\epsilon$ (see Problem (3.0.5)). Recall that the associated quadratic form is (3.0.1) for all $u, v \in V(\Omega_\epsilon) \equiv W^{3,2}(\Omega_\epsilon) \cap W_0^{1,2}(\Omega_\epsilon)$. Let us write $W = W \times \{0\}$ and $Y = Y \times \{0\}$. Set also $H_{\Omega,S}$ for the operator $-\Delta^3 + \mathbb{I}$ subject to strong intermediate boundary conditions on W and weak intermediate boundary conditions on $\partial\Omega \setminus W$. Moreover, we recall that we use the notation $H_{\Omega,D}$ for the operator $-\Delta^3 + \mathbb{I}$ subject to Dirichlet boundary conditions on W and weak intermediate boundary conditions on $\partial\Omega \setminus W$. In the following theorem we analyse the spectral convergence of $H_{\Omega_\epsilon,I}$, depending on the value of the parameter α.

Theorem 3.3.1 (Spectral convergence). *With the notation above, the following statements hold true.*

(i) [Spectral stability] *If $\alpha > 5/2$, then $H^{-1}_{\Omega_\epsilon,I} \xrightarrow{C} H^{-1}_{\Omega,I}$ as $\epsilon \to 0$.*

(ii) [Mild instability] *If $3/2 < \alpha < 5/2$, then $H^{-1}_{\Omega_\epsilon,I} \xrightarrow{C} H^{-1}_{\Omega,S}$ as $\epsilon \to 0$.*

(iii) [Strange term] *If $\alpha = 5/2$, then $H^{-1}_{\Omega_\epsilon,I} \xrightarrow{C} \hat{H}^{-1}_{\Omega}$ as $\epsilon \to 0$, where $\hat{H}_\Omega$ is the operator $-\Delta^3 + \mathbb{I}$ with weak intermediate boundary conditions on $\partial\Omega \setminus W$ and the following boundary conditions on W: $u = 0$, $\Delta(\partial_{x_N^2} u) + 2\Delta_{N-1}(\partial_{x_N^2} u) + K\partial_{x_N} u = 0$, $\partial_{x_N^3} u = 0$, where the factor K is given by*

$$K = \int_Y \left(\Delta^2\left(\frac{\partial V}{\partial y_N}\right) + \Delta_{N-1}\left(\frac{\partial(\Delta V)}{\partial y_N}\right) + \Delta^2_{N-1}\left(\frac{\partial V}{\partial y_N}\right)\right) b(\bar{y})\mathrm{d}\bar{y} = \int_{Y\times(-\infty,0)} |D^3 V|^2 \,\mathrm{d}y,$$

where the function V is Y-periodic in the variables $\bar{y}$ and satisfies the following microscopic problem

$$\begin{cases} \Delta^3 V = 0, & \text{in } Y \times (-\infty, 0), \\ V(\bar{y}, 0) = b(\bar{y}), & \text{on } Y, \\ -\frac{\partial^2(\Delta V)}{\partial x_N^2} + 2\frac{\partial^2}{\partial x_N^2}(\Delta_{N-1} V) = 0, & \text{on } Y, \\ \frac{\partial^3 V}{\partial y_N^3} = 0, & \text{on } Y. \end{cases}$$

(iv) [Strong instability] *If $\alpha \leq 1$, then $H^{-1}_{\Omega_\epsilon,I} \xrightarrow{C} H^{-1}_{\Omega_\epsilon,D}$ as $\epsilon \to 0$.*

Proof. (i) follows from Lemma 2.2.2, choosing $\kappa_\epsilon = \epsilon^{2\tilde{\alpha}/5}$, where $\tilde{\alpha} \in]5/2, \alpha[$.
The proof of (ii) goes as follows. We show that the Condition (C) (see Definition 2.2.1) holds with $V(\Omega_\epsilon) = H^3(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon)$ and $V(\Omega) = H^3_{S,W}(\Omega)$, where $H^3_{S,W}(\Omega)$ is the set of functions $u \in H^3(\Omega) \cap H_0^1(\Omega)$ such that $\frac{\partial u}{\partial x_N} = 0$ on $W \times \{0\}$. With this choice of $V(\Omega_\epsilon)$, $V(\Omega)$, it is not difficult to verify that conditions (C1), (C2)(i), (C2)(iii), (C3)(i) and (C3)(ii) hold true. Then it is sufficient to prove the validity of conditions $(C2)(ii)$ and $(C3)(iii)$. In order to show that $(C2)(ii)$ holds it is sufficient to use the diffeomorphism T_ϵ defined in (2.2.6) mapping $H^3_{S,W}(\Omega)$ to $V(\Omega_\epsilon)$. By using Lemma 2.2.2 we deduce that if $\alpha > 3/2$ then $\lim_{\epsilon\to 0} \|T_\epsilon \varphi\|_{H^3(\Omega_\epsilon \setminus K_\epsilon)} = 0$ for all $\varphi \in V(\Omega)$. In order to show that $(C3)(iii)$ holds it is sufficient to use Lemma 3.1.2(iii).
The proof of (iv) works in a similar way. Namely, we show that Condition (C) holds with $V(\Omega_\epsilon) = H^3(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon)$ and $V(\Omega) = H^3_{0,W}(\Omega)$, where $H^3_{0,W}(\Omega)$ is the set of functions $u \in H^3(\Omega) \cap H_0^1(\Omega)$ such that $\frac{\partial u}{\partial x_N} = \frac{\partial^2 u}{\partial x_N^2} = 0$ on $W \times \{0\}$. We set T_ϵ to be the extension-by-zero operator. Then it is possible to prove that all the conditions (C1)-(C3) hold true. In particular, by choosing E_ϵ to be the restriction to Ω, Condition (C3)(iii) follows directly from Lemma 3.1.2, case $\alpha \leq 1$.
The proof of (iii) will be the object of Sections 3.3.1 and 3.3.2, below. □

3.3.1 Critical case - Macroscopic problem.

In this section we shall consider the case $\alpha = 5/2$ of Theorem 3.3.1. We refer to Section 3.2 for the notation about Φ_ϵ, h_ϵ, T_ϵ, C_ϵ^k, $\hat{u}$, $w^{3,2}_{Per_Y}(Y \times (-\infty, 0))$.

Let $f_\epsilon \in L^2(\Omega_\epsilon)$ and $f \in L^2(\Omega)$ be such that $f_\epsilon \rightharpoonup f$ in $L^2(\mathbb{R}^N)$ as $\epsilon \to 0$, with the understanding that the functions are extended by zero outside their natural domains. Let $v_\epsilon \in V(\Omega_\epsilon) = W^{3,2}(\Omega_\epsilon) \cap W_0^{1,2}(\Omega_\epsilon)$ be such that

$$H_{\Omega_\epsilon, I} v_\epsilon = f_\epsilon, \tag{3.3.1}$$

for all $\epsilon > 0$ small enough. Then $\|v_\epsilon\|_{W^{3,2}(\Omega_\epsilon)} \leq M$ for all $\epsilon > 0$ sufficiently small, hence, possibly passing to a subsequence there exists $v \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$ such that $v_\epsilon \rightharpoonup v$ in $W^{3,2}(\Omega)$ and $v_\epsilon \to v$ in $L^2(\Omega)$.
Let $\varphi \in V(\Omega) = W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$ be a fixed test function. Since $T_\epsilon \varphi \in V(\Omega_\epsilon)$, by (3.3.1) we get

$$\int_{\Omega_\epsilon} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, dx + \int_{\Omega_\epsilon} v_\epsilon T_\epsilon \varphi \, dx = \int_{\Omega_\epsilon} f_\epsilon T_\epsilon \varphi \, dx, \tag{3.3.2}$$

and passing to the limit as $\epsilon \to 0$ we have that

$$\int_{\Omega_\epsilon} v_\epsilon T_\epsilon \varphi \, dx \to \int_\Omega v \varphi \, dx, \quad \int_{\Omega_\epsilon} f_\epsilon T_\epsilon \varphi \, dx \to \int_\Omega f \varphi \, dx.$$

Now consider the first integral in the right-hand side of (3.3.2). Let us define $K_\epsilon = W \times (-1, -\epsilon)$. By splitting the integral in three terms corresponding to $\Omega_\epsilon \setminus \Omega$, $\Omega \setminus K_\epsilon$ and K_ϵ and by arguing as in [19, Section 8.3] one can show that

$$\int_{K_\epsilon} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x \longrightarrow \int_\Omega D^3 v : D^3 \varphi \, \mathrm{d}x, \qquad \int_{\Omega_\epsilon \setminus \Omega} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x \longrightarrow 0,$$

as $\epsilon \longrightarrow 0$. Let Q_ϵ be as in (3.2.5). We split again the remaining integral in two summands,

$$\int_{\Omega_\epsilon \setminus K_\epsilon} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x = \int_{\Omega_\epsilon \setminus (K_\epsilon \cup Q_\epsilon)} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x + \int_{Q_\epsilon} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x. \tag{3.3.3}$$

Again, by arguing as in [19, Section 8.3] it is possible to prove that

$$\int_{\Omega_\epsilon \setminus (K_\epsilon \cup Q_\epsilon)} D^3 v_\epsilon : D^3 T_\epsilon \varphi \, \mathrm{d}x \longrightarrow 0,$$

as $\epsilon \longrightarrow 0$.

Lemma 3.3.2. *For all $y \in Y \times (-1, 0)$ and $i, j, k = 1, \dots, N$ the functions $\hat{h}_\epsilon(\bar{x}, y)$, $\widehat{\frac{\partial h_\epsilon}{\partial x_i}}(\bar{x}, y)$, $\widehat{\frac{\partial^2 h_\epsilon}{\partial x_i \partial x_j}}(\bar{x}, y)$ and $\widehat{\frac{\partial^3 h_\epsilon}{\partial x_i \partial x_j \partial x_k}}(\bar{x}, y)$ are independent of $\bar{x}$. Moreover, $\hat{h}_\epsilon(\bar{x}, y) = O(\epsilon^{5/2})$, $\widehat{\frac{\partial h_\epsilon}{\partial x_i}}(\bar{x}, y) = O(\epsilon^{3/2})$ as $\epsilon \longrightarrow 0$, $\widehat{\frac{\partial^2 h_\epsilon}{\partial x_i \partial x_j}}(\bar{x}, y) = O(\epsilon^{1/2})$ as $\epsilon \longrightarrow 0$, for all $i, j = 1, \dots, N$, uniformly in $y \in Y \times (-1, 0)$, and*

$$\epsilon^{1/2} \widehat{\frac{\partial^3 h_\epsilon}{\partial x_i \partial x_j \partial x_k}}(\bar{x}, y) \longrightarrow \frac{\partial^3 (b(\bar{y})(y_N + 1)^4)}{\partial y_i \partial y_j \partial y_k},$$

as $\epsilon \longrightarrow 0$, for all $i, j, k = 1, \dots, N$, uniformly in $y \in Y \times (-1, 0)$.

Proof. It is a matter of calculations, which can be carried out as in Lemma 3.2.4, with the difference that here $\alpha = 5/2$. □

Lemma 3.3.3. *Let $v_\epsilon \in V(\Omega_\epsilon) = H^3(\Omega_\epsilon) \cap H^1_0(\Omega_\epsilon)$ be such that $\|v_\epsilon\|_{H^3(\Omega_\epsilon)} \leq M$ for all $\epsilon > 0$. Let $v \in H^3(\Omega) \cap H^1_0(\Omega)$ the weak limit of $(v_\epsilon)_\epsilon$ in $H^3(\Omega)$. Let φ be a fixed function in $V(\Omega)$. Let $\hat{v} \in L^2(W, w^{3,2}_{\mathrm{Per}_Y}(Y \times (-\infty, 0)))$ be as in Lemma 3.2.3. Then*

$$\begin{aligned} &\int_{Q_\epsilon} D^3 v_\epsilon : D^3(T_\epsilon \varphi) \, \mathrm{d}x \longrightarrow \\ &- \int_W \int_{Y \times (-1,0)} (D^3_y(\hat{v}) : D^3(b(\bar{y})(1 + y_N)^4) \, \mathrm{d}y \frac{\partial \varphi}{\partial x_N}(\bar{x}, 0) \mathrm{d}\bar{x}. \end{aligned} \tag{3.3.4}$$

Proof. In the following calculations we use the index notation and we drop the summation symbols. We calculate

$$
\begin{aligned}
&\int_{Q_\epsilon} D^3 v_\epsilon : D^3(T_\epsilon \varphi)\,\mathrm{d}x = \int_{Q_\epsilon} \frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h} \frac{\partial^3(\varphi \circ \Phi_\epsilon)}{\partial x_i \partial x_j \partial x_h}\,\mathrm{d}x \\
&= \int_{Q_\epsilon} \frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h} \frac{\partial^3 \varphi}{\partial x_k \partial x_l \partial x_m}(\Phi_\epsilon(x)) \frac{\partial \Phi_\epsilon^{(k)}}{\partial x_i} \frac{\partial \Phi_\epsilon^{(l)}}{\partial x_j} \frac{\partial \Phi_\epsilon^{(m)}}{\partial x_h}\,\mathrm{d}x \\
&+ \int_{Q_\epsilon} \frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h} \frac{\partial^2 \varphi}{\partial x_k \partial x_l}(\Phi_\epsilon(x)) \left[\frac{\partial \Phi_\epsilon^{(k)}}{\partial x_i} \frac{\partial^2 \Phi_\epsilon^{(l)}}{\partial x_j \partial x_h} + \frac{\partial \Phi_\epsilon^{(k)}}{\partial x_j} \frac{\partial^2 \Phi_\epsilon^{(l)}}{\partial x_i \partial x_h} + \frac{\partial \Phi_\epsilon^{(k)}}{\partial x_h} \frac{\partial^2 \Phi_\epsilon^{(l)}}{\partial x_i \partial x_j} \right] \mathrm{d}x, \\
&+ \int_{Q_\epsilon} \frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h} \frac{\partial \varphi}{\partial x_k}(\Phi_\epsilon(x)) \frac{\partial^3 \Phi_\epsilon^{(k)}}{\partial x_i \partial x_j \partial x_h}\,\mathrm{d}x.
\end{aligned}
\tag{3.3.5}
$$

By arguing exactly as in the proof of Lemma 3.2.5 we deduce that the first integral in the right-hand side of (3.3.5) vanishes as $\epsilon \to 0$. We then consider the second integral in the right hand side of (3.3.5). Note that all the terms with $l \neq N$ vanish. Thus, without loss of generality we set $l = N$. About the index k, we prefer to consider separately two cases. We consider first the case $k \neq N$. By the exact integration formula (1.2.4) we obtain

$$
\begin{aligned}
&\left| \int_{Q_\epsilon} \frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h} \frac{\partial^2 \varphi}{\partial x_k \partial x_N}(\Phi_\epsilon(x))\, \delta_{ki} \frac{\partial^2 \Phi_\epsilon^{(N)}}{\partial x_j \partial x_h}\,\mathrm{d}x \right| \\
&\qquad = \epsilon \left| \int_{\widehat{W}_\epsilon} \int_Y \int_{-1}^0 \widehat{\frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h}} \frac{\partial^2 \varphi}{\partial x_k \partial x_N}(\widehat{\Phi}_\epsilon(y)) \delta_{ki} \widehat{\frac{\partial^2 \Phi_\epsilon^{(N)}}{\partial x_j \partial x_h}}\,\mathrm{d}x \right| \\
&\qquad \leq C\epsilon^{1/2} \left| \int_{\widehat{W}_\epsilon} \int_Y \int_{-1}^0 \epsilon^{-5/2} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 \varphi}{\partial x_k \partial x_N}(\widehat{\Phi}_\epsilon(y)) \widehat{\frac{\partial^2 h_\epsilon}{\partial x_j \partial x_h}}\,\mathrm{d}x \right| \\
&\qquad \leq C\epsilon \| \epsilon^{-5/2} \hat{v}_\epsilon \|_{W^{3,2}(\widehat{W}_\epsilon \times Y \times (-1,0))} \left\| \frac{\partial \varphi}{\partial x_k \partial x_N}(\widehat{\Phi}_\epsilon(y)) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \\
&\qquad \leq C\epsilon^{1/2} \| \epsilon^{-5/2} \hat{v}_\epsilon \|_{W^{3,2}(\widehat{W}_\epsilon \times Y \times (-1,0))} \left\| \frac{\partial \varphi}{\partial x_k \partial x_N} \right\|_{L^2(Q_\epsilon)},
\end{aligned}
$$

and the right-hand side tends to zero as $\epsilon \to 0$. Hence, when $k \neq N$ the second integral in the right-hand side of (3.3.5) is vanishing.

If instead $k = N$, then the exact integration formula (1.2.4) yields

$$
\begin{aligned}
&\epsilon^{-5} \int_{\widehat{W}_\epsilon \times Y \times (-1,0)} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 \varphi}{\partial x_N^2}(\widehat{\Phi_\epsilon}(y)) \cdot \\
&\quad \cdot \left[\frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_j \partial y_h} + \frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_j} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i \partial y_h} + \frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_h} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i \partial y_j} \right] \mathrm{d}\bar{x}\mathrm{d}y,
\end{aligned}
\tag{3.3.6}
$$

and since we are summing on the indexes $i, j, h \in 1, \dots, N$, (3.3.6) equals

$$3\epsilon^{-5} \int_{\widehat{W_\epsilon} \times Y \times (-1,0)} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 (\varphi(\widehat{\Phi_\epsilon}(y)))}{\partial x_N^2} \frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_j \partial y_h} \mathrm{d}\bar{x}\mathrm{d}y.$$

Note now that

$$\frac{\partial \widehat{\Phi_\epsilon}^{(k)}}{\partial y_i} = \begin{cases} \epsilon \delta_{ki}, & \text{if } k \neq N, \\ \epsilon \delta_{Ni} - \epsilon \widehat{\frac{\partial h_\epsilon}{\partial x_i}}, & \text{if } k = N. \end{cases}$$

and

$$\frac{\partial^2 \widehat{\Phi_\epsilon}^{(k)}}{\partial y_i \partial y_j} = \begin{cases} 0, & \text{if } k \neq N, \\ -\epsilon^2 \widehat{\frac{\partial^2 h_\epsilon}{\partial x_i \partial x_j}}, & \text{if } k = N. \end{cases}$$

Thus, we have

$$\begin{aligned} &3\epsilon^{-5} \int_{\widehat{W_\epsilon} \times Y \times (-1,0)} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 (\varphi(\widehat{\Phi_\epsilon(y)}))}{\partial x_N^2} \frac{\partial \widehat{\Phi_\epsilon}^{(N)}}{\partial y_i} \frac{\partial^2 \widehat{\Phi_\epsilon}^{(N)}}{\partial y_j \partial y_h} \mathrm{d}\bar{x}\mathrm{d}y \\ &= -3\epsilon^{-2} \int_{\widehat{W_\epsilon} \times Y \times (-1,0)} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 (\varphi(\widehat{\Phi_\epsilon}(y)))}{\partial x_N^2} \left(\delta_{Ni} - \widehat{\frac{\partial h_\epsilon}{\partial x_i}} \right) \widehat{\frac{\partial^2 h_\epsilon}{\partial x_j \partial x_h}} \mathrm{d}\bar{x}\mathrm{d}y. \end{aligned} \tag{3.3.7}$$

The right-hand side of (3.3.7) vanishes as $\epsilon \to 0$. Indeed,

$$\begin{aligned} &\epsilon^{1/2} \left| \int_{\widehat{W_\epsilon} \times Y \times (-1,0)} \epsilon^{-5/2} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 \varphi}{\partial x_N^2} (\widehat{\Phi_\epsilon}(y)) \left(\delta_{Ni} - \widehat{\frac{\partial h_\epsilon}{\partial x_i}} \right) \widehat{\frac{\partial^2 h_\epsilon}{\partial x_j \partial x_h}} \mathrm{d}\bar{x}\mathrm{d}y \right| \\ &\leq C\epsilon^{1/2} \left| \int_{\widehat{W_\epsilon} \times Y \times (-1,0)} \epsilon^{-5/2} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \frac{\partial^2 \varphi}{\partial x_N^2} (\widehat{\Phi_\epsilon}(y)) \, \mathrm{d}\bar{x}\mathrm{d}y \right| \\ &\leq C\epsilon^{1/2} \left\| \epsilon^{-5/2} \frac{\partial^3 \hat{v}_\epsilon}{\partial y_i \partial y_j \partial y_h} \right\|_{L^2(\widehat{W_\epsilon} \times Y \times (-1,0))} \left\| \frac{\partial^2 \varphi}{\partial x_N^2} (\widehat{\Phi_\epsilon}(y)) \right\|_{L^2(\widehat{W_\epsilon} \times Y \times (-1,0))} \\ &\leq C \left\| \frac{\partial^2 \varphi}{\partial x_N^2} \right\|_{L^2(\Phi_\epsilon(Q_\epsilon))} \longrightarrow 0, \end{aligned}$$

as $\epsilon \to 0$, where we have used (1.2.4) and Lemma 3.3.2.
It remains to treat only the third integral in the right hand side of (3.3.5). We

apply the exact integration formula (1.2.4) in order to obtain

$$\epsilon \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \widehat{\frac{\partial^3 v_\epsilon}{\partial x_i \partial x_j \partial x_h}} \frac{\partial \varphi}{\partial x_N}(\widehat{\Phi_\epsilon}(y)) \widehat{\frac{\partial^3 \Phi_\epsilon^{(N)}}{\partial x_i \partial x_j \partial x_h}} \,\mathrm{d}\bar{x}\mathrm{d}y$$

$$= \epsilon^{-2} \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \frac{\partial^3 \widehat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h} \frac{\partial \varphi}{\partial x_N}(\widehat{\Phi_\epsilon}(y)) \widehat{\frac{\partial^3 \Phi_\epsilon^{(N)}}{\partial x_i \partial x_j \partial x_h}} \,\mathrm{d}\bar{x}\mathrm{d}y$$

$$= - \int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \left[\epsilon^{-5/2} \frac{\partial^3 \widehat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h} \right] \left[\frac{\partial \varphi}{\partial x_N}(\widehat{\Phi_\epsilon}(y)) \right] \left[\epsilon^{1/2} \widehat{\frac{\partial^3 h_\epsilon}{\partial x_i \partial x_j \partial x_h}} \right] \mathrm{d}\bar{x}\mathrm{d}y.$$

By Lemma 3.2.3 it is clear that

$$\epsilon^{-5/2} \frac{\partial^3 \widehat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h} \rightharpoonup \frac{\partial^3 \hat{v}}{\partial y_i \partial y_j \partial y_h},$$

weakly in $L^2(W \times Y \times (-\infty, 0))$ as $\epsilon \to 0$. Moreover, by Lemma 3.2.4

$$\epsilon^{1/2} \widehat{\frac{\partial^3 \Phi_\epsilon^{(N)}}{\partial x_i \partial x_j \partial x_h}} \to - \frac{\partial^3 (b(\bar{y})(1+y_N)^4)}{\partial y_i \partial y_j \partial y_h},$$

uniformly in $W \times Y \times (-1, 0)$ as $\epsilon \to 0$. Hence,

$$\int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \left[\epsilon^{-5/2} \frac{\partial^3 \widehat{v_\epsilon}}{\partial y_i \partial y_j \partial y_h} \right] \left[\frac{\partial \varphi}{\partial x_N}(\widehat{\Phi_\epsilon}(y)) \right] \left[\epsilon^{1/2} \widehat{\frac{\partial^3 \Phi_\epsilon^{(N)}}{\partial x_i \partial x_j \partial x_h}} \right] \mathrm{d}\bar{x}\mathrm{d}y$$

$$\to - \int_{W\times Y\times(-1,0)} \frac{\partial^3 \hat{v}}{\partial y_i \partial y_j \partial y_h} \frac{\partial \varphi}{\partial x_N}(\bar{x}, 0) \frac{\partial^3 (b(\bar{y})(1+y_N)^4)}{\partial y_i \partial y_j \partial y_h} \,\mathrm{d}\bar{x}\mathrm{d}y.$$

as $\epsilon \to 0$. □

The previous discussion yields the following

Theorem 3.3.4. *Let $f_\epsilon \in L^2(\Omega_\epsilon)$ and $f \in L^2(\Omega)$ be such that $f_\epsilon \rightharpoonup f$ in $L^2(\Omega)$. Let $v_\epsilon \in W^{3,2}(\Omega_\epsilon) \cap W^{1,2}_0(\Omega_\epsilon)$ be the solutions to $H_{\Omega_\epsilon} v_\epsilon = f_\epsilon$. Then, possibly passing to a subsequence, there exists $v \in W^{3,2}(\Omega) \cap W^{1,2}_0(\Omega)$ and $\hat{v} \in L^2(W, w^{3,2}_{Per_Y}(Y \times (\infty, 0)))$ such that $v_\epsilon \rightharpoonup v$ in $W^{3,2}(\Omega)$, $v_\epsilon \to v$ in $L^2(\Omega)$ and such that statements (a) and (b) in Lemma 3.2.3 hold. Moreover,*

$$- \int_W \int_{Y\times(-1,0)} (D^3_y(\hat{v}) : D^3(b(\bar{y})(1+y_N)^4) \,\mathrm{d}y \frac{\partial \varphi}{\partial x_N}(\bar{x}, 0) \mathrm{d}\bar{x}$$
$$+ \int_\Omega D^3 v : D^3 \varphi + u\varphi \,\mathrm{d}x = \int_\Omega f \varphi \,\mathrm{d}x,$$

for all $\varphi \in W^{3,2}(\Omega) \cap W^{1,2}_0(\Omega)$.

3.3.2 Critical case - Microscopic problem.

Let $\psi \in C^\infty(\overline{W}\times\overline{Y}\times]-\infty,0])$ be such that supp $\psi \subset C\times\overline{Y}\times[d,0]$ for some compact set $C \subset W$ and $d \in]-\infty,0[$ and such that $\psi(\bar{x},\bar{y},0) = 0$ for all $(\bar{x},\bar{y}) \in W \times Y$. Assume also ψ to be Y-periodic in the variable $\bar{y}$. We set

$$\psi_\epsilon(x) = \epsilon^{\frac{5}{2}}\psi\Big(\bar{x}, \frac{\bar{x}}{\epsilon}, \frac{x_N}{\epsilon}\Big),$$

for all $\epsilon > 0$, $x \in W\times]-\infty,0]$. Then $T_\epsilon\psi_\epsilon \in V(\Omega_\epsilon)$ for sufficiently small ϵ, hence we can use it in the weak formulation of the problem in Ω_ϵ, getting

$$\int_{\Omega_\epsilon} D^3 v_\epsilon : D^3 T_\epsilon\psi_\epsilon \,\mathrm{d}x + \int_{\Omega_\epsilon} v_\epsilon T_\epsilon\psi_\epsilon \,\mathrm{d}x = \int_{\Omega_\epsilon} f_\epsilon T_\epsilon\psi_\epsilon \,\mathrm{d}x. \tag{3.3.8}$$

It is not difficult to prove that

$$\int_{\Omega_\epsilon} v_\epsilon T_\epsilon\psi_\epsilon \,dx \longrightarrow 0, \qquad \int_{\Omega_\epsilon} f_\epsilon T_\epsilon\psi\epsilon \,dx \longrightarrow 0, \tag{3.3.9}$$

as $\epsilon \longrightarrow 0$, and by arguing as in [19] and in Subsection 3.2.2 we deduce that

$$\int_{\Omega_\epsilon\setminus\Omega} D^3 v_\epsilon : D^3 T_\epsilon\psi_\epsilon \,dx \longrightarrow 0, \tag{3.3.10}$$

as $\epsilon \longrightarrow 0$. Moreover, by arguing as in [19, Lemma 8.47] it is possible to prove that

$$\int_{\Omega} D^3 v_\epsilon : D^3 T_\epsilon\psi_\epsilon \,\mathrm{d}x \longrightarrow \int_{W\times Y\times(-\infty,0)} D^3_y\hat{v}(\bar{x},y) : D^3_y\psi(\bar{x},y)\,\mathrm{d}\bar{x}\mathrm{d}y, \tag{3.3.11}$$

as $\epsilon \longrightarrow 0$. Then we have the following

Theorem 3.3.5. *Let $\hat{v} \in L^2(W, w^{3,2}_{Per_Y}(Y\times(-\infty,0)))$ be the function from Theorem 3.3.4. Then*

$$\int_{W\times Y\times(-\infty,0)} D^3_y\hat{v}(\bar{x},y) : D^3_y\psi(\bar{x},y)\mathrm{d}\bar{x}\mathrm{d}y = 0,$$

for all $\psi \in L^2(W, w^{3,2}_{Per_Y}(Y\times(-\infty,0)))$ such that $\psi(\bar{x},\bar{y},0) = 0$ on $W\times Y$. Moreover, for any $i,j = 1,\dots,N-1$, we have

$$\frac{\partial^2\hat{v}}{\partial y_i\partial y_j}(\bar{x},\bar{y},0) = -\frac{\partial^2 b}{\partial y_i\partial y_j}(\bar{y})\frac{\partial v}{\partial x_N}(\bar{x},0) \qquad \text{on } W\times Y, \tag{3.3.12}$$

Proof. We need only to prove (3.3.12) since the first part of the statement follows from (3.3.8), (3.3.9), (3.3.10), (3.3.11) (see also the proof of [19, Theorem 8.53]). By applying Lemma 3.1.2, case $\alpha = 5/2$ to $v_\epsilon \in H^3(\Omega_\epsilon)\cap H^1_0(\Omega_\epsilon)$ we deduce the validity of (3.3.12). □

Now we have the following

Lemma 3.3.6. *There exists* $V \in w^{3,2}_{Per_Y}(Y \times (-\infty, 0))$ *satisfying the equation*

$$\int_{Y\times(-\infty,0)} D^3V : D^3\psi \, \mathrm{d}y = 0, \tag{3.3.13}$$

for all $\psi \in w^{3,2}_{Per_Y}(Y \times (-\infty, 0))$ *such that* $\psi(\bar{y}, 0) = 0$ *on* Y*, and the boundary condition*

$$V(\bar{y}, 0) = b(\bar{y}), \qquad \text{on } Y.$$

Function V *is unique up to a sum of a monomial in* y_N *of the form* ay_N^2*. Moreover* $V \in W^{6,2}_{Per_Y}(Y \times (d, 0))$ *for any* $d < 0$ *and it satisfies the equation*

$$\Delta^3 V = 0, \qquad \text{in } Y \times (d, 0),$$

subject to the boundary conditions

$$\begin{cases} \frac{\partial^2(\Delta V)}{\partial x_N^2} + 2\frac{\partial^2}{\partial x_N^2}(\Delta_{N-1}V) = 0, & \text{on } Y, \\ \frac{\partial^3 V}{\partial y_N^3}(\bar{y}, 0) = 0, & \text{on } Y. \end{cases}$$

Proof. Similar to the proof of [19, Lemma 8.60] and to the proof of Lemma 3.2.8. The proof of the existence and the uniqueness of the function V works exactly as in Lemma 3.2.8. Regularity is achieved by standard procedures. Note that in order to find the boundary conditions satisfied by V on Y we need to use the Triharmonic Green Formula (4.1.12) with V in place of f and ψ in place of φ. We choose test functions ψ as in the statement with bounded support in the y_N-direction. We then deduce that

$$\begin{aligned} \int_{Y\times(-\infty,0)} D^3V : D^3\psi \, \mathrm{d}y = &- \int_{Y\times(-\infty,0)} \Delta^3 V\psi \, \mathrm{d}y + \int_Y \frac{\partial^3 V}{\partial y_N^3}\frac{\partial^2 \psi}{\partial y_N^2} \, \mathrm{d}\bar{y} \\ &- \int_Y \left(\frac{\partial^2(\Delta V)}{\partial y_N^2} + 2\Delta_{N-1}\left(\frac{\partial^2 V}{\partial y_N^2}\right)\right)\frac{\partial \psi}{\partial x_N} \, \mathrm{d}\bar{y}, \end{aligned}$$

and by the arbitrariness of ψ we then conclude that V is triharmonic and satisfies the boundary conditions in the statement. □

Theorem 3.3.7 (Characterization of the strange term). *Let* V *be the function defined in Lemma 3.3.6. Let* $v, \hat{v}$ *be as in Theorem 3.3.4. Then*

$$\hat{v}(\bar{x}, y) = -V(y)\frac{\partial v}{\partial x_N}(\bar{x}, 0) + a(\bar{x})y_N^2.$$

for some function $a \in L^2(W)$. Moreover we have the following equalities:

$$\begin{aligned}\int_{Y\times(-\infty,0)} |D^3V|^2 dy &= \int_{Y\times(-\infty,0)} D^3V : D^3(b(\bar{y})(1+y_N^4))dy \\ &= \int_Y \left(\frac{\partial(\Delta^2 V)}{\partial x_N} + \Delta_{N-1}\left(\frac{\partial \Delta V}{\partial x_N}\right) + \Delta_{N-1}^2\left(\frac{\partial V}{\partial x_N}\right)\right) b(\bar{y})d\bar{y}. \quad (3.3.14)\end{aligned}$$

Proof. Let ϕ be the real-valued function defined on $Y\times]-\infty,0]$ by

$$\phi(y) = \begin{cases} b(\bar{y})(1+y_N)^4, & \text{if } -1 \le y_N \le 0, \\ 0, & \text{if } y_N < -1. \end{cases}$$

Then $\phi \in W^{3,2}(Y\times(-\infty,0))$, and $\phi(\bar{y},0) = 0$ for all $\bar{y} \in Y$. Now note that the function $\psi = V - \phi$ is a suitable test-function in equation (3.3.13); by plugging it in we get

$$\int_{Y\times(-\infty,0)} |D^3V|^2\, dy = \int_{Y\times(-\infty,0)} D^3V : D^3(b(\bar{y})(1+y_N)^4)\, dy$$

By applying the triharmonic Green Formula (4.1.12) on the right-hand side of the former equation, and by keeping in account that V is satisfying the boundary conditions listed in Lemma 3.3.6 and $\Delta^3 V = 0$ in $Y\times(d,0)$ for all $d < 0$, we deduce that

$$\int_{Y\times(-\infty,0)} D^3V : D^3(b(\bar{y})(1+y_N)^4)\, dy = \int_Y \left(\frac{\partial(\Delta^2 V)}{\partial x_N} + \Delta_{N-1}\left(\frac{\partial(\Delta V)}{\partial x_N}\right) + \Delta_{N-1}^2\left(\frac{\partial V}{\partial x_N}\right)\right) b(\bar{y})\, d\bar{y}.$$

□

By Lemma 3.3.6 and Theorem 3.3.7 it is now easy to deduce (iii) of Theorem 3.3.1.

Proof of Theorem 3.3.1(iii). Note that the function v of Theorem 3.3.4 satisfies

$$\begin{aligned}-\int_W \int_{Y\times(-1,0)} (D_y^3(\hat{v}) : D^3(b(\bar{y})(1+y_N)^4)\, \mathrm{d}y\, \frac{\partial\varphi}{\partial x_N}(\bar{x},0)\mathrm{d}\bar{x} \\ + \int_\Omega D^3v : D^3\varphi + u\varphi\, \mathrm{d}x = \int_\Omega f\varphi\, \mathrm{d}x, \quad (3.3.15)\end{aligned}$$

for all $\varphi \in H^3(\Omega) \cap H^1_0(\Omega)$. By Theorem 3.3.7 the first integral in the left-hand side of (3.3.15) can be equivalently rewritten as

$$\int_W \left(\int_{Y\times(-\infty,0)} |D^3 V|^2 \, \mathrm{d}y \right) \frac{\partial v}{\partial x_N}(\bar{x},0) \frac{\partial \varphi}{\partial x_N}(\bar{x},0) \mathrm{d}\bar{x},$$

where V is the function defined in Lemma 3.3.6. By the Triharmonic Green Formula (see (4.1.17)) we have that

$$\begin{aligned}\int_\Omega D^3 v : D^3 \varphi \, dx = -\int_\Omega \Delta^3 v \, \varphi \, dx + \int_{\partial\Omega} \frac{\partial^3 f}{\partial n^3} \frac{\partial^2 \varphi}{\partial n^2} \, dS \\ + \int_{\partial\Omega} \left(\left((n^T D^3 v)_{\partial\Omega} : D_{\partial\Omega} n \right) - \frac{\partial^2(\Delta v)}{\partial n^2} - 2 \, \mathrm{div}_{\partial\Omega}(D^3 v[n \otimes n])_{\partial\Omega} \right) \frac{\partial \varphi}{\partial n} \, dS,\end{aligned} \tag{3.3.16}$$

for all $\varphi \in H^3(\Omega) \cap H^1_0(\Omega)$. In particular, we deduce that on $W \times \{0\}$ we have the following boundary integral

$$\int_W \left(-\frac{\partial^2(\Delta v)}{\partial x_N^2}(\bar{x},0) - 2\Delta_{N-1}\left(\frac{\partial^2 v}{\partial x_N^2} \right)(\bar{x},0) + \left(\int_{Y\times(-\infty,0)} |D^3 V|^2 \mathrm{d}y \right) \frac{\partial v}{\partial x_N}(\bar{x},0) \right) \frac{\partial \varphi}{\partial x_N}(\bar{x},0) \mathrm{d}\bar{x}. \tag{3.3.17}$$

Then, by (3.3.15), (3.3.16), (3.3.17) and the arbitrariness of φ we deduce the statement of Theorem 3.3.1, part (iii). □

Chapter 4

Polyharmonic operators on singularly perturbed domains

In this chapter we consider polyharmonic operators $H^m_{\Omega_\epsilon} := (-\Delta)^m + \mathbb{I}$ with strong intermediate boundary conditions on the family of domain Ω_ϵ defined in (3.2.1). We analyse the spectral convergence of $H^m_{\Omega_\epsilon}$ as $\epsilon \longrightarrow 0$, in the spirit of the results proved for the triharmonic operator $(-\Delta^3 + \mathbb{I})$ in §3.2. However, when dealing with polyharmonic operators of arbitrarily high order we face several new difficulties. First of all, we need a 'polyharmonic Green formula' (see (4.1.4)) to shift from the variational formulation to the strong formulation (and viceversa) of the eigenvalue problems for $H^m_{\Omega_\epsilon}$. This turns out to be important in particular in the analysis of the spectral convergence in the critical case $\alpha = 3/2$, see for example the proof of Theorem 4.2.10.

Then we have to deal with a relevant computational complexity in the calculus of higher order derivatives. This problem is overcome by using suitable combinatorial formulae for the computation of the m-th derivative of products of functions and of the m-th derivative of composite functions. See, for example, Lemma 4.2.4.

In the end we are able to prove Theorem 4.2.1, which characterises the spectral convergence of $H^m_{\Omega_\epsilon}$ for all $m \geq 2$.

4.1 A polyharmonic Green formula

In this section we provide a formula which turns out to be useful in recognising the possible natural boundary conditions for polyharmonic operators of any order. Let us begin by stating an easy integration-by-parts formula.

Proposition 4.1.1. *Let Ω be a bounded domain of class $C^{0,1}$ in $\mathbb{R}^N$. Let $m \in \mathbb{N}$, $m \geq 1$ and let $f \in C^{m+1}(\overline{\Omega})$, $\varphi \in C^m(\overline{\Omega})$. Then*

$$\int_\Omega D^m f : D^m \varphi \, \mathrm{d}x = -\int_\Omega D^{m-1}(\Delta f) : D^{m-1}\varphi \, \mathrm{d}x + \int_{\partial\Omega} D^m f : (n \otimes D^{m-1}\varphi) \, \mathrm{d}S, \tag{4.1.1}$$

where the symbol : stands for the Frobenius product, n is the outer unit normal to $\partial\Omega$, and $\otimes$ is the tensor product, defined by

$$(n \otimes D^{m-1}\varphi)_{i,j_1,\cdots,j_{m-1}} = \left(n_i \frac{\partial^{m-1}\varphi}{\partial x_{j_1} \cdots \partial x_{j_{m-1}}} \right)_{i,j_1,\cdots,j_{m-1}}$$

for all $i, j_1, \cdots, j_{m-1} \in \{1, \cdots, N\}$.

Proof. The proof is a simple integration by parts. Indeed, dropping the summation symbols we get

$$\int_\Omega D^m f : D^m \varphi \, \mathrm{d}x = \int_\Omega \frac{\partial^m f}{\partial x_{j_1} \cdots \partial x_{j_m}} \frac{\partial^m \varphi}{\partial x_{j_1} \cdots \partial x_{j_m}} \, \mathrm{d}x$$
$$= -\int_\Omega \frac{\partial^{m+1} f}{\partial x_{j_1}^2 \cdots \partial x_{j_m}} \frac{\partial^{m-1}\varphi}{\partial x_{j_2} \cdots \partial x_{j_m}} \, \mathrm{d}x + \int_{\partial\Omega} \frac{\partial^m f}{\partial x_{j_1} \cdots \partial x_{j_m}} \frac{\partial^{m-1}\varphi}{\partial x_{j_2} \cdots \partial x_{j_m}} n_{j_1} \, \mathrm{d}S$$
$$= -\int_\Omega D^{m-1}(\Delta f) : D^{m-1}\varphi \, \mathrm{d}x + \int_{\partial\Omega} (D^m f) : (n \otimes D^{m-1}\varphi) \, \mathrm{d}S.$$

□

Corollary 4.1.2. *Let $m \in \mathbb{N}$, $m \geq 1$. Let $f \in C^{2m}(\overline{\Omega})$, $\varphi \in C^m(\overline{\Omega})$.*

$$\int_\Omega D^m f : D^m \varphi \, \mathrm{d}x = (-1)^m \int_\Omega \Delta^m f \varphi \, \mathrm{d}x + \sum_{k=0}^{m-1} (-1)^k \int_{\partial\Omega} (D^{m-k}(\Delta^k f)) : (n \otimes D^{m-k-1}\varphi) \, \mathrm{d}S. \tag{4.1.2}$$

In particular, when $\partial\Omega$ is flat (that is, $D_{\partial\Omega} n(x) = 0$ for all $x \in \partial\Omega$), formula (4.1.2) *equals*

$$\int_\Omega D^m f : D^m \varphi \, \mathrm{d}x = (-1)^m \int_\Omega \Delta^m f \varphi \, \mathrm{d}x + \sum_{k=0}^{m-1} (-1)^k \int_{\partial\Omega} \left(D^{m-k-1}\left(\Delta^k \left(\frac{\partial f}{\partial n} \right) \right) \right) : D^{m-k-1}\varphi \, \mathrm{d}S. \tag{4.1.3}$$

Proof. In order to prove (4.1.2) it is sufficient to apply m times the integration by parts argument used in the proof of formula (4.1.1). Then (4.1.3) follows directly from (4.1.2) by noting first that

$$\begin{aligned}(D^{m-k}(\Delta^k f)) : (n \otimes D^{m-k-1}\varphi) &= (n^T D^{m-k}(\Delta^k f)) : D^{m-k-1}\varphi \\ &= \frac{\partial}{\partial n}\Big(D^{m-k-1}(\Delta^k f)\Big) : D^{m-k-1}\varphi,\end{aligned}$$

and then by recalling that when $D_{\partial\Omega} n(x) = 0$ the normal derivative commutes with the other differential operators (we refer to Remark 4.1.6 for further explanations). □

Theorem 4.1.3 (Polyharmonic Green Formula - Flat case). *Let H be the half-space*

$$H = \{(\bar{x}, x_N) \in \mathbb{R}^N : x_N < 0\}.$$

Let $m \in \mathbb{N}$, $m \geq 1$. Let $f \in C^{2m}(\overline{H})$, $\varphi \in C^m(\overline{H})$ with $\operatorname{supp}\varphi \subset \overline{H} \cap B(0, M)$, *for some $M > 0$. Then,*

$$\int_H D^m f : D^m \varphi \, \mathrm{d}x = (-1)^m \int_H \Delta^m f \varphi \, \mathrm{d}x + \sum_{t=0}^{m-1} \int_{\mathbb{R}^{N-1}} B_t(f) \frac{\partial^t \varphi}{\partial x_N^t} \mathrm{d}\bar{x}, \qquad (4.1.4)$$

where $B_t : C^{2m}(\partial H) \to C^{t+1}(\partial H)$ is defined by

$$B_t(f) = \sum_{l=t}^{m-1} (-1)^{m-t-1} \binom{l}{t} \Delta_{N-1}^{l-t} \left(\frac{\partial^{t+1}}{\partial x_N^{t+1}} (\Delta^{m-l-1} f) \right), \qquad (4.1.5)$$

and Δ_{N-1} is the Laplace operator in the first $N-1$ variables.

Proof. First note that as a consequence of the Leibnitz rule we can write

$$\begin{aligned}&\int_{\mathbb{R}^{N-1}} \left(D^{m-k-1}\left(\Delta^k \left(\frac{\partial f}{\partial x_N} \right) \right) \right) : D^{m-k-1}\varphi \, \mathrm{d}\bar{x} \\ &= \sum_{t=0}^{m-k-1} \binom{m-k-1}{t} \int_{\mathbb{R}^{N-1}} \left(D_{\bar{x}}^{m-k-t-1}\left(\Delta^k \left(\frac{\partial^{t+1} f}{\partial x_N^{t+1}} \right) \right) \right) : \left(D_{\bar{x}}^{m-k-t-1}\left(\frac{\partial^t \varphi}{\partial x_N^t} \right) \right) d\bar{x}.\end{aligned} \qquad (4.1.6)$$

Then, by using (4.1.6) in the last integral in the right-hand side of (4.1.3) we get the following as boundary term

$$\sum_{k=0}^{m-1} (-1)^k \sum_{t=0}^{m-k-1} \binom{m-k-1}{t} \int_{\mathbb{R}^{N-1}} D_{\bar{x}}^{m-k-t-1} \left(\frac{\partial^{t+1}(\Delta^k f)}{\partial x_N^{t+1}} \right) : D_{\bar{x}}^{m-k-t-1} \left(\frac{\partial^t \varphi}{\partial x_N^t} \right) \mathrm{d}\bar{x}. \qquad (4.1.7)$$

By switching to coordinates and by dropping the summation symbols, (4.1.7) equals

$$\int_{\mathbb{R}^{N-1}} \frac{\partial^{m-k-t-1}}{\partial x_{i_1} \cdots \partial x_{i_{m-k-t-1}}} \left(\frac{\partial^{t+1}(\Delta^k f)}{\partial x_N^{t+1}} \right) \frac{\partial^{m-k-t-1}}{\partial x_{i_1} \cdots \partial x_{i_{m-k-t-1}}} \left(\frac{\partial^t \varphi}{\partial x_N^t} \right) \mathrm{d}\bar{x}, \qquad (4.1.8)$$

where the indexes i_j run on the first $N-1$ coordinates. By integrating by parts $m-k-t-1$ times in $i_1, \ldots, i_{m-k-t-1}$ in (4.1.8) we deduce that (4.1.7) equals

$$\sum_{k=0}^{m-1} (-1)^{m-t-1} \sum_{t=0}^{m-k-1} \binom{m-k-1}{t} \int_{\mathbb{R}^{N-1}} \frac{\partial^{2(m-k-t-1)}}{\partial^2 x_{i_1} \cdots \partial^2 x_{i_{m-k-t-1}}} \left(\frac{\partial^{t+1}(\Delta^k f)}{\partial x_N^{t+1}} \right) \frac{\partial^t \varphi}{\partial x_N^t} \, \mathrm{d}\bar{x},$$

where we have no other boundary terms because φ has compact support. We rewrite the last expression as

$$\sum_{k=0}^{m-1} (-1)^{m-t-1} \sum_{t=0}^{m-k-1} \binom{m-k-1}{t} \int_{\mathbb{R}^{N-1}} \Delta_{N-1}^{m-k-t-1} \left(\frac{\partial^{t+1}(\Delta^k f)}{\partial x_N^{t+1}} \right) \frac{\partial^t \varphi}{\partial x_N^t} \mathrm{d}\bar{x}. \qquad (4.1.9)$$

We now apply the change of summation index $l = m-k-1$ in the first sum of (4.1.9). We deduce that (4.1.9) equals

$$\sum_{l=0}^{m-1} (-1)^{m-t-1} \sum_{t=0}^{l} \binom{l}{t} \int_{\mathbb{R}^{N-1}} \Delta_{N-1}^{l-t} \left(\frac{\partial^{t+1}(\Delta^{m-l-1} f)}{\partial x_N^{t+1}} \right) \frac{\partial^t \varphi}{\partial x_N^t} \mathrm{d}\bar{x}. \qquad (4.1.10)$$

By exchanging the two sums in (4.1.10) we find (4.1.4). □

Remark 4.1.4. If $m = 2$, then (4.1.4) reads

$$\int_H D^2 f : D^2 \varphi \, \mathrm{d}x = \int_H \Delta^2 f \varphi \, \mathrm{d}x + \int_{\mathbb{R}^{N-1}} \frac{\partial^2 f}{\partial x_N^2} \frac{\partial \varphi}{\partial x_N} \, \mathrm{d}\bar{x} \\ - \int_{\mathbb{R}^{N-1}} \left(\Delta_{N-1} \left(\frac{\partial f}{\partial x_N} \right) + \Delta \left(\frac{\partial f}{\partial x_N} \right) \right) \varphi \, \mathrm{d}\bar{x}, \qquad (4.1.11)$$

which is consistent with the Biharmonic Green Formula provided in [19, Lemma 8.56]. Indeed, if the domain is an hyperplane, the boundary integral

$$\int_{\partial H} \left(\mathrm{div}_{\partial H} \left(D^2 f \cdot n \right)_{\partial \Omega} \right) \varphi \, \mathrm{d}S$$

appearing in [19, Lemma 8.56] coincides with

$$\int_{\mathbb{R}^{N-1}} \Delta_{N-1}\left(\frac{\partial f}{\partial x_N}\right) \mathrm{d}\bar{x},$$

because the tangential divergence coincides with the divergence with respect to the first $N-1$ coordinates, $n = e_N$ (the Nth unit vector of the canonical basis of $\mathbb{R}^N$), and the divergence with respect to $\bar{x}$ of the gradient in $\bar{x}$ is exactly the Laplacian with respect to the first $N-1$ coordinates.

