COMMUNICATION SCHEDULING AND REMOTE ESTIMATION WITH ADDITIVE NOISE CHANNELS

BY

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DISSERTATION

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ABSTRACT

Communication scheduling and remote estimation scenarios arise in the context of wireless sensor networks, which involve monitoring and controlling the state of a dynamical system from remote locations. This entails joint design of transmission and estimation policies, where a sensor (or a group of sensors) observes the state of the system over a given horizon, but has to be selective in what (and when) it transmits due to energy constraints. The estimator, on the other hand, needs to generate real-time estimates of the state regardless of whether there is a transmission from the sensor or not. Hence, a communication scheduling strategy for the sensor and an estimation strategy for the estimator should be jointly designed to minimize the estimation error subject to the energy constraints. Prior works on this topic assumed that the communication channel between the sensor and the estimator is noiseless, which may not be that realistic even though it was an important first step. In this thesis, we study communication scheduling and remote estimation problems with additive noise channels. In particular, we consider a series of four problems as follows. In the first problem, the sensor has two options, namely, not transmitting its observation, or transmitting its observation over an additive noise channel subject to some communication cost. Because of the presence of channel noise, if the sensor decides to transmit its observation over the noisy channel, it needs to encode the message. Furthermore, the estimator needs to decode the noise-corrupted message. Hence, a pair of encoding and decoding strategies should also be jointly designed along with the communication scheduling strategy. In the second problem, the sensor has three options, where two of the options are the same as those in the first problem, and the third one is that the sensor can transmit its observation via a noiseless but more costly channel. The third problem is a variant of the first one, where the encoder has a constraint on its average total power consumption over the time horizon, instead of a constraint on the stage-wise encoding power, which is assumed in the first problem. In the fourth problem, the communication channel noise is generated by an adversary with the objective of maximizing the estimation error. Hence, a game problem instead of an optimization problem is formulated and studied. Under some technical assumptions, we obtain the optimal solutions for the first three problems, and a feedback Stackelberg solution for the fourth problem. We present numerical results illustrating the performances of the proposed solutions. We also discuss possible directions for future research based on the results presented in this thesis.

To my parents and my wife, for their love and support.

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CHAPTER 1

INTRODUCTION

1.1 Background

Communication scheduling and remote state estimation problems arise in applications involving wireless sensor networks, such as environmental monitoring and networked control systems. As an example of environmental monitoring, researchers at the National Aeronautics and Space Administration (NASA) Earth Science group are interested in monitoring the evolution of the soil moisture, which is used in weather forecasts, ecosystem process simulation and so on [1]. In order to achieve this goal, a sensor network is built over an area of interest. The sensors collect data on the soil moisture and send it to the decision unit at NASA via wireless communication. The decision unit at NASA forms estimates on the evolution of the soil moisture based on the messages received from the sensors. Similarly, in a networked control system, where the objective is to control some remote plant, a sensor network is built to measure the state of the remote plant. Sensors transmit their measurements to the controller via a wireless communication network. The controller estimates the state of the plant and then generates a control signal based on that estimate [2]. In both scenarios, the quality of remote state estimation strongly affects the quality of the decision making, that is, weather prediction or control signal generation. The networked sensors are usually constrained by limits on energy. They are not able to communicate with the estimator at every time step and thus, the estimator has to produce its best estimate based on the partial information received from the sensors. Therefore, the communication between the sensors and the estimator should be scheduled judiciously, and the estimator should be designed properly, so that the state estimation error is minimized subject to the communication constraints.

1.2 Related Work and Motivation

Research on the general communication scheduling (sampling) and remote estimation problem dates back to the 1970s, and many results have been derived since then, related to this general topic [3–26]. In particular, the problems considered in [27–29] are closely related and motivate our work. The work in [27] initialized this line of research, where the following problem was studied. Consider the problem of observing in *real-time* a one-dimensional, independent and identically distributed (i.i.d.) stochastic process (call it *source*) over a finite time horizon. There is a sensor network with *one sensor* and *one estimator*. At each time step, the sensor makes a perfect observation of the state of the source. Under an energy constraint, the sensor is able to transmit its observation to the estimator only a limited number of times, call it *hard (communication) constraint*. Hence, after receiving measurement of the source, the sensor needs to decide whether to transmit its observation or not. The communication channel between the sensor and the estimator is perfect, and the estimator will get a notification if there is no transmission. Based on the messages received from the sensor, the estimator generates real-time estimates on the states of the source and is charged for squared estimation error. The underlying optimization problem is to jointly design communication scheduling and remote estimation strategies that minimize the mean squared error over a given time horizon, subject to the hard communication/transmission constraint. To approach the problem, it was assumed that the sensor is restricted to apply a *threshold-based strategy*, namely, the sensor computes the innovation of its actual observation compared to the expected observation, and it decides to transmit the observation if the absolute value of the innovation exceeds some threshold. With the above assumption, it was shown that the optimal threshold depends on time and the remaining communication opportunities, which can be obtained via dynamic programming. Furthermore, the optimal estimator is the conditional mean. The results can be generalized to the case when the source is the state of a linear time-invariant (LTI) system driven by an i.i.d. Gaussian process.

The work in [28] considered a problem setting slightly different from that in [27]. In [28], the sensor is not constrained by communication opportunities, but is charged a cost for each transmission, call it *soft (communication)* *constraint*. In addition, the work in [28] did not restrict the search of communication scheduling strategy to the class of threshold type strategies. It showed that a threshold-based strategy and the conditional mean are jointly optimal if the source is a Gauss-Markov process. Moreover, the conditional mean admits a closed-form expression, which is a Kalman filter-like estimator. The work in [29] extended the results in [28] to a more general class of problems, where the source can be the state of an LTI system driven by any i.i.d. stochastic process with an even and unimodal distribution. The sensor has both soft constraint and hard constraint, and it is also equipped with an energy harvester. It was shown that a threshold-based communication scheduling strategy and a Kalman-like filter are jointly optimal. The results also hold for multidimensional systems under some technical assumptions.

In [27–29], it was assumed that the communication channel between the sensor and the estimator is perfect (call it "noiseless-channel setting"), which may not be realistic even though it is an important first step. The next step would be to study settings with an imperfect channel. Problems with an i.i.d. packet-dropping channel and a Gilbert-Elliott channel (Markov packet-dropping channel) were formulated and studied separately in [30] and [31]. The setting with a random delay channel was considered in [32]. The settings with an adversary who is able to block the communication channel or manipulate the message transmitted by the sensor were studied in [33–36]. In this thesis, we will concentrate on the problems with additive noise channels [37–44].

1.3 Contributions

In this thesis, we consider a series of four problems. We first consider communication scheduling and remote estimation over *an additive noise channel* (call it "single-channel setting"). In this single-channel setting, the source is a one-dimensional i.i.d. stochastic process with an even and log-concave distribution. At each time step, a sensor observes the source and then decides whether or not to transmit its observation. Since the communication channel has an additive channel noise, the sensor may need to encode its observation before transmitting it. Hence, an *encoder* with power constraint is involved in the problem. In this case, the estimator needs to decode the noise-corrupted message, and thus it can also be called *decoder*. It is assumed that there is a side channel between the encoder and the decoder, which enables them to apply different encoding/decoding strategies for different types of source observation, e.g., apply different encoding/decoding strategies for positive and negative observations. We consider soft and hard constraints separately in two sub-problems. In general, the problem is hard to analyze, since it involves both a communication scheduling problem and a zero-delay source-channel coding problem. Regarding the zero-delay source-channel coding problem, it is well known that affine encoding/decoding strategies are jointly optimal if the source and the communication channel noise have the jointly Gaussian distribution. In our problem, however, even though the source has Gaussian distribution, after "thresholding" on it, the input to the encoder will not have Gaussian distribution any more, which makes the problem fairly difficult to solve. To simplify the analysis, we restrict the encoder and the decoder to apply affine encoding and decoding strategies. In addition, we restrict our search of communication scheduling strategies to a class satisfying some symmetry property. We show that under the above assumptions, the optimal communication scheduling strategy is threshold-based. Furthermore, we uncover a rather surprising property, that is, even in the asymptotic case where there is no communication constraint, the sensor should still apply the threshold-based strategy instead of making a transmission at each time step. Numerical results are generated, which show some phenomena not encountered in the noiseless-channel setting.

Next, we consider communication scheduling and remote estimation over *multiple channels* (call it "multi-channel setting"). In this multi-channel setting, the sensor has three options after making an observation. One is that it can choose not to transmit its observation. The second is that it can transmit its observation via a noisy channel under some cost. And the third is that it can transmit its observation via a noiseless but more costly channel. Again, if the sensor decides to transmit its observation via the noisy channel, it will send the observation to an encoder. Different from the single-channel setting, at first we do not assume the existence of a side channel. Instead, we show by constructing a counterexample that without the side channel, the optimal communication scheduling strategy can be non-symmetric, which makes the problem intractable. Hence, the assumption on the existence of a side channel is critical in making the problem tractable.

With this assumption, we show that the optimal communication scheduling strategy is *threshold-in-threshold* based. We also present numerical results, which show some properties inherited from both the single-channel setting and the noiseless-channel setting.

Third, we consider the problem of communication scheduling and remote estimation with power allocation (call it "power allocation setting"). This power allocation setting is formulated based on the single-channel setting. In the single-channel setting, it is assumed that the encoder has a stage-wise constraint on its encoding power. More specifically, when the sensor decides to transmit its observation, the encoder can only utilize limited average power, which is assumed to be the same for each stage, to encode and transmit the observation. In the power allocation setting, however, we consider a more general formulation where the encoder has a constraint on its average total power consumption over the time horizon. Several technical challenges are involved due to the nature of the new constraint. For instance, the average total power should be judiciously distributed across different stages. In addition, at each stage, the communication scheduling policy, the encoding policy, and the decoding policy should be jointly designed to best utilize the average encoding power allocated to that stage. Under some technical assumptions, we show that the optimal communication scheduling policy is still threshold-based. Furthermore, for each stage we obtain a jointly optimal pair of the threshold and the encoding power. Numerical results demonstrate that with this additional flexibility of allocating the average total power, we could achieve lower expected estimation error.

Last but not the least, we consider communication scheduling and remote estimation in the presence of an adversary (call it "adversarial setting"). In this adversarial setting, a remote sensing system consisting of a sensor, an encoder, and a decoder is configured to observe, transmit, and recover a one-dimensional, i.i.d. discrete time stochastic process. At each time step, the sensor makes a measurement of the state variable of the stochastic process, and then it makes a decision as to whether to make a transmission or not. The sensor has both soft and hard communication constraints. If the sensor decides to transmit its observation, it sends the observation to the encoder, which then encodes it and sends a real-valued message to the communication channel. If the sensor decides not to transmit its observation, it maintains silence and the encoder also maintains silence accordingly.

Regardless of whether there is a transmission from the sensor or not, the decoder needs to generate a real-time estimate on the state variable of the stochastic process. The cost charged on the remote sensing system at each time step consists of three terms: the sensor is charged a cost for each transmission (no charge if there is no transmission), the encoder is charged for the encoding power (no charge if there is no transmission from the sensor), and the decoder is charged for the estimation error. All three components of the remote sensing system has the common objective of minimizing the total expected costs of the remote sensing system summed up over the time horizon. In addition, the communication channel between the encoder and the decoder is compromised, through injection of an additive channel noise generated by an adversary, or jammer. Consequently, the encoded message sent by the encoder will be distorted by this channel noise. At each time step, the jammer is charged for the jamming power, and it is rewarded by the estimation error charged on the decoder. The jammer has the objective of minimizing its expected total costs accumulated over the time horizon. As the solution concept between these opposing parties, we adopt the framework of a feedback Stackelberg game, with the sensor, the encoder, and the decoder as the composite leader, and the jammer as the follower. That is, at each time step, the sensor, the encoder, and the decoder first announce (in unison) their communication scheduling policy, the encoding policy, and the decoding policy, respectively. Then, the jammer announces its jamming policy. Under some technical assumptions, we obtain a feedback Stackelberg solution consisting of a threshold-based communication scheduling policy for the sensor, and a pair of piecewise affine encoding and decoding policies for the encoder and the decoder. We also generate numerical results to develop a further understanding of the performance of the remote sensing system compromised by an adversary, under the feedback Stackelberg solution.

1.4 Organization

The rest of this thesis is organized as follows: in Chapter 2, we formulate and consider the single-channel setting, which builds the framework for this series of research. In Chapter 3, we consider the multi-channel setting, which is a mixture between the single-channel setting and the noiseless-channel setting. In Chapter 4, we study the power allocation setting, which is an enhancement of the single-channel setting. In Chapter 5, we formulate and study the adversarial setting to address the recent and emergent issues on cyber physical systems security. In Chapter 6, we discuss possible directions for future research based on this thesis study. Finally in Chapter 7, we draw concluding remarks.

CHAPTER 2

COMMUNICATION SCHEDULING AND REMOTE ESTIMATION OVER AN ADDITIVE NOISE CHANNEL

In this chapter, we consider the communication scheduling and remote estimation problem with an additive noise channel, namely the single-channel setting. Due to the presence of channel noise, the sensor may need to encode its observation before sending it to the communication channel. Hence, the problem involves another decision maker, that is, the *encoder*. On the other hand, the estimator needs to decode the message when there is a transmission from the sensor, and thus we also call it *decoder*. Accordingly, the problem can be viewed as a communication scheduling problem combined with a zero delay source-channel coding problem. For the zero delay source-channel problem, it is well known that affine encoding and decoding policies are jointly optimal when the source and the channel noise have Gaussian distribution. However, in our case the sensor's decision contains some hidden information about the source, which will "reshape" the conditional belief. For example, suppose the sensor applies threshold-based policy. Accordingly, the sensor only transmits an observation falling outside the thresholding interval. Even though the source has Gaussian distribution, after "thresholding" on it (conditioning on the event that it falls outside the thresholding interval), the input to the encoder will not have Gaussian distribution any more, which renders the problem fairly difficult to solve. To overcome this major difficulty, we restrict the encoder and the decoder to apply affine encoding and decoding policies. Furthermore, we assume that the sensor will apply a communication scheduling policy from the class of policies with some symmetry property. We consider two scenarios, which correspond to soft and hard constraints. We show that if the source has an even and log-concave distribution, then the optimal communication scheduling policy is one of the threshold type with a unique optimal threshold. We generate numerical results for the problem with hard constraint, which show interesting phenomena not encountered in the noiseless-channel setting.

The major contributions of this chapter are as follows:

1. We formulate two optimization problems involving an additive noise channel under two types of communication constraints.
2. We show that if the source and noise processes are i.i.d., then the optimization problem with soft constraint can be simplified to a single-stage problem. Furthermore, the optimization problem with hard constraint can be converted to a single-stage problem with soft constraint.
3. Under some technical assumptions, we show that the optimal communication scheduling policy is a threshold-based one with a unique optimal threshold.
4. We generate numerical results for the problem with hard constraint. We uncover two surprising facts: first, the optimal estimation error over the time horizon stays constant if the number of communication opportunities exceeds some threshold. In other words, the communication opportunities above the threshold are redundant in terms of reducing the estimation error. Second, the sensor may not use up all the communication opportunities by the end of the time horizon. We also analyze the reasons for these two interesting phenomena.

The rest of the chapter is organized as follows: in Section 2.1, we formulate the two optimization problems under soft and hard communication constraints. In Section 2.2, we consider the problem with soft constraint. In Section 2.3, we consider the problem with hard constraint. In Section 2.4, we generate and discuss numerical results for the problem with hard constraint.

2.1 Problem Formulation

2.1.1 System Model

Consider a discrete time communication scheduling and remote estimation problem over a finite-time horizon, that is, $t = 1, 2, \ldots, T$. There is one sensor, one encoder and one remote estimator (which is also called "decoder"), as shown in Fig. 2.1. A source process $\{X_t\}$ is a one-dimensional, independent, and identically distributed (i.i.d.) stochastic process, which has density

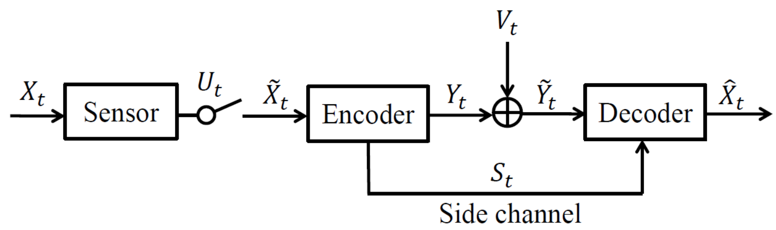

Figure 2.1: System model for single-channel setting

p_X. At time t, the sensor observes X_t. Since the sensor is assumed to have communication constraint (which will be introduced later), it needs to decide whether or not to transmit its observation. Let $U_t \in \{0, 1\}$ be the sensor's decision at time t, where $U_t = 1$ stands for transmission and $U_t = 0$ stands for no transmission. The communication channel is assumed to be noisy. Hence, if the sensor decides to transmit its observation, it sends X_t to the encoder. If the sensor decides not to transmit, it does not send anything to the encoder but a free symbol ϵ stands for its decision. Denote by $\tilde{X}_t$ the message received by the encoder; then

$$\tilde{X}_t = \begin{cases} X_t, & \text{if } U_t = 1 \\ \epsilon, & \text{if } U_t = 0 \end{cases}$$

If the encoder receives X_t from the sensor, it sends an encoded message Y_t to the communication channel. The encoder operates under the average power constraint:

$$\mathbb{E}[Y_t^2 | U_t = 1] \leq P_T$$

where the expectation is taken over Y_t. Furthermore, P_T is known and is invariant of time. The encoded message Y_t is corrupted by an additive channel noise V_t. The noise process $\{V_t\}$ is a one-dimensional i.i.d. stochastic process with density p_V, which is independent of $\{X_t\}$. When sending Y_t to the communication channel, the encoder is able to transmit the sign of X_t to the decoder via a side channel, which is assumed to be noise-free. If the encoder receives ϵ from the sensor, it sends zero to both the communication channel and the side channel. Consequently, the decoder can deduce the sensor's decision from the message conveyed via the side channel. We use $\tilde{Y}_t$ and S_t to denote the messages received by the decoder from the communication channel and the side channel, respectively, that is

$$\tilde{Y}_t = \begin{cases} Y_t + V_t, & \text{if } U_t = 1 \\ V_t, & \text{if } U_t = 0 \end{cases}, \qquad S_t = \begin{cases} \text{sgn}(X_t), & \text{if } U_t = 1 \\ 0, & \text{if } U_t = 0 \end{cases}$$

After receiving $\tilde{Y}_t$ and S_t, the decoder produces an estimate on X_t, denoted by $\hat{X}_t$. The decoder is charged for distortion in estimation. We assume that the distortion function $\rho(X_t, \hat{X}_t)$ is the squared error, $(X_t - \hat{X}_t)^2$.

2.1.2 Communication Constraint

The sensor is said to have a *soft constraint* if there is a non-negative cost function associated with U_t, denoted by $\mathcal{C}(U_t)$. Here, the cost function is assumed to have the form of

$$\mathcal{C}(U_t) = cU_t = \begin{cases} 0, & \text{if } U_t = 0 \\ c, & \text{if } U_t = 1 \end{cases}$$

where c is called the communication cost ($c > 0$), which is known and is invariant of time. The sensor is said to have a *hard constraint* if it is restricted to use the noisy channel for no more than N times ($N < T$).

2.1.3 Decision Strategies

Assume that at time t, the sensor has memory on all its observations up to t, denoted by $X_{1:t}$, and all the decisions it has made up to $t-1$, denoted by $U_{1:t-1}$. The sensor determines whether or not to transmit its observation at time t, based on its current information $(X_{1:t}, U_{1:t-1})$, namely

$$U_t = f_t(X_{1:t}, U_{1:t-1})$$

where f_t is the communication scheduling policy at time t, and $\mathbf{f} = \{f_1, f_2, \ldots, f_T\}$ is the communication scheduling strategy.

Similarly, at time t, the encoder has memory on all the messages received from the sensor up to t, denoted by $\tilde{X}_{1:t}$, and all the messages it has sent to the communication channel and the side channel up to $t-1$, denoted by

$Y_{1:t-1}$ and $S_{1:t-1}$, respectively. The encoder generates the encoded message at time t, based on its current information $(\tilde{X}_{1:t}, Y_{1:t-1}, S_{1:t-1})$, namely

$$Y_t = g_t(\tilde{X}_{1:t}, Y_{1:t-1}, S_{1:t-1})$$

where g_t is the encoding policy at time t, and $\mathbf{g} = \{g_1, g_2, \ldots, g_T\}$ is the encoding strategy.

Finally, we assume that at time t, the decoder has memory on all the messages received from the communication channel up to t, denoted by $\tilde{Y}_{1:t}$, and all the messages received from the side channel up to t, which are $S_{1:t}$. The decoder generates the estimate at time t, based on its current information $(\tilde{Y}_{1:t}, S_{1:t})$, namely

$$\hat{X}_t = h_t(\tilde{Y}_{1:t}, S_{1:t})$$

where h_t is the decoding policy at time t, and $\mathbf{h} = \{h_1, h_2, \ldots, h_T\}$ is the decoding strategy.

Remark 2.1. *Although we do not assume that the decoder has memory on its previous estimates up to t, yet it can deduce them from $(\tilde{Y}_{1:t-1}, S_{1:t-1})$ and $h_1, h_2, \ldots, h_{t-1}$.*

For simplicity, we call the sensor, the encoder, and the decoder as *decision makers*. Correspondingly, we call the communication scheduling policy (strategy), the encoding policy (strategy), and the decoding policy (strategy) as *decision policies (strategies)*.

2.1.4 Optimization Problem

In Sections 2.2 and 2.3, we consider two sub-problems separately, namely the optimization problems with soft and hard constraints. Let the time horizon T, the probability density functions p_X and p_V, and the power constraint P_T be given.

Optimization problem with soft constraint: Given the communication cost c, determine $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ minimizing the cost functional

$$J(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \mathbb{E}\left\{\sum_{t=1}^{T} cU_t + (X_t - \hat{X}_t)^2\right\}$$

Optimization problem with hard constraint: Given the number of transmission opportunities N, determine $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ minimizing, under the hard constraint, the cost functional

$$J(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \mathbb{E}\left\{\sum_{t=1}^{T}(X_t - \hat{X}_t)^2\right\}$$

2.2 Optimization Problem with Soft Constraint

First, we show that the optimization problem with soft constraint can be simplified to a *single-stage* problem, as described in Theorem 2.1.

Theorem 2.1. *Consider the optimization problem formulated in Section 2.1.4 with the soft constraint.*

1. *Without loss of optimality, one can restrict all the decision makers to apply the decision policies* (f_t, g_t, h_t) *in the forms of:*

$$U_t = f_t(X_t),\ Y_t = g_t(\tilde{X}_t),\ \hat{X}_t = h_t(\tilde{Y}_t, S_t) \tag{2.1}$$

2. *Without loss of optimality, one can restrict all the decision makers to apply stationary decision strategies* $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$*, i.e.,* $\boldsymbol{f} = \{f, f, \ldots, f\}$*,* $\boldsymbol{g} = \{g, g, \ldots, g\}$*,* $\boldsymbol{h} = \{h, h, \ldots, h\}$*.*

Proof. Since the source and noise processes are i.i.d., and the communication cost and the power constraint are invariant of time, the above results are quite intuitive. A detailed proof can be found in Appendix A.1.

By Theorem 2.1, the optimization problem with soft constraint can be reduced to a single-stage problem. Therefore, for simplicity we suppress the subscript for time in all the expressions for the rest of this section. To present our main results for the single-stage problem, we need the following four assumptions.

Assumption 2.1. *The source density* p_X *is nonatomic, even, and log-concave with support* $\mathbb{R}$*. Furthermore,* p_X *is continuously differentiable on* $(0, \infty)$ *(and on* $(-\infty, 0)$ *by symmetry).*

Remark 2.2. *There are several probability density functions satisfying Assumption 2.1, e.g., zero-mean Gaussian distribution, zero-mean Laplace distribution, and a few others. For simplicity, we assume here that* p_X *has support* $\mathbb{R}$*. However, the results also hold for the source density with support* $(-a, a), a > 0$*, e.g., uniform distribution. In that case, we require that* p_X *is continuously differentiable on* $(0, a)$*.*

Given any communication scheduling policy f, let $\mathcal{T}_0^f$, $\mathcal{T}_{1+}^f$, and $\mathcal{T}_{1-}^f$ be the *non-transmission region*, the *positive transmission region* and the *negative transmission region*, corresponding to f, where

$$\mathcal{T}_0^f := \{x \in \mathbb{R} | f(x) = 0\},\ \mathcal{T}_{1+}^f := \{x > 0 | f(x) = 1\},\ \mathcal{T}_{1-}^f := \{x < 0 | f(x) = 1\}$$

Note that $\mathcal{T}_0^f$, $\mathcal{T}_1^f$, $\mathcal{T}_2^f$ may not be connected regions. Then, we make the following assumption on the communication scheduling policy.

Assumption 2.2. *The sensor is restricted to apply the communication scheduling policy* f *satisfying*

$$\mathbb{E}[X | X \in \mathcal{T}_{1-}^f] < \mathbb{E}[X | X \in \mathcal{T}_0^f] < \mathbb{E}[X | X \in \mathcal{T}_{1+}^f] \tag{2.2}$$

Remark 2.3. *There is a wide class of communication scheduling policies satisfying inequality* (2.2)*. For example, given any even communication scheduling policy* f*, i.e.,* $f(x) = f(-x) \in \{0, 1\}$*, and any even source density function* p_X*, we have*

$$\mathbb{E}[X | X \in \mathcal{T}_{1-}^f] < 0, \quad \mathbb{E}[X | X \in \mathcal{T}_0^f] = 0, \quad \mathbb{E}[X | X \in \mathcal{T}_{1+}^f] > 0$$

Then, Assumption 2.2 is satisfied.

Assumption 2.3. *The communication channel noise* V *has zero mean, and finite variance, denoted by* σ_V^2*.*

Assumption 2.4. *The encoder and the decoder are restricted to apply piece-*

wise affine policies:

$$g(\tilde{X}) = \begin{cases} S\alpha(S)\big(X - \mathbb{E}\left[X|U=1,S\right]\big), & \text{if } U=1 \\ 0, & \text{if } U=0 \end{cases}$$

$$h(\tilde{Y},S) = \begin{cases} S\dfrac{1}{\alpha(S)}\dfrac{\gamma}{\gamma+1}\tilde{Y} + \mathbb{E}\left[X|U=1,S\right], & \text{if } U=1 \\ \mathbb{E}[X|U=0], & \text{if } U=0 \end{cases}$$

where $\gamma = P_T/\sigma_V^2$ *is the signal-to-noise ration (SNR).* $\alpha(S)$ *is the amplifying ratio, and* $\alpha(S) = \sqrt{P_T/\mathrm{Var}(X|U=1,S)}$*.* $\mathrm{Var}(X|U=1,S)$ *is the conditional variance.*

It can be checked that when applying the encoding policy described above, the power consumption of the encoder meets the average power constraint (more details can be found in [45]). Moreover, the events $U=0$, $(U=1,S=-1)$, and $(U=1,S=1)$ are equivalent to the events $X \in \mathcal{T}_0^f$, $X \in \mathcal{T}_-^f$, and $X \in \mathcal{T}_{1+}^f$, respectively. Therefore, the encoding and decoding policies (g,h) are induced by the source density p_X and the communication scheduling policy f. For simplicity, we use $J(f)$ instead of $J(f,g,h)$ to denote the cost functional in the rest of this section.

Remark 2.4. *Note that the assumption of piece-wise affine encoding policies originates from a prior work [46], which analyzed a memoryless zero-sum jamming game between a pair of transmitter and receiver and an adversary that generates an additive channel noise subject to second order (power) statistical constraints. It was shown in [46] that the saddle-point equilibrium associated with this zero-sum game is achieved by affine encoding/decoding policies for the transmitter-receiver pair. Here, we utilize such piece-wise affine policies, not only because they facilitate a tractable analysis but also because they possess such mini-max robustness properties (see [46] for more details).*

Theorem 2.2. *Consider the single-stage problem under Assumptions 1-4. Then, the optimal communication scheduling policy is of the threshold type:*

$$f(x) = \begin{cases} 0, & \text{if } |x| < \beta \\ 1, & \text{otherwise} \end{cases}$$

where $\beta > 0$ is the threshold. Furthermore, there exists a unique value β^* minimizing the cost functional $J(f)$ among all such thresholds.

To prove Theorem 2.2, we need the following definitions and lemmas. We first introduce a quantization problem.

Quantization Problem: The problem is one of quantizing the realizations (denoted by x) of a real-valued random variable (denoted by X) to $\mathcal{N}$ codepoints ($\mathcal{N}$ is finite and is known) according to some quantization rule (or quantizer) Q, i.e.,

$$Q(x) = q_i, \text{ if } x \in S_i, \quad i \in \{1, 2, \ldots, \mathcal{N}\}$$

where $S_1, S_2, \ldots, S_{\mathcal{N}}$ are called quantization regions and $q_1, q_2, \ldots, q_{\mathcal{N}}$ are the corresponding codepoints. Note that $S_1, S_2, \ldots, S_{\mathcal{N}}$ are mutually disjoint sets and their union equals $\mathbb{R}$. The distortion error between a realization x and the its quantized value $Q(x)$ is $\rho(|x - Q(x)|)$, where $\rho : [0, \infty) \to [0, \infty)$ is called the distortion function. The performance of the quantizer Q is evaluated by its mean distortion error, denoted by $D(Q)$, i.e.,

$$D(Q) := \mathbb{E}\Big[\rho\big(|X - Q(X)|\big)\Big]$$

Then, given the probability distribution of X, the optimization problem is to design a quantizer $Q = Q^*$ (i.e., design $\{S_1, S_2, \ldots, S_{\mathcal{N}}\}$ and $\{q_1, q_2, \ldots, q_{\mathcal{N}}\}$) that minimizes $D(Q)$.

We recall here a result on the regularity of the optimal quantizer, which we will use shortly.

Lemma 2.1 ([47], Theorem 1 and Corollary 1)**.** *Assume that the source X has nonatomic distribution p_X, and $\rho : [0, \infty) \to [0, \infty)$ is convex and nondecreasing. Then, for any $\mathcal{N}$-level quantizer Q with quantization regions $\{S_1, S_2, \ldots, S_{\mathcal{N}}\}$ and the corresponding codepoints $\{q_1, q_2, \ldots, q_{\mathcal{N}}\}$, there exists a quantizer $\hat{Q}$ with quantization regions $\{\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_{\mathcal{N}}\}$ and the corresponding codepoints $\{\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{\mathcal{N}}\}$ such that*

1. *$\hat{S}_i$ is convex, and $\mathbb{P}(X \in \hat{S}_i) = \mathbb{P}(X \in S_i)$, for all $i = 1, \ldots, \mathcal{N}$.*
2. *If $q_i < q_j$, then $\hat{S}_i < \hat{S}_j$, i.e., $x < y$ for any $x \in \hat{S}_i$ and $y \in \hat{S}_j$.*
3. *$D(\hat{Q}) \le D(Q)$.*

Lemma 2.1 says that given any quantizer, we can build another quantizer achieving non-greater mean distortion error, by rearranging the quantization regions. Furthermore, the rearranged quantization regions are connected and have the same probability measure with the original quantization regions.

Now returning to our problem, for any communication scheduling policy f, we can construct a three-level quantizer, denoted by Q^f, with quantizing regions $(\mathcal{T}_0^f,\ \mathcal{T}_{1+}^f,\ \mathcal{T}_{1-}^f)$ and the corresponding codepoints $(\mathbb{E}[X|X \in \mathcal{T}_0^f]$, $\mathbb{E}[X|X \in \mathcal{T}_{1+}^f]$, $\mathbb{E}[X|X \in \mathcal{T}_{1-}^f])$. Let $D(Q^f)$ be the mean squared distortion of Q^f, i.e.,

$$\begin{aligned} D(Q^f) &= \mathbb{E}\Big[\big(X - Q^f(X)\big)^2\Big] \\ &= \sum_{i\in\{0,1+,1-\}} \mathbb{E}\Big[\big(X - \mathbb{E}[X|X \in \mathcal{T}_i^f]\big)^2 | X \in \mathcal{T}_i^f\Big]\ \mathbb{P}(X \in \mathcal{T}_i^f) \\ &= \sum_{i\in\{0,1+,1-\}} \mathrm{Var}(X|X \in \mathcal{T}_i^f)\ \mathbb{P}(X \in \mathcal{T}_i^f) \end{aligned}$$

By Lemma 2.1, we have the following result.

Lemma 2.2. *Suppose the source density p_X is nonatomic and even. Then, for any communication scheduling policy f satisfying Assumption 2.2, we can construct a threshold-based communication scheduling policy $f^{(1)}$ such that*

1. $\mathcal{T}_0^{f^{(1)}} = (-\beta_2, \beta_1), \mathcal{T}_{1+}^{f^{(1)}} = (\beta_1, \infty), \mathcal{T}_{1-}^{f^{(1)}} = (-\infty, -\beta_2)$, *where* $\beta_1 > 0, \beta_2 > 0$ *are thresholds.*
2. $\mathbb{P}(X \in \mathcal{T}_i^{f^{(1)}}) = \mathbb{P}(X \in \mathcal{T}_i^f)$, *for all* $i \in \{0, 1+, 1-\}$.
3. $D(Q^{f^{(1)}}) \leq D(Q^f)$.

