

Prototypes, Poles, and Topological Tessellations of Conceptual Spaces

Abstract. The aim of this paper is to present a general method for constructing natural tessellations of conceptual spaces that is based on their topological structure. This method works for a class of spaces that was defined some 80 years ago by the Russian mathematician Pavel Alexandroff. Alexandroff spaces, as they are called today, are distinguished from other topological spaces by the fact that they exhibit a 1-1 correspondence between their specialization orders and their topological structures.

Recently, Rumfitt used a special case of Alexandroff's method to elucidate the logic of vague concepts in a new way. Elaborating his approach, it can be shown that the color spectrum - and similarly other conceptual spaces - give rise to classical systems of concepts that have the structure of atomic Boolean algebras.

Rumfitt is not the first of having investigated conceptual systems of conceptual spaces. Since some time, due to Gärdenfors and other authors, conceptual spaces have become a popular modeling tool in cognitive psychology, artificial intelligence, linguistics, and philosophy. The core idea of the conceptual space approach is that concepts can be represented geometrically as regions in similarity spaces. Using prototypes and the underlying metric of similarity spaces Gärdenfors constructs a geometrical discretization of conceptual spaces by so-called Voronoi tessellations. As will be shown in this paper, his account can be interpreted as a geometrical version of Alexandroff's topological construction. More precisely, a Voronoi tessellation à la Gärdenfors is extensionally equivalent to a topological discretization constructed by Alexandroff's method. Even more, Rumfitt's as well as Gärdenfors's constructions turn out to be special cases of an approach that works for a more general class of spaces, namely, for weakly scattered Alexandroff spaces. This suggests that the class of these Alexandroff spaces may provide a convenient framework for conceptual spaces in general.

Key words: Conceptual Spaces, Polar Spaces, Alexandroff Spaces, Prototypes, Topological Tessellations, Voronoi Tessellations, McKinsey Axiom.

1. Introduction. The aim of this paper is to present a general method for constructing natural tessellations of conceptual spaces that is based on their topological structure.

This method works for a class of spaces that was defined some 80 years ago by the Russian mathematician Pavel Alexandroff. Alexandroff spaces, as they are called today, are distinguished from other topological spaces by the fact that they exhibit a 1-1 correspondence between their specialization orders and their topological structures.

Recently, Rumfitt used a special case of Alexandroff's method to elucidate the logic of vague concepts in a new way (cf. Rumfitt (2015, chapter 8). Elaborating his approach, it can be shown that the color spectrum - and other similar conceptual spaces - give rise to classical systems of concepts that have the structure of atomic Boolean algebras. More precisely, concepts are represented as regular open regions of an underlying conceptual space in such a way that something is subsumed under a certain concept iff it is represented by an element of the conceptual space that is maximally close to the prototypical element p of the space that is used to define the concept.

Rumfitt is not the first of having proposed to study conceptual systems defined via prototypes of conceptual spaces. Since some time, Gärdenfors and his collaborators have shown that conceptual spaces provide a very useful modeling tool in cognitive psychology, artificial intelligence, linguistics, and philosophy. The core idea of the conceptual space approach is that concepts can be represented geometrically as regions of metrically structured similarity spaces. Using prototypes and the underlying metric of similarity spaces Gärdenfors constructs a geometrical discretization of conceptual spaces by so-called Voronoi tessellations. As will be shown in this paper his account can be interpreted as a geometrical version of Alexandroff's topological construction. More precisely, a Voronoi tessellation à la Gärdenfors is extensionally equivalent to a topological discretization constructed by Alexandroff's method. Even more, Rumfitt's as well as Gärdenfors's constructions turn out to be special cases of an approach that works for a more general class of spaces, namely, for weakly scattered Alexandroff spaces. The Boolean algebras of regular open regions of these spaces yield natural

atomic tessellations of them. This suggests that the class of Alexandroff spaces may provide a convenient framework for conceptual spaces in general.

A conceptual space may be considered as a continuous realm of possible stimuli or experiences. A task of cognitive science is to understand how this realm is rendered a structured space that can serve as a base for the elaboration of a more or less detailed classification of stimuli or experiences.¹ For this task, Gärdenfors and his collaborators have proposed to employ so called Voronoi tessellations based on a Euclidean structure and a finite set of prototypes of the underlying conceptual space:

A Voronoi tessellation based on a set of prototypes is a simple way of classifying a continuous space of stimuli. The partitioning results in a discretization of the space. The prime cognitive effect is that the discretization speeds up learning. ... [A] Voronoi tessellation is a cognitively economical way of representing information about concepts. Furthermore, having a space partitioned into a finite number of classes means that it is possible to give names to the classes. (Gärdenfors (2000, 89))

As we want to show the topological essence of Gärdenfors's (and Rumfitt's) discretizations of continuous conceptual spaces is based on a structural correspondence between order structures and topological structures that was discovered by Alexandroff in the 1930s.