Remark 4.1.5. In the case $m = 3$, the identity (4.1.4) yields the following "Triharmonic Green Formula":

$$\begin{aligned}\int_H D^3 f : D^3\varphi \,\mathrm{d}x = &- \int_H \Delta^3 f \varphi \,\mathrm{d}x + \int_{\mathbb{R}^{N-1}} \frac{\partial^3 f}{\partial x_N^3}\frac{\partial^2 \varphi}{\partial x_N^2} \,\mathrm{d}\bar{x} \\ &- \int_{\mathbb{R}^{N-1}} \left(\frac{\partial^2(\Delta f)}{\partial x_N^2} + 2\Delta_{N-1}\left(\frac{\partial^2 f}{\partial x_N^2}\right)\right)\frac{\partial \varphi}{\partial x_N} \,\mathrm{d}\bar{x} \\ &+ \int_{\mathbb{R}^{N-1}} \left(\frac{\partial(\Delta^2 f)}{\partial x_N} + \Delta_{N-1}\left(\frac{\partial(\Delta f)}{\partial x_N}\right) + \Delta_{N-1}^2\left(\frac{\partial f}{\partial x_N}\right)\right)\varphi \,\mathrm{d}\bar{x}.\end{aligned} \tag{4.1.12}$$

In this way we have identified all the natural boundary conditions for the triharmonic operator $-\Delta^3$ on an hyperplane. Note that if $\varphi \in W^{3,2}(H) \cap W_0^{2,2}(H)$, then all the boundary integrals vanish, with the exception of

$$\int_{\mathbb{R}^{N-1}} \frac{\partial^3 f}{\partial x_N^3}\frac{\partial^2 \varphi}{\partial x_N^2} \,\mathrm{d}\bar{x},$$

which gives the condition

$$\frac{\partial^3 f}{\partial x_N^3} = 0, \qquad \text{on } \partial H.$$

More in general, consider the polyharmonic operator $(-\Delta)^m$ with strong intermediate boundary conditions (namely, $\varphi \in W^{m,2}(H) \cap W_0^{m-1,2}(H)$). Then, the non-vanishing boundary integral in (4.1.4) is given by

$$\int_{\mathbb{R}^{N-1}} B_{m-1}(f)\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}} \,d\bar{x} = \int_{\mathbb{R}^{N-1}} \frac{\partial^m f}{\partial x_N^m}\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}} \,d\bar{x},$$

and by the arbitrariness of $\varphi \in W^{m,2}(H) \cap W_0^{m-1,2}(H)$ we find the boundary condition

$$\frac{\partial^m f}{\partial x_N^m} = 0, \qquad \text{on } \partial H,$$

for all $m \geq 2$.

Remark 4.1.6. In order to prove the analogous of Theorem 4.1.3 on general bounded domains Ω of $\mathbb{R}^N$ we need to decompose higher order differential operators in tangential and normal parts at the boundary. As far as we know, explicit formulas via tangential calculus are not available. Note that this problem is related to the definition of higher order differential operators on Riemannian manifolds, which is not trivial for operators of order bigger or equal to 2 (for example, to define the Hessian matrix on a Riemannian manifold we need to use the covariant derivative and the Levi-Civita connection ∇, via the equality $D^2 f(X,Y) = XY(f) - (\nabla_X Y)(f)$; in particular the Hessian operator on a Riemannian manifold is no more homogeneous of order 2, since it contains the lower order term $(\nabla_X Y)(f)$ which is linked to the curvature of the manifold). By using tangential calculus it is possible to prove that

$$D^2 f(x) = \left(D^2_{\partial\Omega} f(x) + \frac{\partial}{\partial n}\left(\nabla_{\partial\Omega} f(x)\right) \otimes n(x) + n(x) \otimes \nabla_{\partial\Omega}\left(\frac{\partial f(x)}{\partial n}\right) + \frac{\partial^2 f(x)}{\partial n^2} n(x) \otimes n(x)\right) + \frac{\partial f(x)}{\partial n} D_{\partial\Omega} n(x), \quad (4.1.13)$$

for all $x \in \partial\Omega$. In order to prove formula (4.1.13) it is sufficient to compute

$$D^2 f|_{\partial\Omega} = D(\nabla f)|_{\partial\Omega} = (D_{\partial\Omega} + (D[\cdot]n) \otimes n)\left(\nabla_{\partial\Omega} f + \frac{\partial f}{\partial n} n\right).$$

Recall that $D_{\partial\Omega} n$ corresponds to the second fundamental form of $\partial\Omega$, see Theorem 1.4.4. We refer to §1.4 for the precise definition of the tangential operators. Note carefully that the differential operators $\frac{\partial}{\partial n}$ and $D_{\partial\Omega}$ commute if and only if $\partial\Omega$ is flat. More precisely,

$$\frac{\partial}{\partial n}\left(\nabla_{\partial\Omega}\psi\right) = \nabla_{\partial\Omega}\left(\frac{\partial\psi}{\partial n}\right) - (D_{\partial\Omega} n)(\nabla_{\partial\Omega}\psi), \quad (4.1.14)$$

for all $\psi \in C^2(\partial\Omega)$.[1] Then formula (4.1.13) can be equivalently rewritten as

$$D^2 f(x) = \left(D^2_{\partial\Omega} f(x) + \nabla_{\partial\Omega}\left(\frac{\partial f(x)}{\partial n}\right) \otimes n(x) + n(x) \otimes \nabla_{\partial\Omega}\left(\frac{\partial f(x)}{\partial n}\right) + \frac{\partial^2 f(x)}{\partial n^2} n(x) \otimes n(x)\right) - (D_{\partial\Omega} n(x))(\nabla_{\partial\Omega} f(x)) \otimes n(x) + \frac{\partial f(x)}{\partial n} D_{\partial\Omega} n(x), \quad (4.1.15)$$

[1]Formula (4.1.14) follows easily by noting that $\frac{\partial}{\partial n}\left(\nabla_{\partial\Omega}\psi\right) = \nabla(\nabla\psi - \frac{\partial\psi}{\partial n} n) \cdot n = D^2\psi n - \frac{\partial^2\psi}{\partial n^2} n$, and $\nabla_{\partial\Omega}\left(\frac{\partial\psi}{\partial n}\right) = \nabla\left(\frac{\partial\psi}{\partial n}\right) - \frac{\partial^2\psi}{\partial n^2} = D^2\psi n + (D_{\partial\Omega} n)(\nabla_{\partial\Omega} n) - \frac{\partial^2\psi}{\partial n^2} n$.

for all $x \in \partial\Omega$. Finally, note that if we take the trace on both hand sides of (4.1.15) we recover the classical decomposition formula for the Laplacian at the boundary

$$\Delta f(x) = \Delta_{\partial\Omega} f(x) + \frac{\partial^2 f(x)}{\partial n^2} + \mathcal{H}(x)\frac{\partial f(x)}{\partial n},$$

for all $x \in \partial\Omega$, where $\mathcal{H}$ is the curvature of $\partial\Omega$. Indeed, we have that

$$\mathrm{tr}\left(\nabla_{\partial\Omega}\left(\frac{\partial f}{\partial n}\right) \otimes n\right) = \nabla_{\partial\Omega}\left(\frac{\partial f}{\partial n}\right) \cdot n = 0,$$

and

$$\mathrm{tr}\big(D_{\partial\Omega} n(\nabla_{\partial\Omega} f) \otimes n\big) = n^T D_{\partial\Omega} n \nabla_{\partial\Omega} f = 0$$

because $n^T D_{\partial\Omega} n = (D_{\partial\Omega} n)n = 0$ (this follows from the fact that $n \cdot n = 1$, hence $D_{\partial\Omega}(n \cdot n) = 0$).

Theorem 4.1.7 (Triharmonic Green Formula - general domain)**.** *Let Ω be a bounded domain of $\mathbb{R}^N$ of class $C^{0,1}$. Let $f \in C^6(\overline{\Omega})$, $\varphi \in C^3(\overline{\Omega})$. Then*

$$\begin{aligned}\int_\Omega D^3 f : D^3\varphi\, dx = &-\int_\Omega \Delta^3 f\, \varphi\, dx + \int_{\partial\Omega} (n^T D^3 f) : D^2\varphi\, dS \\ &- \int_{\partial\Omega} (n^T D^2(\Delta f))_{\partial\Omega} \cdot \nabla_{\partial\Omega}\varphi\, dS - \int_{\partial\Omega} \frac{\partial^2(\Delta f)}{\partial n^2}\frac{\partial\varphi}{\partial n}\, dS + \int_{\partial\Omega} \frac{\partial(\Delta^2 f)}{\partial n}\varphi\, dS.\end{aligned} \tag{4.1.16}$$

If moreover Ω is of class C^3 then

$$\begin{aligned}\int_\Omega D^3 f : D^3\varphi\, dx = &-\int_\Omega \Delta^3 f\, \varphi\, dx + \int_{\partial\Omega} \frac{\partial^3 f}{\partial n^3}\frac{\partial^2\varphi}{\partial n^2}\, dS \\ &+ \int_{\partial\Omega} \left(\big((n^T D^3 f)_{\partial\Omega} : D_{\partial\Omega} n\big) - \frac{\partial^2(\Delta f)}{\partial n^2} - 2\,\mathrm{div}_{\partial\Omega}(D^3 f[n\otimes n])_{\partial\Omega}\right)\frac{\partial\varphi}{\partial n}\, dS \\ &+ \int_{\partial\Omega} \Bigg(\mathrm{div}^2_{\partial\Omega}\big((n^T D^3 f)_{\partial\Omega}\big) + \mathrm{div}_{\partial\Omega}\big(D_{\partial\Omega} n (D^3 f[n\otimes n])_{\partial\Omega}\big) \\ &\qquad + \frac{\partial(\Delta^2 f)}{\partial n} + \mathrm{div}_{\partial\Omega}\big(n^T D^2(\Delta f)\big)_{\partial\Omega}\Bigg)\varphi\, dS.\end{aligned} \tag{4.1.17}$$

Proof. Standard integration by parts yields

$$
\begin{aligned}
&\int_\Omega D^3 f : D^3\varphi\, dx = \int_\Omega \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}\frac{\partial^3 \varphi}{\partial x_i \partial x_j \partial x_k}\, dx \\
&= -\int_\Omega \frac{\partial^4 f}{\partial x_i^2 \partial x_j \partial x_k}\frac{\partial^2 \varphi}{\partial x_j \partial x_k}\, dx + \int_{\partial\Omega} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}\frac{\partial^2 \varphi}{\partial x_j \partial x_k} n_i\, dS \\
&= \int_\Omega \frac{\partial^5 f}{\partial x_i^2 \partial x_j^2 \partial x_k}\frac{\partial \varphi}{\partial x_k}\, dx + \int_{\partial\Omega} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}\frac{\partial^2 \varphi}{\partial x_j \partial x_k} n_i\, dS - \int_{\partial\Omega} \frac{\partial^4 f}{\partial x_i^2 \partial x_j \partial x_k}\frac{\partial \varphi}{\partial x_k} n_j\, dS \\
&= -\int_\Omega \Delta^3 f \varphi\, dx + \int_{\partial\Omega} (n^T D^3 f) : D^2\varphi\, dS - \int_{\partial\Omega} (n^T D^2(\Delta f)) \cdot \nabla\varphi\, dS + \int_{\partial\Omega} \frac{\partial(\Delta^2 f)}{\partial n}\varphi\, dS,
\end{aligned}
\tag{4.1.18}
$$

where summation symbols on i, j, k from 1 to N have been dropped. Then (4.1.16) follows from (4.1.18) by decomposing the gradient appearing in the third integral on the right-hand side of (4.1.18) in tangential and normal components, see Definition 1.4.1.

In order to prove (4.1.17) we need first to decompose the hessian matrix appearing in the first boundary integral on the right-hand side of (4.1.16). By using formula (4.1.15) on $D^2\varphi$ we deduce that

$$
\begin{aligned}
\int_{\partial\Omega} (n^T D^3 f) : D^2\varphi\, dS &= \int_{\partial\Omega} (n^T D^3 f)_{\partial\Omega} : D^2_{\partial\Omega}\varphi\, dS \\
&\quad + 2\int_{\partial\Omega} (D^3 f[n\otimes n])_{\partial\Omega} \cdot \nabla_{\partial\Omega}\left(\frac{\partial\varphi}{\partial n}\right) dS \\
&\quad - \int_{\partial\Omega} \left(D_{\partial\Omega} n (D^3 f[n\otimes n])_{\partial\Omega}\right) \cdot \nabla_{\partial\Omega}\varphi\, dS \\
&\quad + \int_{\partial\Omega} \left((n^T D^3 f)_{\partial\Omega} : D_{\partial\Omega} n\right)\frac{\partial\varphi}{\partial n}\, dS + \int_{\partial\Omega} \frac{\partial^3 f}{\partial n^3}\frac{\partial^2\varphi}{\partial n^2}\, dS.
\end{aligned}
\tag{4.1.19}
$$

In (4.1.19) the symbol $D^3 f[n\otimes n]$ stands for the vector having as i-th component $\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} n_j n_k$, where sums over j and k are understood. Note also that the third integral on the right-hand side of (4.1.19) is deduced from

$$
-\int_{\partial\Omega} (n^T D^3 f) : \left(D_{\partial\Omega} n(\nabla_{\partial\Omega}\varphi) \otimes n\right) dS,
$$

by using the following equalities

$$
\begin{aligned}
(n^T D^3 f) : \left(D_{\partial\Omega} n(\nabla_{\partial\Omega}\varphi) \otimes n\right) &= \left(D_{\partial\Omega} n(\nabla_{\partial\Omega}\varphi)\right)^T (n^T D^3 f) n \\
&= (\nabla_{\partial\Omega}\varphi)^T \left((D_{\partial\Omega} n)^T (D^3 f[n\otimes n])_{\partial\Omega}\right) \\
&= \left((D_{\partial\Omega} n)(D^3 f[n\otimes n])_{\partial\Omega}\right) \cdot \nabla_{\partial\Omega}\varphi.
\end{aligned}
$$

In the third equality we have used the fact that $D_{\partial\Omega}n$ is a symmetric matrix. Now, since Ω is of class C^2, we plan to apply the Tangential Divergence theorem (see Theorem 1.4.6) to the first, the second, and the third integral in the right-hand side of (4.1.19). We consider separately the first integral. Let us note that for every matrix $A = (a_{ij}(x))_{ij}$ with coefficients $a_{ij} \in C^2(\overline{\Omega})$ and for every function $\psi \in C^2(\overline{\Omega})$, we have

$$\int_{\partial\Omega} \operatorname{div}_{\partial\Omega}\big((A)_{\partial\Omega}(\nabla_{\partial\Omega}\psi)\big)\, dS = 0$$

by (1.4.3). Here $((A)_{\partial\Omega})_{ij} = (a_{ij} \circ p)|_{\partial\Omega}$, where p is defined in Section 1.4. Hence,

$$\int_{\partial\Omega} (\operatorname{div}_{\partial\Omega}(A)_{\partial\Omega}) \cdot \nabla_{\partial\Omega}\psi + (A)_{\partial\Omega} : D^2_{\partial\Omega}\psi\, dS = 0. \tag{4.1.20}$$

Finally, a further application of the Tangential Green formula (see (1.4.4)) on the first summand on the right-hand side of (4.1.20) yields

$$\int_{\partial\Omega} (\operatorname{div}^2_{\partial\Omega}(A)_{\partial\Omega})\psi\, dS = \int_{\partial\Omega} (A)_{\partial\Omega} : D^2_{\partial\Omega}\psi\, dS \tag{4.1.21}$$

for all matrix $A \in C^2(\overline{\Omega})^{N\times N}$, for every function $\psi \in C^2(\overline{\Omega})$. Then, by applying Formula (4.1.21) to the first integral in the right-hand side of (4.1.19) with $A = (n^T D^3 f)$ and $\psi = f$, and by using (1.4.4) on the second and third integral in the right-hand side of (4.1.19) we deduce that

$$\begin{aligned}\int_{\partial\Omega} (n^T D^3 f) : D^2\varphi\, dS = &\int_{\partial\Omega} \operatorname{div}^2_{\partial\Omega}\big((n^T D^3 f)_{\partial\Omega}\big)\varphi\, dS \\ &- 2\int_{\partial\Omega} \operatorname{div}_{\partial\Omega}\big((D^3 f[n\otimes n])_{\partial\Omega}\big)\frac{\partial\varphi}{\partial n}\, dS \\ &+ \int_{\partial\Omega} \operatorname{div}_{\partial\Omega}\Big(D_{\partial\Omega}n(D^3 f[n\otimes n])_{\partial\Omega}\Big)\varphi\, dS \\ &+ \int_{\partial\Omega} \big((n^T D^3 f)_{\partial\Omega} : D_{\partial\Omega}n\big)\frac{\partial\varphi}{\partial n}\, dS \\ &+ \int_{\partial\Omega} \frac{\partial^3 f}{\partial n^3}\frac{\partial^2\varphi}{\partial n^2}\, dS,\end{aligned} \tag{4.1.22}$$

where we have denoted with $(V)_{\partial\Omega}$ the projection[2] of V on the tangent plane to $\partial\Omega$, as defined in §1.4. By applying the Tangential Divergence Theorem to the

[2]Note carefully that if $\psi \in C^2(\overline{\Omega})$ then $\nabla_{\partial\Omega}\psi = (\nabla\psi)_{\partial\Omega}$ whereas, if $\partial\Omega$ is not flat,

$$D^2_{\partial\Omega}\psi = D_{\partial\Omega}(\nabla_{\partial\Omega}\psi) \neq (D^2\psi)_{\partial\Omega} = D^2(\psi\circ p)|_{\partial\Omega}.$$

second boundary integral on the right-hand side of (4.1.16) we finally deduce that

$$-\int_{\partial\Omega}(n^T D^2(\Delta f))_{\partial\Omega}\cdot\nabla_{\partial\Omega}\varphi\, dS=\int_{\partial\Omega}\mathrm{div}_{\partial\Omega}\big(n^T D^2(\Delta f)\big)_{\partial\Omega}\,\varphi\, dS. \tag{4.1.23}$$

By (4.1.22) and (4.1.23) we get (4.1.17), concluding the proof. □

Remark 4.1.8. Note that formula (4.1.17) is consistent with (4.1.12). Indeed, when $\partial\Omega$ is flat the second fundamental form $D_{\partial\Omega}n$ is zero, hence normal derivatives and tangential derivatives commute, and $D^2_{\partial\Omega}\psi=(D^2\psi)_{\partial\Omega}$. Thus, for example, if $\partial\Omega$ is the hyperplane $\mathbb{R}^{N-1}\times\{0\}$ then

$$\int_{\mathbb{R}^{N-1}}\mathrm{div}^2_{\partial\Omega}\big((n^T D^3 f)_{\partial\Omega}\big)\varphi\, d\bar{x}=\int_{\mathbb{R}^{N-1}}\mathrm{div}^2_{\bar{x}}\left(D^2_{\bar{x}}\left(\frac{\partial f}{\partial x_N}\right)\right)\varphi\, d\bar{x}=\int_{\mathbb{R}^{N-1}}\Delta^2_{\bar{x}}\left(\frac{\partial f}{\partial x_N}\right)\varphi\, d\bar{x}.$$

As we have seen in Theorem 4.1.7, it is rather complicated to find a higher order Green Formula. However, if we have more information on the boundary behaviour of the functions involved, easier proofs of the Green Formula may be available. The following Theorem makes a step in this direction.

Theorem 4.1.9. *Let Ω be a bounded domain of $\mathbb{R}^N$ of class $C^{0,1}$, $m\in\mathbb{N}$, $m\geq 2$. Let $f\in H^{2m}(\Omega)\cap H^{m-1}_0(\Omega)$ and $\varphi\in H^m(\Omega)\cap H^{m-1}_0(\Omega)$. Then*

$$\int_\Omega D^m f : D^m\varphi\, dx=(-1)^m\int_\Omega\Delta^m f\varphi dx+\int_{\partial\Omega}\frac{\partial^m f}{\partial n^m}\frac{\partial^{m-1}\varphi}{\partial n^{m-1}}\, dS, \tag{4.1.24}$$

for all $\varphi\in H^m(\Omega)\cap H^{m-1}_0(\Omega)$.

Proof. By (4.1.2) it is easy to see that

$$\int_\Omega D^m f : D^m\varphi\, dx=(-1)^m\int_\Omega\Delta^m f\varphi\, dx+\int_{\partial\Omega}D^m f:(n\otimes D^{m-1}\varphi)\, dS, \tag{4.1.25}$$

for all $\varphi\in H^m(\Omega)\cap H^{m-1}_0(\Omega)$, since $D^l\varphi|_{\partial\Omega}=0$ for all $l\leq m-2$. We note that

$$D^m f:(n\otimes D^{m-1}\varphi)=(n^T D^m f):D^{m-1}\varphi.$$

Moreover we claim that $D^{m-1}\varphi|_{\partial\Omega}=\frac{\partial^{m-1}\varphi}{\partial n^{m-1}}\bigotimes_{i=1}^{m-1}n$. Indeed, if $m=2$ the claim is a direct consequence of the gradient decomposition $\nabla|_{\partial\Omega}=\nabla_{\partial\Omega}+\frac{\partial}{\partial n}n$. Hence we assume $m>2$. Then, note that we trivially have

$$D^{m-2}\varphi|_{\partial\Omega}=\frac{\partial^{m-2}\varphi}{\partial n^{m-2}}\bigotimes_{i=1}^{m-2}n=0,$$

for all $\varphi \in H^m(\Omega) \cap H_0^{m-1}(\Omega)$. Hence,

$$D^{m-1}\varphi|_{\partial\Omega} = D(D^{m-2}\varphi)|_{\partial\Omega} = D_{\partial\Omega}(D^{m-2}\varphi) + D(D^{m-2}\varphi)n \otimes n$$
$$= \left(D\left(\frac{\partial^{m-2}\varphi}{\partial n^{m-2}} \bigotimes_{i=1}^{m-2} n\right)n\right) \otimes n = \frac{\partial^{m-1}\varphi}{\partial n^{m-1}} \bigotimes_{i=1}^{m-1} n,$$

for all $\varphi \in H^m(\Omega) \cap H_0^{m-1}(\Omega)$. This proves the claim. Then we can rewrite (4.1.25) as

$$\int_\Omega D^m f : D^m\varphi \, dx = (-1)^m \int_\Omega \Delta^m f \varphi \, dx + \int_{\partial\Omega} \frac{\partial^{m-1}\varphi}{\partial n^{m-1}} (n^T D^m f) : \left(\bigotimes_{i=1}^{m-1} n\right) dS,$$

and since $(n^T D^m f) : \left(\bigotimes_{i=1}^{m-1} n\right) = D^m f : \left(\bigotimes_{i=1}^{m} n\right) = \frac{\partial^m f}{\partial n^m}$ we deduce (4.1.24). □

4.2 Polyharmonic operators with strong intermediate boundary conditions

Let $m \in \mathbb{N}$, $m \geq 2$. Let $\Omega = W \times (-1, 0)$ where W is a smooth bounded domain of $\mathbb{R}^{N-1}$. Let $g_\epsilon(\bar{x}) = \epsilon^\alpha b(\bar{x}/\epsilon)$ for all $\bar{x} \in W$, and b is a positive, non-constant Y-periodic function of class C^{2m} . Define the perturbed sets

$$\Omega_\epsilon = \{(\bar{x}, x_N) : \bar{x} \in W, -1 < x_N < g_\epsilon(\bar{x})\}. \tag{4.2.1}$$

Consider the polyharmonic operators (in the following we omit the sums on $i_k = 1, \dots, N$ for all $k \leq m$):

$$(-\Delta)^m + \mathbb{I} = (-1)^m \sum_{j_1+\cdots+j_N=m} \frac{m!}{j_1! \cdots j_N!} \frac{\partial^{2m}}{\prod_{k=1}^N \partial x_k^{2j_k}} + \mathbb{I},$$

subject to strong intermediate boundary conditions, corresponding to the energy space $V(\Omega_\epsilon) := W^{m,2}(\Omega_\epsilon) \cap W_0^{m-1,2}(\Omega_\epsilon)$. More precisely, let $H_{\Omega_\epsilon,S}$ be the non-negative self-adjoint operator such that

$$(H_{\Omega_\epsilon,S}u, v)_{L^2(\Omega_\epsilon)} = (H^{1/2}_{\Omega_\epsilon,S}u,\, H^{1/2}_{\Omega_\epsilon,S}v)_{L^2(\Omega_\epsilon)} = Q_{\Omega_\epsilon}(u, v), \tag{4.2.2}$$

for all functions $u, v \in W^{m,2}(\Omega_\epsilon) \cap W_0^{m-1,2}(\Omega_\epsilon)$, where

$$Q_{\Omega_\epsilon}(u, v) = \int_{\Omega_\epsilon} D^m u : D^m v + uv \, dx,$$

is the quadratic form canonically associated with $H_{\Omega_\epsilon,S}$. Under regularity assumptions on f, Ω, we can rewrite

$$\int_{\Omega_\epsilon} D^m u : D^m v + uv\,dx = \int_{\Omega_\epsilon} fv\,dx,$$

for all $v \in V(\Omega_\epsilon)$ into the following Cauchy problem

$$\begin{cases} (-\Delta)^m u + u = f, & \text{in } \Omega_\epsilon, \\ u = 0, & \text{on } \partial\Omega_\epsilon, \\ \frac{\partial^l u}{\partial n^l} = 0, & \text{on } \partial\Omega_\epsilon, \text{ for all } 1 \le l \le m-2, \\ \frac{\partial^m u}{\partial n^m} = 0, & \text{on } \partial\Omega_\epsilon. \end{cases}$$

Let us set $W = W \times \{0\}$ and let $H_{\Omega,D}$ the polyharmonic operator satisfying strong intermediate boundary conditions on $\partial\Omega \setminus \overline{W}$ and Dirichlet boundary conditions on W. Here Dirichlet boundary conditions means that the boundary conditions on W read

$$\frac{\partial^l u}{\partial x_N^l} = 0, \quad \text{on } W, \text{ for all } 0 \le l \le m-1. \tag{4.2.3}$$

Then the following theorem holds.

Theorem 4.2.1 (Spectral convergence). *Let $m \in \mathbb{N}$, $m \ge 2$, Ω_ϵ as in* (4.2.1), *H_{Ω_ϵ} as in* (4.2.2), *for all $\epsilon > 0$. Then the following statements hold true.*

(i) [Spectral stability] *If $\alpha > 3/2$, then $H_{\Omega_\epsilon,S}^{-1} \xrightarrow{C} H_{\Omega,S}^{-1}$ as $\epsilon \to 0$.*

(ii) [Instability] *If $\alpha < 3/2$, then $H_{\Omega_\epsilon,S}^{-1} \xrightarrow{C} H_{\Omega,D}^{-1}$ as $\epsilon \to 0$.*

(iii) [Strange term] *If $\alpha = 3/2$, then $H_{\Omega_\epsilon,I}^{-1} \xrightarrow{C} \hat{H}_\Omega^{-1}$ as $\epsilon \to 0$, where $\hat{H}_\Omega$ is the operator $(-\Delta)^m + \mathbb{I}$ with intermediate boundary conditions on $\partial\Omega \setminus \overline{W}$ and the following boundary conditions on W: $D^l u = 0$, for all $l \le m-2$, $\partial_{x_N}^m u + K\partial_{x_N}^{m-1} u = 0$, where the factor K is given by*

$$K = \int_{Y\times(-\infty,0)} |D^m V|^2\,\mathrm{d}y = -\int_Y \left(\frac{\partial^{m-1}(\Delta V)}{\partial x_N^{m-1}} + (m-1)\Delta_{N-1}\left(\frac{\partial^{m-1} V}{\partial x_N^{m-1}}\right)\right) b(\bar{y})\mathrm{d}\bar{y},$$

and the function V is Y-periodic in the variable $\bar{y}$ and satisfies the following *microscopic problem*

$$\begin{cases} (-\Delta)^m V = 0, & \text{in } Y \times (-\infty, 0), \\ V(\bar{y}, 0) = 0, & \text{on } Y, \\ \frac{\partial^l V}{\partial y_N^l}(\bar{y}, 0) = 0, & \text{on } Y, \text{ for all } 1 \le l \le m-3, \\ \frac{\partial^{m-2} V}{\partial y_N^{m-2}}(\bar{y}, 0) = b(\bar{y}), & \text{on } Y, \\ \frac{\partial^m V}{\partial y_N^m}(\bar{y}, 0) = 0, & \text{on } Y. \end{cases}$$

Proof. (i) has already been proved in Theorem 2.2.4. The proof of (ii) is similar to the one of [19, Theorem 7.3]. The idea is to prove that Condition (C) (see Definition 2.2.1) is satisfied with $V(\Omega) = H^m_{0,W}(\Omega)$, where $H^m_{0,W}(\Omega)$ is the space of functions $u \in H^m(\Omega) \cap H^{m-1}_0(\Omega)$ satisfying the boundary conditions (4.2.3), and $V(\Omega_\epsilon) = H^m(\Omega_\epsilon) \cap H^{m-1}_0(\Omega_\epsilon)$. Let $K_\epsilon = \Omega$ for all $\epsilon > 0$. Then it is easy to see that condition (2.2.1) and condition (C1) are satisfied. We define now T_ϵ as the extension by zero operator through $W \times \{0\}$ and E_ϵ as the restriction operator to Ω. With these definitions it is not difficult to prove that conditions (C2) and (C3)(i),(ii) are satisfied. It remains to prove that condition (C3)(iii) holds. Let $v_\epsilon \in H^m(\Omega_\epsilon) \cap H^{m-1}_0(\Omega_\epsilon)$ be such that $\|v_\epsilon\|_{H^m(\Omega_\epsilon)} \le C$ for all $\epsilon > 0$. Possibly passing to a subsequence there exists a function $v \in H^{m-1}(\Omega)$ such that $v_\epsilon|_\Omega \rightharpoonup v$ in $H^m(\Omega)$ and $v_\epsilon|_\Omega \to v$ in $H^{m-1}(\Omega)$. By considering the sequence of functions $T_\epsilon(v_\epsilon|\Omega)$ it is not difficult to prove that $v \in H^{m-1}_0(\Omega)$. It remains to prove that $\frac{\partial^{m-1} v}{\partial x_N^{m-1}} = 0$ on $W \times \{0\}$. This is proven exactly as in [19, Theorem 7.3] by applying Lemma 4.3 from [43] to the vector field V^i_ϵ defined by

$$V^i_\epsilon = \left(0, \cdots, 0, -\frac{\partial^{m-1} v_\epsilon}{\partial x_N^{m-1}}, 0, \cdots, 0, \frac{\partial^{m-1} v_\epsilon}{\partial x_N^{m-2} \partial x_i}\right),$$

for all $i = 1, \ldots, N-1$, where the only non-zero entries are the i-th and the N-th ones. We remark that it is possible to apply Lemma 4.3 from [43] because by Lemma 2.2.4 the critical threshold for all the polyharmonics operator with strong intermediate boundary conditions is $\alpha = 3/2$, which coincides with the critical value in [43]. We then deduce that

$$\frac{\partial^{m-1} v(\bar{x}, 0)}{\partial x_N^{m-1}} \frac{\partial b(\bar{y})}{\partial y_i} = -\frac{\partial^{m-1} v(\bar{x}, 0)}{\partial x_N^{m-2} \partial x_i} = 0,$$

where the second equality follows from the fact that $v \in H^{m-1}_0(\Omega)$. This concludes the proof of condition $(C3)(iii)$.

We provide a proof of (iii) in the following sections, see Section 4.2.1 and Section 4.2.2. □

4.2.1 Critical case - Macroscopic problem.

Following the approach in [19], we will use the unfolding method from homogenization theory (see §1.2) in order to pass to the limit as $\epsilon \to 0$. Let us define a diffeomorphism Φ_ϵ from Ω_ϵ to Ω by

$$\Phi_\epsilon(\bar{x}, x_N) = (\bar{x}, x_N - h_\epsilon(\bar{x}, x_N)), \quad \text{for all } x = (\bar{x}, x_N) \in \Omega_\epsilon,$$

where h_ϵ is defined by

$$h_\epsilon(\bar{x}, x_N) = \begin{cases} 0, & \text{if } -1 \le x_N \le -\epsilon, \\ g_\epsilon(\bar{x})\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^{m+1}, & \text{if } -\epsilon \le x_N \le g_\epsilon(\bar{x}). \end{cases}$$

With this definition Φ_ϵ is a diffeomorphism of class C^m, even though the L^∞-norm of the higher order derivatives may blow up, as $\epsilon \to 0$. Moreover, one can prove the following

Lemma 4.2.2. *The map Φ_ϵ is a diffeomorphism of class C^m and there exists a constant $c > 0$ independent of ϵ such that*

$$|h_\epsilon| \le c\epsilon^\alpha, \quad \left|D^l h_\epsilon\right| \le c\epsilon^{\alpha-l},$$

for all $l = 1, \dots, m$, $\epsilon > 0$ sufficiently small.

Proof. Follows directly from the definition of h_ϵ. □

As in [19, Section 8.1], we introduce the pullback operator T_ϵ from $L^2(\Omega)$ to $L^2(\Omega_\epsilon)$ given by

$$T_\epsilon u = u \circ \Phi_\epsilon,$$

for all $u \in L^2(\Omega)$. See [19, Section 8.1] for the properties of this operator, with the trivial replacement of $W^{2,2}(\Omega)$ with $W^{m,2}(\Omega)$.

In the sequel we shall use the definitions and the properties of the anisotropic unfolding method introduced in §1.2.

Let $W^{m,2}_{\mathrm{Per}_Y,\mathrm{loc}}(Y \times (-\infty, 0))$ be the space of functions in $W^{m,2}_{\mathrm{loc}}(\mathbb{R}^{N-1} \times (-\infty, 0))$ which are Y-periodic in the first $(N-1)$ variables $\bar{y}$. Then we define $W^{m,2}_{loc}(Y \times (-\infty, 0))$ to be the space of functions in $W^{m,2}_{\mathrm{Per}_Y,\mathrm{loc}}(Y \times (-\infty, 0))$ restricted to $Y \times (-\infty, 0)$. Finally we set

$$w^{m,2}_{\mathrm{Per}_Y}(Y \times (-\infty, 0)) := \Big\{u \in W^{m,2}_{\mathrm{Per}_Y,\mathrm{loc}}(Y \times (-\infty, 0)) : \\ \|D^\gamma u\|_{L^2(Y\times(-\infty,0))} < \infty, \forall |\gamma| = m\Big\}.$$

Let $\mathcal{P}^l_{hom,y}(Y \times (d,0))$ be the set of homogeneous polynomials of degree at most l with domain $(Y \times (d,0))$, for given $d < 0$. Let $\epsilon > 0$ be fixed. We define the projectors P_i from $L^2(\widehat{W}_\epsilon, W^{m,2}(Y\times(-1/\epsilon,0)))$ to $L^2(\widehat{W}_\epsilon, \mathcal{P}^i_{hom,y}(-1/\epsilon,0))$ by setting

$$P_i\psi = \sum_{|\eta|=i} \int_Y D^\eta \psi(\bar{x}, \bar{\zeta}, 0) d\bar{\zeta} \frac{y^\eta}{\eta!} \tag{4.2.4}$$

for all $i = 0, \dots, m-1$, and we then define

$$\begin{aligned} Q_0 &= P_{m-1}; \\ Q_1 &= P_{m-2}(\mathbb{I} - Q_0); \\ Q_2 &= P_{m-3}(\mathbb{I} - Q_0 - Q_1); \\ &\dots \\ Q_{m-1} &= P_0\left(\mathbb{I} - \sum_{j=0}^{m-2} Q_j\right) \end{aligned}$$

Note that Q_{m-1-j}, $j = 0, \dots, m-1$ is a projection on the space of polynomials of degree j, with the property that $Q_{m-1-k}p = 0$ for all polynomials of degree at most $m-1$ such that $k > j$. We finally set

$$\mathcal{P} = Q_0 + Q_1 + \dots + Q_{m-1}, \tag{4.2.5}$$

which is a projector on the space of polynomials in y of degree at most $m-1$. Note that $D^\beta_y \psi_\epsilon(\bar{x}, \bar{y}, 0) - D^\beta_y \mathcal{P}(\psi_\epsilon)(\bar{x}, \bar{y}, 0)$ has null integral mean in $\bar{y}$ for all multiindexes $\beta \in \mathbb{N}^N$ such that $|\beta| = 0, \dots, m-1$.

Lemma 4.2.3. *Let $m \in \mathbb{N}$, $m \geq 2$ be fixed. The following statements hold:*

(i) Let $v_\epsilon \in W^{m,2}(\Omega)$ with $\|v_\epsilon\|_{W^{m,2}(\Omega)} \leq M$, for all $\epsilon > 0$. Let V_ϵ be defined by

$$V_\epsilon(\bar{x}, y) = \hat{v}_\epsilon(\bar{x}, y) - \mathcal{P}(v_\epsilon)(\bar{x}, y),$$

for $(\bar{x}, y) \in \widehat{W_\epsilon} \times Y \times (-1/\epsilon, 0)$, where $\mathcal{P}$ is defined by (4.2.5) *. Then there exists a function $\hat{v} \in L^2(W, w^{m,2}_{\mathrm{Per}_Y}(Y \times (-\infty, 0)))$ such that*

(a) $\frac{D^\gamma_y V_\epsilon}{\epsilon^{m-1/2}} \rightharpoonup D^\gamma_y \hat{v}$ in $L^2(W \times Y \times (d,0))$ as $\epsilon \to 0$, for any $d < 0$, for any $\gamma \in \mathbb{N}^N$, $|\gamma| \leq m-1$.

(b) $\frac{D^\gamma_y V_\epsilon}{\epsilon^{m-1/2}} = \frac{D^\gamma_y \hat{v}_\epsilon}{\epsilon^{m-1/2}} \rightharpoonup D^\gamma_y \hat{v}$ in $L^2(W \times Y \times (-\infty, 0))$ as $\epsilon \to 0$, for any $\gamma \in \mathbb{N}^N$, $|\gamma| = m$,

where it is understood that functions V_ϵ, $D_y^\gamma V_\epsilon$ are extended by zero to the whole of $W\times Y\times(-\infty,0)$ outside their natural domain of definition $\widehat{W_\epsilon}\times Y\times(-1/\epsilon,0)$.

(ii) Let $\psi \in W^{1,2}(\Omega)$. Then

$$\widehat{(T_\epsilon\psi)_{|\Omega}} \overset{\epsilon\to 0}{\to} \psi(\bar{x},0), \quad \text{in } L^2(W\times Y\times(-1,0)).$$

Proof. The proof is similar to the one in [19, Lemma 8.9]. The main idea is to note that $D_y^\gamma V_\epsilon = D_y^\gamma \hat{v}_\epsilon$ for any $|\gamma| = m$ and that

$$\left\| \frac{D_y^\gamma V_\epsilon}{\epsilon^{m-1/2}} \right\|_{L^2(\widehat{W}_\epsilon\times Y\times(-1/\epsilon,0))} \leq \|D^\gamma v_\epsilon\|_{L^2(\Omega)} \leq C,$$

according to the exact integration formula and the Poincaré inequality (of order m). To prove the periodicity of the function $\hat{v}$ we can apply an argument similar to the one contained in Lemma 4.3 in [43] to $D^{m-1}V_\epsilon$ to obtain that $D_y^{m-2}\hat{v}$ is periodic. Then we find out that $\hat{v}$ is also periodic because

$$\int_Y D_y^l \hat{v}(\bar{x},\bar{y},0)\mathrm{d}\bar{y} = 0.$$

for all $l = 1,\dots,m-2$. Indeed, all functions V_ϵ have this property, and the weak limit preserves the integral mean. □

Let $f_\epsilon \in L^2(\Omega_\epsilon)$ and $f \in L^2(\Omega)$ be such that $f_\epsilon \rightharpoonup f$ in $L^2(\mathbb{R}^N)$ as $\epsilon \to 0$, with the understanding that the functions are extended by zero outside their natural domains. Let $v_\epsilon \in V(\Omega_\epsilon) = W^{m,2}(\Omega_\epsilon)\cap W_0^{m-1,2}(\Omega_\epsilon)$ be such that

$$H_{\Omega_\epsilon,S}v_\epsilon = f_\epsilon, \tag{4.2.6}$$

for all $\epsilon > 0$ small enough. Then $\|v_\epsilon\|_{W^{m,2}(\Omega_\epsilon)} \leq M$ for all $\epsilon > 0$ sufficiently small, hence, possibly passing to a subsequence there exists $v \in W^{m,2}(\Omega)\cap W_0^{m-1,2}(\Omega)$ such that $v_\epsilon \rightharpoonup v$ in $W^{m,2}(\Omega)$ and $v_\epsilon \to v$ in $L^2(\mathbb{R}^N)$.
Let $\varphi \in V(\Omega) = W^{m,2}(\Omega)\cap W_0^{m-1,2}(\Omega)$ be a fixed test function. Since $T_\epsilon\varphi \in V(\Omega_\epsilon)$, by (4.2.6) we get

$$\int_{\Omega_\epsilon} D^m v_\epsilon : D^m T_\epsilon\varphi \,\mathrm{d}x + \int_{\Omega_\epsilon} v_\epsilon T_\epsilon\varphi \,\mathrm{d}x = \int_{\Omega_\epsilon} f_\epsilon T_\epsilon\varphi \,\mathrm{d}x, \tag{4.2.7}$$

and passing to the limit as $\epsilon \to 0$ we have that

$$\int_{\Omega_\epsilon} v_\epsilon T_\epsilon\varphi \,\mathrm{d}x \to \int_\Omega v\varphi \,\mathrm{d}x, \quad \int_{\Omega_\epsilon} f_\epsilon T_\epsilon\varphi \,\mathrm{d}x \to \int_\Omega f\varphi \,\mathrm{d}x.$$

Now consider the first integral in the right hand-side of (4.2.7). Set $K_\epsilon = W \times (-1, -\epsilon)$. By splitting the integral in three terms corresponding to $\Omega_\epsilon \setminus \Omega$, $\Omega \setminus K_\epsilon$ and K_ϵ and by arguing as in [19, Section 8.3] one can show that

$$\int_{K_\epsilon} D^m v_\epsilon : D^m \varphi \, \mathrm{d}x \to \int_\Omega D^m v : D^m \varphi \, \mathrm{d}x, \qquad \int_{\Omega_\epsilon \setminus \Omega} D^m v_\epsilon : D^m T_\epsilon \varphi \, \mathrm{d}x \to 0,$$

as $\epsilon \to 0$. Let us define Q_ϵ as in (3.2.5). We split again the remaining integral in two summands,

$$\int_{\Omega_\epsilon \setminus K_\epsilon} D^m v_\epsilon : D^m T_\epsilon \varphi \, \mathrm{d}x = \int_{\Omega_\epsilon \setminus (K_\epsilon \cup Q_\epsilon)} D^m v_\epsilon : D^m T_\epsilon \varphi \, \mathrm{d}x + \int_{Q_\epsilon} D^m v_\epsilon : D^m T_\epsilon \varphi \, \mathrm{d}x. \tag{4.2.8}$$

It is possible to prove (see [19, Section 8.3]) that

$$\int_{\Omega_\epsilon \setminus (K_\epsilon \cup Q_\epsilon)} D^m v_\epsilon : D^m T_\epsilon \varphi \, \mathrm{d}x \to 0,$$

as $\epsilon \to 0$. It remains to analyse the limit as $\epsilon \to 0$ of the last summand in the right-hand side of (4.2.8). Before doing this we state the following:

Lemma 4.2.4. *For all $y \in Y \times (-1, 0)$ and $i_j = 1, \dots, N$ ($j = 1, \dots, l$) the functions $\hat{h}_\epsilon(\bar{x}, y)$, $\widehat{\frac{\partial^l h_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_l}}}(\bar{x}, y)$ for all $l \leq m$, are independent of $\bar{x}$. Moreover, $\|\hat{h}_\epsilon\|_{L^\infty} = O(\epsilon^{3/2})$, $\left\| \widehat{\frac{\partial^l h_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_l}}}(\bar{x}, y) \right\|_{L^\infty} = O(\epsilon^{3/2 - l})$ for all $l \leq m$ as $\epsilon \to 0$, and*

$$\epsilon^{l-3/2} \widehat{\frac{\partial^l h_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_l}}}(\bar{x}, y) \to \frac{\partial^l (b(\bar{y})(y_N + 1)^{m+1})}{\partial y_{i_1} \cdots \partial y_{i_l}},$$

as $\epsilon \to 0$, for all $i_1, \dots, i_l = 1, \dots, N$, uniformly in $y \in Y \times (-1, 0)$, for all $l \in \{2, \dots, m\}$.

Proof. First of all note that the first part of the statement involving the asymptotic behaviour of $\widehat{h_\epsilon}$ as $\epsilon \to 0$ follows directly from Lemma 4.2.2 and the definition of the unfolding operator (see Definition 1.2.9).
Before proceeding we recall some basic notation; namely, we write $\mathcal{P}(A)$ to denote the set of all subsets of a given finite set A and Part(A) to denote the set of all possible partitions of A; that is, each element of $\pi \in$ Part(A) is a set containing as elements pairwise disjoint subsets of A whose union is A. Moreover we use the symbol $|A|$ to denote the cardinality of A; hence, for example $|\pi|$ with $\pi \in$ Part(A) is the number of subsets of A in the partition π. We recall now some calculus formulas regarding derivatives of arbitrary order of a product or a composition of

functions. Let Ω be an open set in $\mathbb{R}^N$, I an open set in $\mathbb{R}$. Let f be a C^n-function from I to $\mathbb{R}$ and Φ be a C^n function from Ω to I. Then we have a multivariate Faa di Bruno formula:

$$\frac{\partial^n f(\Phi(x))}{\partial x_{i_1}\cdots\partial x_{i_n}} = \sum_{\pi\in\mathrm{Part}\{1,\dots,n\}} f^{(|\pi|)}(\Phi(x))\prod_{S\in\pi}\frac{\partial^{|S|}\Phi(x)}{\prod_{j\in S}\partial x_{i_j}}, \tag{4.2.9}$$

and a multidimensional Leibnitz formula given by

$$\frac{\partial^n(uv)}{\partial x_{i_1}\cdots\partial x_{i_n}} = \sum_{S\in\mathcal{P}(\{1,\dots,n\})}\frac{\partial^{|S|}u}{\prod_{j\in S}\partial x_{i_j}}\frac{\partial^{(n-|S|)}v}{\prod_{j\notin S}\partial x_{i_j}}, \tag{4.2.10}$$

where u, v are C^n- functions from $\Omega\subset\mathbb{R}^N$ to $\mathbb{R}$ and the notation $j\notin S$ means that j lies in the complementary of S in $\{1,\dots,n\}$. These formulas can be proved by induction on n. Let $l\in\mathbb{N}$, $l>1$ be fixed. By applying formula (4.2.10) we have that

$$\widehat{\frac{\partial^l h_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_l}}}(\bar{x},y) = \sum_{S\in\mathcal{P}(\{1,\dots,l\})}\epsilon^{\alpha-|S|}\frac{\partial^{|S|}b(\bar{y})}{\prod_{j\in S}\partial y_{i_j}}\widehat{\frac{\partial^{l-|S|}}{\prod_{j\notin S}\partial x_{i_j}}}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^{m+1}. \tag{4.2.11}$$

Moreover, by applying formula (4.2.10) and (4.2.9) we deduce that

$$\begin{aligned}\widehat{\frac{\partial^{l-|S|}}{\prod_{j\notin S}\partial x_{i_j}}}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^{m+1} &= \frac{(m+1)!}{(m+1-l+|S|)!}\epsilon^{-l+|S|}\frac{(y_N+1)^{m+1-l+|S|}}{(\epsilon^{\alpha-1}b(\bar{y})+1)^{m+1}}\prod_{j\notin S}\delta_{i_jN}\\ &+\sum_{\substack{\Lambda\in\mathcal{P}(S^C)\\ \Lambda\neq\emptyset}}\sum_{\pi\in\mathrm{Part}(\{i_j\}_{j\in\Lambda})}\epsilon^{\alpha|\pi|-|\pi|-l+|S|}(-1)^{|\pi|}\frac{(m+|\pi|)!}{m!}\frac{(m+1)!}{(m+1-l+|S|+|\Lambda|)!}\\ &\qquad\frac{(y_N+1)^{m+1-l+|S|+|\Lambda|}}{(\epsilon^{\alpha-1}+1)^{m+1+|\pi|}}\prod_{k\in(S^C\setminus\Lambda)}\delta_{i_kN}\prod_{B\in\pi}\frac{\partial^{|B|}b(\bar{y})}{\prod_{l\in B}\partial y_{i_l}}.\end{aligned} \tag{4.2.12}$$

Indeed, if we set for simplicity $\theta = l-|S|$ we have

$$\begin{aligned}\frac{\partial^\theta}{\prod_{j\notin S}\partial x_{i_j}}\left(\frac{x_N+\epsilon}{g_\epsilon(\bar{x})+\epsilon}\right)^{m+1} &= \sum_{\Lambda\in\mathcal{P}(S^C)}\frac{\partial^{\theta-|\Lambda|}(x_N+\epsilon)^{m+1}}{\prod_{k\notin\Lambda}\partial x_{i_k}}\frac{\partial^{|\Lambda|}(g_\epsilon(\bar{x})+\epsilon)^{-(m+1)}}{\prod_{k\in\Lambda}\partial x_{i_k}}\\ &= \frac{\partial^\theta(x_N+\epsilon)^{m+1}}{\prod_{j\notin S}\partial x_{i_j}}(g_\epsilon(\bar{x})+\epsilon)^{-(m+1)}+\sum_{\substack{\Lambda\in\mathcal{P}(S^C)\\ \Lambda\neq\emptyset}}\frac{\partial^{\theta-|\Lambda|}(x_N+\epsilon)^{m+1}}{\prod_{k\notin\Lambda}\partial x_{i_k}}\frac{\partial^{|\Lambda|}(g_\epsilon(\bar{x})+\epsilon)^{-(m+1)}}{\prod_{k\in\Lambda}\partial x_{i_k}},\end{aligned} \tag{4.2.13}$$

where we have used Formula (4.2.10) and we have written separately the cases in which $\Lambda = \emptyset$ and $\Lambda \neq \emptyset$. We now note that $(g_\epsilon(\bar{x}) + \epsilon)^{-(m+1)} = y^{-(m+1)} \circ (g_\epsilon(\bar{x}) + \epsilon)$ hence we can apply Formula (4.2.9) with $f(y) = y^{-m+1}$ and $\Phi(x) = (g_\epsilon(\bar{x}) + \epsilon)$ in order to get

$$\frac{\partial^{|\Lambda|}(g_\epsilon(\bar{x}) + \epsilon)^{-(m+1)}}{\prod_{k\in\Lambda} \partial x_{i_k}} = \sum_{\pi\in\mathrm{Part}(\{i_k : k\in\Lambda\})} D^{(|\pi|)}\big(y^{-(m+1)}\big)|_{y=(g_\epsilon(\bar{x})+\epsilon)} \prod_{B\in\pi} \frac{\partial^{|B|}(g_\epsilon(\bar{x}) + \epsilon)}{\prod_{l\in B} \partial x_{i_l}}. \tag{4.2.14}$$

By inserting the right-hand side of (4.2.14) in the right-hand side of (4.2.13) and by writing explicitly the derivatives of $(x_N + \epsilon)^{m+1}$ and $(g_\epsilon(\bar{x}) + \epsilon)^{-(m+1)}$ we finally deduce (4.2.12).