PROOF. By Lemma 2.1, given a three-level quantizer Q^f, there exists a three-level quantizer $\hat{Q}$ with quantization regions $(\hat{S}_0, \hat{S}_{1+}, \hat{S}_{1-})$ and corresponding codepoints $(\hat{q}_0, \hat{q}_{1+}, \hat{q}_{1-})$ such that

1. $\hat{S}_i$ is convex, and $\mathbb{P}(X \in \hat{S}_i) = \mathbb{P}(X \in \mathcal{T}_i^f)$, for all $i \in \{0, 1+, 1-\}$.
2. $\hat{S}_{1-} < \hat{S}_0 < \hat{S}_{1+}$.
3. $D(\hat{Q}) \leq D(Q)$.

The second item holds since $\mathbb{E}[X|X \in \mathcal{T}_{1-}^f] < \mathbb{E}[X|X \in \mathcal{T}_0^f] < \mathbb{E}[X|X \in \mathcal{T}_{1+}^f]$ (Assumption 2.2). Note that since $\mathcal{T}_{1+}^f \subseteq (0, \infty)$, $\mathcal{T}_{1-}^f \subseteq (-\infty, 0)$, and the source density p_X is even, we have

$$\mathbb{P}(X \in \hat{S}_{1+}) = \mathbb{P}(X \in \mathcal{T}_{1+}^f) \leq \frac{1}{2}, \quad \mathbb{P}(X \in \hat{S}_{1-}) = \mathbb{P}(X \in \mathcal{T}_{1-}^f) \leq \frac{1}{2}$$

Combining the above inequalities with the second item, we have $\hat{S}_{1+} = (\beta_1, \infty)$, $\hat{S}_{1-} = (-\infty, -\beta_2)$, and $\hat{S}_0 = (-\beta_2,\ \beta_1)$ for some $\beta_1, \beta_2 \geq 0$. We now construct a threshold-based communication scheduling policy $f^{(1)}$ by letting $\mathcal{T}_i^{f^{(1)}} = \hat{S}_i$, $i \in \{0, 1+, 1-\}$. Since the distortion function is the squared error, the optimal codepoints corresponding to quantization regions $(\mathcal{T}_0^{f^{(1)}}, \mathcal{T}_{1+}^{f^{(1)}}, \mathcal{T}_{1-}^{f^{(1)}})$ are $(\mathbb{E}[X|X \in \mathcal{T}_0^{f^{(1)}}], \mathbb{E}[X|X \in \mathcal{T}_{1+}^{f^{(1)}}], \mathbb{E}[X|X \in \mathcal{T}_{1-}^{f^{(1)}}])$. Hence, we have $D(Q^{f^{(1)}}) \leq D(\hat{Q}) \leq D(Q^f)$. □

Note that $f^{(1)}$ constructed in Lemma 2.2 may or may not be symmetric around zero. We now propose the following proposition, which states that based on $f^{(1)}$, we can further construct a threshold-based policy $f^{(2)}$, which is symmetric around zero and has non-greater mean squared distortion. Furthermore, the probability measure over the non-transmission region of $f^{(2)}$ is the same as that of $f^{(1)}$.

Proposition 2.1. *Suppose the source density p_X satisfies Assumption 2.1. Then, for any communication scheduling policy f satisfying Assumption 2.2, we can construct a symmetric threshold-based communication scheduling policy $f^{(2)}$ such that*

1. $\mathcal{T}_0^{f^{(2)}} = (-\beta, \beta), \mathcal{T}_{1+}^{f^{(2)}} = (\beta, \infty), \mathcal{T}_{1-}^{f^{(2)}} = (-\infty, -\beta)$*, where* $\beta > 0$.
2. $\mathbb{P}(X \in \mathcal{T}_0^{f^{(2)}}) = \mathbb{P}(X \in \mathcal{T}_0^f)$.
3. $D(Q^{f^{(2)}}) \leq D(Q^f)$.

To prove Proposition 2.1, we need the following lemma.

Lemma 2.3. *Let p_X be an even and log-concave density. Let $A = [-\tau, \tau]$ be any symmetric closed interval such that $\int_A p_X(x)dx > 0$, and let B be any subset of $\mathbb{R}$ such that $\int_B p_X(x)dx = \int_A p_X(x)dx$. Then,*

$$\mathrm{Var}(X|X \in A) \leq \mathrm{Var}(X|X \in B)$$

PROOF. The proof of the lemma needs results from majorization theory, which are introduced in Appendix A.2.

Furthermore, we also need to apply the property of log-concave distribution, which is introduced below.

Lemma 2.4 ([48], Theorem 6)**.** *Let p_X be a continuously differentiable and log-concave probability density function defined on (a, b). Let β be a variable belonging to interval (a, b). Then, the function $G_X(\beta)$, defined below, is monotone decreasing in β:*

$$G_X(\beta) := \mathbb{E}[X|X > \beta] - \beta \tag{2.3}$$

Note that a and b in Lemma 2.4 can be $-\infty$ and ∞, respectively. In the rest of the chapter we will frequently refer to this function $G_X(\beta)$.[1] We next provide an extension of Lemma 2.4 as follows.

Lemma 2.5. *Let p_X be an even and log-concave probability density function defined on $\mathbb{R}$. Furthermore, let p_X be continuously differentiable on $(0, \infty)$ and $(-\infty, 0)$, and let β take value in $(0, \infty)$. Then, $G_X(\beta)$ as defined by (2.3) is monotone decreasing in β for $\beta \in (0, \infty)$.*

PROOF. Let Y be a random variable such that $Y = |X|$. Denote by p_Y be the probability function of Y. Since the probability density of X, p_X is even, we have

$$p_Y(y) = \begin{cases} 2p_X(y), & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Since p_X is continuously differentiable on $(0, \infty)$, so is p_Y. Furthermore, it can be shown quite readily that for any $\beta \in (0, \infty)$, $\mathbb{E}[Y|Y > \beta] = \mathbb{E}[X|X > \beta]$. Then, we have $G_Y(\beta) = G_X(\beta)$. By Lemma 2.4, $G_Y(\beta)$ is monotone decreasing in β. Hence, we conclude that $G_X(\beta)$ is also monotone decreasing in β. □

We are now in a position to prove Proposition 2.1.

PROOF of Proposition 2.1. By Lemma 2.2, given any communication scheduling policy $f^{(0)}$ satisfying Assumption 2.2, we can construct a threshold-based policy $f^{(1)}$ such that

[1] $G_X(\beta)$ is also called *the mean residual lifetime.*

1. $\mathcal{T}_0^{f^{(1)}} = (-\beta_2, \beta_1), \mathcal{T}_{1+}^{f^{(1)}} = (\beta_1, \infty), \mathcal{T}_{1-}^{f^{(1)}} = (-\infty, -\beta_2)$.
2. $\mathbb{P}(X \in \mathcal{T}_i^{f^{(1)}}) = \mathbb{P}(X \in \mathcal{T}_i^{f^{(0)}})$, for all $i \in \{0, 1+, 1-\}$.
3. $D(Q^{f^{(1)}}) \leq D(Q^{f^{(0)}})$.

Based on policy $f^{(1)}$, we now construct a symmetric threshold-based policy $f^{(2)}$ such that

1. $\mathcal{T}_0^{f^{(2)}} = (-\beta, \beta), \mathcal{T}_{1+}^{f^{(2)}} = (\beta, \infty), \mathcal{T}_{1-}^{f^{(2)}} = (-\infty, -\beta)$.
2. $\mathbb{P}(X \in \mathcal{T}_0^{f^{(2)}}) = \mathbb{P}(X \in \mathcal{T}_0^{f^{(1)}})$.

Then, we only need to show that $D(Q^{f^{(2)}}) \leq D(Q^{f^{(1)}})$. Note that $D(Q^{f^{(1)}})$ and $D(Q^{f^{(2)}})$ can be expressed as

$$\begin{aligned} D(Q^{f^{(1)}}) &= \textstyle\sum_{i \in \{0,1+,1-\}} \mathrm{Var}(X | X \in \mathcal{T}_i^{f^{(1)}}) \, \mathbb{P}(X \in \mathcal{T}_i^{f^{(1)}}) \\ D(Q^{f^{(2)}}) &= \textstyle\sum_{i \in \{0,1+,1-\}} \mathrm{Var}(X | X \in \mathcal{T}_i^{f^{(2)}}) \, \mathbb{P}(X \in \mathcal{T}_i^{f^{(2)}}) \end{aligned}$$

By Lemma 2.3, we obtain $\mathrm{Var}(X | X \in \mathcal{T}_0^{f^{(2)}}) \leq \mathrm{Var}(X | X \in \mathcal{T}_0^{f^{(1)}})$. Since $\mathbb{P}(X \in \mathcal{T}_0^{f^{(2)}}) = \mathbb{P}(X \in \mathcal{T}_0^{f^{(1)}})$, we have

$$\mathrm{Var}(X | X \in \mathcal{T}_0^{f^{(2)}}) \mathbb{P}(X \in \mathcal{T}_0^{f^{(2)}}) \leq \mathrm{Var}(X | X \in \mathcal{T}_0^{f^{(1)}}) \mathbb{P}(X \in \mathcal{T}_0^{f^{(1)}})$$

Hence, we will be done if we show that

$$\begin{aligned} & \textstyle\sum_{i \in \{1+,1-\}} \mathrm{Var}(X | X \in \mathcal{T}_i^{f^{(2)}}) \, \mathbb{P}(X \in \mathcal{T}_i^{f^{(2)}}) \\ \leq & \textstyle\sum_{i \in \{1+,1-\}} \mathrm{Var}(X | X \in \mathcal{T}_i^{f^{(1)}}) \, \mathbb{P}(X \in \mathcal{T}_i^{f^{(1)}}) \end{aligned}$$

Consider the class of threshold-based communication scheduling policies, denoted by $\mathcal{F}$, whose generic element f is in the form of

$$\mathcal{T}_0^f = (-\eta_2, \eta_1), \quad \mathcal{T}_{1+}^f = (\eta_1, \infty), \quad \mathcal{T}_{1-}^f = (-\infty, -\eta_2), \quad \eta_1, \eta_2 \geq 0$$

and

$$\mathbb{P}(X \in \mathcal{T}_0^f) = \mathbb{P}(X \in \mathcal{T}_0^{f^{(0)}}) = k$$

It is clear that $f^{(1)}$ and $f^{(2)}$ are elements of $\mathcal{F}$. Let $PD(Q^f)$ be the sum of the mean squared distortions of Q^f over regions $\mathcal{T}^f_{1+}$ and $\mathcal{T}^f_{1-}$, i.e.,

$$\begin{aligned}
PD(Q^f) &:= \textstyle\sum_{i\in\{1+,1-\}} \mathrm{Var}(X|X\in\mathcal{T}^f_i)\,\mathbb{P}(X\in\mathcal{T}^f_i) \\
&= \mathrm{Var}\big(X|X<-\eta_2\big)\,\mathbb{P}\big(X<-\eta_2\big)+\mathrm{Var}\big(X|X>\eta_1\big)\,\mathbb{P}\big(X>\eta_1\big) \\
&= \mathrm{Var}\big(X|X>\eta_2\big)\,\mathbb{P}\big(X>\eta_2\big)+\mathrm{Var}\big(X|X>\eta_1\big)\,\mathbb{P}\big(X>\eta_1\big)
\end{aligned}$$

where the last equality holds since p_X is even. We now show that $f^{(2)}$ is a global minimizer of $PD(Q^f)$ among all elements in $\mathcal{F}$. Since $\mathbb{P}(X\in\mathcal{T}^f_0)=k$, we have

$$\int_{-\infty}^{-\eta_2} p_X(x)dx+\int_{\eta_1}^{\infty} p_X(x)dx = 1-k$$

Taking the derivatives of both sides with respect to η_1, we have

$$\frac{d\eta_2}{d\eta_1}\cdot\frac{\partial}{\partial\eta_2}\int_{-a}^{-\eta_2} p_X(x)dx+\frac{\partial}{\partial\eta_1}\int_{\eta_1}^{a} p_X(x)dx=0$$

which implies that

$$\frac{d\eta_2}{d\eta_1}=-\frac{p_X(\eta_1)}{p_X(-\eta_2)}=-\frac{p_X(\eta_1)}{p_X(\eta_2)} \tag{2.4}$$

The equality above holds because p_X is even. Now taking the derivative of $PD(Q^f)$ with respect to η_1, we have

$$\begin{aligned}
\frac{d}{d\eta_1}PD(Q^f) &= \frac{d\eta_2}{d\eta_1}\cdot\frac{\partial}{\partial\eta_2}\mathrm{Var}\big(X|X>\eta_2\big)\,\mathbb{P}\big(X>\eta_2\big) \\
&\quad+\frac{\partial}{\partial\eta_1}\mathrm{Var}\big(X|X>\eta_1\big)\,\mathbb{P}\big(X>\eta_1\big)
\end{aligned} \tag{2.5}$$

The second term in (2.5) can be computed as (details of this derivation can be found in [45]):

$$\frac{\partial}{\partial\eta_1}\mathrm{Var}(X|X>\eta_1)\,\mathbb{P}(X>\eta_1)=-p_X(\eta_1)\big(\eta_1-\mathbb{E}[X|X>\eta_1]\big)^2 \tag{2.6}$$

Similarly, we can simplify the first term in (2.5) to:

$$\frac{\partial}{\partial\eta_2}\mathrm{Var}\big(X|X>\eta_2\big)\,\mathbb{P}\big(X>\eta_2\big)=-p_X(\eta_2)\big(\eta_2-\mathbb{E}[X|X>\eta_2]\big)^2 \tag{2.7}$$

Plugging (2.4), (2.6), and (2.7) into (2.5), we have

$$\begin{aligned}
&\frac{d}{d\eta_1} PD(Q^f) \\
= \;& -\frac{p_X(\eta_1)}{p_X(\eta_2)} \cdot -p_X(\eta_2)\big(\eta_2 - \mathbb{E}[X|X > \eta_2]\big)^2 - p_X(\eta_1)\big(\eta_1 - \mathbb{E}[X|X > \eta_1]\big)^2 \\
= \;& p_X(\eta_1)\big[\big(\eta_2 - \mathbb{E}[X|X > \eta_2]\big)^2 - \big(\eta_1 - \mathbb{E}[X|X > \eta_1]\big)^2\big] \\
= \;& p_X(\eta_1)\Big(G_X^2(\eta_2) - G_X^2(\eta_1)\Big)
\end{aligned}$$

By Lemma 2.5, $G_X(\eta)$ is a non-negative and monotone decreasing function. Then, $G_X^2(\eta)$ is monotone decreasing, and

$$\begin{aligned}
&\frac{d}{d\eta_1} PD(Q^f) \geq 0, \text{ if } \eta_1 > \eta_2, \\
&\frac{d}{d\eta_1} PD(Q^f) = 0, \text{ if } \eta_1 = \eta_2, \\
&\frac{d}{d\eta_1} PD(Q^f) \leq 0, \text{ if } \eta_1 < \eta_2
\end{aligned}$$

Hence, $\eta_1 = \eta_2$ is a global minimizer, which corresponds to $f^{(2)}$. □

Now we are in a position to prove Theorem 2.2 by applying Proposition 2.1. The approach of the proof can be summarized as follows:

1. Given any communication scheduling policy f, it can be computed that the cost functional $J(f)$ consists of two parts: the first part is the mean squared distortion of Q^f, and the second part is a cost functional in the noiseless-channel setting.

2. Based on f, we can construct a symmetric threshold-based communication scheduling policy f', which has the same probability measure on the non-transmission region.

3. By Proposition 2.1, if f satisfies Assumption 2.2, then the first part in the cost functional of f' is lower than that of f. By Lemma 2.3, if the source density is even and log-concave (which is also unimodal), then the second part in the cost functional of f' is also lower than that of f.

4. Without loss of optimality, we can consider only the class of symmetric threshold-based policies. By Lemma 2.5, it can be shown that there exists a unique optimal threshold minimizing the cost functional.

PROOF of Theorem 2.2. Consider any communication scheduling policy f. The expected cost corresponding to f can be computed as follows:

$$\begin{aligned} J(f) &= \mathbb{E}\left[cU + (X-\hat{X})^2\right] \\ &= \sum_{i\in\{0,1+,1-\}} \mathbb{E}\left[cU + (X-\hat{X})^2 | X \in \mathcal{T}_i^f\right] \mathbb{P}(X \in \mathcal{T}_i^f) \end{aligned}$$

When $X \in \mathcal{T}_0^f$, we have $U = 0$ and $\hat{X} = \mathbb{E}[X|X \in \mathcal{T}_0^f]$. Hence,

$$\begin{aligned} \mathbb{E}\Big[cU + (X-\hat{X})^2 | X \in \mathcal{T}_0^f\Big] &= \mathbb{E}\Big[\big(X - \mathbb{E}\big[X|X \in \mathcal{T}_0^f\big]\big)^2 | X \in \mathcal{T}_0^f\Big] \\ &= \mathrm{Var}(X|X \in \mathcal{T}_0^f) \end{aligned}$$

When $X \in \mathcal{T}_{1+}^f$, we have $U = 1$, and $Y = \alpha(1)\big(X - \mathbb{E}\big[X|X \in \mathcal{T}_{1+}^f\big]\big)$. Hence,

$$\begin{aligned} \hat{X} &= \frac{1}{\alpha(1)}\frac{\gamma}{\gamma+1}\tilde{Y} + \mathbb{E}\big[X|X \in \mathcal{T}_{1+}^f\big] \\ &= \frac{\gamma}{\gamma+1}X + \frac{1}{\alpha(1)}\frac{\gamma}{\gamma+1}V + \frac{1}{\gamma+1}\mathbb{E}\big[X|X \in \mathcal{T}_{1+}^f\big] \end{aligned}$$

where

$$\alpha(1) = \sqrt{\frac{P_T}{\mathrm{Var}(X|X \in \mathcal{T}_{1+}^f)}}, \quad \gamma = \frac{P_T}{\sigma_V^2}$$

Hence, it can be shown that (for details of the derivation, see [45])

$$\mathbb{E}\big[cU + (X-\hat{X})^2 | X \in \mathcal{T}_{1+}^f\big] = c + \frac{1}{\gamma+1}\mathrm{Var}(X|X \in \mathcal{T}_{1+}^f)$$

Similarly, one can compute that

$$\mathbb{E}\big[cU + (X-\hat{X})^2 | X \in \mathcal{T}_{1-}^f\big] = c + \frac{1}{\gamma+1}\mathrm{Var}(X|X \in \mathcal{T}_{1-}^f)$$

Hence, $J(f)$ can be further expressed as

$$
\begin{aligned}
& J(f) \\
= \; & \mathrm{Var}(X|X \in \mathcal{T}_0^f)\mathbb{P}(X \in \mathcal{T}_0^f) + \frac{1}{\gamma+1}\mathrm{Var}(X|X \in \mathcal{T}_{1+}^f)\mathbb{P}(X \in \mathcal{T}_{1+}^f) \\
& + \frac{1}{\gamma+1}\mathrm{Var}(X|X \in \mathcal{T}_{1-}^f)\mathbb{P}(X \in \mathcal{T}_{1-}^f) + c\,\mathbb{P}(X \in \mathcal{T}_{1-}^f) + c\,\mathbb{P}(X \in \mathcal{T}_{1+}^f) \\
= \; & \frac{1}{\gamma+1}D(Q^f) + \frac{\gamma}{\gamma+1}\mathrm{Var}(X|X \in \mathcal{T}_0^f)\mathbb{P}(X \in \mathcal{T}_0^f) \\
& + c\,\mathbb{P}(X \in \mathcal{T}_{1+}^f) + c\,\mathbb{P}(X \in \mathcal{T}_{1-}^f)
\end{aligned}
\tag{2.8}
$$

Given any communication scheduling policy f, we can construct a symmetric threshold-based communication scheduling policy f' such that

1. $\mathcal{T}_0^{f'} = (-\beta, \beta)$,$\mathcal{T}_{1+}^{f'} = (\beta, \infty)$,$\mathcal{T}_{1-}^{f'} = (-\infty, -\beta)$.
2. $\mathbb{P}(X \in \mathcal{T}_0^{f'}) = \mathbb{P}(X \in \mathcal{T}_0^f)$, or equivalently,
   $\mathbb{P}(X \in \mathcal{T}_{1+}^{f'}) + \mathbb{P}(X \in \mathcal{T}_{1-}^{f'}) = \mathbb{P}(X \in \mathcal{T}_{1+}^f) + \mathbb{P}(X \in \mathcal{T}_{1-}^f)$

By Proposition 2.1 and Lemma 2.3, we have $D(Q^{f'}) \leq D(Q^f)$ and $\mathrm{Var}(X|X \in \mathcal{T}_0^{f'}) \leq \mathrm{Var}(X|X \in \mathcal{T}_0^f)$. Furthermore, we have $\mathbb{P}(X \in \mathcal{T}_0^{f'}) = \mathbb{P}(X \in \mathcal{T}_0^f)$ and thus $c\,\mathbb{P}(X \in \mathcal{T}_{1+}^{f'}) + c\,\mathbb{P}(X \in \mathcal{T}_{1-}^{f'}) = c\,\mathbb{P}(X \in \mathcal{T}_{1+}^f) + c\,\mathbb{P}(X \in \mathcal{T}_{1-}^f)$. Hence, we conclude that $J(f') \leq J(f)$.

The result above implies that without loss of optimality, we can restrict the search of the optimal communication scheduling policy to the class of symmetric threshold type. Denote by $J(\beta)$ the expected cost corresponding to a symmetric threshold-based communication scheduling policy with threshold β, where $\beta \geq 0$. By (2.8), $J(\beta)$ can be computed as

$$
\begin{aligned}
& J(\beta) \\
= \; & \int_{-\beta}^{\beta} x^2 p_X(x)dx + \frac{1}{\gamma+1}\mathrm{Var}(X|X < -\beta)\mathbb{P}(X < -\beta) + c\int_{-\infty}^{-\beta} p_X(x)dx \\
& + \frac{1}{\gamma+1}\mathrm{Var}(X|X > \beta)\mathbb{P}(X > \beta) + c\int_{\beta}^{\infty} p_X(x)dx \\
= \; & 2\int_0^{\beta} x^2 p_X(x)dx + 2\frac{1}{\gamma+1}\mathrm{Var}(X|X > \beta)\mathbb{P}(X > \beta) + 2c\int_{\beta}^{\infty} p_X(x)dx
\end{aligned}
$$

where the second equality holds since p_X is even. Taking the derivative of

$J(\beta)$ with respect to β, and by eq. (2.6), we have

$$\begin{aligned}\frac{d}{d\beta}J(\beta) &= 2p_X(\beta)\Big(\beta^2 - \frac{1}{\gamma+1}\big(\mathbb{E}[X|X>\beta]-\beta\big)^2 - c\Big)\\ &= 2p_X(\beta)\Big(\beta^2 - \frac{1}{\gamma+1}G_X^2(\beta) - c\Big)\end{aligned}$$

Since $c > 0$ and $G_X(\beta)$ is monotone decreasing, there exists a unique β^* in $[0, \infty)$ such that

$$\beta^{*2} = \frac{1}{\gamma+1}G_X^2(\beta^*) + c$$

Furthermore, $dJ(\beta)/d\beta < 0$ when $\beta < \beta^*$ and $dJ(\beta)/d\beta > 0$ when $\beta > \beta^*$. Hence, β^* is the unique global minimizer among all $\beta \geq 0$. □

Remark 2.5. *If the density function p_X has support $(-a, a)$ and $0 < a < \beta^*$, then $dJ(\beta)/d\beta$ is always negative, which implies that the minimizing β is just a. This means the optimal communication scheduling policy is to always choose no transmission regardless of sensor's observation. Such a case can occur when the communication cost is very high.*

2.3 Optimization Problem with Hard Constraint

To present our main results for the problem with the hard constraint, we introduce a number of terms as follows.

First, we let E_t denote the number of remaining communication opportunities at the beginning of the t-th time interval, i.e., $E_t = N - \sum_{i=1}^{t-1} U_i$. Then, evolution of E_t is described by

$$E_t = E_{t-1} - U_{t-1},\ t \geq 2; \quad E_1 = N \tag{2.9}$$

Furthermore, the communication constraint can be described as

$$U_t \leq E_t, \quad \forall\ t = 1, 2, \ldots, T \tag{2.10}$$

Recall that $U_{1:t-1}$ is the common information shared by all the decision makers, and hence E_t is also known by all the decision makers.

Second, we let $J^*(t, E_t)$ be the optimal cost-to-go if the system is initialized at time t with E_t number of communication opportunities. Specifically, we define $J^*(T+1,\cdot) = 0$ for any number of communication opportunities.

Third, for any $E_t > 0$, we let $c(t, E_t)$ denote the difference between two optimal cost-to-go, i.e.,

$$c(t, E_t) = J^*(t+1, E_t - 1) - J^*(t+1, E_t)$$

Remark 2.6. *$c(t, E_t)$ can be interpreted as the opportunity cost for choosing to communicate with the estimator rather than not to communicate.*

The following theorem ensures that without loss of optimality, we can restrict all the decision makers to consider only their current inputs and E_t when making decisions at time t. Furthermore, the optimal decision policies can be obtained via solving a dynamic programming equation.

Theorem 2.3. *Consider the optimization problem with hard constraint as formulated in Section 2.1.4. Without loss of optimality, we can restrict the communication scheduling, the encoding and the decoding policies to the forms:*

$$U_t = f_t(X_t, E_t),\ Y_t = g_t(\tilde{X}_t, E_t),\ \hat{X}_t = h_t(\tilde{Y}_t, S_t, E_t)$$

Furthermore, the optimal cost-to-go $J^*(t, E_t)$ can be obtained by solving the dynamic programming (DP) equation:

$$\begin{aligned} J^*(T+1,\cdot) &= 0 \\ J^*(t, E_t) &= \inf_{f_t, g_t, h_t} \mathbb{E}\left\{(X_t - \hat{X}_t)^2 + J^*(t+1, E_{t+1})\right\} \end{aligned} \tag{2.11}$$

The proof of Theorem 2.3 is similar to that of Theorem 2.1, and hence is not included here; it can be found in [45].

Consider the DP equation (2.11), and we have the following discussion.

1. When $E_t = 0$, by the communication constraint $U_t = f_t(X_t, E_t) = 0$ regardless of the realization of X_t. Consequently, we have $E_{t+1} = 0$. Then, the DP equation can be easily expressed as follows:

   $$J^*(t, 0) = \inf_{f_t, g_t, h_t} \mathbb{E}\left\{(X_t - \hat{X}_t)^2\right\} + J^*(t+1, 0) = \mathrm{Var}(X_t) + J^*(t+1, 0)$$

   The last equality holds since without any information about X_t, the optimal estimator is $\mathbb{E}[X_t]$ and the mean squared error is $\mathrm{Var}(X_t)$.

2. When $E_t > 0$, the DP equation can be written as

$$\begin{aligned} J^*(t, E_t) &= \inf_{f_t, g_t, h_t} \mathbb{E}\left\{(X_t - \hat{X}_t)^2 + J^*(t+1, E_{t+1})\right\} \\ &= J^*(t+1, E_t) + \inf_{f_t, g_t, h_t} \mathbb{E}\left\{c(t, E_t)U_t + (X_t - \hat{X}_t)^2\right\} \end{aligned} \quad (2.12)$$

Note that the minimization in the second line of (2.12) is just the single-stage problem considered in Section 2.2 with communication cost $c(t, E_t)$. This motivates us to make the following two assumptions.

Assumption 2.5. *The sensor is restricted to apply the communication scheduling policies f_t such that for any $1 \le t \le T$ and any $E_t > 0$,*

$$\mathbb{E}[X_t | U_t = 1, E_t, X_t < 0] < \mathbb{E}[X_t | E_t, U_t = 0] < \mathbb{E}[X_t | U_t = 1, E_t, X_t > 0]$$

Assumption 2.6. *The encoder and the decoder are restricted to apply piecewise affine policies:*

$$g_t(\tilde{X}_t, E_t) = \begin{cases} S_t \alpha(S_t) \left(X_t - \mathbb{E}\left[X_t | U_t = 1, E_t, S_t\right]\right), & \text{if } U_t = 1 \\ 0, & \text{if } U_t = 0 \end{cases}$$

$$h_t(\tilde{Y}_t, S_t, E_t) = \begin{cases} S_t \dfrac{1}{\alpha(S_t)} \dfrac{\gamma}{\gamma + 1} \tilde{Y}_t + \mathbb{E}\left[X_t | U_t = 1, E_t, S_t\right], & \text{if } U_t = 1 \\ \mathbb{E}[X_t | U_t = 0, E_t], & \text{if } U_t = 0 \end{cases}$$

where $\gamma = P_T / \sigma_V^2$, $\alpha(S_t) = \sqrt{P_T / \mathrm{Var}(X_t | U_t = 1, E_t, S_t)}$, and $\mathrm{Var}(X_t | U_t = 1, E_t, S_t)$.

Then, we have the following theorem on the optimality of symmetric threshold-based communication scheduling strategy. Its proof involves simply an application of Theorem 2.2, and hence is not included here.

Theorem 2.4. *Consider the problem with hard constraint under Assumptions 2.1, 2.3, 2.5 and 2.6, the optimal communication scheduling policy f_t^* for the sensor is*

$$f_t^*(X_t, E_t) = \begin{cases} 1, & \text{if } E_t > 0 \text{ and } |X_t| > \beta_t^*(E_t) \\ 0, & \text{if } E_t = 0 \text{ or } |X_t| \le \beta_t^*(E_t) \end{cases} \quad (2.13)$$

where $\beta_t^*(E_t)$ *is non-negative and is the unique solution to the fixed-point equation:*

$$\beta^2 = \frac{1}{\gamma+1} G_X^2(\beta) + c(t, E_t), \quad \beta \geq 0 \tag{2.14}$$

where

$$G_X\big(\beta\big) = \mathbb{E}\big[X_t | X_t > \beta\big] - \beta$$

Remark 2.7. *Consider the case where* $E_t > T - t$*, that is, the sensor is always allowed to communicate with the estimator for the remaining time steps. First, we note that the opportunity cost* $c(t, E_t)$ *is zero. Furthermore, since* $G_X(0) = \mathbb{E}[X|X > 0] > 0$*, the solution to* (2.14) *is non-zero. Then, even though the sensor can always communicate with the estimator, the optimal communication policy is still the threshold-based policy with threshold* $\beta_t^*(E_t) > 0$*, which might seem counter-intuitive: why would the sensor not transmit its observation although it is allowed to do so? This surprising result is due to the fact that threshold information, i.e., whether or not the state sample belongs to a fixed, known interval, might be more informative than a noisy observation of the state at the output of the noisy channel. Hence, it might be better not to communicate explicitly over the noisy channel but rely on the side channel which signals where the sample lies. For example, at the extreme case of a very noisy channel (*$\gamma \to 0$*) the output of the communication channel,* $\tilde{Y}_t$*, is effectively useless, irrespective of the realization* X_t*. However, depending on the threshold and the realization* X_t*, thresholding information could be significantly more informative.*

2.4 Numerical Results

In this section, we present numerical results for the problem with hard constraint. We select the source density to be Laplace distribution with zero-mean and parameter λ, i.e.,

$$p_X(x) = \begin{cases} \frac{1}{2}\lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ \frac{1}{2}\lambda e^{\lambda x}, & \text{if } x < 0 \end{cases}$$

Then, it is easy to see that

$$G_X(\beta) = \mathbb{E}[X_t | X_t > \beta] - \beta = \frac{1}{\lambda}, \quad \forall\, \beta \geq 0$$

Hence, the solution to (2.14) is

$$\beta_t^*(E_t) = \sqrt{\frac{1}{\gamma+1}\frac{1}{\lambda^2} + c(t, E_t)} = \sqrt{m + c(t, E_t)}$$

where $m := 1/((\gamma+1)\lambda^2)$. Then, the optimal communication scheduling policy is described by (2.13). Furthermore, the optimal encoding/decoding policies (g_t^*, h_t^*) are as follows:

$$\begin{aligned} g_t(\tilde{X}_t, E_t) &= \begin{cases} \alpha\big(|X_t| - \beta_t^*(E_t) - \lambda^{-1}\big), & \text{if } U_t = 1 \\ 0, & \text{if } U_t = 0 \end{cases} \\ h_t(\tilde{Y}_t, S_t, E_t) &= \begin{cases} S_t\Big(\frac{1}{\alpha}\frac{\gamma}{\gamma+1}\tilde{Y}_t + \beta_t^*(E_t) + \lambda^{-1}\Big), & \text{if } U_t = 1 \\ 0, & \text{if } U_t = 0 \end{cases} \end{aligned}$$

where $\gamma = P_T/\sigma_V^2$, and $\alpha = \sqrt{P_T/\lambda^{-2}}$. By plugging the optimal communication scheduling, the encoding, and the decoding policies (f_t^*, g_t^*, h_t^*) into the DP equation (2.11), we obtain the explicit update rule for the optimal cost-to-go $J^*(t, E_t)$, as shown below

$$\begin{aligned} &J^*(t, E_t) = J^*(t+1, E_t) + 2\lambda^{-2}, && \text{if } E_t = 0 \\ &J^*(t, E_t) = J^*(t+1, E_t) + 2\lambda^{-2} - 2\big(\beta_t^*(E_t)\lambda^{-1} + \lambda^{-2}\big)e^{-\lambda\beta_t^*(E_t)}, && \text{if } E_t > 0 \end{aligned} \tag{2.15}$$

We choose the parameters as follows: $T = 100$, $\lambda = 1$, and the signal-to-noise ratio (SNR) $\gamma = 0.1, 1, 10$. We solve the optimal cost-to-go $J^*(t, E_t)$ by applying the update rule (2.15). We have plotted the optimal 100-stage estimation error $J^*(1, N)$ versus the number of communication opportunities N under different SNRs, as shown in Fig. 2.2.