A Voronoi tessellation of a conceptual space uniquely determines a topological tessellation that is extensionally equivalent to a regular open tessellation constructed by Alexandroff's method. The constructions of Rumfitt and Gärdenfors boil down to very special cases of Alexandroff's construction. Thus it makes sense to consider the far-reaching generalizations that are suggested by Alexandroff's original construction. This

¹ With some good will, Carnap's „attribute spaces“ may be considered as forerunners of conceptual spaces in Gärdenfors's sense. For a more detailed comparison of the similarities and differences of the two approaches see Sznajder (2016, section 6). In particular, in contrast to attribute spaces, the regions of conceptual spaces that correspond to concepts are non-homogeneous in the sense that some (generating) points are more prototypical than others.

generalization suggests that Alexandroff spaces should be considered as a natural topological habitat of conceptual spaces. They provide the natural framework for conceptual spaces that are interested in those concepts that are empirically meaningful. This claim can be explicated as follows:

- (1) An empirically meaningful concept has to be stable in the sense that, if it applies to a situation x , it also applies to small variations x' of x . This stability is accompanied with a certain conceptual vagueness. Stable concepts do not hold with absolute precision. They are unable to single out empirical objects with absolute precision. This should be considered as virtue rather than a vice. Otherwise, concepts would no longer be empirically applicable.
- (2) Arbitrary conjunctions of stable concepts should be stable. This requirement is the expression of a reasonable conceptual modesty. Otherwise, we could get rid of the inherent vagueness of empirical concepts by purely logical means, namely, by piling up more and more concepts that eventually result in an absolutely precise conceptualization of reality.
- (3) Topologically, the requirements (1) and (2) can be satisfied by the requirement that a conceptual space S has the structure of an Alexandroff topological space $(S, \mathcal{O}S)$ such that concepts are characterized as elements of the Boolean algebra \mathcal{O}^*S of regular open subsets of S .

In this paper we rely on a topological account of concepts, i.e., concepts are characterized as topologically well-formed regions of a topological space $(X, \mathcal{O}X)$, namely, as elements of \mathcal{O}^*X . This is similar to Gärdenfors's geometrical approach of conceptual spaces according to which concepts are represented by convex regions of an underlying conceptual space that is usually assumed to be an Euclidean space endowed with a metric. In comparison with the geometrical account the topological one to be presented

in this paper is more austere insofar as one and the same topological structure may give rise to different geometrical structures.

The outline of this paper is as follows: The preparatory section 2 recalls the basic topological concepts needed in the rest of this paper. Then, in section 3, a special class of conceptual spaces is discussed in detail, namely, “polar spaces” and their topological structures. These spaces were recently defined by Ian Rumfitt in his book *The Boundary Stones of Reason* (Rumfitt 2015) as a convenient framework for dealing with the logic of vague concepts. This class of spaces may be considered as an elementary but paradigmatic example for the general topological account elaborated in this paper. In section 4 the relation between the topologically defined tessellations of polar spaces and the better known geometrically defined Voronoi tessellations of Gärdenfors’s conceptual spaces is explicated. Section 5 deals with the topology and order structure of an especially subclass of Alexandroff spaces, namely, weakly scattered Alexandroff spaces. It is shown that this class of spaces may be considered as the most general class of spaces that gives rise to well-behaved classifications and categorizations of objects.

2. Elements of Topology. In this section we recall some topological concepts that are necessary for understanding the rest of this paper. Let us start right-on with the basic definitions:

(2.1) Definition. Let X be a set with power set PX . A topological space is a relational structure (X, OX) with $OX \subseteq PX$ satisfying two requirements that $\emptyset, X \in OX$ and finite intersections and arbitrary unions of elements of OX are elements of OX . The elements of OX are called the open sets of X . The set-theoretical complements CA of open sets A are

called closed sets, and the set of closed sets is denoted by CX .² If there is no danger of confusion a topological space (X, OX) is simply denoted by X . ♦

If X has more than one point several different topological structures OX exist on X . In particular, there are two extreme topological structures (X, O_0X) and (X, O_1X) defined by $O_0X := \{\emptyset, X\}$ and $O_1X := PX$. The topology (X, O_0X) is called the indiscrete topology on X , and the topology (X, O_1X) is called the discrete topology. With respect to set-theoretical inclusion \subseteq all topological structures (X, OX) on X lie between these two (rather uninteresting) topologies: $O_0X \subseteq OX \subseteq O_1X$. ♦

