Problem 11402: Squares on Graphs

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By the uniqueness of the minimum, $t' = \phi(\lambda)$. Since $\langle t_n \rangle$ is bounded we conclude that $\langle t_n \rangle$ converges to $\phi(\lambda)$. This shows that ϕ is continuous. ■

Lemma 2. $\lim_{\lambda\to+\infty} \phi(\lambda) = 0$ *and* $\lim_{\lambda\to 0^+} |\phi(\lambda)| = +\infty$.

Proof. Let $\langle \lambda_n \rangle$ be a sequence such that $\lim_{n\to\infty} \lambda_n = +\infty$, and let $t_n = \phi(\lambda_n)$. For $t \in \mathbb{R}$, we have $f(t_n)/\lambda_n + |t_n| \le f(t)/\lambda_n + |t|$, and in particular $f(t_n)/\lambda_n + |t_n| \le f(0)/\lambda_n$. Let λ_0 be a fixed positive value, and let $m = \inf_{\mathbb{R}}[f(t) + \lambda_0|t|]$. Now $f(t_n) \ge m - \lambda_0|t_n|$, so $(1 - \lambda_0/\lambda_n)|t_n| \le \big(f(0) - m\big)/\lambda_n$. Therefore $\lim_{n\to\infty} t_n = 0$.

For the other claim of the lemma, let $\langle \lambda_n \rangle$ be a positive sequence that tends to zero, let $t_n = \phi(\lambda_n)$, and let t' be a limit point of $\langle t_n \rangle$ (if one exists). The argument of Lemma 1 proves that for any real t, $f(t) \ge f(t')$. That makes $f(t')$ a global minimum for f, contrary to the hypothesis. Since $\langle t_n \rangle$ has no limit point, $\lim_{n\to\infty} |t_n| = +\infty$. ■

From these two lemmas, we see that the range of ϕ contains $(0, \infty)$ or $(-\infty, 0)$ (but not both). We will show that in the first case conclusion (a) holds. Similarly, the second case leads to (b).

Assume the range contains $(0, \infty)$, and let $m(\lambda) = \inf_{\mathbb{R}} \big(f(t) + \lambda|t|\big)$. Now $f(t) \ge \sup_\lambda \big(m(\lambda) - \lambda|t|\big)$. If $t = \phi(\lambda)$, then $f(\phi(\lambda)) = m(\lambda) - \lambda|\phi(\lambda)|$. Thus f is the pointwise supremum of a family of affine functions on $(0, \infty)$, so f is convex there. We claim that f is actually *strictly* convex. Indeed, if f is affine on some interval $[a, b]$ with $0 < a < b$, then we can choose λ such that the function f_λ given by $f_\lambda(t) = f(t) + \lambda|t|$ reaches its infimum at a point of (a, b). Since f_λ is is affine on this interval, it is minimized at an interior point only if it is constant on that interval, which contradicts the uniqueness of the minimum point.

Let s, t be given with $t > 0$ and $-t \le s < t$. There exists λ such that $t = \phi(\lambda)$. Thus

$$f(s) + \lambda|s| > f(t) + \lambda|t| \ge f(t) + \lambda|s|.$$

We obtain $f(s) > f(t)$. (If $-t \le s \le t$, we obtain $f(s) \ge f(t)$.) For the integral inequality, we have $-|u(x)| \le u(x) \le |u(x)|$. So $f(u(x)) \ge f\big(|u(x)|\big)$. Since f is convex, Jensen's inequality yields

$$\int_\Omega f(u) \ge \int_\Omega f\big(|u|\big) \ge f\left(\int_\Omega |u|\right).$$

It is a strict inequality since u is not essentially constant and f is strictly convex.

Also solved by R. Stong.

Squares On Graphs

11402 [2008, 949]. *Proposed by Doru Catalin Barboianu, Infarom Publishing, Craiova, Romania* Let $f : [0, 1] \to [0, \infty)$ be a continuous function such that $f(0) = f(1) = 0$ and $f(x) > 0$ for $0 < x < 1$. Show that there exists a square with two vertices in the interval (0,1) on the x-axis and the other two vertices on the graph of f.

