Mathematical Models of Games of Chance: Epistemological Taxonomy and Potential in Problem Gambling Research

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Introduction

What is the current place of mathematics in problem gambling research, and how can mathematics contribute toward minimizing the harmful effects of excessive gambling? In this paper, I answer the first question; then I draw upon the main principles and propose further research in the matter of the second question.

First, mathematics is strongly connected to gambling through the mathematical models[1] underlying any game of chance. Games of chance are developed structurally and physically around abstract mathematical models, which are their mere essence, and the applications within these mathematical models represent the premises of their functionality. For instance, within statistical models, the house edge is ensured through precise calculations regarding expected value; if such calculations were not possible, the game would never run. Since in the research, treatment, and prevention of problem gambling we cannot separate the gambler from the game he plays, it follows that an optimal psychological intervention cannot disregard mathematics. Call this the gambling-math indispensability principle (Bărboianu, 2013b).

Determinants of the decision to gamble include not only the gambler's biological and psychological constitution, but also the structural characteristics of the gambling activity itself (Griffiths, 1993), among which games' structures are strictly related to the mathematical models of the games. Games' structures directly influence their outcomes in an idealized mathematically-modeled way - for instance, outcome volatility (see Turner, 2011) - and the behavior of outcomes is determinant for gamblers' decisions.

A second premise is the specificity of the gambling addiction through the goals of the player and the monetary reward. Although addiction is a pathological issue (and thus a medical one), the existence of the goal of winning distinguishes gambling addiction from other types of addiction and relates it to mathematics.

Thus far, the interventions involving mathematics to problem-gamblers were limited to *didactical* interventions, either school based or in experimental research. Past

[1] I will refer to *mathematical model* throughout this paper not according to a general use, or to a dictionary/encyclopedic definition, but to its specific defintion within epistemology of applied mathematics, namely the ensamble formed by 1) empirical context idealized as relational structures; 2) structure-preserving maps from the empirical structures to the mathematical structures within the governing (mathematical) theory; 3) interpretations of the empirical objects and relationships between them within the mathematical structures and of the mathematical formal resuls back within the empirical context. Such sense based on structural analogy expressed in mathematical terms (iso/homo-morphic structures) is different from the sense of "model" used in other disciplines, for instance psychology.

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studies on the impact of a mathematical didactic intervention with gamblers, testing whether learning about mathematics of gambling does change gambling behavior, were mainly empirical (see Abbott & Volberg, 2000; Gerstein et al., 1999; Hertwig et al., 2004; Lambros & Delfabbro, 2007; Pelletier & Ladouceur, 2007; Steenbergh et al., 2004; Williams & Connolly, 2006; Peard, 2008; Turner et al., 2008a, 2008b). The content of most of the teaching modules fell within *Introduction to and Basics of Probability and Statistics*, covering definition and properties of probability, basics of descriptive and inferential statistics, discrete random variables, expected value, classical probability distributions, and central limit theorem. The modules were packed with examples and applications from games of chance and had lessons dedicated to demystifying mathematically the common gambling fallacies. These studies have yielded contradictory, non-conclusive results, and many of them tended unexpectedly to answer *no* to the hypothesis that gamblers receiving such specific mathematical education show a significant change in gambling behavior after the intervention. These studies are problematic from the standpoint of the experimental setup in three important areas: sampling, evaluation, and testing of hypotheses (Bărboianu, 2013a); either of these issues may provide an explanation for the contradictory results. Aside from these issues, the following question arises: What mathematical knowledge would an optimal teaching module contain, with respect to the intended effect of limiting excessive gambling? In other words, what is missing (if anything) in the current didactic interventions? As I will show in this paper, the mathematical models and the act of mathematical modeling in gambling hold a *potential* for providing at least a partial answer to this question through further research.

The main goals of this paper are: turning attention of the researchers in this field on the mathematical models underlying games of chance and the quantifiable gambling activity; an objective classification of these models by epistemological criteria and evidentiating a current disinterest on a particular category of models (the functional ones) in favor of the statistical models; detecting a potential of all these mathematical models to contribute in problem gambling prevention and treatment (as generating a particular kind of knowledge – the epistemic knowledge – related to and deriving from these models and the act of mathematical modeling).

I begin by identifying two main categories of mathematical models – statistical & probabilistic and functional – which represent games and any quantifiable gambling activity; then I discuss the prevalence of models of the first category in the interest of all parties involved in the study of gambling – researchers, game producers and operators, and players.