Topological structures (X, OX) can be defined in many equivalent ways. For our purposes particularly useful is the definition in terms of closure operators cl or interior kernel operators int . These operators have to satisfy the so-called Kuratowski axioms:

(2.2) Definition. Let X be a (non-empty) set with power set PX . A topological closure operator is an operator $PX \xrightarrow{cl} PX$ satisfying the four requirements (1) - (4) below. Dually, a topological interior kernel operator is a map $PX \xrightarrow{int} PX$ satisfying the four requirements (1)* - (4)*:

(1) $cl(A \cup B) = cl(A) \cup cl(B)$.	(1)* $int(A \cap B) = int(A) \cap int(B)$.
(2) $cl(cl(A)) = cl(A)$.	(2)* $int(int(A)) = int(A)$.
(3) $A \subseteq cl(A)$.	(3)* $int(A) \subseteq A$.
(4) $cl(\emptyset) = \emptyset$.	(4)* $int(X) = X$.

Closure operators cl and interior kernel operators int are interdefinable: $cl(A) = \mathbf{C}int\mathbf{C}(A)$, and $int(B) = \mathbf{C}cl\mathbf{C}(B)$, \mathbf{C} being the set-theoretical complement with respect to X .

Every topological closure operator cl uniquely defines a topological structure (X, OX) and viceversa: Given cl define the class of open sets $OX \subseteq PX$ by $OX := \{B; B = \mathbf{C}cl(A); A \subseteq X\}$.

² A set may be open and closed. For instance, the sets \emptyset and X are open and closed for any topological structure (X, OX) .

Dually, given a topological interior operator int the corresponding topological structure OX is defined by $OX := \{A; A = \text{int}(A), A \subseteq X\}$.

For every $A \subseteq X$ the boundary $\text{bd}(A)$ of A (in (X, OX)) is defined as $\text{bd}(A) = \text{cl}(A) \cap \text{cl}(\complement A)$. ♦

For the rest of this paper the following topological concepts are essential:

(2.3) Definition. Let (X, OX) be a topological space.

- (i) An element $x \in X$ is isolated (with respect to (X, OX)) iff $\{x\} \in OX$. The set of isolated points of X is denoted by $\text{ISO}(X)$.
- (ii) A subset $A \subseteq X$ is dense in X iff $\text{cl}(A) = X$.
- (iii) (X, OX) is weakly scattered iff $\text{ISO}(X)$ is dense in X , i.e., $\text{cl}(\text{ISO}(X)) = X$.
- (iv) (X, OX) satisfies the McKinsey axiom iff $\text{int}(\text{cl}(A)) \subseteq \text{cl}(\text{int}(A))$ for all $A \subseteq X$. ♦

(2.4) Lemma. Let (X, OX) be a topological space. The following assertions are equivalent:

- (i) X is weakly scattered.
- (ii) X satisfies the McKinsey axiom.
- (iii) For all $A \subseteq X$ iff $\text{bd}(\text{bd}(A)) = \text{bd}(A)$.

Proof: Proposition (2.1), Proposition (2.4) (Bezhanishvili, Mines, Morandi (2003)). ♦

(2.5) Proposition. Let (X, OX) be a topological space, cl and int its closure and interior kernel operator, respectively. An open subset $A \in OX$ is regular open iff $A = \text{int}(\text{cl}(A))$. The set of all regular open subsets of X is denoted by O^*X . O^*X is well known to be a complete Boolean algebra. There is a canonical map $OX \xrightarrow{j} O^*X$ defined by $j(A) := \text{int}(\text{cl}(A))$ and an inclusion $O^*X \xrightarrow{i} OX$ such that $j \cdot i = \text{id}_{O^*X}$. ♦

The following definition is essential for the class of topological spaces we will deal with in this paper, namely, Alexandroff spaces.

(2.6) Definition (Specialization Order of a Topology). Let X be a set. A quasi-order on X is a binary relation \leq such that for all $x, y, z \in X$ (i) $x \leq x$, (ii) $x \leq y, y \leq z$ imply $x \leq z$. If the relation \leq also satisfies (iii) $x \leq y$ and $y \leq x$ imply $x = y$ (anti-symmetry) the quasiorder is called a partial order. Then the structure (X, \leq) is called a poset.