One can see that, for each fixed SNR, the optimal 100-stage estimation error is non-increasing in terms of the number of communication opportunities. To be more specific, there exists a threshold on the number of communication opportunities (call it *opportunity threshold*) such that the optimal 100-stage

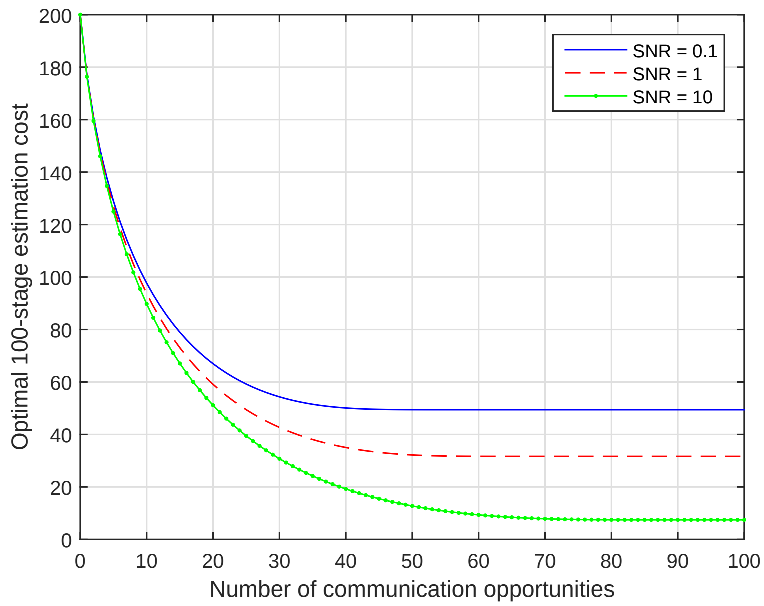

Figure 2.2: 100-stage estimation error vs. the number of communication opportunities

estimation error decreases when the number of communication opportunities is below the threshold, and it stays constant above the threshold. Figure 2.2 shows that when the SNR increases, the opportunity threshold increases.

The existence of an opportunity threshold was not observed in the noiseless channel setting (see [27], Figure 5), where the optimal estimation error strictly decreases to zero as the number of communication opportunities increases to the length of time horizon. This surprising phenomenon can be interpreted as follows: since the sensor applies the symmetric threshold-based policy with threshold $\beta_t^*(E_t) = \sqrt{c(t, E_t) + m} \geq \sqrt{m}$, the expectation of the consumed communication opportunities is upper bounded by $T \cdot \mathbb{P}(|X_t| \geq \sqrt{m}) = Te^{-\lambda\sqrt{m}}$. When the number of communication opportunities is greater than $Te^{-\lambda\sqrt{m}}$, the additional communication opportunities will not be consumed (in the expected sense), and thus the optimal expected estimation error will not further decrease. It can also be checked from Fig. 2.2 that the opportunity thresholds under different signal-to-noise ratios are roughly $Te^{-\lambda\sqrt{m}}$. Moreover, $m = \frac{1}{\gamma+1}\frac{1}{\lambda^2}$ and $Te^{-\lambda\sqrt{m}} = Te^{-1/\sqrt{\gamma+1}}$, which is an increasing function of the SNR γ. Hence, the opportunity threshold increases with the SNR.

Figure 2.3 depicts a sample path of the number of remaining communication opportunities versus time. When generating the plot, we have chosen $T = 100$, $\lambda = 1$, $\gamma = 0.1$, and the number of communication opportunities $N = 50$. One can see that the communication opportunities are not used up by the end of the time horizon. The reason has been discussed in Remark 2.7.

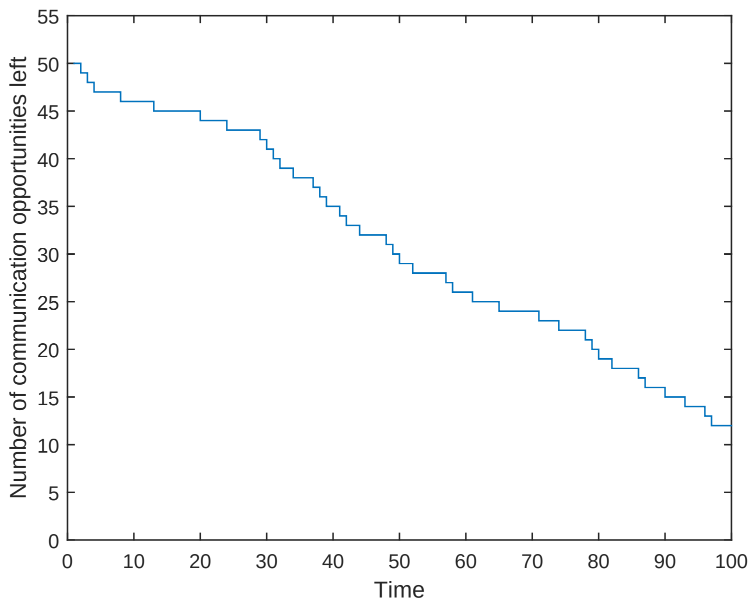

Figure 2.3: A sample path of the number of remaining communication opportunities vs. time

When the number of communication opportunities is larger than the opportunity threshold, the optimal 100-stage estimation error does not change with respect to the number of communication opportunities. We call it *minimal error*. Figure 2.2 shows that the minimal error decreases as the SNR increases. Without loss of generality, we can assume that the sensor is allowed to communicate at each step, that is, $N = T$. Then, the opportunity cost is $c(t, E_t) = 0$. Recall that $\beta_t^*(E_t) = \sqrt{c(t, E_t) + m}$ and $m = \frac{1}{\gamma+1}\frac{1}{\lambda^2}$. Hence, the update rule for the cost function can be simplified as follows:

$$J^*(t,T) = J^*(t+1,T) + \left(\frac{2}{\lambda^2} - (\frac{2\sqrt{m}}{\lambda} + \frac{2}{\lambda^2}) \cdot e^{-\lambda\sqrt{m}}\right)$$

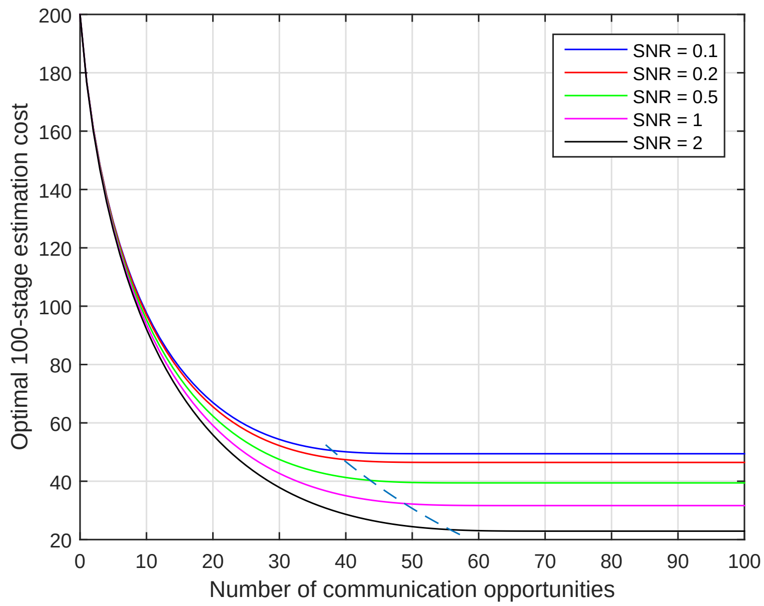

Figure 2.4: Opportunity threshold vs. minimal error under different signal to noise ratios

with $J^*(T+1,T)=0$, which implies that

$$\begin{aligned} J^*(1,T) &= T\left(\tfrac{2}{\lambda^2} - \left(\tfrac{2\sqrt{m}}{\lambda} + \tfrac{2}{\lambda^2}\right)\cdot e^{-\lambda\sqrt{m}}\right) \\ &= T\,2\lambda^{-2}\left(1 - \left(\tfrac{1}{\sqrt{1+\gamma}} + 1\right)\cdot e^{-\frac{1}{\sqrt{1+\gamma}}}\right) \end{aligned}$$

It is straightforward to check that $J^*(1,T)$ is a decreasing function of the SNR γ. Hence, the minimal error decreases as the SNR increases.

Plotting the opportunity threshold $Te^{-\lambda m}$ versus minimal error $J^*(1,T)$ under different SNRs (dash line) in Fig. 2.2, we arrive at Fig. 2.4. One can see that the intersection between the dash line and each solid line is roughly the turning point of the solid line. Therefore, the plot of opportunity threshold versus minimal error under different SNRs is an important one. In fact, the plot suggests the lowest capacity of the battery that one should choose when building a physical system so that the expected estimation error is minimized. In addition, the plot predicts the minimal estimation error.

Consider the asymptotic case where the SNR $\gamma \to \infty$, and thus $m = \frac{1}{\gamma+1}\frac{1}{\lambda^2} \to 0$. Then the opportunity threshold $Te^{-\lambda m} \to T$, and the minimal error $J^*(1,T) \to 0$. Hence, the optimal 100-stage estimation error will be strictly decreasing in terms of the number of communication opportunities in the asymptotic case, as also noted in the prior work (see [27], Figure 5). Moreover, the estimation error will reach zero when the number of communication opportunities is equal to the length of the time horizon.

CHAPTER 3

COMMUNICATION SCHEDULING AND REMOTE ESTIMATION OVER MULTIPLE CHANNELS

In this chapter, we consider the communication scheduling and remote estimation problem with two channels, that is, a noiseless channel and an additive noise channel. After making an observation of the source, the sensor can choose among the three options of non-transmission, transmission over the noisy channel, and transmission over the noiseless channel. Similar to the single-channel setting, if the sensor decides to transmit its observation via the noisy channel, it will encode its message and the estimator (or decoder) will decode the noise-corrupted message. Furthermore, we restrict the encoder and the decoder to apply affine encoding and decoding policies. Different from the single-channel setting, we do not assume that there exists a side channel between the encoder and the decoder when formulating the problems. Next, we assume that the sensor will apply a symmetric communication scheduling policy, and we show that the optimal one is of the threshold-in-threshold type. However, we show by constructing a counterexample that the symmetry assumption is in fact not valid in terms of globally optimality. In other words, the globally optimal communication scheduling policy can be non-symmetric, which renders the problem intractable. Analysis on this counter intuitive case shows that symmetric communication scheduling policy cannot be optimal since it has disconnected the noisy transmission region. Then, we argue that this issue can be resolved by assuming the existence of a side channel. Hence, the side-channel assumption is critical to make the problem tractable. With this additional assumption, we show that the optimal communication scheduling policy is of threshold-in-threshold type. The optimal thresholds can be obtained for some specific source distributions. Moreover, we generate numerical results for the problem with hard constraint.

The main contributions of this chapter are given as follows:

1. We formulate two optimization problems under two types of commu-

nication constraints, namely soft constraint and hard constraint.

2. We show by constructing a counterexample that without the side channel, the optimal communication scheduling policy can be non-symmetric even if the source has symmetric and unimodal distribution. Therefore, we justify the assumption of using the side channel.

3. Under some technical assumptions, we show that the optimal communication scheduling policy is of the symmetric threshold-in-threshold type. When the source has Laplace distribution, the optimal thresholds can be uniquely determined.

4. We generate numerical results for the problem with hard constraint, which show some properties inherited from both the noiseless-channel setting and the single-channel setting.

The rest of this chapter is organized as follows: in Section 3.1, we formulate the optimization problems with soft and hard constraints. In Section 3.2.1, we justify the importance of the side channel. In Section 3.2.2, we consider the modified problem (with the side channel) under the soft constraint. In Section 3.3, we consider the modified problem under the hard constraint. In Section 3.4, we generate and analyze numerical results for the modified problem with hard constraint.

3.1 Problem Formulation

3.1.1 System Model

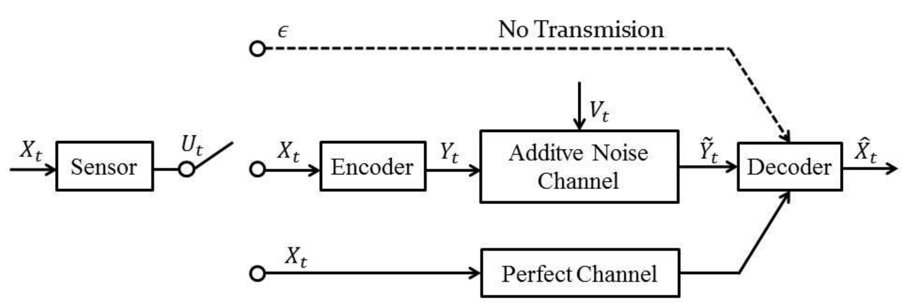

Figure 3.1: System model for multi-channel setting

Consider a discrete time communication scheduling and remote estimation problem over a finite time horizon, i.e., $t = 1, 2, \ldots, T$. A one-dimensional source process $\{X_t\}$ is an independent identically distributed (i.i.d.) stochastic process with probability density function p_X. At time t, the sensor, as shown in Fig. 3.1, observes the state of the source X_t. Then, it decides whether and how to transmit its observation to the remote estimator (which is also called "decoder"). Let $U_t \in \{0, 1, 2\}$ be the sensor's decision at time t. $U_t = 0$ means that the sensor chooses not to transmit its observation to the decoder, and hence it sends a free symbol ϵ to the decoder representing that nothing is transmitted. $U_t = 1$ means that the sensor chooses to transmit its observation to the decoder over an additive noise channel. Therefore, the sensor sends X_t to an encoder, which then sends an encoded message, call it Y_t, to the communication channel. Y_t is corrupted by an additive channel noise V_t. The noise process $\{V_t\}$ is a one-dimensional i.i.d. stochastic process with density p_V, which is independent of $\{X_t\}$. The encoder has average power constraint, that is,

$$\mathbb{E}[Y_t^2 | U_t = 1] \leq P_T$$

where P_T is known and constant for all t. When $U_t = 2$, the sensor chooses to transmit its observation over a noiseless channel. Hence, the decoder will receive X_t. Let $\tilde{Y}_t$ be the message received by decoder at time t, we have

$$\tilde{Y}_t = \begin{cases} \epsilon, & \text{if } U_t = 0 \\ Y_t + V_t, & \text{if } U_t = 1 \\ X_t, & \text{if } U_t = 2 \end{cases}$$

After receiving $\tilde{Y}_t$, the decoder generates an estimate on X_t, denoted by $\hat{X}_t$. The decoder is charged for squared distortion $(X_t - \hat{X}_t)^2$.

3.1.2 Communication Constraints

We consider the optimization problems under two kinds of communication constraints, separately. In the first scenario, at each time t, the sensor is charged for its decision, i.e., there is a cost function associated with U_t, denoted by $c(U_t)$, such that

$$c(U_t) = \begin{cases} 0, & \text{if } U_t = 0 \\ c_1, & \text{if } U_t = 1 \\ c_2, & \text{if } U_t = 2 \end{cases}$$

Here, we have $c_2 > c_1 > 0$, which means that usage of the noiseless channel is more costly than that of the noisy channel. c_1, c_2 are called the communication costs for using the noisy channel and the perfect channel, respectively. This kind of communication constraint is called *soft constraint*. In the second scenario, the sensor is not charged for transmitting its observations. Instead, the sensor is able to use the noisy channel and the perfect channel for no more than N_1 and N_2 times, respectively, i.e.,

$$\sum_{t=1}^{T} \mathbf{1}_{\{U_t=1\}} \leq N_1, \quad \sum_{t=1}^{T} \mathbf{1}_{\{U_t=2\}} \leq N_2$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function, and N_1, N_2 are positive integers. Such a kind of communication constraint is called *hard constraint*.

3.1.3 Decision Strategies

Assume that at time t, the sensor has memory of all its measurements by t, denoted by $X_{1:t}$, and all the decisions it has made by $t-1$, denoted by $U_{1:t-1}$. The sensor makes decision U_t based on its current information $(X_{1:t}, U_{1:t-1})$, that is,

$$U_t = f_t(X_{1:t}, U_{1:t-1})$$

where f_t is the communication scheduling policy at time t and $\mathbf{f} = \{f_1, f_2, \ldots, f_T\}$ is the communication scheduling strategy.

Assume that at time t, no matter whether and how the sensor decides to transmit the source output, it always transmits its decision U_t to the encoder.[1] Let $\tilde{X}_t$ be the message received by the encoder at time t. Then,

$$\tilde{X}_t = \begin{cases} (X_t, U_t), & \text{if } U_t = 1 \\ U_t, & \text{otherwise} \end{cases}$$

Denote by $\tilde{X}_{1:t}$ the messages received by the encoder up to time t. Similar to the above, we assume that the encoder has memory on $\tilde{X}_{1:t}$, and all the encoded messages it has sent to the communication channel by $t-1$, denoted by $Y_{1:t-1}$.[2] The encoder generates the encoded message Y_t based on its current information $(\tilde{X}_{1:t}, Y_{1:t-1})$, that is,

$$Y_t = g_t(\tilde{X}_{1:t}, Y_{1:t-1})$$

where g_t is the encoding policy at time t and $\mathbf{g} = \{g_1, g_2, \ldots, g_T\}$ is the encoding strategy.

Finally, assume that the decoder can deduce U_t from $\tilde{Y}_t$. Furthermore, it is assumed that at time t, the decoder has memory on all the messages received by t, denoted by $\tilde{Y}_{1:t}$, and all the estimates it has generated by $t-1$, denoted by $\hat{X}_{1:t-1}$. The decoder produces the estimate $\hat{X}_t$ based on its current information $(\tilde{Y}_{1:t}, \hat{X}_{1:t-1})$, namely,

$$\hat{X}_t = h_t(\tilde{Y}_{1:t}, \hat{X}_{1:t-1})$$

where h_t is the decoding policy at time t and $\mathbf{h} = \{h_1, h_2, \ldots, h_T\}$ is the decoding strategy.

In particular, we call the sensor, the encoder, and the decoder the *decision makers*. We call (f_t, g_t, h_t) the *decision making policies* at time t, and $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ the *decision making strategies*.

Remark 3.1. *At time t, the sensor's decisions by $t-1$, namely $U_{1:t-1}$, is the common information shared by all the decision makers. This is an important property, which will be applied when solving the optimization problem with hard constraint.*

[1] Physically, the sensor and the encoder are built together.

[2] If the sensor decides not to transmit its observation over the noisy channel, the encoder will send $Y_t = 0$ to the communication channel.

3.1.4 Optimization Problems

Consider the setting described above, with the time horizon T, probability density functions p_X and p_V, and power constraint P_T as given.

Optimization problem with soft constraint: Given the communication cost function $c(\cdot)$, determine the decision-making strategies $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ minimizing the cost functional

$$J(\mathbf{f}, \mathbf{g}, \mathbf{h}) := \mathbb{E}\left\{\sum_{t=1}^{T} c(U_t) + (X_t - \hat{X}_t)^2\right\}$$

Optimization problem with hard constraint: Given the numbers of communication opportunities N_1 and N_2, determine the decision-making strategies $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ minimizing the cost functional

$$J(\mathbf{f}, \mathbf{g}, \mathbf{h}) := \mathbb{E}\left\{\sum_{t=1}^{T} (X_t - \hat{X}_t)^2\right\}$$

under the hard constraint.

3.2 Optimization Problem with Soft Constraint

3.2.1 Counter Intuitive Property of the Optimal Communication Scheduling Policy

Since the source and the noise processes are i.i.d., by an argument similar to that in Theorem 2.1, the optimization decision strategies can be obtained by solving a single-stage problem, as described in the following theorem.

Theorem 3.1. *Consider the optimization problem with soft constraint formulated in Section 3.1.4. Without loss of optimality, the decision makers can apply decision-making policies in the form of*

$$U_t = f_t(X_t), \quad Y_t = g_t(\tilde{X}_t), \quad \hat{X}_t = h_t(\tilde{Y}_t), \quad t = 1, 2, \ldots, T$$

where (f_t, g_t, h_t) *are designed to minimize the instantaneous cost*

$$J_t(f_t, g_t, h_t) := \mathbb{E}[cU_t + (X_t - \hat{X}_t)^2]$$

Furthermore,

$$f_1 = f_2 = \ldots = f_T, \quad g_1 = g_2 = \ldots = g_T, \quad h_1 = h_2 = \ldots = h_T$$

For simplicity, we henceforth suppress the subscript for time in this subsection. We further make the following assumptions on the optimization problem.

Assumption 3.1. *The source density p_X is symmetric and unimodal around zero, i.e.,*

$$p_X(x) = p_X(-x), \ \forall \ x \in \mathbb{R}$$
$$p_X(x_1) \geq p_X(x_2), \ \forall \ |x_1| \geq |x_2|$$

Here, we make a weaker assumption on the source compared to Assumption 2.1 in Chapter 2, which assumes the source density is even and log-concave. The reason is that the proof techniques in the two settings are different, which require different conditions.

Assumption 3.2. *The communication channel noise V is zero-mean and has finite variance, denoted by σ_V^2.*

Assumption 3.3. *When the sensor decides to transmit its observation via the noisy channel, the encoder and decoder are restricted to apply affine policies in the form of*

$$\begin{aligned} g(X) &= \alpha(X - \mathbb{E}[X|U=1]) \\ h(\tilde{Y}) &= \frac{1}{\alpha}\frac{\gamma}{\gamma+1}\tilde{Y} + \mathbb{E}[X|U=1] \end{aligned}$$

where $\gamma := P_T/\sigma_V^2$ is the signal-to-noise ratio (SNR). α is the amplifying ratio, and $\alpha = \sqrt{P_T/\mathrm{Var}(X|U=1)}$. $\mathrm{Var}(X|U=1)$ is the variance of X conditioning on the event that the sensor transmits the source output over the noisy channel.

Assumption 3.3 is inherited from Assumption 2.4 in Chapter 2. A detailed explanation on why we make such an assumption can be found in Remark 2.4.

Note that the source density is symmetric around zero. Moreover, the distortion metric is the squared error, which is also symmetric around zero. It is intuitive to have a guess that the optimal communication scheduling policy is symmetric around zero. Also note that in an asymptotic case where the communication channel is noiseless, the optimal communication scheduling policy is symmetric around zero (as shown in [28]). Hence, we make the following assumption.

Assumption 3.4. *The sensor is restricted to apply the communication scheduling policy in the form of:*

$$f(x) = f(-x), \quad \forall\, x \in \mathbb{R}$$

The following corollary is a consequence of Assumptions 3.1-3.4.

Corollary 3.1. *Consider the single-stage problem with Assumptions 3.1-3.4 hold, the optimal communication scheduling policy is of the symmetric threshold-in-threshold type:*

$$f(x) = \begin{cases} 0, & \text{if } |x| \leq \beta_1 \\ 1, & \text{if } \beta_1 < |x| \leq \beta_2 \\ 2, & \text{if } |x| > \beta_2 \end{cases} \tag{3.1}$$

The parameters β_1 and β_2 are called "thresholds", and $0 < \beta_1 \leq \beta_2 < \infty$.

Before proving Corollary 3.1, we first introduce some notations. Let $\mathcal{T}_0^f$, $\mathcal{T}_1^f$, $\mathcal{T}_2^f$ be the *non-transmission region*, the *noisy transmission region*, and the *perfect transmission region*, respectively, according to communication policy f, i.e.,

$$\mathcal{T}_i^f := \{x \in \mathbb{R} | f(x) = i\}, \;\; i \in \{0, 1, 2\}$$

Consider the cost functional $J(f, g, h)$ associated with any group of decision policies (f, g, h) satisfying Assumption 3.3[3] and any communication channel noise satisfying Assumption 3.2, we have

$$\begin{aligned} J(f,g,h) &= \mathbb{E}\big[c(U) + (X - \hat{X})^2\big] \\ &= \textstyle\sum_{i\in\{0,1,2\}} \mathbb{E}\big[c(U) + (X - \hat{X})^2 \big| X \in \mathcal{T}_i^f\big] \cdot \mathbb{P}(X \in \mathcal{T}_i^f) \end{aligned}$$

Then, we have the following discussions.

[3]Here we do not place any restriction on f, which may or may not be symmetric around zero.

1. When $X \in \mathcal{T}_0^f$, the sensor decides not to transmit its observation. Then, the optimal estimator is the conditional mean $\mathbb{E}[X|X \in \mathcal{T}_0^f]$. Moreover, we have

$$\begin{aligned} \mathbb{E}[(X-\hat{X})^2|X \in \mathcal{T}_0^f] &= \mathbb{E}\Big[\big(X-\mathbb{E}[X|X \in \mathcal{T}_0^f]\big)^2|X \in \mathcal{T}_0^f\Big] \\ &= \mathrm{Var}(X|X \in \mathcal{T}_0^f) \end{aligned}$$

2. When $X \in \mathcal{T}_1^f$, the sensor decides to transmit its observation over the noisy channel. By Assumptions 3.3, we have

$$\begin{aligned} \hat{X} &= \frac{1}{\alpha}\frac{\gamma}{\gamma+1}\tilde{Y} + \mathbb{E}[X|X \in \mathcal{T}_1^f] \\ &= \frac{\gamma}{\gamma+1}X + \frac{1}{\gamma+1}\mathbb{E}[X|X \in \mathcal{T}_1^f] + \frac{1}{\alpha}\frac{\gamma}{\gamma+1}V \end{aligned}$$

Furthermore, the mean squared error conditioned on $X \in \mathcal{T}_1^f$ can be computed as

$$\mathbb{E}[(X-\hat{X})^2|X \in \mathcal{T}_1^f] = \frac{1}{\gamma+1}\mathrm{Var}(X|X \in \mathcal{T}_1^f) \tag{3.2}$$

3. When $X \in \mathcal{T}_2^f$, the sensor decides to transmit its observation over the perfect channel, and thus the decoder simply reports $\hat{X} = X$.

Combining the three cases together, we have

$$\begin{aligned} & J(f,g,h) \\ = \; & \mathrm{Var}(X|X \in \mathcal{T}_0^f)\mathbb{P}(X \in \mathcal{T}_0^f) + c_1\mathbb{P}(X \in \mathcal{T}_1^f) \\ & + \frac{1}{\gamma+1}\mathrm{Var}(X|X \in \mathcal{T}_1^f)\mathbb{P}(X \in \mathcal{T}_1^f) + c_2\mathbb{P}(X \in \mathcal{T}_2^f) \end{aligned} \tag{3.3}$$

With the notations and discussions above, we are able to prove Corollary 3.1 (see Appendix A.3).

Although Assumption 3.4 and Corollary 3.1 seem very intuitive at first glance, the following counterexample renders them not valid from the point of global optimality.

Counterexample: Consider the case where X has uniform distribution over

$[-L, L]$, namely,

$$p_X(x) = \frac{1}{2L},\ x \in [-L, L]$$

Assume the parameters satisfy

$$\frac{\gamma+1}{\gamma}c_1 < c_2; \quad \sqrt{(c_2-c_1)(\gamma+1)} < L \tag{3.4}$$

By Corollary 3.1, the single-stage problem admits a solution including a communication scheduling policy f^* of symmetric threshold-in-threshold type with thresholds β_1, β_2, and a pair of encoding/decoding policies (g^*, h^*) induced by f^* according to Assumption 3.3. By (3.4), we have $0 < \beta_1 < \beta_2 < L$. Hence, the non-transmission region, the noisy transmission region, and the perfect transmission region corresponding to f^* are as follows:

$$\mathcal{T}_0^{f^*} = [-\beta_1, \beta_1], \quad \mathcal{T}_1^{f^*} = [-\beta_2, -\beta_1) \cup (\beta_1, \beta_2], \quad \mathcal{T}_2^{f^*} = [-L, -\beta_2) \cup (\beta_2, L]$$

We now construct another communication scheduling policy f' by specifying its non-transmission region, noisy transmission region, and perfect transmission region:

$$\mathcal{T}_0^{f'} = \mathcal{T}_0^{f^*}, \quad \mathcal{T}_1^{f'} = (\beta_1, 2\beta_2 - \beta_1], \quad \mathcal{T}_2^{f'} = [-L, -\beta_1) \cup (2\beta_2 - \beta_1, L]$$

Since the source is uniformly distributed, we have

$$\mathbb{P}(X \in \mathcal{T}_1^{f'}) = \mathbb{P}(X \in \mathcal{T}_1^{f^*}) = \frac{\beta_2 - \beta_1}{L}$$

Essentially, we rearrange the noisy transmission region, without changing its probability measure, to make the region connected. This procedure is illustrated in Fig. 3.2. Induced by f', we obtain the encoding and decoding policies (g', h') satisfying Assumption 3.3. Furthermore, by (3.3), we have

$$J(f', g', h') - J(f^*, g^*, h^*) = \frac{\mathbb{P}(X \in \mathcal{T}_1^{f'})}{\gamma+1}\big(\mathrm{Var}(X|X \in \mathcal{T}_1^{f'}) - \mathrm{Var}(X|X \in \mathcal{T}_1^{f^*})\big)$$

The regions $\mathcal{T}_1^{f'}$ and $\mathcal{T}_1^{f^*}$ have the same probability measure under uniform distribution, while $\mathcal{T}_1^{f'}$ is connected. Apparently, we have $\mathrm{Var}(X|X \in \mathcal{T}_1^{f'}) < \mathrm{Var}(X|X \in \mathcal{T}_1^{f^*})$, which implies $J(f') < J(f^*)$. Hence, the symmetric communication scheduling policy f^* together with the encoding/decoding

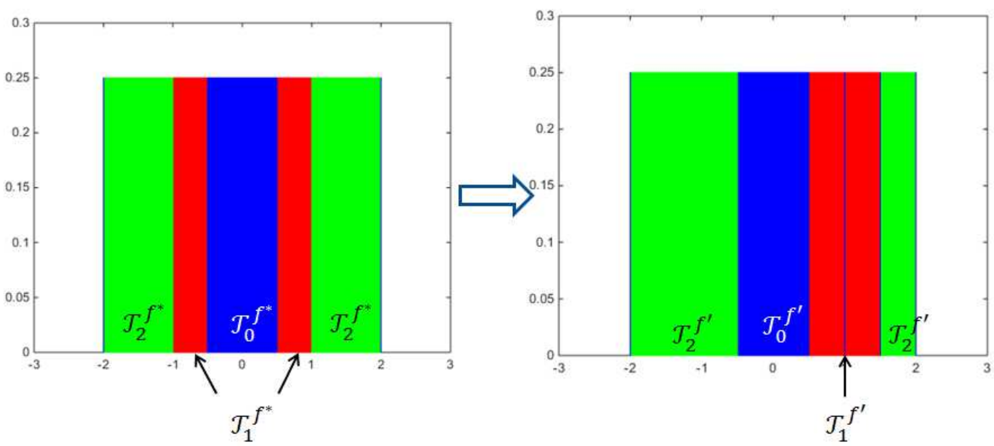

Figure 3.2: The counterexample

policies (g^*, h^*) are not globally optimal.

Remark 3.2. *The counterexample above uncovers a counter intuitive result, namely, with the existence of a noisy channel, the common folklore that the optimal communication scheduling policy is symmetric threshold-in-threshold based does not hold. As illustrated in the example, the noisy transmission region under symmetric communication policy is disconnected, which results in large conditional variance. Therefore, symmetric communication policy does not take full advantage of the noisy channel.*

The non-symmetric property of the optimal communication scheduling policy makes the problem fairly difficult to solve. In order to fix this issue and render the problem tractable, we further assume the existence of a side channel.

3.2.2 Modified Problem

We assume there exists a side channel between the encoder and the decoder. Recall that at time t, if the sensor decides to transmit its observation X_t via the noisy channel, it sends the observation to the encoder. Then, the encoder sends an encoded message Y_t to the noisy channel. We now assume the encoder additionally sends the sign of X_t, denoted by S_t, to the decoder over the side channel, which is illustrated in Fig. 3.3. Assume that the side

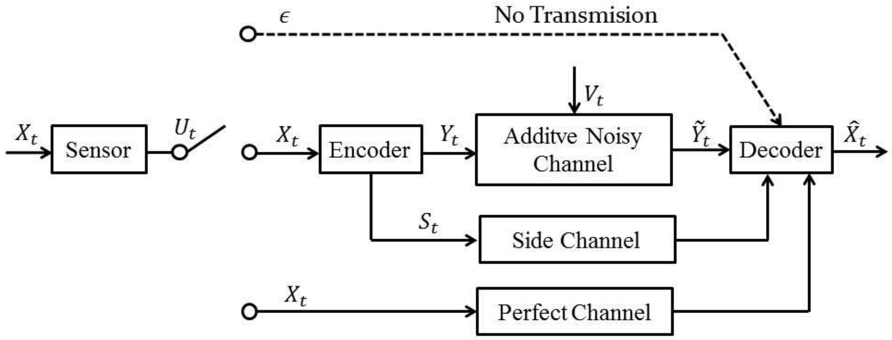

Figure 3.3: Modified system

channel is noise-free.[4] Let $S_{1:t}$ be the collections of the side information up to t. Now, the information available to the encoder and the decoder at time t is $(\tilde{X}_{1:t}, S_{1:t}, Y_{1:t-1})$ and $(\tilde{Y}_{1:t}, S_{1:t}, \hat{X}_{1:t-1})$, respectively. The encoder and the decoder generate the encoded message Y_t and the estimate $\hat{X}_t$, respectively, according to

$$Y_t = g_t(\tilde{X}_{1:t}, S_{1:t}, Y_{1:t-1}), \quad \hat{X}_t = h_t(\tilde{Y}_{1:t}, S_{1:t}, \hat{X}_{1:t-1})$$

Similar to Theorem 3.1, it can be shown that the encoder and the decoder can consider only their current inputs (without loss of optimality) when making decisions, namely,

$$Y_t = g_t(\tilde{X}_t, S_t), \quad \hat{X}_t = h_t(\tilde{Y}_t, S_t)$$

Furthermore, the optimal decision strategies $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ can be obtained by solving the single-stage problem, and hence we keep suppressing the subscript representing time in this subsection. It is important to note that the side channel enables the encoder/decoder to apply different encoding/decoding policies for the positive and negative realizations of the source. Hence, we need to modify Assumption 3 (but keep Assumptions 1 and 2).