I then go on to analyze the content categories and basic structure of gambling-mathematics knowledge available to be taught, which is identifiable around the mathematical models. I call attention to the epistemic knowledge, which is attached to the mathematical concepts per se, but also to the functions of a mathematical model, especially that of *representation*. Next, I argue for the potential of this epistemic knowledge in both didactical and cognitive interventions to gamblers. For the clinical cognitive interventions, I suggest the use of the *reduction-to-models* principle in conjunction with the *facing-the-odds* principle for an objective representation (of games of chance and gambling) as basic mathematical models free of gaming risk factors[2], begging for further theoretical and empirical research within psychology which to establish their effectiveness and actual implementation.

[2] In the sense of the definition from note 1, "free of" in the mathematical-modeling context refers to the idealization for the various purposes within the mathematical model.

Mathematical models of games of chance and gambling

Games of chance are developed in their physical consumer-ready form on the basis of mathematical models, which stand as the premises of their existence and represent their physical processes (Bărboianu, 2013b). As an example (of physical-process representation), roulette betting is represented as follows:

A roulette (complex) bet is defined as a finite family $B=(A_i, p_i, s_i)_{i\in I}$, where A_i is a subset of the set *R* of roulette numbers, which allows a single placement according to the configuration of the roulette table (such as for straight, split, corner column, color bet, etc.), p_i is the payout of A_i, and $s_i > 0$ is the stake of the placement A_i. All bets *B* so defined form the bet space. Then, for each bet *B*, a profit function W_B is defined as follows: $W_B : R \to \mathbf{R}$, $W_B(e)=\sum_{i\in I} s_i[1_{A_i}(e)+1_{A_i}(e)p_i-1]$, where **R** is the set of real numbers and 1_A is the characteristic function of set *A*. Variable $e\in R$ stands for the outcome of a game. The value of function W_B is called the profit of bet B (Bărboianu, 2007). For each bet B, function W_B may take negative or non-negative values. The profit function expresses the net amount the player wins or loses after the spin as a result of the player's bet (applying the convention that profit can also be negative, that is, a loss). This formal system is part of a mathematical model of the roulette game (there is more than one model involved in the game of roulette) representing some of the physical processes of betting, namely chip placements and their overall financial result after the spin. Using the mathematical structure of the bet space and the properties of the profit function, a relation of equivalence between bets can be defined; in the real world, the relation represents a natural financial equivalence between bets. The inferred properties of the equivalence of the bets in the mathematical model yield criteria of optimization of roulette betting with respect to personal money and time management[3].

Mathematical models can additionally represent gambling systems related to several plays, several games, and generally any quantifiable gambling activity. For instance, still from roulette, the Martingale system (keeping the same bet and raising its stake with the same multiplier successively until the first success) is modeled through a geometrical progression whose partial sums of terms obey a certain inequality, which in fact ensure an overall positive profit when a bet is won. In the Martingale example, not only do we have a *representation* of a physical process (the described betting system) through a mathematical model, but also a mathematical explanation (through the same model) of the trust this roulette system is granted by gamblers[4].

Probabilistic and statistical models and functional models

Games of chance and gambling as a quantifiable activity are represented in applied mathematics through specific mathematical models which can be distinguished through two main categories with respect to the purposes they serve, which I call:

1. Probabilistic and statistical models
2. Functional models

While probabilistic and statistical models serve for the applications related to the games' outcomes occurring under conditions of uncertainty, functional models serve to

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[3] This mathematical model of a roulette bet can be generalized and adapted further to other types of bets specific to other games.

[4] Still, the Martingale system fails, in certain specific instances, against practical factors such as consuming the entire fund available before the winning bet, low profit rates, or generally against the probability laws, which always favor the house over the player in the long run, no matter the gambling system used.

represent the physical systems and processes that make the games actually function as well as for applications related to the functioning.

As examples, computing the probability of hitting a specific number or at least a specific quantity of winning numbers at a specific lottery is workable within a *probabilistic model*, which assumes establishing the right probability field within which to work with the appropriate discrete probability distribution. Computing statistical means and errors (in the statistical sense) is workable within a *statistical model* by establishing the distribution of the random variable which describes the outcome and using mathematical means and measures such as expected value, deviation, dispersion, or variance. Card movement in poker is described through a *functional model* dealing with combinations of symbols and values specific to cards. Roulette complex bets are represented in a functional model as elements of a mathematical structure with vectorial and topological features. Paylines of a multiline slot machine are represented as lines in a Cartesian grid or paths in a graph, and their mutual independence is described through topological properties, all these still within a functional model (Bărboianu, 2013c).