A topological structure $(X, \mathcal{O}X)$ defines a quasi-order (X, \leq) on X by $x \leq y := x \in \text{cl}(y)$. This quasi-order is called the specialization quasi-order of $(X, \mathcal{O}X)$. The set of maximal elements of (X, \leq) is denoted by M .³♦

For traditional topological spaces like Euclidean spaces $(E, \mathcal{O}E)$ the specialization order (E, \leq) is trivial, i.e. $x \leq y$ iff $x = y$. In contrast, for the topological structures to be considered in the following, the specialization order is non-trivial and of great importance. Under appropriate conditions, the topological structure $(X, \mathcal{O}X)$ of a topological space can be reconstructed from its specialization order (X, \leq) .

(2.7) Definition (Upper topology defined by a partial order (X, \leq)). Let (X, \leq) be partial order. For $A \subseteq X$ define $\uparrow A := \{x; a \leq x \text{ for some } a \in A\}$. The set $\uparrow A$ is called the upper set of A . Analogously, the lower set $\downarrow A$ of A is defined by $\downarrow A := \{y; y \leq a \text{ for some } a \in A\}$. The upper topology $(X, \mathcal{O}X)$ corresponding to (X, \leq) is defined by $\mathcal{O}X := \{\uparrow A; A \subseteq X\}$.♦

(2.8) Definition. A topological space X is an Alexandroff topological space iff arbitrary intersections of open sets are open. Equivalently (invoking the Kuratowski axioms) a topological space X is an Alexandroff space iff arbitrary unions of closed sets are closed.

³ For the Euclidean topology $(\mathbf{R}, \mathcal{O}\mathbf{R})$ the specialization order \leq is trivial, i.e., \leq is just the identity $=$, and one has $M = \mathbf{R}$. For the Alexandroff space $(N, \mathcal{O}\mathbf{N})$ (see (2.8)(iv)) one has $M = \emptyset$.

Clearly, if X is Alexandroff for every $A \subseteq X$ there exist a smallest open set $V(A)$ containing A , namely, the intersection of all open sets that contain A . ♦

In order to distinguish between different classes of topological spaces separation axioms have turned out to be a convenient means. A few important instances of separation axioms relevant for those topological spaces considered in this paper are given in the following definition.

(2.9) Definition (Separation axioms). Let X be a topological space, and x, y different points of X .

- (i) X is a T_0 -space iff for every $x \in X$ there exists an open set $a \in \mathcal{O}X$ such that either $x \in a$ and $y \notin a$, or $x \notin a$ and $y \in a$.
- (ii) X is a $T_{1/2}$ -space iff if every point $x \in X$ is either open or closed.
- (iii) X is a T_1 -space iff if every point $x \in X$ is closed.
- (iv) $(X, \mathcal{O}X)$ is a T_2 -space iff there exists open sets $a \in \mathcal{O}X$ and $b \in \mathcal{O}X$ containing x and y , such that $x \notin b$ and $y \notin a$. ♦

(2.10) Examples of Topological Spaces and their Separation Axioms.

- (i) If X has more than one point, the indiscrete topology $(X, \mathcal{O}_0 X)$ is not T_0 . The discrete topology $(X, \mathcal{O}_1 X)$ is T_2 .

(ii) The standard Euclidean topology \mathbf{OR} of the real line \mathbf{R} is generated by open intervals $(a, b) = \{x; a < x < b\}$. Two distinct points x and y can be separated by open intervals $U(x)$ and $U(y)$ that are disjoint to each other. Hence $(\mathbf{R}, \mathbf{OR})$ is a T_2 -space. All points are closed, no point is open.

- (iii) Let $\mathbf{Q} \subseteq \mathbf{R}$ be the subset of rational numbers. The set \mathbf{Q} as a subset of \mathbf{R} inherits a topology from \mathbf{R} by declaring a subset A of \mathbf{Q} to be open iff $A = \mathbf{Q} \cap B$, for some $B \in \mathbf{OR}$. The topological space $(\mathbf{Q}, \mathcal{O}\mathbf{Q})$ has many clopen (= open and closed) subsets.

(iv) Let (\mathbf{N}, \leq) be the set of natural numbers endowed with their natural order \leq . A topological space $(\mathbf{N}, \mathbf{O}\mathbf{N})$ is defined by stipulating that \emptyset, \mathbf{N} and the sets $\uparrow n := \{m; n \leq m\}$ are open for each $n \in \mathbf{N}$. Then $(\mathbf{N}, \mathbf{O}\mathbf{N})$ is an Alexandroff space that satisfies T_0 but not T_1 . No point of $(\mathbf{N}, \mathbf{O}\mathbf{N})$ is open, the only closed point of $(\mathbf{N}, \mathbf{O}\mathbf{N})$ is 0.