Assumption 3.5. *The encoder and the decoder are restricted to apply piece-*

[4]When there is no transmission over the noisy channel, there is also no transmission over the side channel. In this case, we write $S_t = 0$. The side information S_t is only a one-bit message, which can be sent reliably.

wise affine encoding and decoding policies, respectively, i.e.,

$$\begin{aligned} g(X,S) &= S\alpha(S)\left(X - \mathbb{E}\left[X|U=1,S\right]\right) \\ h(\tilde{Y},S) &= \frac{1}{\alpha(S)}\frac{\gamma}{\gamma+1}S\tilde{Y} + \mathbb{E}\left[X|U=1,S\right] \end{aligned}$$

The parameter $\gamma = P_T/\sigma_V^2$ *is the signal-to-noise ratio.* $\alpha(S)$ *is the amplifying ratio, and* $\alpha(S) = \sqrt{P_T/\mathrm{Var}(X|U=1,S)}$. $\mathbb{E}[X|U=1,S]$ *and* $\mathrm{Var}(X|U=1,S)$ *are the conditional mean and variance, respectively.*

We now compute the cost functional $J(f,g,h)$ associated with any communication scheduling policy f and the encoding/decoding policies (g,h) induced by f under Assumption 3.5. Let $\mathcal{T}_{1+}^f$, $\mathcal{T}_{1-}^f$ be the *positive noisy transmission region* and the *negative noisy transmission region*, respectively, according to communication policy f, i.e.,

$$\mathcal{T}_{1+}^f := \{x > 0 | f(x) = 1\}, \;\; \mathcal{T}_{1-}^f := \{x < 0 | f(x) = 1\}$$

Similar to (3.2) and (3.3), it can be computed that

$$\begin{aligned} \mathbb{E}[(X-\hat{X})^2|X \in \mathcal{T}_{1+}^f] &= \frac{1}{\gamma+1}\mathrm{Var}(X|X \in \mathcal{T}_{1+}^f), \\ \mathbb{E}[(X-\hat{X})^2|X \in \mathcal{T}_{1-}^f] &= \frac{1}{\gamma+1}\mathrm{Var}(X|X \in \mathcal{T}_{1-}^f) \end{aligned}$$

Moreover,

$$\begin{aligned} & J(f,g,h) \\ = \; & \mathrm{Var}(X|X \in \mathcal{T}_0^f)\mathbb{P}(X \in \mathcal{T}_0^f) + c_2\mathbb{P}(X \in \mathcal{T}_2^f) \\ & + \frac{1}{\gamma+1}\mathrm{Var}(X|X \in \mathcal{T}_{1+}^f)\mathbb{P}(X \in \mathcal{T}_{1+}^f) + c_1\mathbb{P}(X \in \mathcal{T}_{1-}^f) \\ & + \frac{1}{\gamma+1}\mathrm{Var}(X|X \in \mathcal{T}_{1-}^f)\mathbb{P}(X \in \mathcal{T}_{1-}^f) + c_1\mathbb{P}(X \in \mathcal{T}_{1+}^f) \end{aligned} \tag{3.5}$$

Comparing (3.3) with (3.5), the conditional variance over the noisy transmission region $\mathrm{Var}(X|X \in \mathcal{T}_1^f)$ is replaced by two conditional variances over the positive/negative noisy transmission regions, that is, $\mathrm{Var}(X|X \in \mathcal{T}_{1+}^f)$ and $\mathrm{Var}(X|X \in \mathcal{T}_{1-}^f)$. As discussed in Remark 3.2, symmetric communication schedule policy results in disconnected noisy transmission region $\mathcal{T}_1^f$, which further leads to a large conditional variance $\mathrm{Var}(X|X \in \mathcal{T}_1^f)$. However, with the existence of the side channel, we may still have connected positive/negative noisy transmission regions $\mathcal{T}_{1+}^f$ and $\mathcal{T}_{1-}^f$, which would result in small conditional variances $\text{Var}(X|X \in \mathcal{T}_{1+}^f)$ and $\text{Var}(X|X \in \mathcal{T}_{1-}^f)$. Therefore, Assumption 3.4 is a reasonable assumption for the modified problem. We keep this assumption and we further establish the optimality of threshold-in-threshold based policy, as stated in the following theorem.

Theorem 3.2. *Consider the modified problem with Assumptions 3.1, 3.2, 3.4 and 3.5 hold. Without loss of optimality, the sensor can apply the communication scheduling policy in the symmetric threshold-in-threshold form described by* (3.1)*.*

To prove Theorem 3.2, we need the following proposition.

Proposition 3.1. *Let p_X be the probability density function of random variable X. p_X is symmetric and unimodal around zero. Consider two open intervals (β_1, β_2) and (β_1', β_2') such that $0 \leq \beta_1 \leq \beta_1'$ and $\mathbb{P}(X \in (\beta_1, \beta_2)) = \mathbb{P}(X \in (\beta_1', \beta_2'))$, then*

$$\text{Var}(X|X \in (\beta_1, \beta_2)) \leq \text{Var}(X|X \in (\beta_1', \beta_2'))$$

PROOF. See Appendix A.4.

We are now in a position to prove Theorem 3.2. The idea of the proof is as follows: given any symmetric communication scheduling policy f, we can construct another symmetric communication scheduling policy $\tilde{f}$ achieving non-greater cost. Analysis on $\tilde{f}$ shows that it is either threshold-in-threshold based or "threshold-in-threshold-in-threshold" based. For the second case, we can further construct another communication scheduling policy f' of threshold-in-threshold type, which achieves non-greater cost.

Proof of Theorem 3.2. We prove the theorem by showing that given any group of decision policies (f, g, h) satisfying Assumptions 3.4 and 3.5, there exist a symmetric threshold-in-threshold based communication scheduling policy and a pair of induced encoding/decoding policies achieving no greater cost.

Let (f, g, h) satisfying Assumptions 3.4 and 3.5 be given. Since $f(x)$ and $p_X(x)$ are symmetric around zero, we have

$$\mathbb{E}\left[X|X \in \mathcal{T}_0^f\right] = 0$$

Furthermore, it can be checked that

$$\begin{aligned}\mathbb{E}\big[X|X \in \mathcal{T}^f_{1+}\big] &= -\mathbb{E}\big[X|X \in \mathcal{T}^f_{1-}\big] =: b \\ \mathbb{P}(X \in \mathcal{T}^f_{1+}) &= \mathbb{P}(X \in \mathcal{T}^f_{1-}) \\ \mathrm{Var}(X|X \in \mathcal{T}^f_{1+}) &= \mathrm{Var}(X|X \in \mathcal{T}^f_{1-})\end{aligned}$$

Then, (3.5) can be further expressed as

$$\begin{aligned}& J(f,g,h) \\ = \;& 2\int_{x\in\, \mathcal{T}^f_0 \,\cap\, [0,\infty)} x^2 p_X(x)dx + 2\int_{x\in\mathcal{T}^f_{1+}} \big(c_1 + \frac{1}{\gamma+1}(x-b)^2\big)p_X(x)dx \\ & + 2\int_{x\in\, \mathcal{T}^f_2 \,\cap\, [0,\infty)} c_2 p_X(x)dx \\ =: \;& 2\int_{x\in(0,\infty)} \tilde{J}(x, f(x))p_X(x)dx\end{aligned}$$

where $\tilde{J}(x,u)$ is defined on $x \in [0,\infty)$ and

$$\tilde{J}(x,u) = \begin{cases} x^2, & \text{if } u = 0 \\ c_1 + \dfrac{1}{\gamma+1}(x-b)^2, & \text{if } u = 1 \\ c_2, & \text{if } u = 2 \end{cases}$$

We now construct a communication scheduling policy $\tilde{f}$ such that

$$\tilde{f}(x) = \begin{cases} \underset{u\in\{0,1,2\}}{\arg\min}\; \tilde{J}(x,u), & \text{if } x \geq 0 \\ f(-x), & \text{if } x < 0 \end{cases}$$

Denote by $\tilde{b} := \mathbb{E}[X|X \in \mathcal{T}^{\tilde{f}}_{1+}]$ the conditional mean over $\mathcal{T}^{\tilde{f}}_{1+}$. Moreover, let $(\tilde{g}, \tilde{h})$ be the encoding/decoding policies induced by $\tilde{f}$ by Assumption 3.5.

Then,

$$
\begin{aligned}
& J(f,g,h) \\
\geq\ & 2\int_{x\in[0,\infty)} \tilde{J}(x,\tilde{f}(x))p_X(x)dx \\
=\ & 2\int_{x\in\ \mathcal{T}_0^{\tilde{f}}\ \cap\ [0,\infty)} x^2 p_X(x)dx + 2\int_{x\in\mathcal{T}_{1+}^{\tilde{f}}} \big(c_1 + \frac{1}{\gamma+1}(x-b)^2\big)p_X(x)dx \\
& + 2\int_{x\in\ \mathcal{T}_2^{\tilde{f}}\ \cap\ [0,\infty)} c_2 p_X(x)dx \\
\geq\ & 2\int_{x\in\ \mathcal{T}_0^{\tilde{f}}\ \cap\ [0,\infty)} x^2 p_X(x)dx + 2\int_{x\in\mathcal{T}_{1+}^{\tilde{f}}} \big(c_1 + \frac{1}{\gamma+1}(x-\tilde{b})^2\big)p_X(x)dx \\
& + 2\int_{x\in\ \mathcal{T}_2^{\tilde{f}}\ \cap\ [0,\infty)} c_2 p_X(x)dx \\
=\ & J(\tilde{f},\tilde{g},\tilde{h})
\end{aligned}
$$

The first inequality holds dues to the way that $\tilde{f}$ is constructed. The second inequality holds since

$$
\begin{aligned}
\int_{x\in\mathcal{T}_{1+}^{\tilde{f}}} (x-b)^2 p_X(x)dx &= \mathbb{E}\big[(X-b)^2 | X\in\mathcal{T}_{1+}^{\tilde{f}}\big]\mathbb{P}(X\in\mathcal{T}_{1+}^{\tilde{f}}) \\
&\geq \mathbb{E}\big[(X-\tilde{b})^2 | X\in\mathcal{T}_{1+}^{\tilde{f}}\big]\mathbb{P}(X\in\mathcal{T}_{1+}^{\tilde{f}}) \\
&= \int_{x\in\mathcal{T}_{1+}^{\tilde{f}}} (x-\tilde{b})^2 p_X(x)dx
\end{aligned}
$$

where the inequality further dues to the fact that $\tilde{b}$ is the conditional mean and thus the minimum mean squared error estimator.

We now analyze the structure of $\tilde{f}$, and we only need to consider $x \geq 0$. It is easy to check that there exists $\beta_1 > 0$ such that

$$
\begin{aligned}
&\tilde{J}(x,0) \leq \min\{\tilde{J}(x,1),\tilde{J}(x,2)\},\ x\in[0,\beta_1], \\
&\tilde{J}(x,0) > \min\{\tilde{J}(x,1),\tilde{J}(x,2)\},\ x\in(\beta_1,\infty)
\end{aligned}
$$

Hence, $\tilde{f}(x) = 0$ when $x \in [0,\beta_1]$, and we only need to compare $\tilde{J}(x,1)$ with $\tilde{J}(x,2)$ when $x\in(\beta_1,\infty)$. Note that $\tilde{J}(x,1)$ is a porabolic opening upward, and $\tilde{J}(x,2)$ is constant. Hence, either of the following three cases occurs when $x\in(\beta_1,\infty)$:

Case I: $\tilde{J}(x,1)$ and $\tilde{J}(x,2)$ do not intersect, which implies

$$\tilde{J}(x,1) > \tilde{J}(x,2), x \in (\beta_1, \infty)$$

Therefore, $\tilde{f}(x) = 2$ when $x \in (\beta_1, \infty)$, and $\tilde{f}$ is of threshold-in-threshold type with $\beta_1 = \beta_2$.

Case II: $\tilde{J}(x,1)$ and $\tilde{J}(x,2)$ intersect only once at $x = \beta_2$, and

$$\tilde{J}(x,1) \leq \tilde{J}(x,2),\ x \in (\beta_1, \beta_2]; \quad \tilde{J}(x,1) > \tilde{J}(x,2),\ x \in (\beta_2, \infty)$$

Then, $\tilde{f}(x) = 1$ when $x \in (\beta_1, \beta_2]$ and $\tilde{f}(x) = 2$ when $x \in (\beta_2, \infty)$. Hence, $\tilde{f}$ is of threshold-in-threshold type.

Case III: $\tilde{J}(x,1)$ and $\tilde{J}(x,2)$ intersect twice at β_l and β_r, which implies

$$\tilde{J}(x,1) \leq \tilde{J}(x,2),\ x \in (\beta_l, \beta_r); \quad \tilde{J}(x,1) > \tilde{J}(x,2),\ x \in (\beta_1, \beta_l] \cup [\beta_r, \infty)$$

Hence, $\tilde{f}(x) = 1$ when $x \in (\beta_l, \beta_r)$, and $\tilde{f}(x) = 2$ when $x \in (\beta_1, \beta_l] \cup [\beta_r, \infty)$. Although $\tilde{f}$ is not in threshold-in-threshold form, yet we can construct a policy f' of threshold-in-threshold type based on $\tilde{f}$, which achieves non-greater cost. Let f' be as follows:

$$\mathcal{T}_0^{f'} = [-\beta_1, \beta_1], \mathcal{T}_{1+}^{f'} = (\beta_1, \beta_2], \mathcal{T}_{1-}^{f'} = [-\beta_2, -\beta_1), \mathcal{T}_2^{f'} = (-\infty, -\beta_2) \cup (\beta_2, \infty)$$

where β_2 is selected such that

$$\int_{\beta_1}^{\beta_2} p_X(x)dx = \int_{\beta_l}^{\beta_r} p_X(x)dx$$

As illustrated in Fig. 3.4, we shifted the positive and the negative transmission regions without changing the probability measure over each region. Let (g', h') be the encoding and the decoding policies induced by f' following

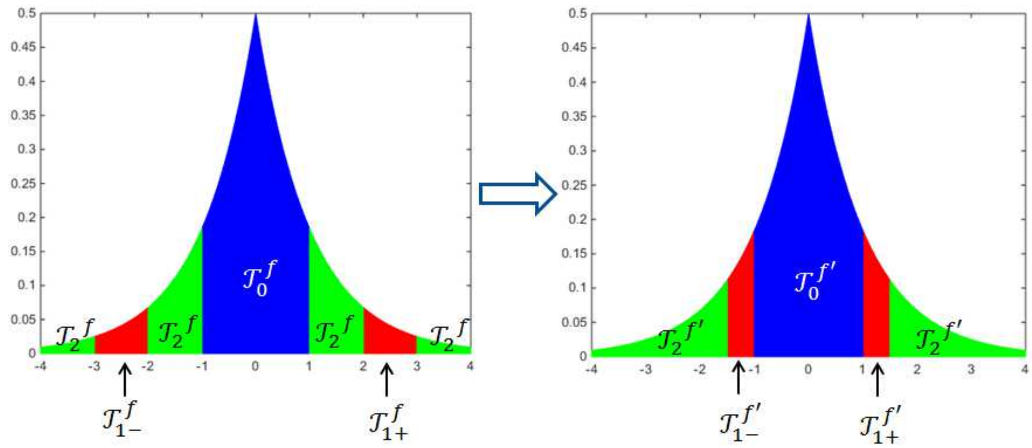

Figure 3.4: Construction of f' based on $\tilde{f}$

Assumption 3.5. By (3.5), it can be computed that

$$\begin{aligned}
& J(f',g',h') - J(\tilde{f},\tilde{g},\tilde{h}) \\
= \quad & \frac{1}{\gamma+1}\big(\mathrm{Var}(X|X\in\mathcal{T}_{1+}^{f'}) - \mathrm{Var}(X|X\in\mathcal{T}_{1+}^{\tilde{f}})\big)\mathbb{P}(X\in\mathcal{T}_{1+}^{\tilde{f}}) \\
& +\frac{1}{\gamma+1}\big(\mathrm{Var}(X|X\in\mathcal{T}_{1-}^{f'}) - \mathrm{Var}(X|X\in\mathcal{T}_{1-}^{\tilde{f}})\big)\mathbb{P}(X\in\mathcal{T}_{1-}^{\tilde{f}}) \\
= \quad & \frac{2}{\gamma+1}\big(\mathrm{Var}(X|X\in\mathcal{T}_{1+}^{f'}) - \mathrm{Var}(X|X\in\mathcal{T}_{1+}^{\tilde{f}})\big)\mathbb{P}(X\in\mathcal{T}_{1+}^{\tilde{f}})
\end{aligned}$$

Moreover, by Proposition 3.1, we have $\mathrm{Var}(X|X\in\mathcal{T}_{1+}^{f'}) \leq \mathrm{Var}(X|X\in\mathcal{T}_{1+}^{\tilde{f}})$. Hence,

$$J(f',g',h') \leq J(\tilde{f},\tilde{g},\tilde{h}) \leq J(f,g,h)$$

and f' is a threshold-in-threshold based policy. □

With Theorem 3.2, we simply an optimization problem over a function space to an optimization problem over a two-dimensional space. Hence, we can compute the optimal thresholds β_1 and β_2 via a standard approach. Once the communication scheduling policy f is of threshold-in-threshold type with thresholds β_1 and β_2 (consider the interior point first, i.e., $\beta_1 < \beta_2$), the cost function (3.5) can be expressed as

$$\begin{aligned}
J(f,g,h) \quad = \quad & 2\int_0^{\beta_1} x^2 p_X(x)dx + 2c_2\int_{\beta_2}^{\infty} p_X(x)dx + 2c_1\int_{\beta_1}^{\beta_2} p_X(x)dx \\
& +\frac{2}{\gamma+1}\mathrm{Var}\big(X|X\in(\beta_1,\beta_2)\big)\int_{\beta_1}^{\beta_2} p_X(x)dx
\end{aligned}$$

Taking partial derivative of $J(\beta_1, \beta_2)$ with respect to β_1, we have

$$\begin{aligned}
&\frac{\partial J(f,g,h)}{\partial \beta_1} \\
&= 2\beta_1^2 p_X(\beta_1) - 2c_1 p_X(\beta_1) + \frac{2}{\gamma+1}\frac{\partial}{\partial \beta_1}\Big(\mathrm{Var}(X|X \in (\beta_1, \beta_2)) \int_{\beta_1}^{\beta_2} p_X(x)dx\Big)
\end{aligned}$$

Similar to (A.2), it can be checked that

$$\frac{\partial}{\partial \beta_1}\big(\mathrm{Var}(X|X \in (\beta_1, \beta_2)) \int_{\beta_1}^{\beta_2} p_X(x)dx\big) = -p_X(\beta_1)\big(\beta_1 - \mathbb{E}[X|X \in (\beta_1, \beta_2)]\big)^2$$

We also compute the partial derivative of $J(f, g, h)$ with respect to β_2. By the first-order optimality condition, the locally optimal thresholds (β_1, β_2) should satisfy

$$\begin{aligned}
&\beta_1^2 - \frac{1}{\gamma+1}\big(\beta_1 - \mathbb{E}[X|X \in (\beta_1, \beta_2)]\big)^2 - c_1 = 0, \\
&\frac{1}{\gamma+1}\big(\beta_2 - \mathbb{E}[X|X \in (\beta_1, \beta_2)]\big)^2 + c_1 - c_2 = 0
\end{aligned} \tag{3.6}$$

Once we obtain solution(s) of (3.6), which are extrema of $J(f, g, h)$, we need to compare $J(f, g, h)$ evaluated at the inner extrema with that evaluated at the boundary, i.e. $\beta_1 = \beta_2$. The one achieving the lowest cost is the global optimal solution.

The existence and uniqueness of solution to (3.6) are difficult to analyze for general symmetric and unimodal densities. The reason is $\mathbb{E}[X|X \in (\beta_1, \beta_2)]$ depends on the source density p_X, which might be complex. To deal with this issue, we specify the source to have Laplace distribution with parameters $(0, \lambda^{-1})$, namely,

$$p_X(x) = \begin{cases} \frac{1}{2}\lambda e^{-\lambda x}, & x \geq 0 \\ \frac{1}{2}\lambda e^{\lambda x}, & x < 0 \end{cases}$$

Then, it can be computed that

$$\begin{aligned}
\mathbb{E}\big[X|X \in (\beta_1, \beta_2)\big] &= \frac{1}{\lambda} + \beta_1 + \frac{(\beta_2 - \beta_1)e^{-\lambda(\beta_2-\beta_1)}}{e^{-\lambda(\beta_2-\beta_1)} - 1} \\
&=: \frac{1}{\lambda} + \beta_1 + \frac{\Delta\beta}{1 - e^{\lambda\Delta\beta}}
\end{aligned} \tag{3.7}$$

where $\Delta\beta = \beta_2 - \beta_1$. Plugging (3.7) into (3.6), we have

$$\beta_1 = \sqrt{c_1 + \frac{1}{\gamma+1}\left(\frac{1}{\lambda} + \frac{\Delta\beta}{1-e^{\lambda\Delta\beta}}\right)^2},$$
$$\beta_2 - \beta_1 - \frac{\Delta\beta}{1-e^{\lambda\Delta\beta}} = \frac{1}{\lambda} + \sqrt{(\gamma+1)(c_2-c_1)}$$

which can be further simplified to

$$\begin{aligned} &\beta_1 = \sqrt{c_1 + \frac{1}{\gamma+1}\left(\Delta\beta - \sqrt{(c_2-c_1)(1+\gamma)}\right)^2}, \\ &\frac{\Delta\beta e^{\lambda\Delta\beta}}{e^{\lambda\Delta\beta}-1} = \frac{1}{\lambda} + \sqrt{(c_2-c_1)(1+\gamma)} \end{aligned} \tag{3.8}$$

Define a function $\varphi(x)$ in terms of x as follows

$$\varphi(x) := \frac{xe^{\lambda x}}{e^{\lambda x}-1} = \frac{x}{1-e^{-\lambda x}}, \quad x \in (0,\infty)$$

Then,

$$\frac{d\varphi(x)}{dx} = \frac{1-e^{-\lambda x}}{(1-e^{-\lambda x})^2} = \frac{1}{1-e^{-\lambda x}} > 0, \quad \forall\, x \in (0,\infty)$$

Furthermore, it can be verified that $\varphi(x)$ ranges over $(1/\lambda, \infty)$ when $x \in (0,\infty)$. Hence, the second equation in (3.8) has a unique solution, which determines β_1 by the first equation in (3.8), and $\beta_2 = \Delta\beta + \beta_1$.

As discussed earlier, we need to compare the performance of the inner extremum obtained from (3.8) with that of the boundary. Consider any $\beta_1 = \beta_2, \beta_1 > 0$ on the boundary, we first fix β_1 and minimize $J(f,g,h)$ over $\beta_2^* \in [\beta_1, \infty)$. It can be shown (by analyzing $\partial J(f,g,h)/\partial\beta_2^*$) that the minimizing $\beta_2^* = \beta_1 + \Delta\beta$, where $\Delta\beta$ satisfies the second equation in (3.8). Then, we keep $\beta_2^* = \beta_1 + \Delta\beta$ and minimize $J(f,g,h)$ over $\beta_1 \in (0,\infty)$. Taking the derivative of $J(f,g,h)$ with respect to β_1, we have

$$\frac{dJ(f,g,h)}{d\beta_1} = \frac{\partial J(f,g,h)}{\partial\beta_1} + \frac{\partial J(f,g,h)}{\partial\beta_2^*}\frac{d\beta_2^*}{d\beta_1} = \frac{\partial J(f,g,h)}{\partial\beta_1}$$

where the second equality holds since $\partial J(f,g,h)/\partial\beta_2^* = 0$ when $\beta_2^* = \beta_1 + \Delta\beta$. By analyzing $\partial J(f,g,h)/\partial\beta_1$, it can be shown that the minimizing β_1^* is the one satisfying the first equation in (3.8). Hence, the inner extremum (β_1^*, β_2^*) obtained from (3.8) outperforms any boundary point $\beta_1 = \beta_2$, which implies

(β_1^*, β_2^*) is the global minimum.

Remark 3.3. *When formulating the problem in Section 3.1, we assume that $c_1 < c_2$. If $c_1 \geq c_2$, that is, the noisy channel is more costly than the noiseless channel, the sensor should always use the noiseless channel if it decides to transmit its observation. Therefore, the problem collapses to noiseless-channel setting. By the results from [29], the optimal communication scheduling policy is still threshold-in-threshold based, with optimal thresholds $\beta_1 = \beta_2 = \sqrt{c_2}$.*

3.3 Optimization Problem with Hard Constraint

In this section, we continue our focus on the modified problem, but this time with hard constraint. We first introduce E_t^n and E_t^p as the remaining communication opportunities at time t for the noisy channel and the perfect channel, respectively. Then, E_t^n and E_t^p can be obtained from the sensor's decisions up to $t-1$, namely,

$$E_t^n = N_1 - \sum_{i=1}^{t-1} \mathbf{1}_{\{U_i=1\}}, \quad E_t^p = N_2 - \sum_{i=1}^{t-1} \mathbf{1}_{\{U_i=2\}}$$

As discussed in Remark 3.1, $U_{1:t-1}$ is the common information shared by all the decision makers. Hence, E_t^n and E_t^p are also known by the sensor, the encoder, and the decoder. With a little abuse of notation, we introduce $J(t, E_t^n, E_t^p)$ as the optimal cost-to-go when the system is initialized at time t with E_t^n and E_t^p communication opportunities for noisy channel and perfect channel, respectively. Then, we have the following theorem on the structure of the optimal decision policies. Its proof is similar to that of Theorem 2.3 and hence is omitted here.

Theorem 3.3. *Without loss of optimality, the sensor, the encoder, and the decoder can apply the following types of decision policies:*

$$U_t = f_t(X_t, E_t^n, E_t^p), \quad Y_t = g_t(X_t, S_t, E_t^n, E_t^p), \quad \hat{X}_t = h_t(\tilde{Y}_t, S_t, E_t^n, E_t^p)$$

Furthermore, the optimal cost-to-go $J(t, E_t^n, E_t^p)$ can be obtained by solving *the dynamic programming (DP) equation:*

$$J^*(t, E_t^n, E_t^p) = \inf_{f_t, g_t, h_t} \Big\{ \mathbb{E}\big[(X_t - \hat{X}_t)^2 + J^*(t+1, E_{t+1}^n, E_{t+1}^p)\big] \Big\}$$

with boundary condition that $J^*(T+1, \cdot, \cdot) = 0$.

Depending on the realization of X_t, E_{t+1}^n may be E_t^n or $E_t^n - 1$, and E_{t+1}^p may be E_t^p or $E_t^p - 1$. Hence, the DP equation can be written as

$$\begin{aligned}
& J^*(t, E_t^n, E_t^p) \\
= \; & \inf_{f_t, g_t, h_t} \Big\{ \mathbb{E}\big[(X_t - \hat{X}_t)^2 + J^*(t+1, E_{t+1}^n, E_{t+1}^p)\big] \Big\} \\
= \; & J^*(t+1, E_t^n, E_t^p) \\
& + \inf_{f_t, g_t, h_t} \Big\{ \mathbb{E}\Big[(X_t - \hat{X}_t)^2 + c_1(t, E_t^n, E_t^p)\mathbf{1}_{\{U_t=1\}} + c_2(t, E_t^n, E_t^p)\mathbf{1}_{\{U_t=2\}}\Big] \Big\} \\
= \; & J^*(t+1, E_t^n, E_t^p) + \inf_{f_t, g_t, h_t} \Big\{ \mathbb{E}\big[(X_t - \hat{X}_t)^2 + c(t, E_t^n, E_t^p, U_t)\big] \Big\}
\end{aligned}$$

where

$$\begin{aligned}
c_1(t, E_t^n, E_t^p) &= J^*(t+1, E_t^n - 1, E_t^p) - J^*(t+1, E_t^n, E_t^p), \\
c_2(t, E_t^n, E_t^p) &= J^*(t+1, E_t^n, E_t^p - 1) - J^*(t+1, E_t^n, E_t^p)
\end{aligned}$$

and

$$c(t, E_t^n, E_t^p, U_t) = \begin{cases} 0, & \text{if } U_t = 0 \\ c_1(t, E_t^n, E_t^p), & \text{if } U_t = 1 \\ c_2(t, E_t^n, E_t^p), & \text{if } U_t = 2 \end{cases}$$

Then the problem inside $\inf\{\cdot\}$ is a single-stage problem with soft constraint. Hence, we make the assumptions analogous to those we have made in Section 3.2.2.

Assumption 3.6. *The source has Laplace distribution with parameters* $(0, \lambda^{-1})$. *The noise has zero mean and finite variance* σ_V^2.

Assumption 3.7. *The sensor is restricted to apply the communication scheduling policy in the form of*

$$f_t(x, \cdot, \cdot) = f_t(-x, \cdot, \cdot), \quad \forall\, x \in \mathbb{R}$$

Assumption 3.8. *The encoder and the decoder are restricted to apply piecewise affine encoding and decoding policies, respectively, i.e.,*

$$\begin{aligned} g_t(X_t, S_t, E_t^n, E_t^p) &= S_t\alpha_t\left(X_t - \mathbb{E}\left[X_t|U_t = 1, S_t, E_t^n, E_t^p\right]\right) \\ h_t(\tilde{Y}_t, S_t, E_t^n, E_t^p) &= \frac{1}{\alpha_t}\frac{\gamma}{\gamma+1}S_t\tilde{Y}_t + \mathbb{E}\left[X_t|U_t = 1, S_t, E_t^n, E_t^p\right] \end{aligned}$$

The parameters $\gamma = P_T/\sigma_V^2$, *and* $\alpha_t = \sqrt{P_T/\mathrm{Var}(X_t|U_t = 1, S_t, E_t^n, E_t^p)}$.

Then, we have the following theorem by applying Theorem 3.2.

Theorem 3.4. *Consider the modified problem with Assumptions 3.6-3.8 hold, the optimal communication scheduling policy is of the symmetric threshold-in-threshold type as follows:*

$$f_t(X_t, E_t^n, E_t^p) = \begin{cases} 0, & \text{if} \quad |X_t| \le \beta_1(t, E_t^n, E_t^p) \\ 1, & \text{if} \quad \beta_1(t, E_t^n, E_t^p) < |X_t| < \beta_2(t, E_t^n, E_t^p) \\ 2, & \text{if} \quad |X_t| \ge \beta_2(t, E_t^n, E_t^p) \end{cases}$$

where the optimal thresholds $\beta_1(t, E_t^n, E_t^p)$ *and* $\beta_2(t, E_t^n, E_t^p)$ *can be obtained from* (3.8) *if* $c_2(t, E_t^n, E_t^p) > c_1(t, E_t^n, E_t^p)$. *Otherwise, both* $\beta_1(t, E_t^n, E_t^p)$ *and* $\beta_2(t, E_t^n, E_t^p)$ *are equal to* $\sqrt{c_2(t, E_t^n, E_t^p)}$.

3.4 Numerical Results

In this section, we present numerical results for the problem with hard constraint. Similar to what we have done in Section 2.4, we solve the DP equation numerically with $\lambda = 1$, $\gamma = 1$ and $T = 100$. We plot the optimal 100-stage estimation error versus the numbers of communication opportunities for the perfect channel and the noisy channel separately on two figures. We also generate a sample path of the numbers of remaining communication opportunities, E_t^n and E_t^p, versus time. The numerical results have properties inheriting from both the noiseless-channel setting and the single-channel setting.

In Fig. 3.5, we fix the number of communication opportunities for the noiseless channel as $N_2 = 0, 10, 20$, and we plot the optimal 100-stage estimation error versus the number of communication opportunities for the

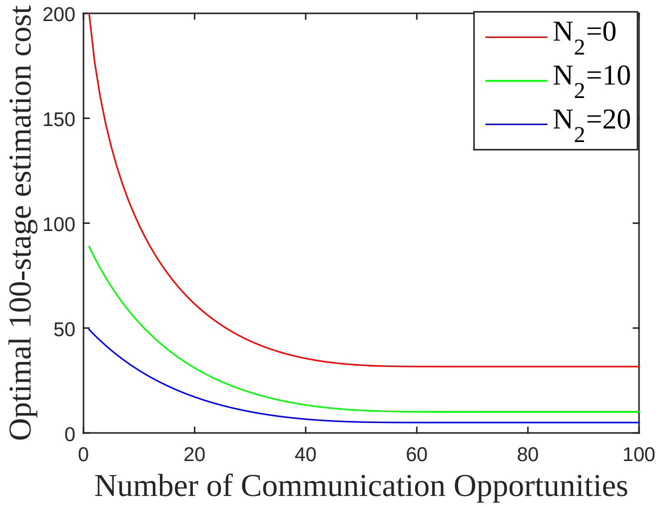

Figure 3.5: Optimal 100-stage estimation error vs. number of communication opportunities for the noisy channel

noisy channel N_1. When $N_2 = 0$, there is no communication opportunity for the noiseless channel, the problem collapses to the single-channel setting. As discussed in Section 2.4, there exists an opportunity threshold such that the optimal 100-stage estimation error decreases when the number of communication opportunities is below the threshold, and stays constant above the threshold. One can see that the existence of opportunity threshold remains in the case when there is an additional noiseless channel.