Below are the purposes and mathematical governing theories[5] involved in each of the two categories of mathematical models used in gambling.

1. Probabilistic and statistical models. Purposes: quantification of the gambling uncertainty, measuring risk and possibility, prediction, computation of the parameters characterizing games, of the means and statistical errors, providing practical statistics (collections of data), optimization, providing strategy and optimal play. Governing theories: Measure Theory, Probability Theory, Mathematical Statistics, Real Analysis, Decision Theory.

2. Functional models. Purposes: description of the gaming processes and of the functioning of the games, optimization, providing strategy and optimal play, providing the necessary theoretical support for the probabilistic and statistical models. Governing theories: Set Theory, Combinatorics, Number Theory, Algebra, Topology, Geometry, Graph Theory, Real Analysis.

A game and the gambling activity related to that game are represented by several mathematical models, each of them serving a purpose for the applied mathematician. This multimodel feature holds even for the same subcategory of models. For instance, computing the strength of a poker hand in terms of probabilities assumes establishing several different probability fields for the events related to one's own hand and those events of type "at least one" related to the opponents' hands, which means a different probabilistic model for each application. Every game of chance is represented by models from both the first and second categories, even though a model from the latter category may be trivial. This happens because outcomes and uncertainty are specific to games of chance by definition (thus explaining the existence of first-category models) and every probabilistic/statistical model needs a functional model in order to ensure the grounding mathematical structures necessary for the governing theories of the probabilistic/statistical model to be applied. For example, any probability computation within a probabilistic model needs *a priori* a grounding model representing the gaming events to be measured, which must belong to a Boolean structure, and this latter model is a functional one.

Prevalence of the probabilistic and statistical models.

There is a prevalence of models of the first category in the interest of all parties involved in the study of gambling – researchers, game producers and operators, and players. This prevalence is explainable first through the fact that these models provide

[5] Applying to *existent* games of chance. Other mathematical theories may be included in the future with the development of other types of games as well as with the development of pure mathematics itself.

measures, estimations, and predictions for the financial results of the gambling activity, which in turn generate the most important indicators for the commercial aspect of the phenomenon. These models actually provide game producers and operators with a mathematical "guarantee" that a certain game can be run with no risk of ruin for the house over the long run; for the players, probabilities and statistical indicators are the most important mathematical criteria in making gaming decisions. Second, games of chance have simple processes of functioning. For commercial reasons, they are designed to be as undemanding as possible with straightforward sets of rules and short timeframes of the sessions; according to this uncomplicated design, the functional models representing them are usually simple, sometimes trivial (unlike other types of games – for instance strategy games such as chess – whose complexity is modeled through richer mathematical models). There are also exceptions to this functional-model-simplicity rule – that is, games whose apparently simple functioning hides complex mathematical models. Such is the case with roulette betting, in which complex bets are represented as elements of a mathematical structure generating vectorial and topological spaces and classes of equivalence within which various further applications can be developed. This exception applies also to multiline slots, where probability applications related to groups of paylines rely on a representation of the display as part of a discrete mathematical structure (Cartesian grid or graph) generating a metric space (and implicitly a topology), within which properties such as connection, neighboring, and independence are defined to serve the probabilistic models (Bărboianu, 2013c).

This prevalence of the models of the first category in the study of gambling is also reflected in the content of the existent courses of gambling mathematics – either experimental or school based – where gambling is presented exclusively as a plain, direct application of Probability Theory and Mathematical Statistics. As I shall argue further in this paper, considering *all* the mathematical models behind a game of chance – even having those of the second category limited to the functions of representation and description – as well as the act of modeling itself can enhance any didactical mathematical intervention or a psychological cognitive intervention.