By (4.2.11) and (4.2.12) we deduce that

$$\begin{aligned}
&\epsilon^{l-\alpha} \frac{\widehat{\partial^l h_\epsilon}}{\partial x_{i_1} \cdots \partial x_{i_l}}(\bar{x}, y) \\
&= \epsilon^{l-\alpha} \sum_{S\in\mathcal{P}(\{1,\dots,l\})} \epsilon^{\alpha-|S|} \frac{\partial^{|S|} b(\bar{y})}{\prod_{j\in S} \partial y_{i_j}} \frac{(m+1)!}{(m+1-l+|S|)!} \epsilon^{-l+|S|} \frac{(y_N+1)^{m+1-l+|S|}}{(\epsilon^{\alpha-1} b(\bar{y}) + 1)^{m+1}} \prod_{j\notin S} \delta_{i_j N} \\
&+ \epsilon^{l-\alpha} \sum_{S\in\mathcal{P}(\{1,\dots,l\})} \epsilon^{\alpha-|S|} \frac{\partial^{|S|} b(\bar{y})}{\prod_{j\in S} \partial y_{i_j}} \sum_{\substack{\Lambda\in\mathcal{P}(S^C)\\ \Lambda\neq\emptyset}} \sum_{\pi\in\mathrm{Part}(\{i_j\}_{j\in\Lambda})} \epsilon^{\alpha|\pi|-|\pi|-l+|S|} (-1)^{|\pi|} \frac{(m+|\pi|)!}{m!} \\
&\frac{(m+1)!}{(m+1-l+|S|+|\Lambda|)!} \frac{(y_N+1)^{m+1-l+|S|+|\Lambda|}}{(\epsilon^{\alpha-1}+1)^{m+1+|\pi|}} \prod_{k\in(S^C\setminus\Lambda)} \delta_{i_k N} \prod_{B\in\pi} \frac{\partial^{|B|} b(\bar{y})}{\prod_{l\in B} \partial y_{i_l}}.
\end{aligned} \tag{4.2.15}$$

Now note that

$$\begin{aligned}
&\epsilon^{l-\alpha} \sum_{S\in\mathcal{P}(\{1,\dots,l\})} \epsilon^{\alpha-|S|} \frac{\partial^{|S|} b(\bar{y})}{\prod_{j\in S} \partial y_{i_j}} \frac{(m+1)!}{(m+1-l+|S|)!} \epsilon^{-l+|S|} \frac{(y_N+1)^{m+1-l+|S|}}{(\epsilon^{\alpha-1} b(\bar{y}) + 1)^{m+1}} \prod_{j\notin S} \delta_{i_j N} \\
&= \sum_{S\in\mathcal{P}(\{1,\dots,l\})} \frac{\partial^{|S|} b(\bar{y})}{\prod_{j\in S} \partial y_{i_j}} \frac{(m+1)!}{(m+1-l+|S|)!} \frac{(y_N+1)^{m+1-l+|S|}}{(\epsilon^{\alpha-1} b(\bar{y}) + 1)^{m+1}} \prod_{j\notin S} \delta_{i_j N},
\end{aligned}$$

hence it is of order 0 in ϵ. On the contrary, all the other summands appearing in the right-hand side of (4.2.15) are of order

$$\epsilon^{l-\alpha} \epsilon^{\alpha-|S|} \epsilon^{\alpha|\pi|-|\pi|-l+|S|} = \epsilon^{\alpha|\pi|-|\pi|},$$

and since $|\pi| \geq 1$ (because $\pi \in \mathrm{Part}(\{i_j\}_{j\in\Lambda})$, $\Lambda \neq \emptyset$) and $\alpha = 3/2 > 1$ we deduce that all the summands of order $\epsilon^{\alpha|\pi|-|\pi|}$ vanish as $\epsilon \to 0$. By letting $\epsilon \to 0$ in

(4.2.15) we see that

$$\lim_{\epsilon\to 0}\epsilon^{l-\alpha}\widehat{\frac{\partial^l h_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_l}}}(\bar{x},y)$$
$$= \sum_{S\in\mathcal{P}(\{1,\dots,l\})}\frac{\partial^{|S|}b(\bar{y})}{\prod_{j\in S}\partial y_{i_j}}\frac{(m+1)!}{(m+1-l+|S|)!}(y_N+1)^{m+1-l+|S|}\prod_{j\notin S}\delta_{i_jN}$$
$$= \frac{\partial^l}{\partial y_{i_1}\cdots\partial y_{i_l}}\big(b(\bar{y})(y_N+1)^{m+1}\big),$$

concluding the proof. □

Now we have the tools to prove the following

Proposition 4.2.5. *Let $v_\epsilon\in V(\Omega_\epsilon)$ be such that $\|v_\epsilon\|_{W^{m,2}(\Omega_\epsilon)}\le M$ for all $\epsilon>0$. Let $v\in V(\Omega)$ be the weak limit of $(v_\epsilon|_\Omega)_\epsilon$ in $W^{m,2}(\Omega)$. Let $g(y)=b(\bar{y})(1+y_N)^{m+1}$ for all $y\in Y\times(-1,0)$. Let $\hat{v}\in L^2(W,w^{m,2}_{Per_Y}(Y\times(-\infty,0)))$ be as in Lemma 4.2.3. Then*

$$\int_{Q_\epsilon}D^m v_\epsilon : D^m(T_\epsilon\varphi)\,\mathrm{d}x\to$$
$$-\sum_{l=1}^{m-1}\binom{m}{l+1}\int_W\int_{Y\times(-1,0)}\frac{y_N^{l-1}}{(l-1)!}D_y^{l+1}\left(\frac{\partial^{m-l-1}\hat{v}(\bar{x},y)}{\partial y_N^{m-l-1}}\right):D_y^{l+1}g(y)\,\mathrm{d}y\,\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\mathrm{d}\bar{x},$$

for all $\varphi\in W^{m,2}(\Omega)\cap W^{m-1,2}(\Omega)$, as $\epsilon\to 0$.

Proof. In the following calculations we use the index notation and we drop the summation symbol $\sum_{i_1,\cdots,i_m=1}^N$. Moreover we will use the notation Part(A) as in the proof of Lemma 4.2.4 and we define

$$P_1(t)=\{\pi=(S_1,\dots,S_t)\in\mathrm{Part}(\{1,\dots,m\}):\exists! S_k\text{ with }|S_k|>1\},$$

$$P_2(t)=\{\pi\in\mathrm{Part}(\{1,\dots,m\}):|\pi|=t,\pi\notin P_1(t)\}.$$

We note that in the definition of $P_1(t)$ we may assume without loss of generality that the only element S_k with cardinality strictly bigger than 1 is S_1. In the following calculations we always assume that a given partition π of cardinality t

is given by $\pi = \{S_1, \dots, S_t\}$. With the help of (4.2.9) we calculate

$$\begin{aligned}
\int_{Q_\epsilon} D^m v_\epsilon : D^m(T_\epsilon \varphi)\, \mathrm{d}x &= \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^m (\varphi \circ \Phi_\epsilon)}{\partial x_{i_1} \cdots \partial x_{i_m}}\, \mathrm{d}x \\
&= \sum_{\substack{\pi \in \mathrm{Part}(\{1,\dots,m\}) \\ \pi = \{S_1,\dots,S_{|\pi|}\}}} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^{|\pi|} \varphi}{\prod_{k=1}^{|\pi|} \partial x_{j_k}} (\Phi_\epsilon(x)) \prod_{k=1}^{|\pi|} \frac{\partial^{|S_k|} \Phi_\epsilon^{(j_k)}}{\prod_{l \in S_k} \partial x_{i_l}} \mathrm{d}x \\
&= \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^m \varphi}{\partial x_{j_1} \cdots \partial x_{j_m}} (\Phi_\epsilon(x)) \frac{\partial \Phi_\epsilon^{(j_1)}}{\partial x_{i_1}} \cdots \frac{\partial \Phi_\epsilon^{(j_m)}}{\partial x_{i_m}} \mathrm{d}x, \\
&+ \sum_{t=1}^{m-1} \sum_{\pi \in P_1(t)} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^t \varphi}{\prod_{k=1}^{t} \partial x_{j_k}} (\Phi_\epsilon(x)) \prod_{k=1}^{t} \frac{\partial^{|S_k|} \Phi_\epsilon^{(j_k)}}{\prod_{l \in S_k} \partial x_{i_l}} \mathrm{d}x \\
&+ \sum_{t=2}^{m-2} F_t(v_\epsilon, \varphi, \Phi_\epsilon),
\end{aligned} \tag{4.2.16}$$

where $F_t(v_\epsilon, \varphi, \Phi_\epsilon)$ is defined by

$$F_t(v_\epsilon, \varphi, \Phi_\epsilon) = \sum_{\pi \in P_2(t)} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^t \varphi}{\prod_{k=1}^{t} \partial x_{j_k}} \prod_{k=1}^{t} \frac{\partial^{|S_k|} \Phi_\epsilon^{(j_k)}}{\prod_{l \in S_k} \partial x_{i_l}} dx.$$

We consider separately the three summands in the right hand side of (4.2.16). Let us remark for future use that

$$\frac{\partial \Phi_\epsilon^{(k)}}{\partial x_i} = \begin{cases} \delta_{ki}, & \text{if } k \neq N, \\ \delta_{Ni} - \frac{\partial h_\epsilon}{\partial x_i}, & \text{if } k = N. \end{cases}$$

whence

$$\frac{\partial^l \Phi_\epsilon^{(k)}}{\partial x_{i_1} \cdots \partial x_{i_l}} = \begin{cases} 0, & \text{if } k \neq N, \\ -\frac{\partial^l h_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_l}}, & \text{if } k = N. \end{cases}$$

for all $2 \leq l \leq m$. Consider now the first term in the right hand side of (4.2.16) .

We unfold it by taking into account (1.2.4) in order to obtain

$$
\begin{aligned}
&\left|\epsilon\int_{\hat{W}_\epsilon}\int_{Y\times(-1,0)}\widehat{\frac{\partial^m v_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_m}}}\frac{\partial^m\varphi}{\partial x_{j_1}\cdots\partial x_{j_m}}(\hat{\Phi}_\epsilon(y))\,\widehat{\frac{\partial\Phi_\epsilon^{(j_1)}}{\partial x_{i_1}}}\cdots\widehat{\frac{\partial\Phi_\epsilon^{(j_m)}}{\partial x_{i_m}}}\,dyd\bar{x}\right|\\
&=\epsilon^{-2m+1}\left|\int_{\hat{W}_\epsilon}\int_{Y\times(-1,0)}\frac{\partial^m\hat{v}_\epsilon}{\partial y_{i_1}\cdots\partial y_{i_m}}\frac{\partial^m\varphi}{\partial x_{j_1}\cdots\partial x_{j_m}}(\hat{\Phi}_\epsilon(y))\,\frac{\partial\widehat{\Phi}_\epsilon^{(j_1)}}{\partial y_{i_1}}\cdots\frac{\partial\widehat{\Phi}_\epsilon^{(j_m)}}{\partial y_{i_m}}\,dyd\bar{x}\right|\\
&\le C\epsilon^{-2m+1}\epsilon^{m-1/2}\epsilon^m\int_{\hat{W}_\epsilon}\int_{Y\times(-1,0)}\left|\epsilon^{-m+1/2}\frac{\partial^m\hat{v}_\epsilon}{\partial y_{i_1}\cdots\partial y_{i_m}}\frac{\partial^m\varphi}{\partial x_{j_1}\cdots\partial x_{j_m}}(\hat{\Phi}_\epsilon(y))\right|dyd\bar{x}\\
&\le C\epsilon^{1/2}\left\|\epsilon^{-m+1/2}\frac{\partial^m\hat{v}_\epsilon}{\partial y_{i_1}\cdots\partial y_{i_m}}\right\|_{L^2(\hat{W}_\epsilon\times Y\times(-1,0))}\left\|\frac{\partial^m\varphi}{\partial x_{j_1}\cdots\partial x_{j_m}}(\hat{\Phi}_\epsilon(y))\right\|_{L^2(\hat{W}_\epsilon\times Y\times(-1,0))}\\
&\le C\epsilon^{1/2}\left\|\frac{\partial^m\varphi}{\partial x_{j_1}\cdots\partial x_{j_m}}(\hat{\Phi}_\epsilon(y))\right\|_{L^2(\hat{W}_\epsilon\times Y\times(-1,0))}\\
&\le C\left\|\frac{\partial^m\varphi}{\partial x_{j_1}\cdots\partial x_{j_m}}\right\|_{L^2(\Phi_\epsilon(Q_\epsilon))},
\end{aligned}
$$

which vanishes as $\epsilon\longrightarrow 0$. In the first inequality we have used the fact that

$$
\frac{\partial\hat{\Phi}_\epsilon^{(k)}}{\partial y_i}=\begin{cases}\epsilon\delta_{ki}, & \text{if } k\neq N,\\ \epsilon\delta_{Ni}-\epsilon\widehat{\frac{\partial h_\epsilon}{\partial x_i}}, & \text{if } k=N.\end{cases}
$$

hence in particular

$$
\left|\frac{\partial\hat{\Phi}_\epsilon^{(k)}}{\partial y_i}\right|\le C\epsilon,
$$

for sufficiently small $\epsilon>0$. We have also used the fact that

$$
\left\|\epsilon^{-m+1/2}\frac{\partial^m\hat{v}_\epsilon}{\partial y_{i_1}\cdots\partial y_{i_m}}\right\|_{L^2(\hat{W}_\epsilon\times Y\times(-1,0))}
$$

is bounded uniformly in ϵ, which is a consequence of Lemma 4.2.3. Let now $1\le t\le m-1$ be fixed and consider

$$
\begin{aligned}
&\sum_{\pi\in P_1(t)}\int_{Q_\epsilon}\frac{\partial^m v_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_m}}\frac{\partial^t\varphi}{\prod_{k=1}^t\partial x_{j_k}}(\Phi_\epsilon(x))\prod_{k=1}^t\frac{\partial^{|S_k|}\Phi_\epsilon^{(j_k)}}{\prod_{l\in S_k}\partial x_{i_l}}\mathrm{d}x\\
&\quad=\sum_{\pi\in P_1(t)}\int_{Q_\epsilon}\frac{\partial^m v_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_m}}\frac{\partial^t\varphi}{\prod_{k=1}^t\partial x_{j_k}}(\Phi_\epsilon(x))\frac{\partial^{m-t+1}\Phi_\epsilon^{(j_1)}}{\prod_{l\in S_1}\partial x_{i_l}}\frac{\partial\Phi_\epsilon^{(j_2)}}{\partial x_{i_{S_2}}}\cdots\frac{\partial\Phi_\epsilon^{(j_t)}}{\partial x_{i_{S_t}}}\mathrm{d}x,
\end{aligned}
\tag{4.2.17}
$$

where to shorten the notation we have identified $S_2, \dots, S_t$ with the only element they contain. Note that if $j_1 \neq N$ then the integral in (4.2.17) is zero. Thus, without loss of generality we set $j_1 = N$.

It is also possible to prove that as $\epsilon \to 0$ the lowest order terms in ϵ appear when $j_2 = \cdots = j_t = N$ and $\frac{\partial \Phi_\epsilon^{(N)}}{\partial x_{i_t}} = \delta_{N i_t}$ (note that we have $\frac{\partial \Phi_\epsilon^{(N)}}{\partial x_{i_t}} = \delta_{N i_t} + \frac{\partial h_\epsilon}{\partial x_{i_t}}$ and $\left|\frac{\partial h_\epsilon}{\partial x_{i_t}}\right| \leq C\epsilon^{1/2}$ as $\epsilon \to 0$). We give a general explanation of this fact here, and we refer to Lemma 3.2.5 in Chapter 3 where more details and computations are provided in the case of the triharmonic operator. The difference in the rate of decay in ϵ is due to the fact that $\varphi \in H^m(\Omega) \cap H_0^{m-1}(\Omega)$, hence the derivatives in the form

$$\frac{\partial^t \varphi}{\partial x_{j_1} \cdots \partial x_{j_t}},$$

with $1 \leq t \leq m-1$ and with at least one index $j_t \neq N$, $t = 1, \dots, m-l$ are in $H_0^{m-t}(\Omega)$. Thus, they will satisfy a one-dimensional H_0^{m-t}-Poincaré inequality in the x_N direction, namely

$$\begin{aligned}\left\|\frac{\partial^t \varphi}{\partial x_{j_1} \cdots \partial x_{j_t}}\right\|_{L^2(W\times(-\epsilon,0))} &\leq C\epsilon^{m-t}\left\|\frac{\partial^m \varphi}{\partial x_{j_1} \cdots \partial x_{j_t} \partial x_N^{m-t}}\right\|_{L^2(W\times(-\epsilon,0))} \\ &\leq C\epsilon^{m-t}\|\varphi\|_{H^m(\Omega)}.\end{aligned}$$

Note that instead the normal derivative of order t, $\partial_{x_N}^t \varphi$ is not in $H_0^{m-t}(\Omega)$, but in $H^{m-t}(\Omega) \cap H_0^{m-t-1}(\Omega)$; hence, it satisfies at most an inequality in the following form

$$\left\|\frac{\partial^t \varphi}{\partial x_N^t}\right\|_{L^2(W\times(-\epsilon,0))} \leq C\epsilon^{m-t-1/2}\|\varphi\|_{H^m(\Omega)},$$

that follows from the one-dimensional Poincaré inequality in $H_0^{m-t-1}(\Omega)$ and the standard boundedness of H^1 functions along almost all lines parallel to the e_N-axis. We deduce that the normal derivatives of order t have a slower decay than the other derivatives of order t. In order to simplify the expressions we will not write down the higher order terms in ϵ. Hence, by setting $j_1 = \cdots = j_t = N$ in (4.2.17) we deduce that

$$\begin{aligned}&\sum_{\pi \in P_1(t)} \int_{Q_\epsilon} \frac{\partial^m v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_m}} \frac{\partial^t \varphi}{\partial x_N^t}(\Phi_\epsilon(x)) \frac{\partial^{m-t+1}\Phi_\epsilon^{(N)}}{\prod_{l \in S_1} \partial x_{i_l}} \delta_{i_{S_2} N} \cdots \delta_{i_{S_t} N} \mathrm{d}x \\ &= \sum_{\pi \in P_1(t)} \int_{Q_\epsilon} \frac{\partial^t \varphi}{\partial x_N^t}(\Phi_\epsilon(x)) \frac{\partial^m v_\epsilon}{\prod_{l \in S_1} \partial x_{i_l} \partial x_N^{t-1}} \frac{\partial^{m-t+1}\Phi_\epsilon^{(N)}}{\prod_{l \in S_1} \partial x_{i_l}} \mathrm{d}x \\ &= \binom{m}{t-1} \int_{Q_\epsilon} \frac{\partial^t \varphi}{\partial x_N^t}(\Phi_\epsilon(x)) \frac{\partial^m v_\epsilon}{\prod_{l \in S_1} \partial x_{i_l} \partial x_N^{t-1}} \frac{\partial^{m-t+1}\Phi_\epsilon^{(N)}}{\prod_{l \in S_1} \partial x_{i_l}} \mathrm{d}x,\end{aligned} \tag{4.2.18}$$

where in the last equality in (4.2.18) we have used the fact that each of the summands

$$\int_{Q_\epsilon} \frac{\partial^t \varphi}{\partial x_N^t}(\Phi_\epsilon(x)) \frac{\partial^m v_\epsilon}{\prod_{l\in S_1} \partial x_{i_l} \partial x_N^{t-1}} \frac{\partial^{m-t+1}\Phi_\epsilon^{(N)}}{\prod_{l\in S_1} \partial x_{i_l}} \,\mathrm{d}x$$

equals

$$\int_{Q_\epsilon} \frac{\partial^t \varphi}{\partial x_N^t}(\Phi_\epsilon(x))\, D^{m-t+1}\left(\frac{\partial^{t-1} v_\epsilon}{\partial x_N^{t-1}}\right) : D^{m-t+1}\Phi_\epsilon^{(N)} \,\mathrm{d}x.$$

In particular they do not depend on the choice of π. Note moreover that the cardinality of $P_1(t)$ equals the number of choices of $m-t+1$ differentiation indexes among m elements without repetitions, that is $|P_1(t)| = \binom{m}{m-t+1} = \binom{m}{t-1}$. By unfolding the right-hand side of (4.2.18) we have that

$$\begin{aligned} &\binom{m}{t-1}\epsilon \int_{\hat{W}_\epsilon}\int_{Y\times(-1,0)} \frac{\partial^t \varphi}{\partial x_N^t}(\hat{\Phi}_\epsilon(y)) \widehat{\frac{\partial^m v_\epsilon}{\prod_{l\in S_1} \partial x_{i_l} \partial x_N^{t-1}}}\, \widehat{\frac{\partial^{m-t+1}\Phi_\epsilon^{(N)}}{\prod_{l\in S_1} \partial x_{i_l}}} dy d\bar{x} \\ &= -\binom{m}{t+1}\epsilon^{1/2} \int_{\hat{W}_\epsilon}\int_{Y\times(-1,0)} \epsilon^{-m+1/2} \frac{\partial^m \hat{v}_\epsilon}{\prod_{l\in S_1} \partial y_{i_l} \partial y_N^{t-1}} \frac{\partial^t \varphi}{\partial x_N^t}(\hat{\Phi}_\epsilon(y)) \frac{\widehat{\partial^{m-t+1} h_\epsilon}}{\prod_{l\in S_1} \partial x_{i_l}} dy d\bar{x}. \end{aligned} \tag{4.2.19}$$

It is easy to see that the final expression appearing in the right-hand side of (4.2.19) equals

$$\begin{aligned} -\binom{m}{t+1} \int_{\hat{W}_\epsilon}\int_{Y\times(-1,0)} \left[\epsilon^{-m+1/2} \frac{\partial^m \hat{v}_\epsilon}{\prod_{l\in S_1} \partial y_{i_l} \partial y_N^{t-1}}\right]\left[\frac{1}{\epsilon^{m-t-1}} \frac{\partial^t \varphi}{\partial x_N^t}(\hat{\Phi}_\epsilon(y))\right] \\ \left[\epsilon^{m-t+1-3/2} \frac{\widehat{\partial^{m-t+1} h_\epsilon}}{\prod_{l\in S_1} \partial x_{i_l}}\right] dy d\bar{x}. \end{aligned} \tag{4.2.20}$$

Now

$$\epsilon^{-m+1/2} \frac{\partial^m \hat{v}_\epsilon}{\prod_{l\in S_1} \partial y_{i_l} \partial y_N^{t-1}} \rightharpoonup \frac{\partial^m \hat{v}}{\prod_{l\in S_1} \partial y_{i_l} \partial y_N^{t-1}},$$

weakly in $L^2(\widehat{W}_\epsilon \times Y \times (-1,0))$ as $\epsilon \to 0$, by Lemma 4.2.3, and

$$\epsilon^{m-t+1-3/2} \frac{\widehat{\partial^{m-t+1} h_\epsilon}}{\prod_{l\in S_1} \partial x_{i_l}} \to \frac{\partial^{m-t+1}(b(\bar{y})(1+y_N)^{m+1})}{\prod_{l\in S_1} \partial y_{i_l}},$$

in $L^\infty(\widehat{W}_\epsilon \times Y \times (-1,0))$ as $\epsilon \to 0$, by Lemma 4.2.4. Moreover it is possible to prove that

$$\frac{1}{\epsilon^{m-t-1}} \frac{\partial^t \varphi}{\partial x_N^t}(\hat{\Phi}_\epsilon(y)) \to \frac{y_N^{m-t-1}}{(m-t-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0),$$

strongly in $L^2(\widehat{W}_\epsilon \times Y \times (-1,0))$ as $\epsilon \to 0$ (we give a proof of this fact in Lemma 4.2.6 below). Hence (4.2.20) tends to

$$-\binom{m}{t+1}\int_{\hat{W}_\epsilon}\int_{Y\times(-1,0)} \frac{y_N^{m-t-1}}{(m-t-1)!} D_y^{m-t+1}\left(\frac{\partial^{t-1}\hat{v}}{\partial y_N^{t-1}}\right): \\ D_y^{m-t+1}\Big(b(\bar{y})(1+y_N)^{m+1}\Big)dy\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)d\bar{x}.$$

By setting $m-t=l$ we recover the thesis of the proposition. Then, in order to conclude the proof it is sufficient to prove that the integrals in $F_t(v_\epsilon,\varphi,\Phi_\epsilon)$ vanish as $\epsilon \to 0$. We will show this by comparing each integral appearing in the definition of $F_t(v_\epsilon,\varphi,\Phi_\epsilon)$ with the corresponding integral of the form (4.2.17), which is convergent as $\epsilon \to 0$, hence it is uniformly bounded in ϵ. Note that

$$\epsilon^{-m}\frac{\partial^{m-t+1}\hat{\Phi}_\epsilon^{(j_1)}}{\prod_{l\in S_1}\partial y_{i_l}}\frac{\partial\hat{\Phi}_\epsilon^{(j_2)}}{\partial y_{i_{S_2}}}\cdots\frac{\partial\hat{\Phi}_\epsilon^{(j_t)}}{\partial y_{i_{S_t}}} = O(\epsilon^{3/2-m+t-1}) = O(\epsilon^{1/2-m-t}),$$

for all $\pi \in P_1(t)$, by Lemma 4.2.4, whereas as soon as we consider a partition $\pi' \in P_2(t)$ with $|S_1'| = m-t < m-t+1$ there must exists S_k', $k>1$ with $|S_k'| = 2$. Let us assume that $k=2$. Then we have

$$\epsilon^{-m}\frac{\partial^{m-t}\hat{\Phi}_\epsilon^{(j_1)}}{\prod_{l\in S_1'}\partial y_{i_l}}\frac{\partial^2\hat{\Phi}_\epsilon^{(j_2)}}{\prod_{l\in S_2'}\partial y_{i_l}}\frac{\partial\hat{\Phi}_\epsilon^{(j_3)}}{\partial y_{i_{S_3'}}}\cdots\frac{\partial\hat{\Phi}_\epsilon^{(j_t)}}{\partial y_{i_{S_t'}}} = O(\epsilon^{3/2-m+t}\epsilon^{3/2-2}) = O(\epsilon^{1-m+t}),$$

and since $\epsilon^{1-m+t} = o\big(\epsilon^{1/2-m+t}\big)$ as $\epsilon \to 0$ and the integral (4.2.17) is bounded, we deduce that the integral in $F_t(v_\epsilon,\varphi,\Phi_\epsilon)$ involving

$$\frac{\partial^m v_\epsilon}{\partial x_{i_1}\cdots\partial x_{i_m}}\frac{\partial^t\varphi}{\prod_{k=1}^t\partial x_{j_k}}\frac{\partial^{m-t}\hat{\Phi}_\epsilon^{(j_1)}}{\prod_{l\in S_1'}\partial y_{i_l}}\frac{\partial^2\hat{\Phi}_\epsilon^{(j_2)}}{\prod_{l\in S_2'}\partial y_{i_l}}\frac{\partial\hat{\Phi}_\epsilon^{(j_3)}}{\partial y_{i_{S_3'}}}\cdots\frac{\partial\hat{\Phi}_\epsilon^{(j_t)}}{\partial y_{i_{S_t'}}},$$

for all $\pi' \in P_2(t)$ defined above, vanishes as $\epsilon \to 0$. By arguing in a similar way for all the terms in $F_t(v_\epsilon,\varphi,\Phi_\epsilon)$ we deduce the validity of the statement. □

Lemma 4.2.6. *Let m, t as in the proof of Proposition 4.2.5. Define $l := m-t \in \mathbb{N}$, $1 \le l \le m-1$. Then*

$$\frac{1}{\epsilon^{l-1}}\frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\hat{\Phi}_\epsilon(y)) \to \frac{y_N^{l-1}}{(l-1)!}\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0),$$

in $L^2(W \times Y \times (-1,0)$ as $\epsilon \to 0$.

Proof. Note that for $l = 1$ the statement is trivial. Then without loss of generality we assume $l > 1$. Assume $\varphi \in H^m(\Omega) \cap H_0^{m-1}(\Omega) \cap C^\infty(\Omega)$. Then

$$
\begin{aligned}
&\int_{\widehat{W_\epsilon}\times Y\times(-1,0)} \left| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\hat{\Phi}_\epsilon(y)) - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0) \right|^2 d\bar{x}dy \\
&= \int_{-1}^0 \sum_{k\in I_{W,\epsilon}} \int_{C_\epsilon^k} \int_Y \Bigg| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}} \Big(\epsilon\Big[\frac{\bar{x}}{\epsilon}\Big] + \epsilon\bar{y}, \epsilon y_N - h_\epsilon\Big(\epsilon\Big[\frac{\bar{x}}{\epsilon}\Big] + \epsilon\bar{y}, \epsilon y_N\Big)\Big) \\
&\qquad\qquad - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0) \Bigg|^2 d\bar{y}d\bar{x}dy_N \\
&= \int_{-1}^0 \sum_{k\in I_{W,\epsilon}} \int_{C_\epsilon^k} \int_{C_\epsilon^k} \Bigg| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}} \Big(\bar{z}, \epsilon y_N - h_\epsilon\Big(\bar{z}, \epsilon y_N\Big)\Big) \\
&\qquad\qquad - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0) \Bigg|^2 d\bar{x}\frac{d\bar{z}}{\epsilon^{N-1}}dy_N.
\end{aligned}
\tag{4.2.21}
$$

Now, let $\bar{z} \in C_\epsilon^k$ be fixed. By expanding φ in Taylor's series with remainder in Lagrange form we deduce that

$$
\begin{aligned}
\frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}\big(\bar{z}, \epsilon y_N - h_\epsilon(\bar{z}, \epsilon y_N)\big) &= \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}\big(\bar{z}, \epsilon y_N - h_\epsilon(\bar{z}, \epsilon y_N)\big) - \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\bar{z},0) \\
&= \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{z},\xi) \frac{(\epsilon y_N - h_\epsilon(\bar{z},\epsilon y_N))^{l-1}}{(l-1)!},
\end{aligned}
$$

for some $\xi \in (0, \epsilon y_N - h_\epsilon(\bar{z}, \epsilon y_N))$. We then deduce that the term appearing in the right-hand side of (4.2.21) can be rewritten as

$$
\begin{aligned}
&\int_{-1}^0 \sum_{k\in I_{W,\epsilon}} \int_{C_\epsilon^k} \int_{C_\epsilon^k} \Bigg| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{z},\xi) \frac{(\epsilon y_N - h_\epsilon(\bar{z},\epsilon y_N))^{l-1}}{(l-1)!} \\
&\qquad\qquad - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0) \Bigg|^2 d\bar{x}\frac{d\bar{z}}{\epsilon^{N-1}}dy_N.
\end{aligned}
\tag{4.2.22}
$$

Then the right-hand side of (4.2.22) is estimated from above by

$$\begin{aligned}
&\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\Bigg|\left(\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right)\frac{y_N^{l-1}}{(l-1)!}\\
&\qquad+\sum_{s=1}^{l-1}\binom{l-1}{s}\frac{1}{\epsilon^{l-1}}\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)(\epsilon y_N)^{l-1-s}(-h_\epsilon(\bar z,\epsilon y_N))^s\Bigg|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N.\\
&\le C\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,0)\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&+C\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,0)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&+C\sum_{s=1}^{l-1}\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)\right|^2\left|\frac{1}{\epsilon^{l-1}}(\epsilon y_N)^{l-1-s}|h_\epsilon(\bar z,\epsilon y_N)|^s\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N.
\end{aligned}\tag{4.2.23}$$

Now we consider separately the three integrals on the right-hand side of (4.2.22). For what concerns the first integral we first note that by the Fundamental Theorem of Calculus

$$\begin{aligned}
&\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,0)\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&=\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\int_0^{\xi}\frac{\partial^{m}\varphi}{\partial x_N^{m}}(\bar z,t)dt\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N,
\end{aligned}\tag{4.2.24}$$

and the right-hand side of equality (4.2.24) is estimated from above by

$$\begin{aligned}
&\le\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}|\xi|\int_{\xi}^{0}\left|\frac{\partial^{m}\varphi}{\partial x_N^{m}}(\bar z,t)\right|^2 dtd\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&\le C\epsilon\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\int_{-c\epsilon}^{0}\left|\frac{\partial^{m}\varphi}{\partial x_N^{m}}(\bar z,t)\right|^2 dt\frac{d\bar z}{\epsilon^{N-1}}d\bar x dy_N\\
&\le C\epsilon\epsilon^{N-1}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{-c\epsilon}^{0}\left|\frac{\partial^{m}\varphi}{\partial x_N^{m}}(\bar z,t)\right|^2 dt\frac{d\bar z}{\epsilon^{N-1}}\le C\epsilon\left\|\frac{\partial^{m}\varphi}{\partial x_N^{m}}\right\|^2_{L^2(W\times(-c\epsilon,0))},
\end{aligned}\tag{4.2.25}$$

where in the first inequality of (4.2.25) we have used Holder's inequality. Now consider the second integral in (4.2.23). We have the following estimate

$$\begin{aligned}
&C\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,0)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&=C\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,0)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right|^2\frac{|\bar z-\bar x|^N}{|\bar z-\bar x|^N}d\bar x\frac{d\bar z}{\epsilon^{N-1}}\\
&\le C\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,0)-\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)}{|\bar z-\bar x|^{N/2}}\right|^2\epsilon^N d\bar x\frac{d\bar z}{\epsilon^{N-1}}\\
&\le C\epsilon\left\|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right\|^2_{B_2^{1/2}(W)},
\end{aligned}\tag{4.2.26}$$

where $B_2^{1/2}(W)$ is the Besov space of parameters $2,1/2$ (note that the norm appearing in the right-hand of (4.2.26) is estimated from above by the $H^1(\Omega)$- norm of $\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}$, due to the Trace Theorem). Finally we consider the third integral, which is easily estimated as follows:

$$\begin{aligned}
&C\sum_{s=1}^{l-1}\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)\right|^2\left|\frac{1}{\epsilon^{l-1}}(\epsilon y_N)^{l-1-s}|h_\epsilon(\bar z,\epsilon y_N)|^s\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&\le C\epsilon^{N-1}\sum_{s=1}^{l-1}\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)\right|^2\left(\frac{1}{\epsilon^{l-1}}(\epsilon)^{l-1-s}|C\epsilon^{3/2}|^s\right)^2\frac{d\bar z}{\epsilon^{N-1}}dy_N\\
&\le C\sum_{s=1}^{l-1}\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\left|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)\right|^2\epsilon^s d\bar z dy_N\le C\epsilon\left\|\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}\right\|^2_{H^1(\Omega)}.
\end{aligned}\tag{4.2.27}$$

By using (4.2.25), (4.2.26), (4.2.27) in (4.2.22) we deduce that

$$\begin{aligned}
&\int_{-1}^{0}\sum_{k\in I_{W,\epsilon}}\int_{C_\epsilon^k}\int_{C_\epsilon^k}\left|\frac{1}{\epsilon^{l-1}}\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar z,\xi)\frac{(\epsilon y_N-h_\epsilon(\bar z,\epsilon y_N))^{l-1}}{(l-1)!}\right.\\
&\qquad\left.-\frac{y_N^{l-1}}{(l-1)!}\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar x,0)\right|^2 d\bar x\frac{d\bar z}{\epsilon^{N-1}}dy_N\le C\epsilon\|\varphi\|_{H^m(\Omega)}\to 0,
\end{aligned}\tag{4.2.28}$$

as $\epsilon\to 0$. This concludes the proof in the case of smooth functions. Now, if $\varphi\in H^m(\Omega)\cap H_0^{m-1}(\Omega)$, by [35, Theorem 9, p.77] there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}\subset$

$H^m(\Omega) \cap H_0^{m-1}(\Omega) \cap C^\infty(\Omega)$ such that

$$\varphi_n \longrightarrow \varphi, \qquad \text{in } H^m(\Omega_\epsilon),$$

as $n \longrightarrow \infty$ and $\mathrm{Tr}_{\partial\Omega} D^\eta \varphi_n = \mathrm{Tr}_{\partial\Omega} D^\eta \varphi$ for all $|\eta| \leq m-1$. Then

$$\begin{aligned}
&\left\| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\widehat{\Phi}_\epsilon(y)) - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \\
&\leq \left\| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\widehat{\Phi}_\epsilon(y)) - \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi_n}{\partial x_N^{m-l}}(\widehat{\Phi}_\epsilon(y)) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \\
&+ \left\| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi_n}{\partial x_N^{m-l}}(\widehat{\Phi}_\epsilon(y)) - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi_n}{\partial x_N^{m-1}}(\bar{x},0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \\
&+ \left\| \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi_n}{\partial x_N^{m-1}}(\bar{x},0) - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))}
\end{aligned} \tag{4.2.29}$$

We first consider

$$\left\| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\widehat{\Phi}_\epsilon(y)) - \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi_n}{\partial x_N^{m-l}}(\widehat{\Phi}_\epsilon(y)) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))}.$$

By the exact integration formula (see Lemma 1.2.10) we can directly consider

$$\begin{aligned}
&\frac{1}{\epsilon^{2l-2}} \frac{1}{\epsilon} \int_W \int_{-\epsilon}^0 \left| \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\Phi_\epsilon(x)) - \frac{\partial^{m-l}\varphi_n}{\partial x_N^{m-l}}(\Phi_\epsilon(x)) \right|^2 d\bar{x}dx_N \\
&\leq \frac{C}{\epsilon^{2l-1}} \int_W \int_{-\epsilon}^{-h_\epsilon(\bar{x},0)} \left| \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\bar{x},x_N) - \frac{\partial^{m-l}\varphi_n}{\partial x_N^{m-l}}(\bar{x},x_N) \right|^2 d\bar{x}dx_N \\
&\leq \frac{C}{\epsilon^{2l-1}} \int_W \int_{-\epsilon}^{0} \left| \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\bar{x},x_N) - \frac{\partial^{m-l}\varphi_n}{\partial x_N^{m-l}}(\bar{x},x_N) \right|^2 d\bar{x}dx_N.
\end{aligned} \tag{4.2.30}$$

Now by the Poincaré inequality and the fact that $\frac{\partial^{m-k}\varphi}{\partial x_N^{m-k}}(\bar{x},0) - \frac{\partial^{m-k}\varphi_n}{\partial x_N^{m-k}}(\bar{x},0) = 0$ for all $k = l, \dots, m-1$ by the choice of φ_n, we deduce that the right-hand side of (4.2.30) is less or equal than

$$\frac{C\epsilon^{2l}}{\epsilon^{2l-1}} \int_W \int_{-\epsilon}^0 \left| \frac{\partial^m\varphi}{\partial x_N^m}(\bar{x},x_N) - \frac{\partial^m\varphi_n}{\partial x_N^m}(\bar{x},x_N) \right|^2 d\bar{x}dx_N \leq C\epsilon \|\varphi - \varphi_n\|^2_{H^m(\Omega)}.$$

Going back to (4.2.29) we have

$$
\begin{aligned}
&\left\| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\widehat{\Phi}_\epsilon(y)) - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \\
&\leq C\|\varphi - \varphi_n\|_{H^m(\Omega)} + \left\| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi_n}{\partial x_N^{m-l}}(\widehat{\Phi}_\epsilon(y)) - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi_n}{\partial x_N^{m-1}}(\bar{x},0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \\
&+ \left\| \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi_n}{\partial x_N^{m-1}}(\bar{x},0) - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))}
\end{aligned}
\tag{4.2.31}
$$

Fix $\delta > 0$ arbitrarily small. Choose n big enough so that $\|\varphi - \varphi_n\|_{H^m(\Omega)} \leq \delta$,

$$
\left\| \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi_n}{\partial x_N^{m-1}}(\bar{x},0) - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \leq \delta.
$$

Note that this is possible by the Trace Theorem and the convergence of φ_n to φ in H^m. Now, with the choice of n and δ above take $\epsilon > 0$ small enough in such a way that

$$
\left\| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi_n}{\partial x_N^{m-l}}(\widehat{\Phi}_\epsilon(y)) - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi_n}{\partial x_N^{m-1}}(\bar{x},0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \leq \delta.
$$

This is possible by the previous discussion on the behaviour of smooth functions. Finally we deduce that for every δ there exist $\epsilon > 0$ such that

$$
\left\| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\widehat{\Phi}_\epsilon(y)) - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} \leq 3\delta. \tag{4.2.32}
$$

By the arbitrariness of δ in (4.2.32) we deduce that

$$
\lim_{\epsilon \to 0} \left\| \frac{1}{\epsilon^{l-1}} \frac{\partial^{m-l}\varphi}{\partial x_N^{m-l}}(\widehat{\Phi}_\epsilon(y)) - \frac{y_N^{l-1}}{(l-1)!} \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0) \right\|_{L^2(\widehat{W}_\epsilon \times Y \times (-1,0))} = 0,
$$

concluding the proof.

□

We summarise the previous discussion in the following

Theorem 4.2.7. *Let $f_\epsilon \in L^2(\Omega_\epsilon)$ and $f \in L^2(\Omega)$ be such that $f_\epsilon \rightharpoonup f$ in $L^2(\Omega)$. Let $g(y) = b(\bar{y})(1 + y_N)^{m+1}$ for all $y \in Y \times (-1, 0)$. Let $v_\epsilon \in H^m(\Omega_\epsilon) \cap H_0^{m-1}(\Omega_\epsilon)$ be the solutions to $H_{\Omega_\epsilon,S} v_\epsilon = f_\epsilon$. Then, possibly passing to a subsequence, there exists $v \in H^m(\Omega) \cap H_0^{m-1}(\Omega)$ and $\hat{v} \in L^2(W, w^{m,2}_{Per_Y}(Y \times (\infty, 0)))$ such that $v_\epsilon \rightharpoonup v$ in $H^m(\Omega)$, $v_\epsilon \to v$ in $L^2(\mathbb{R}^N)$, statements (a) and (b) in Lemma 4.2.3 hold, and such that*

$$-\sum_{l=1}^{m-1} \binom{m}{l+1} \int_W \int_{Y\times(-1,0)} \left[\frac{y_N^{l-1}}{(l-1)!} D_y^{l+1} \left(\frac{\partial^{m-l-1} \hat{v}(\bar{x}, y)}{\partial y_N^{m-l-1}} \right) : D_y^{l+1} g(y) \right] \mathrm{d}y \, \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x}, 0) \mathrm{d}\bar{x} + \int_\Omega D^m v : D^m \varphi + u\varphi \, \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}x. \quad (4.2.33)$$

for all $\varphi \in H^m(\Omega) \cap H_0^{m-1}(\Omega)$.

Notation. In the sequel we will refer to

$$-\sum_{l=1}^{m-1} \binom{m}{l+1} \int_W \int_{Y\times(-1,0)} \left[\frac{y_N^{l-1}}{(l-1)!} D_y^{l+1} \left(\frac{\partial^{m-l-1} \hat{v}(\bar{x}, y)}{\partial y_N^{m-l-1}} \right) : D_y^{l+1} g(\bar{y}) \right] \mathrm{d}y \, \frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x}, 0) \, \mathrm{d}\bar{x} \quad (4.2.34)$$

as the *strange term* appearing in the homogenization.

4.2.2 Critical case - Microscopic problem.

Let $\psi \in C^\infty(\overline{W} \times \overline{Y} \times]-\infty, 0])$ be such that $\operatorname{supp} \psi \subset C \times \overline{Y} \times [d, 0]$ for some compact set $C \subset W$ and for some $d \in (-\infty, 0)$. Moreover, assume that $\psi(\bar{x}, \bar{y}, 0) = D^l \psi(\bar{x}, \bar{y}, 0) = 0$ for all $(\bar{x}, \bar{y}) \in W \times Y$, for all $1 \leq l \leq m-2$. Let also ψ be Y-periodic in the variable $\bar{y}$. We set

$$\psi_\epsilon(x) = \epsilon^{m-\frac{1}{2}} \psi\left(\bar{x}, \frac{\bar{x}}{\epsilon}, \frac{x_N}{\epsilon}\right),$$

for all $\epsilon > 0$, $x \in W \times]-\infty, 0]$. Then $T_\epsilon \psi_\epsilon \in V(\Omega_\epsilon)$ for sufficiently small ϵ, hence we can use it in the weak formulation of the problem in Ω_ϵ, getting

$$\int_{\Omega_\epsilon} D^m v_\epsilon : D^m T_\epsilon \psi_\epsilon \, \mathrm{d}x + \int_{\Omega_\epsilon} v_\epsilon T_\epsilon \psi_\epsilon \, \mathrm{d}x = \int_{\Omega_\epsilon} f_\epsilon T_\epsilon \psi_\epsilon \, \mathrm{d}x.$$

It is not difficult to prove that

$$\int_{\Omega_\epsilon} v_\epsilon T_\epsilon \psi_\epsilon \, \mathrm{d}x \to 0, \qquad \int_{\Omega_\epsilon} f_\epsilon T_\epsilon \psi_\epsilon \, \mathrm{d}x \to 0 \quad (4.2.35)$$

as $\epsilon \to 0$. By arguing as in [19, §8.4], it is possible to prove that

$$\int_{\Omega_\epsilon \setminus \Omega} D^m v_\epsilon : D^m T_\epsilon \psi_\epsilon \, \mathrm{d}x \to 0, \quad (4.2.36)$$

as $\epsilon \longrightarrow 0$. Moreover, a suitable modification of [19, Lemma 8.47] yields

$$\int_{\Omega} D^m v_\epsilon : D^m T_\epsilon \psi_\epsilon \, \mathrm{d}x \longrightarrow \int_{W\times Y\times(-\infty,0)} D^m_y \hat{v}(\bar{x}, y) : D^m_y \psi(\bar{x}, y) \, \mathrm{d}\bar{x}\mathrm{d}y. \tag{4.2.37}$$

Now we have the following

Theorem 4.2.8. *Let $\hat{v} \in L^2(W, w^{m,2}_{Per_Y}(Y \times (\infty, 0)))$ be the function from Theorem 4.2.7. Then*

$$\int_{W\times Y\times(-\infty,0)} D^m_y \hat{v}(\bar{x}, y) : D^m_y \psi(\bar{x}, y) \mathrm{d}\bar{x}\mathrm{d}y = 0, \tag{4.2.38}$$

for all $\psi \in L^2(W, w^{m,2}_{Per_Y}(Y \times (\infty, 0)))$ such that $\psi(\bar{x}, \bar{y}, 0) = D^l \psi(\bar{x}, \bar{y}, 0) = 0$ for all $(\bar{x}, \bar{y}) \in W \times Y$, for all $1 \leq l \leq m-2$. Moreover, for any $j = 1, \dots, N-1$, we have

$$\frac{\partial^{m-1}\hat{v}}{\partial y_j \partial y_N^{m-2}}(\bar{x}, \bar{y}, 0) = -\frac{\partial b}{\partial y_j}(\bar{y})\frac{\partial^{m-1} v}{\partial x_N^{m-1}}(\bar{x}, 0) \qquad \text{on } W \times Y, \tag{4.2.39}$$

and

$$\frac{\partial^{m-1}\hat{v}}{\partial y_{i_1} \cdots \partial y_{i_{m-1}}}(\bar{x}, \bar{y}, 0) = 0 \qquad \text{on } W \times Y, \tag{4.2.40}$$

for all $i_1, \dots, i_{m-1} = 1, \dots, N-1$.