(v) An Alexandroff space $(X, \mathbf{O}X)$ satisfies T_1 iff it is discrete. ♦

(2.11) Proposition. The topological space $(X, \mathbf{O}X)$ is a T_0 -Alexandroff space iff its quasi-specialization order (X, \leq) is a partial order.

Proof. Check the definitions. ♦

The equivalence between the topological structure $(X, \mathbf{O}X)$ and the order-theoretical structure of the specialization (quasi-)order (X, \leq) renders modal logic a rich source of Alexandroff spaces:

(2.12) Proposition. Let (W, \leq) an S4 Kripke frame of possible worlds W such that \leq is reflexive, transitive, and anti-symmetric. Then the upper topology on W defines a T_0 -Alexandroff space $(W, \mathbf{O}W)$. ♦

Now we can formulate the main formal result of this paper:

(2.13) Theorem. Let $(X, \mathbf{O}X)$ be a weakly scattered Alexandroff space, $\text{ISO}(X)$ the set of isolated elements of (X, \leq) . Then the Boolean lattice \mathbf{O}^*X of regular open subsets of X is an atomic Boolean algebra, i.e., $\mathbf{O}^*X = 2^L$, L being the set of atoms of \mathbf{O}^*X . More precisely, the elements of L are the regular open sets $\text{int}(\text{cl}(p))$, $p \in \text{ISO}(X)$. The Boolean algebra \mathbf{O}^*X defines a regular open atomic tessellation of the space X by $X = \{\text{intcl}(p); p \in \text{ISO}(X)\} = \text{VL}$. ♦

We will deal with this theorem on two different levels of generality: First, as an especially simple case, we will consider so-called polar spaces, recently introduced by Rumfitt (2015). These spaces are characterized by specialization orders (X, \leq) of depth 2. Then we consider general weakly scattered Alexandroff spaces (X, OX) .

3. The Topological Structure of Polar Spaces. In this section we explicate the topological structure of the polar spaces (cf. Rumfitt (2015, chapter (8.4))). In particular, we show that these spaces are very simple Alexandroff spaces.

Rumfitt presents his approach by way of example, discussing the polar topology of the well-known color spectrum as a conceptual space of color experiences. He does not, however, calculate its topology in any detail. This will be undertaken in this section.

Let X be a set of coloured objects that is to serve as the underlying set of a conceptual space for color experiences. We are looking for a discretization of X , i. e., a partition of X that allows us to classify color experiences in different categories. Usually, this is done with the aid of certain paradigmatic or prototypical experiences of red, of blue, yellow and so on. (cf. Gärdenfors (2000)). Rumfitt argues that the task of classification of colors is best conceptualized as a procedure based on a comparison with certain color experiences that are to be considered as paradigmatic or prototypical:

The spectrum enables us to attach senses to colour terms not because it shows boundaries, but because it displays colour paradigms or poles. Sainsbury likens colour paradigms to ‘magnetic poles exerting various degrees of influence: some objects cluster firmly to one pole, some to another, and some, though sensitive to the forces, join no cluster’. ... I prefer a simpler analogy which likens paradigms to gravitational poles, that is, massive bodies. If a small body is sufficiently close to a gravitational pole, it will be drawn towards it, rather as we are drawn to classify as red those objects that are sufficiently close in colour to a paradigm, or pole, of red. Rumfitt (2015, 236)

The essential mathematical structure to be extracted from this example is the following: Assume that there is given a set X of objects to be classified, and a subset P of X to be considered as a set of distinguished elements that are “paradigmatic” or “prototypical” objects. In the terminology of Rumfitt (2015) they are called poles. These poles are used to classify the ordinary objects.

More precisely, the set of poles is used to endow X with the structure of a topologically structured conceptual space. Let us assume that for every object $x \in X$ there is a non-empty set $m(x) \subseteq P$ of poles p such that all the $p \in m(x)$ are maximally close to x . It may well be that for some x the set $m(x)$ comprises more than one element, but $m(x)$ is assumed to be always non-empty. It seems plausible to assume that for a pole $p \in P$ one has $m(p) = \{p\}$. This is to be interpreted as the assumption that for a paradigmatic object p the unique maximally close paradigmatic object to it is p itself. The attribution of maximally close poles to each object $x \in X$ can be conceived as a function $X \rightarrow 2^P$ satisfying two conditions:

$$(3.1) \quad (i) \text{ For all } x \in X \ m(x) \neq \emptyset \quad (ii) \quad \text{For all } p \in P \ m(p) = \{p\}.$$