Epistemic knowledge and mathematical knowledge

Representation and explanation, along with *prediction* are the main functions of a mathematical model in physical sciences, among other functions like description, optimization, measuring, approximation, or simulation (Newton & al, 1990; Morton, 1993; Saatsi, 2011), and those functions also apply to the study of games and gambling. Mathematical modeling is the main mean of inference in science, and modern to contemporary accounts of the role of mathematics in scientific explanation of physical phenomena have argued for its indispensability (see Quine, 1981; Colyvan, 1998; Baker, 2009; Saatsi, 2011). The inference based on mathematical models, which is the core method of scientific reasoning, is possible first because mathematics is a rich source of structures, and second because of the representation function of a mathematical model, which allows mathematical structures to be recognized as embedded in the physical world. Such inference works in three steps (immersion, derivation and interpretation, where the first and third steps assume establishing mappings from the empirical setup to a convenient mathematical structure), in terms of Bueno and Colyvan's (2011) inferential conception of mathematical modeling, an extension of Pincock's (2004) mapping account. The mapping account establishes isomorphic or homomorphic relations[6] between mathematical structures from within a mathematical theory and mathematical structures recognized in the idealized physical system (Pincock, 2004; Bueno & Colyvan, 2011). This iso/homo-morphic feature of the mapping account

[6] In algebraic categories, a *homorphism* is a structure-preserving map in terms of relations between the objects of the structures. If the map is bijective, it is called *isomorphism*.

justifies epistemologically the inference based on the mathematical model. The practical execution of the three steps above is the object of *applied mathematics*, and the setup of the mappings from steps 1 and 3 is called *mathematical modeling.*

Mathematical modeling and the inference based on models, although they provide the central method of investigation in science, are subject to epistemic questions, criticisms, and claims related to ontology, causality, truth, and trustworthiness. For instance, modeling assumptions (which revert to idealizing the physical system) are generally false relative to a standard governing theory, and the need for a model usually arises from some unmanageability of this governing theory. These assumptions are far from arbitrary, being made on the basis of physical intuition, inductive reasoning, and intelligent guesswork (Morton, 1993). Bas van Fraasen (2004) treats the mathematical model as a mediating model of convenience, something we adopt only because we are not capable of managing the empirical facts in their totality in any other way.

These topics illustrate just a small portion of the subjects of debates among philosophers of science regarding epistemic issues raised by mathematical models and the inferences based on them. All knowledge related to the applicability of the mathematical models to physical systems, including the act of modeling itself and the debating of the issues above, is *epistemic knowledge*[7].

Coming now to the *mathematical knowledge* that a mathematical model carries, this knowledge is not limited to the mathematical theories where the derivation step takes place, (such knowledge can be categorized as pure *mathematics knowledge*), but this mathematical knowledge extends to the applications within these theories (particularizations), the final applications in terms of the initial physical system (interpretations), which can then be categorized as applied *mathematics knowledge*.

Gambling-mathematics knowledge and related epistemic knowledge

The structure of gambling-mathematics knowledge can be identified entirely within and around the mathematical models of games and gambling, because all the mathematical actions, facts, and information related to gambling are derived exclusively from these models.

First, the mathematical (governing) theories or parts of them addressed specifically to gambling (that the models use in the derivation step) contain formal systems, propositions, and theorems, and the logical flow between them yields the theoretical results of any gambling-mathematics application. All this theoretical knowledge is *pure-mathematics knowledge*, which is accessible to and manageable by gamblers with a high level of mathematical education.

Second, the general applications that set up the specific theoretical framework and limit the mathematical theories to the modeling needs of each game in part (the theoretical support of what we usually call the *mathematics of roulette, of blackjack, of poker,* and so on) are included in *applied-mathematics knowledge*, still theoretical, and still accessible to and manageable by gamblers with a level of mathematical education lower than the pure-math one, but still high.

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[7] Epistemic knowledge is that knowledge acquired from the field of Epistemology, which gathers the various theories of knowledge (what is knowledge, how knowledge is acquired, how concepts like belief and truth relate to the objective knowledge and what is their place in the rational judgements, etc.). Epistemology has a diffuse border with Philosophy of Science. For instance, the epistemology of mathematical models and modeling falls within the latter. Naturalists such as W.O. Quine take epistemology to be a branch of psychology (Quine, 1969). The concerns of the epistemology of mathematical models and modeling are: rational justifications of the use of mathematical models, issues of construction and representation, functions of the models, <u>the relationships of the models as abstract things with the real world</u>, the acquirement of knowledge through models, the nature of this knowledge, various other issues concerning the dichotomy abstract-empirical. Each category of mathematical model raises specific epistemological issues.