Proof. The first part of the statement follows from (4.2.35), (4.2.36) and (4.2.37) by arguing as in [19, Theorem 8.53]. In order to prove formulas (4.2.39) and (4.2.40) we note that, since $D^{m-2} v_\epsilon(\bar{x}, g_\epsilon(\bar{x})) = 0$ for all $\bar{x} \in W$, we have

$$\frac{\partial^{m-2} v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_{m-2}}}(\bar{x}, g_\epsilon(\bar{x})) = 0, \qquad \text{for all } i_1, \dots, i_{m-2} = 1, \dots, N, \text{ for all } \bar{x} \in W .$$

An iterated differentiation with respect to x_j, $j \in \{1, \dots, N-1\}$ yields

$$\frac{\partial^{m-1} v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_{m-2}} \partial x_j}(\bar{x}, g_\epsilon(\bar{x})) + \frac{\partial^{m-1} v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_{m-2}} \partial x_N}(\bar{x}, g_\epsilon(\bar{x}))\frac{\partial g_\epsilon(\bar{x})}{\partial x_j} = 0,$$

for all $\bar{x} \in W$. Hence, by setting

$$V^{ij}_\epsilon = \left(0, \dots, 0, -\frac{\partial^{m-1} v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_{m-2}} \partial x_N}, 0, \dots, 0, \frac{\partial^{m-1} v_\epsilon}{\partial x_j \partial x_{i_1} \cdots \partial x_{i_{m-2}}}\right),$$

for all $i_1, \dots, i_{m-2} = 1, \dots, N$, $j = 1, \dots, N-1$, where the only non-zero entries are the i-th and the N-th, we obtain that

$$V^{ij}_\epsilon \cdot n_\epsilon = 0, \qquad \text{on } \Gamma_\epsilon,$$

where n_ϵ is the outer normal to $\Gamma_\epsilon \equiv \{(\bar{x}, g_\epsilon(\bar{x})) : \bar{x} \in W\}$. We note that by Lemma 4.2.3

$$\frac{1}{\sqrt{\epsilon}} \frac{\partial}{\partial y_k} \left(\widehat{\frac{\partial^{m-1} v_\epsilon}{\partial x_{i_1} \cdots \partial x_{i_{m-2}} \partial x_j}} \right) \overset{\epsilon \to 0}{\rightharpoonup} \frac{\partial^m \hat{v}}{\partial y_{i_1} \cdots \partial y_{m-2} \partial y_j \partial y_k},$$

in $L^2(W \times Y \times]-\infty, 0[)$. By [43, Lemma 4.3], we deduce that

$$\frac{\partial^{m-1} \hat{v}}{\partial y_{i_1} \cdots \partial y_{i_{m-2}} \partial y_j}(\bar{x}, \bar{y}, 0) = -\frac{\partial b}{\partial y_j}(\bar{y}) \frac{\partial^{m-1} v}{\partial x_N \partial x_{i_1} \cdots \partial x_{i_{m-2}}}(\bar{x}, 0), \qquad \text{on } W \times Y,$$

for all $i_1, \dots, i_{m-2} = 1, \dots, N$, $j = 1, \dots, N-1$. Since $v \in W^{m,2}(\Omega) \cap W_0^{m-1,2}(\Omega)$, $D^{m-2} v(\bar{x}, 0) = 0$ for all $x \in W$. This implies that all the derivatives

$$\frac{\partial^{m-1} v}{\partial x_N \partial x_{i_1} \cdots \partial x_{i_{m-2}}}(\bar{x}, 0),$$

where one of the indexes i_k is different from N are zero. This concludes the proof. □

Now we have the following

Lemma 4.2.9. *There exists $V \in w^{m,2}_{Per_Y}(Y \times (-\infty, 0))$ satisfying the equation*

$$\int_{Y \times (-\infty, 0)} D^m V : D^m \psi \, dy = 0, \tag{4.2.41}$$

for all $\psi \in w^{m,2}_{Per_Y}(Y \times (-\infty, 0))$ such that $D^l \psi(\bar{y}, 0) = 0$ on Y, for all $0 \le l \le m-2$, and the boundary conditions

$$\begin{cases} V(\bar{y}, 0) = 0, & \text{on } Y, \\ \frac{\partial^{m-2} V}{\partial y_N^{m-2}}(\bar{y}, 0) = b(\bar{y}), & \text{on } Y. \end{cases}$$

Function V is unique up to the sum of a polynomial in y_N of degree $m-1$. Moreover $V \in W^{2m,2}_{Per_Y}(Y \times (d, 0))$ for any $d < 0$ and it satisfies the equation

$$(-\Delta)^m V = 0, \qquad \text{in } Y \times (d, 0),$$

subject to the boundary conditions

$$\begin{cases} V(\bar{y}, 0) = 0, & \text{on } Y, \\ \frac{\partial^l V}{\partial y_N^l}(\bar{y}, 0) = 0, & \text{on } Y, \text{ for all } 1 \le l \le m-3 \\ \frac{\partial^{m-2} V}{\partial y_N^{m-2}}(\bar{y}, 0) = b(\bar{y}), & \text{on } Y \\ \frac{\partial^m V}{\partial y_N^m}(\bar{y}, 0) = 0, & \text{on } Y. \end{cases}$$

Proof. Similar to the proof of [19, Lemma 8.60] (see also the proofs of Lemma 3.2.8 and Lemma 3.3.6). We just note that in order to deduce the classical formulation of problem (4.2.41) it is sufficient to choose test functions ψ as in the statement with bounded support in the y_N direction. By using the Polyharmonic Green Formula (see (4.1.4)) we then deduce that

$$\int_{Y\times(-\infty,0)} D^m V : D^m\psi \, \mathrm{d}y = (-1)^m \int_{Y\times(-\infty,0)} \Delta^m V\psi \, \mathrm{d}y + \int_Y \frac{\partial^m V}{\partial y_N^m}\frac{\partial^{m-1}\psi}{\partial y_N^{m-1}}\, d\bar{y}.$$

By the arbitrariness of ψ it is then easy to deduce the statement of Lemma 4.2.9. □

Theorem 4.2.10. *Let V be as in Lemma 4.2.9. Then*

$$\sum_{l=1}^{m-1}\binom{m}{l+1}\int_{Y\times(-1,0)}\left[\frac{y_N^{l-1}}{(l-1)!}D^{l+1}\left(\frac{\partial^{m-l-1}V(y)}{\partial y_N^{m-l-1}}\right) : D^{l+1}(b(\bar{y})(1+y_N)^{m+1})\right] dy = \int_{Y\times(-\infty,0)} |D^m V|^2 \, \mathrm{d}y. \quad (4.2.42)$$

Furthermore

$$\int_{Y\times(-\infty,0)} |D^m V|^2 \, \mathrm{d}y = -\int_Y \left(\frac{\partial^{m-1}(\Delta V)}{\partial x_N^{m-1}} + (m-1)\Delta_{N-1}\left(\frac{\partial^{m-1} V}{\partial x_N^{m-1}}\right)\right) b(\bar{y})\, \mathrm{d}\bar{y}. \quad (4.2.43)$$

Proof. Let ϕ be the real-valued function defined on $Y\times]-\infty,0]$ by

$$\phi(y) = \begin{cases} \frac{y_N^{m-2}}{(m-2)!} b(\bar{y})(1+y_N)^{m+1}, & \text{if } -1 \le y_N \le 0, \\ 0, & \text{if } y_N < -1. \end{cases}$$

Then $\phi \in W^{m,2}(Y\times(-\infty,0))$, $\frac{\partial^l\phi}{\partial y_N^l}(\bar{y},0) = 0$ for all $0 \le l \le m-3$, and

$$\frac{\partial^{m-2}\phi}{\partial y_N^{m-2}}(\bar{y},0) = b(\bar{y}), \qquad \text{for all } y \in Y. \quad (4.2.44)$$

Now note that the function $\psi = V - \phi$ is a suitable test-function in equation (4.2.41); by plugging it in we deduce that

$$\int_{Y\times(-\infty,0)} |D^m V|^2 \, \mathrm{d}y = \int_{Y\times(-1,0)} D^m V : D^m\phi \, \mathrm{d}y.$$

For the sake of readiness we set $g(y) = b(\bar{y})(1+y_N)^{m+1}$. By the Leibnitz rule we have that

$$\int_{Y\times(-1,0)} D^m V : D^m\phi \, \mathrm{d}y = \int_{Y\times(-1,0)} D^m V : \left(\sum_{k=0}^{m}\binom{m}{k} D^{m-k}\left(\frac{y_N^{m-2}}{(m-2)!}\right) D^k g\right) dy. \quad (4.2.45)$$

Note that

$$D^{m-k}\frac{y_N^{m-2}}{(m-2)!}=\frac{1}{(m-2)!}\frac{\partial^{m-k}y_N^{m-2}}{\partial x_{i_1}\cdots\partial x_{i_{m-k}}}=\begin{cases}0, & \text{if } k=0,1;\\ \frac{y_N^{k-2}}{(k-2)!}\delta_{i_1N}\cdots\delta_{i_{m-k}N}, & \text{for } k\geq 2.\end{cases}$$

Hence, we can rewrite the left-hand side of (4.2.45) as follows

$$\begin{aligned}&\sum_{k=2}^{m}\binom{m}{k}\int_{Y\times(-1,0)}D^k\left(\frac{\partial^{m-k}V(y)}{\partial y_N^{m-k}}\right):\left(\frac{y_N^{k-2}}{(k-2)!}D^k g(y)\right)dy\\ &=\sum_{k=2}^{m}\binom{m}{k}\int_{Y\times(-1,0)}\frac{y_N^{k-2}}{(k-2)!}D^k\left(\frac{\partial^{m-k}V(y)}{\partial y_N^{m-k}}\right):D^k g(y)\,dy,\end{aligned}$$

which coincides with the left-hand side of (4.2.42) up to the change of summation index defined by $k=l+1$. Finally, (4.2.43) follows by applying the polyharmonic Green formula (4.1.4) on $\int_{Y\times(-1,0)}D^mV:D^m\phi\,\mathrm{d}y$. Indeed, we note that the boundary integrals on $\partial Y\times(-1,0)$ are zero, due to the periodicity of V and b. Moreover the boundary integral on $\partial Y\times\{-1\}$ is zero since ϕ vanishes there together with all its derivatives. Then, the only non-trivial boundary integral is supported on $Y\times\{0\}$. More precisely, we have

$$\int_{Y\times(-1,0)}D^mV:D^m\phi\,\mathrm{d}y=(-1)^m\int_{Y\times(-1,0)}\Delta^mV\phi\,\mathrm{d}y+\sum_{t=0}^{m-1}\int_Y B_t(V)(\bar{y},0)\frac{\partial^t\phi(\bar{y},0)}{\partial y_N^t}\,\mathrm{d}\bar{y},$$

and by recalling that $\Delta^mV=0$ in $Y\times(-1,0)$, $\frac{\partial^mV}{\partial y_N^m}=0$ on $Y\times\{0\}$, $\frac{\partial^l\phi}{\partial y_N^l}=0$ on $Y\times\{0\}$, for all $0\leq l\leq m-3$ and by (4.2.44), we deduce that

$$\int_{Y\times(-1,0)}D^mV:D^m\phi\,\mathrm{d}y=\int_Y B_{m-2}(V)(\bar{y},0)b(\bar{y})\,\mathrm{d}\bar{y}$$

and by formula (4.1.4)

$$B_{m-2}(V)(\bar{y},0)=-\sum_{l=m-2}^{m-1}\binom{l}{m-2}\Delta_{N-1}^{l-m+2}\left(\frac{\partial^{m-1}}{\partial y_N^{m-1}}(\Delta^{m-l-1}V)\right),$$

form which we deduce (4.2.43). □

Theorem 4.2.11. *Let $m\in\mathbb{N}$, $m\geq 2$. Let V be as in Lemma 4.2.9. Let v, $\hat{v}$ be the functions defined in Theorem 4.2.7. Let also $g(y)=b(\bar{y})(1+y_N)^{m+1}$ for all $y\in Y\times(-1,0)$. Then*

$$\hat{v}(\bar{x},y)=-V(y)\frac{\partial^{m-1}v}{\partial x_N^{m-1}}(\bar{x},0)+F(\bar{x},y_N),$$

where $F(\bar{x}, y_N)$ is a polynomial of degree at most $m-1$ in y_N with coefficients $a_j(\bar{x}) \in L^2(W)$, for all $j = 0, \dots, m-2$ and $a_0 \equiv 0$. Moreover, the strange term (4.2.34) *is given by*

$$-\sum_{l=1}^{m-1}\binom{m}{l+1}\int_W\int_{Y\times(-1,0)}\left[\frac{y_N^{l-1}}{(l-1)!}D_y^{l+1}\left(\frac{\partial^{m-l-1}\hat{v}(\bar{x},y)}{\partial y_N^{m-l-1}}\right):D_y^{l+1}g(\bar{y})\right]\mathrm{d}y\,\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\mathrm{d}\bar{x}$$
$$=\int_{Y\times(-\infty,0)}|D^mV|^2dy\int_W\frac{\partial^{m-1}v}{\partial x_N^{m-1}}(\bar{x},0)\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\mathrm{d}\bar{x}$$
$$=-\int_Y\left(\frac{\partial^{m-1}(\Delta V)}{\partial x_N^{m-1}}+(m-1)\Delta_{N-1}\left(\frac{\partial^{m-1}V}{\partial x_N^{m-1}}\right)\right)b(\bar{y})\mathrm{d}\bar{y}\int_W\frac{\partial^{m-1}v}{\partial x_N^{m-1}}(\bar{x},0)\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\mathrm{d}\bar{x}.$$

Proof. Follows from Theorem 4.2.8, Lemma 4.2.9 and by Theorem 4.2.10 by observing that $-V(y)\frac{\partial^{m-2}v}{\partial x_N^{m-2}}(\bar{x},0)$ satisfies problem (4.2.38) with the boundary conditions (4.2.39). □

We are now ready to conclude the proof of (iii) of Theorem 4.2.1.

Proof of Theorem 4.2.1(iii). Let us set $g(y) = b(\bar{y})(1+y_N)^{m+1}$ for all $y \in Y\times(-1,0)$. The function v in Theorem 4.2.7 satisfies

$$\sum_{l=1}^{m-1}\binom{m}{l+1}\int_W\int_{Y\times(-1,0)}\left[\frac{y_N^{l-1}}{(l-1)!}D^{l+1}\left(\frac{\partial^{m-l-1}V(y)}{\partial y_N^{m-l-1}}\right):D^{l+1}g(y)\right]\mathrm{d}y\,\frac{\partial^{m-1}v}{\partial x_N^{m-1}}(\bar{x},0)\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\mathrm{d}\bar{x}$$
$$+\int_\Omega D^mv:D^m\varphi+u\varphi\,\mathrm{d}x=\int_\Omega f\varphi\,\mathrm{d}x. \quad (4.2.46)$$

for all $\varphi \in H^m(\Omega)\cap H_0^{m-1}(\Omega)$. By Theorem 4.2.11 we can rewrite the first integral on the left-hand side of (4.2.46) as

$$\int_{Y\times(-\infty,0)}|D^mV|^2dy\int_W\frac{\partial^{m-1}v}{\partial x_N^{m-1}}(\bar{x},0)\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\,\mathrm{d}\bar{x}$$

and by the Polyharmonic Green Formula (see (4.1.24)) we have that

$$\int_\Omega D^mv:D^m\varphi\,dx=(-1)^m\int_\Omega\Delta^mv\varphi+\int_{\partial\Omega}\frac{\partial^mv}{\partial n^m}\frac{\partial^{m-1}\varphi}{\partial n^{m-1}}dS, \quad (4.2.47)$$

for all $\varphi \in H^m(\Omega)\cap H_0^{m-1}(\Omega)$. Hence, on $W\times\{0\}$ we find the following boundary integral

$$\int_W\left(\frac{\partial^mv}{\partial x_N^m}(\bar{x},0)+\left(\int_{Y\times(-\infty,0)}|D^mV|^2\,dy\right)\frac{\partial^{m-1}v}{\partial x_N^{m-1}}(\bar{x},0)\right)\frac{\partial^{m-1}\varphi}{\partial x_N^{m-1}}(\bar{x},0)\,\mathrm{d}\bar{x}, \quad (4.2.48)$$

for all $\varphi \in H^m(\Omega)\cap H_0^{m-1}(\Omega)$. By (4.2.46), (4.2.47), (4.2.48) and the arbitrariness of φ we deduce the statement of Theorem 4.2.1, part (iii). □

Chapter 5

Biharmonic operator on dumbbell domains

5.1 Introduction

This chapter is devoted to a spectral analysis of the biharmonic operator subject to Neumann boundary conditions on a domain which undergoes a singular perturbation. The focus is on planar dumbbell-shaped domains Ω_ϵ, with $\epsilon > 0$, described in Figure 5.1. Namely, given two bounded smooth domains Ω_L, Ω_R in $\mathbb{R}^2$ with $\Omega_L \cap \Omega_R = \emptyset$ such that $\partial\Omega_L \supset \{(0,y) \in \mathbb{R}^2 : -1 < y < 1\}$, $\partial\Omega_R \supset \{(1,y) \in \mathbb{R}^2 : -1 < y < 1\}$, and $(\Omega_R \cup \Omega_L) \cap ([0,1] \times [-1,1]) = \emptyset$, we set

$$\Omega = \Omega_L \cup \Omega_R, \quad \text{and} \quad \Omega_\epsilon = \Omega \cup R_\epsilon \cup L_\epsilon\,,$$

for all $\epsilon > 0$ small enough. Here $R_\epsilon \cup L_\epsilon$ is a thin channel connecting Ω_L and Ω_R defined by

$$R_\epsilon = \{(x,y) \in \mathbb{R}^2 : x \in (0,1), 0 < y < \epsilon g(x)\}, \tag{5.1.1}$$
$$L_\epsilon = (\{0\} \times (0, \epsilon g(0)) \cup (\{1\} \times (0, \epsilon g(1)))),$$

where $g \in C^2[0,1]$ is a positive real-valued function. Note that Ω_ϵ collapses to the limit set $\Omega_0 = \Omega \cup ([0,1] \times \{0\})$ as $\epsilon \to 0$.

We consider the eigenvalue problem

$$\begin{cases} \Delta^2 u - \tau\Delta u + u = \lambda\, u, & \text{in } \Omega_\epsilon, \\ (1-\sigma)\frac{\partial^2 u}{\partial n^2} + \sigma\Delta u = 0, & \text{on } \partial\Omega_\epsilon, \\ \tau\frac{\partial u}{\partial n} - (1-\sigma)\, \mathrm{div}_{\partial\Omega_\epsilon}(D^2 u \cdot n)_{\partial\Omega_\epsilon} - \frac{\partial(\Delta u)}{\partial n} = 0, & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{5.1.2}$$

where $\tau \geq 0$, $\sigma \in (-1,1)$ are fixed parameters, and we analyse the behaviour of the eigenvalues and of the corresponding eigenfunctions as $\epsilon \to 0$. Here $(\cdot)_{\partial\Omega_\epsilon}$ is

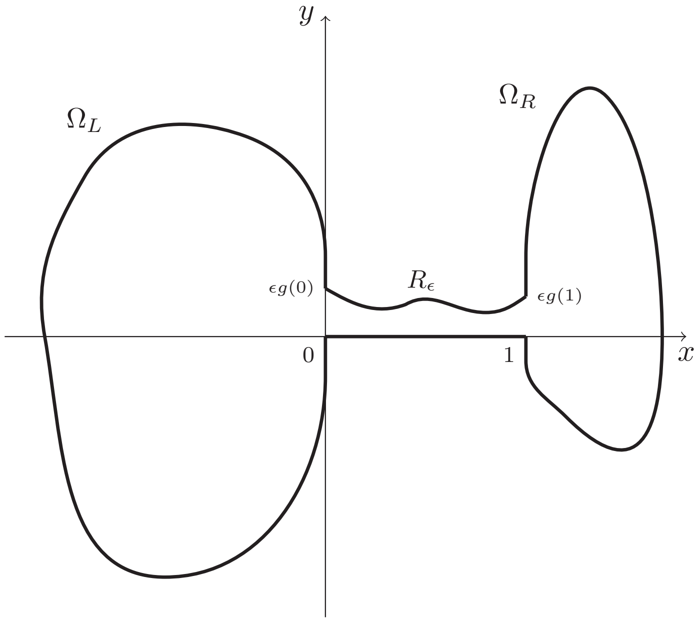


Figure 5.1: The dumbbell domain Ω_ϵ.

the projection on the tangent line to $\partial\Omega_\epsilon$, and we refer to §1.4 for the definition of the tangential divergence $\mathrm{div}_{\partial\Omega_\epsilon}$. The corresponding Poisson problem reads

$$\begin{cases} \Delta^2 u - \tau\Delta u + u = f, & \text{in } \Omega_\epsilon, \\ (1-\sigma)\frac{\partial^2 u}{\partial n^2} + \sigma\Delta u = 0, & \text{on } \partial\Omega_\epsilon, \\ \tau\frac{\partial u}{\partial n} - (1-\sigma)\,\mathrm{div}_{\partial\Omega_\epsilon}(D^2u\cdot n)_{\partial\Omega_\epsilon} - \frac{\partial(\Delta u)}{\partial n} = 0, & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{5.1.3}$$

with datum $f \in L^2(\Omega_\epsilon)$.

Since $\partial\Omega_\epsilon$ has corner singularities at the junctions $(0,0)$, $(0,\epsilon g(0))$, $(1,0)$, $(1,\epsilon g(1))$ and H^4 regularity does not hold around those points, we shall always understand problems (5.1.2), (5.1.3), (as well as analogous problems) in a weak (variational) sense, in which case only H^2 regularity is required.

Namely, the variational formulation of problem (5.1.3) is the following: find $u \in H^2(\Omega_\epsilon)$ such that

$$\int_{\Omega_\epsilon} (1-\sigma)D^2u : D^2\varphi + \sigma\Delta u\Delta\varphi + \tau\nabla u\cdot\nabla\varphi + u\varphi\,dx = \int_{\Omega_\epsilon} f\varphi\,dx\,, \tag{5.1.4}$$

for all $\varphi \in H^2(\Omega_\epsilon)$. The quadratic form associated with the left-hand side of (5.1.4) - call it $B_{\Omega_\epsilon}(u,\varphi)$ - is coercive for all $\tau \geq 0$ and $\sigma \in (-1,1)$, see e.g. [47], [48]. In particular, by standard spectral theory this quadratic form allows to define a non-negative self-adjoint operator $T = (\Delta^2 - \tau\Delta + I)_{N(\sigma)}$ in $L^2(\Omega_\epsilon)$ which plays the role of the classical operator $\Delta^2 - \tau\Delta + I$ subject to the boundary conditions above. More precisely, T is uniquely defined by the relation

$$B_{\Omega_\epsilon}(u,\varphi) =< T^{1/2}u, T^{1/2}\varphi >_{L^2(\Omega_\epsilon)},$$

for all $u, \varphi \in H^2(\Omega_\epsilon)$, as described in Theorem 1.1.6 and in Section 2.1.

The operator T is densely defined and its eigenvalues and eigenfunctions are exactly those of problem (5.1.2). Moreover, since the embedding $H^2(\Omega_\epsilon) \subset L^2(\Omega_\epsilon)$ is compact (see, for example, [35]), $(\Delta^2 - \tau\Delta + I)_{N(\sigma)}$ has compact resolvent, hence the spectrum is discrete and consists of a divergent increasing sequence of positive eigenvalues $\lambda_n(\Omega_\epsilon),\ n \in \mathbb{N}$, with finite multiplicity (here each eigenvalue is repeated as many times as its multiplicity).

Problem (5.1.2) arises in linear elasticity in connection with the Kirchhoff-Love model for the study of vibrations and deformations of free plates, in which case σ represents the Poisson ratio of the material and τ the lateral tension. In this sense, the dumbbell domain Ω_ϵ could represent a plate and R_ϵ a part of it which degenerates to the segment $[0,1] \times \{0\}$.

We note that problem (5.1.2) can be considered as a natural fourth order version of the corresponding eigenvalue problem for the Neumann Laplacian $-\Delta_N$. namely

$$\begin{cases} -\Delta u + u = \lambda\, u, & \text{in } \Omega_\epsilon, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{5.1.5}$$

It is known that the eigenelements of the Neumann Laplacian on a typical dumbbell domain as above have a singular behaviour, see [10], [11], [12], [14], [15], [18], and the references therein. For example, it is known that not all the eigenvalues of $-\Delta_N$ on Ω_ϵ converge to the eigenvalues of $-\Delta_N$ in Ω; indeed, some of the eigenvalues of the dumbbell domain are asymptotically close to the eigenvalues of a boundary value problem defined in the channel R_ϵ. This allows the appearance in the limit of extra eigenvalues associated with an ordinary differential equation in the segment $(0,1)$, which are generally different from the eigenvalues of $-\Delta_N$ in Ω. Such singular behaviour reflects a general characteristic of boundary value problems with Neumann boundary conditions, the stability of which requires rather strong assumptions on the admissible domain perturbations, see e.g., [14], [19], [88]. We refer to [57, p. 420] for a classical counterexample.

The aim of the present chapter is to clarify how the Neumann boundary conditions affect the spectral behaviour of the operator $\Delta^2 - \tau\Delta$ on dumbbell domains, by extending the validity of some results known for the second order operator $-\Delta_N$ to the fourth-order operator $(\Delta^2 - \tau\Delta)_{N(\sigma)}$.

First of all, we prove that the eigenvalues of problem (5.1.2) can be asymptotically decomposed into two families of eigenvalues as

$$(\lambda_n(\Omega_\epsilon))_{n\geq 1} \approx (\omega_k)_{k\geq 1} \cup (\theta_l^\epsilon)_{l\geq 1}, \quad \text{as } \epsilon \to 0, \tag{5.1.6}$$

where $(\omega_k)_{k\geq 1}$ are the eigenvalues of problem

$$\begin{cases} \Delta^2 w - \tau\Delta w + w = \omega_k\, w, & \text{in } \Omega, \\ (1-\sigma)\frac{\partial^2 w}{\partial n^2} + \sigma\Delta w = 0, & \text{on } \partial\Omega, \\ \tau\frac{\partial w}{\partial n} - (1-\sigma)\,\mathrm{div}_{\partial\Omega}(D^2 w\cdot n)_{\partial\Omega} - \frac{\partial(\Delta w)}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.1.7}$$

and $(\theta_l^\epsilon)_{l\geq 1}$ are the eigenvalues of problem

$$\begin{cases} \Delta^2 v - \tau\Delta v + v = \theta_l^\epsilon\, v, & \text{in } R_\epsilon, \\ (1-\sigma)\frac{\partial^2 v}{\partial n^2} + \sigma\Delta v = 0, & \text{on } \Gamma_\epsilon, \\ \tau\frac{\partial v}{\partial n} - (1-\sigma)\,\mathrm{div}_{\Gamma_\epsilon}(D^2 v\cdot n)_{\Gamma_\epsilon} - \frac{\partial(\Delta v)}{\partial n} = 0, & \text{on } \Gamma_\epsilon, \\ v = 0 = \frac{\partial v}{\partial n}, & \text{on } L_\epsilon. \end{cases} \tag{5.1.8}$$

The decomposition (5.1.6) is proved under the assumption that a certain condition on R_ϵ, called H-Condition, is satisfied. We provide in particular a simple condition on the profile function g which guarantees the validity of the H-Condition.

Thus, in order to analyse the behaviour of $\lambda_n(\Omega_\epsilon)$ as $\epsilon \to 0$, it suffices to study θ_l^ϵ as $\epsilon \to 0$. To do so, we need to pass to the limit in the variational formulation of problem (5.1.8). Since the domain R_ϵ collapses to a segment as $\epsilon \to 0$, we use thin domain techniques in order to find the appropriate limiting problem. As in the case of the Laplace operator, the limiting problem depends on the shape of the channel R_ϵ via the profile function $g(x)$. More precisely it can be written as follows

$$\begin{cases} \frac{1-\sigma^2}{g}(gh'')'' - \frac{\tau}{g}(gh')' + h = \theta h, & \text{in } (0,1), \\ h(0) = h(1) = 0, \\ h'(0) = h'(1) = 0. \end{cases} \tag{5.1.9}$$

This allows to prove convergence results for the eigenvalues and eigenfunctions of problem (5.1.3). The precise statement can be found in Theorem 5.6.1.

We also note that the Dirichlet problem for the operator $\Delta^2 u - \tau\Delta u + u$, namely

$$\begin{cases} \Delta^2 u - \tau\Delta u + u = \lambda\, u, & \text{in } \Omega_\epsilon, \\ u = 0, & \text{on } \partial\Omega_\epsilon, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega_\epsilon \end{cases} \tag{5.1.10}$$

is stable in the sense that its eigenelements converge to those of the operator $\Delta^2 - \tau\Delta + I$ in Ω as $\epsilon \to 0$. In other words, as for the Laplace operator, in the case of Dirichlet boundary conditions, no eigenvalues from the channel R_ϵ appear in the limit as $\epsilon \to 0$. In fact, it is well known that Dirichlet eigenvalues on thin domains diverge to $+\infty$ as $\epsilon \to 0$, because of the Poincaré inequality.

However, the presence of the thin channel changes the rate of convergence of the eigenvalues, as proved in the interesting article [1] (see also [2] and [66]).

In order to prove our results, we study the convergence of the resolvent operators $(\Delta^2 - \tau\Delta + I)^{-1}_{N(\sigma,\tau)}$ and this is done by using the notion of $\mathcal{E}$-convergence (see §1.3).

5.2 Decomposition of the eigenvalues

The main goal of this section is to prove the decomposition of the eigenvalues of problem (5.1.2) into the two families of eigenvalues coming from (5.1.7) and (5.1.8). First of all we note that, since Ω_ϵ, Ω and R_ϵ are sufficiently regular, by standard spectral theory for differential operators it follows that the operators associated with the quadratic forms appearing in the weak formulation of problems (5.1.2), (5.1.7), (5.1.8) have compact resolvents. Thus, the spectra of such problems are discrete and consist of positive eigenvalues of finite multiplicity. The eigenpairs of problems (5.1.2), (5.1.7), (5.1.8) will be denoted by $(\lambda_n(\Omega_\epsilon), \varphi_n^\epsilon)_{n\geq 1}$, $(\omega_n, \varphi_n^\Omega)_{n\geq 1}$, $(\theta_n^\epsilon, \gamma_n^\epsilon)_{n\geq 1}$ respectively, where the three families of eigenfunctions φ_n^ϵ, φ_n^Ω, γ_n^ϵ are complete orthonormal bases of the spaces $L^2(\Omega_\epsilon)$, $L^2(\Omega)$, $L^2(R_\epsilon)$ respectively. Moreover we set $(\lambda_n^\epsilon)_{n\geq 1} = (\omega_k)_{k\geq 1} \cup (\theta_l^\epsilon)_{l\geq 1}$, where it is understood that the eigenvalues are arranged in increasing order and repeated according to their multiplicity. In particular if $\omega_k = \theta_l^\epsilon$ for some $k, l \in \mathbb{N}$, then such an eigenvalue is repeated in the sequence $(\lambda_n^\epsilon)_{n\geq 1}$ as many times as the sum of the multiplicities of ω_k and θ_l^ϵ. Let us note explicitly that the order in the sequence $(\lambda_n^\epsilon)_{n\geq 1}$ depends on ϵ. For each λ_n^ϵ we define the function $\phi_n^\epsilon \in H^2(\Omega) \oplus H^2(R_\epsilon)$ in the following way:

$$\phi_n^\epsilon = \begin{cases} \varphi_k^\Omega, & \text{in } \Omega, \\ 0, & \text{in } R_\epsilon, \end{cases} \tag{5.2.1}$$

if $\lambda_n^\epsilon = \omega_k$, for some $k \in \mathbb{N}$; otherwise

$$\phi_n^\epsilon = \begin{cases} 0, & \text{in } \Omega, \\ \gamma_l^\epsilon, & \text{in } R_\epsilon, \end{cases} \tag{5.2.2}$$

if $\lambda_n^\epsilon = \theta_l^\epsilon$, for some $l \in \mathbb{N}$. We observe that in the case $\lambda_n^\epsilon = \omega_k = \theta_l^\epsilon$ for some $k, l \in \mathbb{N}$, with ω_k of multiplicity m_1 and θ_l^ϵ of multiplicity m_2 we agree to order the eigenvalues (and the corresponding functions ϕ_n^ϵ) by listing first the m_1 eigenvalues ω_k, then the remaining m_2 eigenvalues θ_l^ϵ.

Note that $(\phi_i^\epsilon, \phi_j^\epsilon)_{L^2(\Omega_\epsilon)} = \delta_{ij}$ where δ_{ij} is the Kronecker symbol, that is $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. Note also that although ϕ_n^ϵ defined by (5.2.2) are in $H^2(\Omega_\epsilon)$ (due to the Dirichlet boundary condition imposed in L_ϵ), the function

ϕ_n^ϵ defined by (5.2.1) do not lie in $H^2(\Omega_\epsilon)$. To bypass this problem we define a sequence of functions in $H^2(\Omega_\epsilon)$ by setting

$$\xi_n^\epsilon = \begin{cases} E\varphi_k^\Omega, & \text{if } \lambda_n^\epsilon = \omega_k, \\ \phi_n^\epsilon, & \text{if } \lambda_n^\epsilon = \theta_l^\epsilon, \end{cases}$$

where E is a linear continuous extension operator mapping $H^2(\Omega)$ to $H^2(\mathbb{R}^N)$. Then it is easy to verify that for fixed i, j, we have $(\xi_i^\epsilon, \xi_j^\epsilon)_{L^2(\Omega_\epsilon)} = \delta_{ij} + o(1)$ as $\epsilon \to 0$. Then for fixed n and for ϵ small enough, $\xi_1^\epsilon, \dots, \xi_n^\epsilon$ are linearly independent.

Now we prove an upper bound for the eigenvalues $\lambda_n(\Omega_\epsilon)$.

Theorem 5.2.1 (Upper bound). *Let $n \geq 1$ be fixed. The eigenvalues λ_n^ϵ are uniformly bounded in ϵ and*

$$\lambda_n(\Omega_\epsilon) \leq \lambda_n^\epsilon + o(1), \quad \text{as } \epsilon \to 0. \tag{5.2.3}$$

Proof. The fact that λ_n^ϵ remains bounded as $\epsilon \to 0$ is an easy consequence of the inequality

$$\lambda_n^\epsilon \leq \omega_n < \infty, \tag{5.2.4}$$

which holds by definition of λ_n^ϵ. In the sequel we write $\perp$ to denote the orthogonality in L^2, and $[f_1, \dots, f_m]$ for the linear span of the functions $f_1, \dots, f_m$.

By the variational characterization of the eigenvalues $\lambda_n(\Omega_\epsilon)$ we have

$$\lambda_n(\Omega_\epsilon) = \min \left\{ \frac{\displaystyle\int_{\Omega_\epsilon} (1-\sigma)|D^2\psi|^2 + \sigma|\Delta\psi|^2 + \tau|\nabla\psi|^2 + |\psi|^2}{\displaystyle\int_{\Omega_\epsilon} |\psi|^2} \right. \\ \left. : \psi \in H^2(\Omega_\epsilon), \psi \not\equiv 0 \text{ and } \psi \perp \varphi_1^\epsilon, \dots, \varphi_{n-1}^\epsilon \right\}. \tag{5.2.5}$$

Since the functions $\xi_1^\epsilon, \dots, \xi_n^\epsilon$ are linearly independent, by a dimension argument there exists $\xi^\epsilon \in [\xi_1^\epsilon, \dots, \xi_n^\epsilon]$ such that $\|\xi^\epsilon\|_{L^2(\Omega_\epsilon)} = 1$, and $\xi^\epsilon \perp \varphi_1^\epsilon, \dots, \varphi_{n-1}^\epsilon$.

We can write $\xi^\epsilon = \sum_{i=1}^n \alpha_i \xi_i^\epsilon$, for some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ depending on ϵ such that $\sum_{i=1}^n \alpha_i^2 = 1 + o(1)$ as $\epsilon \to 0$. By using ξ^ϵ as a test function in (5.2.5) we get

$$\begin{aligned} \lambda_n(\Omega_\epsilon) &\leq \int_{\Omega_\epsilon} (1-\sigma)|D^2\xi^\epsilon|^2 + \sigma|\Delta\xi^\epsilon|^2 + \tau|\nabla\xi^\epsilon|^2 + |\xi^\epsilon|^2 \\ &= \sum_{i=1}^n \alpha_i^2 \left(\int_{\Omega_\epsilon} (1-\sigma)|D^2\xi_i^\epsilon|^2 + \sigma|\Delta\xi_i^\epsilon|^2 + \tau|\nabla\xi_i^\epsilon|^2 + |\xi_i^\epsilon|^2 \right) \\ &+ \sum_{i\neq j} \alpha_i\alpha_j \left(\int_{\Omega_\epsilon} (1-\sigma)(D^2\xi_i^\epsilon : D^2\xi_j^\epsilon) + \sigma\Delta\xi_i^\epsilon\Delta\xi_j^\epsilon + \tau\nabla\xi_i^\epsilon \cdot \nabla\xi_j^\epsilon + \xi_i^\epsilon\xi_j^\epsilon \right). \end{aligned} \tag{5.2.6}$$

By definition of ξ_i^ϵ and the absolute continuity of the Lebesgue integral, we have

$$\int_{\Omega_\epsilon}(1-\sigma)|D^2\xi_i^\epsilon|^2+\sigma|\Delta\xi_i^\epsilon|^2+\tau|\nabla\xi_i^\epsilon|^2+|\xi_i^\epsilon|^2=\begin{cases}\omega_k+o(1), & \text{if } \exists\, k \text{ s.t. } \lambda_i^\epsilon=\omega_k,\\ \theta_\epsilon^l, & \text{if } \exists\, l \text{ s.t. } \lambda_i^\epsilon=\theta_\epsilon^l,\end{cases}$$

which implies that $\int_{\Omega_\epsilon}(1-\sigma)|D^2\xi_i^\epsilon|^2+\sigma|\Delta\xi_i^\epsilon|^2+\tau|\nabla\xi_i^\epsilon|^2+|\xi_i^\epsilon|^2\le\lambda_n^\epsilon+o(1)$.

Note that

$$\sum_{i\neq j}\alpha_i\alpha_j\left(\int_{\Omega_\epsilon}(1-\sigma)(D^2\xi_i^\epsilon:D^2\xi_j^\epsilon)+\sigma\Delta\xi_i^\epsilon\Delta\xi_j^\epsilon+\tau\nabla\xi_i^\epsilon\cdot\nabla\xi_j^\epsilon+\xi_i^\epsilon\xi_j^\epsilon\right)=o(1).$$

Hence, $\lambda_n(\Omega_\epsilon)\le\sum_{i=1}^n\alpha_i^2(\lambda_n^\epsilon+o(1))+o(1)\le\lambda_n^\epsilon+o(1)$ which concludes the proof of (5.2.3). □

Remark 5.2.2. Note that the shape of the channel R_ϵ does not play any role in establishing the upper bound. The only fact needed is that the measure of R_ϵ tends to 0 as $\epsilon\to 0$.

In the sequel we shall provide a lower bound for the eigenvalues $\lambda_n(\Omega_\epsilon)$. Before doing so, let us introduce some notation.

Definition 5.2.3. Let $\sigma\in(-1,1)$, $\tau\ge 0$. We denote by $H^2_{L_\epsilon}(R_\epsilon)$ the space obtained as the closure in $H^2(R_\epsilon)$ of $C^\infty(\overline{R_\epsilon})$ functions which vanish in a neighborhood of L_ϵ. Furthermore, for any Lipschitz bounded open set U we define

$$[f]_{H^2_{\sigma,\tau}(U)}=\left|(1-\sigma)\|D^2f\|^2_{L^2(U)}+\sigma\|\Delta f\|^2_{L^2(U)}+\tau\|\nabla f\|^2_{L^2(U)}+\|f\|^2_{L^2(U)}\right|^{1/2},$$

for all $f\in H^2(U)$.

Note the functions u in $H^2_{L_\epsilon}(R_\epsilon)$ satisfy the conditions $u=0$ and $\nabla u=0$ on L_ϵ in the sense of traces.

Proposition 5.2.4. *Let $n\in\mathbb{N}$ be such that the following two conditions are satisfied:*

(i) For all $i=1,\dots,n$,

$$|\lambda_i^\epsilon-\lambda_i(\Omega_\epsilon)|\to 0 \qquad \text{as } \epsilon\to 0, \tag{5.2.7}$$

(ii) There exists $\delta>0$ such that

$$\lambda_n^\epsilon\le\lambda_{n+1}(\Omega_\epsilon)-\delta \tag{5.2.8}$$

for any $\epsilon>0$ small enough.

Let P_n be the projector from $L^2(\Omega_\epsilon)$ onto the linear span $[\phi_1^\epsilon, \dots, \phi_n^\epsilon]$ defined by

$$P_n g = \sum_{i=1}^{n} (g, \phi_i^\epsilon)_{L^2(\Omega_\epsilon)} \phi_i^\epsilon \,, \tag{5.2.9}$$

for all $g \in L^2(\Omega_\epsilon)$, where ϕ_i^ϵ is defined in (5.2.1), (5.2.2). *Then*

$$\|\varphi_i^\epsilon - P_n \varphi_i^\epsilon\|_{H^2(\Omega)\oplus H^2(R_\epsilon)} \longrightarrow 0, \tag{5.2.10}$$

as $\epsilon \longrightarrow 0$, for all $i = 1, \dots, n$.

Proof. By (5.2.3) and (5.2.4) we can extract a subsequence from both the sequences $(\lambda_i^\epsilon)_{\epsilon>0}$ and $(\lambda_i(\Omega_\epsilon))_{\epsilon>0}$ such that

$$\lambda_i^{\epsilon_k} \longrightarrow \lambda_i, \text{ and } \lambda_i(\Omega_{\epsilon_k}) \longrightarrow \widehat{\lambda}_i,$$

as $k \longrightarrow \infty$, for all $i = 1, \dots, n+1$.
By assumption we have $\lambda_i = \widehat{\lambda}_i$ for all $i = 1, \dots, n$. Thus, by passing to the limit as $\epsilon \longrightarrow 0$ in (5.2.3) (with n replaced by $n+1$) and in (5.2.8), we get

$$\lambda_n \le \widehat{\lambda}_{n+1} - \delta \le \lambda_{n+1} - \delta.$$

We rewrite $\lambda_1, \dots, \lambda_n$ without repetitions due to multiplicity in order to get a new sequence

$$\widetilde{\lambda}_1 < \widetilde{\lambda}_2 < \dots < \widetilde{\lambda}_s = \lambda_n \tag{5.2.11}$$

and set $\widetilde{\lambda}_{s+1} := \widehat{\lambda}_{n+1} \le \lambda_{n+1}$. Thus, by assumption (5.2.8) we have that

$$\widetilde{\lambda}_s < \widetilde{\lambda}_{s+1}. \tag{5.2.12}$$

For each $r = 1, \dots, s$, let $\widetilde{\lambda}_r = \lambda_{i_r} = \dots = \lambda_{j_r}$, for some $i_r \le j_r$, $i_r, j_r \in \{1, \dots, n\}$, where it is understood that $j_r - i_r + 1$ is the multiplicity of $\widetilde{\lambda}_r$. Furthermore, we define the eigenprojector Q_r from $L^2(\Omega_\epsilon)$ onto the linear span $[\varphi_{i_r}^\epsilon, \dots, \varphi_{j_r}^\epsilon]$ by

$$Q_r g = \sum_{i=i_r}^{j_r} (g, \varphi_{i_r}^\epsilon)_{L^2(\Omega_\epsilon)} \varphi_{i_r}^\epsilon. \tag{5.2.13}$$

We now proceed to prove the following

Claim: $\|\xi_i^{\epsilon_k} - Q_r \xi_i^{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} \longrightarrow 0$ as $\epsilon \longrightarrow 0$, for all $i_r \le i \le j_r$ and $r \le s$.
Let us prove it by induction on $1 \le r \le s$.
If $r = 1$, we define the function

$$\chi_{\epsilon_k} = \xi_i^{\epsilon_k} - Q_1 \xi_i^{\epsilon_k} = \xi_i^{\epsilon_k} - \sum_{l=1}^{j_1} (\xi_i^{\epsilon_k}, \varphi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \varphi_l^{\epsilon_k}.$$

Then $\chi_{\epsilon_k} \in H^2(\Omega_{\epsilon_k})$, $(\chi_{\epsilon_k}, \varphi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = 0$ for all $l = 1, \dots, j_1$ and by the min-max representation of $\lambda_2(\Omega_{\epsilon_k})$ we have that

$$[\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} \geq \lambda_2(\Omega_{\epsilon_k}) \|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} \geq \widetilde{\lambda}_2 \|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} - o(1). \tag{5.2.14}$$

On the other hand, it is easy to prove by definition of χ_{ϵ_k} that

$$\int_{\Omega_{\epsilon_k}} (1-\sigma)\big(D^2\chi_{\epsilon_k} : D^2\psi\big) + \sigma\Delta\chi_{\epsilon_k}\Delta\psi + \tau\nabla\chi_{\epsilon_k}\cdot\nabla\psi + \chi_{\epsilon_k}\psi\, dx$$
$$= \lambda_1(\Omega_{\epsilon_k}) \int_{\Omega_{\epsilon_k}} \chi_{\epsilon_k}\psi\, dx + o(1) \tag{5.2.15}$$

for all $\psi \in H^2(\Omega_{\epsilon_k})$. This in particular implies that

$$[\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} = \lambda_1(\Omega_{\epsilon_k}) \|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} + o(1) \tag{5.2.16}$$

and consequently,

$$[\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} \leq \widetilde{\lambda}_1 \|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} + o(1). \tag{5.2.17}$$

Hence, inequalities (5.2.14), (5.2.17) imply that

$$\widetilde{\lambda}_2 \|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} - o(1) \leq \widetilde{\lambda}_1 \|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} + o(1),$$

which implies that $\|\chi_{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k})} = o(1)$ (otherwise we would have $\widetilde{\lambda}_2 \leq \widetilde{\lambda}_1 + o(1)$, against (5.2.11)). Finally, equation (5.2.16) implies that $[\chi_{\epsilon_k}]_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} = o(1)$, so that also $\|\chi_{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} = o(1)$.

Let $r > 1$ and assume by induction hypothesis that

$$\|\xi_i^{\epsilon_k} - Q_t\xi_i^{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} \to 0 \tag{5.2.18}$$

as $k \to \infty$, for all $i_t \leq i \leq j_t$ and for all $t = 1, \dots, r-1$. We have to prove that (5.2.18) holds also for $t = r$. Let $i_r \leq i \leq j_r$ and let $\chi_{\epsilon_k} = \xi_i^{\epsilon_k} - Q_r\xi_i^{\epsilon_k}$. Then

$$(\chi_{\epsilon_k}, \varphi_h^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \to 0 \quad \text{as } k \to \infty, \text{ for all } h = 1, \dots, j_r\,. \tag{5.2.19}$$

Indeed, if $h \in \{i_r, \dots, j_r\}$ then by definition of χ_{ϵ_k}, $(\chi_{\epsilon_k}, \varphi_h^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = 0$. Otherwise, if $h < i_r$, note that the function $\varphi_h^{\epsilon_k}$ satisfies

$$\int_{\Omega_{\epsilon_k}} (1-\sigma)\left(D^2\varphi_h^{\epsilon_k} : D^2\psi\right) + \sigma\Delta\varphi_h^{\epsilon_k}\Delta\psi + \tau\nabla\varphi_h^{\epsilon_k}\nabla\psi + \varphi_h^{\epsilon_k}\psi\, dx$$
$$= \lambda_h(\Omega_{\epsilon_k}) \int_{\Omega_{\epsilon_k}} \varphi_h^{\epsilon_k}\psi\, dx\,,$$

for all $\psi \in H^2(\Omega_{\epsilon_k})$, briefly $B_{\Omega_{\epsilon_k}}(\varphi_h^{\epsilon_k}, \psi) = \lambda_h(\Omega_{\epsilon_k})(\varphi_h^{\epsilon_k}, \psi)_{L^2(\Omega_{\epsilon_k})}$, for all $\psi \in H^2(\Omega_{\epsilon_k})$, where B_U denotes the quadratic form associated with the operator $\Delta^2 - \tau\Delta + I$ on an open set U. Similarly, $B_{\Omega_{\epsilon_k}}(\xi_i^{\epsilon_k}, \psi) = \lambda_i^{\epsilon_k}(\xi_i^{\epsilon_k}, \psi)_{L^2(\Omega_{\epsilon_k})} + o(1)$ for all $\psi \in H^2(\Omega_{\epsilon_k})$. Thus, $\lambda_h(\Omega_{\epsilon_k})(\varphi_h^{\epsilon_k}, \xi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = \lambda_i^{\epsilon_k}(\xi_i^{\epsilon_k}, \varphi_h^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} + o(1)$ which implies

$$(\lambda_h(\Omega_{\epsilon_k}) - \lambda_i^{\epsilon_k})(\varphi_h^{\epsilon_k}, \xi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = o(1) \tag{5.2.20}$$

and since $(\lambda_h(\Omega_{\epsilon_k}) - \lambda_i^{\epsilon_k}) \to (\widetilde{\lambda}_h - \widetilde{\lambda}_i) \neq 0$ by assumption, by (5.2.20) we deduce that $(\varphi_h^{\epsilon_k}, \xi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = o(1)$ as $\epsilon_k \to 0$, for all $h = 1, \dots, j_r$, which implies (5.2.19).

As in the case $r = 1$ we may deduce that

$$[\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} \geq \widetilde{\lambda}_{r+1}\|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} - o(1). \tag{5.2.21}$$

On the other hand, by definition of χ_{ϵ_k} we have

$$[\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} \leq \widetilde{\lambda}_{r}\|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} + o(1). \tag{5.2.22}$$

By (5.2.21), (5.2.22) and (5.2.11) it must be $\|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} = o(1)$ and by (5.2.22) we deduce that $[\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} = o(1)$, hence $\|\chi_{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} \to 0$, as $k \to \infty$. This concludes the proof of the Claim.

Now define the projector $\widetilde{Q}_n$ from $L^2(\Omega_\epsilon)$ into the linear span $[\varphi_1^\epsilon, \dots, \varphi_n^\epsilon]$ by

$$\widetilde{Q}_n g = \sum_{i=1}^{n} (g, \varphi_i^\epsilon)_{L^2(\Omega_\epsilon)} \varphi_i^\epsilon.$$

Then, as a consequence of the Claim we have that

$$\|\xi_i^{\epsilon_k} - \widetilde{Q}_n \xi_i^{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} \to 0 \tag{5.2.23}$$

as $k \to \infty$, for all $i = 1, \dots, n$. Indeed for all indexes $i = 1, \dots, n$ there exists $1 \leq r \leq s$ such that $i_r \leq i \leq j_r$; let assume for simplicity that $r = 1$. Then we have $\|\xi_i^{\epsilon_k} - Q_1 \xi_i^{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} \to 0$ as $k \to \infty$; and also

$$\|\xi_i^{\epsilon_k} - \widetilde{Q}_n \xi_i^{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} \leq \|\xi_i^{\epsilon_k} - Q_1 \xi_i^{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} + \sum_{l > j_1}^{n} \left|(\xi_i^{\epsilon_k}, \varphi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})}\right| \|\varphi_l^{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})}$$

and the right-hand side tends to 0 as $k \to \infty$ because $\|\varphi_l^{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})}$ is uniformly bounded in k and $(\xi_i^{\epsilon_k}, \varphi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \to 0$ as $k \to \infty$ (to see this it is sufficient to argue as in the proof of (5.2.20)). Moreover, since $\|\xi_i^{\epsilon_k} - \phi_i^{\epsilon_k}\|_{H^2(\Omega) \oplus H^2(R_{\epsilon_k})} \to 0$ as $k \to \infty$

for all $i = 1, \dots, n$, we also have $\|\phi_i^{\epsilon_k} - \widetilde{Q}_n \phi_i^{\epsilon_k}\|_{H^2(\Omega)\oplus H^2(R_{\epsilon_k})} \to 0$ as $k \to \infty$, for all $i = 1, \dots, n$. Thus $(\widetilde{Q}_n \phi_1^{\epsilon_k}, \dots, \widetilde{Q}_n \phi_n^{\epsilon_k})$ is a basis in $(L^2(\Omega_{\epsilon_k})^n)$ for $[\varphi_1^{\epsilon_k}, \dots, \varphi_n^{\epsilon_k}]$. Hence, $\varphi_i^{\epsilon_k} = \sum_{l=1}^n a_{li}^{\epsilon_k} \widetilde{Q}_n \phi_l^{\epsilon_k}$ for some coefficients $a_{li}^{\epsilon_k} = (\varphi_i^{\epsilon_k}, \phi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} + o(1)$ as $k \to \infty$. Then for all $i = 1, \dots, n$ we have

$$
\begin{aligned}
&\|\varphi_i^{\epsilon_k} - P_n \varphi_i^{\epsilon_k}\|_{H^2(\Omega)\oplus H^2(R_{\epsilon_k})} \\
&\qquad = \left\| \sum_{l=1}^n (\varphi_i^{\epsilon_k}, \phi_l^{\epsilon_k})_{L^2} [\phi_l^{\epsilon_k} - \widetilde{Q}_n \phi_l^{\epsilon_k}] + o(1) \sum_{l=1}^n \widetilde{Q}_n \phi_l^{\epsilon_k} \right\|_{H^2(\Omega)\oplus H^2(R_{\epsilon_k})}
\end{aligned}
$$

and the right-hand side tends to 0 as $k \to \infty$. □

Remark 5.2.5. In the former proof one could prove that the matrix $A = (a_{li}^{\epsilon_k})_{l,i=1,\dots,n}$ is almost orthogonal, in the sense that $AA^t = A^t A = \mathbb{I} + o(1)$ as $k \to \infty$. Indeed, it is sufficient to show that the matrix $\tilde{A} = \big((\phi_l, \varphi_m^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})}\big)_{l,m=1,\dots,n}$ is almost orthogonal. Let l be fixed and note that $\phi_l = \sum_{m=1}^n (\phi_l, \varphi_m^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \varphi_m + (\mathbb{I} - \widetilde{Q}_m)\phi_l$, hence, by (5.2.23) we deduce that

$$
\delta_{li} = (\phi_l, \phi_m)_{L^2(\Omega_{\epsilon_k})} = \sum_{m=1}^n (\phi_l, \varphi_m)_{L^2(\Omega_{\epsilon_k})} (\varphi_m, \phi_i)_{L^2(\Omega_{\epsilon_k})} + o(1)\,, \tag{5.2.24}
$$

as $k \to \infty$. Note that we can rewrite (5.2.24) as $\tilde{A}\tilde{A}^t = \mathbb{I} + o(1)$, and in a similar way we also get that $\tilde{A}^t \tilde{A} = \mathbb{I} + o(1)$, concluding the proof.