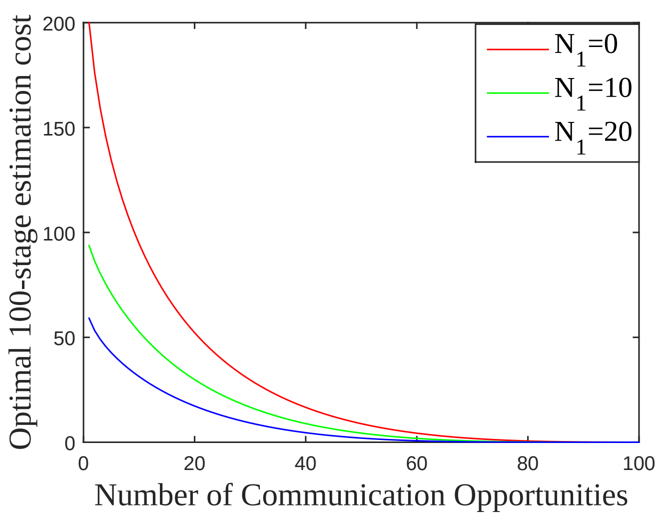

Figure 3.6: Optimal 100-stage estimation error vs. number of communication opportunities for the noiseless channel

Figure 3.6 illustrates the performances of the decision strategies when N_1 is fixed as $N_1 = 0, 10, 20$, and N_2 varies over $\{0, 1, \dots, 100\}$. When $N_1 = 0$, there is no communication opportunity for the noisy channel, and the problem collapses to that in the noiseless-setting. The plot recovers the result in [27]. As shown in [27], the optimal 100-stage estimation error decreases to zero as the number of communication opportunities for the noiseless channel increases to the length of the time horizon. This trend remains for the case when there is an additional noisy channel.

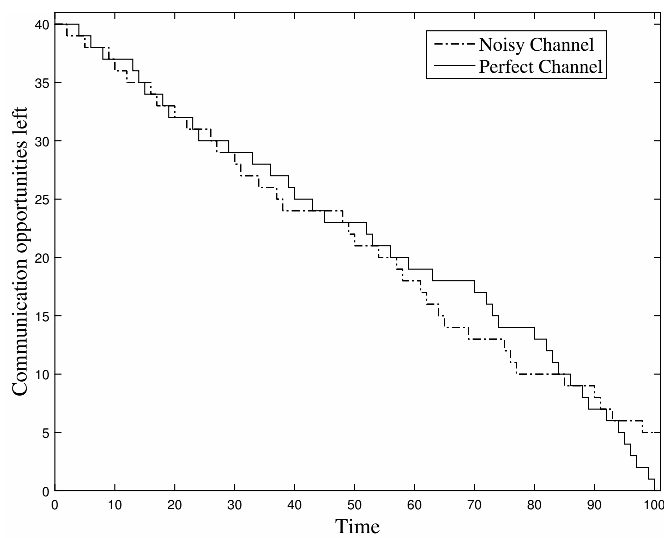

Figure 3.7: A sample path

Figure 3.7 depicts a sample path illustrating the evolution of the remaining communication opportunities for the noisy and the noiseless channels, i.e., E_t^n and E_t^p. When generating the plot, we chose $N_1 = N_2 = 40$. One can observe that by the end of time horizon, the sensor used up all the communication opportunities for the noiseless channel (inheriting from [27], Figure 6), but not all the communication opportunities for the noisy channel (inhereting from Fig. 2.3 in Section 2.4).

CHAPTER 4

REMOTE ESTIMATION WITH COMMUNICATION SCHEDULING AND POWER ALLOCATION

In this chapter, we consider a communication scheduling and remote estimation problem over an additive noise channel. At each time, the sensor makes an observation of the state of a one-dimensional stochastic process, and then decides whether to transmit its observation to the remote estimator or not. The sensor is restricted to transmit its observations for no more than a fixed number of times. In addition, the sensor is charged a cost for each transmission. If the sensor decides to transmit its observation, it sends it to the encoder, who then sends an encoded message over the communication channel. The encoded message is distorted by an additive communication channel noise, and is received by the remote estimator. If the sensor decides not to transmit its observation, the remote estimator will receive a notification about the sensor's decision. The remote estimator generates a real-time estimate on the state variable, and it is charged for the estimation error. Different from the problems considered in the earlier chapters, which assumed that the encoder has a stage-wise constraint on the average encoding power when there is a transmission from the sensor, we consider in this chapter a more general setting where the encoder has a constraint on its average total power consumption over the time horizon. In this scenario, the encoding power should be wisely allocated to each stage. In addition, the communication scheduling policy, the encoding policy, and the decoding policy should be jointly designed to best utilize the encoding power. Under some technical assumptions, we show that the optimal communication scheduling policy is still threshold-based, and we jointly optimize the threshold together with the stage-wise average encoding power. We generate numerical results to demonstrate that with the additional flexibility on choosing the state-wise encoding power, the estimation error accumulated over the time horizon can be further reduced.

The contributions of this chapter are listed as follows:

1. We formulate a communication scheduling and remote estimation problem over an additive noise channel. Different from what was covered by Chapters 2 and 3, the encoder has a constraint on its average total power consumption over the time horizon.

2. Under some technical assumptions, we show that the optimal communication scheduling policy is threshold-based. In addition, we jointly design the optimal threshold together with the stage-wise encoding power when the sensor decides to transmit its observation.

3. We generate numerical results to demonstrate the performance of the designed policies, and we compare it with the performance of the policies proposed in the prior work.

The rest of the chapter is organized as follows: in Section 4.1, we formulate the optimization problem. In Section 4.2.1, we solve a single-stage problem, whose results are applied in Section 4.2.2 to solve the multi-stage problem. In Section 4.3, we present and illustrate the numerical results.

4.1 Problem Formulation

4.1.1 System Model

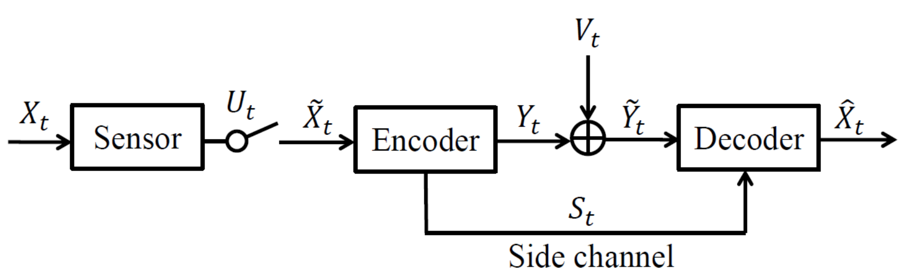

Figure 4.1: System model

We are interested in measuring a one-dimensional discrete time stochastic process $\{X_t\}$ over a finite time horizon, i.e., $t = 1, 2, \ldots, T$. In order to achieve this goal, a sensor is placed. At each time t, the sensor, as shown in Fig. 4.1, perfectly observes X_t. The sensor is assumed to have limited energy for communication such that it is able to transmit its observations only a limited number of times. Thus, after observing X_t, the sensor needs to decide whether to transmit its observation or not. Let $U_t \in \{0, 1\}$ be the sensor's decision at time t, where 1 means to transmit and 0 means not to transmit. Moreover, suppose that the communication channel is noisy. Hence, if the sensor decides to transmit its observation, it sends the observation X_t to an encoder. Denote by $\tilde{X}_t$ be the message received by the encoder at time t. Then,

$$\tilde{X}_t = \begin{cases} X_t, & \text{if } U_t = 1 \\ \epsilon, & \text{if } U_t = 0 \end{cases}$$

where ϵ is a free symbol representing that the encoder does not receive any message from the sensor. Once the encoder receives X_t, it sends an encoded message to the communication channel, denoted by $Y_t, Y_t \in \mathbb{R}$. The encoded message Y_t is corrupted by an additive channel noise $V_t, V_t \in \mathbb{R}$. The encoder will not send any message to the communication channel if it does not receive any message from the sensor, denoted by $Y_t = 0$. Denote by $\tilde{Y}_t$ the message received by the decoder. Then,

$$\tilde{Y}_t = \begin{cases} Y_t + V_t, & \text{if } \tilde{X}_t \neq \epsilon \\ V_t, & \text{if } \tilde{X}_t = \epsilon \end{cases}$$

When sending the encoded message Y_t to the communication channel, the encoder is able to send a two-bit message, denoted by S_t, to the decoder via a side channel. $S_t \in \{0, 1, -1\}$, where $\{1, -1\}$ is the sign of X_t and 0 means no message is sent via the communication channel. Furthermore, it is assumed that the side channel is noise-free. Based on the messages received from the encoder, the decoder generates an estimate on X_t, denoted by $\hat{X}_t$.

4.1.2 Communication Constraints

There are two types of communication constraints: The sensor is able to transmit its observations for no more than N times, where $N < T$, or equivalently,

$$\sum_{t=1}^{T} U_t \leq N$$

Furthermore, the encoder is assumed to have an *average total power constraint*, that is

$$\sum_{t=1}^{T} \mathbb{E}\big[Y_t^2\big] \leq P_{total} \tag{4.1}$$

where P_{total} is the *total power*. Note that when there is no transmission, $Y_t = 0$ and thus there is no power consumption.

4.1.3 Decision Strategies

Assume that at time t, the information available to the sensor is all its observations up to t, denoted by $X_{1:t}$. The sensor makes decision U_t based on its current information $X_{1:t}$ and the communication scheduling policy at time t, denoted by f_t, i.e.,

$$U_t = f_t(X_{1:t})$$

Define $\mathbf{f} := \{f_1, f_2, \ldots, f_T\}$; we call $\mathbf{f}$ the communication scheduling strategy.

Similarly, assume that at time t, the information available to the encoder is the collection of all the messages received from the sensor up to t, denoted by $\tilde{X}_{1:t}$. The encoder generates the encoded message Y_t based on its current information $\tilde{X}_{1:t}$ and the encoding policy at time t, denoted by g_t, i.e.,

$$Y_t = g_t(\tilde{X}_{1:t})$$

Define $\mathbf{g} := \{g_1, g_2, \ldots, g_T\}$; we call $\mathbf{g}$ the encoding strategy.

Finally, assume that at time t, the information available to the decoder is the collection of all the messages received from the encoder up to t, denoted by $\tilde{Y}_{1:t}, S_{1:t}$. The decoder generates the estimate $\hat{X}_t$ based on its current information $(\tilde{Y}_{1:t}, S_{1:t})$ and the decoding policy at time t, denoted by h_t, i.e.,

$$\hat{X}_t = h_t(\tilde{Y}_{1:t}, S_{1:t})$$

Define $\mathbf{h} := \{h_1, h_2, \ldots, h_T\}$. We call $\mathbf{h}$ the decoding strategy.

We denote by $U_{1:t-1}$ the sensor's decisions up to $t-1$. Although we do not assume that the sensor, the encoder and the decoder have memory on $U_{1:t-1}$, yet the sensor can deduce this information from $X_{1:t-1}$ and $\mathbf{f}$. The encoder and the decoder can deduce $U_{1:t-1}$ from $\tilde{X}_{1:t-1}$ and $S_{1:t-1}$, respectively. Hence, $U_{1:t-1}$ is the common information among all the decision makers, and this property will be used later.

4.1.4 Optimization Problem

At time t, the sensor is charged a cost c if and only if it transmits its observation, call it *communication cost*. Furthermore, the decoder is charged for squared estimation error, $(X_t - \hat{X}_t)^2$, call it *estimation cost*.

The cost functional associated with $(\mathbf{f}, \mathbf{g}, \mathbf{h})$, denoted by $J(\mathbf{f}, \mathbf{g}, \mathbf{h})$, is the expected value of the sum of communication cost and estimation cost over the time horizon, i.e.,

$$J(\mathbf{f}, \mathbf{g}, \mathbf{h}) := \mathbb{E}\Big[\sum_{t=1}^{T} cU_t + (X_t - \hat{X}_t)^2\Big]$$

Hence, the optimization problem is to design $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ to minimize the above cost functional subjected to the communication constraints.

We would like to clarify the difference between the problems studied in this chapter and the one studied in Chapter 2. For the problem studied in Chapter 2, it is assumed that the encoder has a stage-wise constraint on its average encoding power when the sensor decides to transmit its observation, i.e.,

$$\mathbb{E}[Y_t^2 | U_t = 1] \leq \bar{P} \quad \forall\, t = 1, \ldots, T$$

where $\bar{P}$ is assumed to be a constant. Due to such a constraint, when there is a transmission from the sensor, the encoder will use the same power,[1] which is predefined, to encode and transmit the message. In this chapter, however, the encoder is assumed to have a constraint on its average total power consumption over the time horizon, as described by (4.1). Then, several technical challenges are involved due to such a constraint. First, the encoding power, $\mathbb{E}[Y_t^2]$, should be wisely allocated to each stage. Second, at each stage t, the communication scheduling policy, f_t, and the encoding policy, g_t, should be jointly designed such that the encoding power allocated to that stage is best utilized. To be more specific, the sensor may transmit its observation with low probability yet the encoder may use high power to encode and transmit the observation, or vice versa. Both challenges will be

[1]The encoder should utilized the maxmium allowable power so that the estimation error can be minimized.

addressed in this chapter.

4.2 Main Results

We first solve a single-stage problem. Then, we will apply the results for the single-stage problem to solve the multi-stage problem.

4.2.1 The Single-stage Problem

Consider the optimization problem formulated in Section 4.1. We assume that the length of the time horizon, $T = 1$, i.e., the problem is a single-stage problem. For simplicity, we suppress the subscript for time in all the expressions during this section. Furthermore, we assume that $N = T$, i.e., the sensor is always allowed to transmit its observation. However, the sensor is still charged a cost if it makes a transmission. Let $J(f, g, h)$ be the cost functional associated with communication scheduling, encoding and decoding policies (f, g, h), i.e.,

$$J(f, g, h) := \mathbb{E}[cU + (X - \hat{X})^2]$$

In this scenario, the encoder has the average total power constraint

$$\mathbb{E}[Y^2] \leq P_{total}$$

We attach this constraint to the cost functional, $J(f, g, h)$, via the Lagrange multiplier, λ, where $\lambda > 0$. We denote by $J_\lambda(f, g, h)$ the augmented cost functional, i.e.,

$$J_\lambda(f, g, h) = \mathbb{E}[cU + (X - \hat{X})^2] + \lambda\, \mathbb{E}[Y^2]$$

The augmented cost functional $J_\lambda(f,g,h)$ can also be written as

$$\begin{aligned} & J_\lambda(f,g,h) \\ = \;& \mathbb{E}[cU + (X-\hat{X})^2] + \lambda\, \mathbb{E}[Y^2] \\ = \;& c\,\mathbb{E}[U] + \mathbb{E}[(X-\hat{X})^2] + \lambda\, \mathbb{E}[Y^2|U=1]\, \mathbb{P}(U=1) \\ = \;& c\,\mathbb{E}[U] + \mathbb{E}[(X-\hat{X})^2] + \lambda\, \mathbb{E}[Y^2|U=1]\, \mathbb{E}[U] \\ = \;& \big(c + \lambda \mathbb{E}[Y^2|U=1]\big)\mathbb{E}[U] + \mathbb{E}[(X-\hat{X})^2] \end{aligned}$$

The second equality holds since $Y^2 = 0$ when $U = 0$, and the third equality holds since U is a binary variable.

To minimize $J_\lambda(f,g,h)$, we may first fix $\mathbb{E}[Y^2|U=1] = P$, and derive the minimizing (f,g,h) subject to this constraint. Then, we choose the minimizing P over $[0,\infty)$. We call $\mathbb{E}[Y^2|U=1] = P$ the *average encoding power constraint* and P is the *encoding power*. For any $P \geq 0$, we define

$$J_{\lambda,P}(f,g,h) := \mathbb{E}[(c+\lambda P)U + (X-\hat{X})^2]$$

We are now deriving the optimal decision policies (f,g,h) minimizing $J_{\lambda,P}(f, g,h)$ subject to $\mathbb{E}[Y^2|U=1] = P$, call it *Sub-problem*, and we make the following assumptions.

Assumption 4.1. *The source variable X is a continuous random variable with an even and unimodal density function $p_X(\cdot)$. That is,*

$$\begin{aligned} p_X(x) &= p_X(-x),\ \forall\, x \in \mathbb{R} \\ p_X(a) &\geq p_X(b),\ \forall\, |a| \leq |b| \end{aligned}$$

The communication channel noise V has zero mean and finite variance, denoted by σ_V^2. In addition, the source variable X and the channel noise V are independent.

Assumption 4.2. *The encoder and the decoder are restricted to apply piece-*

wise affine encoding and decoding policies in the form of

$$g(\tilde{X}) = \begin{cases} S\,\alpha(S)\,\big(X - \mathbb{E}[X|U=1,S]\big), & \text{if } U = 1 \\ 0, & \text{if } U = 0 \end{cases}$$
$$h(\tilde{Y}, S) = \begin{cases} \dfrac{S}{\alpha(S)}\dfrac{\gamma}{\gamma+1}\tilde{Y} + \mathbb{E}[X|U=1,S], & \text{if } U = 1 \\ \mathbb{E}[X|U=0], & \text{if } U = 0 \end{cases} \tag{4.2}$$

where

$$\alpha(S) = \sqrt{\frac{P}{\mathrm{Var}(X|U=1,S)}}, \quad \gamma = \frac{P}{\sigma_V^2}$$

The expressions $\mathbb{E}[X|U=1,S]$, $\mathbb{E}[X|U=0]$ and $\mathrm{Var}(X|U=1,S)$ are the conditional means and variance, respectively, which depend on the distribution of X and the choice of f. Furthermore, it can be shown that the constraint on the encoding power, i.e., $\mathbb{E}[Y^2|U=1] = P$, is satisfied if the encoder applies the piecewise affine encoding policy described in (4.2).

Assumption 4.3. *The sensor is restricted to apply the communication scheduling policy in the form of*

$$f(x) = f(-x) \in \{0, 1\}, \quad \forall\, x \in \mathbb{R}$$

Remark 4.1. *Assumption 4.3 is not only an assumption but also a conjecture on the optimal communication scheduling policy. Since the source has symmetric distribution, and we can encode/decode positive and negative realizations of the source separately, it is natural to conjecture that the optimal communication scheduling policy is symmetric about zero.*

With the assumptions described above, we have the following theorem.

Theorem 4.1. *Consider the Sub-problem with Assumptions 4.1-4.3 hold, the optimal communication scheduling policy, f, is threshold-based. That is, f is in the form of*

$$f(x) = \begin{cases} 0, & \text{if } |x| \le \beta \\ 1, & \text{if } |x| > \beta \end{cases} \tag{4.3}$$

where $\beta \ge 0$ is the threshold.

Proof. The Sub-problem is just the communication scheduling and remote estimation problem over an additive noise channel, which has been studied in Chapter 2. Correspondingly, Theorem 4.1 is a restatement of Theorem 2.2. We include it here so that this chapter is self-contained. □

With Theorem 4.1, minimizing $J_{\lambda,P}(f,g,h)$ over (f,g,h,P) is equivalent to minimizing $J_{\lambda,P}(f,g,h)$ over (β,P).[2] Correspondingly, an infinite-dimensional optimization problem has been simplified to a finite-dimensional one. Once the sensor applies the threshold-based policy f with threshold β, the encoder and the decoder apply the piecewise affine encoding and decoding policies, respectively, with the encoding power P, the cost functional $J_{\lambda,P}(f,g,h)$ can be expressed as follows:

$$\begin{aligned} J_{\lambda,P}(f,g,h) &= 2\big(c+\lambda P\big)\int_{\beta}^{\infty} p_X(x)dx + 2\int_0^{\beta} x^2 p_X(x)dx \\ &\quad + \frac{2\sigma_V^2}{P+\sigma_V^2}\mathrm{Var}(X|X>\beta)\int_{\beta}^{\infty} p_X(x)dx \end{aligned}$$

Taking the partial derivatives of $J_{\lambda,P}(f,g,h)$ with respect to β and P, respectively, we have

$$\begin{aligned} \frac{\partial J_{\lambda,P}(f,g,h)}{\partial\beta} &= 2p_X(\beta)\left(\beta^2 - \frac{\sigma_V^2}{P+\sigma_V^2}\big(\beta-\mathbb{E}[X|X>\beta]\big)^2 - (c+\lambda P)\right) \\ \frac{\partial J_{\lambda,P}(f,g,h)}{\partial P} &= 2\int_{\beta}^{\infty} p_X(x)dx\cdot\left(\lambda - \frac{\sigma_V^2}{(P+\sigma_V^2)^2}\mathrm{Var}(X|X>\beta)\right) \end{aligned}$$

By the first-order optimality condition, the optimal solution (β,P) should satisfy

$$\begin{aligned} \beta^2 - \frac{\sigma_V^2}{P+\sigma_V^2}(\beta-\mathbb{E}[X|X>\beta])^2 - (c+\lambda P) = 0 \\ \lambda - \frac{\sigma_V^2}{(P+\sigma_V^2)^2}\mathrm{Var}(X|X>\beta) = 0 \end{aligned} \tag{4.4}$$

In general, $\mathbb{E}[X|X>\beta]$ and $\mathrm{Var}(X|X>\beta)$ depend on β and the distribution of X, which might be hard to compute. Hence, we make the following assumption specifying the distribution of the source variable.

Assumption 4.4. *The source variable X has the Laplace distribution with*

[2] As discussed before, the encoding and the decoding policies, g and h, are induced by the source density p_X and the communication scheduling policy f.

parameters $(0, k^{-1})$, *i.e.*,

$$p_X(x) = \begin{cases} \frac{1}{2}ke^{-kx}, & \textit{if } x \geq 0 \\ \frac{1}{2}ke^{kx}, & \textit{if } x < 0 \end{cases}$$

We note that $p_X(x)$ is even and unimodal, which satisfies Assumption 4.1. In addition, condition on the event that $X > \beta$, X has a shifted exponential distribution with parameter k, i.e.,

$$p_{X|X>\beta}(x) = ke^{-k(x-\beta)},\ x \geq 0$$

which implies that

$$\mathbb{E}[X|X > \beta] = \beta + \frac{1}{k}, \quad \mathrm{Var}(X|X > \beta) = \frac{1}{k^2}$$

Consequently, the first-order optimality condition, described by (4.4), can be further expressed as

$$\begin{aligned} &\beta^2 - \frac{\sigma_V^2}{P + \sigma_V^2}\frac{1}{k^2} - (c + \lambda P) = 0 \\ &\lambda - \frac{\sigma_V^2}{(P + \sigma_V^2)^2}\frac{1}{k^2} = 0 \end{aligned} \tag{4.5}$$

Depending on the value of λ, we have the following discussion.

1. When $0 < \lambda < \frac{1}{\sigma_V^2 k^2}$, (4.5) admits a unique solution

$$\begin{aligned} &\beta^* = \sqrt{c + \frac{\sigma_V^2}{P^* + \sigma_V^2}\frac{1}{k^2}(1 + \frac{P^*}{P^* + \sigma_V^2})} \\ &P^* = \sqrt{\frac{\sigma_V^2}{\lambda k^2}} - \sigma_V^2 \end{aligned} \tag{4.6}$$

   It can be shown that when fixing β, $J_{\lambda,P}(f, g, h)$ attains global minimum over $P \in [0, \infty)$ at P^*, which is independent of β. It can also be shown that when plugging in $P = P^*$, $J_{\lambda,P^*}(f, g, h)$ attains global minimum over $\beta \in [0, \infty)$ at β^*. Hence, the pair (β^*, P^*), described by (4.6), is jointly optimal over $[0, \infty) \times [0, \infty)$.

2. When $\lambda \geq \frac{1}{\sigma_V^2 k^2}$, (4.5) does not admit a solution, which implies that

the jointly optimal solution may be on the boundary. By a similar argument as above, it can be shown that $J_{\lambda,P}(f,g,h)$ attains global minimum at $(\beta^*, 0)$.

We combine the above two cases and claim that $J_{\lambda,P}(f,g,h)$ attains global minimum at

$$\begin{aligned}\beta^* &= \sqrt{c + \frac{\sigma_V^2}{P^* + \sigma_V^2}\frac{1}{k^2}(1 + \frac{P^*}{P^* + \sigma_V^2})} \\ P^* &= \max\left\{\sqrt{\frac{\sigma_V^2}{\lambda k^2}} - \sigma_V^2, 0\right\}\end{aligned} \tag{4.7}$$

We have so far shown that with Assumptions 4.1-4.3, the optimal communication scheduling policy f is threshold-based with threshold β^*, and the optimal encoding and decoding policies (g,h) are of the form described by (4.2) with the encoding power P^*. The pair (β^*, P^*) is captured by (4.7). We are now in a position to determine the Lagrange multiplier, λ, or equivalently, determine P^*. Recall that the encoder should satisfy the total power constraint. Namely,

$$\begin{aligned}\mathbb{E}[Y^2] &= \mathbb{E}[Y^2|U=1]\cdot\mathbb{P}(U=1) \\ &= P^*\cdot\mathbb{P}(|X|>\beta^*) \\ &= P^*\cdot\exp(-k\beta^*) \\ &= P^*\cdot\exp\left(-k\sqrt{c + \frac{\sigma_V^2}{P^* + \sigma_V^2}\frac{1}{k^2}\left(1 + \frac{P^*}{P^* + \sigma_V^2}\right)}\right) \\ &=: F(P^*) \\ &= P_{total}\end{aligned} \tag{4.8}$$

We take $F(P^*) = P_{total}$ (instead of $F(P^*) \leq P_{total}$) since the encoder should take the maximum allowable total power to achieve the lowest estimation error. It can be shown that $F(P)$ is a strictly increasing function of P ranging over $[0,\infty)$. Hence, for any $P_{total} \geq 0$, there exists a unique P^* satisfying (4.8).

4.2.2 The Multi-stage Problem

We now consider the optimization problem over multiple time steps as formulated in Section 4.1. Similar to the approach applied to solve the single-stage problem, we attach the constraint on the average total power to the cost functional via the Lagrange multiplier, λ, and we denote by $J_\lambda(\mathbf{f}, \mathbf{g}, \mathbf{h})$ the argumented cost functional, i.e.,

$$J_\lambda(\mathbf{f}, \mathbf{g}, \mathbf{h}) := \sum_{t=1}^{T} \mathbb{E}\Big[cU_t + \lambda Y_t^2 + (X_t - \hat{X}_t)^2\Big]$$

We make the following assumption on the source and the noise processes.

Assumption 4.5. *The source and the noise processes $\{X_t\}$ and $\{V_t\}$ are independent and identically distributed (i.i.d.). Furthermore, $\{X_t\}$ is independent of $\{V_t\}$.*

Let E_t be the number of remaining communication opportunities at time step t, i.e.,

$$E_t = N - \sum_{i=1}^{t-1} U_i$$

Recall that at time t, information about $U_{1:t-1}$ is shared among the sensor, the encoder, and the decoder. Hence, E_t is also known by all the decision makers. Let $J^*(t, E_t)$ be the optimal cost-to-go when the system is initialized at time t with E_t number of communication opportunities, and denote by (f_t^*, g_t^*, h_t^*) the optimal decision policies at time t. By a similar argument with Theorem 2.3 in Chapter 2, we have the following proposition.

Proposition 4.1. *(1) Without any loss of optimality, we can restrict the sensor, the encoder, and the decoder to apply, respectively, the communication scheduling policy, the encoding policy, and the decoding policy in the form of:*

$$U_t = f_t(X_t, E_t),\ Y_t = g_t(\tilde{X}_t, E_t),\ \hat{X}_t = h_t(\tilde{Y}_t, S_t, E_t)$$

(2) The optimal cost-to-go $J^*(t, E_t)$ and the optimal decision policies (f_t^*, g_t^*, h_t^*) can be obtained by solving the dynamic programming (DP) equation

$$J^*(t, E_t) = J^*(t+1, E_t) + \inf_{f_t, g_t, h_t} \Big\{\mathbb{E}\big[\big(c + c_t(E_t)\big)U_t + \lambda Y_t^2 + (X_t - \hat{X}_t)^2\big]\Big\}$$

with the boundary conditions $J^*(T+1,\cdot)=0$*, where*

$$c_t(E_t) = J(t+1, E_t - 1) - J(t+1, E_t)$$

is the opportunity cost.

The minimization problem in the braces is a single-stage problem with the communication cost $c + c_t(E_t)$. Hence, we make the following assumptions.

Assumption 4.6. *X_t has Laplace distribution with parameters $(0, k^{-1})$. In addition, V_t has zero mean and finite variance σ_V^2.*

Assumption 4.7. *The encoder and the decoder are restricted to apply piece-wise affine encoding and decoding policies in the form of*

$$\begin{aligned}
Y_t &= \begin{cases} S_t\ \alpha_t(S_t, E_t)\ (X_t - \mathbb{E}[X_t|U_t = 1, S_t, E_t]), & \text{if } U_t = 1 \\ 0, & \text{if } U_t = 0 \end{cases} \\
\hat{X}_t &= \begin{cases} \dfrac{S_t}{\alpha_t(S_t, E_t)} \dfrac{\gamma_t}{\gamma_t + 1} \tilde{Y}_t + \mathbb{E}[X_t|U_t = 1, S_t, E_t], & \text{if } U_t = 1 \\ \mathbb{E}[X_t|U_t = 0, E_t], & \text{if } U_t = 0 \end{cases}
\end{aligned} \tag{4.9}$$

where

$$\alpha_t(S_t, E_t) = \sqrt{\frac{P_t(E_t)}{\mathrm{Var}(X_t|U_t = 1, S_t)}}, \quad \gamma_t = \frac{P_t(E_t)}{\sigma_V^2}$$

and $P_t(E_t)$ is the encoding power at time t.

Assumption 4.8. *The sensor is restricted to apply the communication scheduling policies in the form of*

$$f_t(x, e_t) = f_t(-x, e_t) \quad \forall\, x \in \mathbb{R}, e_t \in \{1, 2, \ldots, N\}$$

In view of the above, the result of the single-stage problem leads to the following.

Theorem 4.2. *Consider the optimization problem formulated in Section 4.1 and assume that Assumptions 4.5-4.8 hold. Then, the optimal communica-*

tion scheduling policy is

$$f_t^*(X_t, E_t) = \begin{cases} 0, \ \textit{if}\ E_t = 0\ \textit{or}\ |X_t| \leq \beta_t^*(E_t) \\ 1, \ \textit{if}\ E_t > 0\ \textit{and}\ |X_t| > \beta_t^*(E_t) \end{cases}$$

The optimal encoding and decoding policies are in the form of (4.9) *with encoding power the* $P_t^*(E_t)$. *In addition,* $\beta_t^*(E_t)$ *and* $P_t^*(E_t)$ *are as follows*

$$\beta_t^*(E_t) = \sqrt{c + c_t(E_t) + \frac{\sigma_V^2}{P_t^*(E_t) + \sigma_V^2} \frac{1}{k^2} \left(1 + \frac{P_t^*(E_t)}{P_t^*(E_t) + \sigma_V^2}\right)}$$

$$P_t^*(E_t) = \max\left\{\sqrt{\frac{\sigma_V^2}{\lambda k^2}} - \sigma_V^2, 0\right\}$$

By Theorem 4.2, the encoding power, $P_t^*(E_t)$, depends only on λ and σ_V^2, which is invariant of time t and the remaining communication opportunities for the sensor, E_t. This is a rather surprising result. An explanation for this phenomena is as follows. Condition on the event that $U_t = 1$, $S_t = 1$ (or $S_t = -1$), X_t has shifted (and reflected) exponential distribution, whose variance is $1/k^2$. Thus, if the sensor decides to transmit its observation, the input signal to the encoder always has variance $1/k^2$. Furthermore, the encoder and the decoder are restricted to apply affine encoding and decoding policies, and thus the minimum mean squared error (MMSE) depends only on the encoding power, the variance of the input signal, and the variance of the noise. Therefore, given that σ_V^2 and the variance of the input signal are independent of t and E_t, it is natural for the encoder to allocate the same encoding power, $P_t^*(E_t)$, regardless of t and E_t. We denote by $P^* := P_t^*(E_t)$ for the rest of the chapter.