Finally, the results of practical applications specific for each game or any quantifiable gambling activity, obtained through particularizations of the general applications and computations, yielding practical and numerical results such as game parameters, approximations, odds, statistical indicators, directions of optimization and recommendations, still fall within the *applied-mathematics knowledge* category; however, they are accessible to and manageable by gamblers with lower levels of mathematical education, under the condition of being properly defined, presented, and explained, not simply delivered as plain information.

This pure and applied gambling mathematics knowledge (hereafter abbreviated as PAGMK) may be acquired by gamblers via instructional means (gambling-mathematics courses in schools or private organizations, experimental interventions) or specific media (books, journals, magazines, and websites). Of course, the structure and content of such gambling-mathematics resources vary, and the existent courses usually follow the curricula of regular Introduction to/Basics of Probability and Statistics classes in post-secondary schools, with the focus on applications of these disciplines in gambling. Regarding the latter category of resources, the plethora of popular literature on gambling mathematics published in the last two decades raises the necessity of critical selection and professional certification when it comes to a recommendation (Turner et al., 2003). This is because such commercial publications serve various scopes, wide or narrow, and the information delivered by them can be useful even if incomplete, but also misleading (as is the case with most of the so-called “how to win” or “strategy” titles, in which the mathematical information or systems described are not mathematical at all). In any case, credentials of the authors and publishers of such publications must be verified.

Coming now to the epistemic knowledge related to gambling mathematics, it can be found surrounding PAGMK, as mathematical knowledge is itself one of the objects of epistemology (Colyvan, 2012). Every mathematical concept and mathematical act generates epistemic knowledge regarding ontology, truth, explanation, interpretation, and critique, and this also applies to the particular field of PAGMK. A relevant example is the epistemic knowledge attached to the concept of *probability*, which is the central mathematical concept around which gambling mathematics revolves. It happens that probability is one of the mathematical concepts highly predisposed to philosophical interpretation, questions regarding existence, and how probability represents the empirical world of uncertainty (Dubucs, 2010).

Past and current courses on the mathematics of gambling, either in the curricula of some schools or in experimental didactic interventions in problem gambling research, lack epistemic components. I argue in a forthcoming paper that these missing components would enhance any didactic intervention or psychological intervention—or a mixture of these—with respect to achieving the goal of preventing and limiting excessive gambling. Furthermore, this issue suggests a possible explanation for the contradictory results of previous empirical studies on whether *learning* about the mathematics of gambling does, in fact, change gambling behavior. For the current paper, I restrict my arguments toward the side of clinical cognitive interventions.

How mathematical models were considered thus far in gambling

Statistical and probabilistic models are of interest to players on one hand, because of their function of prediction under uncertainty and because they offer mathematical measures and indicators that are seen as the only stable “certain” facts in an uncertain experimental environment such as gambling. On the other hand, those models concerned researchers of various profiles, for the same reason, and also for the reasons that motivated their prevalence I talked about in a previous section of this paper.

Among such researchers, problem gambling specialists developed and tested mathematical teaching modules applicable to gamblers. These modules include knowledge attached to these particular models. The aim of such knowledge was to limit excessive gambling through a better understanding of the probabilistic/statistical facts of the games and gambling. Such modules, however, barely touched knowledge attached to the functional models and lacked entirely the epistemic knowledge related to PAGMK.

It worth mentioning here two contributions to the cognitive assets of gamblers with respect to gambling mathematics, namely the Harvard Medical School's Division on Addictions' module called *Facing the Odds: The Mathematics of Gambling and Other Risks* (Shaffer & Vander Bilt, 1996) and Centre for Addiction and Mental Health's curriculum called *Youth making choices: Gambling Prevention Program* (Turner & al., 2010), two middle-school curricula on probability, statistics and number sense designed to increase young people's mathematics literacy while concurrently preventing or reducing their participation in risky and potentially addictive behaviors. More oriented on practical (empirical) statistics, works of K. Harrigan (2009), N. Turner (1998, 2011), and Turner & al. (2003) on the statistical indicators of the games in relation to their mathematical design and real outcomes also contributed to these assets.

Facing the odds is in fact a principle that has been tested. Probability, as the central concept of gambling mathematics, ought to receive greatest attention. However, *facing* is not enough and the studies mentioned at the end of the previous section (following the publication and application of the module) confirmed that premise. More knowledge related to the concept of probability is required beyond the standard curricula, which points out that if we teach the mathematics of gambling with the goal of changing gambling behavior, we must do it in a different manner, with respect to both content and approach, from the customary methodology.