In the sequel we shall need the following lemma.

Lemma 5.2.6. *Let $1 \le i \le j \le n$. Assume that $\widehat{\lambda} \in \mathbb{R}$ is such that, possibly passing to a subsequence, $\lambda_m(\Omega_\epsilon) \to \widehat{\lambda}$ as $\epsilon \to 0$ for all $m \in \{i, \dots, j\}$. If $\chi_\epsilon \in [\varphi_i^\epsilon, \dots, \varphi_j^\epsilon]$, $\|\chi_\epsilon\|_{L^2(\Omega_\epsilon)} = 1$ and $\chi_\epsilon|_\Omega \rightharpoonup \chi$ in $H^2(\Omega)$ then*

$$
\int_\Omega (1-\sigma)(D^2\chi : D^2\psi) + \sigma \Delta\chi \Delta\psi + \tau \nabla\chi \cdot \nabla\psi + \chi\psi \, dx = \widehat{\lambda} \int_\Omega \chi\psi \, dx\,, \tag{5.2.25}
$$

for all $\psi \in H^2(\Omega)$.

Proof. Since $\chi_\epsilon \in [\varphi_i^\epsilon, \dots, \varphi_j^\epsilon]$ and $\|\chi_\epsilon\|_{L^2(\Omega_\epsilon)} = 1$ there exist coefficients $(a_l(\epsilon))_{l=i}^j$ such that $\chi_\epsilon = \sum_{l=i}^j a_l(\epsilon)\varphi_l^\epsilon$ and $\sum_{l=i}^j a_l^2(\epsilon) = 1$. Note that for all $m \in \{i, \dots, j\}$, possibly passing to a subsequence, there exists $\widehat{\varphi}_m \in H^2(\Omega)$ such that $\varphi_m^\epsilon|_\Omega \rightharpoonup \widehat{\varphi}_m$ in $H^2(\Omega)$. Since $\chi_\epsilon|_\Omega \rightharpoonup \chi$ in $H^2(\Omega)$ by assumption, we get that $\chi = \sum_{l=i}^j a_l \widehat{\varphi}_l$ in Ω for some coefficients $(a_l)_{l=i}^j$. Let $\psi \in H^2(\Omega)$ be fixed and consider an extension

$\widetilde{\psi} = E\psi \in H^2(\mathbb{R}^N)$. Then

$$\begin{aligned}&\int_{\Omega_\epsilon} (1-\sigma)\big(D^2\chi_\epsilon : D^2\widetilde{\psi}\big) + \sigma\Delta\chi_\epsilon\Delta\widetilde{\psi} + \tau\nabla\chi_\epsilon\nabla\widetilde{\psi} + \chi_\epsilon\widetilde{\psi}\\ &= \sum_{l=i}^{j} a_l(\epsilon)\left[\int_{\Omega_\epsilon} (1-\sigma)\big(D^2\varphi_l^\epsilon : D^2\widetilde{\psi}\big) + \sigma\Delta\varphi_l^\epsilon\Delta\widetilde{\psi} + \tau\nabla\varphi_l^\epsilon\nabla\widetilde{\psi} + \varphi_l^\epsilon\widetilde{\psi}\right] \\ &= \sum_{l=i}^{j} a_l(\epsilon)\lambda_l(\Omega_\epsilon)\int_{\Omega_\epsilon} \varphi_l^\epsilon\widetilde{\psi}.\end{aligned} \tag{5.2.26}$$

Then it is possible to pass to the limit in both sides of (5.2.26) by splitting the integrals over Ω_ϵ into an integral over R_ϵ (that tends to 0 as $\epsilon \to 0$) and an integral over Ω. Moreover, the integrals over Ω will converge to the corresponding integrals in (5.2.25) as $\epsilon \to 0$, because of the weak convergence of χ_ϵ in $H^2(\Omega)$ and the strong convergence of $E\psi$ to ψ in $H^2(\Omega)$. □

We proceed to prove the lower bound for $\lambda_n(\Omega_\epsilon)$. To do so, we need to add an extra assumption on the shape of Ω_ϵ. Hence, we introduce the following condition in the spirit of what is known for the Neumann Laplacian (see e.g., [10], [11], [18]).

Definition 5.2.7 (H-Condition)**.** We say that the family of dumbbell domains Ω_ϵ, $\epsilon > 0$, satisfies the H-Condition if, given functions $u_\epsilon \in H^2(\Omega_\epsilon)$ such that $\|u_\epsilon\|_{H^2(\Omega_\epsilon)} \le R$ for all $\epsilon > 0$, there exist functions $\bar{u}_\epsilon \in H^2_{L_\epsilon}(R_\epsilon)$ such that

(i) $\|u_\epsilon - \bar{u}_\epsilon\|_{L^2(R_\epsilon)} \to 0$ as $\epsilon \to 0$,

(ii) $[\bar{u}_\epsilon]^2_{H^2_{\sigma,\tau}(R_\epsilon)} \le [u_\epsilon]^2_{H^2_{\sigma,\tau}(\Omega_\epsilon)} + o(1)$ as $\epsilon \to 0$.

Recall that $[\cdot]_{H^2_{\sigma,\tau}}$ is defined above in Definition 5.2.3. We will show in Section 5.3 that a wide class of channels R_ϵ satisfies the H-Condition.

Theorem 5.2.8 (Lower bound)**.** *Assume that the family of dumbbell domains Ω_ϵ, $\epsilon > 0$, satisfies the H-Condition. Then for every $n \in \mathbb{N}$ we have $\lambda_n(\Omega_\epsilon) \ge \lambda_n^\epsilon - o(1)$ as $\epsilon \to 0$.*

Proof. By Theorem 5.2.1 and its proof we know that both $\lambda_i(\Omega_\epsilon)$ and λ_i^ϵ are uniformly bounded in ϵ. Then, for each subsequence ϵ_k we can find a subsequence (which we still call ϵ_k), sequences of real numbers $(\lambda_i)_{i\in\mathbb{N}}$, $(\widehat{\lambda}_i)_{i\in\mathbb{N}}$, and sequences of $H^2(\Omega)$ functions $(\phi_i)_{i\in\mathbb{N}}$, $(\widehat{\varphi}_i)_{i\in\mathbb{N}}$, such that the following conditions are satisfied:

(i) $\lambda_i^{\epsilon_k} \longrightarrow \lambda_i$, for all $i \ge 1$;

(ii) $\lambda_i(\Omega_{\epsilon_k}) \longrightarrow \widehat{\lambda}_i$, for all $i \ge 1$;

(iii) $\xi_i^{\epsilon_k}|_{\Omega} \longrightarrow \phi_i$ strongly in $H^2(\Omega)$, for all $i \geq 1$;

(iv) $\varphi_i^{\epsilon_k}|_{\Omega} \longrightarrow \widehat{\varphi}_i$ weakly in $H^2(\Omega)$, for all $i \geq 1$;

Note that (iii) immediately follows by recalling that $\xi_i^{\epsilon_k}|_{\Omega}$ either it is zero or it coincides with φ_i^{Ω}. Then (iv) is deduced by the estimate $\|\varphi_i^{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} \leq c\,\lambda_i(\Omega_{\epsilon_k})$ and by the boundedness of the sequence $\lambda_i(\Omega_{\epsilon_k})$, $k \in \mathbb{N}$.

We plan to prove that $\widehat{\lambda}_i = \lambda_i$ for all $i \geq 1$. We do it by induction. For $i = 1$ we clearly have $\lambda_1 = \lambda_1(\Omega) = 1 = \lambda_1(\Omega_{\epsilon_k})$ for all k; hence, by passing to the limit as $k \to \infty$ in the right-hand side of the former equality we get $\lambda_1 = \widehat{\lambda}_1$. Then, we assume by induction hypothesis that $\widehat{\lambda}_i = \lambda_i$ for all $i = 1, \dots, n$ and we prove that $\widehat{\lambda}_{n+1} = \lambda_{n+1}$. There are two possibilities: either $\lambda_n = \lambda_{n+1}$ or $\lambda_n < \lambda_{n+1}$. In the first case we deduce by (5.2.3) that

$$\lambda_n = \widehat{\lambda}_n \leq \widehat{\lambda}_{n+1} \leq \lambda_{n+1} = \lambda_n,$$

hence all the inequalities are equalities and in particular $\widehat{\lambda}_{n+1} = \lambda_{n+1}$. Consequently we can assume without loss of generality that $\lambda_n < \lambda_{n+1}$. In this case we must have $\widehat{\lambda}_{n+1} \in [\lambda_n, \lambda_{n+1}]$ because $\lambda_n = \widehat{\lambda}_n$ and $\lambda_n(\Omega_{\epsilon_k}) \leq \lambda_{n+1}(\Omega_{\epsilon_k}) \leq \lambda_{n+1}^{\epsilon_k} + o(1)$ as $k \to \infty$. Let $r = \max\{\lambda_i : i < n, \lambda_i < \lambda_n\}$. Then $\lambda_r < \lambda_{r+1} = \cdots = \lambda_n < \lambda_{n+1}$. In particular we can apply Proposition 5.2.4 with n replaced by r in order to get

$$\|\varphi_i^{\epsilon_k} - P_r\varphi_i^{\epsilon_k}\|_{H^2(\Omega)\oplus H^2(R_{\epsilon_k})} \to 0 \tag{5.2.27}$$

as $k \to \infty$, for all $i = 1, \dots, r$. We now divide the proof in two steps.

<u>Step 1</u>: we prove that $\lambda_n < \widehat{\lambda}_{n+1}$.
Let us assume by contradiction that $\lambda_n = \widehat{\lambda}_{n+1}$; then $\widehat{\lambda}_{r+1} = \cdots = \widehat{\lambda}_n = \widehat{\lambda}_{n+1}$. Define the subspace S by $S = [\varphi_{r+1}^{\epsilon_k}, \dots, \varphi_{n+1}^{\epsilon_k}]$. Hence, S is $(n-r+1)$-dimensional. We then choose $\chi_{\epsilon_k} \in S$ with the following properties:

(I) $\|\chi_{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k})} = 1$.

(II) $\chi_{\epsilon_k} \perp \phi_{r+1}^{\epsilon_k}, \dots, \phi_n^{\epsilon_k}$ in $L^2(\Omega_{\epsilon_k})$.

This choice is possible because $[\phi_{r+1}^{\epsilon_k}, \dots, \phi_n^{\epsilon_k}]$ is $(n-r)$-dimensional. Moreover, we have that

$$(\chi_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0, \tag{5.2.28}$$

as $k \to \infty$, for all $i = 1, \dots, r$. To see this, recall that $\chi_{\epsilon_k} \in S$, hence

$$(\chi_{\epsilon_k}, \varphi_j^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = 0, \quad \forall j \leq r. \tag{5.2.29}$$

By (5.2.27) and (5.2.29), we have

$$(\chi_{\epsilon_k}, P_r\varphi_j^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0, \quad \forall j \leq r,$$

as $k \longrightarrow \infty$. Thus,

$$\sum_{l=1}^{r}(\varphi_j^{\epsilon_k}, \phi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})}(\chi_{\epsilon_k}, \phi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0, \quad \forall j \leq r, \tag{5.2.30}$$

as $k \longrightarrow \infty$. We can rewrite (5.2.30) as $A^t b \to 0$ as $k \longrightarrow \infty$, where A is the matrix defined in Remark 5.2.5 and $b \in \mathbb{R}^r$ is the vector defined by $b_l = ((\chi_{\epsilon_k}, \phi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})})_l$ for all $l \in \{1, \dots, r\}$. Hence, also $AA^t b \to 0$ as $k \to \infty$ and by Remark 5.2.5 we deduce that $AA^t b = \big(\mathbb{I} + o(1)\big)b = b + o(1) \to 0$ as $k \longrightarrow \infty$, since b is bounded in k. This implies that each component of b, which is $(\chi_{\epsilon_k}, \phi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})}$ tends to zero as $k \longrightarrow \infty$, which is (5.2.28).
It is now clear that (5.2.28) and property (II) of χ_{ϵ_k} yield

$$(\chi_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0, \quad \text{for all } i = 1, \dots, n, \tag{5.2.31}$$

as $k \longrightarrow \infty$. Since $\|\chi_{\epsilon_k}\|_{H^2(\Omega)} \leq C \max_{r+1 \leq j \leq n+1} \|\varphi_j^{\epsilon_k}\|_{H^2(\Omega)} < \infty$ there exists a function $\chi \in H^2(\Omega)$ such that possibly passing to a subsequence

$$\chi_{\epsilon_k}|_{\Omega} \rightharpoonup \chi \quad \text{in } H^2(\Omega), \tag{5.2.32}$$

as $k \longrightarrow \infty$. By (5.2.31) and (5.2.32) we deduce that $(\chi, \phi_i)_{L^2(\Omega)} = 0$, for all $i = 1, \dots, n$. By Lemma 5.2.6 χ is a n-th eigenfunction of $(\Delta^2 - \tau\Delta + \mathbb{I})_{N(\sigma)}$ in Ω associated with $\widetilde{\lambda}_n$ which is orthogonal to $\phi_1, \dots, \phi_n$, among which there are all the possible n-th eigenfunctions. Since $\lambda_n < \lambda_{n+1}$, the only way to avoid a contradiction is that $\chi \equiv 0$ in Ω, that is

$$\|\chi_{\epsilon_k}\|_{L^2(\Omega)} \to 0, \qquad \|\chi_{\epsilon_k}\|_{L^2(R_{\epsilon_k})} \to 1, \tag{5.2.33}$$

as $k \longrightarrow \infty$. We use now the H-Condition; let us choose a sequence of functions $\overline{\chi}_{\epsilon_k} \in H^2_{L_{\epsilon_k}}(R_{\epsilon_k})$ such that $\|\chi_{\epsilon_k} - \overline{\chi}_{\epsilon_k}\|_{L^2(R_{\epsilon_k})} \to 0$ as $k \to \infty$ and

$$[\overline{\chi}_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(R_{\epsilon_k})} \leq [\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} + o(1), \tag{5.2.34}$$

as $k \longrightarrow \infty$. Then we can extend by zero $\overline{\chi}_{\epsilon_k}$ to get a function (that we still call $\overline{\chi}_{\epsilon_k}$) in $H^2(\Omega_{\epsilon_k})$. Hence,

$$\begin{aligned}(\overline{\chi}_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} &= (\overline{\chi}_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(R_{\epsilon_k})} \\ &= (\overline{\chi}_{\epsilon_k} - \chi_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(R_{\epsilon_k})} + (\chi_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} - (\chi_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega)},\end{aligned} \tag{5.2.35}$$

for all $i = 1, \dots, n$. By (5.2.31), (5.2.33), and the definition of $\overline{\chi}_{\epsilon_k}$ the right hand side of (5.2.35) tends to 0 as $k \longrightarrow \infty$, for all $i = 1, \dots, n$. Thus, $\overline{\chi}_{\epsilon_k}$ is asymptotically orthogonal to $\phi_1^{\epsilon_k}, \dots, \phi_n^{\epsilon_k}$. In particular, by the variational characterization of the eigenvalues $\lambda_i^{\epsilon_k}$ we get that

$$[\overline{\chi}_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(R_{\epsilon_k})} \geq \lambda_{n+1}^{\epsilon_k} \|\overline{\chi}_{\epsilon_k}\|_{L^2(R_{\epsilon_k})} - o(1) \geq \lambda_{n+1} \|\overline{\chi}_{\epsilon_k}\|_{L^2(R_{\epsilon_k})} - o(1). \qquad (5.2.36)$$

On the other hand, by (5.2.34) we deduce that

$$\begin{aligned}[\overline{\chi}_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(R_{\epsilon_k})} &\leq [\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} + o(1) \\ &= \lambda_n \|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} + o(1) = \lambda_n \|\overline{\chi}_{\epsilon_k}\|^2_{L^2(R_{\epsilon_k})} + o(1).\end{aligned}$$

This is a contradiction to (5.2.36) because $\lambda_n < \lambda_{n+1}$. Step 1 is complete.

Step 2: we prove that $\widehat{\lambda}_{n+1} = \lambda_{n+1}$.
Assume by contradiction that $\widehat{\lambda}_{n+1} < \lambda_{n+1}$. Let us note that as a consequence of Step 1 we can use Proposition 5.2.4 for the n-th eigenvalues in order to obtain

$$\|\varphi_i^{\epsilon_k} - P_n \varphi_i^{\epsilon_k}\|_{H^2(\Omega)\oplus H^2(R_{\epsilon_k})} \to 0, \qquad (5.2.37)$$

for all $i = 1, \dots, n$. Then we can use the same argument we used in Step 1 for χ_{ϵ_k} to show that

$$\|\varphi_{n+1}^{\epsilon_k}\|_{L^2(\Omega)} \longrightarrow 0, \qquad (5.2.38)$$

as $k \to \infty$. To see this, just note that $\varphi_{n+1}^{\epsilon_k}$ is orthogonal to $\varphi_1^{\epsilon_k}, \dots, \varphi_n^{\epsilon_k}$, and by (5.2.37) we deduce that $(\varphi_{n+1}^{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \to 0$, as $k \to \infty$, for all $i = 1, \dots, n$. Moreover,

$$(\varphi_{n+1}^{\epsilon_k}, \phi_{n+1}^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0, \qquad (5.2.39)$$

as $k \to \infty$. Indeed, looking at the weak formulation of problem (5.1.2) and denoting by B_U denotes the quadratic problem associated with the operator $\Delta^2 - \tau\Delta + I$ on an open set U, we deduce both

$$B_{\Omega_{\epsilon_k}}(\varphi_{n+1}^{\epsilon_k}, \phi_{n+1}^{\epsilon_k}) = \lambda_{n+1}(\Omega_{\epsilon_k})(\varphi_{n+1}^{\epsilon_k}, \phi_{n+1}^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} + o(1),$$

and

$$B_{\Omega_{\epsilon_k}}(\phi_{n+1}^{\epsilon_k}, \varphi_{n+1}^{\epsilon_k}) = \lambda_{n+1}^{\epsilon_k}(\phi_{n+1}^{\epsilon_k}, \varphi_{n+1}^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} + o(1),$$

and subtracting the above equalities and passing to the limit as $k \to \infty$ we obtain $(\widehat{\lambda}_{n+1} - \lambda_{n+1}) \lim_{k\to\infty} (\phi_{n+1}^{\epsilon_k}, \varphi_{n+1}^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = 0$, which implies (5.2.39). Then

$$(\varphi_{n+1}^{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0, \qquad (5.2.40)$$

as $k \to \infty$, for all $i = 1, \dots, n+1$. Passing to the limit in k we have $(\widehat{\varphi}_{n+1}, \phi_i)_{L^2(\Omega)} = 0$ for all $i = 1, \dots, n+1$. However, as in Step 1 we would have $[\widehat{\varphi}_{n+1}]^2_{H^2_{\sigma,\tau}(\Omega)} =$

$\widehat{\lambda}_{n+1}\|\widehat{\varphi}_{n+1}\|_{L^2(\Omega)}$, which contradicts the assumption $\widehat{\lambda}_{n+1} < \lambda_{n+1}$ unless $\widehat{\varphi}_{n+1} \equiv 0$, which gives (5.2.38).
Now we use the H-Condition and (5.2.38) in order to find a function $\overline{\varphi}^{\epsilon_k}_{n+1} \in H^2_{L_{\epsilon_k}}(R_{\epsilon_k})$ such that $\|\overline{\varphi}^{\epsilon_k}_{n+1}\|_{L^2(R_{\epsilon_k})} = 1 + o(1)$ and

$$[\overline{\varphi}^{\epsilon_k}_{n+1}]^2_{H^2_{\sigma,\tau}(R_{\epsilon_k})} \le [\varphi^{\epsilon_k}_{n+1}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} + o(1) = \lambda_{n+1}(\Omega_{\epsilon_k}) + o(1) \le \widehat{\lambda}_{n+1} + o(1),$$

as $k \to \infty$. On the other hand, by the variational characterization of $\lambda^{\epsilon_k}_{n+1}$ and by (5.2.38), (5.2.40) we deduce that $[\overline{\varphi}^{\epsilon_k}_{n+1}]^2_{H^2_{\sigma,\tau}(R_{\epsilon_k})} \ge \lambda^{\epsilon_k}_{n+1}\|\overline{\varphi}^{\epsilon_k}_{n+1}\|_{L^2(R_{\epsilon_k})} - o(1) \ge \lambda_{n+1} - o(1)$, as $k \to \infty$, hence $\lambda_{n+1} \le \widehat{\lambda}_{n+1}$, a contradiction. Thus it must be $\lambda_{n+1} = \widehat{\lambda}_{n+1}$. □

We will say that $x_\epsilon \in (0, \infty)$ *divides the spectrum* of a family of nonnegative self-adjoint operators A_ϵ, $\epsilon > 0$, with compact resolvents in $L^2(\Omega_\epsilon)$ if there exist $\delta, M, N, \epsilon_0 > 0$ such that

$$[x_\epsilon - \delta, x_\epsilon + \delta] \cap \{\lambda^\epsilon_n\}_{n=1}^\infty = \emptyset, \quad \forall \epsilon < \epsilon_0 \tag{5.2.41}$$

$$x_\epsilon \le M, \quad \forall \epsilon < \epsilon_0 \tag{5.2.42}$$

$$N(x_\epsilon) := \#\{\lambda^\epsilon_i : \lambda^\epsilon_i \le x_\epsilon\} \le N < \infty. \tag{5.2.43}$$

If x_ϵ divides the spectrum we define the projector P_{x_ϵ} from $L^2(\Omega_\epsilon)$ onto the linear span $[\phi^\epsilon_1, \dots, \phi^\epsilon_{N(x_\epsilon)}]$ of the first $N(x_\epsilon)$ eigenfunctions by

$$P_{x_\epsilon} g = \sum_{i=1}^{N(x_\epsilon)} (g, \phi^\epsilon_i)_{L^2(\Omega_\epsilon)} \phi^\epsilon_i\,,$$

for all $g \in L^2(\Omega_\epsilon)$. Then, recalling Theorem 5.2.1 and Theorem 5.2.8 we deduce the following.

Theorem 5.2.9 (Decomposition of the eigenvalues). *Let Ω_ϵ, $\epsilon > 0$, be a family of dumbbell domains satisfying the H-Condition. Then the following statements hold:*

(i) $\lim_{\epsilon\to 0} |\lambda_n(\Omega_\epsilon) - \lambda^\epsilon_n| = 0$, *for all $n \in \mathbb{N}$.*

(ii) For any x_ϵ dividing the spectrum, $\lim_{\epsilon\to 0} \|\varphi^\epsilon_{r_\epsilon} - P_{x_\epsilon}\varphi^\epsilon_{r_\epsilon}\|_{H^2(\Omega)\oplus H^2(R_\epsilon)} = 0$, *for all $r_\epsilon = 1, \dots, N(x_\epsilon)$.*

5.3 Proof of the H-Condition for regular dumbbells

The goal of this section is to prove that the H-Condition holds for regular dumbbell domains. More precisely, we will consider channels R_ϵ such that the profile function g has the following monotonicity property:

(MP): *there exists* $\delta \in]0, 1/2[$ *such that* g *is decreasing on* $[0, \delta)$ *and increasing on* $(1-\delta, 1]$.

If (MP) is satisfied then the set

$$A_\epsilon = \{(x, y) \in \mathbb{R}^2 : x \in (0, \delta) \cup (1-\delta, 1), 0 < y < \epsilon g(x)\}$$

is contained in the union of the two rectangles $[0, \delta] \times [0, \epsilon g(0)]$ and $[1-\delta, 1] \times [0, \epsilon g(1)]$. This fact will be used in the proof of the following theorem in order to control the H^2 norm of the candidate function $\overline{u}_\epsilon$ appearing in the H-Condition.

Theorem 5.3.1. *Assume that the dumbbell* $\Omega_\delta = \Omega \cup R_\delta$ *is such that* R_δ *satisfies property (MP). Then* Ω_δ *satisfies the H-Condition.*

Before writing the proof of this theorem we need to introduce some notation. First, for the sake of clarity we will consider a "one-sided" dumbbell $\Omega_\epsilon = \Omega \cup R_\epsilon$ where Ω is a smooth bounded domain in $\mathbb{R}^2$ such that the segment $\{0\} \times [-1, 1]$ is contained in the boundary of Ω, $\Omega \cap \{x \geq 0\} = \emptyset$ and R_ϵ is defined as in (5.1.1). We will assume that R_ϵ satisfies the (MP) condition on $0 < x < \delta$ only. Let L_ϵ be the segment $\{0\} \times (0, \epsilon g(0))$.

For any $\gamma \in (0, 1)$, we define a function $\chi_\epsilon^\gamma \in C^{1,1}[-\epsilon^\gamma, 1]$, such that $\chi_\epsilon^\gamma(-\epsilon^\gamma) = (\chi_\epsilon^\gamma)'(-\epsilon^\gamma) = 0$, $\chi_\epsilon^\gamma(x) \equiv 1$ for all $0 \leq x \leq 1$ and such that the following bounds on the derivatives

$$\|(\chi_\epsilon^\gamma)'\|_{L^\infty(-\epsilon^\gamma, 0)} \leq \frac{c_1}{\epsilon^\gamma}, \quad \|(\chi_\epsilon^\gamma)''\|_{L^\infty(-\epsilon^\gamma, 0)} \leq \frac{c_2}{\epsilon^{2\gamma}},$$

are satisfied for some positive real numbers c_1, c_2. A possible choice for χ_ϵ^γ is

$$\chi_\epsilon^\gamma(x) = \begin{cases} -2\left(\dfrac{x+\epsilon^\gamma}{\epsilon^\gamma}\right)^3 + 3\left(\dfrac{x+\epsilon^\gamma}{\epsilon^\gamma}\right)^2, & x \in (-\epsilon^\gamma, 0), \\ \\ 1, & x \in (0, 1), \end{cases}$$

which gives the (non-optimal) bounds $c_1 = 3/2$, $c_2 = 6$. For any $\gamma, \beta > 0$ we define the function $f_{\gamma,\beta} \in C^{1,1}(0, 1)$ by setting

$$f = f_{\gamma,\beta}(x) = \begin{cases} -\epsilon^\gamma \left(\frac{x}{\epsilon^\beta}\right)^2 + (\epsilon^\beta + 2\epsilon^\gamma)\left(\frac{x}{\epsilon^\beta}\right) - \epsilon^\gamma, & x \in (0, \epsilon^\beta), \\ x, & x \in (\epsilon^\beta, 1). \end{cases} \tag{5.3.1}$$

Note that f is a $C^{1,1}$-diffeomorphism from $(0, \epsilon^\beta)$ onto $(-\epsilon^\gamma, \epsilon^\beta)$. Then,

$$f'(x) = \begin{cases} 1 + 2\epsilon^{\gamma-\beta}(1 - \frac{x}{\epsilon^\beta}), & x \in (0, \epsilon^\beta), \\ 1, & x \in (\epsilon^\beta, 1), \end{cases}$$

and

$$f''(x) = \begin{cases} -2\epsilon^{\gamma-2\beta}, & x \in (0, \epsilon^\beta), \\ 0, & x \in (\epsilon^\beta, 1), \end{cases}$$

which implies that $|f'(x) - 1| \leq 2\epsilon^{\gamma-\beta}$, for all $x \in (0,1)$, and $|f''(x)| \leq 2\epsilon^{\gamma-2\beta}$, for all $x \in (0,1)$. Thus, if $\gamma > \beta$ then

$$f'(x) = 1 + o(1) \quad \text{as } \epsilon \to 0. \tag{5.3.2}$$

For any $\theta \in (0,1)$, we define the following sets:

$$\begin{aligned} K_\epsilon^\theta &= \{(x,y) \in \Omega : -\epsilon^\theta < x < 0,\ 0 < y < \epsilon g(0)\}, \\ \Gamma_\epsilon^\theta &= \{(-\epsilon^\theta, y) : 0 < y < \epsilon g(0)\}, \\ J_\epsilon^\theta &= \{(x,y) \in R_\epsilon : 0 < x < \epsilon^\theta\}, \\ Q_\epsilon^\theta &= \{(x,y) \in \mathbb{R}^2 : 0 < x < \epsilon^\theta, 0 < y < \epsilon g(0)\}. \end{aligned}$$

Finally, if $\gamma/3 < \beta < \gamma/2$, for every $u_\epsilon \in H^2(\Omega_\epsilon)$ we define the function $\overline{u}_\epsilon \in H^2(R_\epsilon)$ by setting

$$\overline{u}_\epsilon(x,y) = u_\epsilon(f(x), y)\, \chi_\epsilon^\gamma(f(x)), \tag{5.3.3}$$

for all $(x,y) \in R_\epsilon$. Function $\overline{u}_\epsilon$ will be used to prove the validity of the H-Condition. Before doing so, we need to prove the following proposition.

Proposition 5.3.2. *Let $\Omega_\epsilon = \Omega \cup R_\epsilon$ with R_ϵ satisfying the (MP) condition. Let $u_\epsilon \in H^2(\Omega_\epsilon)$ be such that $\|u_\epsilon\|_{H^2(\Omega_\epsilon)} \leq R$ for all $\epsilon > 0$. Then, with the notation above and for $0 < \theta < \frac{1}{3}$, we have*

$$\|u_\epsilon\|_{L^2(J_\epsilon^\theta)} = O(\epsilon^{2\theta}), \quad \|\nabla u_\epsilon\|_{L^2(J_\epsilon^\theta)} = O(\epsilon^\theta), \quad \textit{as } \epsilon \to 0. \tag{5.3.4}$$

Proof. We define the function $u_\epsilon^s \in H^2(J_\epsilon^\theta)$ by setting

$$u_\epsilon^s(x,y) = -3u_\epsilon(-x,y) + 4u_\epsilon\Big(-\frac{x}{2}, y\Big),$$

for all $(x,y) \in J_\epsilon^\theta$. The function u_ϵ^s can be viewed as a higher order reflection of u_ϵ with respect to the y-axis. Let us note that we can estimate the L^2 norm of u_ϵ^s, of its gradient and of its derivatives of order 2, in the following way:

$$\|u_\epsilon^s\|_{L^2(J_\epsilon^\theta)} \leq C\|u_\epsilon\|_{L^2(K_\epsilon^\theta)}, \tag{5.3.5}$$

$$\|\nabla u_\epsilon^s\|_{L^2(J_\epsilon^\theta)} \leq C\|\nabla u_\epsilon\|_{L^2(K_\epsilon^\theta)}, \tag{5.3.6}$$

$$\|D^\alpha u_\epsilon^s\|_{L^2(J_\epsilon^\theta)} \leq C\|D^\alpha u_\epsilon\|_{L^2(K_\epsilon^\theta)}, \tag{5.3.7}$$

for any multiindex α of length 2 and for some constant C independent of ϵ. To obtain the three inequalities above, we are using that the image of K_ϵ^θ under the reflexion about the y-axis contains J_ϵ^θ. This is a consequence of (MP). Since the L^2 norms on the right-hand sides of the inequalities above are taken on a subset of Ω, we can improve the estimate of (5.3.5) and (5.3.6) using Hölder's inequality and Sobolev embeddings to obtain

$$\|u_\epsilon\|_{L^2(K_\epsilon^\theta)} \le |K_\epsilon^\theta|^{1/2}\|u_\epsilon\|_{L^\infty(\Omega)} \le c\big(\epsilon^{\theta+1}\big)^{1/2}\|u_\epsilon\|_{H^2(\Omega)}, \tag{5.3.8}$$

and in a similar way

$$\|\nabla u_\epsilon\|_{L^2(K_\epsilon^\theta)} \le |K_\epsilon^\theta|^{\frac{1}{2}-\frac{1}{p}}\|\nabla u_\epsilon\|_{L^p(\Omega)} \le c\big(\epsilon^{\theta+1}\big)^{\frac{1}{2}-\frac{1}{p}}\|u_\epsilon\|_{H^2(\Omega)}, \tag{5.3.9}$$

for any $2 < p < \infty$. Thus

$$\|u_\epsilon^s\|_{L^2(J_\epsilon^\theta)} \le C\epsilon^{\frac{\theta+1}{2}}\|u_\epsilon\|_{H^2(\Omega)}, \text{ and } \|\nabla u_\epsilon^s\|_{L^2(J_\epsilon^\theta)} \le C\big(\epsilon^{\theta+1}\big)^{\frac{1}{2}-\frac{1}{p}}\|u_\epsilon\|_{H^2(\Omega)}. \tag{5.3.10}$$

We also get

$$\|D^\alpha u_\epsilon^s\|_{L^2(J_\epsilon^\theta)} \le C\|u_\epsilon\|_{H^2(\Omega)}\,. \tag{5.3.11}$$

We define now the function

$$\psi_\epsilon = (u_\epsilon - u_\epsilon^s)|_{J_\epsilon^\theta} \in H^2(J_\epsilon^\theta).$$

Then $\psi_\epsilon = 0 = \nabla\psi_\epsilon$ on L_ϵ. Let us first estimate $\|\nabla u_\epsilon\|_{L^2(J_\epsilon^\theta)}$. Since we have

$$\|\nabla u_\epsilon\|^2_{L^2(J_\epsilon^\theta)} = \sum_{i=1}^2 \int_{J_\epsilon^\theta}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^2 dx,$$

we can directly estimate the L^2-norm of the partial derivatives. Since $\partial_{x_i}\psi_\epsilon = 0$ on L_ϵ for all $i = 1, 2$ we apply a one-dimensional Poincaré inequality in the x-direction. We proceed as follows. For each $x_2 \in (\epsilon g(\epsilon^\theta), \epsilon g(0))$ we denote by $h_\epsilon(x_2)$ the unique number such that $\epsilon g(h_\epsilon(x_2)) = x_2$ (that is, the inverse function of $\epsilon g(\cdot)$, which exists because of hypothesis (MP)). For $x_2 \in (0, \epsilon g(\epsilon^\theta))$ we define $h_\epsilon(x_2) = \epsilon^\theta$. Observe that $0 \le h_\epsilon(x_2) \le \epsilon^\theta$ and that J_ϵ^θ can be expressed as $J_\epsilon^\theta = \{(x_1, x_2) : 0 < x_2 < \epsilon g(0); 0 < x_1 < h_\epsilon(x_2)\}$. Hence, for $i = 1, 2$ we have

$$\left\|\frac{\partial\psi_\epsilon}{\partial x_i}(\cdot, x_2)\right\|^2_{L^2(0,h_\epsilon(x_2))} \le \frac{1}{\lambda_1(h_\epsilon(x_2))}\left\|\frac{\partial}{\partial x_1}\left(\frac{\partial\psi_\epsilon}{\partial x_i}(\cdot, x_2)\right)\right\|^2_{L^2(0,h_\epsilon(x_2))} \tag{5.3.12}$$

where $\lambda_1(\rho) = \left(\frac{\pi}{2}\right)^2\rho^{-2}$ is the first eigenvalue of the problem

$$\begin{cases} -v'' = \lambda v, & \text{in } (0, \rho), \\ v(0) = 0, \\ v'(\rho) = 0. \end{cases}$$

Since $0 \leq h_\epsilon(x_2) \leq \epsilon^\theta$, we get the bound $\lambda_1(h_\epsilon(x_2)) \geq \left(\frac{\pi}{2\epsilon^\theta}\right)^2$ and integrating in (5.3.12) with respect to $x_2 \in (0, \epsilon g(0))$, we get

$$\left\|\frac{\partial \psi_\epsilon}{\partial x_i}\right\|^2_{L^2(J^\theta_\epsilon)} \leq \left(\frac{2\epsilon^\theta}{\pi}\right)^2 \left\|\frac{\partial^2 \psi_\epsilon}{\partial x \partial x_i}\right\|^2_{L^2(J^\theta_\epsilon)}. \tag{5.3.13}$$

Now note that $\left\|\frac{\partial^2 \psi_\epsilon}{\partial x \partial x_i}\right\|_{L^2(J^\theta_\epsilon)} \leq C\|u_\epsilon\|_{H^2(\Omega_\epsilon)} \leq CR$ for all $\epsilon > 0$, where we have used (5.3.11). Hence we rewrite inequality (5.3.13) in the following way:

$$\left\|\frac{\partial \psi_\epsilon}{\partial x_i}\right\|_{L^2(J^\theta_\epsilon)} \leq \frac{2}{\pi}\epsilon^\theta(CR + o(1)) = O(\epsilon^\theta), \tag{5.3.14}$$

as $\epsilon \longrightarrow 0$, for $i = 1, 2$.

Finally, by the inequalities (5.3.10), (5.3.14) we deduce that

$$\begin{aligned} \|\nabla u_\epsilon\|_{L^2(J^\theta_\epsilon)} &\leq \|\nabla \psi_\epsilon\|_{L^2(J^\theta_\epsilon)} + \|\nabla u^s_\epsilon\|_{L^2(J^\theta_\epsilon)} \\ &\leq O(\epsilon^\theta) + C\big(\epsilon^{\theta+1}\big)^{\frac{1}{2}-\frac{1}{p}}\|u_\epsilon\|_{H^2(\Omega)} \leq O(\epsilon^\theta), \end{aligned} \tag{5.3.15}$$

where we have used that $(\theta + 1)(1/2 - 1/p) > \theta$ for large enough p.

It remains to prove that $\|u_\epsilon\|_{L^2(J^\theta_\epsilon)} = O(\epsilon^{2\theta})$ as $\epsilon \longrightarrow 0$. We can repeat the argument for u_ϵ instead of $\partial_{x_i} u_\epsilon$, with the difference that now we can improve the decay of $\|\psi_\epsilon\|_{L^2(J^\theta_\epsilon)}$ by using the one-dimensional Poincaré inequality twice. More precisely we have that

$$\|\psi_\epsilon\|_{L^2(J^\theta_\epsilon)} \leq \left(\frac{2}{\pi}\right)^2 \epsilon^{2\theta} \left\|\frac{\partial^2 \psi_\epsilon}{\partial x^2}\right\|_{L^2(J^\theta_\epsilon)}$$

from which we deduce $\|\psi_\epsilon\|_{L^2(J^\theta_\epsilon)} = O(\epsilon^{2\theta})$, as $\epsilon \longrightarrow 0$. Hence,

$$\|u_\epsilon\|_{L^2(J^\theta_\epsilon)} \leq \|\psi_\epsilon\|_{L^2(J^\theta_\epsilon)} + \|u^s_\epsilon\|_{L^2(J^\theta_\epsilon)} \leq O(\epsilon^{2\theta}) + C\epsilon^{\frac{\theta+1}{2}}\|u_\epsilon\|_{H^2(\Omega)} = O(\epsilon^{2\theta}), \tag{5.3.16}$$

as $\epsilon \longrightarrow 0$, concluding the proof. □

We can now give a proof of Theorem 5.3.1.

Proof of Theorem 5.3.1. Let $u_\epsilon \in H^2(\Omega_\epsilon)$ be such that $\|u_\epsilon\|_{H^2(\Omega_\epsilon)} \leq R$ for any $\epsilon > 0$. We prove that the H-Condition holds if we choose $\overline{u}_\epsilon$ as in (5.3.3) with $\gamma < 1/3$. Note that $u_\epsilon \equiv \overline{u}_\epsilon$ on $R_\epsilon \setminus J^\beta_\epsilon$. Let us first estimate $\|\overline{u}_\epsilon\|_{L^2(J^\beta_\epsilon)}$. By a change of variable and by (5.3.2) we deduce that

$$\begin{aligned}
\|\overline{u}_\epsilon\|^2_{L^2(J^\beta_\epsilon)} &= \int_0^{\epsilon^\beta}\int_0^{\epsilon g(x)} |(u_\epsilon \chi^\gamma_\epsilon)(f(x),y)|^2\, dydx \\
&= \int_{-\epsilon^\gamma}^{\epsilon^\beta}\int_0^{\epsilon g(f^{-1}(z))} |(u_\epsilon \chi^\gamma_\epsilon)(z,y)|^2 |f'(f^{-1}(z))|^{-1}\, dydz \\
&\le (1+o(1)) \int_{-\epsilon^\gamma}^{\epsilon^\beta}\int_0^{\epsilon g(f^{-1}(z))} |(u_\epsilon \chi^\gamma_\epsilon)(z,y)|^2 dydz \\
&\le (1+o(1))\|u_\epsilon\|^2_{L^2(Z^\gamma_\epsilon)},
\end{aligned} \tag{5.3.17}$$

where $Z^\gamma_\epsilon = \{(x,y) \in \Omega_\epsilon : -\epsilon^\gamma < x < \epsilon^\beta, 0 < y < \epsilon g(f^{-1}(x))\}$. Note that since the function g is non increasing, then $Z^\gamma_\epsilon \subset K^\gamma_\epsilon \cup J^\beta_\epsilon$. Hence,

$$\|\overline{u}_\epsilon\|^2_{L^2(J^\beta_\epsilon)} \le (1+o(1))\Big(\|u_\epsilon\|^2_{L^2(K^\gamma_\epsilon)} + \|u_\epsilon\|^2_{L^2(J^\beta_\epsilon)}\Big). \tag{5.3.18}$$

Note that the last summand in the right-hand side of (5.3.18) behaves as $O(\epsilon^{4\beta})$ as $\epsilon \longrightarrow 0$ because of Proposition 5.3.2. Also by (5.3.8) with θ replaced by γ, we get

$$\|u_\epsilon\|_{L^2(K^\gamma_\epsilon)} \le c\epsilon^{\frac{\gamma+1}{2}}\|u_\epsilon\|_{H^2(\Omega)},$$

Thus,

$$\|\overline{u}_\epsilon\|^2_{L^2(J^\beta_\epsilon)} \le (1+o(1))(O(\epsilon^{4\beta}) + O(\epsilon^{\gamma+1}) = O(\epsilon^{4\beta}),$$

as $\epsilon \longrightarrow 0$. We then have by Proposition 5.3.2 that

$$\|u_\epsilon - \overline{u}_\epsilon\|_{L^2(R_\epsilon)} = \|u_\epsilon - \overline{u}_\epsilon\|_{L^2(J^\beta_\epsilon)} \le \|u_\epsilon\|_{L^2(J^\beta_\epsilon)} + \|\overline{u}_\epsilon\|_{L^2(J^\beta_\epsilon)} = O(\epsilon^{2\beta}),$$

as $\epsilon \longrightarrow 0$. This concludes the proof of (i) in the H-Condition.

In order to prove (ii) from Definition 5.2.7, we first need to compute $\|\nabla\overline{u}_\epsilon\|_{L^2(J^\beta_\epsilon)}$ and $\|D^2\overline{u}_\epsilon\|_{L^2(J^\beta_\epsilon)}$. We have

$$\begin{aligned}
\frac{\partial\overline{u}_\epsilon}{\partial x}(x,y) &= \left[\left(\frac{\partial u_\epsilon}{\partial x}\chi^\gamma_\epsilon\right)(f(x),y) + (u_\epsilon(\chi^\gamma_\epsilon)')(f(x),y)\right] f'(x), \\
\frac{\partial\overline{u}_\epsilon}{\partial y}(x,y) &= \left(\frac{\partial u_\epsilon}{\partial y}\chi^\gamma_\epsilon\right)(f(x),y).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\nabla\overline{u}_\epsilon\|_{L^2(J^\beta_\epsilon)} &\le \|f'\|_{L^\infty}\big(\|\nabla u_\epsilon(f(\cdot),\cdot)\|_{L^2(J^\beta_\epsilon)} + \|(u_\epsilon(\chi^\gamma_\epsilon)')(f(\cdot),\cdot)\|_{L^2(J^\beta_\epsilon)}\big) \\
&\le \|f'\|_{L^\infty}\|f'\|^{-1/2}_{L^\infty}\big(\|\nabla u_\epsilon\|_{L^2(K^\gamma_\epsilon\cup J^\beta_\epsilon)} + c_1\|\epsilon^{-\gamma}u_\epsilon\|_{L^2(K^\gamma_\epsilon)}\big) \\
&\le (1+o(1))\big(\|\nabla u_\epsilon\|_{L^2(K^\gamma_\epsilon)} + \|\nabla u_\epsilon\|_{L^2(J^\beta_\epsilon)} + c_1\epsilon^{-\gamma}\|u_\epsilon\|_{L^2(K^\gamma_\epsilon)}\big),
\end{aligned} \tag{5.3.19}$$

where we have used the definition of χ_ϵ^γ and the change of variables $(f(x), y) \mapsto (x, y)$. By Proposition 5.3.2 we know that $\|\nabla u_\epsilon\|_{L^2(J_\epsilon^\beta)} = O(\epsilon^\beta)$ as $\epsilon \to 0$. Moreover, by (5.3.8), (5.3.9) with θ replaced by γ, we deduce that

$$\|u_\epsilon\|_{L^2(K_\epsilon^\gamma)} = O(\epsilon^{\frac{\gamma+1}{2}}), \qquad \|\nabla u_\epsilon\|_{L^2(K_\epsilon^\gamma)} = O(\epsilon^{\gamma_p}),$$

for any $p < \infty$, where we have set

$$\gamma_p = \left(\frac{1}{2} - \frac{1}{p}\right)(\gamma + 1).$$

Finally, we deduce by (5.3.19) that

$$\|\nabla \overline{u}_\epsilon\|_{L^2(J_\epsilon^\beta)} \leq (1 + o(1))(O(\epsilon^{\gamma_p}) + O(\epsilon^\beta) + \epsilon^{-\gamma} O(\epsilon^{\gamma_p})) = O(\epsilon^\beta), \tag{5.3.20}$$

because $\gamma_p - \gamma > \beta$, for sufficiently large p (note that $\beta < (1-\gamma)/2$ for $\gamma < 1/3$).

We now estimate the L^2 norm of $D^2\overline{u}_\epsilon$. In order to simplify our notation we write $F(x, y) = (f(x), y)$, $\chi_\epsilon^\gamma = \chi$, $\bar{u}_\epsilon = \bar{u}$, $u_\epsilon = u$ and we use the subindex notation for the partial derivatives, that is, $u_x = \frac{\partial u}{\partial x}$ and so on. First, note that

$$\begin{aligned}
\bar{u}_{xx} &= \Big[\Big(u_{xx}\chi + 2u_x\chi' + u\chi''\Big) \circ F\Big] \cdot |f'|^2 + \Big[\Big(u_x\chi + u\chi'\Big) \circ F\Big] \cdot f'', \\
\bar{u}_{xy} &= \Big[\Big(u_{xy}\chi + u_y\chi'\Big) \circ F\Big] \cdot f', \\
\bar{u}_{yy} &= \Big(u_{yy}\chi\Big) \circ F,
\end{aligned} \tag{5.3.21}$$

and we may write

$$\bar{u}_{xx} = [u_{xx}\chi \circ F] \cdot |f'|^2 + R_1, \quad \bar{u}_{xy} = [u_{xy}\chi \circ F] \cdot f' + R_2, \quad \bar{u}_{yy} = u_{yy}\chi \circ F.$$

where

$$\begin{aligned}
R_1 &= \Big[\Big(2u_x\chi' + u\chi''\Big) \circ F\Big] \cdot |f'|^2 + \Big[\Big(u_x\chi + u\chi'\Big) \circ F\Big] \cdot f'', \\
R_2 &= u_y\chi' \circ F \cdot f'.
\end{aligned}$$

We now show that $\|R_1\|_{L^2(J_\epsilon^\beta)} = o(1)$, $\|R_2\|_{L^2(J_\epsilon^\beta)} = o(1)$ as $\epsilon \to 0$. For this, we will prove that each single term in R_1 and R_2 is $o(1)$ as $\epsilon \to 0$. Recall that $f'(x) = 1 + o(1)$ and $f''(x) = o(1)$, $\chi' = O(\epsilon^{-\gamma})$ and $\chi'' = O(\epsilon^{-2\gamma})$ for $x \in (0, \epsilon^\beta)$. By a change of variables, by the Sobolev Embedding Theorem and the definition of χ it is easy to deduce that

$$\begin{aligned}
&\|(u_x\chi') \circ F\|_{L^2(J_\epsilon^\beta)} \leq (1 + o(1))\|u_x\chi'\|_{L^2(K_\epsilon^\gamma)} \leq CR\epsilon^{\gamma_p - \gamma} = O(\epsilon^\beta), \\
&\|(u\chi'') \circ F\|_{L^2(J_\epsilon^\beta)} \leq c_2(1 + o(1))\|u\epsilon^{-2\gamma}\|_{L^2(K_\epsilon^\gamma)} \leq CR\epsilon^{\frac{1-3\gamma}{2}}, \\
&\|(u_y\chi') \circ F\|_{L^2(J_\epsilon^\beta)} \leq c_1(1 + o(1))\|\epsilon^{-\gamma}u_y\|_{L^2(K_\epsilon^\gamma)} \leq CR\epsilon^{\gamma_p - \gamma} = O(\epsilon^\beta).
\end{aligned}$$

By (5.3.20) we also have

$$\|(u_x\chi + u\chi')\circ F\|_{L^2(J_\epsilon^\beta)} \le (1+o(1))\|\nabla\overline{u}_\epsilon\|_{L^2(J_\epsilon^\beta)} = O(\epsilon^\beta). \tag{5.3.22}$$

Hence the L^2 norms of R_1, R_2 vanish as $\epsilon \longrightarrow 0$. In particular,

$$\|D^2\overline{u}_\epsilon\|_{L^2(J_\epsilon^\beta)} = (1+o(1))\|D^2u_\epsilon\|_{L^2(K_\epsilon^\gamma\cup J_\epsilon^\beta)} + O(\epsilon^{\frac{1-3\gamma}{2}}) + O(\epsilon^\beta),$$

as $\epsilon \longrightarrow 0$. In a similar way we can also prove that

$$\|\Delta\overline{u}_\epsilon\|_{L^2(J_\epsilon^\beta)} = (1+o(1))\|\Delta u_\epsilon\|_{L^2(K_\epsilon^\gamma\cup J_\epsilon^\beta)} + O(\epsilon^{\frac{1-3\gamma}{2}}) + O(\epsilon^\beta),$$

as $\epsilon \longrightarrow 0$. Hence,

$$\begin{aligned}(1-\sigma)\|D^2\overline{u}_\epsilon\|^2_{L^2(J_\epsilon^\beta)} + \sigma\|\Delta\overline{u}_\epsilon\|^2_{L^2(J_\epsilon^\beta)} + \tau\|\nabla\overline{u}_\epsilon\|^2_{L^2(J_\epsilon^\beta)} \\ = (1-\sigma)\|D^2u_\epsilon\|^2_{L^2(K_\epsilon^\gamma\cup J_\epsilon^\beta)} + \sigma\|\Delta u_\epsilon\|^2_{L^2(K_\epsilon^\gamma\cup J_\epsilon^\beta)} + o(1).\end{aligned} \tag{5.3.23}$$

By adding to both handsides of (5.3.23) $(1-\sigma)\|D^2\overline{u}_\epsilon\|^2_{L^2(R_\epsilon\setminus J_\epsilon^\beta)}$, $\sigma\|\Delta\overline{u}_\epsilon\|^2_{L^2(R_\epsilon\setminus J_\epsilon^\beta)}$ and $\tau\|\nabla\overline{u}_\epsilon\|^2_{L^2(R_\epsilon\setminus J_\epsilon^\beta)}$, and keeping in account that $\overline{u}_\epsilon \equiv u_\epsilon$ on $R_\epsilon \setminus J_\epsilon^\beta$ we deduce that

$$\begin{aligned}(1-\sigma)\|D^2\overline{u}_\epsilon\|^2_{L^2(R_\epsilon)} + \sigma\|\Delta\overline{u}_\epsilon\|^2_{L^2(R_\epsilon)} + \tau\|\nabla\overline{u}_\epsilon\|^2_{L^2(R_\epsilon)} \\ = (1-\sigma)\|D^2u_\epsilon\|^2_{L^2(K_\epsilon^\gamma\cup R_\epsilon)} + \sigma\|\Delta u_\epsilon\|^2_{L^2(K_\epsilon^\gamma\cup R_\epsilon)} + \tau\|\nabla u_\epsilon\|^2_{L^2(R_\epsilon\setminus J_\epsilon^\beta)} + o(1) \\ \le (1-\sigma)\|D^2u_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \sigma\|\Delta u_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \tau\|\nabla u_\epsilon\|^2_{L^2(\Omega_\epsilon)} + o(1),\end{aligned} \tag{5.3.24}$$

as $\epsilon \longrightarrow 0$, concluding the proof of (ii) in the H-Condition. Note that in (5.3.24), we have used the monotonicity of the quadratic form with respect to inclusion of sets. Such property is straightforward for $\sigma \in [0, 1)$. In the case $\sigma \in (-1, 0)$ it follows by observing that

$$\begin{aligned}(1-\sigma)\left[u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2\right] + \sigma\left[u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2\right] \\ \ge u_{xx}^2 + u_{yy}^2 + \sigma(u_{xx}^2 + u_{yy}^2) = (1+\sigma)(u_{xx}^2 + u_{yy}^2) > 0,\end{aligned}$$

for all $u \in H^2(\Omega_\epsilon)$. □

5.4 Asymptotic analysis on the thin domain

The purpose of this section is to study the convergence of the eigenvalue problem (5.1.8) as $\epsilon \longrightarrow 0$. Since the thin domain R_ϵ is shrinking to the segment $(0, 1)$

as $\epsilon \to 0$, we plan to identify the limiting problem in $(0,1)$ and to prove that the resolvent operator of problem (5.1.8) converges as $\epsilon \to 0$ to the resolvent operator of the limiting problem in a suitable sense which guarantees the spectral convergence.