The function m is called a pole distribution for X and is denoted by (X, m, P) . The requirements (3.1) (i) and (ii) guarantee that poles do some classificatory work by classifying the elements of X according to the poles that are maximally close to them: First, poles are distinguished from non-poles as those elements that are, so to speak, “self-classifying”, i.e., $m(p) = \{p\}$.⁴ Secondly, pole distributions (X, m, P) define a topology on X with the help of the following interior kernel operator $2^X \rightarrow 2^X$:

(3.2) Proposition. Let (X, m, P) be a pole distribution, $A \subseteq X$ and define the operator by

⁴ Note that it is not required that m is defined with the aid of a fully-fledged metric on X as Gärdenfors seems to assume.

$$(3.2)_1 \quad x \in \text{int}(A) \text{ iff } x \in A \ \& \ \forall_{p \in P} (p \in m(x) \Rightarrow p \in A)$$

Then int is a topological interior kernel operator that defines an Alexandroff topology. Informally formulated, $x \in \text{int}(A)$ iff $x \in A$ and moreover all poles that are maximally close to x also belong to A . In other words, the interior of A comprises those elements of A whose maximally close poles also belong to A . Thereby elements of $\text{int}(A)$ “have no connection to elements outside A ”.

Equivalently, the topology corresponding to a pole distribution (X, m, P) can be defined by the closure operator cl

$$(3.2)_2 \quad x \in \text{cl}(A) \text{ iff } x \in A \text{ or } \exists_{p \in P} (p \in A \text{ and } p \in m(x)).$$

That is, the closure $\text{cl}(A)$ of a set A comprises the members of A together with all objects for which at least one of their maximally close poles is in A . In other words, $\text{cl}(A)$ comprises all elements of A that are in A or have at least connection to elements of A . The topological space (X, OX) defined by the operators int or cl is called the polar space of the distribution (X, m, P) .♦

Proof. The proof that int and cl are topological operators consists in a routine check that these operators satisfy the Kuratowski axioms (2.2), see Rumfitt (2015, 243 - 246). A closer inspection of the definitions $(3.2)_1$ or $(3.2)_2$ reveals that they even satisfy the stronger Alexandroff condition (2.6).♦

Although Rumfitt introduces the topology (X, OX) given by a pole distribution (X, m, P) he does not describe the topology in any further detail. In particular, he does not mention that (X, OX) is an Alexandroff topology. Nor does he show explicitly that O^*X is atomic. All this could have been proven easily with the formal apparatus he has at his disposal. Endowed with the topology defined by a pole distribution (X, m, P) the color spectrum (X, OX) is a very special Alexandroff space, namely, (X, OX) is a T_0 -space such

that the singletons $\{p\} \subseteq P$ are open and all the singletons $\{x\}$ for $x \in X - P$ are closed.⁵

More precisely, the following proposition obtains:

(3.3) Proposition. Let (X, OX) be the topological space defined by a pole distribution (X, m, P) . Then X is a T_0 -space. For all $x \in X$, the smallest open set that contains x is the set $V(x) := \{x\} \cup m(x)$. For the elements $p \in P$ and $x \in X - P$ one calculates:

$$\begin{aligned} \text{int}(p) &= \{p\} & \text{int}(x) &= \emptyset & V(x) &= \{x\} \cup m(x) \\ \text{cl}(x) &= \{x\} & \text{cl}(p) &= \{x; p \in m(x)\}, & \text{int}(\text{cl}(p)) &= \{x; \{p\} = m(x)\} \end{aligned}$$

Proof. In order to prove these assertions one just has to check the definitions. Be it sufficient to prove that $V(x) = \{x\} \cup m(x)$. According to the definition of the interior operator int one has

$$y \in \text{int}(\{x\} \cup m(x)) \Leftrightarrow y \in \{x\} \cup m(x) \ \& \ \forall p(p \in m(y) \Rightarrow p \in \{x\} \cup m(x)).$$

Clearly every element in $\{x\} \cup m(x)$ satisfies this condition. On the other hand, any smaller set, properly contained $\{x\} \cup m(x)$, does not satisfy the condition. For two different elements x and y $V(x)$ and $V(y)$ are different. Hence X is a T_0 -space. Thus the sets $\{\{x\} \cup m(x), x \in X\}$ form a unique minimal base for the topology on X .

Proposition (3.3) provides the topological data that can be used to characterize Rumfitt's polar spaces as a special class of topological spaces (cf. Bezhanishvili, Esakia, and Gabelaia (2003) and Bezhanishvili, Mines, and Morandi (2003)).