Solution by Byron Schmuland and Peter Hooper, University of Alberta, Edmonton, AB, Canada. Extend f by letting $f(x) = 0$ for $x \ge 1$. Define $g(x) = f(x + f(x)) - f(x)$ for $x \ge 0$. If there exists $x \in (0, 1)$ with $g(x) = 0$, then a square as required exists with vertices

$$(x, 0), \quad (x + f(x), 0), \quad (x, f(x)), \quad (x + f(x), f(x)).$$

Now g is continuous, so to show that such x exists we will show that $y, z \in (0, 1)$ exist with $g(y) \geq 0$ and $g(z) \leq 0$. Let z be a value where f takes its maximum. Then $f(z) \geq f(z + f(z))$, so that $g(z) \leq 0$. Since $0 + f(0) = 0 < z < z + f(z)$, by continuity there is a value $y \in (0, z)$ so that $y + f(y) = z$. Hence $g(y) = f(y + f(y)) - f(y) = f(z) - f(y) \geq 0$.

Editorial comment. Pál Péter Dályay (Hungary) noted a generalization: Given any $p > 0$, there exists a rectangle with base-to-height ratio p having two vertices on the x-axis and the other two vertices on the graph of f.

Also solved by B. M. Ábrego & S. Fernández-Merchant, F. D. Ancel, K. F. Andersen (Canada), R. Bagby, N. Caro (Brazil), D. Chakerian, R. Chapman (U.K.), B. Cipra, P. Corn, C. Curtis, P. P. Dályay (Hungary), C. Diminnie & R. Zarnowski, P. J. Fitzsimmons, D. Fleischman, T. Forgács, O. Geupel (Germany), D. Grinberg, J. Grivaux (France), J. M. Groah, E. A. Herman, S. J. Herschkorn, E. J. Ionascu, A. Kumar & C. Gibbard (U.S.A. & Canada), S. C. Locke, O. P. Lossers (Netherlands), R. Martin (Germany), K. McInturff, M. McMullen, M. D. Meyerson R. Mortini M. J. Nielsen, M. Nyenhuis (Canada), Á. Plaza & S. Falcón (Spain), K. A. Ross, T. Rucker, J. Schaer (Canada), K. Schilling, E. Shrader, A. Stadler (Switzerland), R. Stong, B. Taber, M. Tetiva (Romania), T. Thomas (U.K.), J. B. Zacharias & K. Greeson, BSI Problems Group (Germany), GCHQ Problem Solving Group (U.K.), Lafayette College Problem Group, Microsoft Research Problems Group, Missouri State University Problem Solving Group, Northwestern University Math Problem Solving Group, NSA Problems Group, and the proposer.

A Trig Series Rate

11410 [2009, 83]. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* For $0 < \phi < \pi/2$, find

$$\lim_{x\to 0} x^{-2}\left(\frac{1}{2}\log\cos\phi + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{\sin^2(nx)}{(nx)^2}\sin^2(n\phi)\right).$$

Solution by Otto B. Ruehr, Michigan Technological University, Houghton, MI. We begin with three elementary identities. The first is

$$\sum_{n=1}^{\infty} r^n \sin^2 n\phi = \frac{r(r+1)\sin^2\phi}{(1-r)\left[(1-r)^2 + 4r\sin^2\phi\right]}. \tag{i}$$

This is derived by writing $\sin^2 n\phi$ in terms of exponentials and summing the resulting geometric series. Now divide (i) by r and integrate with respect to r to get

$$\sum_{n=1}^{\infty}\frac{r^n}{n}\sin^2 n\phi = \frac{1}{4}\log\left[\frac{(1-r)^2 + 4r\sin^2\phi}{(1-r)^2}\right]. \tag{ii}$$

Differentiate (i) with respect to r to obtain

$$\sum_{n=1}^{\infty} nr^{n-1}\sin^2 n\phi = \frac{1}{2(1-r)^2} - \frac{1}{2}\left[\frac{(r-1)^2 - 2(r^2+1)\sin^2\phi}{[(1-r)^2 + 4r\sin^2\phi]^2}\right]. \tag{iii}$$

The limit at $r = -1$ in (ii) gives us

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin^2 n\phi = -\frac{1}{2}\log\cos\phi.$$

Now we can write the requested limit as

$$\lim_{x\to 0} x^{-2} \lim_{r\to -1^+}\sum_{n=1}^{\infty}\frac{r^n}{n}\left[1 - \frac{\sin^2 nx}{n^2x^2}\right]\sin^2 n\phi.$$