More precisely, we shall prove that the the limiting eigenvalue problem in $[0,1]$ is

$$\begin{cases} \frac{1-\sigma^2}{g}(gh'')'' - \frac{\tau}{g}(gh')' + h = \theta h, & \text{in } (0,1), \\ h(0) = h(1) = 0, \\ h'(0) = h'(1) = 0. \end{cases} \tag{5.4.1}$$

Note that the weak formulation of (5.4.1) is

$$(1-\sigma^2)\int_0^1 h''\psi''g\,dx + \tau\int_0^1 h'\psi'g\,dx + \int_0^1 h\psi g\,dx = \theta\int_0^1 h\psi g\,dx,$$

for all $\psi \in H_0^2(0,1)$, where h is to be found in the Sobolev space $H_0^2(0,1)$. In the sequel, we shall denote by $L_g^2(0,1)$ the Hilbert space $L^2((0,1);g(x)dx)$.

5.4.1 Finding the limiting problem

In order to use thin domain techniques in the spirit of [74], we need to fix a reference domain R_1 and pull-back the eigenvalue problem defined on R_ϵ onto R_1 by means of a suitable diffeomorphism.

Let R_1 be the rescaled domain obtained by setting $\epsilon = 1$ in the definition of R_ϵ (see (5.1.1)). For any fixed $\epsilon > 0$, let Φ_ϵ be the map from R_1 to R_ϵ defined by $\Phi_\epsilon(x',y') = (x',\epsilon y') = (x,y)$ for all $(x',y') \in R_1$. We consider the composition operator T_ϵ from $L^2(R_\epsilon;\epsilon^{-1}dxdy)$ to $L^2(R_1)$ defined by

$$T_\epsilon u(x',y') = u \circ \Phi_\epsilon(x',y') = u(x',\epsilon y'),$$

for all $u \in L^2(R_\epsilon)$, $(x',y') \in R_1$. We also endow the spaces $H^2(R_1)$ and $H^2(R_\epsilon)$ with the norms defined by

$$\begin{aligned} \|\varphi\|^2_{H^2_{\epsilon,\sigma,\tau}(R_1)} = \int_{R_1}\Bigg((1-\sigma)\left[\left|\frac{\partial^2\varphi}{\partial x^2}\right|^2 + \frac{2}{\epsilon^2}\left|\frac{\partial^2\varphi}{\partial x\partial y}\right|^2 + \frac{1}{\epsilon^4}\left|\frac{\partial^2\varphi}{\partial y^2}\right|^2\right] \\ + \sigma\left|\frac{\partial^2\varphi}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2\varphi}{\partial y^2}\right|^2 + \tau\left[\left|\frac{\partial\varphi}{\partial x}\right|^2 + \frac{1}{\epsilon}\left|\frac{\partial\varphi}{\partial y}\right|^2\right] + |\varphi|^2\Bigg)dxdy, \end{aligned} \tag{5.4.2}$$

$$\|\varphi\|^2_{H^2_{\sigma,\tau}(R_\epsilon)} = \int_{R_\epsilon} \Bigg((1-\sigma)\left[\left|\frac{\partial^2\varphi}{\partial x^2}\right|^2 + 2\left|\frac{\partial^2\varphi}{\partial x\partial y}\right|^2 + \left|\frac{\partial^2\varphi}{\partial y^2}\right|^2 \right] + \sigma\left|\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right|^2 + \tau\left[\left|\frac{\partial\varphi}{\partial x}\right|^2 + \left|\frac{\partial\varphi}{\partial y}\right|^2 \right] + |\varphi|^2 \Bigg) dxdy\,. \quad (5.4.3)$$

It is not difficult to see that if $\varphi \in H^2(R_\epsilon)$ then

$$\|T_\epsilon\varphi\|^2_{H^2_{\epsilon,\sigma,\tau}(R_1)} = \epsilon^{-1}\|\varphi\|^2_{H^2_{\sigma,\tau}(R_\epsilon)}.$$

We consider the following Poisson problem with datum $f_\epsilon \in L^2(R_\epsilon)$:

$$\begin{cases} \Delta^2 v_\epsilon - \tau\Delta v_\epsilon + v_\epsilon = f_\epsilon, & \text{in } R_\epsilon, \\ (1-\sigma)\frac{\partial^2 v_\epsilon}{\partial n_\epsilon^2} + \sigma\Delta v_\epsilon = 0, & \text{on } \Gamma_\epsilon, \\ \tau\frac{\partial v_\epsilon}{\partial n_\epsilon} - (1-\sigma)\,\mathrm{div}_{\partial\Omega_\epsilon}(D^2 v_\epsilon \cdot n_\epsilon)_{\partial\Omega_\epsilon} - \frac{\partial(\Delta v_\epsilon)}{\partial n_\epsilon} = 0, & \text{on } \Gamma_\epsilon, \\ v = 0 = \frac{\partial v_\epsilon}{\partial n_\epsilon}, & \text{on } L_\epsilon. \end{cases} \quad (5.4.4)$$

Note that the energy space associated with Problem (5.4.4) is exactly $H^2_{L_\epsilon}(R_\epsilon)$. By setting $\tilde{v}_\epsilon = v_\epsilon(x', \epsilon y')$, $\tilde{f}_\epsilon = f(x', \epsilon y')$ and pulling-back problem (5.4.4) to R_1 by means of Φ_ϵ, we get the following equivalent problem in R_1 in the unknown $\tilde{v}_\epsilon$ (we use again the variables (x, y) instead of (x', y') to simplify the notation):

$$\begin{cases} \frac{\partial^4\tilde{v}_\epsilon}{\partial x^4} + \frac{2}{\epsilon^2}\frac{\partial^4\tilde{v}_\epsilon}{\partial x^2\partial y^2} + \frac{1}{\epsilon^4}\frac{\partial^4\tilde{v}_\epsilon}{\partial y^4} - \tau\left(\frac{\partial^2\tilde{v}_\epsilon}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2\tilde{v}_\epsilon}{\partial y^2}\right) + \tilde{v}_\epsilon = \tilde{f}_\epsilon, & \text{in } R_1, \\ (1-\sigma)\left(\frac{\partial^2\tilde{v}_\epsilon}{\partial x^2}\tilde{n}_x^2 + \frac{2}{\epsilon}\frac{\partial^2\tilde{v}_\epsilon}{\partial x\partial y}\tilde{n}_x\tilde{n}_y + \frac{1}{\epsilon^2}\frac{\partial^2\tilde{v}_\epsilon}{\partial y^2}\tilde{n}_y^2\right) + \sigma\left(\frac{\partial^2\tilde{v}_\epsilon}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2\tilde{v}_\epsilon}{\partial y^2}\right) = 0, & \text{on } \Gamma_1, \\ \tau\left(\frac{\partial\tilde{v}_\epsilon}{\partial x}\tilde{n}_x + \frac{1}{\epsilon}\frac{\partial\tilde{v}_\epsilon}{\partial y}\tilde{n}_y\right) - (1-\sigma)\,\mathrm{div}_{\Gamma_{1,\epsilon}}(D^2_\epsilon\tilde{v}_\epsilon \cdot \tilde{n})_{\Gamma_{1,\epsilon}} - \nabla_\epsilon(\Delta_\epsilon\tilde{v}_\epsilon)\cdot\tilde{n} = 0, & \text{on } \Gamma_1, \\ \tilde{v}_\epsilon = 0 = \frac{\partial\tilde{v}_\epsilon}{\partial x}n_x + \frac{1}{\epsilon}\frac{\partial\tilde{v}_\epsilon}{\partial y}\tilde{n}_y, & \text{on } L_1. \end{cases} \quad (5.4.5)$$

Here $\tilde{n} = (\tilde{n}_x, \tilde{n}_y) = (n_x, \epsilon^{-1}n_y)$ and the operators $\Delta_\epsilon, \nabla_\epsilon$ are the standard differential operators associated with $(\partial_x, \epsilon^{-1}\partial_y)$. Moreover,

$$\mathrm{div}_{\Gamma_{1,\epsilon}} F = \frac{\partial F_1}{\partial x} + \frac{1}{\epsilon}\frac{\partial F_2}{\partial y} - \tilde{n}_\epsilon\nabla_\epsilon F\,\tilde{n}_\epsilon,$$

and $(F)_{\Gamma_{1,\epsilon}} = F - (F, \tilde{n})\,\tilde{n}$ for any vector field $F = (F_1, F_2)$.

Assume now that the data f_ϵ, $\epsilon > 0$ are such that $(\tilde{f}_\epsilon)_{\epsilon>0}$ is an equibounded family in $L^2(R_1)$, i.e.,

$$\int_{R_1} |\tilde{f}_\epsilon|^2\, dxdy' \le c, \quad \text{or equivalently} \quad \int_{R_\epsilon} |f_\epsilon|^2 dxdy \le c\epsilon\,, \quad (5.4.6)$$

for all $\epsilon > 0$, where c is a positive constant not depending on ϵ.

We plan to pass to the limit in (5.4.5) as $\epsilon \to 0$ by arguing as follows. If $\tilde{v}_\epsilon \in H^2_{L_1}(R_1)$ is the solution to problem (5.4.5), then we have the following integral equality

$$(1-\sigma)\int_{R_1} \frac{\partial^2 \tilde{v}_\epsilon}{\partial x^2}\frac{\partial^2 \varphi}{\partial x^2} + \frac{2}{\epsilon^2}\frac{\partial^2 \tilde{v}_\epsilon}{\partial x \partial y}\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{1}{\epsilon^4}\frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2}\frac{\partial^2 \varphi}{\partial y^2} dx \\ + \sigma \int_{R_1} \Big(\frac{\partial^2 \tilde{v}_\epsilon}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2}\Big)\Big(\frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2 \varphi}{\partial y^2}\Big) dx \\ + \tau \int_{R_1} \frac{\partial \tilde{v}_\epsilon}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{1}{\epsilon^2}\frac{\partial \tilde{v}_\epsilon}{\partial y}\frac{\partial \varphi}{\partial y} dx + \int_{R_1} \tilde{v}_\epsilon \varphi \, dx = \int_{R_1} \tilde{f}_\epsilon \varphi \, dx, \quad (5.4.7)$$

for all $\varphi \in H^2_{L_1}(R_1)$. By choosing $\varphi = \tilde{v}_\epsilon$ we deduce the following apriori estimate:

$$(1-\sigma)\int_{R_1} \left|\frac{\partial^2 \tilde{v}_\epsilon}{\partial x^2}\right|^2 + \frac{2}{\epsilon^2}\left|\frac{\partial^2 \tilde{v}_\epsilon}{\partial x \partial y}\right|^2 + \frac{1}{\epsilon^4}\left|\frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2}\right|^2 dx + \sigma \int_{R_1} \left|\frac{\partial^2 \tilde{v}_\epsilon}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2}\right|^2 dx \\ + \tau \int_{R_1} \left|\frac{\partial \tilde{v}_\epsilon}{\partial x}\right|^2 + \frac{1}{\epsilon^2}\left|\frac{\partial \tilde{v}_\epsilon}{\partial y}\right|^2 dx + \int_{R_1} |\tilde{v}_\epsilon|^2 dx \le \frac{1}{2}\int_{R_1} |\tilde{f}_\epsilon|^2 \, dx + \frac{1}{2}\int_{R_1} |\tilde{v}_\epsilon|^2 \, dx, \quad (5.4.8)$$

for all $\epsilon > 0$. This implies that $\|\tilde{v}_\epsilon\|_{H^2_{\epsilon,\sigma,\tau}(R_1)} \le C$ for all $\epsilon > 0$, in particular $\|\tilde{v}_\epsilon\|_{H^2(R_1)} \le C(\sigma, \tau)$ for all $\epsilon > 0$; hence, there exists $v \in H^2(R_1)$ such that, up to a subsequence $\tilde{v}_\epsilon \to v$, weakly in $H^2(R_1)$, strongly in $H^1(R_1)$. Moreover from (5.4.8) we deduce that

$$\left\|\frac{\partial^2 \tilde{v}_\epsilon}{\partial x \partial y}\right\|_{L^2(R_1)} \le C\epsilon, \qquad \left\|\frac{\partial \tilde{v}_\epsilon}{\partial y}\right\|_{L^2(R_1)} \le C\epsilon, \quad (5.4.9)$$

$$\left\|\frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2}\right\|_{L^2(R_1)} \le C\epsilon^2, \quad (5.4.10)$$

for all $\epsilon > 0$, hence there exists $u \in L^2(R_1)$ such that, up to a subsequence

$$\frac{1}{\epsilon^2}\frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2} \rightharpoonup u, \text{ weakly in } L^2(R_1), \quad (5.4.11)$$

as $\epsilon \to 0$. By (5.4.9) we deduce that the limit function v is constant in y. Indeed, if we choose any function $\phi \in C^\infty_c(R_1)$, then

$$\int_{R_1} v \frac{\partial \phi}{\partial y} = \lim_{\epsilon \to 0} \int_{R_1} \tilde{v}_\epsilon \frac{\partial \phi}{\partial y} = -\lim_{\epsilon \to 0} \int_{R_1} \frac{\partial \tilde{v}_\epsilon}{\partial y}\phi = 0,$$

hence $\frac{\partial v}{\partial y} = 0$ and then $v(x, y) \equiv v(x)$ for almost all $(x, y) \in R_1$. This suggests to choose test functions ψ depending only on x in the weak formulation (5.4.7). Possibly passing to a subsequence, there exists $f \in L^2(R_1)$ such that

$$\tilde{f}_\epsilon \rightharpoonup f, \qquad \text{in } L^2(R_1), \text{ as } \epsilon \to 0.$$

Let $\psi \in H_0^2(0,1)$. Then $\psi \in H^2(R_1)$ (here it is understood that the function is extended to the whole of R_1 by setting $\psi(x,y) = \psi(x)$ for all $(x,y) \in R_1$) and clearly $\psi \equiv 0$ on L_1. Use ψ as a test function in (5.4.7), pass to the limit as $\epsilon \to 0$ and consider (5.4.11) to get

$$\int_0^1 \Big(\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 \psi}{\partial x^2} + \sigma \mathcal{M}(u)\frac{\partial^2 \psi}{\partial x^2} + \tau \frac{\partial v}{\partial x}\frac{\partial \psi}{\partial x} + v\psi\Big) g(x)\,dx = \int_0^1 \mathcal{M}(f)\psi g(x)\,dx, \tag{5.4.12}$$

for all $\psi \in H_0^2(0,1)$. Here, the averaging operator $\mathcal{M}$ is defined from $L^2(R_1)$ to $L_g^2(0,1)$ by

$$\mathcal{M}h(x) = \frac{1}{g(x)}\int_0^{g(x)} h(x,y)\,dy\,,$$

for all $h \in L^2(R_1)$ and for almost all $x \in (0,1)$.

From (5.4.12) we deduce that

$$\frac{1}{g}(v''g)'' + \frac{\sigma}{g}(\mathcal{M}(u)g)'' - \frac{\tau}{g}(v'g)' + v = \mathcal{M}(f), \qquad \text{in } (0,1),$$

where the equality is understood in the sense of distributions.
Coming back to (5.4.7) we may also choose test functions $\varphi(x,y) = \epsilon^2\zeta(x,y)$, where $\zeta \in H_{L_1}^2(R_1)$. Using (5.4.9), (5.4.10) and letting $\epsilon \to 0$ we deduce

$$(1-\sigma)\int_{R_1} u\frac{\partial^2 \zeta}{\partial y^2} + \sigma\int_{R_1}\Big(\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 \zeta}{\partial y^2} + u\frac{\partial^2 \zeta}{\partial y^2}\Big) = 0,$$

which can be rewritten as

$$\int_{R_1}\Big(u + \sigma\frac{\partial^2 v}{\partial x^2}\Big)\frac{\partial^2 \zeta}{\partial y^2} = 0, \tag{5.4.13}$$

for all $\zeta \in H_{L_1}^2(R_1)$. In particular this holds for all $\zeta \in C_c^\infty(R_1)$, hence there exists the second order derivative

$$\frac{\partial^2}{\partial y^2}\Big(u + \sigma\frac{\partial^2 v}{\partial x^2}\Big) = 0. \tag{5.4.14}$$

Hence, $u(x,y) + \sigma\frac{\partial^2 v}{\partial x^2} = \psi_1(x) + \psi_2(x)y$ for almost all $(x,y) \in R_1$ and for some functions $\psi_1, \psi_2 \in L^2(R_1)$, and then (5.4.13) can be written as

$$\int_{R_1}(\psi_1(x) + y\psi_2(x))\frac{\partial^2 \zeta}{\partial y^2} = 0, \tag{5.4.15}$$

Integrating twice by parts in y in equation (5.4.15) we deduce that

$$-\int_{\partial R_1}\psi_2(x)\zeta n_y dS + \int_{\partial R_1}(\psi_1(x) + y\psi_2(x))\frac{\partial \zeta}{\partial y}n_y dS = 0, \tag{5.4.16}$$

for all $\zeta \in H^2_{L_1}(R_1)$. We are going to choose now particular functions ζ in (5.4.16). Consider first $b = \frac{1}{2}\min_{x\in[0,1]} g(x) > 0$ so that the rectangle $(0,1)\times(0,b) \subset R_1$ and consider a function $\eta = \eta(y)$ with $\eta \in C^\infty(0,b)$ such that $\eta(y) = 1 + \alpha y$ for $0 < y < b/4$, where $\alpha \in \mathbb{R}$ is a parameter, and $\eta(y) \equiv 0$ for $y \in (\frac{3}{4}b, b)$. If we define $\zeta(x,y) = \theta(x)\eta(y)$ for $(x,y) \in (0,1)\times(0,b)$ where $\theta \in C^\infty_c(0,1)$ and we extend this function ζ by 0 to all of R_1, then we can use ζ in (5.4.16) in order to obtain

$$\alpha \int_0^1 \psi_1(x)\theta(x)dx - \int_0^1 \psi_2(x)\theta(x)dx = 0,$$

for all $\alpha \in \mathbb{R}$ and all $\theta \in C^\infty_0(0,1)$. But this easily implies that $\psi_1 \equiv \psi_2 \equiv 0$. Thus, we obtain

$$u(x,y) = u(x) = -\sigma \frac{\partial^2 v(x)}{\partial x^2},$$

for almost all $(x,y) \in R_1$, i.e., $\frac{1}{\epsilon^2}\frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2} \rightharpoonup -\sigma\frac{\partial^2 v(x)}{\partial x^2}$ in $L^2(R_1)$. Hence v solves the following limit problem

$$\begin{cases} \frac{1-\sigma^2}{g}(gv'')'' - \frac{\tau}{g}(gv')' + v = \mathcal{M}(f), & \text{in } (0,1), \\ v(0) = v(1) = 0, \\ v'(0) = v'(1) = 0, \end{cases} \tag{5.4.17}$$

and then by regularity theory we deduce that $v \in H^4(0,1)$.

5.4.2 Spectral convergence

We aim at proving the spectral convergence of the eigenvalues and eigenfunctions of problem (5.1.8) to the corresponding eigenvalues and eigenfunctions of the one dimensional problem (5.1.9). To do so we shall prove the compact convergence of the associated resolvent operators combined with the computations carried out in the previous section. Note that the domain R_ϵ varies with ϵ, hence the corresponding Hilbert spaces vary as well. To bypass this problem we will use the notion of $\mathcal{E}$-convergence of the resolvent operators in L^2, introduced in §1.3. In particular, we apply Theorem 1.3.5 to problem (5.1.8). To do so, we consider the following Hilbert spaces

$$\mathscr{H}_\epsilon = L^2(R_\epsilon; \epsilon^{-1}dxdy), \text{ and } \mathscr{H}_0 = L^2_g(0,1),$$

and we denote by $\mathcal{E}_\epsilon$ the extension operator from $L^2_g(0,1)$ to $L^2(R_\epsilon; \epsilon^{-1}dxdy)$, defined by

$$(\mathcal{E}_\epsilon v)(x,y) = v(x), \tag{5.4.18}$$

for all $v \in L^2_g(0,1)$, for almost all $(x,y) \in R_\epsilon$. Clearly $\|E_\epsilon u_0\|_{(R_\epsilon;\epsilon^{-1}dxdy)} = \|u_0\|_{L^2_g(0,1)}$, hence $\mathcal{E}_\epsilon$ trivially satisfies property (1.3.1).

We consider the operators $A_\epsilon = (\Delta^2 - \tau\Delta + I)_{L_\epsilon}$, $A_0 = (\Delta^2 - \tau\Delta + I)_D$ on $\mathscr{H}_\epsilon$ and $\mathscr{H}_0$ respectively, associated with the eigenvalue problems (5.1.8) and (5.1.9), respectively. Namely, $(\Delta^2 - \tau\Delta + I)_{L_\epsilon}$ is the operator $\Delta^2 - \tau\Delta + I$ on R_ϵ subject to Dirichlet boundary conditions on L_ϵ and Neumann boundary conditions on $\partial R_\epsilon \setminus L_\epsilon$ as described in (5.1.8). Similarly, $(\Delta^2 - \tau\Delta + I)_D$ is the operator $\Delta^2 - \tau\Delta + I$ on $(0,1)$ subject to Dirichlet boundary conditions as described in (5.1.9).

Then we can prove the following

Theorem 5.4.1. *The operators* $(\Delta^2 - \tau\Delta + I)_{L_\epsilon}$ *spectrally converge to* $(\Delta^2 - \tau\Delta + I)_D$ *as* $\epsilon \to 0$, *in the sense of Theorem 1.3.5.*

Proof. In view of Theorem 1.3.5, it is sufficient to prove the following two facts:

(1) if $f_\epsilon \in L^2(R_\epsilon;\epsilon^{-1}dxdy)$ is such that $\epsilon^{-1/2}\|f_\epsilon\|_{L^2(R_\epsilon)} = 1$ for any $\epsilon > 0$, and v_ϵ is the corresponding solutions of Problem (5.4.4), then there exists a subsequence $\epsilon_k \to 0$ as $k \to \infty$ and $\bar{v} \in L^2_g(0,1)$ such that v_{ϵ_k} $\mathcal{E}$-converge to $\bar{v}$ as $k \to \infty$.

(2) if $f_\epsilon \in L^2(R_\epsilon;\epsilon^{-1}dxdy)$ and $f_\epsilon \xrightarrow{E} f$ as $\epsilon \to 0$, then the corresponding solutions v_ϵ of Problem (5.4.4) $\mathcal{E}$-converge to the solution of Problem (5.4.17) with datum f.

Note that (1) follows immediately from the computations in Section 5.4.1. Indeed, if $f_\epsilon \in L^2(R_\epsilon;\epsilon^{-1}dxdy)$ is as in (1), up to a subsequence, $\tilde{f}_\epsilon \rightharpoonup f$ in $L^2(R_1)$, which implies that $\tilde{v}_\epsilon \rightharpoonup v_0 \in H^2_0(0,1)$ in $H^2(R_1)$, where v_0 is the solution of Problem (5.4.17). This implies that $\|v_\epsilon - \mathcal{E}v_0\|_{L^2(R_\epsilon;\epsilon^{-1}dxdy)} \to 0$, hence (1) is proved.
In order to show (2) we take a sequence of functions $f_\epsilon \in L^2(R_\epsilon;\epsilon^{-1}dxdy)$ and $f \in L^2_g(0,1)$ such that $\epsilon^{-1/2}\|f_\epsilon - \mathcal{E}_\epsilon f\|_{L^2(R_\epsilon)} \to 0$ as $\epsilon \to 0$. After a change of variable, this is equivalent to $\|\tilde{f}_\epsilon - \mathcal{E}f\|_{L^2(R_1)} \to 0$ as $\epsilon \to 0$. Arguing as in Section 5.4.1, one show that the $\tilde{v}_\epsilon \rightharpoonup v \in L^2_g(0,1)$ in $H^2(R_1)$ and that v solves problem (5.4.17). Hence $\|\tilde{v}_\epsilon - \mathcal{E}v\|_{L^2(R_1)} \to 0$ as $\epsilon \to 0$, or equivalently, $\|v_\epsilon - \mathcal{E}_\epsilon v\|_{L^2(R_\epsilon;\epsilon^{-1}dxdy)} \to 0$ as $\epsilon \to 0$, proving (2).

□

5.5 A theorem on the rate of convergence

The aim of this section is to prove an estimate to control the $\mathcal{E}$-convergence of the solutions of Problem (5.1.8) to the solutions of Problem (5.1.9). We prove this estimate only in the case $\sigma = 0$. For the sake of simplicity, since $\sigma = 0$, the norms

$\|\cdot\|_{H^2_{\epsilon,\sigma,\tau}(R_1)}$, $\|\cdot\|_{H^2_{\sigma,\tau}(R_\epsilon)}$ will be denoted by $\|\cdot\|_{H^2_{\epsilon,\tau}(R_1)}$, $\|\cdot\|_{H^2_{\tau}(R_\epsilon)}$. We also write $\mathcal{E}$ to denote the extension operator defined in (5.4.18) in the case $\epsilon = 1$.

Theorem 5.5.1. *Let $\tau \geq 0$ and let $(f_\epsilon)_{\epsilon>0}$ satisfy the assumption* (5.4.6). *Let $\tilde{v}_\epsilon$ be the solution of* (5.4.5) *and h_ϵ be the solution of*

$$\begin{cases} \frac{1}{g}(gh'')'' - \frac{\tau}{g}(gh')' + h = \mathcal{M}_\epsilon f_\epsilon, & \text{in } (0,1) \\ h(0) = h(1) = 0, \\ h'(0) = h'(1) = 0, \end{cases} \tag{5.5.1}$$

Then

$$\|\tilde{v}_\epsilon - \mathcal{E}h_\epsilon\|_{H^2_\epsilon(R_1)} \leq c\epsilon\|\tilde{f}_\epsilon\|_{L^2(R_1)} \tag{5.5.2}$$

Proof. We give the proof only in the case $\tau > 0$, since the case $\tau = 0$ is easier. Note that by the equivalence between problems (5.4.4) and (5.4.5), the estimate (5.5.2) is equivalent to the following

$$\|v_\epsilon - \mathcal{E}_\epsilon h_\epsilon\|_{H^2(R_\epsilon)} \leq c\epsilon\|f_\epsilon\|_{L^2(R_\epsilon)}. \tag{5.5.3}$$

By the variational formulation of problem (5.4.4) we know that

$$\lambda_\epsilon = \min\Big\{\frac{1}{2}\Big(\int_{R_\epsilon} |D^2\varphi|^2 + \tau|\nabla\varphi|^2 + |\varphi|^2\,dx\Big) - \int_{R_\epsilon} f_\epsilon\varphi\,dx \\ : \varphi \in H^2(R_\epsilon),\ \varphi = 0 = \frac{\partial\varphi}{\partial n} \text{ on } L_\epsilon\Big\} \tag{5.5.4}$$

is unique and attained at v_ϵ. Similarly the minimum

$$\kappa_\epsilon = \min_{\varphi\in H^2_0(0,1)} \Big\{\frac{1}{2}\Big(\int_0^1 g|\varphi''| + \tau g|\varphi'|^2 + g|\varphi|^2\,dx\Big) - \int_0^1 g\mathcal{M}_\epsilon f_\epsilon\varphi\,dx\Big\} \tag{5.5.5}$$

is unique and attained at h_ϵ. Let us write $\mathfrak{h}_\epsilon = \mathcal{E}_\epsilon h_\epsilon$. Note that $\mathfrak{h}_\epsilon \in H^2(R_\epsilon)$ because $\|\mathfrak{h}_\epsilon\|^2_{H^2(R_\epsilon)} = \epsilon\|\mathfrak{h}_\epsilon\|^2_{H^2_g(0,1)} < \infty$. Moreover, by definition of $\mathcal{E}_\epsilon$ the function $\mathfrak{h}_\epsilon$ is such that

$$\mathfrak{h}_\epsilon = 0 = \frac{\partial\mathfrak{h}_\epsilon}{\partial n} \quad \text{on } L_\epsilon.$$

Thus, $\mathfrak{h}_\epsilon$ is an admissible candidate in the variational formulation (5.5.4), hence

$$\begin{aligned}\lambda_\epsilon &\leq \frac{1}{2}\Big(\int_{R_\epsilon} |D^2\mathfrak{h}_\epsilon|^2 + \tau|\nabla\mathfrak{h}_\epsilon|^2 + |\mathfrak{h}_\epsilon|^2\,dx\Big) - \int_{R_\epsilon} f_\epsilon\mathfrak{h}_\epsilon\,dx \\ &= \frac{1}{2}\Big(\int_0^1 \epsilon g(x)\Big[|\mathfrak{h}_\epsilon''|^2 + \tau|\mathfrak{h}_\epsilon'|^2 + |\mathfrak{h}_\epsilon|^2\Big]\,dx\Big) - \int_0^1\int_0^{\epsilon g(x)} f_\epsilon(x,y)\,dy\,\mathfrak{h}_\epsilon(x)\,dx \\ &= \frac{\epsilon}{2}\Big(\int_0^1 g(x)\Big[|\mathfrak{h}_\epsilon''|^2 + \tau|\mathfrak{h}_\epsilon'|^2 + |\mathfrak{h}_\epsilon|^2\Big]\,dx\Big) - \epsilon\int_0^1 g(x)\,\mathcal{M}_\epsilon f_\epsilon(x)\,\mathfrak{h}_\epsilon(x)\,dx \\ &= \epsilon\,\kappa_\epsilon. \end{aligned} \tag{5.5.6}$$

Now we plan to find a lower bound for λ_ϵ in terms of κ_ϵ. By algebraic calculation we have

$$\begin{aligned}
\lambda_\epsilon &= \frac{1}{2}\Big(\int_{R_\epsilon} |D^2 v_\epsilon|^2 + \tau|\nabla v_\epsilon|^2 + |v_\epsilon|^2\,dx\Big) - \int_{R_\epsilon} f_\epsilon v_\epsilon\,dx\\
&= \frac{1}{2}\Big(\int_{R_\epsilon} |D^2(v_\epsilon - \mathfrak{h}_\epsilon + \mathfrak{h}_\epsilon)|^2 + \tau|\nabla(v_\epsilon - \mathfrak{h}_\epsilon + \mathfrak{h}_\epsilon)|^2 + |v_\epsilon - \mathfrak{h}_\epsilon + \mathfrak{h}_\epsilon|^2\,dx\Big)\\
&- \int_{R_\epsilon} f_\epsilon(v_\epsilon - \mathfrak{h}_\epsilon + \mathfrak{h}_\epsilon)\,dx\\
&= \epsilon\kappa_\epsilon + \frac{1}{2}\Big(\int_{R_\epsilon} |D^2(v_\epsilon - \mathfrak{h}_\epsilon)|^2 + \tau|\nabla(v_\epsilon - \mathfrak{h}_\epsilon)|^2 + |v_\epsilon - \mathfrak{h}_\epsilon|^2\,dx\Big)\\
&+ \int_{R_\epsilon} D^2(v_\epsilon - \mathfrak{h}_\epsilon) : D^2\mathfrak{h}_\epsilon\,dx + \tau\int_{R_\epsilon} \nabla(v_\epsilon - \mathfrak{h}_\epsilon)\cdot\nabla\mathfrak{h}_\epsilon\,dx\\
&+ \int_{R_\epsilon} (v_\epsilon - \mathfrak{h}_\epsilon)\mathfrak{h}_\epsilon\,dx - \int_{R_\epsilon} f_\epsilon(v_\epsilon - \mathfrak{h}_\epsilon)\,dx,
\end{aligned} \tag{5.5.7}$$

and we define:

$$\begin{aligned}
I_1 &= \int_{R_\epsilon} D^2(v_\epsilon - \mathfrak{h}_\epsilon) : D^2\mathfrak{h}_\epsilon\,dx,\\
I_2 &= \int_{R_\epsilon} \nabla(v_\epsilon - \mathfrak{h}_\epsilon)\cdot\nabla\mathfrak{h}_\epsilon\,dx,\\
I_3 &= \int_{R_\epsilon} (v_\epsilon - \mathfrak{h}_\epsilon)\mathfrak{h}_\epsilon\,dx,\\
I_4 &= \int_{R_\epsilon} f_\epsilon(v_\epsilon - \mathfrak{h}_\epsilon)\,dx.
\end{aligned} \tag{5.5.8}$$

Hence, we can rewrite (5.5.7) as

$$\lambda_\epsilon = \epsilon\kappa_\epsilon + \frac{1}{2}\|v_\epsilon - \mathfrak{h}_\epsilon\|^2_{H^2_\tau(R_\epsilon)} + I_1 + \tau I_2 + I_3 - I_4. \tag{5.5.9}$$

The idea is to estimate all the terms I_j, $j = 1, \dots, 4$ in terms of $\|v_\epsilon - \mathfrak{h}_\epsilon\|^2_{H^2_\tau(R_\epsilon)}$ and $\epsilon^2\|f_\epsilon\|^2_{L^2(R_\epsilon)}$ in order to obtain an estimate of the type

$$\lambda_\epsilon \geq \epsilon\kappa_\epsilon + c_1\|v_\epsilon - \mathfrak{h}_\epsilon\|^2_{H^2_\tau(R_\epsilon)} - c_2\epsilon^2\|f_\epsilon\|^2_{L^2(R_\epsilon)},$$

and by coupling this estimate with the upper bound (5.5.6) we get

$$\|v_\epsilon - \mathfrak{h}_\epsilon\|_{H^2_\tau(R_\epsilon)} \leq c\epsilon\|f_\epsilon\|_{L^2(R_\epsilon)},$$

which is (5.5.3).

Let us first analyse I_1:

$$\begin{aligned} I_1 &= \int_{R_\epsilon} \Big(\frac{\partial^2 v_\epsilon}{\partial x^2} - \mathfrak{h}_\epsilon''\Big)\mathfrak{h}_\epsilon''\, dxdy = \int_0^1 \int_0^{\epsilon g(x)} \Big(\frac{\partial^2 v_\epsilon}{\partial x^2} - \mathfrak{h}_\epsilon''\Big) dy \mathfrak{h}_\epsilon''\, dx \\ &= \int_0^1 \epsilon\Big(\mathcal{M}_\epsilon\Big(\frac{\partial^2 v_\epsilon}{\partial x^2}\Big) - \mathfrak{h}_\epsilon''\Big)\, \mathfrak{h}_\epsilon''\, g dx = \int_{R_\epsilon} \Big(\mathcal{M}_\epsilon\Big(\frac{\partial^2 v_\epsilon}{\partial x^2}\Big) - \mathfrak{h}_\epsilon''\Big)\, \mathfrak{h}_\epsilon''\, dx \\ &= \int_{R_\epsilon} \Big(\mathcal{M}_\epsilon\Big(\frac{\partial^2 v_\epsilon}{\partial x^2}\Big) - (\mathcal{M}_\epsilon v_\epsilon)''\Big)\mathfrak{h}_\epsilon''\, dxdy + \int_{R_\epsilon} (\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon)''\mathfrak{h}_\epsilon''\, dxdy. \end{aligned}$$

Let us define

$$J_1 = \int_{R_\epsilon} \Big(\mathcal{M}_\epsilon\Big(\frac{\partial^2 v_\epsilon}{\partial x^2}\Big) - \frac{\partial^2}{\partial x^2}(\mathcal{M}_\epsilon v_\epsilon)\Big)\mathfrak{h}_\epsilon''\, dxdy.$$

Then

$$I_1 = J_1 + \epsilon \int_0^1 g(x)(\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon)''\, \mathfrak{h}_\epsilon''\, dx. \tag{5.5.10}$$

In a similar way we have

$$\begin{aligned} I_2 &= \int_{R_\epsilon} \Big(\frac{\partial v_\epsilon}{\partial x} - \mathfrak{h}_\epsilon'\Big)\, \mathfrak{h}_\epsilon'\, dxdy \\ &= \int_{R_\epsilon} \Big(\mathcal{M}_\epsilon\Big(\frac{\partial v_\epsilon}{\partial x}\Big) - \frac{\partial}{\partial x}(\mathcal{M}_\epsilon v_\epsilon)\Big)\, \mathfrak{h}_\epsilon'\, dxdy + \epsilon \int_0^1 (\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon)'\, \mathfrak{h}_\epsilon'\, g dx \\ &= J_2 + \epsilon \int_0^1 (\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon)'\, \mathfrak{h}_\epsilon'\, dx \end{aligned}$$

where

$$J_2 = \int_{R_\epsilon} \Big(\mathcal{M}_\epsilon\Big(\frac{\partial v_\epsilon}{\partial x}\Big) - \frac{\partial}{\partial x}(\mathcal{M}_\epsilon v_\epsilon)\Big)\, \mathfrak{h}_\epsilon'\, dxdy.$$

Also,

$$I_3 = \epsilon \int_0^1 (\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon)\, \mathfrak{h}_\epsilon\, g\, dx.$$

Finally,

$$\begin{aligned} I_4 &= \int_{R_\epsilon} (f_\epsilon - \mathcal{M}_\epsilon f_\epsilon)(v_\epsilon - \mathfrak{h}_\epsilon)\, dxdy + \int_{R_\epsilon} \mathcal{M}_\epsilon f_\epsilon (v_\epsilon - \mathfrak{h}_\epsilon)\, dxdy \\ &= J_4 + \epsilon \int_0^1 \mathcal{M}_\epsilon f_\epsilon\, (\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon)\, g dx, \end{aligned}$$

where we have defined

$$J_4 = \int_{R_\epsilon} (f_\epsilon - \mathcal{M}_\epsilon f_\epsilon)(v_\epsilon - \mathfrak{h}_\epsilon)\, dxdy.$$

It is possible to prove that $\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon \in H^2_{0,g}(0,1)$ (see Remark 5.5.2 after the proof), hence we can plug it as test-function in the weak formulation of problem (5.5.1) in order to get

$$\int_0^1 \left[(\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon)''\mathfrak{h}_\epsilon'' + \tau(\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon)'\mathfrak{h}_\epsilon' + (\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon)\mathfrak{h}_\epsilon\right] g dx = \int_0^1 (\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon)\, \mathcal{M}_\epsilon f_\epsilon \; g dx,$$

hence

$$I_1 + \tau I_2 + I_3 - I_4 = J_1 + \tau J_2 - J_4. \tag{5.5.11}$$

Hence we have reduced the problem to estimating the integrals J_k, $k = 1, 2, 4$. We plan first to treat the terms containing $(\mathcal{M}_\epsilon v_\epsilon)'$ and $(\mathcal{M}_\epsilon v_\epsilon)''$. The change of coordinates $z = \epsilon y g(x)$ and a differentiation under the integral sign yield

$$\begin{aligned}(\mathcal{M}_\epsilon v_\epsilon)' &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\epsilon g(x)}\int_0^{\epsilon g(x)} v_\epsilon(x,z)\,dz\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^1 v_\epsilon(x,\epsilon y g(x))dy\right)\\ &= \int_0^1 \frac{\partial v_\epsilon}{\partial x}(x,\epsilon y g(x))\,dy + \int_0^1 \frac{\partial v_\epsilon}{\partial y}(x,\epsilon y g(x))\,\epsilon y g'(x)\,dy \\ &= \mathcal{M}_\epsilon\left(\frac{\partial v_\epsilon}{\partial x}\right) + \int_0^1 \frac{\partial v_\epsilon}{\partial y}(x,\epsilon y g(x))\,\epsilon y g'(x)\,dy.\end{aligned} \tag{5.5.12}$$

Hence,

$$J_2 = -\int_{R_\epsilon}\int_0^1 \frac{\partial v_\epsilon}{\partial y}(x,\epsilon y g(x))\,\epsilon y g'(x)\,dy \mathfrak{h}_\epsilon'\,dx.$$

This implies the estimate

$$|J_2| \le C\epsilon \int_{R_\epsilon} \mathcal{M}_\epsilon\left(\left|\frac{\partial v_\epsilon}{\partial y}\right|\right)\mathfrak{h}_\epsilon'\,dxdy \le C\epsilon \left\|\frac{\partial v_\epsilon}{\partial y}\right\|_{L^2(R_\epsilon)} \|\mathfrak{h}_\epsilon'\|_{L^2(R_\epsilon)}. \tag{5.5.13}$$

Since $\mathfrak{h}_\epsilon$ is the solution of problem (5.5.1), we can improve the estimate (5.5.13) using an apriori estimate on $\mathfrak{h}_\epsilon$. Namely, $\mathfrak{h}_\epsilon$ solves

$$\int_0^1 g\,\mathfrak{h}_\epsilon''\varphi'' + \tau\, g\,\mathfrak{h}_\epsilon'\varphi' + g\,\mathfrak{h}_\epsilon\varphi\,dx = \int_0^1 g\,\mathcal{M}_\epsilon f_\epsilon \varphi\,dx, \tag{5.5.14}$$

for all $\varphi \in H^2_{0,g}(0,1)$. By choosing $\varphi = \mathfrak{h}_\epsilon$ in (5.5.14) and by classical inequalities we get

$$\int_0^1 \left[|\mathfrak{h}_\epsilon''|^2 + \tau|\mathfrak{h}_\epsilon'|^2 + |\mathfrak{h}_\epsilon|^2\right] g dx \le \frac{1}{2}\|\mathfrak{h}_\epsilon\|^2_{L^2_g(0,1)} + \frac{1}{2}\|\mathcal{M}_\epsilon f_\epsilon\|^2_{L^2_g(0,1)},$$

hence,

$$\int_0^1 g\Big[|\mathfrak{h}_\epsilon''|^2 + \tau|\mathfrak{h}_\epsilon'|^2 + |\mathfrak{h}_\epsilon|^2\Big]\,dx \le c\|\mathcal{M}_\epsilon f_\epsilon\|^2_{L^2_g(0,1)}, \tag{5.5.15}$$

for some positive constant $c > 0$. By multiplying both sides of (5.5.15) by $\epsilon > 0$ we finally get

$$\int_{R_\epsilon} \big[|\mathfrak{h}_\epsilon''|^2 + \tau|\mathfrak{h}_\epsilon'|^2 + |\mathfrak{h}_\epsilon|^2\big]\,dxdy \le c\epsilon\|\mathcal{M}_\epsilon f_\epsilon\|^2_{L^2_g(0,1)}. \tag{5.5.16}$$

Hence, in particular,

$$\tau\|\mathfrak{h}_\epsilon'\|^2_{L^2(R_\epsilon)} \le c\epsilon\|\mathcal{M}_\epsilon f_\epsilon\|^2_{L^2_g(0,1)}. \tag{5.5.17}$$

By keeping into account (5.5.17), we can improve the estimate (5.5.13) as follows

$$\begin{aligned}
\tau|J_2| &\le c\sqrt{\tau}\epsilon^{3/2}\|\mathcal{M}_\epsilon f_\epsilon\|_{L^2_g(0,1)}\left\|\frac{\partial v_\epsilon}{\partial y}\right\|_{L^2(R_\epsilon)}\\
&\le c\epsilon^3\|\mathcal{M}_\epsilon f_\epsilon\|^2_{L^2_g(0,1)} + \frac{\tau}{8}\left\|\frac{\partial v_\epsilon}{\partial y}\right\|^2_{L^2(R_\epsilon)}\\
&\le c\epsilon^3\|\mathcal{M}_\epsilon f_\epsilon\|^2_{L^2_g(0,1)} + \frac{\tau}{8}\|\nabla(v_\epsilon - \mathfrak{h}_\epsilon)\|^2_{L^2(R_\epsilon)},
\end{aligned} \tag{5.5.18}$$

where we used the trivial inequality

$$\|\nabla(v_\epsilon - \mathfrak{h}_\epsilon)\|^2_{L^2(R_\epsilon)} = \left\|\frac{\partial v_\epsilon}{\partial y}\right\|^2_{L^2(R_\epsilon)} + \left\|\frac{\partial v_\epsilon}{\partial x} - \mathfrak{h}_\epsilon'\right\|^2_{L^2(R_\epsilon)} \ge \left\|\frac{\partial v_\epsilon}{\partial y}\right\|^2_{L^2(R_\epsilon)}.$$

We now compute $(\mathcal{M}_\epsilon v_\epsilon)''$ which appears in J_1. Keeping in mind the previous computations for $(\mathcal{M}_\epsilon v_\epsilon)'$ we have that

$$\begin{aligned}
(\mathcal{M}_\epsilon v_\epsilon)'' &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^1 \frac{\partial v_\epsilon}{\partial x}(x,\epsilon y g(x))\,dy + \int_0^1 \frac{\partial v_\epsilon}{\partial y}(x,\epsilon y g(x))\epsilon y g'(x)\,dy\right)\\
&= \mathcal{M}_\epsilon\Big(\frac{\partial^2 v_\epsilon}{\partial x^2}\Big) + 2\epsilon\int_0^1 \frac{\partial^2 v_\epsilon}{\partial x\partial y}(x,\epsilon y g(x)) y g'(x)\,dy\\
&+ \epsilon^2\int_0^1 \frac{\partial^2 v_\epsilon}{\partial y^2}(x,\epsilon y g(x))(y g'(x))^2 dy + \epsilon\int_0^1 \frac{\partial v_\epsilon}{\partial y}(x,\epsilon y g(x)) y g''(x)\,dy,
\end{aligned}$$

and since g is a bounded function in $C^2(0,1)$ we deduce the existence of a positive constant $a > 0$ depending only on g such that

$$\left|\mathcal{M}_\epsilon\Big(\frac{\partial^2 v_\epsilon}{\partial x^2}\Big) - (\mathcal{M}_\epsilon v_\epsilon)''\right| \le a\epsilon\mathcal{M}_\epsilon\left(\left|\frac{\partial^2 v_\epsilon}{\partial x\partial y}\right|\right) + a\epsilon^2\mathcal{M}_\epsilon\left(\left|\frac{\partial^2 v_\epsilon}{\partial y^2}\right|\right) + a\epsilon\mathcal{M}_\epsilon\left(\left|\frac{\partial v_\epsilon}{\partial y}\right|\right).$$

Hence,

$$
\begin{aligned}
|J_1| &= \left| \int_{R_\epsilon} \Big(\mathcal{M}_\epsilon \Big(\frac{\partial^2 v_\epsilon}{\partial x^2} \Big) - (\mathcal{M}_\epsilon v_\epsilon)'' \Big) \mathfrak{h}_\epsilon''\, dxdy \right| \\
&\le a \int_{R_\epsilon} \Big[\epsilon \mathcal{M}_\epsilon \Big(\Big| \frac{\partial^2 v_\epsilon}{\partial x \partial y} \Big| \Big) + \epsilon^2 \mathcal{M}_\epsilon \Big(\Big| \frac{\partial^2 v_\epsilon}{\partial y^2} \Big| \Big) + \epsilon \mathcal{M}_\epsilon \Big(\Big| \frac{\partial v_\epsilon}{\partial y} \Big| \Big) \Big] |\mathfrak{h}_\epsilon''| dxdy \\
&\le a\epsilon \|\mathfrak{h}_\epsilon''\|_{L^2(R_\epsilon)} \left\| \frac{\partial^2 v_\epsilon}{\partial x \partial y} \right\|_{L^2(R_\epsilon)} + a\epsilon^2 \|\mathfrak{h}_\epsilon''\|_{L^2(R_\epsilon)} \left\| \frac{\partial^2 v_\epsilon}{\partial y^2} \right\|_{L^2(R_\epsilon)} \\
&+ a\epsilon \|\mathfrak{h}_\epsilon''\|_{L^2(R_\epsilon)} \left\| \frac{\partial v_\epsilon}{\partial y} \right\|_{L^2(R_\epsilon)} \\
&\le c \Big(\epsilon \left\| \frac{\partial^2 v_\epsilon}{\partial x \partial y} \right\|_{L^2(R_\epsilon)} + \epsilon^2 \left\| \frac{\partial^2 v_\epsilon}{\partial y^2} \right\|_{L^2(R_\epsilon)} + \epsilon \left\| \frac{\partial v_\epsilon}{\partial y} \right\|_{L^2(R_\epsilon)} \Big) \epsilon^{1/2} \|\mathcal{M}_\epsilon f_\epsilon\|_{L^2_g(0,1)},
\end{aligned}
$$

where in the last inequality we used the apriori estimate (5.5.16). By applying Young inequality and the trivial inequalities

$$
\begin{aligned}
\left\| \frac{\partial^2 v_\epsilon}{\partial x \partial y} \right\|_{L^2(R_\epsilon)} &\le \|D^2(v_\epsilon - \mathfrak{h}_\epsilon)\|_{L^2(R_\epsilon)}, \\
\left\| \frac{\partial^2 v_\epsilon}{\partial y^2} \right\|_{L^2(R_\epsilon)} &\le \|D^2(v_\epsilon - \mathfrak{h}_\epsilon)\|_{L^2(R_\epsilon)},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
|J_1| \le c\epsilon^3 \|\mathcal{M}_\epsilon f_\epsilon\|^2_{L^2_g(0,1)} &+ \frac{1}{8} \|D^2(v_\epsilon - \mathfrak{h}_\epsilon)\|^2_{L^2(R_\epsilon)} + c\epsilon^5 \|\mathcal{M}_\epsilon f_\epsilon\|^2_{L^2_g(0,1)} \\
&+ \frac{1}{8} \|D^2(v_\epsilon - \mathfrak{h}_\epsilon)\|^2_{L^2(R_\epsilon)} + \frac{\tau}{8} \|\nabla(v_\epsilon - \mathfrak{h}_\epsilon)\|^2_{L^2(R_\epsilon)}. \qquad (5.5.19)
\end{aligned}
$$

We finally estimate J_4. First note that

$$
\begin{aligned}
J_4 &= \int_{R_\epsilon} (f_\epsilon - \mathcal{M}_\epsilon f_\epsilon)(v_\epsilon - \mathfrak{h}_\epsilon)\, dxdy \\
&= \int_{R_\epsilon} (f_\epsilon - \mathcal{M}_\epsilon f_\epsilon)(v_\epsilon - \mathcal{M}_\epsilon v_\epsilon)\, dxdy + \int_{R_\epsilon} (f_\epsilon - \mathcal{M}_\epsilon f_\epsilon)(\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon)\, dxdy,
\end{aligned}
$$

and

$$
\int_{R_\epsilon} (f_\epsilon - \mathcal{M}_\epsilon f_\epsilon)(\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon)\, dxdy = \int_0^1 \left(\int_0^{\epsilon g(x)} f_\epsilon - \mathcal{M}_\epsilon f_\epsilon dy \right) (\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon)\, dx = 0.
$$

Thus,

$$
|J_4| \le \|f_\epsilon - \mathcal{M}_\epsilon f_\epsilon\|_{L^2(R_\epsilon)} \|v_\epsilon - \mathcal{M}_\epsilon v_\epsilon\|_{L^2(R_\epsilon)} \le c\epsilon \left\| \frac{\partial v_\epsilon}{\partial y} \right\|_{L^2(R_\epsilon)} \|f_\epsilon - \mathcal{M}_\epsilon f_\epsilon\|_{L^2(R_\epsilon)}, \qquad (5.5.20)
$$

where we used the Poincaré inequality in the y-variable.