(3.4) Proposition. Let (X, OX) be a polar space defined by a pole distribution (X, m, P) .

Then the following holds:

- (i) X is weakly scattered.⁶

⁵ A space having this property is sometimes said to satisfy the $T_{1/2}$ -axiom located halfway between T_0 and T_1 .

⁶ One can easily show that polar spaces are not only weakly scattered but even scattered. This will, however, not be needed in the following.

- (ii) X satisfies the McKinsey axiom.
- (iii) For all $A \subseteq X$ $\text{bd}(\text{bd}(A)) = \text{bd}(A)$.

Proof. By (2.4) (i) – (iii) are equivalent. Hence it is enough to prove just (i). By (3.3) the set $\text{ISO}(X)$ of isolated points of X is just the set P of poles. Hence every x is contained in at least one $\text{cl}(p)$ by definition of the polar distribution m . Hence $X = \bigcup_{p \in P} \text{cl}(p) \subseteq \text{cl}(\text{ISO}(X))$, i.e., X is weakly scattered. ♦

(3.5) Proposition. Let (X, OX) be the polar topological space defined by (X, m, P) . The Boolean algebra O^*X of regular open sets of (X, OX) is atomic. The atoms of O^*X are the sets $\text{intcl}(p) = \{x; \{p\} = m(x)\}$. In O^*X two atoms $\text{int}(\text{cl}(p))$ and $\text{int}(\text{cl}(p^*))$ generate the regular open set $\{x; \{p, p^*\} \cap m(x) \neq \emptyset\}$.

Proof. Check the definitions and apply (3.3). ♦

In section 5 we will show that the “correct” generalization of polar spaces, for which the Boolean algebra O^*X is still atomic, is the class of weakly scattered Alexandroff spaces.

4. Topological and Geometrical Tessellations. The aim of this section is to discuss several types of tessellations of conceptual spaces that can be used to set up conceptual classifications based on geometrical and/or topological structure of conceptual spaces. Due to the fact that for polar spaces (X, OX) the Boolean algebra O^*X is atomic the resulting tessellation is particularly simple and essentially unique.

(4.1) Definition. A regular open tessellation T of a topological space (X, OX) is a set of disjoint regular open subsets $A_i \in \text{O}^*X$ such that $X = \text{VA}_i$. The supremum VA_i of the A_i taken here in O^*X , not in OX . If the sets A_i are atoms of O^*X the tessellation T is called an atomic tessellation. Points of X that are not in the interior of any cell A_i are said to belong to the boundary $\text{bd}(T)$ of the tessellation T . ♦

(4.2) Examples.

(i) Let (X, OX) be a topological space, $A \in O^*X$, $A \neq \emptyset$, X . Denote the Boolean complement of A in O^*X by A^* ($A^* = \text{int}(\mathbf{c}A)$) Then $\{A, A^*\}$ is a regular open tessellation of X with the two cells A and A^* . The boundary $\text{bd}(T)$ of T is just the boundary $\text{bd}(A)$ ($= \text{bd}(A^*)$).

(ii) More generally, let A_1, \dots, A_n be n regular open subsets of X . The set $\{A_1, \dots, A_n\}$ generates a regular open tessellation of X that has 2^m cells, $m \leq 2^n$. ♦

These examples show that regular open tessellations of topological spaces abound. A more interesting problem is to prove whether or a space has a (unique) atomic regular open tessellation. As is well known Euclidean spaces (E, OE) lack atomic tessellations. On the other hand, polar spaces have atomic regular open tessellations:

(4.3) Proposition. Let (X, m, P) be a pole distribution for X . Then the topological space (X, OX) has a canonical regular open atomic tessellation T by the atoms of the Boolean algebra O^*X , i.e., $T = \bigvee_{p \in P} \text{int}(\text{cl}(p))$.

Proof. Let (X, OX) be defined by (X, m, P) . As proved in (3.8) the Boolean algebra O^*X is isomorphic to the powerset 2^P . Thus, the atoms of O^*X generate a regular open tessellation T of X . The atoms of T are the regular open sets $\text{int}(\text{cl}(p)) = \{x; \{p\} = m(x)\}$, $p \in P$. Elements of X to which more than one pole is maximally close are located in the boundary of T , i.e., $\text{bd}(T) = \{x; \{p, p'\} \subseteq m(x) \text{ for some } p, p' \in P \text{ and } p \neq p'\}$. ♦

As mentioned already in the introduction, topologically defined polar spaces a la Rumfitt are not the only example of conceptual spaces. Probably better known are the geometrically defined conceptual spaces a la Gärdenfors that have become an increasingly popular modeling tool in many empirical sciences and also in philosophy.