Similar to the single-stage problem, we may need to determine the Lagrange multiplier, λ, or equivalently, the encoding power, P^*. Since the encoder must satisfy the average total power constraint, we have

$$\begin{aligned} \textstyle\sum_{t=1}^{T} \mathbb{E}[Y_t^2] &= \textstyle\sum_{t=1}^{T} \mathbb{P}(U_t = 1) \cdot \mathbb{E}[Y_t^2 | U_t = 1] \\ &= \textstyle\sum_{t=1}^{T} \mathbb{P}(|X_t| > \beta_t^*(E_t)) \cdot P^* \\ &=: G(P^*)P^* \\ &= P_{total} \end{aligned} \tag{4.10}$$

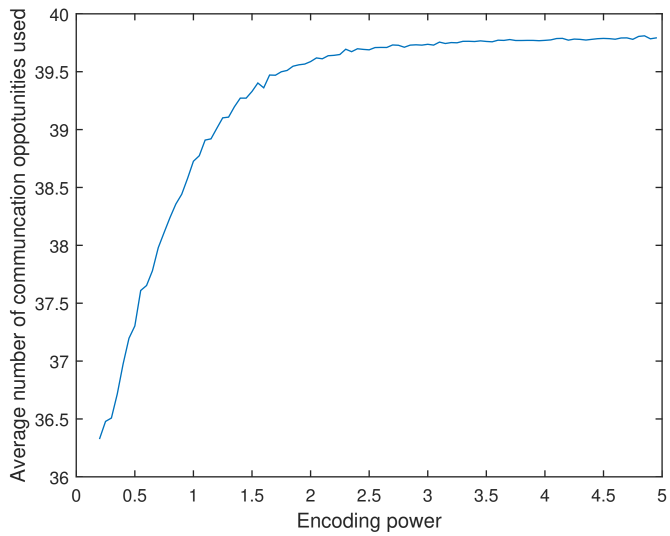

Figure 4.2: Average number of expended communication opportunities vs. encoding power

Apparently, $G(P^*)P^*$ is a continuous function of P^*. In addition, numerical results show that $G(P^*)$ is non-decreasing in P^*, which will be discussed in Section 4.3. Then, $G(P^*)P^*$ is a strictly increasing function of P^* ranging over $[0, \infty)$. Therefore, the solution to (4.10) is unique.

4.3 Numerical Results

We first show by simulation result that $G(P^*)$ is a non-decreasing function of P^*. Then, we are able to claim that $G(P^*)P^*$ is a strictly increasing function of P^* ranging over $[0, \infty)$. Note that

$$G(P^*) = \sum_{t=1}^{T} \mathbb{P}(U_t = 1) = \sum_{t=1}^{T} \mathbb{E}[U_t]$$

which is the average number of expended communication opportunities over the time horizon. Hence, $G(P^*)$ can be computed by the Monte Carlo method. We choose $N = 40$, $T = 100$, $c = 0$, $k = 1$, and $\sigma_V^2 = 1$, and we plot $G(P^*)$ versus P^*. As illustrated in Fig. 4.2, $G(P^*)$ is a non-decreasing function of P^*.

The optimal decision policies described in Theorem 4.2, call it *new algo-*

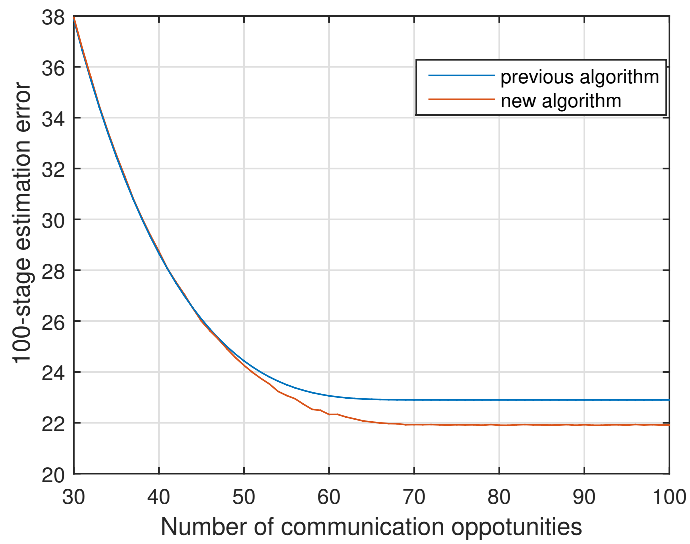

Figure 4.3: Comparison between new algorithm vs. old algorithm

rithm, are in the same form with the ones described in Theorem 2.4 in Chapter 2, call it *previous algorithm*. However, the encoding power, $\mathbb{E}[Y_t^2|U_t = 1]$, and the threshold, $\beta_t(E_t)$, are different for the two groups of policies. To compare the performance of the two algorithms, we need to make sure that the two algorithms are tested under the same average total power consumption over the time horizon, and under the same number of communication opportunities. We choose $T = 100$, $c = 0$, $k = 1$, and $\sigma_V^2 = 1$. Furthermore, we choose the encoding power, $P_{prev} = 2$, for the previous algorithm. Then, for each $N \in \{0, \ldots, T\}$, we compute the 100-stage estimation error and the average number of expended communication opportunities under the previous algorithm, denoted by $J_{prev}(1, N)$ and $\mathbb{E}_{prev}[\sum_{t=1}^{T} U_t]$, respectively. Next, we compute the total power consumed by the previous algorithm, denoted by P_{total}, where

$$P_{total} = P_{prev} \cdot \mathbb{E}_{prev}\left[\sum_{t=1}^{T} U_t\right]$$

Next, we solve (4.10) numerically with the P_{total} obtained in the last step, to get the encoding power, P_{new}, for the new algorithm. By doing this, it is guaranteed that the average total power consumed by the new algorithm is the same with the old algorithm. Finally, we compute the 100-stage estimation error achieved by the new algorithm with the encoding power P_{new}, denoted by $J_{new}(1, N)$. We plot $J_{prev}(1, N)$ and $J_{new}(1, N)$ vs. N on the same figure, as illustrated in Fig. 4.3. One can see that, with the same number of communication opportunities and the same average total power consumption, the new algorithm achieves lower estimation error. The reason is that in the problem considered in Chapter 2, the encoding power is fixed and the threshold is designed under this fixed power, while in the problem studied in this chapter, the encoding power and the threshold are jointly designed.

CHAPTER 5

COMMUNICATION SCHEDULING AND REMOTE ESTIMATION IN THE PRESENCE OF AN ADVERSARY

In this chapter, we study a communication scheduling and remote estimation problem via the worst-case approach. Specifically, a remote sensing system consisting of a sensor, an encoder and a decoder is configured to observe, transmit, and recover a discrete time stochastic process. At each time step, the sensor makes an observation on the state variable of the stochastic process. The sensor is constrained by the number of transmissions over the time horizon, and thus it needs to decide whether to transmit its observation or not after making each measurement. If the sensor decides to transmit, it sends the observation to the encoder, which then encodes and transmits the observation to the decoder. Otherwise, the sensor and the encoder maintain silence. The decoder is required to generate a real-time estimate on the state variable. The sensor, the encoder, and the decoder collaborate to minimize the sum of the communication cost for the sensor, the encoding cost for the encoder, and the estimation error for the decoder. There is also a jammer interfering with the communication between the encoder and the decoder, by injecting an additive channel noise to the communication channel. The jammer is charged for the jamming power and is rewarded for the estimation error generated by the decoder, and it aims to minimize its net cost. We consider a feedback Stackelberg game with the sensor, the encoder, and the decoder as the composite leader, and the jammer as the follower. Under some technical assumptions, we obtain a feedback Stackelberg solution consisting of a threshold-based communication scheduling policy for the sensor, and a pair of piecewise affine encoding and decoding policies for the encoder and the decoder, respectively. We also generate numerical results to demonstrate the performance of the remote sensing system under the feedback Stackelberg solution. The contributions of this chapter are listed as follows:

1. We formulate a dynamic game problem capturing the scenario of communication scheduling and remote estimation in the presence of an

adversary.

2. We show that the feedback Stackelberg solution to the dynamic game can be computed iteratively, by solving a single-stage Stackelberg game at each time step.

3. Under some technical assumptions, we obtain the Stackelberg solution to the single-stage Stackelberg game.

4. We generate and present numerical results to illustrate the performance of the remote sensing system under the feedback Stackelberg solution, e.g., how much the encoding power is allocated to each time step.

The rest of this chapter is organized as follows. In Section 5.1, we present the mathematical formulation of the problem. In Section 5.2, we present the technical assumptions, the main results, and the proofs. In Section 5.3, we provide numerical results.

5.1 Problem Formulation

5.1.1 Remote Sensing System with an Adversary

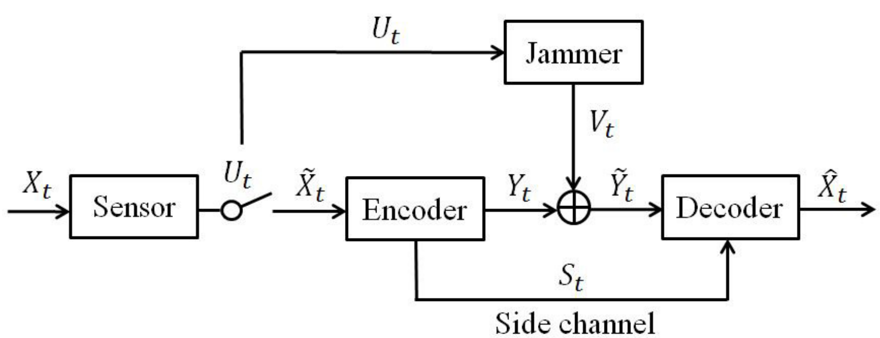

Figure 5.1: Remote sensing system with adversary

Consider a system, as described in Fig. 5.1, which involves observing the state of a remote plant over a finite time horizon $t = 1, \dots, T$, and recovering that information at the other end. The states of the plant over the time horizon are characterized by a one-dimensional, independent and identically distributed (i.i.d.) stochastic process, denoted by $\{X_t\}$, which we call the *source process*. Assume that X_t, for each t, is a continuous random variable, and denote by p_{X_t} its probability density function. The system consists of three components: a sensor, an encoder, and a decoder. In addition, there is an adversary, call it "jammer", that injects a disturbing input into the communication channel with the objective of maximizing the distortion of the recovered state. The functionalities of the system components and the jammer are described as follows.

Sensor: At each time t, the sensor makes a perfect measurement of X_t, which is assumed to be noiseless. Then, the sensor makes a binary decision, denoted by U_t, on whether the measured quantity should be transmitted or not. Here, $U_t = 1$ (respectively, $U_t = 0$) means to transmit (respectively, not to transmit). The sensor is charged a cost, c, for each transmission and there is no charge for non-transmission. In addition, the sensor is able to make no more than $N (N < T)$ times of transmission over the time horizon, that is,

$$\sum_{t=1}^{T} U_t \leq N \tag{5.1}$$

If the sensor decides to transmit its measurement, it sends the observation, X_t, to the encoder. Otherwise, it sends a free symbol ϵ representing that there is no transmission. Let $\tilde{X}_t$ be the message received by the encoder; then

$$\tilde{X}_t = \begin{cases} X_t, & \text{if } U_t = 1 \\ \epsilon, & \text{if } U_t = 0 \end{cases}$$

Encoder: After receiving the message, $\tilde{X}_t$, from the sensor, the encoder communicates with the decoder via two communication channels. One channel, call it the *additive noise channel*, allows the encoder to transmit a real-valued message, Y_t, yet it corrupts the message by adding a channel noise V_t, and V_t is generated by the jammer. The other communication channel, call it the *side channel*, is noiseless, yet it allows the encoder to send only a 2-bit message, $S_t \in \{0, 1, -1\}$, to the decoder. The encoder is charged for the encoding power, λY_t^2 ($\lambda > 0$). Due to this charge, the messages (Y_t, S_t) sent by the encoder depend on the sensor's decision, U_t. To be more specific, if the sensor decides to transmit its observation (i.e., $U_t = 1$), the encoder sends $Y_t \in \mathbb{R}$ to the additive noise channel, and it sends the sign of the source, i.e., $S_t = \mathrm{sgn}(X_t)$, to the side channel. If the sensor decides not to transmit its observation (i.e., $U_t = 0$), the encoder sends $Y_t = 0$ to the additive noise channel to save its encoding power, and it sends $S_t = 0$ to the side channel to inform the decoder that there is no transmission from the sensor.

Jammer: As mentioned above, the communication channel noise, V_t, which is a real-valued and continuous random variable, is generated by the jammer. Specifically, at each time t, the jammer has access to the sensor's decision, U_t. Then, the jammer decides on a probability density function, denoted by p_{V_t}, and generates V_t correspondingly. Similar to the encoder, the jammer is charged for the jamming power, ηV_t^2 $(\eta > 0)$. Due to this charge, when there is no transmission from the sensor (i.e., $U_t = 0$), the jammer generates $V_t = 0$ with probability one to save its jamming power.

Decoder: After receiving the noise-corrupted message from the additive noise channel, denoted by $\tilde{Y}_t$, and the message from the side channel, S_t, the encoder generates an estimate on the state of the plant, denoted by $\hat{X}_t$. As described above, the dependency of $\tilde{Y}_t$ and S_t on the sensor's decision, U_t, is as follows:

$$\tilde{Y}_t = \begin{cases} Y_t + V_t, & \text{if } U_t = 1 \\ 0, & \text{if } U_t = 0 \end{cases} \qquad S_t = \begin{cases} \mathrm{sgn}(X_t), & \text{if } U_t = 1 \\ 0, & \text{if } U_t = 0 \end{cases}$$

The decoder is charged for the squared estimation error, $(X_t - \hat{X}_t)^2$. Furthermore, the jammer is rewarded $(X_t - \hat{X}_t)^2$ as it has the objective of maximizing the estimation error.

5.1.2 Information Structure and Decision Strategy

We call the sensor, the encoder, the decoder, and the jammer the *decision makers*. We introduce E_t as the remaining communication opportunities for the sensor at time t, that is,

$$E_t := N - \sum_{i=1}^{t-1} U_i$$

The evolution of E_t can be described by

$$\begin{aligned} E_1 &= N, \\ E_{t+1} &= E_t - U_t \end{aligned}$$

Furthermore, the constraint that the sensor can make no more than N transmissions, described in (5.1), can also be expressed as

$$U_t \leq E_t \quad \forall\, t = 1, \ldots, T$$

where $E_t = 0$ renders $U_t = 0$.

Since the sensor's decision at each time is common information shared among all the decision makers, E_t can also be computed by all the decision makers. We assume that at each time t, all the decision makers take only their current inputs and E_t into account when making decisions. Specifically, at time t, the sensor, the encoder, and the decoder make decisions according to

$$U_t = f_t(X_t, E_t), \quad Y_t = g_t(\tilde{X}_t, E_t), \quad \hat{X}_t = h_t(\tilde{Y}_t, S_t, E_t)$$

and the jammer picks the probability density function of V_t according to

$$p_{V_t} = d_t(U_t, E_t)$$

Here, f_t, g_t, h_t, and d_t are, respectively, the *communication scheduling policy*, the *encoding policy*, the *decoding policy*, and the *jamming policy*, at time t.

Remark 5.1. *The intuition supporting the assumption above is as follows: since the source process is i.i.d., the cost functional is stage-additive, and the communication constraint on the sensor at time t, i.e., $U_t \leq E_t$, depends only on E_t, it would be sufficient for a decision maker to consider only its current input(s) and E_t when making a decision. Note that for the setting with i.i.d. noise process instead of adversary generated noise process, studied in Chapter 2, it has already been proved that all the decision makers can consider only their current inputs ant E_t without any loss of optimality.*

We denote by

$$\begin{aligned} \boldsymbol{f} &:= \{f_1, \ldots, f_T\}, \quad & \boldsymbol{g} &:= \{g_1, \ldots, g_T\}, \\ \boldsymbol{h} &:= \{h_1, \ldots, h_T\}, \quad & \boldsymbol{d} &:= \{d_1, \ldots, d_T\} \end{aligned}$$

the *communication scheduling strategy*, the *encoding strategy*, the *decoding strategy*, and the *jamming strategy*, over the decision horizon. In addition, we use

$$\boldsymbol{f}_{a:b} := \{f_a, \ldots, f_b\}, \qquad \boldsymbol{g}_{a:b} := \{g_a, \ldots, g_b\},$$
$$\boldsymbol{h}_{a:b} := \{h_a, \ldots, h_b\}, \qquad \boldsymbol{d}_{a:b} := \{d_a, \ldots, d_b\}$$

to denote the subsets of $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$, and $\boldsymbol{d}$, respectively, where $a, b \in \{1, \ldots, T\}$ and $a \leq b$. In the special case when a equals b, we write

$$\boldsymbol{f}_a := \{f_a\}, \quad \boldsymbol{g}_a := \{g_a\}, \quad \boldsymbol{h}_a := \{h_a\}, \quad \boldsymbol{d}_a := \{d_a\}$$

Given two disjoint sets of policies, e.g., $\boldsymbol{f}_{a:b}$ and $\boldsymbol{f}^*_{c:d}$, $b < c$, we denote by

$$\boldsymbol{f}_{a:b} + \boldsymbol{f}^*_{c:d} := \{\boldsymbol{f}_a, \ldots, \boldsymbol{f}_b, \boldsymbol{f}^*_c, \ldots, \boldsymbol{f}^*_d\}$$

the union of the sets.

5.1.3 Cost Functional

Recall that at each time t, the sensor is charged cU_t for the transmission, the encoder is charged λY_t^2 for the encoding power, and the decoder is charged $(X_t - \hat{X}_t)^2$ for the estimation error. The sensor, the encoder, and the decoder have the common objective of minimizing the expected value of the sum of their costs accumulated over the time horizon, namely, minimizing

$$J_S(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{d}) := \sum_{t=1}^{T} \mathbb{E}\left\{cU_t + \lambda Y_t^2 + (X_t - \hat{X}_t)^2\right\}$$

$J_S(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{d})$ is the cost functional of the remote sensing system. The expectation in $J_S(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{d})$ is taken over the probability density functions $p_{X_1}, \ldots, p_{X_T}$ and $p_{V_1}, \ldots, p_{V_T}$.

Similarly, at each time t, the jammer is charged ηV_t^2 for the jamming power and rewarded $(X_t - \hat{X}_t)^2$ for the estimation error. Hence, the jammer has the objective of minimizing the cost functional, denoted by $J_A(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{d})$, given as follows:

$$J_A(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{d}) := \sum_{t=1}^{T} \mathbb{E}\left\{\eta V_t^2 - (X_t - \hat{X}_t)^2\right\}$$

5.1.4 Problem Definition

Given the time horizon, T, the probability density functions of the source process, $p_{X_1}, \ldots, p_{X_T}$, the communication cost for one transmission, c, the number of transmission opportunities, N, the unit price of the encoding power, λ, and the unit price of the jamming power, η, our goal is to find a *feedback Stackelberg solution*[1] [49, Definition 3.29] for the dynamic game with the team of the sensor, the encoder and the decoder as the composite leader, and the jammer as the follower. Namely, we want to find $(\boldsymbol{f}^*, \boldsymbol{g}^*, \boldsymbol{h}^*)$ and $\boldsymbol{d}^*$ such that the following equations hold for all $t = 1, \ldots, T$. First,

$$\begin{aligned} & J_S(\boldsymbol{f}_{1:t-1} + \boldsymbol{f}^*_{t:T}, \boldsymbol{g}_{1:t-1} + \boldsymbol{g}^*_{t:T}, \boldsymbol{h}_{1:t-1} + \boldsymbol{h}^*_{t:T}, \boldsymbol{d}_{1:t-1} + \boldsymbol{d}^*_{t:T}) \\ = & \min_{\boldsymbol{f}_t, \boldsymbol{g}_t, \boldsymbol{h}_t} \max_{\boldsymbol{d}_t \in \mathcal{R}_t(\boldsymbol{f}_{1:t} + \boldsymbol{f}^*_{t+1:T}, \boldsymbol{g}_{1:t} + \boldsymbol{g}^*_{t+1:T}, \boldsymbol{h}_{1:t} + \boldsymbol{h}^*_{t+1:T}, \boldsymbol{d}_{1:t-1} + \boldsymbol{d}^*_{t+1:T})} \\ & J_S(\boldsymbol{f}_{1:t-1} + \boldsymbol{f}_t + \boldsymbol{f}^*_{t+1:T}, \boldsymbol{g}_{1:t-1} + \boldsymbol{g}_t + \boldsymbol{g}^*_{t+1:T}, \boldsymbol{h}_{1:t-1} + \boldsymbol{h}_t + \boldsymbol{h}^*_{t+1:T}, \\ & \quad \boldsymbol{d}_{1:t-1} + \boldsymbol{d}_t + \boldsymbol{d}^*_{t+1:T}) \end{aligned}$$

where

$$\begin{aligned} & \mathcal{R}_t(\boldsymbol{f}_{1:t} + \boldsymbol{f}^*_{t+1:T}, \boldsymbol{g}_{1:t} + \boldsymbol{g}^*_{t+1:T}, \boldsymbol{h}_{1:t} + \boldsymbol{h}^*_{t+1:T}, \boldsymbol{d}_{1:t-1} + \boldsymbol{d}^*_{t+1:T}) \\ & := \Big\{ \tilde{\boldsymbol{d}}_t \Big| J_A(\boldsymbol{f}_{1:t} + \boldsymbol{f}^*_{t+1:T}, \boldsymbol{g}_{1:t} + \boldsymbol{g}^*_{t+1:T}, \boldsymbol{h}_{1:t} + \boldsymbol{h}^*_{t+1:T}, \boldsymbol{d}_{1:t-1} + \tilde{\boldsymbol{d}}_t + \boldsymbol{d}^*_{t+1:T}) \\ & \quad = \min_{\boldsymbol{d}_t} J_A(\boldsymbol{f}_{1:t} + \boldsymbol{f}^*_{t+1:T}, \boldsymbol{g}_{1:t} + \boldsymbol{g}^*_{t+1:T}, \boldsymbol{h}_{1:t} + \boldsymbol{h}^*_{t+1:T}, \boldsymbol{d}_{1:t-1} + \boldsymbol{d}_t + \boldsymbol{d}^*_{t+1:T}) \Big\} \end{aligned}$$

Second, $\mathcal{R}_t(\boldsymbol{f}_{1:t-1} + \boldsymbol{f}^*_{t:T}, \boldsymbol{g}_{1:t-1} + \boldsymbol{g}^*_{t:T}, \boldsymbol{h}_{1:t-1} + \boldsymbol{h}^*_{t:T}, \boldsymbol{d}_{1:t-1} + \boldsymbol{d}^*_{t+1:T})$ is a singleton, and

$$\mathcal{R}_t(\boldsymbol{f}_{1:t-1} + \boldsymbol{f}^*_{t:T}, \boldsymbol{g}_{1:t-1} + \boldsymbol{g}^*_{t:T}, \boldsymbol{h}_{1:t-1} + \boldsymbol{h}^*_{t:T}, \boldsymbol{d}_{1:t-1} + \boldsymbol{d}^*_{t+1:T}) = \{\boldsymbol{d}^*_t\}$$

[1] Feedback Stackelberg solution, also called stage-wise Stackelberg solution, means that the leader is not able to enforce the decision policies at all stages of the game on the follower before the start of the game. Instead, the leader is able to enforce the decision policy on the follower at each stage of the game. In the definition and the formulation given below, we assume at the outset that the stage-wise reaction function of the follower is a singleton, which will subsequently be shown to be the case for the problem at hand.

5.2 Main Results

The feedback Stackelberg solution has an important property that it can be computed iteratively, as described by the following lemma.

Lemma 5.1 ([49, Theorem 7.2])**.** *For the dynamic game formulated in Section 5.1.4, the set of communication scheduling, encoding, and decoding strategies* $(\boldsymbol{f}^*, \boldsymbol{g}^*, \boldsymbol{h}^*)$ *together with the jamming strategy* $\boldsymbol{d}^*$ *provides a feedback Stackelberg solution, if there exist real-valued functions* $V_S(\cdot,\cdot)$ *and* $V_A(\cdot,\cdot)$ *defined over* $\{1,\dots,T+1\}\times\{0,\dots,N\}$*, such that for all* $t=1,\dots,T$ *and* $E_t = 0,\dots,N$*,*

$$V_S(T+1,\cdot) = V_A(T+1,\cdot) = 0$$

$$V_S(t,E_t) = \min_{f_t,g_t,h_t,\, d_t\in R_t(f_t,g_t,h_t)} \mathcal{C}_S(t,E_t,f_t,g_t,h_t,d_t) = \mathcal{C}_S(t,E_t,f_t^*,g_t^*,h_t^*,d_t^*) \tag{5.2}$$

where $R_t(f_t,g_t,h_t)$ *is a singleton set defined by*

$$\begin{aligned} R_t(f_t,g_t,h_t) := \Big\{\tilde{d}_t \mid \mathcal{C}_A(t,E_t,f_t,g_t,h_t,\tilde{d}_t) \\ = \min_{d_t}\ \mathcal{C}_A(t,E_t,f_t,g_t,h_t,d_t)\ \forall\ E_t = 0,\dots,N\Big\}\end{aligned}$$

In addition, $\mathcal{C}_S(t,E_t,f_t,g_t,h_t,d_t)$ *and* $\mathcal{C}_A(t,E_t,f_t,g_t,h_t,d_t)$ *are defined by*

$$\begin{aligned} \mathcal{C}_S(t,E_t,f_t,g_t,h_t,d_t) &:= \mathbb{E}\big[cU_t + \lambda Y_t^2 + (X_t-\hat{X}_t)^2 + V_S(t+1,E_{t+1})\big],\\ \mathcal{C}_A(t,E_t,f_t,g_t,h_t,d_t) &:= \mathbb{E}\big[\eta V_t^2 - (X_t-\hat{X}_t)^2 + V_A(t+1,E_{t+1})\big] \end{aligned} \tag{5.3}$$

with the expectation taken over X_t *and* V_t*, and*

$$\begin{aligned} &U_t = f_t(X_t,E_t),\quad Y_t = g_t(\tilde{X}_t,E_t),\quad \hat{X}_t = h_t(\tilde{Y}_t,S_t,E_t),\quad p_{V_t} = d_t(U_t,E_t),\\ &E_{t+1} = E_t - U_t \end{aligned}$$

We now show that $(f_t^*, g_t^*, h_t^*, d_t^*)$ satisfying (5.2) for stage t can be obtained by solving a single-stage Stackelberg game.

Theorem 5.1. *The communication scheduling policy, encoding policy, decoding policy, and jamming policy* $(f_t^*, g_t^*, h_t^*, d_t^*)$ *satisfy* (5.2) *in Lemma 5.1 if and only if for all* $t=1,\dots,T$ *and* $E_t = 0,\dots,N$*,*

$$\min_{f_t,g_t,h_t,\, d_t\in\bar{R}_t(f_t,g_t,h_t)} \bar{\mathcal{C}}_S(t,E_t,f_t,g_t,h_t,d_t) = \bar{\mathcal{C}}_S(t,E_t,f_t^*,g_t^*,h_t^*,d_t^*) \tag{5.4}$$

where $\bar{R}_t(f_t, g_t, h_t)$ *is a singleton set defined by*

$$\bar{R}_t(f_t, g_t, h_t) := \Big\{ \tilde{d}_t \mid \bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, \tilde{d}_t) \\ = \min_{d_t} \ \bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t) \ \forall \ E_t = 0, \ldots, N \Big\}$$

In addition, $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ *and* $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$ *are defined by*

$$\begin{aligned} \bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t) &:= \mathbb{E}\left[\big(c + c_t(E_t)\big) U_t + \lambda Y_t^2 + (X_t - \hat{X}_t)^2 \right], \\ \bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t) &:= \mathbb{E}\left[\eta V_t^2 - (X_t - \hat{X}_t)^2 \right] \end{aligned} \tag{5.5}$$

with

$$c_t(E_t) = \begin{cases} V_S(t+1, E_t - 1) - V_S(t+1, E_t), & \text{if } E_t > 0 \\ 0, & \text{if } E_t = 0 \end{cases} \tag{5.6}$$

Proof. We first show that for any (f_t, g_t, h_t),

$$R_t(f_t, g_t, h_t) = \bar{R}_t(f_t, g_t, h_t) \tag{5.7}$$

By the definitions of $\mathcal{C}_A(t, E_t, f_t, g_t, h_t, d_t)$ and $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$ described in (5.3) and (5.5), we have

$$\begin{aligned} & \mathcal{C}_A(t, E_t, f_t, g_t, h_t, d_t) \\ = \ & \bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t) + \mathbb{E}\big[V_A(t+1, E_{t+1})\big] \\ = \ & \bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t) + \mathbb{E}\big[V_A(t+1, E_t - U_t)\big] \\ = \ & \bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t) + \mathbb{E}\Big[V_A\big(t+1, E_t - f_t(X_t, E_t)\big)\Big] \end{aligned}$$

The second term, $\mathbb{E}\big[V_A\big(t+1, E_t - f_t(X_t, E_t)\big)\big]$, is independent of d_t. Therefore, with (f_t, g_t, h_t) given, the jamming policy minimizing $\mathcal{C}_A(t, E_t, f_t, g_t, h_t, d_t)$ also minimizes $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$, and vice versa. Then, we have that (5.7) holds.

We next show that for any (f_t, g_t, h_t),

$$\mathcal{C}_S(t, E_t, f_t, g_t, h_t, d_t) = \bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t) + V_S(t+1, E_t) \quad \forall \ E_t = 0, \ldots, N \tag{5.8}$$

Case I: When $E_t > 0$, we have

$$\begin{aligned}
& \mathcal{C}_S(t, E_t, f_t, g_t, h_t, d_t) \\
= \ & \mathbb{E}\big[cU_t + \lambda Y_t^2 + (X_t - \hat{X}_t)^2 + V_S(t+1, E_{t+1})\big] \\
= \ & \mathbb{E}\big[cU_t + \lambda Y_t^2 + (X_t - \hat{X}_t)^2 + V_S(t+1, E_t - U_t)\big] \\
= \ & \mathbb{E}\Big[cU_t + \lambda Y_t^2 + (X_t - \hat{X}_t)^2 + V_S(t+1, E_t) \\
& \quad + \big(V_S(t+1, E_t - 1) - V_S(t+1, E_t)\big)U_t\Big] \\
= \ & \mathbb{E}\big[cU_t + \lambda Y_t^2 + (X_t - \hat{X}_t)^2 + c_t(E_t)U_t\big] + V_S(t+1, E_t) \\
= \ & \bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t) + V_S(t+1, E_t)
\end{aligned}$$

The first and the last equalities hold due to the definitions of $\mathcal{C}_S(t, E_t, f_t, g_t, h_t, d_t)$ and $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ described in (5.3) and (5.5). The second equality holds due to the evolution of E_t. The third equality holds since

$$\begin{aligned}
& V_S(t+1, E_t - U_t) \\
= \ & \begin{cases} V_S(t+1, E_t) & \text{if } U_t = 0 \\ V_S(t+1, E_t - 1) & \text{if } U_t = 1 \end{cases} \\
= \ & V_S(t+1, E_t) + \Big(V_S(t+1, E_t - 1) - V_S(t+1, E_t)\Big)U_t
\end{aligned}$$

The fourth equality holds since $V_S(t+1, E_t)$ is a constant.

Case II: When $E_t = 0$, we have

$$\begin{aligned}
& \mathcal{C}_S(t, E_t, f_t, g_t, h_t, d_t) \\
= \ & \mathbb{E}\big[cU_t + \lambda Y_t^2 + (X_t - \hat{X}_t)^2 + V_S(t+1, 0)\big] \\
= \ & \mathbb{E}\big[cU_t + \lambda Y_t^2 + (X_t - \hat{X}_t)^2 + c_t(E_t)U_t\big] + V_S(t+1, 0) \\
= \ & \bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t) + V_S(t+1, 0)
\end{aligned}$$

The above equalities hold since $E_t = 0$ renders $U_t = 0$, and then $E_{t+1} = 0$. Combining cases I and II, we have that (5.8) holds.

Finally, we are in a position to prove the theorem. Given that the 4-tuple

$(f_t^*, g_t^*, h_t^*, d_t^*)$ satisfies (5.2) in Lemma 5.1, we have

$$\min_{f_t, g_t, h_t,\, d_t \in \bar{R}_t(f_t, g_t, h_t)} \mathcal{C}_S(t, E_t, f_t, g_t, h_t, d_t) = \mathcal{C}_S(t, E_t, f_t^*, g_t^*, h_t^*, d_t^*)$$

since $R_t(f_t, g_t, h_t) = \bar{R}_t(f_t, g_t, h_t)$. Then, we have

$$\begin{aligned} & -V_S(t+1, E_t) + \min_{f_t, g_t, h_t,\, d_t \in \bar{R}_t(f_t, g_t, h_t)} \mathcal{C}_S(t, E_t, f_t, g_t, h_t, d_t) \\ = & -V_S(t+1, E_t) + \mathcal{C}_S(t, E_t, f_t^*, g_t^*, h_t^*, d_t^*) \\ = & \bar{\mathcal{C}}_S(t, E_t, f_t^*, g_t^*, h_t^*, d_t^*) \end{aligned} \tag{5.9}$$

where the second equality holds because of (5.8). Furthermore, we have

$$\begin{aligned} & -V_S(t+1, E_t) + \min_{f_t, g_t, h_t,\, d_t \in \bar{R}_t(f_t, g_t, h_t)} \mathcal{C}_S(t, E_t, f_t, g_t, h_t, d_t) \\ = & \min_{f_t, g_t, h_t,\, d_t \in \bar{R}_t(f_t, g_t, h_t)} \Big\{ \mathcal{C}_S(t, E_t, f_t, g_t, h_t, d_t) - V_S(t+1, E_t) \Big\} \\ = & \min_{f_t, g_t, h_t,\, d_t \in \bar{R}_t(f_t, g_t, h_t)} \bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t) \end{aligned} \tag{5.10}$$

where the first equality holds since $V_S(t+1, E_t)$ is a constant that is independent of (f_t, g_t, h_t, d_t), and the second equality holds because of (5.8). Combining (5.9) and (5.10), we have $(f_t^*, g_t^*, h_t^*, d_t^*)$ satisfy (5.4).