By collecting (5.5.7), (5.5.11), (5.5.18), (5.5.19), (5.5.20) we deduce that

$$\begin{aligned}\lambda_\epsilon \geq \epsilon\kappa_\epsilon + \frac{1}{2}\|v_\epsilon - \mathfrak{h}_\epsilon\|^2_{H^2_\tau(R_\epsilon)} - c\epsilon^3(1+\epsilon^2)\|\mathcal{M}_\epsilon f_\epsilon\|^2_{L^2_g(0,1)} \\ - \frac{1}{4}\|D^2(v_\epsilon - \mathfrak{h}_\epsilon)\|^2_{L^2(R_\epsilon)} - \frac{\tau}{4}\|\nabla(v_\epsilon - \mathfrak{h}_\epsilon)\|^2_{L^2(R_\epsilon)} \\ - c\epsilon\|f_\epsilon - \mathcal{M}_\epsilon f_\epsilon\|_{L^2(R_\epsilon)}\left\|\frac{\partial v_\epsilon}{\partial y}\right\|_{L^2(R_\epsilon)}.\end{aligned}$$

Since $\|f_\epsilon - \mathcal{M}_\epsilon f_\epsilon\|_{L^2(R_\epsilon)} \leq \|f_\epsilon\|_{L^2(R_\epsilon)}$ we deduce that

$$\begin{aligned}c\epsilon\|f_\epsilon - \mathcal{M}_\epsilon f_\epsilon\|_{L^2(R_\epsilon)}\left\|\frac{\partial v_\epsilon}{\partial y}\right\|_{L^2(R_\epsilon)} &\leq \frac{\tau}{8}\left\|\frac{\partial v_\epsilon}{\partial y}\right\|^2_{L^2(R_\epsilon)} + c\epsilon^2\|f_\epsilon\|^2_{L^2(R_\epsilon)} \\ &\leq \frac{\tau}{8}\|\nabla(v_\epsilon - \mathfrak{h}_\epsilon)\|^2_{L^2(R_\epsilon)} + c\epsilon^2\|f_\epsilon\|^2_{L^2(R_\epsilon)}.\end{aligned}$$

Thus,

$$\lambda_\epsilon \geq \epsilon\kappa_\epsilon + \frac{1}{8}\|v_\epsilon - \mathfrak{h}_\epsilon\|^2_{H^2_\tau(R_\epsilon)} - C\epsilon^2(1+\epsilon^2)\|f_\epsilon\|^2_{L^2(R_\epsilon)}. \tag{5.5.21}$$

Then (5.5.6) and (5.5.21) imply that

$$\|v_\epsilon - \mathfrak{h}_\epsilon\|^2_{H^2_\tau(R_\epsilon)} \leq C\epsilon^2\|f_\epsilon\|^2_{L^2(R_\epsilon)},$$

for some positive constant C not depending on τ. This concludes the proof.

□

Remark 5.5.2. In the proof we have used the fact that $\mathcal{M}_\epsilon v_\epsilon - \mathfrak{h}_\epsilon \in H^2_0(0,1)$. To prove this it is sufficient to show that $\mathcal{M}_\epsilon v_\epsilon \in H^2_0(0,1)$ for all $\epsilon > 0$. In the following computations we write v in place of v_ϵ. As done in the proof of Thm 5.5.1, one proves that

$$\begin{aligned}(\mathcal{M}_\epsilon v)'' = \frac{1}{\epsilon g(x)}\int_0^{\epsilon g(x)} v_{xx}(x,y)\,dy + \frac{2g'(x)}{g^2(x)}\int_0^{\epsilon g(x)} v_{xy}(x,y)y\,dy \\ + \frac{(g'(x))^2}{g(x)^3}\int_0^{\epsilon g(x)} v_{yy}(x,y)y\,dy + \frac{g''(x)}{\epsilon g(x)^2}\int_0^{\epsilon g(x)} v_y(x,y)y\,dy.\end{aligned} \tag{5.5.22}$$

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $C^\infty_{L_\epsilon}(R_\epsilon)$ such that $\varphi_n \to v$ in $H^2(R_\epsilon)$. Then

$$\int_0^{\epsilon g(x)} |v - \varphi_n|^2 + |\nabla v - \nabla\varphi_n|^2 + \left|D^2 v - D^2\varphi_n\right|^2 dy \to 0,$$

for almost all $x \in (0,1)$, as $n \to \infty$. As a consequence of this one can prove that all the integrals appearing in the right-hand side of (5.5.22) can be approximated in $L^2(0,1)$ by the same integrals where the function v is replaced by φ_n. For example, by Jensen inequality we get

$$\int_0^1 \left| \frac{1}{\epsilon g(x)} \int_0^{\epsilon g(x)} (\varphi_n - v)\, dy \right|^2 dx \leq \int_0^1 \frac{1}{\epsilon g(x)} \int_0^{\epsilon g(x)} |\varphi_n - v|^2 \to 0,$$

as $n \to \infty$. Note that the function $\psi_n := \frac{1}{\epsilon g(x)} \int_0^{\epsilon g(x)} \varphi_n\, dy$ is in $C_c^\infty(0,1)$ for all $n \in \mathbb{N}$, and $\psi_n \to \frac{1}{\epsilon g(x)} \int_0^{\epsilon g(x)} v dy$ in $L^2(0,1)$. Hence $\mathcal{M}_\epsilon v$ can be approximated in $H^2(0,1)$ by a sequence of function $C_c^\infty(0,1)$, and this implies that $\mathcal{M}_\epsilon v \in H_0^2(0,1)$.

5.6 Final convergence results

Recall that the eigenpairs of problems (5.1.2), (5.1.7) are denoted by $(\lambda_n(\Omega_\epsilon), \varphi_n^\epsilon)$, $(\omega_n, \varphi_n^\Omega)_{n\geq 1}$ respectively, where the two families of eigenfunctions φ_n^ϵ, φ_n^Ω are complete orthonormal bases of the spaces $L^2(\Omega_\epsilon)$, $L^2(\Omega)$, respectively. Denote now by $(h_n, \theta_n)_{n\geq 1}$ the eigenpairs of problem (5.1.9) where the eigenfunctions h_n define an orthonormal basis of the space $L_g^2(0,1)$. In the spirit of the definition of λ_n^ϵ given in Section 2, we set now

$$(\lambda_n^0)_{n\geq 1} = (\omega_k)_{k\geq 1} \cup (\theta_l)_{l\geq 1}, \tag{5.6.1}$$

where it is understood that the eigenvalues are arranged in increasing order and repeated according to their multiplicity. For each λ_n^0 we define the function $\phi_n^0 \in H^2(\Omega) \oplus H^2(R_\epsilon)$ in the following way:

$$\phi_n^0 = \begin{cases} \varphi_k^\Omega, & \text{in } \Omega, \\ 0, & \text{in } R_\epsilon, \end{cases}$$

if $\lambda_n^0 = \omega_k$, for some $k \in \mathbb{N}$; otherwise

$$\phi_n^0 = \begin{cases} 0, & \text{in } \Omega, \\ \epsilon^{-1/2} \mathcal{E}_\epsilon h_l, & \text{in } R_\epsilon \end{cases}$$

if $\lambda_n^\epsilon = \theta_l$, for some $l \in \mathbb{N}$ (here we agree to order the eigenvalues and the eigenfunctions following the same rule used in the definition of λ_n^ϵ and ϕ_n^ϵ in Section 2).

Finally, if $x > 0$ divides the spectrum $\lambda_n(\Omega_\epsilon)$ for all $\epsilon > 0$ sufficiently small (see the end of Section 2) and $N(x)$ is the number of eigenvalues with $\lambda_n(\Omega_\epsilon) \leq x$

(counting their multiplicity), we define the projector P_x^0 from $L^2(\Omega_\epsilon)$ onto the linear span $[\phi_1^0, \dots, \phi_{N(x)}^0]$ by setting

$$P_x^0 u = \sum_{i=1}^{N(x)} (u, \phi_i^0)_{L^2(\Omega_\epsilon)} \phi_i^0,$$

for all $u \in L^2(\Omega_\epsilon)$. (Note that choosing x independent of ϵ is possible by the limiting behaviour of the eigenvalues.) Then, using Theorems 5.2.9 and 5.4.1 we deduce the following.

Theorem 5.6.1. *Let Ω_ϵ, $\epsilon > 0$, be a family of dumbbell domains satisfying the H-Condition. Then the following statements hold:*

(i) $\lim_{\epsilon\to 0} |\lambda_n(\Omega_\epsilon) - \lambda_n^0| = 0$, *for all $n \in \mathbb{N}$.*

(ii) For any x dividing the spectrum, $\lim_{\epsilon\to 0} \|\varphi_n^\epsilon - P_x^0 \varphi_n^\epsilon\|_{H^2(\Omega)\oplus L^2(R_\epsilon)} = 0$, *for all $n = 1, \dots, N(x)$.*

Proof. The convergence of the eigenvalues follows directly by Theorems 5.2.9 and 5.4.1. Indeed, by Theorem 5.2.9 we know that $|\lambda_n(\Omega_\epsilon) - \lambda_n^\epsilon| \to 0$ as $\epsilon \to 0$. If $\lambda_n^\epsilon = \omega_k$ for some $k \in \mathbb{N}$, for all sufficiently small ϵ, then we are done; otherwise, if $\lambda_n^\epsilon = \theta_l^\epsilon$ for some $l \in \mathbb{N}$, definitely in ϵ, by Theorem 5.4.1 we deduce that $\theta_l^\epsilon \to \theta_l$ as $\epsilon \to 0$, hence $|\lambda_n(\Omega_\epsilon) - \theta_l| \le |\lambda_n(\Omega_\epsilon) - \theta_l^\epsilon| + |\theta_l^\epsilon - \theta_l| \to 0$ as $\epsilon \to 0$.
Consider now the convergence of the eigenfunctions. By Theorem 5.4.1 it follows that for any $\epsilon > 0$ there exists an orthonormal sequence of eigenfunctions δ_j^ϵ in $L^2(R_\epsilon, \epsilon^{-1}dxdy)$ associated with the eigenvalues θ_j^ϵ of problem (5.1.8) such that

$$\|\delta_j^\epsilon - \mathcal{E}_\epsilon h_j\|_{L^2(R_\epsilon, \epsilon^{-1}dxdy)} \to 0, \tag{5.6.2}$$

as $\epsilon \to 0$, for all $j \in \mathbb{N}$. We set $\gamma_j^\epsilon = \epsilon^{-1/2}\delta_j^\epsilon$ and we note that γ_j^ϵ is a sequence of eigenfunctions of Problem (5.1.8) which is orthonormal in $L^2(R_\epsilon)$, as required in Theorem 5.2.9. Thus by Theorem 5.2.9 (ii), we deduce that

$$\begin{aligned}
&\left\| \varphi_n^\epsilon - \sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \epsilon^{-1/2}\mathcal{E}_\epsilon h_i)_{L^2(R_\epsilon)} \epsilon^{-1/2}\mathcal{E}_\epsilon h_i \right\|_{L^2(R_\epsilon)} \\
&\le \left\| \varphi_n^\epsilon - \sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \gamma_i^\epsilon)_{L^2(R_\epsilon)} \gamma_i^\epsilon \right\|_{L^2(R_\epsilon)} \\
&+ \left\| \sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \gamma_i^\epsilon)_{L^2(R_\epsilon)} \gamma_i^\epsilon - \sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \epsilon^{-1/2}\mathcal{E}_\epsilon h_i)_{L^2(R_\epsilon)} \epsilon^{-1/2}\mathcal{E}_\epsilon h_i \right\|_{L^2(R_\epsilon)},
\end{aligned} \tag{5.6.3}$$

and

$$\left\| \varphi_n^\epsilon - \sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \gamma_i^\epsilon)_{L^2(R_\epsilon)} \gamma_i^\epsilon \right\|_{L^2(R_\epsilon)} \longrightarrow 0, \tag{5.6.4}$$

as $\epsilon \longrightarrow 0$. Moreover,

$$\begin{aligned}
&\left\| \sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \gamma_i^\epsilon)_{L^2(R_\epsilon)} \gamma_i^\epsilon - \sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \epsilon^{-1/2}\mathcal{E}_\epsilon h_i)_{L^2(R_\epsilon)} \epsilon^{-1/2}\mathcal{E}_\epsilon h_i \right\|_{L^2(R_\epsilon)} \\
&\leq \left\| \sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \epsilon^{-1/2}\mathcal{E}_\epsilon h_i)_{L^2(R_\epsilon)} (\gamma_i^\epsilon - \epsilon^{-1/2}\mathcal{E}_\epsilon h_i) \right\|_{L^2(R_\epsilon)} \\
&+ \left\| \sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \gamma_i^\epsilon - \epsilon^{-1/2}\mathcal{E}_\epsilon h_i)_{L^2(R_\epsilon)} \gamma_i^\epsilon \right\|_{L^2(R_\epsilon)} \\
&\leq C \sum_{i=1}^{N(x)} \|\gamma_i^\epsilon - \epsilon^{-1/2}\mathcal{E}_\epsilon h_i\|_{L^2(R_\epsilon)}.
\end{aligned} \tag{5.6.5}$$

By (5.6.3), (5.6.4) and (5.6.5) we deduce that

$$\left\| \varphi_n^\epsilon - \sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \epsilon^{-1/2}\mathcal{E}_\epsilon h_i)_{L^2(R_\epsilon)} \epsilon^{-1/2}\mathcal{E}_\epsilon h_i \right\|_{L^2(R_\epsilon)} \leq o(1) + C \sum_{i=1}^{N(x)} \|\delta_i^\epsilon - \mathcal{E}_\epsilon h_i\|_{L^2(R_\epsilon, \epsilon^{-1}dxdy)}. \tag{5.6.6}$$

Since the right-hand side of (5.6.6) goes to zero as $\epsilon \longrightarrow 0$ by (5.6.2), we conclude that $\lim_{\epsilon\to 0} \|\varphi_n^\epsilon - P_x^0 \varphi_n^\epsilon\|_{L^2(R_\epsilon)} = 0$. Finally, the fact that $\lim_{\epsilon\to 0} \|\varphi_n^\epsilon - P_x^0 \varphi_n^\epsilon\|_{H^2(\Omega)} = 0$ follows directly from Theorem 5.2.9. □

Chapter 6

Reissner-Mindlin system with free boundary conditions on dumbbell domains

In this chapter we shall analyse the spectral behaviour of the Reissner-Mindlin system on dumbbell domains. The Reissner-Mindlin model for elastic plates of thickness $t > 0$ is a strongly elliptic system of three partial differential equations of the second order depending on the parameter t. It is defined by

$$\begin{cases} -\frac{\mu_1}{12}\Delta\beta - \frac{\mu_1+\mu_2}{12}\nabla(\operatorname{div}\beta) - \frac{\mu_1 k}{t^2}(\nabla w - \beta) = \lambda\frac{t^2}{12}\beta, & \text{in } \Omega_\delta, \\ -\frac{\mu_1 k}{t^2}(\Delta w - \operatorname{div}\beta) = \lambda w, & \text{in } \Omega_\delta, \end{cases} \tag{6.0.1}$$

where $\beta \in V(\Omega_\delta)^2$, $w \in V(\Omega_\delta)$ are the unknowns and λ is the eigenvalue. Here $V(\Omega_\delta)$ is a given subspace of $H^1(\Omega_\delta)$ containing $H^1_0(\Omega_\delta)$. System (6.0.1) models the vibrations of a plate with reference configuration given by $\Omega_\delta \times \left(-\frac{t}{2}, \frac{t}{2}\right)$. In our setting Ω_δ is a dumbbell domain, i.e.,the union of a bounded smooth disconnected open set Ω and a thin channel R_δ connecting the connected components of Ω, as already defined in (5.1.1) (see also (6.2.6) below, where we recall the geometrical setting) . As $\delta \to 0$ the channel R_δ collapses to a one-dimensional manifold. We assume that the deformation of the plate corresponding to the midplane Ω_δ is described by the Reissner-Mindlin model in terms of the rotations $\beta = (\beta_1, \beta_2)$ of the fibers and in terms of the transverse displacement w of the midplane. Our main purpose is to show that the results obtained in Chapter 5 (in particular, the limiting differential problem in the channel) are compatible with the Reissner-Mindlin model, which is known to have a spectrum convergent to the spectrum of the biharmonic operator as the thickness parameter t of the plate tends to zero. According to the spectral decomposition proved in Theorem

5.2.9, to confirm the compatibility between the two plate models we need, first, to prove an asymptotic spectral decomposition result for the Reissner-Mindlin system. Then it will be sufficient to prove that the differential problem in the channel R_δ for the Reissner-Mindlin system converges to the differential problem (5.1.8) as $t \longrightarrow 0$.

We recall now how the Reissner-Mindlin model (6.0.1) is usually deduced from the classical laws of the theory of elasticity. In the following we assume that $\Omega \subset \mathbb{R}^2$ is the midplane of the plate and $t > 0$ is the thickness. Let $u = (u_1, u_2, u_3)$ be the displacement of the plate at a point $x = (x_1, x_2, x_3) \in \Omega \times [-t/2, t/2]$. The constitutive laws for u can be expressed as follows:

$$\begin{cases} u_1(x) = -x_3\beta_1(x_1, x_2), \\ u_2(x) = -x_3\beta_2(x_1, x_2), \\ u_3(x_1, x_2, 0) = w(x_1, x_2). \end{cases} \tag{6.0.2}$$

Here β_1 and β_2 denotes the angles between the vertical direction and the projections of the normal direction to the middle-surface on the x_1x_3-plane and x_2x_3-plane respectively; moreover w denotes the vertical displacement of the middle surface. Let $\epsilon(u) = (e_{ij})_{i,j=1}^3$ be the linearized strain tensor associated with u. Recall that

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

for all $i, j \in \{1, 2, 3\}$. As a consequence of (6.0.2) we deduce that

$$e_{11} = -x_3\frac{\partial \beta_1}{\partial x_1}, \qquad e_{22} = -x_3\frac{\partial \beta_2}{\partial x_2}, \qquad e_{12} = e_{21} = -\frac{x_3}{2}\left(\frac{\partial \beta_1}{\partial x_2} + \frac{\partial \beta_2}{\partial x_1}\right)$$

$$e_{13} = e_{31} = \frac{1}{2}\left(\frac{\partial w}{\partial x_1} - \beta_1\right), \qquad e_{23} = e_{32} = \frac{1}{2}\left(\frac{\partial w}{\partial x_2} - \beta_2\right).$$

Actually, to the components $e_{13}, e_{31}, e_{23}, e_{32}$ it is applied a correction factor k which gives

$$e_{13} = e_{31} = \frac{\sqrt{k}}{2}\left(\frac{\partial w}{\partial x_1} - \beta_1\right), \qquad e_{23} = e_{32} = \frac{\sqrt{k}}{2}\left(\frac{\partial w}{\partial x_2} - \beta_2\right).$$

Let now $\Sigma = (\sigma_{ij})$, $i, j \in \{1, 2, 3\}$ be the stress tensor associated with u. In order to recover the component e_{33} of the linearized strain tensor, as in the Kirchhoff-Love model for a thin plate we make the assumption that $\sigma_{33} = 0$. This is made by applying the Hooke's Law:

$$\Sigma = \lambda \mathrm{tr}(\epsilon(u))\mathbb{I} + 2\mu\epsilon(u), \tag{6.0.3}$$

where μ, λ are the Lamé coefficients. The coefficient μ is also known as modulus of rigidity or shear modulus, and λ is such that $K = \lambda + \frac{2}{3}\mu$ is the modulus of hydrostatic compression, see for example [89]. The Lamé coefficients may be also expressed in terms of the Young modulus E (or modulus of extension) and the Poisson ratio σ via the relations

$$\lambda = \frac{\sigma E}{(1-2\sigma)(1+\sigma)}, \qquad \mu = \frac{E}{2(1+\sigma)}.$$

From the Hooke's Law (6.0.3) we deduce that

$$\sigma_{33} = \lambda(e_{11} + e_{22} + e_{33}) + 2\mu e_{33} = \frac{E}{(1+\sigma)(1-2\sigma)}[\sigma\,(e_{11} + e_{22}) + (1-\sigma)e_{33}].$$

Hence if we assume that $\sigma_{33} = 0$, then it follows that

$$e_{33} = -\frac{\sigma}{1-\sigma}(e_{11} + e_{22}) = \frac{\sigma}{1-\sigma} x_3 \operatorname{div}(\beta).$$

In this way we have expressed all the components of the strain tensor $\epsilon(u)$ in terms of $x = (x_1, x_2, x_3)$, $\beta = (\beta_1, \beta_2)$, w and their derivatives. We may then proceed to compute the total elastic energy E_{el} of the plate corresponding to the general configuration determined by β, w.

It is well known that the elastic energy density W can be expressed as a function of the components of the strain tensor $\epsilon(u)$ in the following way

$$W = \frac{E}{2(1+\sigma)}\left[\sum_{i,j=1}^{3} e_{ij}^2 + \frac{\sigma}{1-2\sigma}(e_{11} + e_{22} + e_{33})^2\right], \tag{6.0.4}$$

see for example [82, 89]. Indeed, if we consider W as a function of the variables e_{ij}, $i, j = 1, 2, 3$ then

$$\begin{aligned}\frac{\partial W}{\partial e_{ij}} &= \frac{\sigma E}{(1-2\sigma)(1+\sigma)}(e_{11} + e_{22} + e_{33})\delta_{ij} + \frac{E}{(1+\sigma)}e_{ij} \\ &= \lambda(e_{11} + e_{22} + e_{33})\delta_{ij} + 2\mu e_{ij} = \sigma_{ij},\end{aligned}$$

for all $i, j = 1, 2, 3$. By replacing in the expression (6.0.4) for W the representations of the components of the strain tensor we obtain

$$\begin{aligned} W &= \frac{Ex_3^2}{2(1+\sigma)}\left[\left(\frac{\partial\beta_1}{\partial x_1}\right)^2+\left(\frac{\partial\beta_2}{\partial x_2}\right)^2+\frac{\sigma}{1-\sigma}(\operatorname{div}\beta)^2+\frac{1}{2}\left(\frac{\partial\beta_1}{\partial x_2}+\frac{\partial\beta_2}{\partial x_1}\right)^2\right] \\ &+\frac{E}{2(1+\sigma)}\left[\frac{k}{2}\left(\frac{\partial w}{\partial x_1}-\beta_1\right)^2+\frac{k}{2}\left(\frac{\partial w}{\partial x_2}-\beta_2\right)^2\right] \\ &= \frac{Ex_3^2}{2(1-\sigma^2)}[(1-\sigma)|\epsilon(\beta)|^2+\sigma(\operatorname{div}(\beta))^2] \\ &+\frac{E}{2(1+\sigma)}\left[\frac{k}{2}\left(\frac{\partial w}{\partial x_1}-\beta_1\right)^2+\frac{k}{2}\left(\frac{\partial w}{\partial x_2}-\beta_2\right)^2\right]. \end{aligned}$$

Hence the total elastic energy is given by

$$\begin{aligned} E_{el}(\beta,w) &= \int_{\Omega\times(-\frac{t}{2},\frac{t}{2})} W\, dx_1dx_2dx_3 \\ &= \frac{Et^3}{12(1-\sigma^2)}\int_\Omega \frac{1}{2}\left[(1-\sigma)|\epsilon(\beta)|^2+\sigma(\operatorname{div}(\beta))^2\right]dx_1dx_2 \\ &+\frac{Etk}{2(1+\sigma)}\int_\Omega\frac{1}{2}\left[\left(\frac{\partial w}{\partial x_1}-\beta_1\right)^2+\left(\frac{\partial w}{\partial x_2}-\beta_2\right)^2\right]dx_1dx_2. \end{aligned}$$

According to the notation used in (6.2.3) below, we may write

$$E_{el}(\beta,w)=\frac{t^3}{2}a(\beta,\beta)+\frac{Etk}{2(1+\sigma)}\int_\Omega\frac{1}{2}\left[\left(\frac{\partial w}{\partial x_1}-\beta_1\right)^2+\left(\frac{\partial w}{\partial x_2}-\beta_2\right)^2\right]dx_1dx_2.$$

If $\beta\in H^2(\Omega)^2\cap H^1_0(\Omega)^2$, by integration by parts it is easy to see that

$$\begin{aligned} a(\beta,\beta) &= -\frac{E}{24(1+\sigma)}\int_\Omega\Delta\beta\cdot\beta\, dx_1dx_2-\frac{E}{24(1-\sigma)}\int_\Omega\nabla(\operatorname{div}(\beta))\cdot\beta\, dx_1dx_2 \\ &= -\frac{\mu_1}{12}\int_\Omega\Delta\beta\cdot\beta\, dx_1dx_2-\frac{\mu_1+\mu_2}{12}\int_\Omega\nabla(\operatorname{div}(\beta))\cdot\beta\, dx_1dx_2 \end{aligned}$$

with $\mu_1=\frac{E}{2(1+\sigma)}$ and $\mu_2=\frac{\sigma E}{1-\sigma^2}$, see also Section 6.2.1. Hence, the total elastic energy E_{el} is the functional associated with the Reissner-Mindlin system (6.0.1).

Note that on the right-hand side of the first equation in (6.0.1) it appears a coefficient $\frac{t^2}{12}$, which induces a change in the natural $L^2(\Omega)^2$-inner product. In order to clarify the motivation behind this it is necessary to say something about the evolution problem associated with the system (6.0.1). In what follows we denote the time variable with τ and the mass density with ρ.

The kinetic energy is given by

$$E_K(\beta, w) = \frac{\rho t^3}{24} \int_\Omega \left| \frac{\partial \beta}{\partial \tau} \right|^2 dx_1 dx_2 + \frac{\rho t}{2} \int_\Omega \left(\frac{\partial w}{\partial \tau} \right)^2 dx_1 dx_2.$$

It follows that the evolution equations for the vibration of the plate are given by

$$\begin{cases} \frac{t^3 \rho}{12} \frac{\partial^2 \beta}{\partial \tau^2} - \frac{t^3 \mu_1}{12} \Delta \beta - \frac{t^3(\mu_1+\mu_2)}{12} \nabla(\mathrm{div}(\beta)) - \mu_1 k t (\nabla w - \beta) = 0, & \text{in } \Omega, \\ t\rho \frac{\partial^2 w}{\partial \tau^2} - \mu_1 k t \Delta w + \mu_1 k t \,\mathrm{div}(\beta) = 0, & \text{in } \Omega. \end{cases} \tag{6.0.5}$$

If one looks for a stationary wave solution of (6.0.5) in the form

$$\begin{cases} \beta(x_1, x_2, \tau) = \sin(\omega\tau)\, \beta_0(x_1, x_2), \\ w(x_1, x_2, \tau) = \sin(\omega\tau)\, w_0(x_1, x_2), \end{cases}$$

we see that (β_0, w_0) solves

$$\begin{cases} -\frac{t^3 \mu_1}{12} \Delta \beta_0 - \frac{t^3(\mu_1+\mu_2)}{12} \nabla(\mathrm{div}(\beta_0)) - \mu_1 k t (\nabla w_0 - \beta_0) = \omega^2 \frac{t^3 \rho}{12} \beta_0, & \text{in } \Omega, \\ -\mu_1 k t \Delta w_0 + \mu_1 k t \,\mathrm{div}\, \beta_0 = \omega^2 t \rho w_0, & \text{in } \Omega. \end{cases}$$

By setting $\lambda = \frac{\rho\omega^2}{t^2}$ and by dividing both equations by t^3 we find (6.0.1).

6.1 A general Korn's inequality

A fundamental tool to prove coercivity results for the Reissner-Mindlin system on standard Sobolev spaces is the Korn's inequality. The classical version of this inequality is given by

$$\|\nabla \beta\|_{L^2(\Omega)^N} \leq C_\Omega \|\epsilon(\beta)\|_{L^2(\Omega)^N},$$

for all $\beta \in H^1_0(\Omega)^N$, where $\epsilon(\beta) = \frac{1}{2}(\nabla\beta + (\nabla\beta)^T)$ is the symmetric part of the Jacobian matrix associated with β. However, in applications is sometimes useful to have a Korn's inequality for functions satisfying different costraints on the boundary of Ω. We recall here a result in this direction.

Let us define $so(N)$ to be the subset of $\mathbb{R}^{N\times N}$ of skew-symmetric matrices with constant entries. Then we have the following

Theorem 6.1.1. *Let $1 < p < \infty$. Let Ω be a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary and let $u \in W^{1,p}(\Omega)^N$. Let V be a weakly closed linear space of $W^{1,p}(\Omega)^N$ such that $V \cap \mathfrak{M} = \emptyset$, where*

$$\mathfrak{M} = \{\Psi(x) \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N) : \Psi(x) = Ax + b, A \in so(N), b \in \mathbb{R}^N\}.$$

Assume that $u \in V$. Then

$$\int_{\Omega} |\nabla u|^p dx \leq C \int_{\Omega} |\epsilon(u)|^p, \tag{6.1.1}$$

where the constant C depends only on Ω.

Proof. We refer to [81, Theorem 2, §2]. □

6.2 The Reissner-Mindlin system

Let Ω be a bounded and Lipschitz domain of $\mathbb{R}^2$. Let $B \subset \partial\Omega$ be a non-empty open set. Let us define

$$H^1_B(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } B\}. \tag{6.2.1}$$

Note that by Poincaré inequality the norm $\|\nabla u\|_{L^2(\Omega)}$ is equivalent to the standard $H^1(\Omega)$ norm on $H^1_B(\Omega)$. We define the following problem:
Find a non-trivial $(\beta, w) \in H^1_B(\Omega)^2 \times H^1_B(\Omega)$ such that

$$a(\beta, \eta) + \frac{Ek}{2(1+\sigma)t^2}(\nabla w - \beta, \nabla v - \eta) = \lambda \left[(w, v) + \frac{t^2}{12}(\beta, \eta) \right], \tag{6.2.2}$$

for all $(\eta, v) \in H^1_B(\Omega)^2 \times H^1_B(\Omega)$. Here $a(\cdot,\cdot)$ from $H^1_B(\Omega)^2 \times H^1_B(\Omega)^2$ to $\mathbb{R}$ is the elliptic bilinear form defined by

$$a(\beta, \eta) = \frac{E}{12(1-\sigma^2)} \int_{\Omega} \Big((1-\sigma)\epsilon(\beta) : \epsilon(\eta) + \sigma \operatorname{div}(\beta) \operatorname{div}(\eta) \Big) dx, \tag{6.2.3}$$

where $\epsilon(\cdot)$ is the linear strain tensor defined by

$$\epsilon(\beta)_{ij} = \frac{1}{2} \left(\frac{\partial \beta^i}{\partial x_j} + \frac{\partial \beta^j}{\partial x_i} \right)_{i,j},$$

for all $i, j \in \{1, 2\}$, E is the Young modulus and σ is the Poisson ratio. Recall that by the *Korn's inequality* the bilinear form $a(\cdot,\cdot)$ is coercive on $H^1_B(\Omega)$ for all $\sigma \in (-1, 1)$. Indeed, we first note that we have the following Korn's inequality:

$$\|\nabla \beta\|_{L^2(\Omega)} \leq C_{\Omega} \|\epsilon(\beta)\|_{L^2(\Omega)}, \tag{6.2.4}$$

for all $\beta \in H^1_B(\Omega)^2$, where the constant $C_{\Omega} > 0$ depends only on Ω. We remark that inequality (6.2.4) is a consequence of Theorem 6.1.1, because the rigid translations are not in $H^1_B(\Omega)$. Then the coercivity of a for $\sigma \in [0, 1[$ follows directly from

(6.2.4). The coercivity of a for negative values of the Poisson ratio is obtained instead by noticing that if $\sigma < 0$, then

$$\begin{aligned}\int_\Omega (1-\sigma)|\epsilon(\beta)|^2 + \sigma|\operatorname{div}\beta|^2 dx &\geq \int_\Omega (1-\sigma)|\epsilon(\beta)|^2 + 2\sigma|\epsilon(\beta)|^2 dx \\ &= \int_\Omega (1+\sigma)|\epsilon(\beta)|^2 dx,\end{aligned}$$

for all $\beta \in H^1_B(\Omega)^2$, due to the elementary inequality

$$|\operatorname{div}\beta|^2 \leq 2|\epsilon(\beta)|^2.$$

Then, for $\sigma < 0$, the bilinear form $a(\cdot,\cdot)$ is coercive when $\sigma > -1$, because with this choice of σ, a is estimated from below by a positive multiple of the quadratic form $\int_\Omega |\epsilon(\beta)|^2 dx$ which is coercive (it coincides with a when $\sigma = 0$).
Let n be the unit outer normal vector to $\partial\Omega$ and let s be the tangential unit vector obtained by rotating n by $\pi/2$ anticlockwise. By integration by parts (see §6.2.1 below for more details) one can prove that the classical formulation of Problem (6.2.2) is given by

$$\begin{cases} -\frac{\mu_1}{12}\Delta\beta - \frac{\mu_1+\mu_2}{12}\nabla(\operatorname{div}\beta) - \frac{\mu_1 k}{t^2}(\nabla w - \beta) = \lambda\frac{t^2}{12}\beta, & \text{in } \Omega, \\ -\frac{\mu_1 k}{t^2}(\Delta w - \operatorname{div}\beta) = \lambda w, & \text{in } \Omega, \\ \frac{\mu_1}{6}(n^T\epsilon(\beta)n) + \frac{\mu_2}{12}(\operatorname{div}\beta) = 0, & \text{on } \partial\Omega\setminus B, \\ s^T\epsilon(\beta)n + n^T\epsilon(\beta)s = 0, & \text{on } \partial\Omega\setminus B, \\ \frac{\mu_1 k}{t^2}(\nabla w - \beta)\cdot n = 0, & \text{on } \partial\Omega\setminus B, \\ \beta = w = 0, & \text{on } B, \end{cases} \tag{6.2.5}$$

where $\mu_1 = \frac{E}{2(1+\sigma)}$ is one of the Lamé coefficients (which in literature is usually denoted by μ), while $\mu_2 = \frac{\sigma E}{1-\sigma^2}$ is related to the other Lamé coefficient λ via the equality $\mu_2 = \lambda\frac{1-2\sigma}{1-\sigma}$.

We now recall the geometrical setting of the problem in the dumbbell domain Ω_δ, see also §5.1. Let $\delta \in \mathbb{R}$, $\delta > 0$, a small positive real number. Given two bounded smooth domains Ω_L, Ω_R in $\mathbb{R}^2$ with $\Omega_L \cap \Omega_R = \emptyset$ such that

$$\partial\Omega_L \supset \{(0,y) \in \mathbb{R}^2 : -1 < y < 1\}, \qquad \partial\Omega_R \supset \{(1,y) \in \mathbb{R}^2 : -1 < y < 1\},$$

and $(\Omega_R \cup \Omega_L) \cap ([0,1] \times [-1,1]) = \emptyset$, we set

$$\Omega = \Omega_L \cup \Omega_R, \text{ and } \Omega_\delta = \Omega \cup R_\delta \cup L_\delta\,,$$

for all $\delta > 0$ small enough. Here $R_\delta \cup L_\delta$ is a thin channel connecting Ω_L and Ω_R defined by

$$R_\delta = \{(x,y) \in \mathbb{R}^2 : x \in (0,1), 0 < y < \delta g(x)\}, \tag{6.2.6}$$

$$L_\delta = (\{0\} \times (0, \delta g(0)) \cup (\{1\} \times (0, \delta g(1)))),$$

where $g \in C^2[0,1]$ is a positive real-valued function. Let us also define

$$\Gamma_\delta = \partial R_\delta \setminus \overline{L_\delta}.$$

Note that Ω_δ collapses to the limit set $\Omega_0 = \Omega \cup ([0,1] \times \{0\})$ as $\delta \to 0$.

6.2.1 Possible boundary conditions

In this section we briefly recall various possible boundary conditions for the Reissner-Mindlin system discussed in the literature (see e.g., [9]). Let us define the shear modulus κ by

$$\kappa = \frac{Ek}{2(1+\sigma)}.$$

We will consider

$$a(\beta,\eta) + \frac{\kappa}{t^2}(\nabla w - \beta, \nabla v - \eta) = \lambda \left[(w,v) + \frac{t^2}{12}(\beta,\eta) \right],$$

where a is the continuous bilinear defined in (6.2.3), $\beta, \eta \in V$, $w, v \in W$, and V is a suitable subspace of $H^1_B(\Omega)^2$, W is a subspace of $H^1_B(\Omega)$, both containing $C^\infty_c(\Omega)$. By integration by parts we then deduce that

$$\begin{aligned}
&-\frac{E}{24(1+\sigma)}\int_\Omega \Delta\beta\cdot\eta - \frac{E}{24(1-\sigma)}\int_\Omega \nabla(\operatorname{div}\beta)\cdot\eta \\
&+\frac{E}{24(1+\sigma)}\int_{\partial\Omega}\left(\eta^T\epsilon(\beta)n + n^T\epsilon(\beta)\eta\right)dS + \frac{E\sigma}{12(1-\sigma^2)}\int_{\partial\Omega}\operatorname{div}\beta(\eta\cdot n)dS \\
&-\frac{\kappa}{t^2}\int_\Omega(\Delta w - \operatorname{div}\beta)v\,dx - \frac{\kappa}{t^2}\int_\Omega(\nabla w - \beta)\cdot\eta\,dx + \frac{\kappa}{t^2}\int_{\partial\Omega}(\nabla w - \beta)\cdot n\,v\,dS \\
&= \lambda\left(\int_\Omega wv + \frac{t^2}{12}\beta\cdot\eta\,dx\right) \qquad (6.2.7)
\end{aligned}$$

for all $\eta \in V$, $v \in W$. Depending on the choice of the subspaces V, W we now find different boundary conditions.

Case I : Hard clamped boundary conditions.
In this case $V = (H^1_0(\Omega))^2$, $W = H^1_0(\Omega)$. In particular, since both η and v vanish on $\partial\Omega$, the boundary integrals in (6.2.7) vanish. We deduce that the strong formulation of Problem (6.2.7) is

$$\begin{cases}
-\frac{E}{24(1+\sigma)}\Delta\beta - \frac{E}{24(1-\sigma)}\nabla(\operatorname{div}\beta) - \frac{\kappa}{t^2}(\nabla w - \beta) = \frac{\lambda t^2}{12}\beta, & \text{in } \Omega, \\
-\frac{\kappa}{t^2}(\Delta w - \operatorname{div}\beta) = \lambda w, & \text{in } \Omega, \\
\beta = 0 = w, & \text{on } \partial\Omega.
\end{cases}$$

Case II : Soft clamped boundary conditions
In this case $V = \{\Phi \in (H^1_B(\Omega))^2 : \Phi \cdot n = 0 \text{ on } \partial\Omega\}$, $W = H^1_0(\Omega)$. The boundary integral involving v vanishes since $v = 0$ on $\partial\Omega$. The boundary integral involving $\eta \cdot n$ vanishes as well. We are then left with the other boundary integral, which gives the condition

$$s^T \epsilon(\beta) n + n^T \epsilon(\beta) s = 0, \qquad \text{on } \partial\Omega \setminus B.$$

Hence the strong formulation of Problem (6.2.7) is

$$\begin{cases} -\frac{E}{24(1+\sigma)}\Delta\beta - \frac{E}{24(1-\sigma)}\nabla(\operatorname{div}\beta) - \frac{\kappa}{t^2}(\nabla w - \beta) = \frac{\lambda t^2}{12}\beta, & \text{in } \Omega, \\ -\frac{\kappa}{t^2}(\Delta w - \operatorname{div}\beta) = \lambda w, & \text{in } \Omega, \\ s^T \epsilon(\beta) n + n^T \epsilon(\beta) s = 0, & \text{on } \partial\Omega \setminus B, \\ w = 0 = \beta \cdot n, & \text{on } \partial\Omega \setminus B, \\ w = 0 = \beta, & \text{on } B, \end{cases}$$

Case III: Hard simply supported boundary conditions
In this case $V = \{\Phi \in (H^1_B(\Omega))^2 : \Phi \cdot s = 0 \text{ on } \partial\Omega\}$, $W = H^1_0(\Omega)$. The boundary integral involving v vanishes since $v = 0$ on $\partial\Omega$. The other two boundary integrals do not vanish, hence they yield the following boundary condition (note that $\eta \in V$, hence $\eta|_{\partial\Omega\setminus B}$ is normal to $\partial\Omega$):

$$(1-\sigma) n^T \epsilon(\beta) n + \sigma \operatorname{div}\beta = 0, \qquad \text{on } \partial\Omega \setminus B.$$

Hence the strong formulation of Problem (6.2.7) is

$$\begin{cases} -\frac{E}{24(1+\sigma)}\Delta\beta - \frac{E}{24(1-\sigma)}\nabla(\operatorname{div}\beta) - \frac{\kappa}{t^2}(\nabla w - \beta) = \frac{\lambda t^2}{12}\beta, & \text{in } \Omega, \\ -\frac{\kappa}{t^2}(\Delta w - \operatorname{div}\beta) = \lambda w, & \text{in } \Omega, \\ (1-\sigma) n^T \epsilon(\beta) n + \sigma \operatorname{div}\beta = 0, & \text{on } \partial\Omega \setminus B, \\ w = 0 = \beta \cdot s, & \text{on } \partial\Omega \setminus B, \\ w = 0 = \beta, & \text{on } B. \end{cases}$$

Case IV: Soft simply supported boundary conditions
In this case $V = (H^1_B(\Omega))^2$, $W = H^1_0(\Omega)$. The boundary integral involving v vanishes since $v = 0$ on $\partial\Omega$. The other two boundary integrals do not vanish, hence they yield the following boundary conditions:

$$(1-\sigma) n^T \epsilon(\beta) n + \sigma \operatorname{div}\beta = 0, \qquad s^T \epsilon(\beta) n + n^T \epsilon(\beta) s = 0, \qquad \text{on } \partial\Omega \setminus B.$$

To see this it is sufficient to decompose the vector field η in its tangential and normal components using the formula $\eta|_{\partial\Omega} = (\eta \cdot s)s + (\eta \cdot n)n$. Hence the strong formulation of Problem (6.2.7) is

$$\begin{cases} -\frac{E}{24(1+\sigma)}\Delta\beta - \frac{E}{24(1-\sigma)}\nabla(\operatorname{div}\beta) - \frac{\kappa}{t^2}(\nabla w - \beta) = \frac{\lambda t^2}{12}\beta, & \text{in } \Omega,\\ -\frac{\kappa}{t^2}(\Delta w - \operatorname{div}\beta) = \lambda w, & \text{in } \Omega,\\ (1-\sigma)n^T\epsilon(\beta)n + \sigma \operatorname{div}\beta = 0, & \text{on } \partial\Omega \setminus B,\\ s^T\epsilon(\beta)n + n^T\epsilon(\beta)s = 0, & \text{on } \partial\Omega \setminus B,\\ w = 0, & \text{on } \partial\Omega,\\ \beta = 0, & \text{on } B. \end{cases}$$

Case V : Neumann boundary conditions
In this case $V = (H^1_B(\Omega))^2$, $W = H^1_B(\Omega)$. This implies that all the boundary integrals in (6.2.7) are non-vanishing, hence (arguing as in Case IV for η) they give the boundary conditions

$$(1-\sigma)n^T\epsilon(\beta)n + \sigma \operatorname{div}\beta = 0, \qquad s^T\epsilon(\beta)n + n^T\epsilon(\beta)s = 0, \qquad \text{on } \partial\Omega \setminus B,$$

and

$$(\nabla w - \beta)\cdot n = 0, \qquad \text{on } \partial\Omega \setminus B.$$

We deduce that the strong formulation of Problem (6.2.7) is

$$\begin{cases} -\frac{E}{24(1+\sigma)}\Delta\beta - \frac{E}{24(1-\sigma)}\nabla(\operatorname{div}\beta) - \frac{\kappa}{t^2}(\nabla w - \beta) = \frac{\lambda t^2}{12}\beta, & \text{in } \Omega,\\ -\frac{\kappa}{t^2}(\Delta w - \operatorname{div}\beta) = \lambda w, & \text{in } \Omega,\\ (1-\sigma)n^T\epsilon(\beta)n + \sigma \operatorname{div}\beta = 0, & \text{on } \partial\Omega \setminus B,\\ s^T\epsilon(\beta)n + n^T\epsilon(\beta)s = 0, & \text{on } \partial\Omega \setminus B,\\ \frac{\partial w}{\partial n} - \beta\cdot n = 0, & \text{on } \partial\Omega \setminus B,\\ w = 0 = \beta, & \text{on } B. \end{cases}$$

Remark 6.2.1. Keeping in mind that Problem (6.2.2) behaves like the biharmonic operator as $t \to 0$ (see [9]) we can classify the boundary conditions for the Reissner-Mindlin in three big families. Boundary conditions of type I and II are of Dirichlet type because the associated problem converges as $t \to 0$ to the Dirichlet problem for the biharmonic operator (limiting energy space $H^2_0(\Omega)$). Boundary conditions of type III and IV are of intermediate type because the associated problem converges as $t \to 0$ to the intermediate boundary value problem for the biharmonic operator (limiting energy space $H^2(\Omega) \cap H^1_0(\Omega)$). Finally boundary conditions of type V are of Neumann type because the associated problem converges as $t \to 0$ to the Neumann problem for the biharmonic operator (limiting energy space $H^2(\Omega)$).
According to this distinction it is possible to prove that only the Reissner-Mindlin system with boundary conditions of type V exhibits spectral instability phenomena in the dumbbell domain Ω_δ as $\delta \to 0$, allowing the appearance of extra eigenvalues coming from the channel as $\delta \to 0$.