Another category of researchers (statisticians, economists, psychologists, etc.) focusing on statistical and probabilistic models are those who study the games with regard to their outcomes and economic impact, and the gambling phenomenon as a social issue.

Functional models were ignored in the field of problem gambling; they concerned mostly the gaming mathematicians who model and study the games and gambling in depth, as well as the math-inclined players, who see the inner mathematical facts of the games as necessary mainly to improve playing skills and strategy and to get knowledge on optimal play[8].

The potential of the mathematical models in psychological cognitive interventions

I identified two main elements related to mathematical models and the act of modeling that will *potentially* have positive effects in a non-standard psychological intervention using them: 1) the epistemic knowledge attached to PAGMK; and 2) the placement of the gaming risk factors with respect to the mathematical models, for whose potential I argue in this section.

The potential of the epistemic knowledge attached to PAGMK

Epistemic knowledge is active knowledge, in sense that it is followed in the learner's mind by mental processes not limited to the object of learning, but relating to other objects, going into adjacent disciplines, and interposing the personal psychological background of established knowledge, beliefs, and methods of investigation. Such processes are questioning, critiquing, transforming, adapting, and accommodating with contradictorv knowledge or beliefs previously assimilated. Thus, epistemic knowledge

[8] Both terms *strategy* and *optimal play* do not have an absolute character in sense of practical results. Any gaming strategy is relative by definition, as relating to the personal goals of the player, which can also be subjective. An optimal play can be mathematically defined and provided for several games, however the optimal play does not guarantee any ultimate winning, as the practical results still obey the general probability laws, and the optimal play involves criteria based on probability. The same limitation applies for a strategy.

implies psychological processes of a higher intensity and number than in the case of the standard taught knowledge, and consequently the inclusion of epistemic knowledge in a cognitive intervention will establish a highly active psychological background for the student, who will become more receptive and connected to the psychological components of the intervention. This is a first general argument in favor of epistemic knowledge.

Another general argument is the reinforcement of critical thinking. Critical thinking is recognized in the literature on scholastic education as an important positive factor in the educational assessment and the quality of learning (see Ennis, 1993; Pithers & Soden, 2000). A similar attribute of critical thinking also applies in problem gambling (Turner et al., 2008): studies have found that prevention of excessive gambling, based on didactical/cognitive interventions focusing on improving the gambler's critical thinking on issues related to randomness, probability, and statistics, with the effect of correcting gambling misconceptions and fallacies, tends to have practical effects expressed through the decrease of gambling activity of the subjects of the interventions (Gerstein et al., 1999; Abbot & Volberg, 2000). Taught mathematics develops over time a mathematical thinking of the student (Jaworski, 2006; Aizikovitsh & Amit, 2010) and mathematical thinking is *per se* critical thinking in ongoing development (see Tall, 1991, 1995; Dreyfus & Eisenberg, 1996).

Given the wide critical component of epistemic knowledge (Ichikawa & Steup, 2013) (through ideas related to pragmatism, skepticism, justification, scientific truth, etc.), the inclusion of epistemic knowledge in a didactical/cognitive intervention on gambling mathematics would just enlarge the volume of knowledge that can develop critical thinking during the process of learning. Since critical thinking is something that is *developed* rather than directly taught (Siegel, 1989), enlarging the volume of taught knowledge through the addition of epistemic knowledge would consequently increase the development of critical thinking in both intensity and duration (of the sessions).

In particular, cognitive interventions using epistemic knowledge attached to the concept of probability – for instance, familiarizing the learner with other scientific *interpretations* of probability[9], and with the epistemic knowledge connected to the scientific *design* of the mathematical probability and of each of the scientific interpretations of probability individually and connected to the way these formal accounts represent the physical reality – will potentially[10] help the gambler to distinguish between probability and physical possibility, by taking probability as a mathematical or pseudo-mathematical non-absolute measure of possibility, understanding that probability measures only events belonging to ideal structures that only approximate reality through representation. This distinction between probability and possibility is also facilitated by the knowledge about the existence of various scientific interpretations of probability, which can play a role in correcting gambling fallacies and adopting critical thinking.