The above argument still holds if we reverse the direction. That is, given that the 4-tuple $(f_t^*, g_t^*, h_t^*, d_t^*)$ satisfies (5.4), we have $(f_t^*, g_t^*, h_t^*, d_t^*)$ satisfying (5.2) in Lemma 5.1, which completes the proof of the theorem. □

By Theorem 5.1, we are able to obtain $(f_t^*, g_t^*, h_t^*, d_t^*)$ satisfying (5.2) in Lemma 5.1, by solving a single-stage Stackelberg game with the sensor, the encoder, and the decoder as the composite leader, and the jammer as the follower. The cost functionals for the leader and the follower are, respectively, $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ and $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$ described in (5.5). However, this single-stage Stackelberg game is generally difficult to solve, as minimizing $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ or $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$ is an infinite-dimensional optimization problem. Hence, we need to make some (mild) assumptions. To introduce these assumptions, we need some additional notation. Denote by $P_e(t, E_t)$ and $P_j(t, E_t)$ the encoding power and jamming power, respectively, given that there is a transmission from the sensor. To be more specific,

$$P_e(t, E_t) := \mathbb{E}[Y_t^2 | U_t = 1, E_t],$$
$$P_j(t, E_t) := \mathbb{E}[V_t^2 | U_t = 1, E_t]$$

where the event $(U_t = 1, E_t)$ is equivalent to $f_t(X_t, E_t) = 1$.

Assumption 5.1. *The source density, $p_X(\cdot)$, is even and unimodal,*[2] *i.e.,*

$$p_X(x) = p_X(-x) \quad \forall\, x \in \mathbb{R},$$
$$p_X(a) \geq p_X(b) \quad \forall\, |a| \leq |b|$$

Furthermore, $p_X(\cdot)$ is log-concave and continuously differentiable on $(-\infty, 0) \cup (0, \infty)$.[3]

Assumption 5.2. *At each time t, the sensor applies a symmetric communication scheduling policy, f_t, satisfying*

$$f_t(x, e) = f_t(-x, e) \quad \forall\, x \in \mathbb{R},\ e \in \{0, \ldots, N\}$$

Assumption 5.3. *At each time t, if there is a transmission from the sensor (i.e., $U_t = 1$), the encoder and the decoder apply piecewise affine encoding and decoding policies, respectively, in the form of*

$$\begin{aligned} g_t(X_t, E_t) &= S_t \cdot \alpha(t, S_t, E_t) \cdot \big(X_t - \mathbb{E}[X_t | U_t = 1, S_t, E_t]\big) \\ h_t(\tilde{Y}_t, S_t, E_t) &= \frac{S_t}{\alpha(t, S_t, E_t)} \frac{P_e(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)} \tilde{Y}_t + \mathbb{E}[X_t | U_t = 1, S_t, E_t] \end{aligned}$$

where

$$\alpha(t, S_t, E_t) = \sqrt{\frac{P_e(t, E_t)}{\mathrm{Var}(X_t | U_t = 1, S_t, E_t)}} \tag{5.11}$$

Assumption 5.4. *At each time t, if there is a transmission from the sensor (i.e., $U_t = 1$), the jammer generates a zero-mean jamming noise, V_t.*

We have several remarks on the above assumptions, which are listed below.

[2] We write p_X instead of p_{X_t} as the source process is an i.i.d. stochastic process.

[3] If p_X has support $[-a, a]$ instead of $\mathbb{R}$, then it is required that p_X be continuously differentiable on $(-a, 0) \cup (0, a)$.

1. There are many source densities satisfying Assumption 5.1, e.g., density functions of zero-mean Gaussian distribution, zero-mean Laplace distribution, zero-mean uniform distribution, and so on.

2. Since the source density is symmetric around zero, it is intuitive that the sensor should apply a symmetric communication scheduling policy, i.e., the one described in Assumption 5.2.

3. The piecewise affine assumption on the encoding and decoding policies, i.e., Assumption 5.3, is not conservative in the setting with an adversary. It is well known (see [50, 51]) that for a zero-delay source-channel coding problem with a jammer, if the source density is Gaussian, then under the saddle-point solution the encoder and the decoder should apply affine encoding and decoding policies, and the jammer should generate a Gaussian jamming noise. This result has been generalized in [46] to non-Gaussian source densities such that the encoder and the decoder should still apply affine policies at the saddle point. Here, we make the "piecewise affine" assumption instead of the "affine" assumption due to the existence of the side channel, which enables the encoder and the decoder to apply different encoding and decoding policies for the positive and the negative realizations of the source.

4. For Assumption 5.4, since the jammer is charged for jamming power, it should generate a zero-mean jamming noise to save its power (with the same variance, the zero-mean noise has the lowest power).

With Assumptions 5.1-5.4 holding, we are able to obtain neat expressions for $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ and $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$, as described in the following lemma.

Lemma 5.2. *Under Assumptions 5.1-5.4, the cost functionals* $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ *and* $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$ *defined in* (5.5) *admit the fol-*

lowing expressions:

$$
\begin{aligned}
& \bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t) \\
= \;& 2\,\mathbb{P}(U_t = 1, S_t = 1, E_t)\Big[c + c_t(E_t) + \lambda P_e(t, E_t) \\
& + \frac{P_j(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)} \mathrm{Var}(X_t | U_t = 1, S_t = 1, E_t)\Big] \\
& + \mathbb{P}(U_t = 0, S_t = 0, E_t) \mathrm{Var}(X_t | U_t = 0, S_t = 0, E_t) \\
& \bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t) \\
= \;& 2\,\mathbb{P}(U_t = 1, S_t = 1, E_t)\Big[\eta P_j(t, E_t) \\
& - \frac{P_j(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)} \mathrm{Var}(X_t | U_t = 1, S_t = 1, E_t)\Big] \\
& - \mathbb{P}(U_t = 0, S_t = 0, E_t) \mathrm{Var}(X_t | U_t = 0, S_t = 0, E_t)
\end{aligned}
\tag{5.12}
$$

Proof. We compute $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ and $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$ term by term under Assumptions 5.1-5.4. First, we have

$$
\begin{aligned}
& \mathbb{E}\Big[\big(c + c_t(E_t)\big) U_t\Big] \\
= \;& 0 \cdot \mathbb{P}(U_t = 0, S_t = 0, E_t) + \big(c + c_t(E_t)\big)\, \mathbb{P}(U_t = 1, S_t = 1, E_t) \\
& + \big(c + c_t(E_t)\big)\, \mathbb{P}(U_t = 1, S_t = -1, E_t) \\
= \;& 2\, \big(c + c_t(E_t)\big)\, \mathbb{P}(U_t = 1, S_t = 1, E_t)
\end{aligned}
\tag{5.13}
$$

where the events $(U_t = 0, S_t = 0, E_t)$, $(U_t = 1, S_t = 1, E_t)$, and $(U_t = 1, S_t = -1, E_t)$ are equivalent to $f_t(X_t, E_t) = 0$, $(f_t(X_t, E_t) = 1, X_t > 0)$, and $(f_t(X_t, E_t) = 1, X_t < 0)$, respectively. Then, the first equality holds due to the law of total expectation,[4] and the second equality holds since the source density, $p_X(\cdot)$, and the communication scheduling policy, $f_t(\cdot, E_t)$, are symmetric around zero.

[4] There is actually another event, $(U_t = 1, S_t = 0, E_t)$, or equivalently, $(f_t(X_t, E_t) = 1, X_t = 0)$. However, this event occurs with zero probability as X_t is a continuous random variable. Hence, we do not consider this event in our analysis.

Then, we have

$$
\begin{aligned}
& \mathbb{E}[Y_t^2] \\
= \;& \mathbb{E}[Y_t^2|U_t=0,E_t]\;\mathbb{P}(U_t=0,E_t)+\mathbb{E}[Y_t^2|U_t=1,E_t]\;\mathbb{P}(U_t=1,E_t) \\
= \;& \mathbb{E}[Y_t^2|U_t=1,E_t]\;\big(\mathbb{P}(U_t=1,S_t=1,E_t)+\mathbb{P}(U_t=1,S_t=-1,E_t)\big) \\
= \;& 2\;P_e(t,E_t)\;\mathbb{P}(U_t=1,S_t=1,E_t)
\end{aligned}
\tag{5.14}
$$

where the first equality holds due to the law of total expectation. The second equality holds since the encoder sends $Y_t = 0$ when $U_t = 0$ to save its encoding power.

Similarly, we have

$$
\begin{aligned}
& \mathbb{E}[V_t^2] \\
= \;& \mathbb{E}[V_t^2|U_t=0,E_t]\;\mathbb{P}(U_t=0,E_t)+\mathbb{E}[V_t^2|U_t=1,E_t]\;\mathbb{P}(U_t=1,E_t) \\
= \;& \mathbb{E}[V_t^2|U_t=1,E_t]\;\big(\mathbb{P}(U_t=1,S_t=1,E_t)+\mathbb{P}(U_t=1,S_t=-1,E_t)\big) \\
= \;& 2\;P_j(t,E_t)\;\mathbb{P}(U_t=1,S_t=1,E_t)
\end{aligned}
\tag{5.15}
$$

Again, the second equality holds since the jammer generates $V_t = 0$ with probability one when $U_t = 0$ to save its jamming power.

Finally, we compute $\mathbb{E}[(X_t - \hat{X}_t)^2]$. By the law of total expectation, we have

$$
\begin{aligned}
& \mathbb{E}\big[(X_t-\hat{X}_t)^2\big] \\
= \;& \mathbb{E}\big[(X_t-\hat{X}_t)^2|U_t=0,S_t=0,E_t\big]\;\mathbb{P}(U_t=0,S_t=0,E_t) \\
& +\mathbb{E}\big[(X_t-\hat{X}_t)^2|U_t=1,S_t=1,E_t\big]\;\mathbb{P}(U_t=1,S_t=1,E_t) \\
& +\mathbb{E}\big[(X_t-\hat{X}_t)^2|U_t=1,S_t=-1,E_t\big]\;\mathbb{P}(U_t=1,S_t=-1,E_t)
\end{aligned}
$$

We compute the three conditional expectations as follows:

1. When $U_t = 0$, there is no transmission from the sensor. Hence, the minimum mean squared error estimator, $\hat{X}_t$, is the conditional mean, i.e.,

$$\hat{X}_t = \mathbb{E}\big[X_t|U_t=0,S_t=0,E_t\big]$$

Then,

$$\mathbb{E}\big[(X_t - \hat{X}_t)^2 | U_t = 0, S_t = 0, E_t\big] = \mathrm{Var}\big(X_t | U_t = 0, S_t = 0, E_t\big)$$

2. When $U_t = 1$, by Assumption 5.3, we have

$$\begin{aligned}
\hat{X}_t &= \frac{S_t}{\alpha(t, S_t, E_t)} \frac{P_e(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)} \tilde{Y}_t + \mathbb{E}\big[X_t | U_t = 1, S_t, E_t\big] \\
&= \frac{S_t}{\alpha(t, S_t, E_t)} \frac{P_e(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)} \\
&\quad \cdot \Big(S_t \cdot \alpha(t, S_t, E_t) \ \cdot \big(X_t - \mathbb{E}[X_t | U_t = 1, S_t, E_t]\big) + V_t\Big) \\
&\quad + \mathbb{E}\big[X_t | U_t = 1, S_t, E_t\big] \\
&= \frac{P_e(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)} X_t \\
&\quad + \frac{P_j(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)} \mathbb{E}[X_t | U_t = 1, S_t, E_t] \\
&\quad + \frac{S_t}{\alpha(t, S_t, E_t)} \frac{P_e(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)} V_t
\end{aligned}$$

Then above expression holds for both $S_t = 1$ and $S_t = -1$. Correspondingly,

$$\begin{aligned}
&\mathbb{E}\big[(X_t - \hat{X}_t)^2 | U_t = 1, S_t, E_t\big] \\
= \ &\mathbb{E}\Bigg[\bigg(\frac{P_j(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)} \big(X_t - E[X_t | U_t = 1, S_t, E_t]\big) \\
&\qquad - \frac{S_t}{\alpha(t, S_t, E_t)} \frac{P_e(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)} V_t\bigg)^2 \bigg| U_t = 1, S_t, E_t\Bigg] \\
= \ &\frac{P_j^2(t, E_t)}{\big(P_e(t, E_t) + P_j(t, E_t)\big)^2} \mathrm{Var}(X_t | U_t = 1, S_t, E_t) \\
&+ \frac{1}{\alpha^2(t, S_t, E_t)} \frac{P_e^2(t, E_t)}{\big(P_e(t, E_t) + P_j(t, E_t)\big)^2} \mathbb{E}[V_t^2 | U_t = 1, S_t, E_t] \\
&- \frac{2S_t}{\alpha(t, S_t, E_t)} \frac{P_e(t, E_t) P_j(t, E_t)}{\big(P_e(t, E_t) + P_j(t, E_t)\big)^2}
\end{aligned}$$

$$\cdot\, \mathbb{E}\Big[\big(X_t - \mathbb{E}[X_t|U_t = 1, S_t, E_t]\big)V_t|U_t = 1, S_t, E_t\Big] \tag{5.16}$$

Note that the only information the jammer has about X_t and S_t is the hidden information contained in U_t. Therefore, conditioned on $U_t = 1$, the jamming noise, V_t, is independent of S_t. Then, we have

$$\mathbb{E}[V_t^2|U_t = 1, S_t, E_t] = \mathbb{E}[V_t^2|U_t = 1, E_t] = P_j(t, E_t) \tag{5.17}$$

In addition, conditioned on $(U_t = 1, S_t)$, V_t is independent of X_t. Hence, we have

$$\begin{aligned} & \mathbb{E}\Big[\big(X_t - \mathbb{E}[X_t|U_t = 1, S_t, E_t]\big)V_t|U_t = 1, S_t, E_t\Big] \\ = \; & \mathbb{E}\Big[\big(X_t - \mathbb{E}[X_t|U_t = 1, S_t, E_t]\big)|U_t = 1, S_t, E_t\Big]\mathbb{E}\Big[V_t|U_t = 1, S_t, E_t\Big] \\ = \; & 0 \end{aligned} \tag{5.18}$$

Plugging in (5.17), (5.18), and the expression for $\alpha(t, S_t, E_t)$, described by (5.11), into (5.16), we have

$$\begin{aligned} & \mathbb{E}\big[(X_t - \hat{X}_t)^2|U_t = 1, S_t, E_t\big] \\ = \; & \frac{P_j(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)}\mathrm{Var}(X_t|U_t = 1, S_t, E_t) \end{aligned}$$

which holds for both $S_t = 1$ and $S_t = -1$.

Combining the results for the above two cases, i.e., $U_t = 0$ and $U_t = 1$, we have

$$\begin{aligned} & \mathbb{E}\big[(X_t - \hat{X}_t)^2\big] \\ = \; & \mathrm{Var}\big(X_t|U_t = 0, S_t = 0, E_t\big)\, \mathbb{P}(U_t = 0, S_t = 0, E_t) \\ & + \frac{P_j(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)}\mathrm{Var}(X_t|U_t = 1, S_t = 1, E_t)\, \mathbb{P}(U_t = 1, S_t = 1, E_t) \\ & + \frac{P_j(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)}\mathrm{Var}(X_t|U_t = 1, S_t = -1, E_t)\, \mathbb{P}(U_t = 1, S_t = -1, E_t) \\ = \; & \mathrm{Var}\big(X_t|U_t = 0, S_t = 0, E_t\big)\, \mathbb{P}(U_t = 0, S_t = 0, E_t) \\ & + 2\frac{P_j(t, E_t)}{P_e(t, E_t) + P_j(t, E_t)}\mathrm{Var}(X_t|U_t = 1, S_t = 1, E_t)\, \mathbb{P}(U_t = 1, S_t = 1, E_t) \end{aligned} \tag{5.19}$$

where the last equality holds since the source density, $p_X(\cdot)$, and the communication scheduling policy, $f_t(\cdot, E_t)$, are symmetric around zero.

Combining (5.13), (5.14), (5.15) and (5.19), we reach the expressions for $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ and $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$, as described by (5.12). $\square$

We would like to clarify how the choices of the communication scheduling policy, f_t, the encoding policy, g_t, the decoding policy, h_t, and the jamming policy, d_t, affect $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ and $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$. In the expressions for $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ and $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$, described by (5.12), the terms $\mathbb{P}(U_t = 1, S_t = 1, E_t)$, $\mathbb{P}(U_t = 0, S_t = 0, E_t)$, $\mathrm{Var}(X_t | U_t = 1, S_t = 1, E_t)$, and $\mathrm{Var}(X_t | U_t = 0, S_t = 0, E_t)$ depend on the communication scheduling policy, f_t. The encoding power, $P_e(t, E_t)$, depends on the encoding policy, g_t. The jamming power, $P_j(t, E_t)$, depends on the jamming policy d_t. Note that the decoding policy, h_t, is induced by f_t, g_t, and d_t by Assumption 5.3.

We also note that $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ and $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$ depend on the density of the jamming noise, p_{V_t}, only through the jamming power, $P_j(t, E_t)$, but not the type of density (e.g., Gaussian, Laplace, uniform, etc.). This is due to the fact that the encoder and the decoder apply piecewise affine encoding and decoding policies (Assumption 5.3). Therefore, without loss of generality, we assume that the jammer generates the jamming noise, V_t, with zero-mean Gaussian distribution, denoted by $\mathcal{N}\big(0, P_j(t, E_t)\big)$. More specifically, we assume that

$$d_t(U_t, E_t) = \mathcal{N}\big(0, P_j(t, E_t)\big) \tag{5.20}$$

Under Assumptions 5.1-5.4, we are able to solve the single-stage Stackelberg game and obtain $(f_t^*, g_t^*, h_t^*, d_t^*)$ satisfying (5.4) in Theorem 5.1, as described in the following theorem.

Theorem 5.2. *Consider the single-stage Stackelberg game with the sensor, the encoder, and the decoder as the composite leader, and the jammer as the follower. The cost functionals for the leader and the follower are, respectively,* $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ *and* $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$ *defined in* (5.5). *With Assumptions 5.1-5.4 holding, the 4-tuple of policies* $(f_t^*, g_t^*, h_t^*, d_t^*)$ *listed below provides a Stackelberg equilibrium solution and satisfies* (5.4) *in Theorem 5.1.*

1. *The communication scheduling policy f_t^* is threshold-based, i.e.,*

$$f_t^*(X_t, E_t) = \begin{cases} 1, & \text{if } |X_t| > \beta_t^*(E_t) \text{ and } E_t > 0 \\ 0, & \text{if } |X_t| \le \beta_t^*(E_t) \text{ or } E_t = 0 \end{cases}$$

   where $\beta_t^*(E_t)$ is called "threshold". Furthermore, $\beta_t^*(E_t)$ is the unique solution to the fixed-point equation:

$$\beta^2 = \xi \cdot G_X^2(\beta) + c + c_t(E_t), \quad \beta \ge 0 \tag{5.21}$$

   where $G_X(\beta)$, usually called "mean residual lifetime", is defined by

$$G_X(\beta) := \mathbb{E}[X_t | X_t > \beta] - \beta$$

   $c_t(E_t)$ is defined in (5.6)*, and*

$$\xi = \begin{cases} 1 - \dfrac{\eta}{4\lambda}, & \text{if } \dfrac{\lambda}{\eta} \ge \dfrac{1}{2} \\ \dfrac{\lambda}{\eta}, & \text{if } \dfrac{\lambda}{\eta} < \dfrac{1}{2} \end{cases}$$

2. *The encoding and the decoding policies g_t^* and h_t^* are the piecewise affine ones described in Assumption 5.3, with the encoding power, $P_e^*(t, E_t)$, being as follows:*

$$P_e^*(t, E_t) = \begin{cases} \dfrac{\eta}{4\lambda^2} \operatorname{Var}(X_t | U_t = 1, S_t, E_t), & \text{if } \dfrac{\lambda}{\eta} \ge \dfrac{1}{2} \\ \dfrac{1}{\eta} \operatorname{Var}(X_t | U_t = 1, S_t, E_t), & \text{if } \dfrac{\lambda}{\eta} < \dfrac{1}{2} \end{cases}$$

3. *The jamming policy, d_t^*, is the one described by* (5.20) *with the jamming power, $P_j^*(t, E_t)$, being as follows:*[5]

$$P_j^*(t, E_t) = \begin{cases} \left(\dfrac{2\lambda}{\eta} - 1\right) P_e^*(t, E_t), & \text{if } \dfrac{\lambda}{\eta} \ge \dfrac{1}{2} \\ 0, & \text{if } \dfrac{\lambda}{\eta} < \dfrac{1}{2} \end{cases}$$

[5]When $P_j^*(t, E_t) = 0$, the jamming noise density, $\mathcal{N}(0, 0)$, is a Dirac delta function, which means that the jammer generates $V_t = 0$ with probability one.

Remark 5.2. *For simplicity, we do not provide the expressions for the encoding policy, the decoding policy, and the jamming policy for the case when there is no transmission from the sensor. For that case, we have already specified in the problem formulation that the encoder will send* $Y_t = 0$*, and the jammer will send* $V_t = 0$ *with probability one. Correspondingly, the minimum mean squared error estimate that the decoder should generate is the conditional mean,* $\mathbb{E}[X_t|U_t = 0, E_t]$*, which is zero since the source density and the communication scheduling policy are symmetric around zero.*

Proof. Recall that with Assumptions 5.1-5.4 holding, there are three things needed to be determined, i.e., the communication scheduling policy, f_t, the encoding power, $P_e(t, E_t)$, and the jamming power, $P_j(t, E_t)$. Correspondingly, the proof consists of the three following steps:

Step 1. We show that there is a unique best response policy for the follower (i.e., the jammer) in reaction to the policies for the leader (i.e., the group of the sensor, the encoder, and the decoder). Equivalently, we show that there is a unique jamming power, $P_j(t, E_t)$, minimizing $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$ over $[0, \infty)$ corresponding to the communication scheduling policy, f_t, and the encoding power, $P_e(t, E_t)$. Taking the partial derivative of $\bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)$, expressed by (5.12), with respect to $P_j(t, E_t)$, we have

$$
\begin{aligned}
& \frac{\partial \bar{\mathcal{C}}_A(t, E_t, f_t, g_t, h_t, d_t)}{\partial P_j(t, E_t)} \\
= \;& 2\,\mathbb{P}(U_t = 1, S_t = 1, E_t)\Big[\eta \\
& - \frac{P_e(t, E_t)}{\big(P_e(t, E_t) + P_j(t, E_t)\big)^2}\mathrm{Var}(X_t|U_t = 1, S_t = 1, E_t)\Big]
\end{aligned}
$$

Note that $P_e(t, E_t) \geq 0$ and $P_j(t, E_t) \geq 0$ as the encoding power and the jamming power cannot be negative. Then, we have the following lines of reasoning.

1. When
$$
P_e(t, E_t) \leq \frac{\mathrm{Var}(X_t|U_t = 1, S_t = 1, E_t)}{\eta}
$$

it can be checked that

$$\frac{\partial \bar{\mathcal{C}}_A(t,E_t,f_t,g_t,h_t,d_t)}{\partial P_j(t,E_t)} \begin{cases} < 0, & if\ P_j(t,E_t) \in \big(0,\bar{P}\big), \\ = 0, & if\ P_j(t,E_t) = \bar{P}, \\ > 0, & if\ P_j(t,E_t) \in \big(\bar{P},\infty\big) \end{cases}$$

where

$$\bar{P} = \sqrt{\frac{P_e(t,E_t)}{\eta}\mathrm{Var}(X_t|U_t=1,S_t=1,E_t)} - P_e(t,E_t)$$

Therefore, $P_j(t,E_t)=\bar{P}$ is the unique one that minimizes $\mathcal{C}_A(t,E_t,f_t,g_t,h_t,d_t)$ over $[0,\infty)$.

2. When

$$P_e(t,E_t) > \frac{\mathrm{Var}(X_t|U_t=1,S_t=1,E_t)}{\eta}$$

we have that

$$\frac{P_e(t,E_t)}{\big(P_e(t,E_t)+P_j(t,E_t)\big)^2} < \frac{1}{P_e(t,E_t)} < \frac{\eta}{\mathrm{Var}(X_t|U_t=1,S_t=1,E_t)}$$

hold for all $P_j(t,E_t) \in (0,\infty)$, and then

$$\frac{\partial \bar{\mathcal{C}}_A(t,E_t,f_t,g_t,h_t,d_t)}{\partial P_j(t,E_t)} > 0 \quad \forall\ P_j(t,E_t) \in (0,\infty)$$

Therefore, $P_j(t,E_t)=0$ is the unique one that minimizes $\bar{\mathcal{C}}_A(t,E_t,f_t,g_t,h_t,d_t)$ over $[0,\infty)$.

We combine the above two cases, and obtain the unique jamming power $P_j(t,E_t)$ minimizing $\bar{\mathcal{C}}_A(t,E_t,f_t,g_t,h_t,d_t)$ as follows:

$$P_j(t,E_t) = \begin{cases} \sqrt{\frac{P_e(t,E_t)}{\eta}\mathrm{Var}(X_t|U_t=1,S_t=1,E_t)} - P_e(t,E_t), \\ \qquad\qquad \text{if } P_e(t,E_t) \le \frac{\mathrm{Var}(X_t|U_t=1,S_t=1,E_t)}{\eta} \\ 0, \qquad\qquad \text{if } P_e(t,E_t) > \frac{\mathrm{Var}(X_t|U_t=1,S_t=1,E_t)}{\eta} \end{cases} \tag{5.22}$$

Step 2. We assume that the jammer always applies the best response policy derived in Step 1, and we would like to jointly design f_t and $P_e(t, E_t)$ to minimize $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$. To solve the joint optimization problem, we first fix the communication scheduling policy, f_t, and design $P_e(t, E_t)$ correspondingly, which is done at this step. In the next step (Step 3), we find the minimizing f_t.

We plug (5.22) into $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$, captured by (5.12), and have

$$\begin{aligned} &\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t) \\ =& \begin{cases} \bar{\mathcal{C}}_S^1\big(f_t, P_e(t, E_t)\big), & \text{if } P_e(t, E_t) \leq \dfrac{\mathrm{Var}(X_t|U_t = 1, S_t = 1, E_t)}{\eta}, \\ \bar{\mathcal{C}}_S^2\big(f_t, P_e(t, E_t)\big), & \text{if } P_e(t, E_t) > \dfrac{\mathrm{Var}(X_t|U_t = 1, S_t = 1, E_t)}{\eta} \end{cases} \end{aligned} \tag{5.23}$$

where

$$\begin{aligned} &\bar{\mathcal{C}}_S^1\big(f_t, P_e(t, E_t)\big) \\ =\;& 2\,\mathbb{P}(U_t = 1, S_t = 1, E_t)\big[c + c_t(E_t) + \lambda P_e(t, E_t) \\ &- \sqrt{\eta P_e(t, E_t)\mathrm{Var}(X_t|U_t = 1, S_t = 1, E_t)} + \mathrm{Var}(X_t|U_t = 1, S_t = 1, E_t)\big] \\ &+ \mathbb{P}(U_t = 0, S_t = 0, E_t)\mathrm{Var}(X_t|U_t = 0, S_t = 0, E_t), \\ &\bar{\mathcal{C}}_S^2\big(f_t, P_e(t, E_t)\big) \\ =\;& 2\,\mathbb{P}(U_t = 1, S_t = 1, E_t)\Big[c + c_t(E_t) + \lambda P_e(t, E_t)\Big] \\ &+ \mathbb{P}(U_t = 0, S_t = 0, E_t)\mathrm{Var}(X_t|U_t = 0, S_t = 0, E_t) \end{aligned}$$

We note that $\bar{\mathcal{C}}_S^1\big(f_t, P_e(t, E_t)\big) = \bar{\mathcal{C}}_S^2\big(f_t, P_e(t, E_t)\big)$ when $P_e(t, E_t) = \mathrm{Var}(X_t|U_t = 1, S_t = 1, E_t)/\eta$. In addition, $\bar{\mathcal{C}}_S^2\big(f_t, P_e(t, E_t)\big)$ is increasing in $P_e(t, E_t)$. Therefore, minimizing $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ over $P_e(t, E_t) \in [0, \infty)$ is equivalent to minimizing $\bar{\mathcal{C}}_S^1\big(f_t, P_e(t, E_t)\big)$ over $P_e(t, E_t) \in [0, \mathrm{Var}(X_t|U_t = 1, S_t = 1, E_t)/\eta]$. Taking derivative of $\bar{\mathcal{C}}_S^1\big(f_t, P_e(t, E_t)\big)$ with respect to $P_e(t, E_t)$, we have

$$\frac{\partial \bar{\mathcal{C}}_S^1\big(f_t, P_e(t,E_t)\big)}{\partial P_e(t,E_t)}$$

$$= \quad 2\,\mathbb{P}(U_t=1, S_t=1, E_t)\left[\lambda - \frac{1}{2}\sqrt{\frac{\eta \mathrm{Var}(X_t|U_t=1,S_t=1,E_t)}{P_e(t,E_t)}}\right]$$

This leads to

$$\frac{\partial \bar{\mathcal{C}}_S^1\big(f_t, P_e(t,E_t)\big)}{\partial P_e(t,E_t)} \begin{cases} < 0, & \text{if } P_e(t,E_t) \in (0,\tilde{P}), \\ = 0, & \text{if } P_e(t,E_t) = \tilde{P}, \\ > 0, & \text{if } P_e(t,E_t) \in (\tilde{P},\infty) \end{cases}$$

where

$$\tilde{P} = \frac{\eta \mathrm{Var}(X_t|U_t=1,S_t=1,E_t)}{4\lambda^2}$$

It can be seen that $\bar{\mathcal{C}}_S^1\big(f_t, P_e(t,E_t)\big)$ attains global minimum over $P_e(t,E_t) \in [0,\infty)$ at $P_e(t,E_t) = \tilde{P}$. However, our goal is to minimize $\bar{\mathcal{C}}_S^1\big(f_t, P_e(t,E_t)\big)$ over $P_e(t,E_t) \in [0, \mathrm{Var}(X_t|U_t=1,S_t=1,E_t)/\eta]$. Toward that end, we have the following two cases:

1. When

   $$\tilde{P} = \frac{\eta \mathrm{Var}(X_t|U_t=1,S_t=1,E_t)}{4\lambda^2} \le \frac{\mathrm{Var}(X_t|U_t=1,S_t=1,E_t)}{\eta}$$

   or equivalently,

   $$\eta \le 2\lambda$$

   the encoding power $P_e(t,E_t)$ that minimizes $\bar{\mathcal{C}}_S^1\big(f_t, P_e(t,E_t)\big)$ over $[0, \mathrm{Var}(X_t|U_t=1,S_t=1,E_t)/\eta]$ is $\tilde{P}$.

2. When

   $$\tilde{P} = \frac{\eta \mathrm{Var}(X_t|U_t=1,S_t=1,E_t)}{4\lambda^2} > \frac{\mathrm{Var}(X_t|U_t=1,S_t=1,E_t)}{\eta}$$

   or equivalently,

   $$\eta > 2\lambda$$

   the encoding power $P_e(t,E_t)$ that minimizes $\bar{\mathcal{C}}_S^1\big(f_t, P_e(t,E_t)\big)$ over

$[0, \mathrm{Var}(X_t|U_t = 1, S_t = 1, E_t)/\eta]$ is $\mathrm{Var}(X_t|U_t = 1, S_t = 1, E_t)/\eta$.

Combining the above two cases, we obtain the encoding power $P_e(t, E_t)$ that minimizes $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ with the communication scheduling policy f_t being fixed, which is as follows:

$$P_e(t, E_t) = \begin{cases} \dfrac{\eta}{4\lambda^2} \mathrm{Var}(X_t|U_t = 1, S_t, E_t), & \text{if } \dfrac{\lambda}{\eta} \geq \dfrac{1}{2} \\ \dfrac{1}{\eta} \mathrm{Var}(X_t|U_t = 1, S_t, E_t), & \text{if } \dfrac{\lambda}{\eta} < \dfrac{1}{2} \end{cases} \tag{5.24}$$

In addition, we can further compute the jamming power $P_j(t, E_t)$ under the best response policy, by plugging (5.24) into (5.22):

$$P_j(t, E_t) = \begin{cases} \left(\dfrac{2\lambda}{\eta} - 1\right) P_e(t, E_t), & \text{if } \dfrac{\lambda}{\eta} \geq \dfrac{1}{2} \\ 0, & \text{if } \dfrac{\lambda}{\eta} < \dfrac{1}{2} \end{cases} \tag{5.25}$$

Note that (5.24) and (5.25) match the expressions for the encoding power and the jamming power, respectively, in items 2 and 3 of the theorem.