6.3 Spectral decomposition results

In this Section we show that the eigenvalues and the eigenfunction of the Reissner-Mindlin system with free boundary conditions defined by

$$\begin{cases} -\frac{E}{24(1+\sigma)}\Delta\beta - \frac{E}{24(1-\sigma)}\nabla(\operatorname{div}\beta) - \frac{\kappa}{t^2}(\nabla w - \beta) = \frac{\lambda t^2}{12}\beta, & \text{in } \Omega_\delta, \\ -\frac{\kappa}{t^2}(\Delta w - \operatorname{div}\beta) = \lambda w, & \text{in } \Omega_\delta, \\ (1-\sigma)n^T\epsilon(\beta)n + \sigma\operatorname{div}\beta = 0, & \text{on } \partial\Omega_\delta \setminus B, \\ s^T\epsilon(\beta)n + n^T\epsilon(\beta)s = 0, & \text{on } \partial\Omega_\delta \setminus B, \\ \frac{\partial w}{\partial n} - \beta\cdot n = 0, & \text{on } \partial\Omega_\delta \setminus B, \\ w = 0 = \beta, & \text{on } B, \end{cases} \tag{6.3.1}$$

with eigenpairs $(\lambda_n(\Omega_\delta), (\varphi_n^\delta, w_n^\delta))_{n\geq1}$ are asymptotically close as $\delta \to 0$ to either the eigenpairs of

$$\begin{cases} -\frac{E}{24(1+\sigma)}\Delta\beta - \frac{E}{24(1-\sigma)}\nabla(\operatorname{div}\beta) - \frac{\kappa}{t^2}(\nabla w - \beta) = \frac{\omega t^2}{12}\beta, & \text{in } \Omega, \\ -\frac{\kappa}{t^2}(\Delta w - \operatorname{div}\beta) = \omega w, & \text{in } \Omega, \\ (1-\sigma)n^T\epsilon(\beta)n + \sigma\operatorname{div}\beta = 0, & \text{on } \partial\Omega \setminus B, \\ s^T\epsilon(\beta)n + n^T\epsilon(\beta)s = 0, & \text{on } \partial\Omega \setminus B, \\ \frac{\partial w}{\partial n} - \beta\cdot n = 0, & \text{on } \partial\Omega \setminus B, \\ w = 0 = \beta, & \text{on } B, \end{cases} \tag{6.3.2}$$

denoted by $(\omega_n, (\varphi_n^\Omega, w_n^\Omega))_{n\geq1}$, or to the eigenpairs $(\theta_n^\delta, (\gamma_n^\delta, u_n^\delta))_{n\geq1}$ of

$$\begin{cases} -\frac{E}{24(1+\sigma)}\Delta\beta - \frac{E}{24(1-\sigma)}\nabla(\operatorname{div}\beta) - \frac{\kappa}{t^2}(\nabla w - \beta) = \frac{\theta t^2}{12}\beta, & \text{in } R_\delta, \\ -\frac{\kappa}{t^2}(\Delta w - \operatorname{div}\beta) = \theta w, & \text{in } R_\delta, \\ \sigma\operatorname{div}\beta + (1-\sigma)n^T\epsilon(\beta)n = 0, & \text{on } \Gamma_\delta, \\ s^T\epsilon(\beta)n + n^T\epsilon(\beta)s = 0, & \text{on } \Gamma_\delta, \\ \frac{\partial w}{\partial n} - \beta\cdot n = 0, & \text{on } \Gamma_\delta, \\ w = 0 = \beta, & \text{on } L_\delta. \end{cases} \tag{6.3.3}$$

Note that, since Ω_δ, Ω and R_δ are sufficiently regular, by standard spectral theory for differential operators it follows that the operators associated with the quadratic forms appearing in the weak formulation of problems (6.3.1), (6.3.2), (6.3.3) have compact resolvents, hence it makes sense to define the sequences of eigenvalues $\lambda_n(\Omega_\delta)$, ω_n, and θ_n^δ as we did above. We define $(\lambda_n^\delta)_{n\geq1} = (\omega_k)_{k\geq1} \cup (\theta_l^\delta)_{l\geq1}$, where it is understood that the eigenvalues are arranged in increasing order and repeated according to their multiplicity. For each λ_n^δ we define the functions

$(\phi_n^\delta, v_n^\delta) \in (H^1(\Omega)^2 \times H^1(\Omega)) \oplus ((H^1(R_\epsilon))^2 \times H^1(R_\epsilon))$ in the following way:

$$(\phi_n^\delta, v_n^\delta) = \begin{cases} (\varphi_k^\Omega, w_k^\Omega), & \text{in } \Omega, \\ 0, & \text{in } R_\delta, \end{cases} \tag{6.3.4}$$

if $\lambda_n^\delta = \omega_k$, for some $k \in \mathbb{N}$; otherwise

$$(\phi_n^\delta, v_n^\delta) = \begin{cases} 0, & \text{in } \Omega, \\ (\gamma_l^\delta, u_n^\delta), & \text{in } R_\delta, \end{cases} \tag{6.3.5}$$

if $\lambda_n^\delta = \theta_l^\delta$, for some $l \in \mathbb{N}$. Moreover, we define a sequence of functions in $H^1(\Omega_\delta)^2 \times H^1(\Omega_\delta)$ by setting

$$(\xi_n^\delta, \zeta_n^\delta) = \begin{cases} (E\varphi_k^\Omega, Ew_k^\Omega), & \text{if } \lambda_n^\delta = \omega_k, \\ (\phi_n^\delta, w_n^\delta), & \text{if } \lambda_n^\delta = \theta_l^\delta, \end{cases}$$

where E is a linear continuous extension operator mapping $H^1(\Omega)^3$ to $H^1(\mathbb{R}^N)^3$.

Definition 6.3.1. We denote by $(H^1_{L_\delta}(R_\delta))^3$ the space obtained as the closure in $(H^1(R_\delta))^3$ of $(C^\infty(\overline{R_\delta}))^3$ functions which vanish in a neighbourhood of L_δ. Furthermore, for any Lipschitz bounded open set U we define

$$[f]_{(H^1_{RM}(U))^3} = \left| \frac{E}{12(1-\sigma^2)} \Big((1-\sigma) \|\epsilon(\bar{f})\|\|^2_{(L^2(U))^2} + \sigma \|\mathrm{div}(\bar{f})\|^2_{(L^2(U))^2} \Big) + \frac{Ek}{2(1+\sigma)t^2} \|\nabla f^3 - \bar{f}\|^2_{L^2(U)} \right|^{1/2},$$

for all $f = (\bar{f}, f^3) \in (H^1(U))^3$.

Note the functions u in $(H^1_{L_\delta}(R_\delta))^3$ satisfy the condition $u = 0$ on L_δ in the sense of traces.

Definition 6.3.2 (H-Condition)**.** We say that the family of dumbbell domains Ω_δ, $\delta > 0$, satisfies the H-Condition if, given functions $u_\delta \in H^1(\Omega_\delta)^3$ such that $\|u_\delta\|_{H^1(\Omega_\delta)^3} \le R$ for all $\delta > 0$, there exist functions $\mathscr{U}_\delta \in H^1_{L_\delta}(R_\delta)^3$ such that

(i) $\|u_\delta - \mathscr{U}_\delta\|_{L^2(R_\delta)^3} \to 0$ as $\delta \to 0$,

(ii) $[\mathscr{U}_\delta]^2_{H^1_{RM}(R_\delta)} \le [u_\delta]^2_{H^1_{RM}(\Omega_\delta)} + o(1)$ as $\delta \to 0$.

Recall that $[\cdot]_{H^1_{RM}}$ is defined above in Definition 6.3.1. We remark that with arguments similar to those in Chapter 5 it is possible to prove that channels R_δ with the (MP) property (see §5.3) satisfy the H-Condition.

If x_δ divides the spectrum we define the projector P_{x_δ} from $L^2(\Omega_\delta)^3$ onto the linear span $[\phi_1^\delta, \dots, \phi_{N(x_\delta)}^\delta]$ of the first $N(x_\delta)$ eigenfunctions by

$$P_{x_\delta} g = \sum_{i=1}^{N(x_\delta)} (g, \phi_i^\delta)_{L^2(\Omega_\delta)} \phi_i^\delta ,$$

for all $g \in L^2(\Omega_\delta)^3$.

With these definitions it is possible to prove the following result

Theorem 6.3.3 (Decomposition of the eigenvalues)**.** *Let Ω_δ, $\delta > 0$, be a family of dumbbell domains satisfying the H-Condition. Then the following statements hold:*

(i) $\lim_{\delta \to 0} |\lambda_n(\Omega_\delta) - \lambda_n^\delta| = 0$, *for all $n \in \mathbb{N}$.*

(ii) For any x_δ dividing the spectrum,

$$\lim_{\delta \to 0} \|(\varphi_{r_\delta}^\delta, w_{r_\delta}^\delta) - P_{x_\delta}(\varphi_{r_\delta}^\delta, w_{r_\delta}^\delta)\|_{H^1(\Omega)^3 \oplus H^1(R_\delta)^3} = 0,$$

for all $r_\delta = 1, \dots, N(x_\delta)$.

Proof. The proof follows from a straightforward adaptation of the methods in Chapter 5. Indeed, it is sufficient to replace the quadratic form associated with the biharmonic operator with the quadratic form associated with the Reissner-Mindlin system and to replace the scalar eigenfunctions of the biharmonic operator with the vectors of eigenfunctions of the Reissner-Mindlin system. □

6.4 The effect of shrinking the channel at fixed thickness

In this section we pass to the limit as $\delta \to 0$ in Problem (6.3.3), while $t > 0$ remains fixed. Since R_δ is a thin domain, it is convenient to apply rescaling techniques in the spirit of [73, 74, 77]. We transform Problem (6.3.3) to another differential problem in the fixed domain R_1 by means of the pullback associated with the function $(x, y) \mapsto (x, y/\delta)$. Moreover, we denote with the symbol $\sim$ the pullback of the functions and of the operators with respect to this map. Namely, we write $\tilde{\beta} = \beta(x, y/\delta)$ for all $(x, y) \in R_\delta$, $\tilde{\nabla} = (\partial_x, \delta^{-1}\partial_y)$ and $\tilde{\varepsilon} = \frac{1}{2}(\tilde{\nabla} + \tilde{\nabla}^T)$. We obtain the following problem

$$\begin{cases} -\frac{\mu_1}{12}\left(\partial_{xx}\tilde{\beta}+\frac{1}{\delta^2}\partial_{yy}\tilde{\beta}\right)-\frac{\mu_1+\mu_2}{12}\tilde{\nabla}(\partial_x\tilde{\beta}^1+\frac{1}{\delta}\partial_y\tilde{\beta}^2)-\frac{\mu_1 k}{t^2}(\tilde{\nabla}\tilde{w}-\tilde{\beta})=\theta\frac{t^2}{12}\tilde{\beta}, & \text{in } R_1,\\ -\frac{\mu_1 k}{t^2}(\partial_{xx}\tilde{w}+\frac{1}{\delta^2}\partial_{yy}\tilde{w}-\partial_x\tilde{\beta}^1-\frac{1}{\delta}\partial_y\tilde{\beta}^2)=\theta\tilde{w}, & \text{in } R_1,\\ \frac{\mu_1}{6}(\tilde{n}^T\tilde{\epsilon}(\tilde{\beta})\tilde{n})+\frac{\mu_1+\mu_2}{12}(\partial_x\tilde{\beta}^1+\frac{1}{\delta}\partial_y\tilde{\beta}^2)=0, & \text{on } \Gamma_1,\\ \tilde{s}^T\tilde{\epsilon}(\tilde{\beta})\tilde{n}+\tilde{n}^T\tilde{\epsilon}(\tilde{\beta})\tilde{s}=0, & \text{on } \Gamma_1,\\ \frac{\mu_1 k}{t^2}(\tilde{\nabla}\tilde{w}-\tilde{\beta})\cdot\tilde{n}=0, & \text{on } \Gamma_1,\\ \tilde{\beta}=\tilde{w}=0, & \text{on } L_1, \end{cases} \tag{6.4.1}$$

which can be more explicitly written as a system of three PDEs in the unknowns $(\tilde{\beta}^1,\tilde{\beta}^2,\tilde{w})$ given by

$$\begin{cases} -\frac{\mu_1}{12}\left(\partial_{xx}\tilde{\beta}^1+\frac{1}{\delta^2}\partial_{yy}\tilde{\beta}^1\right)-\frac{\mu_1+\mu_2}{12}\left(\partial_{xx}\tilde{\beta}^1+\frac{1}{\delta}\partial_{xy}\tilde{\beta}^2\right)-\frac{\mu_1 k}{t^2}(\partial_x\tilde{w}-\tilde{\beta}^1)=\theta\frac{t^2}{12}\tilde{\beta}^1, & \text{in } R_1,\\ -\frac{\mu_1}{12}\left(\partial_{xx}\tilde{\beta}^2+\frac{1}{\delta^2}\partial_{yy}\tilde{\beta}^2\right)-\frac{\mu_1+\mu_2}{12}\left(\frac{1}{\delta}\partial_{xy}\tilde{\beta}^1+\frac{1}{\delta^2}\partial_{yy}\tilde{\beta}^2\right)-\frac{\mu_1 k}{t^2}(\frac{1}{\delta}\partial_y\tilde{w}-\tilde{\beta}^2)=\theta\frac{t^2}{12}\tilde{\beta}^2, & \text{in } R_1,\\ -\frac{\mu_1 k}{t^2}(\partial_{xx}\tilde{w}+\frac{1}{\delta^2}\partial_{yy}\tilde{w}-\partial_x\tilde{\beta}^1-\frac{1}{\delta}\partial_y\tilde{\beta}^2)=\theta\tilde{w}, & \text{in } R_1,\\ \frac{\mu_1}{6}(\tilde{n}^T\tilde{\epsilon}(\tilde{\beta})\tilde{n})+\frac{\mu_1+\mu_2}{12}(\partial_x\tilde{\beta}^1+\frac{1}{\delta}\partial_y\tilde{\beta}^2)=0, & \text{on } \Gamma_1,\\ \tilde{s}^T\tilde{\epsilon}(\tilde{\beta})\tilde{n}+\tilde{n}^T\tilde{\epsilon}(\tilde{\beta})\tilde{s}=0, & \text{on } \Gamma_1,\\ \frac{\mu_1 k}{t^2}(\tilde{\nabla}\tilde{w}-\tilde{\beta})\cdot\tilde{n}=0, & \text{on } \Gamma_1,\\ \tilde{\beta}=\tilde{w}=0, & \text{on } L_1. \end{cases} \tag{6.4.2}$$

The weak formulation of this problem is

$$\tilde{a}(\tilde{\beta},\eta)+\frac{Ek}{2(1-\sigma)t^2}(\tilde{\nabla}\tilde{w}-\tilde{\beta},\tilde{\nabla}\tilde{v}-\eta)=\theta\left[(\tilde{w},v)+\frac{t^2}{12}(\tilde{\beta},\eta)\right], \tag{6.4.3}$$

for all $\eta\in(H^1_{L_1}(R_1))^2$, for all $w\in H^1_{L_1}(R_1)$ where

$$\tilde{a}(\tilde{\beta},\eta)=\frac{E}{12(1-\sigma^2)}\int_{R_1}(1-\sigma)\tilde{\epsilon}\,(\tilde{\beta}):\tilde{\epsilon}(\eta)+\sigma\,\tilde{\operatorname{div}}\tilde{\beta}\,\tilde{\operatorname{div}}\eta\,dx, \tag{6.4.4}$$

which can be rewritten as

$$\begin{aligned}\tilde{a}(\tilde{\beta},\eta)=\frac{E}{12(1-\sigma^2)}\int_{R_1}(1-\sigma)\Bigg[&\frac{\partial\tilde{\beta}^1}{\partial x}\frac{\partial\eta^1}{\partial x}+\frac{1}{2}\left(\frac{\partial\tilde{\beta}^2}{\partial x}+\frac{1}{\delta}\frac{\partial\tilde{\beta}^1}{\partial y}\right)\left(\frac{\partial\eta^2}{\partial x}+\frac{1}{\delta}\frac{\partial\eta^1}{\partial y}\right)\\ &+\frac{1}{\delta^2}\frac{\partial\tilde{\beta}^2}{\partial y}\frac{\partial\eta^2}{\partial y}\Bigg]+\sigma\left[\left(\frac{\partial\tilde{\beta}^1}{\partial x}+\frac{1}{\delta}\frac{\partial\tilde{\beta}^2}{\partial y}\right)\left(\frac{\partial\eta^1}{\partial x}+\frac{1}{\delta}\frac{\partial\eta^2}{\partial y}\right)\right]dxdy.\end{aligned} \tag{6.4.5}$$

Let $f\in L^2(R_1)$ and $F\in(L^2(R_1))^2$. Let us consider the Poisson problem given by

$$\tilde{a}(\tilde{\beta},\eta)+\frac{Ek}{2(1-\sigma)t^2}(\tilde{\nabla}\tilde{w}-\tilde{\beta},\tilde{\nabla}\tilde{v}-\eta)=(f,v)+\frac{t^2}{12}(F,\eta). \tag{6.4.6}$$

By (6.4.6) we deduce the apriori estimate

$$\frac{E}{12(1-\sigma^2)}\int_{R_1}\left((1-\sigma)|\tilde{\epsilon}(\tilde{\beta})|^2+\sigma|\tilde{\operatorname{div}}\tilde{\beta}|^2+\frac{Ek}{2(1-\sigma)t^2}|\tilde{\nabla}\tilde{w}-\tilde{\beta}|^2\right)dxdy \\ \leq C\left(\|f\|^2_{L^2(R_1)}+\frac{t^2}{12}\|F\|^2_{L^2(R_1)}\right). \quad (6.4.7)$$

In particular, by Korn's inequality (6.1.1), we have

$$\|\tilde{\nabla}\beta\|^2_{L^2(R_1)} \leq C\Big((1-\sigma)\|\tilde{\epsilon}(\tilde{\beta})\|^2_{L^2(R_1)}+\sigma\|\tilde{\operatorname{div}}\tilde{\beta}\|^2_{L^2(R_1)}\Big),$$

for all $\delta > 0$. By the Poincaré inequality, this implies that $(\tilde{\beta}_\delta)_\delta$ is a bounded sequence in $H^1(R_1)^2$, hence it has a weakly convergent subsequence to $\beta_0 \in H^1(R_1)^2$. Moreover, since

$$\int_{R_1}\left|\frac{1}{\delta}\frac{\partial\tilde{\beta}^i}{\partial y}\right|^2 dxdy < C, \quad \text{for all } \delta > 0,\, i = 1, 2,$$

we deduce that

$$\int_{R_1}\left|\frac{\partial\tilde{\beta}^i}{\partial y}\right|^2 dxdy = O(\delta^2), \quad \text{as } \delta \to 0,\, i = 1, 2,$$

and then we deduce that β_0 must be constant in y. Going back to the apriori estimate we deduce that

$$\int_{R_1}|\tilde{\nabla}\tilde{w}|^2\, dxdy \leq 2\int_{R_1}|\tilde{\nabla}\tilde{w}-\tilde{\beta}|^2 dxdy + 2\int_{R_1}|\tilde{\beta}|^2 dxdy \leq C,$$

for all $\delta > 0$. Thus, $(\tilde{w}_\delta)_\delta$ is a uniformly bounded sequence in $H^1(R_1)$ and we can extract a weakly convergent subsequence with limit $w_0 \in H^1(R_1)$. Again, since

$$\int_{R_1}\left|\frac{\partial\tilde{w}}{\partial y}\right|^2 dxdy = O(\delta^2), \quad \text{as } \delta \to 0,$$

we deduce that w_0 is constant in y. Note also that the sequence of functions $\delta^{-1}\partial_y\tilde{\beta}$ is a bounded sequence in $(L^2(R_1))^2$, hence there exists a function $u \in (L^2(R_1))^2$ such that, possibly passing to a subsequence, $\delta^{-1}\tilde{\beta} \rightharpoonup u$ in $(L^2(R_1))^2$ as $\delta \to 0$. In a similar way there exists a function $W \in L^2(R_1)$ such that, possibly passing to a subsequence, $\delta^{-1}\partial_y\tilde{w} \rightharpoonup W$ in $L^2(R_1)$ as $\delta \to 0$.

Now we are ready to pass to the limit as $\delta \to 0$ in (6.4.6). Since the limit functions

β_0 and w_0 are constant in y we are induced to choose test-functions not depending on y in (6.4.6). With this choice we deduce that

$$A(\tilde{\beta},\eta)+\frac{Ek}{2(1-\sigma)t^2}\int_{R_1}\left(\frac{\partial\tilde{w}}{\partial x}-\tilde{\beta}^1\right)\left(\frac{\partial v}{\partial x}-\eta^1\right)-\left(\frac{1}{\delta}\frac{\partial\tilde{w}}{\partial y}-\tilde{\beta}^2\right)\eta^2\,dxdy$$
$$=(f,v)+\frac{t^2}{12}(F,\eta), \quad (6.4.8)$$

for all $\eta \in (H^1((0,1);g(x)dx))^2$, for all $v \in H^1((0,1);g(x)dx)$ where we have defined

$$A(\tilde{\beta},\eta)=\frac{E(1-\sigma)}{12(1-\sigma^2)}\int_{R_1}\left[\frac{\partial\tilde{\beta}^1}{\partial x}\frac{\partial\eta^1}{\partial x}+\frac{1}{2}\left(\frac{\partial\tilde{\beta}^2}{\partial x}+\frac{1}{\delta}\frac{\partial\tilde{\beta}^1}{\partial y}\right)\frac{\partial\eta^2}{\partial x}\right]dxdy$$
$$+\frac{E\sigma}{12(1-\sigma^2)}\int_{R_1}\left[\left(\frac{\partial\tilde{\beta}^1}{\partial x}+\frac{1}{\delta}\frac{\partial\tilde{\beta}^2}{\partial y}\right)\frac{\partial\eta^1}{\partial x}\right]dxdy,$$

for all $\eta \in (H^1((0,1);g(x)dx))^2$, for all $v \in H^1((0,1);g(x)dx)$. We define the averaging operator $\mathcal{M}$ mapping $L^2(R_1)$ to $L^2_g(0,1)$ by

$$\mathcal{M}h(x)=\frac{1}{g(x)}\int_0^{g(x)}h(x,y)\,dy\,,$$

for all $h \in L^2(R_1)$ and for almost all $x \in (0,1)$. By taking the limit as $\delta \to 0$ in (6.4.8) we deduce that

$$\frac{E}{12(1-\sigma^2)}\int_0^1\left(\frac{\partial\beta_0^1}{\partial x}\frac{\partial\eta^1}{\partial x}+\frac{(1-\sigma)}{2}\frac{\partial\beta_0^2}{\partial x}\frac{\partial\eta^2}{\partial x}+\frac{(1-\sigma)}{2}u^1\frac{\partial\eta^2}{\partial x}+\sigma u^2\frac{\partial\eta^1}{\partial x}\right)g(x)dx$$
$$+\frac{Ek}{2(1-\sigma)t^2}\int_0^1\left(\left(\frac{\partial w_0}{\partial x}-\beta_0^1\right)\left(\frac{\partial v}{\partial x}-\eta^1\right)-\left(W-\beta^2\right)\eta^2\right)g(x)dx$$
$$=\int_0^1\left(\mathcal{M}(f)v+\frac{t^2}{12}\mathcal{M}(F)\cdot\eta\right)g(x)dx, \quad (6.4.9)$$

for all $\eta \in (H^1((0,1);g(x)dx))^2$, for all $v \in H^1((0,1);g(x)dx)$.

We aim at identifying the functions W and u appearing in (6.4.9). To accomplish this plan we use different test functions in (6.4.6). Namely, we choose $\eta(x,y) = \delta\mu(x,y)$ and $v(x,y) = \delta\xi(x,y)$, where $\mu \in (H^1(R_1))^2$, $\xi \in H^1(R_1)$ are given functions. By substituting η and μ in (6.4.6) and by taking the limit as $\delta \to 0$ we deduce that

$$\frac{E}{12(1-\sigma^2)}\int_0^1\left(\frac{(1-\sigma)}{2}\frac{\partial\beta_0^2}{\partial x}\frac{\partial\mu^1}{\partial y}+\frac{(1-\sigma)}{2}u^1\frac{\partial\mu^1}{\partial y}+u^2\frac{\partial\mu^2}{\partial y}+\sigma\frac{\partial\beta_0^1}{\partial x}\frac{\partial\mu^2}{\partial y}\right)g(x)dx$$
$$+\frac{Ek}{2(1-\sigma)t^2}\int_0^1\left(W-\beta_0^2\right)\frac{\partial\xi}{\partial y}\,g(x)dx=0, \quad (6.4.10)$$

for all $\mu \in H^1(R_1)^2$, $\xi \in H^1(R_1)$. By the arbitrariness of $\mu = (\mu^1, \mu^2)$, ξ and by the surjectivity of the Trace operator we deduce that

$$W = \beta_0^2, \quad u^2 = -\sigma \frac{\partial \beta_0^1}{\partial x}, \quad u^1 = -\frac{\partial \beta_0^2}{\partial x}.$$

By substituting these expressions for $u = (u^1, u^2)$ and W in (6.4.9), we obtain the following limit problem for the beam

$$\frac{E}{12}\int_0^1 \frac{\partial \beta_0^1}{\partial x}\frac{\partial \eta^1}{\partial x} g(x)dx + \frac{Ek}{2(1-\sigma)t^2}\int_0^1 \left(\frac{\partial w_0}{\partial x} - \beta_0^1\right)\left(\frac{\partial v}{\partial x} - \eta^1\right) g(x)dx$$
$$= (\mathcal{M}(f), v)_{L^2((0,1);g(x)dx)} + \frac{t^2}{12}(\mathcal{M}(F), \eta)_{L^2((0,1);g(x)dx)}, \quad (6.4.11)$$

for all $\eta \in (H^1((0,1);g(x)dx))^2$, for all $v \in H^1((0,1);g(x)dx)$. Note that the second component β_0^2 does not give any contribution in the limit equation, accordingly to the loss of dimensionality of the problem.

Let us define the eigenpairs $(\theta_l, h_l) = (h_l^1, h_l^2)$ of the eigenvalue problem associated with (6.4.11), defined by

$$\begin{cases} -\frac{E}{12g}((h^1)'g)' - \frac{Ek}{2(1-\sigma)t^2}\left((h^2)' - h^1\right) = \frac{\theta t^2}{12}h^1, & \text{in } (0,1),\\ \frac{Ek}{2(1-\sigma)t^2}[(h^2)'' - (h^1)'] = \theta h^2, & \text{in } (0,1),\\ h(0) = h(1) = 0, & \\ h'(0) = h'(1) = 0, & \end{cases} \quad (6.4.12)$$

We define now

$$(\lambda_n^0(t))_{n\geq 1} = (\omega_k)_{k\geq 1} \cup (\theta_l)_{l\geq 1}, \quad (6.4.13)$$

where it is understood that the eigenvalues are arranged in increasing order and repeated according to their multiplicity. For each $\lambda_n^0(t)$ we define the function $\psi_n = (\psi_n^1, \psi_n^2, \psi_n^3) \in (H^1(\Omega))^3 \oplus (H^1_{L_\delta}(R_\delta))^3$ in the following way:

$$\psi_n = \begin{cases} (\phi_k^\Omega, w_k^\Omega), & \text{in } \Omega\\ 0, & \text{in } R_\delta, \end{cases}$$

if $\lambda_n^0(t) = \omega_k$, for some $k \in \mathbb{N}$; otherwise

$$\psi_n = \begin{cases} 0, & \text{in } \Omega,\\ \delta^{-1/2}(\mathcal{E}_\epsilon h_l^1, 0, \mathcal{E}_\epsilon h_l^2), & \text{in } R_\delta, \end{cases}$$

if $\lambda_n^0(t) = \theta_l$, for some $l \in \mathbb{N}$, where $\mathcal{E}$ is the extension operator defined in (5.4.18) (just set $\epsilon = 1$ and replace ϵ by δ).

Finally, if $x > 0$ divides the spectrum $\lambda_n(\Omega_\delta)$ for all $\delta > 0$ sufficiently small nd $N(x)$ is the number of eigenvalues with $\lambda_n(\Omega_\delta) \leq x$ (counting their multiplicity), we define the projector P_x^0 from $(L^2(\Omega_\delta))^3$ onto the linear span $[\psi_1, \ldots, \psi_{N(x)}]$ by setting

$$P_x^0 u = \sum_{i=1}^{N(x)} (u, \psi_i)_{L^2(\Omega_\delta)^3} \psi_i,$$

for all $u \in L^2(\Omega_\epsilon)^3$. Then, by using compact convergence results (see §5.4.2) it is possible to prove the following result:

Theorem 6.4.1. *Let Ω_δ, $\delta > 0$, be a family of dumbbell domains satisfying the H-Condition. Then the following statements hold:*

(i) $\lim_{\delta \to 0} |\lambda_n(\Omega_\delta) - \lambda_n^0(t)| = 0$, *for all* $n \in \mathbb{N}$.

(ii) For any x dividing the spectrum,

$$\lim_{\delta \to 0} \left\| (\varphi_n^\delta, w_n^\delta) - P_x^0(\varphi_n^\delta, w_n^\delta) \right\|_{H^1(\Omega)^3 \oplus L^2(R_\epsilon)^3} = 0,$$

for all $n = 1, \ldots, N(x)$.

6.5 Spectral convergence for vanishing thickness

The spectral convergence of the Reissner-Mindlin system as $t \to 0$ is nowadays a well understood topic. We refer to the Appendix to this chapter (see §6.6) for a proof of the convergence of the eigenvalues and eigenfunctions of the Reissner-Mindlin system with hard clamped boundary conditions in at least a non-trivial open subset $B \subset \partial\Omega$. Here we want to remark instead the relation between the two parameters δ and t. According to the results proved in this chapter we know that as $\delta \to 0$ there holds an asymptotic decomposition of the eigenvalues in two families

$$(\lambda_n(t, \Omega_\delta))_n \approx (\omega_k(t))_k \cup (\theta_l^\delta(t))_l,$$

as $\delta \to 0$ where we recall that $\omega_k(t)$ is the k-th eigenvalue of Problem (6.3.2) and $\theta_l^\delta(t)$ is the l-th eigenvalue of Problem (6.3.3), with fixed thickness parameter t. Moreover we know that the eigenvalues $(\theta_l^\delta(t))_l$ of problem (6.3.3) converge (with preservation of the generalized multiplicity) to the eigenvalues $(\theta_l(t))_l$ of the one-dimensional problem (6.4.12) as $\delta \to 0$, independently of $t > 0$.

Moreover it is possible to pass to the limit in equation (6.4.11) as $t \to 0$. Indeed, we are allowed to choose $\beta_0^1 = \partial_x w_0$ and $\eta^1 = \partial_x v$ as test function in (6.4.11) since we know that (6.4.11) satisfies a regularity estimate in the following form

$$\|\beta_0\|_{H^2(\Omega)} + \|w_0\|_{H^2(\Omega)} + \left\| \frac{1}{t^2}(\nabla w_0 - \beta_0) \right\|_{L^2(\Omega)} \leq C(\|\mathcal{M}(f)\|_{L^2(\Omega)} + \frac{t^2}{12}\|\mathcal{M}(F)\|_{L^2(\Omega)}).$$

We refer to the Appendix to this Chapter for a proof of this regularity estimate (see §6.6). Hence we deduce that as $t \longrightarrow 0$ problem (6.4.11) converges to

$$\frac{E}{12}\int_0^1 \frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 v}{\partial x^2}\, g(x)dx = (\mathcal{M}(f), v)_{L^2((0,1);g(x)dx)},$$

for all $w \in H_0^2((0,1); g(x)dx)$, which is the weak formulation of

$$\begin{cases} -\frac{E}{12g}(w_0''g)'' = \mathcal{M}(f), & \text{in } (0,1),\\ w_0(0) = w_0(1) = 0, \\ w_0'(0) = w_0'(1) = 0, \end{cases} \tag{6.5.1}$$

Let us remark that

$$\frac{E}{12} = (1-\sigma^2)\frac{2\mu_1 + \mu_2}{12},$$

and by recalling (0.0.8), we note that the one dimensional Problem (6.5.1) is compatible with (5.1.9). In particular, the coefficient $(1-\sigma^2)$ in Problem (5.4.1) appears also in Problem (6.5.1). By using suitable $\mathcal{E}$-convergence result in the spirit of Section 5.4.2, from the convergence of the Poisson problem (6.4.11) as $t \longrightarrow 0$ we deduce that the eigenvalues $\theta_l(t)$ converge to the eigenvalues τ_l of problem

$$\begin{cases} -\frac{E}{12g}(w_0''g)'' = \tau w_0, & \text{in } (0,1),\\ w_0(0) = w_0(1) = 0, \\ w_0'(0) = w_0'(1) = 0, \end{cases} \tag{6.5.2}$$

as $t \longrightarrow 0$.

It is also possible to prove (see the Appendix below and Remark 6.6.2) that the eigenvalues $(\lambda_n(t,\Omega_\delta))_n$ converge to the eigenvalues of the biharmonic operator on the dumbbell domain as $t \longrightarrow 0$, corresponding to the Cauchy problem

$$\begin{cases} \frac{E}{12(1-\sigma^2)}\Delta^2 w = \lambda(0,\Omega_\delta)w, & \text{in } \Omega_\delta,\\ (1-\sigma)\frac{\partial^2 w}{\partial n^2} + \sigma\Delta w = 0, & \text{on } \Gamma_\delta,\\ (1-\sigma)\ \mathrm{div}_{\partial\Omega_\delta}(D^2 w\cdot n)_{\partial\Omega_\delta} + \frac{\partial(\Delta w)}{\partial n} = 0, & \text{on } \Gamma_\delta,\\ w = \frac{\partial w}{\partial n} = 0, & \text{on } B. \end{cases} \tag{6.5.3}$$

In a similar way we can prove that, once we have fixed $\delta > 0$, the eigenvalues $(\theta_l^\delta(t))_l$ of Problem (6.3.3) converge to the eigenvalues $(\theta_l^\delta(0))_l$ of

$$\begin{cases} \frac{E}{12(1-\sigma^2)}\Delta^2 w = \theta^\delta(0)w, & \text{in } R_\delta,\\ (1-\sigma)\frac{\partial^2 w}{\partial n^2} + \sigma\Delta w = 0, & \text{on } \Gamma_\delta,\\ (1-\sigma)\ \mathrm{div}_{\partial\Omega_\delta}(D^2 w\cdot n)_{\partial\Omega_\delta} + \frac{\partial(\Delta w)}{\partial n} = 0, & \text{on } \Gamma_\delta,\\ w = 0 = \frac{\partial w}{\partial n}, & \text{on } L_\delta, \end{cases} \tag{6.5.4}$$

as $t \longrightarrow 0$. Let us define with $(\lambda_n^0)_n$ the eigenvalues λ_n^0 associated with the limit of the biharmonic Neumann problem in the dumbbell, as defined in (5.6.1), whereas $(\lambda_n^0(t))_n$ are the eigenvalues defined by (6.4.13). In summary, we find out that

$$\lambda_n(t, \Omega_\delta) \xrightarrow{t \to 0} \lambda(0, \Omega_\delta) \xrightarrow{\delta \to 0} \lambda_n^0,$$

and

$$\lambda_n(t, \Omega_\delta) \xrightarrow{\delta \to 0} \lambda_n^0(t) \xrightarrow{t \to 0} \lambda_n^0.$$

We remark that the case in which both t and δ tend to zero simultaneously is far more difficult. For example the best constant in Korn's inequality may blow up as $\delta \longrightarrow 0$, giving problems to the coercivity of the quadratic form (6.2.3). The analysis of this interesting case is however beyond the scope of this thesis.

6.6 Appendix to Chapter 6

In this section, for the sake of completeness, we collect several results concerning the passage to the limit as $t \longrightarrow 0$ in the Reissner-Mindlin system with partially clamped boundary conditions. We remark that a similar result for completely free boundary conditions is true but it is much more complicated because of the absence of a Korn's inequality in the form (6.1.1). In the sequel we shall consider without loss of generality the passage to the limit for a plate which is clamped on the whole of $\partial\Omega$. Indeed, Korn's inequality and Poincaré's inequality are known to hold for Sobolev functions satisfying a Dirichlet boundary condition on a (possibly small) part of the boundary. Where it is needed we will underline what changes in the case of $B \subset \partial\Omega$, $B \neq \partial\Omega$.
It is widely known (see e.g., [65]) that the Reissner-Mindlin plate system spectrally converge as $t \longrightarrow 0$ to the biharmonic equation

$$\frac{E}{12(1-\sigma^2)}\Delta^2 w = \lambda w, \qquad \text{in } \Omega.$$

We recall that the main ingredient to pass to the limit as $t \longrightarrow 0$ in (6.2.2) in the case of the clamped plate (i.e., $w \in H^1_0(\Omega)$, $\beta \in (H^1_0(\Omega))^2$) is the regularity estimate

$$\|\beta\|_{H^2(\Omega)} + \|w\|_{H^2(\Omega)} + \left\|\frac{1}{t^2}(\nabla w - \beta)\right\|_{L^2(\Omega)} \leq C\left(\|f\|_{L^2(\Omega)} + \frac{t^2}{12}\|F\|_{L^2(\Omega)}\right), \quad (6.6.1)$$

where the constant C does not depend on t, and f, F are the data in the Poisson problem (see (6.6.2) below). Let us define for the sake of simplicity

$$\gamma(t) = \frac{\kappa}{t^2}(\nabla w - \beta),$$

for all $t > 0$. We point out that once we have the estimate (6.6.1) the passage to the limit is easily achieved. Indeed, one considers

$$(RM)\begin{cases} a(\beta(t),\eta) + (\gamma(t), \nabla v - \eta) = (f, v) + \frac{t^2}{12}(F,\eta), \\ \gamma(t) = \frac{\kappa}{t^2}(\nabla w(t) - \beta(t)), \end{cases} \qquad (6.6.2)$$

for all $v \in H^1_0(\Omega)$, for all $\eta \in (H^1_0(\Omega))^2$, where $f \in L^2(\Omega)$ and $\theta \in (L^2(\Omega))$ are given functions. We define the problem

$$(KL)\begin{cases} a(\beta_0,\eta) + (\gamma_0, \nabla v - \eta) = (f, v), \\ \nabla w_0 - \beta_0 = 0. \end{cases} \qquad (6.6.3)$$

for all $v \in H^1_0(\Omega)$, for all $\eta \in (H^1_0(\Omega))^2$, where γ_0 is the L^2-weak limit of the sequence $\gamma(t)$ as $t \to 0$ (note that $\gamma(t)$ is bounded in $L^2(\Omega)$ due to (6.6.1)). By subtracting (KL) to (RM) we deduce that

$$\begin{cases} a(\beta(t) - \beta_0, \eta) + (\gamma(t) - \gamma_0, \nabla v - \eta) = \frac{t^2}{12}(F, \eta), \\ \gamma(t) = \frac{\kappa}{t^2}\Big[\nabla(w(t) - w_0) - (\beta(t) - \beta_0)\Big], \end{cases} \tag{6.6.4}$$

for all $v \in H^1_0(\Omega)$ and $\eta \in (H^1_0(\Omega))^2$. Choose $\eta = \beta(t) - \beta_0$, $v = w(t) - w_0$ in (6.6.4). Then

$$a(\beta(t) - \beta_0, \beta(t) - \beta_0) = \frac{t^2}{12}(F, \eta) + \frac{t^2}{\kappa}(\gamma(t) - \gamma_0, \gamma(t)),$$

from which we deduce by Korn's inequality, Poincare's inequality, Cauchy-Schwarz inequality and the estimate (6.6.1) that

$$\begin{aligned} &\|\beta(t) - \beta_0\|^2_{H^1(\Omega)} \\ &\le Ct^2\|F\|_{L^2(\Omega)}\|\beta(t) - \beta_0\|_{L^2(\Omega)} + Ct^2(\|\gamma(t)\|_{L^2(\Omega)} + \|\gamma_0\|_{L^2(\Omega)})\|\gamma(t)\|_{L^2(\Omega)} \\ &\le Ct\left(\|f\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)}\right)^{1/2}\|\beta(t) - \beta_0\|_{L^2(\Omega)} + Ct^2\left(\|f\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)}\right). \end{aligned}$$

With the help of a weighted Young's inequality we then deduce that

$$\|\beta(t) - \beta_0\|_{H^1(\Omega)} \le Ct\left(\|f\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)}\right)^{1/2},$$

and since

$$\nabla(w(t) - w_0) = (\beta(t) - \beta_0) + \frac{t^2}{\kappa}\gamma(t),$$

by using again the a priori estimate (6.6.1) for $\|\gamma(t)\|_{L^2(\Omega)}$ and the Poincaré inequality we deduce that

$$\begin{aligned} \|w(t) - w_0\|_{H^1(\Omega)} &\le \|\beta(t) - \beta_0\|_{H^1(\Omega)} + t^2\left(\|f\|_{L^2(\Omega)} + \frac{t^2}{12}\|F\|_{L^2(\Omega)}\right)^{1/2} \\ &\le C(t + t^2)\left(\|f\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)}\right)^{1/2}. \end{aligned} \tag{6.6.5}$$

Hence, the solution $(\beta(t), w(t))$ of (RM) is converging to the solution (β_0, w_0) of (KL) strongly in $H^1(\Omega)$ as $t \to 0$. Moreover, since $\|\gamma(t)\|_{L^2(\Omega)^2} \le Ct$, by passing to the limit as $t \to 0$ we deduce that $\nabla w_0 = \beta_0$ a.e. in Ω, and since $\beta_0 \in H^1(\Omega)^2$, $w_0 \in H^2(\Omega)$.

Note that since (KL) holds for all $\eta \in (H^1_0(\Omega))^2$, $v \in H^1_0(\Omega)$, in particular it holds for all $(\nabla v, v) \in (H^1_0(\Omega))^2 \times H^2_0(\Omega)$. With this choice we see that (KL) equals

$$a(\nabla w_0, \nabla v) = (f, v),$$

for all $v \in H^2_0(\Omega)$, and $a(\nabla w_0, \nabla v)$ is exactly

$$\frac{E}{12(1-\sigma^2)} \int_\Omega (1-\sigma)\, D^2 w_0 : D^2 v + \sigma\, \Delta w_0 \Delta v \, dx,$$

which is the classical quadratic form associated with the Dirichlet problem for the biharmonic operator

$$\begin{cases} \frac{E}{12(1-\sigma^2)} \Delta^2 w_0 = f, & \text{in } \Omega, \\ w_0 = \frac{\partial w_0}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{6.6.6}$$

Thus to pass to the limit as $t \to 0$ it is sufficient to prove the regularity estimate (6.6.1).

Proposition 6.6.1. *Assume that the boundary of Ω is $C^{1,1}$ or that Ω is a convex polygon of $\mathbb{R}^2$ and $f \in L^2(\Omega)$, $F \in L^2(\Omega)^2$. Then Problem* (6.6.2) *admits the regularity estimate* (6.6.1).

Proof. Here we follow the approach of [26]. We first apply a Helmholtz type decomposition to the vector field $\frac{1}{t^2}(\nabla w(t) - \beta(t)) \in (L^2(\Omega))^2$. Namely we write

$$\frac{1}{t^2}(\nabla w(t) - \beta(t)) = \nabla \varphi(t) \oplus \mathrm{curl}(p(t)),$$

where $\varphi(t) \in H^1_0(\Omega)$ and $p(t) \in H^1(\Omega)/\mathbb{R}$. Note that this is an orthogonal decomposition in $L^2(\Omega)$. The existence of such a decomposition is a classical topic and can be found for example in [62]. Then by arguing as in [26] it is not difficult to prove that Problem (6.6.2) admits the following equivalent formulation

$$\begin{cases} a(\beta(t), \eta) - \kappa(\nabla\varphi + \mathrm{curl}\, p(t), \eta) = \frac{t^2}{12}(F, \eta), & \text{for all } \eta \in (H^1_0(\Omega))^2, \\ \kappa(\nabla\varphi, \nabla\xi) = (f, \xi), & \text{for all } \xi \in H^1_0(\Omega), \\ (\nabla w(t) - \beta(t), \nabla\xi) = t^2(\nabla\varphi, \nabla\xi), & \text{for all } \xi \in H^1_0(\Omega), \\ -(\beta(t), \mathrm{curl}\, q) = t^2(\mathrm{curl}\, p(t), \mathrm{curl}\, q), & \text{for all } q \in H^1(\Omega)/\mathbb{R}. \end{cases} \tag{6.6.7}$$

Note that the second equation in (6.6.7) is exactly the weak formulation of

$$\begin{cases} -\kappa\Delta\varphi = f, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.6.8}$$

for which we know that the solution φ lies in $H^2(\Omega)$ with the estimate

$$\|\varphi\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \tag{6.6.9}$$

see for example [72].[1]

Now we consider the following auxiliary problem

$$\begin{cases} a(\tilde{\beta}(t), \eta) - \kappa(\operatorname{curl}\tilde{p}(t), \eta) = \kappa(\nabla\varphi, \eta) + \frac{t^2}{12}(F, \eta), & \text{for all } \eta \in (H^1_0(\Omega))^2, \\ -(\tilde{\beta}(t), \operatorname{curl} q) = 0, & \text{for all } q \in H^1(\Omega)/\mathbb{R}, \end{cases} \tag{6.6.10}$$

in the unknowns $\tilde{\beta}(t) \in H^1_0(\Omega)^2$ and $\tilde{p}(t) \in H^1(\Omega)/\mathbb{R}$. This problem is similar to the Stokes system with the standard condition $\operatorname{div}\tilde{\beta} = 0$ replaced by $\operatorname{curl}\tilde{\beta} = 0$. By well-known regularity results (see e.g., [83]), Poincaré's inequality and (6.6.9) we deduce that

$$\begin{aligned} \|\tilde{\beta}(t)\|^2_{H^2(\Omega)} + \|\operatorname{curl}\tilde{p}(t)\|^2_{L^2(\Omega)} &\leq C\|\nabla\varphi\|^2_{H^1(\Omega)} + \frac{Ct^2}{12}\|F\|^2_{L^2(\Omega)} \\ &\leq C\left(\|f\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)}\right), \end{aligned} \tag{6.6.11}$$

with the constant C not depending on t.

Finally we set $\beta^*(t) = \beta(t) - \tilde{\beta}(t)$ and $p^*(t) = p(t) - \tilde{p}(t)$. Then $(\beta^*(t), p^*(t))$ satisfies the following system

$$\begin{cases} a(\beta^*(t), \eta) = \kappa(\operatorname{curl} p^*(t), \eta), & \forall \eta \in (H^1_0(\Omega))^2, \\ -(\beta^*(t), \operatorname{curl} q) = t^2(\operatorname{curl} p^*(t), \operatorname{curl} q) + t^2(\operatorname{curl}\tilde{p}(t), \operatorname{curl} q), & \forall q \in H^1(\Omega)/\mathbb{R}, \end{cases} \tag{6.6.12}$$

Take $\eta = \beta^*$, $q = p^*$ in order to deduce the estimate

$$\begin{aligned} a(\beta^*, \beta^*) + \kappa t^2\|\operatorname{curl} p^*\|^2_{L^2(\Omega)} &= -\kappa t^2(\operatorname{curl}\tilde{p}, \operatorname{curl} p^*) \\ &\leq Ct^2\|\operatorname{curl} p^*(t)\|_{L^2(\Omega)}\left(\|f\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)}\right)^{1/2}. \end{aligned}$$

Hence,

$$\|\operatorname{curl} p^*(t)\|_{L^2(\Omega)} \leq Ct\left(\|f\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)}\right)^{1/2},$$

[1] Actually, this result holds with less regularity at the boundary for $\partial\Omega$ (a Lipschitz domain is sufficient). However, if we replace $H^1_0(\Omega)$ with $H^1_B(\Omega)$ in (6.6.8), then in general we need a convex domain or a sufficiently smooth domain to prove that $\varphi \in H^2(\Omega)$. If we want to consider more general domains (for example, Lipschitz domains), then it is still possible to prove a regularity estimate in a suitable H^s space with $2 > s > 1$ (e.g., in a non-convex polygon $s > 3/2$).

and consequently, by definition of $p*$ and (6.6.11), we deduce that

$$\|\mathrm{curl}\, p(t)\|_{L^2(\Omega)} \leq C\left(\|f\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)}\right)^{1/2}. \tag{6.6.13}$$

By recalling the first equation in (6.6.7), we find that

$$a(\beta, \eta) = \kappa(\nabla\varphi(t) + \mathrm{curl}\, p(t), \eta) + \frac{t^2}{12}(F, \eta), \quad \text{for all } \eta \in (H^1_0(\Omega))^2,$$

where we remark that the functions $\nabla\varphi(t)$ and $\mathrm{curl}\, p(t)$ are in $L^2(\Omega)$ with uniform estimates in t. By standard regularity procedures we then deduce that

$$\begin{aligned} \|\beta(t)\|^2_{H^2(\Omega)} &\leq C\|\nabla\varphi(t) + \mathrm{curl}\, p(t)\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)^2} \\ &\leq C'\left(\|f\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)^2}\right), \end{aligned} \tag{6.6.14}$$

where the second inequality on the right-hand side follows from (6.6.9) and (6.6.13).

Finally we note that the third equation in (6.6.7) is the variational formulation of the classical problem

$$\begin{cases} -\Delta w = -\,\mathrm{div}\,\beta - t^2\Delta\varphi \in L^2(\Omega), & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases}$$

for which we know that the H^2 regularity holds. Hence,

$$\|w\|_{H^2(\Omega)} \leq C\left(\|\mathrm{div}\,\beta + t^2\Delta\varphi\|_{L^2(\Omega)}\right) \leq C\left(\|f\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)^2}\right)^{1/2},$$

where we have used (6.6.14) and (6.6.9). It is also easy to deduce from the last equation of (6.6.7) that $t\|\mathrm{curl}\,\mathrm{curl}\, p\|_{L^2(\Omega)} \leq C\left(\|f\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)^2}\right)^{1/2}$. Hence we have

$$\begin{aligned} \|\beta\|_{H^2(\Omega)} + \|w\|_{H^2(\Omega)} + \|\varphi\|_{H^2(\Omega)} + \|\mathrm{curl}\, p\|_{L^2(\Omega)} + t\|\mathrm{curl}\,\mathrm{curl}\, p\|_{L^2(\Omega)} \\ \leq C\left(\|f\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)^2}\right)^{1/2}, \end{aligned}$$

which is a more accurate estimate than the one in (6.6.1), since $\frac{1}{t^2}(\nabla w(t) - \beta(t))$ equals $\nabla\varphi(t) + \mathrm{curl}(p(t))$. □

Remark 6.6.2. Let Ω be a domain in $\mathbb{R}^2$ of class $C^{0,1}$ such that $\partial\Omega \setminus \{p_0, \dots, p_k\}$ is $C^{1,1}$, where $p_0, \dots, p_k \in \partial\Omega$. Let $B \subset \partial\Omega$, $B \neq \partial\Omega$, where B is as in (6.2.1). Then it is still possible to prove a regularity estimate similar to (6.6.1). Indeed, we have

$$\|\beta\|_{H^s(\Omega)^2} + \|w\|_{H^s(\Omega)} + \|\varphi\|_{H^s(\Omega)} + \|\operatorname{curl} p\|_{L^2(\Omega)} + t\|\operatorname{curl}\operatorname{curl} p\|_{L^2(\Omega)} \\ \leq C\left(\|f\|^2_{L^2(\Omega)} + \frac{t^2}{12}\|F\|^2_{L^2(\Omega)^2}\right)^{1/2}, \quad (6.6.15)$$

for a suitable $1 < s \leq 2$. Note that if Ω is convex we can choose $s = 2$. Otherwise, by using regularity results in the spirit of [72, §4] it is possible to prove that the functions β, w, φ have singularities in a neighbourhood of the convex 'corners'. In order to pass to the limit as $t \to 0$ in problem (RM) defined in (6.6.2), estimate (6.6.15) is enough. Indeed, from (6.6.15) we deduce that $\|\gamma(t)\|_{L^2(\Omega)^2} \leq Ct$ for all $t > 0$, hence in the limit $\nabla w_0 = \beta_0 \in H^1(\Omega)^2$ a.e. in Ω, and arguing as in (6.6.5) we also deduce that $\beta(t) \to \beta_0$ in $H^1(\Omega)^2$. Hence, $w_0 \in H^2(\Omega)$ and solves problem (6.6.6). Note carefully that we are not claiming that $w_0 \in H^4(\Omega)$. As a consequence of (6.6.15), the passage to the limit as $t \to 0$ in the dumbbell Ω_δ (see (6.5.3)) is justified.

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