There is a natural non-formal interpretation and development of the concept of probability for every gambler, just as there is for any other formalized concept, resulting from the natural cognitive processes of observation, generalization, and abstraction (Murphy, 2004). This personal interpretation, which depends on each psychological profile, might match partially or totally – consciously or not – one of the

[9] To cite the classical scientific interpretations of probability, we mention Laplace's account for *classical probability*, Carnap's for *logical probability*, Ramsey's and de Finetti's for *subjective probability*, Venn, de Finetti and von Mises for *frequential probability*, and Popper's and Pierce's for *propensitic probability*.

[10] The potential is given by the mere content of such knowledge. The *real* aimed effects of an intervention which includes epistemic knowledge would manifest only if certain *practical* conditions are met regarding the entire design of the learning module and process (content, structure, differentiation, and accommodation with the student's level of mathematical knowledge, etc.). The real effects can be detected only through empirical studies on the students after the intervention.

scientific interpretations of probability. In this natural interpretation, the gambler takes probability to be an objective measure, while assigning it for a personal (subjective, non-mathematical) degree of belief in the occurrence of an event. Depending on the gambler's mathematical education, but also on his/her psychological profile, the concepts of measure and event may be interpreted in various senses, some of them pseudo-mathematical (for example, one can take the measure as obeying only the additivity axiom, or one can refer to event, even though said event does not belong to a Boolean structure, and still assign it a probability).

For instance, there are gamblers who take the probability of an event as being the *relative frequency* of the occurrence of that event in past experimental setups which were conducted under the same conditions; this perception matches the *finite frequentist interpretation* of probability. It can be assumed that these are persons who rely on practical (past) statistics, implicitly on physical evidence (since relative frequency refers to experiments that have already *happened* and whose outcomes were registered; by contrast, mathematical probability refers to infinite series of experiments, most of them in the *future*). Such a gambler might be a prudent person and might tend to trust the odds calculators based on partial simulations. (Empirical study is required to test this hypothesis.) Despite prudence, the finite frequentist gambler might be predisposed to gambling fallacies and erroneous beliefs more than other profiles when s/he establishes his/her own degree of belief on the basis of relative frequency. For instance, a fallacy occurs when assigning a "probability" – based on partial simulations given by an odds calculator whose base of records is still small – to a rare/frequent (in sense of mathematical probability) event; this approach can yield a big difference (positive or negative) between that "probability" and the real mathematical probability.

As another example, there might be gamblers who see the probability of an event as a *physical* property of the experimental setup, that is, its tendency to yield the occurrence of that event or a certain relative frequency of such an occurrence. In other words, the fact that the (mathematical) probability of a die to land a certain number is 1/6 (or, in frequentist terms, the relative frequency of that number landing approximates 1/6) is seen as a physical property of the die itself (or of the entire experimental setup of throwing the die). This view matches the *propensitic interpretation* of probability. It can be assumed that these are analytical realistic persons, who believe in determinism, and who tend to relate the abstract and the empirical to unification. When basing a gambling decision on a physical tendency, such a gambler might not make a big distinction between probability and physical possibility (unless educated in this epistemic matter) and thus might establish a stronger personal degree of belief in the occurrence of an event than would other profiles, which makes him/her predisposed to gambling fallacies. For instance, s/he might have a stronger belief in the "compensation rule" of the Martingale system, according to which a long series of consecutive identical outcomes should stop at some point sooner or later for the relative frequency to match the probability of that outcome.

The scientific interpretations of probability are just a small part of the epistemic knowledge attached to the concept of probability and the above examples show that such knowledge could influence positively a psychological methodology designed within problem gambling interventions. Other epistemic knowledge attached to other mathematical concepts from gambling mathematics may have other influences (especially that of expected value), but this is not the place to develop this issue; the point I want to make is that there exists a potential for such positive influences, which can be developed in the psychological practice.

Mathematical models free of gaming risk factors as objects of representation

In the process of mathematical modeling, the games are idealized through removal of their physical components unessential for the modeling purposes, and reduced to pure mathematical structures. Without such removal, the mathematical structures would not be recognized any more in the investigated physical system. In such idealizations, only the parametric design of the game or machine is essential. This physical surplus that is removed includes (but is not limited to) cases, external design, interface, commands, motion of the mechanical components, and visual effects.