Step 3. Finally, we find the communication scheduling policy f_t that minimizes $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$, with the encoding power $P_e(t, E_t)$ and the jamming power $P_j(t, E_t)$ being captured by (5.24) and (5.25), respectively. Plugging (5.24) and (5.25) into $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$, we have

$$\begin{aligned} & \bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t) \\ = \; & 2\,\mathbb{P}(U_t = 1, S_t = 1, E_t)\big[c + c_t(E_t) + \xi \cdot \mathrm{Var}(X_t|U_t = 1, S_t = 1, E_t)\big] \\ & + \mathbb{P}(U_t = 0, S_t = 0, E_t)\mathrm{Var}(X_t|U_t = 0, S_t = 0, E_t) \end{aligned} \tag{5.26}$$

where

$$\xi = \begin{cases} 1 - \dfrac{\eta}{4\lambda}, & \text{if } \dfrac{\lambda}{\eta} \geq \dfrac{1}{2} \\ \dfrac{\lambda}{\eta}, & \text{if } \dfrac{\lambda}{\eta} < \dfrac{1}{2} \end{cases}$$

It can be checked that $\xi \in (0, 1)$ for both cases. Then, the cost functional, $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$, captured by (5.26), has the same form with (2.8) in Chapter 2 (page 24). Minimizing such a cost functional is eventually a communication scheduling and remote estimation problem with an additive noise channel, and this problem has been studied in Chapter 2. By Theorem 2.2 in Chapter 2, we have that the communication scheduling policy minimizing $\bar{\mathcal{C}}_S(t, E_t, f_t, g_t, h_t, d_t)$ is the threshold-based one described in item 1 of the theorem. □

We now highlight some salient aspects of the main results. First, we see that when $\eta \geq 2\lambda$, the jamming power under the Stackelberg solution is zero. This is due to the fact that when the unit price for the jamming power (i.e., η) is high, the rewards that the jammer gains from the estimation error cannot compensate the cost of generating the jamming noise. Therefore, the jammer would rather generate zero. Second, when $\eta < 2\lambda$, the signal-to-noise ratio (SNR) under the Stackelberg solution is a constant that is independent of time, t, and remaining communication opportunities, E_t. This indicates that the most efficient way for the jammer to allocate its power is to select the jamming power proportional to the encoding power.

5.3 Numerical Results

In this section, we conduct numerical analysis to develop further understanding into the performance of the remote sensing system under the feedback Stackelberg solution. Let the source have Laplace distribution with parameters $(0, 1)$, i.e.,

$$p_X(x) = \frac{1}{2}e^{-|x|}$$

In addition, let the length of the time horizon be $T = 50$, the communication cost be $c = 0$, and the unit price of the encoding power be $\lambda = 1$. We compute $V_S(t, E_t)$ iteratively via (5.2) and (5.3) for all $t = 1, \ldots, T$ and $E_t = 0, \ldots, N$. Note that $V_S(1, N)$ equals the cost functional of the remote sensing system evaluated at the feedback Stackelberg solution, i.e., $J_S(\boldsymbol{f}^*, \boldsymbol{g}^*, \boldsymbol{h}^*, \boldsymbol{d}^*)$. We plot $V_S(1, N)$ versus N under different ratios of λ to η (by choosing different η), as shown in Fig. 5.2. We note that the cost of the remote sensing system decreases as the ratio of λ to η decreases. This is in line with the intuition that as the jamming power gets more expensive (relative to the encoding power), the jammer would utilize less power when generating the jamming noise, which results in lower mean squared error and/or lower consumption on the encoding power. We also note that for each fixed λ/η, there exists a threshold

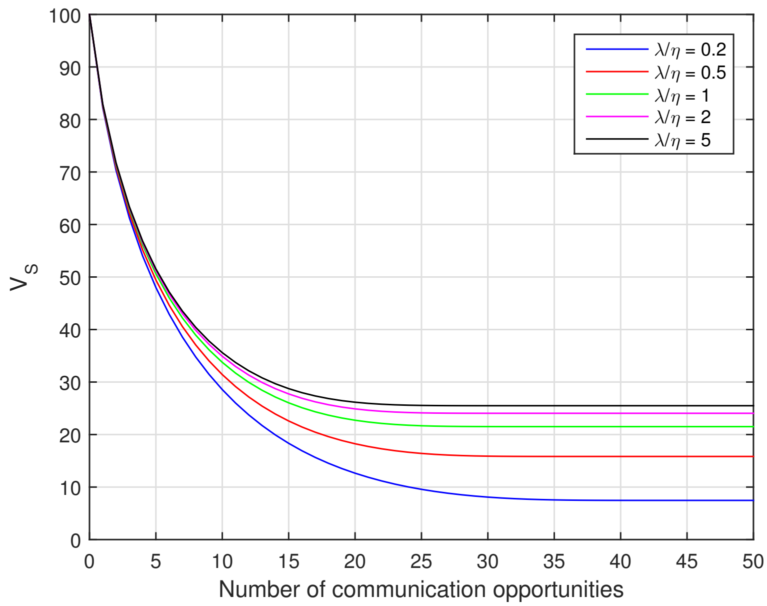

Figure 5.2: Cost of remote sensing system vs. number of transmission opportunities

on N such that $V_S(1, N)$ is a decreasing function in N when N is below the threshold, and $V_S(1, N)$ is held at a constant when N is above the threshold. This phenomenon has also been observed in Chapter 2, Fig. 2.2. Although the problem studied in Chapter 2 is different from the one here,[6] the explanation for the phenomenon is similar. Recall that the sensor applies a threshold-based policy, described in Theorem 5.2 item 1, when making the decision, U_t, on whether to transmit or not. In addition, it can be shown that there exists a lower bound on the threshold. Hence, the expected number of transmissions over the time horizon, $\sum_{t=1}^{T} \mathbb{E}[U_t]$, is upper bounded. Once the number of transmission opportunities, N, exceeds this upper bound, the additional transmission opportunities will not be utilized (in the average sense), as shown in Fig. 5.3. Hence, these additional transmission opportunities will not contribute toward reducing the expected cost of the remote sensing system.

We are also interested in how the encoding power, $\mathbb{E}[Y_t^2]$, is allocated over the time horizon. Toward that end, we choose $\lambda/\eta = 1$, and we compute $\mathbb{E}[Y_t^2]$ numerically (via the Monte Carlo method) for all $t = 1, \ldots, T$. We

[6]In Chapter 2, a communication scheduling and remote estimation problem with an i.i.d. noise process was studied, which does not involve a jammer.

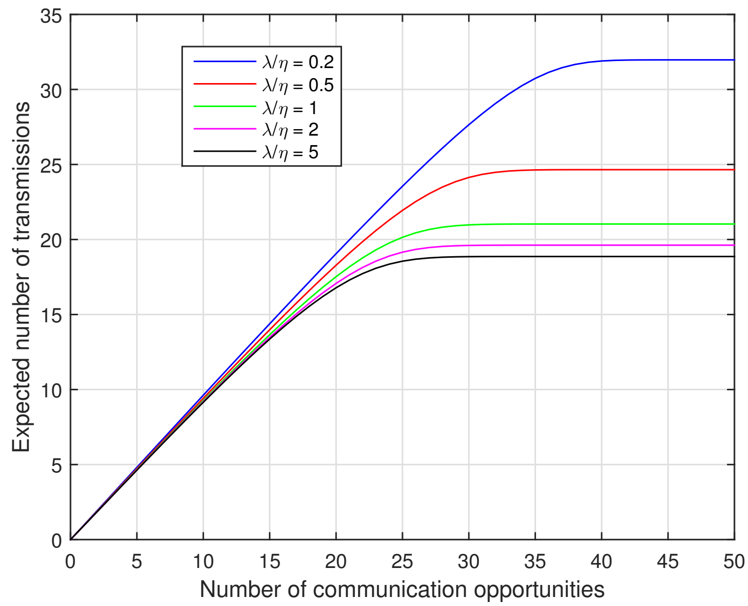

Figure 5.3: Expected number of transmissions vs. number of transmission opportunities

plot $\mathbb{E}[Y_t^2]$ versus t in Fig. 5.4,[7] which reveals several interesting phenomena.

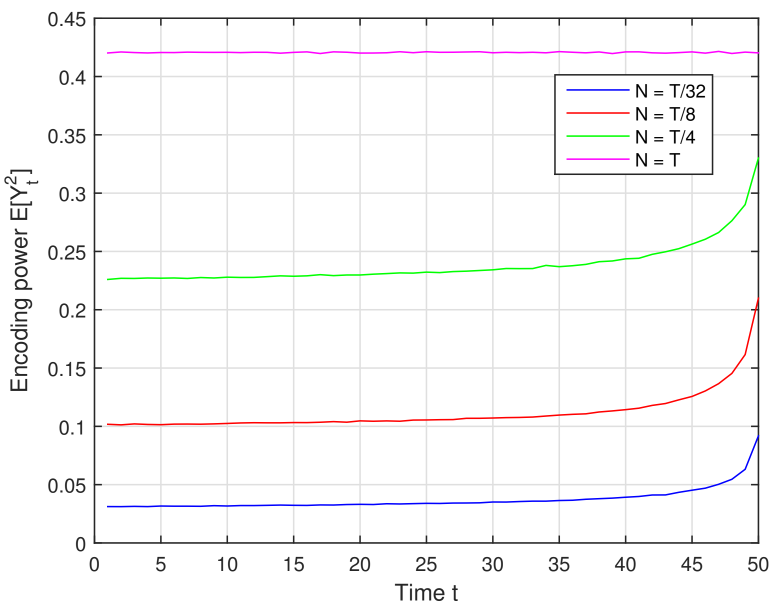

Figure 5.4: Encoding power vs. time

[7] $N = T/32$ is not an integer, and thus we round it to the nearest integer. Same for the other Ns.

First, the encoding power, $\mathbb{E}[Y_t^2]$, increases as t increases. On the other hand, when the number of transmission opportunities is $N = T$ (i.e., the sensor is always allowed to transmit), the encoding power $\mathbb{E}[Y_t^2]$ is the same for all t. Second, the larger the transmission opportunities, N, the larger is the encoding power, $\mathbb{E}[Y_t^2]$. To illustrate the phenomena, we need to expand $\mathbb{E}[Y_t^2]$ as follows:

$$\mathbb{E}[Y_t^2] = \mathbb{P}(U_t = 1)\cdot\mathbb{E}[Y_t^2|U_t = 1] = \mathbb{P}(E_t > 0)\cdot\mathbb{P}(U_t = 1|E_t > 0)\cdot\mathbb{E}[Y_t^2|U_t = 1]$$

The above expansion implies that the encoding power at time t, $\mathbb{E}[Y_t^2]$, depends on three terms: the probability that the sensor has opportunity(s) for communication, the probability that the sensor decides to transmit its observation, and the power the encoder will use to transmit the message. We first consider the third term, $\mathbb{E}[Y_t^2|U_t = 1]$. By Theorem 5.2, the encoder will use the power $P^*(t, E_t) = \eta/4\lambda^2 \cdot \mathrm{Var}(X_t|U_t = 1, S_t, E_t)$ to transmit the message. Note that conditioned on the event $(U_t = 1, S_t, E_t)$, X_t is a shifted exponential distribution with unit variance, which is due to the memoryless property. Therefore, $P^*(t, E_t)$ is a constant that is independent of t and E_t. Then, we have that $\mathbb{E}[Y_t^2|U_t = 1]$ is a constant. Next, we consider the second term, $\mathbb{P}(U_t = 1|E_t > 0)$. The probability that the sensor makes a transmission depends on the threshold, $\beta_t^*(E_t)$, which is the solution to the fixed point equation (5.21). The larger the threshold, the smaller the transmission probability. Furthermore, it can be seen that in (5.21), the larger the opportunity cost, $c_t(E_t)$, the larger is the solution $\beta_t^*(E_t)$ that the equation admits. The opportunity cost $c_t(E_t)$, defined in (5.6), describes the cost the sensor incurs in the future by choosing to transmit its observation rather than reserve the opportunity for future use. The closer the time step t to the end of the time horizon, the smaller is the opportunity cost. Therefore, as t increases, the opportunity cost $c_t(E_t)$ and the threshold $\beta_t^*(E_t)$ decrease, and the transmission probability $\mathbb{P}(U_t = 1|E_t > 0)$ increases. In particular, when $N = T$, i.e., the sensor is always allowed to transmit, the opportunity cost equals zero. Therefore, the transmission probability is the same for all t. This partially[8] explains the first phenomenon. Finally, we consider the first term, i.e., the probability that the sensor has opportunity(s) for com-

[8]The first term is non-increasing in t, yet we assume that the increase in the second term dominates the decrease in the first term as t increases.

munication, $\mathbb{P}(E_t > 0)$. Intuitively, the more opportunities the sensor has at the beginning of the time horizon, the more probable the sensor is able to make a transmission at time t. Therefore, for each fixed t, $\mathbb{P}(E_t > 0)$ should be increasing in N. This partially[9] explains the second phenomenon.

[9]It is difficult to analyze the dependency of the second term in N, yet we assume that the first term dominates in this case.

CHAPTER 6

FUTURE WORK

In this chapter, we discuss possible directions for future research based on the framework of this thesis, which are listed as follows:

1. In the problems considered in this thesis, the source processes were assumed to be i.i.d.. We may generalize the results to Markov source processes. For example, the source variable can be the state of an LTI system driven by an i.i.d. stochastic process. In this case, we may need to assume that the decoder sends a feedback signal to the sensor, which carries information on the noise-corrupted message the decoder receives from the communication channel. Similar to the affineness assumption we made on the encoding and decoding policies, we may restrict the encoder to apply an affine encoding policy, and restrict the decoder to apply a Kalman filter-like estimation policy.

2. In the problems considered in this thesis, the source variable is one-dimensional. We may consider the setting(s) with a multidimensional source. In order to measure different components of the source, multiple sensors would be placed. Each sensor may measure the source only in one dimension (one component), and different components of the source would be correlated. The sensors may send their measurements to one estimator or multiple estimators, which will produce estimate(s) on the source. Related works on this direction can be found in [52] and the references therein.

3. In the thesis we assumed that the sensor always makes perfect measurements on the source variables. We may consider a more general case where there is an observation noise. Related works on this direction can be found in [17–19].

4. As an extension to the multi-channel setting, we may consider the set-

ting with two additive noise channels, where one channel is cheap but noisy, and the other one is costly but less noisy, and in addition, the sensor still has the option of not transmitting its observation. Here, if the sensor decides to transmit its observation, it always sends the observation to the encoder. The encoder generates an encoded message, but may send it to the noisy channel or the less noisy channel, depending on the sensor's decision. We expect that under suitable assumptions (similar to the ones made in Theorem 3.2), the optimal communication scheduling policy is still threshold-in-threshold based.

5. We may also consider the setting(s) where the sensor is equipped with an energy harvester (as in [25, 29]) such that it could obtain additional transmission opportunities over the time horizon. We may assume that numbers of transmission opportunities obtained at different stages are captured an i.i.d. stochastic process (call it "harvesting process"), which is independent of the source process and the noise process. We expect that the optimal communication scheduling policy is threshold-based (or threshold-in-threshold based), and the optimal threshold(s) would depend on the distribution of the harvesting process.

CHAPTER 7

CONCLUSION

In this thesis, we have presented results to date on communication scheduling and remote estimation with additive noise channels. The research was motivated by the prior work in [27–29], where several settings with a noiseless communication channel were studied. In particular, we considered a series of four settings in this thesis, namely the single-channel setting, the multi-channel setting, the power allocation setting, and the adversarial setting. In the single-channel setting, the communication channel between the sensor and the estimator was an additive noise channel. Therefore, if the sensor decides to transmit its observation, it needs to send the observation to the encoder, who then encodes and transmits the message. In the multi-channel setting, the sensor had an additional option (compared with its options in the single-channel setting) of transmitting its observation over a noiseless yet more costly channel. In the power allocation setting, the encoder had a constraint on its average total power consumption over the time horizon, instead of a constraint on the stage-wise encoding power, which was assumed in the single-channel setting. In the adversarial setting, the communication channel noise was generated by an adversary with the objective maximizing the estimation error. Under some technical assumptions, we obtained the optimal solutions for the first three settings, and a feedback Stackelberg solution for the adversarial setting. We also presented numerical results illustrating the performance of the proposed solutions, and we discussed possible directions for future research.

APPENDIX A

SOME PROOFS

A.1 Proof of Theorem 2.1

Before proving Theorem 2.1, we first introduce the following notations. For any $1 \leq a \leq b \leq T$, let $\mathbf{f}_{a:b}, \mathbf{g}_{a:b}, \mathbf{h}_{a:b}$ denote the subsets of $\mathbf{f}, \mathbf{g}, \mathbf{h}$ such that

$$\begin{aligned} \mathbf{f}_{a:b} &= \{f_a, f_{a+1}, \ldots, f_b\}, \\ \mathbf{g}_{a:b} &= \{g_a, g_{a+1}, \ldots, g_b\}, \\ \mathbf{h}_{a:b} &= \{h_a, h_{a+1}, \ldots, h_b\} \end{aligned}$$

Furthermore, let I_{st}, I_{et}, I_{dt} denote the information about the past system states available to the sensor, the encoder, and the decoder, respectively, at time t ($t > 1$), i.e.,

$$I_{st} = \{X_{1:t-1}, U_{1:t-1}\}, \quad I_{et} = \{\tilde{X}_{1:t-1}, Y_{1:t-1}, S_{1:t-1}\}, \quad I_{dt} = \{\tilde{Y}_{1:t-1}, S_{1:t-1}\}$$

Let I_t be union of I_{st}, I_{et}, and I_{dt}, i.e.,

$$I_t = \{X_{1:t-1}, U_{1:t-1}, \tilde{X}_{1:t-1}, Y_{1:t-1}, S_{1:t-1}, \tilde{Y}_{1:t-1}\}$$

PROOF of Theorem 2.1. It is easy to see the validity of the following sequence of equalities:

$$
\begin{aligned}
& \inf_{\mathbf{f},\mathbf{g},\mathbf{h}} J(\mathbf{f},\mathbf{g},\mathbf{h}) \\
= \; & \inf_{\mathbf{f},\mathbf{g},\mathbf{h}} \mathbb{E}\left\{\sum_{t=1}^{T} cU_t + (X_t - \hat{X}_t)^2\right\} \\
= \; & \inf_{f_1,g_1,h_1} \mathbb{E}\left\{cU_1 + (X_1 - \hat{X}_1)^2 + \inf_{\mathbf{f}_{2:T},\mathbf{g}_{2:T},\mathbf{h}_{2:T}} \mathbb{E}\left\{\sum_{t=2}^{T} cU_t + (X_t - \hat{X}_t)^2\right\}\right\} \\
= \; & \inf_{f_1,g_1,h_1} \mathbb{E}\Big\{cU_1 + (X_1 - \hat{X}_1)^2 + \inf_{f_2,g_2,h_2} \mathbb{E}\Big\{cU_2 + (X_2 - \hat{X}_2)^2 + \ldots \\
& \qquad + \inf_{f_T,g_T,h_T} \mathbb{E}\big\{cU_T + (X_T - \hat{X}_T)^2\big\} \ldots \Big\}\Big\}
\end{aligned}
$$

Then, at time $t = T$, the optimization problem is to design (f_T, g_T, h_T) minimizing

$$J_1(f_T, g_T, h_T) := \mathbb{E}\left\{cU_T + (X_T - \hat{X}_T)^2\right\}$$

call it *Problem 1*. Recall that the decisions at time T are generated by $U_T = f_T(X_T, I_{sT})$, $Y_T = g_T(\tilde{X}_T, I_{eT})$, $\hat{X}_T = h_T(\tilde{Y}_T, S_T, I_{dT})$. We will show that using information about the past (I_{sT}, I_{eT}, I_{dT}) when making decisions cannot help improve the performance (that is, reduce the expected cost). Consider another problem, call it *Problem 2*, where I_T is available to all the decision makers, and one needs to design (f'_T, g'_T, h'_T) minimizing

$$J_2(f'_T, g'_T, h'_T) := \mathbb{E}\left\{cU_T + (X_T - \hat{X}_T)^2\right\}$$

where $U_T = f'_T(X_T, I_T)$, $Y_T = g'_T(\tilde{X}_T, I_T)$, $\hat{X}_T = h'_T(\tilde{Y}_T, S_T, I_T)$. Since the sensor, the encoder, and the decoder can always ignore the redundant information and behave as if they only know I_{sT}, I_{eT}, I_{dT}, respectively, the optimal cost in *Problem 2* is upper bounded by that in *Problem 1*, i.e.,

$$\inf_{(f'_T,g'_T,h'_T)} J_2(f'_T, g'_T, h'_T) \leq \inf_{(f_T,g_T,h_T)} J_1(f_T, g_T, h_T)$$

Similarly, consider a third problem, call it *Problem 3*, where I_{sT}, I_{eT}, I_{dT} are not available to the sensor, the encoder, and the decoder, respectively. One needs to design (f''_T, g''_T, h''_T) to minimize

$$J_3(f''_T, g''_T, h''_T) = \mathbb{E}\left\{cU_T + (X_T - \hat{X}_T)^2\right\}$$

where $U_T = f''_T(X_T)$, $Y_T = g''_T(\tilde{X}_T)$, $\hat{X}_T = h''_T(\tilde{Y}_T, S_T)$. By a similar argument as above, the optimal cost in *Problem 1* cannot be greater than that in *Problem 3*. Hence,

$$\inf_{(f_T,g_T,h_T)} J_1(f_T, g_T, h_T) \leq \inf_{(f''_T,g''_T,h''_T)} J_3(f''_T, g''_T, h''_T)$$

Let us come back to *Problem 2*. One can observe that the communication cost c, the distortion function $\rho(\cdot,\cdot)$, and the power constraint of the encoder do not depend on I_T. Furthermore, since $\{X_t\}$ and $\{V_t\}$ are i.i.d. stochastic processes, X_T and V_T are also independent of I_T. Therefore, there is no loss of optimality in restricting $U_T = f'_T(X_T)$, $Y_T = g'_T(\tilde{X}_T)$, $\hat{X}_T = h'_T(\tilde{Y}_T, S_T)$, and thus

$$\inf_{(f'_T,g'_T,h'_T)} J_2(f'_T, g'_T, h'_T) = \inf_{(f''_T,g''_T,h''_T)} J_3(f''_T, g''_T, h''_T)$$

The equality above indicates that in *Problem 1*, the sensor, the encoder, and the decoder can safely ignore their information about the past, namely I_{sT}, I_{eT}, and I_{dT}, when making decisions.

Since (f_T, g_T, h_T) do not take I_T as a parameter, the design of (f_T, g_T, h_T) is independent of the design of $(f_{1:T-1}, g_{1:T-1}, h_{1:T-1})$. Consequently, the problem can be viewed as a $(T-1)$-stage problem and a single-stage problem. By induction, we can show that (f_1, g_1, h_1), (f_2, g_2, h_2), $\ldots$, (f_T, g_T, h_T) can be designed independently, and (f_t, g_t, h_t) is designed to minimize the stage-wise cost $\mathbb{E}\{cU_t + (X_t - \hat{X}_t)^2\}$. Hence, the optimal decision policies (f_t, g_t, h_t) are in the form of (2.1). Furthermore, since $\{X_t\}$ and $\{V_t\}$ are i.i.d. stochastic processes, the optimal decision policies (f_t, g_t, h_t) should be the same for all $t = 1, 2, \ldots, T$. □

A.2 Proof of Lemma 2.3

To prove Lemma 2.3, we first introduce results from majorization theory. Given a Borel measurable set A, we use A^σ to denote its symmetric rearrangement, i.e., $A^\sigma = [-a, a]$, and $\mathcal{L}(A^\sigma) = \mathcal{L}(A)$ (same Lebesgue measure). Given a non-negative integrable function $p : \mathbb{R} \to \mathbb{R}$, we use p^σ to denote its symmetric rearrangement, which is described as follows,

$$p^{\sigma}(x) := \int_0^{\infty} \mathbf{1}_{\{z\in\mathbb{R}|p(z)\geq\rho\}^{\sigma}}(x)d\rho, \quad x\in\mathbb{R}$$

$\mathbf{1}_{\{z\in\mathbb{R}|p(z)\geq\rho\}^{\sigma}}(x)$ is the indicator function on whether or not x is an element of $\{z \in \mathbb{R}|p(z) \geq \rho\}^{\sigma}$, i.e.,

$$\mathbf{1}_{\{z\in\mathbb{R}|p(z)\geq\rho\}^{\sigma}}(x) = \begin{cases} 1, & \text{if } x \in \{z\in\mathbb{R}|p(z)\geq\rho\}^{\sigma} \\ 0, & \text{otherwise} \end{cases}$$

Definition A.1. *Given two probability densities p and q defined on $\mathbb{R}$, we say p majorizes q, denoted by $p \succ q$, if*

$$\int_{|x|<t} q^{\sigma}(x)dx \leq \int_{|x|<t} p^{\sigma}(x)dx, \quad \text{for all } t \geq 0$$

Lemma A.1 ([28], Lemma 4)**.** *Let p_X and $p_{X'}$ be probability density functions defined on $\mathbb{R}$. Assume that p_X is even and log-concave, and $p_X \succ p_{X'}$. Then,*

$$\int_{-\infty}^{\infty} x^2 p_X(x)dx \leq \int_{-\infty}^{\infty} (x-y)^2 p_{X'}(x)dx, \quad \text{for all } \ y\in\mathbb{R}$$

or equivalently, $\mathrm{Var}(X) \leq \mathrm{Var}(X')$.

Lemma A.2 ([28], Lemma 2)**.** *Let p_X and $p_{X'}$ be probability density functions defined on $\mathbb{R}$. Assume that p_X is even and log-concave, and $p_X \succ p_{X'}$. Let $A = [-\tau, \tau]$ be any symmetric closed interval such that $\int_A p_X(x)dx > 0$ and let $h : \mathbb{R} \to [0, 1]$ be any function such that $\int_{\mathbb{R}} h(x)p_{X'}(x)dx = \int_A p_X(x)dx$. Then,*

$$p_{X|X\in A} \succ \frac{h \cdot p_{X'}}{\int_{\mathbb{R}} h(x)p_{X'}(x)dx}$$

We are now in the position to prove Lemma 2.3.

PROOF of Lemma 2.3. One can see that p_X majorizes itself. Furthermore, we choose $h(x)$ to be the indicator function on whether x belongs to B or not, i.e., $h(x) = \mathbf{1}_{\{x\in B\}}$. Then, $\int_{\mathbb{R}} h(x)p_X(x)dx = \int_B p_X(x)dx = \int_A p_X(x)dx$. By Lemma A.2, the conditional density $p_{X|X\in A}$ majorizes the conditional density $p_{X|X\in B}$. Since A is symmetric about zero, and p_X is even and log-concave, we have $p_{X|X\in A}$ is also even and log-concave. By Lemma A.1, we conclude that $\mathrm{Var}(X|X \in A) \leq \mathrm{Var}(X|X \in B)$. □

A.3 Proof of Corollary 3.1

PROOF of Corollary 3.1. Assumption 3.4 states that $\mathcal{T}_0^f$, $\mathcal{T}_1^f$, $\mathcal{T}_2^f$ are symmetric around zero.[1] Combining Assumptions 3.1 and 3.4, it is easy to see that

$$\mathbb{E}[X|X \in \mathcal{T}_0^f] = \mathbb{E}[X|X \in \mathcal{T}_1^f] = \mathbb{E}[X|X \in \mathcal{T}_2^f] = 0$$

Then, the expected cost $J(f, g, h)$ in eq. (3.3) can be further expressed as

$$\begin{aligned} J(f,g,h) &= \int_{x \in \mathcal{T}_0^f} x^2 p_X(x)dx + \int_{x \in \mathcal{T}_1^f} (c_1 + \frac{1}{\gamma+1}x^2) p_X(x)dx \\ &\quad + \int_{x \in \mathcal{T}_2^f} c_2 p_X(x)dx \\ &=: \int_{x \in \mathbb{R}} \tilde{J}(x, f(x)) p_X(x)dx \end{aligned}$$

where

$$\tilde{J}(x, f(x)) = \begin{cases} x^2, & \text{if } f(x) = 0 \\ c_1 + \dfrac{1}{\gamma+1}x^2, & \text{if } f(x) = 1 \\ c_2, & \text{if } f(x) = 2 \end{cases}$$

Hence, $J(f, g, h)$ can be minimized by f^* satisfying

$$f^*(x) = \arg\min_{u \in \{0,1,2\}} \tilde{J}(x, u)$$

and (g^*, h^*) induced by f^* according to Assumption 3.3. Since $\tilde{J}(x, 0)$, $\tilde{J}(x, 1)$, and $\tilde{J}(x, 2)$ are symmetric around zero, we only need to consider the case when $x \geq 0$. Let $\beta_{01} = \sqrt{(\gamma + 1)c_1/\gamma}$ and $\beta_{02} = \sqrt{c_2}$. Since $1/(\gamma + 1) < 1$, it is easy to check that

$$\begin{aligned} &\tilde{J}(x,0) \leq \tilde{J}(x,1),\ x \in [0, \beta_{01}]; \quad \tilde{J}(x,0) > \tilde{J}(x,1),\ x \in (\beta_{01}, \infty) \\ &\tilde{J}(x,0) \leq \tilde{J}(x,2),\ x \in [0, \beta_{02}]; \quad \tilde{J}(x,0) > \tilde{J}(x,2),\ x \in (\beta_{02}, \infty) \end{aligned}$$

[1] However, $\mathcal{T}_0^f$, $\mathcal{T}_1^f$, $\mathcal{T}_2^f$ may or may not be connected.

Let $\beta_1 = \min\{\beta_{01}, \beta_{02}\}$, and we have

$$\tilde{J}(x,0) \leq \min\{\tilde{J}(x,1), \tilde{J}(x,2)\},\ x \in [0, \beta_1];$$
$$\tilde{J}(x,0) > \min\{\tilde{J}(x,1), \tilde{J}(x,2)\},\ x \in (\beta_1, \infty)$$

Hence, $f^*(x) = 0$ when $x \in [0, \beta_1]$. Furthermore, when $x \in (\beta_1, \infty)$, we only need to compare $\tilde{J}(x,1)$ with $\tilde{J}(x,2)$, and either of the following cases occurs:

Case I: $c_1 + \frac{1}{\gamma+1}\beta_1^2 > c_2$, and then

$$\tilde{J}(x,1) > \tilde{J}(x,2),\ \forall\, x \in (\beta_1, \infty)$$

which implies that $f^*(x) = 2$ when $x \in (\beta_1, \infty)$. Hence, f^* is of the threshold-in-threshold type described by (3.1), with thresholds $\beta_1 = \beta_2$.

Case II: $c_1 + \frac{1}{\gamma+1}\beta_1^2 \leq c_2$. Let $\beta_2 = \sqrt{(c_2 - c_1)(\gamma+1)}$. It can be checked that

$$\tilde{J}(x,1) \leq \tilde{J}(x,2),\ x \in (\beta_1, \beta_2]; \quad \tilde{J}(x,1) > \tilde{J}(x,2),\ x \in (\beta_2, \infty)$$

Hence, $f^*(x) = 1$ when $x \in (\beta_1, \beta_2]$, and $f^*(x) = 2$ when $x \in (\beta_2, \infty)$. f^* is of the threshold-in-threshold type. □

A.4 Proof of Proposition 3.1

PROOF of Proposition 3.1. Let $k := \mathbb{P}(X \in (\beta_1, \beta_2))$. Consider any open interval $(\eta_1, \eta_2), \eta_1 \geq 0$ such that $\mathbb{P}(X \in (\eta_1, \eta_2)) = k$. Since

$$\mathbb{P}(X \in (\eta_1, \eta_2)) = \int_{\eta_1}^{\eta_2} p_X(x)dx = k$$

taking derivative with respect to η_1, we have

$$-p_X(\eta_1) + \frac{d\eta_2}{d\eta_1}\, p_X(\eta_2) = 0 \tag{A.1}$$

Now consider the partial derivative of $\mathrm{Var}(X|X \in (\eta_1, \eta_2))\mathbb{P}(X \in (\eta_1, \eta_2))$

with respect to η_1. It can be computed that

$$\frac{\partial}{\partial \eta_1} \text{Var}\big(X|X \in (\eta_1, \eta_2)\big) \int_{\eta_1}^{\eta_2} p_X(x)dx = -p_X(\eta_1)\big(\eta_1 - \mathbb{E}[X|X \in (\eta_1, \eta_2)]\big)^2 \tag{A.2}$$

Similarly, we have

$$\frac{\partial}{\partial \eta_2} \text{Var}\big(X|X \in (\eta_1, \eta_2)\big) \int_{\eta_1}^{\eta_2} p_X(x)dx = p_X(\eta_2)\big(\eta_2 - \mathbb{E}[X|X \in (\eta_1, \eta_2)]\big)^2 \tag{A.3}$$

Combining (A.1)-(A.3), we obtain

$$\begin{aligned}
& \frac{d}{d\eta_1} \text{Var}(X|X \in (\eta_1, \eta_2))\mathbb{P}(X \in (\eta_1, \eta_2)) \\
= \quad & \frac{\partial}{\partial \eta_1} \text{Var}(X|X \in (\eta_1, \eta_2))\mathbb{P}(X \in (\eta_1, \eta_2)) \\
& + \frac{d\eta_2}{d\eta_1} \frac{\partial}{\partial \eta_2} \text{Var}(X|X \in (\eta_1, \eta_2))\mathbb{P}(X \in (\eta_1, \eta_2)) \\
= \quad & p_X(\eta_1)\big((\eta_2 - \mathbb{E}[X|X \in (\eta_1, \eta_2)])^2 - (\eta_1 - \mathbb{E}[X|X \in (\eta_1, \eta_2)])^2\big)
\end{aligned}$$

Since $p_X(x)$ is monotone decreasing when $x \geq 0$, it is easy to see that

$$\eta_2 - \mathbb{E}[X|X \in (\eta_1, \eta_2)] > \mathbb{E}[X|X \in (\eta_1, \eta_2)] - \eta_1$$

Hence,

$$\frac{d}{d\eta_1} \text{Var}(X|X \in (\eta_1, \eta_2))\mathbb{P}(X \in (\eta_1, \eta_2)) = \frac{d}{d\eta_1} k\text{Var}(X|X \in (\eta_1, \eta_2)) > 0$$

The inequality above implies that when preserving the probability measure over (η_1, η_2),

$$\frac{d}{d\eta_1} \text{Var}(X|X \in (\eta_1, \eta_2)) > 0$$

Integrating both sides from β_1 to β_1' and by comparison principle, we establish the desired inequality. □

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