Determinants of the decision to gamble include not only the gambler's biological and psychological constitution, but also the structural characteristics of the gambling activity itself (Griffiths, 1993). Such structural characteristics include addictive elements *of the games* – also categorized as risk factors in the literature – among which the *near-miss effect* and the *illusion of control* are the most important (harmful). Other less interactive risk factors would be the sound and image effects coming from the physical design of the games. Obviously, the illusion of control through a stop button (at slot machines) or similar devices for other types of games and sound/image effects fall within the physical surplus removed in the idealization required by the *mathematical* models describing the games. Regarding the near-miss effect (as well as the near-miss as a gaming phenomenon), it holds the same status with respect to a mathematical model. That is because, although the near-miss (and consequently the near-miss psychological effect) is a direct consequence of the parametric design of the game (yielding a relatively high probability of an outcome near a winning one and implicitly a high frequency of such event), the purposes of a probabilistic model are to provide quantifications under uncertainty of the *winning* outcomes and not particular losing ones. There are forms of near-miss in all games of chance. For example, in roulette, what I call a physical near-miss is the landing of the ball on the wheel (physically) near a winning number. In that trivial case, the parametric configuration of the game yields the same probability of occurrence for any outcome, so any near-miss is not favored. Depending on the type of game, the parametric configuration is or is not manipulated by the game producer such that to yield frequent near-misses. As a manipulation example we have slots, as a non-manipulating one we have card games, where the near-miss is just accidental. In this latter case, we don't have the same probability for all outcomes (combinations of cards), so some near-miss events may be more frequent than other events; however this situation is not created with any intention. In all cases, the near-miss event has no place in a probabilistic or statistical model, due to the rational arguments of the applied mathematician and the principle that all that is irrelevant for the modeling purposes is ignored, in order to have the right mathematical structures available. For example, a probabilistic model of a slot game will show the parametric configuration of the reels (number of stops and the exact distribution of symbols on each reel) and then the computed probabilities of the winning combinations of symbols. Nobody will see in this model near-miss combinations because nobody needs them; the near-miss combinations will hold the same status with the rest within a functional model (based on sets of combinations), although they have different (higher) probabilities in comparison to the rest of the combinations; all that count within the probabilistic model are the probabilities of the winning combinations. A gambler sees and is influenced by near-miss in the real game; the gambler won't see it in the mathematical model of that game if properly and effectively counseled in this matter, as well as s/he won't see any visual effects or any other gaming risk factors.

Having the risk factors outside the mathematical models, the potential of a cognitive intervention based on knowledge related to mathematical models could manifest in both a didactical intervention – which would focus only on the mathematical facts of the games and induce defocus on the gaming risk factors – and a clinical intervention developed so as to create for the patient an objective *representation* of the games s/he plays as pure mathematical structures free of risk factors..

Such interventions based on the principle of *reduction to models* would be based largely on functional models, not just on the probabilistic and statistical ones, the latter still remaining important. The reduction-to-models principle would be a completion of the facing (and interpreting)-the-odds principle, and their expected positive effects on decreasing excessive gambling can be tested only through empirical studies following this research.

Conclusions

In this paper I have made a structural analysis of the mathematical and epistemic knowledge available for gamblers, as being attached to the mathematical models of games of chance and the act of modeling. I have put in evidence two categories of these models and have shown that only one of these two has received interest from problem gambling researchers. I argued that considering functional models and the epistemic knowledge attached to gambling mathematics can enable the potential of such knowledge in both didactical interventions and clinical cognitive interventions, with respect to the aim of limiting excessive gambling.

Further research, both theoretical and empirical, is necessary in various directions for establishing the following:

- whether the learning principles presented are practicably applicable to gamblers, either didactically or clinically;
- what would be the optimal content and structure of the teaching modules and therapists' modules enhanced with such principles;
- whether the potential of this non-standard knowledge will actually manifest, given the various levels of education of the gamblers;
- whether such knowledge can be reduced to warning messages and how such warning messages differ from the warning messages specific to other addictions.

Problem gambling research has been focused from its very beginning on the biological/psychological make-up of the individual (Griffiths, 2009), and this is explainable at least partially through the fact that this domain is mainly comprised of medical doctors and psychologists. The proposed research moves the focus to the games themselves, by appealing to the unexploited potential of mathematics. Given that gambling is a complex domain which involves not only gamblers, but also the related gaming environment, the interdisciplinary research employing the full and direct contribution of mathematics is unavoidable.

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