Hence, after having dealt with polar spaces in some detail, it may be expedient now to deal in some more detail with that class of conceptual spaces, and to explain how Rumfitt's and Gärdenfors' conceptual spaces may be conceived both as Alexandroff spaces.

Gärdenfors' conceptual spaces are similarity spaces. Similarity spaces are multi-dimensional structures with a metric defined on them. Distances in the space are meant to measure similarity: the smaller the distance between objects the larger the similarity between them. The distances in a similarity space are meant to measure dissimilarities between objects. The metrics most frequently encountered are the familiar Euclidean distance and the so-called Manhattan distance that simply adds up distances along all the dimensions of a space.

Neither the shape of a similarity space nor the choice of a metric defined on it are arbitrary. Usually, both the structure and the metric of a similarity space are determined on the basis of a large set of similarity ratings, often obtained in an empirical study, which serve as input for one of several related statistical techniques that turn similarities into geometric objects.

Now, let us consider in some more detail, the arguably most prominent class of tessellations, namely, the so-called Voronoi tessellations (cf. Gärdenfors (2000), Decock and Douven (2015), Zenker and Gärdenfors (2015)). Voronoi tessellations may be characterized as geometrical tessellations. They are defined by using the underlying geometrical structure of Euclidean conceptual spaces, more precisely, their metrical structure. The general definition is as follows:

Given a set P of two or more but a finite number of distinct points in the Euclidean plane, we associate all locations in that space with the closest member(s) of P with respect to the Euclidean distance. The result is a tessellation of the plane into a set of the regions associated with members of P . We call this tessellation the planar ordinary Voronoi diagram generated by P , and the regions constituting the Voronoi diagram ordinary Voronoi polygons.

(Okabe et al. (1992, p. 44))

As an illustration of this general definition let us consider the simplest Voronoi tessellation of the Euclidean plane E :

(4.4) Example. Let E be the Euclidean plane endowed with a Cartesian coordinate system (x, y) . Choose two points $p_L = (-1, 0)$ and $p_R = (1, 0)$. Then the Voronoi tessellation in the sense of the above quoted definition taking p_L and p_R as generators is given by the two open cells

$$L := \{(x,y); d((x,y), p_L) < d((x,y), p_R)\} \quad R := \{(x,y); d((x,y), p_R) < d((x,y), p_L)\}$$

The boundary of this tessellation is given by the topological boundary $bd(L) = bd(R)$ of the two half-planes L and R . It is the line

$$\{(x,y); d((x,y), p_L) = d((x,y), p_R)\} = \{(x, y); x = 0\}. \blacklozenge$$

Following the procedure of (4.5) this construction is easily generalized to a partition of the plane into at most 2^n regular open polygons (and their boundaries), n finite number. Let p_1, \dots, p_n a finite set of different points. Then a general Voronoi tessellation may be conceived of as the result of the intersection of the $n!/2! (n-2)!$ pairs of half-planes each defined by the bisectors of pairs (p_i, p_j) of different points $p_i \neq p_j$ in such a way that the plane is divided into convex open cells together with their boundaries.

Clearly, a geometrically defined Voronoi tessellation of Euclidean space gives rise to a regular open tessellation in the sense of (4.1): By construction all the open Voronoi cells are disjoint to each other, and, due to fact that they are convex (cf. Gärdenfors (2000, 88) they are not only open but even regular open. By the very definition of the Voronoi cells the points not in the interior of a cell are those points that have equal distance to two (or more) paradigmatic points p_i . Hence they are located on the topological boundary of those cells.

A Voronoi tessellation based on the metrical structure of a Euclidean space E not only

defines a topological tessellation in the sense of (4.1), it also yields a polar distribution (E, m, P) :

(4.5) Proposition. Let T be a Voronoi tessellation of the Euclidean plane E defined by a finite set P of prototypes p_1, \dots, p_n . Then a pole distribution (X, m, P) is defined as follows: Take the Voronoi generators p_1, \dots, p_n as the set P of poles of a pole distribution $X \xrightarrow{m} 2^P$ defined as

$$m(x) = \{p_i; x \in \text{cl}(\langle p_i \rangle); \langle p_i \rangle \text{ the open Voronoi cell defined by the generators } p_i\}.$$

The resulting polar topological space (E, m, P) defines a regular open tessellation of E by the regular open atoms $\text{int}(\text{cl}(p_i))$ of O^*E such that the $\text{int}(\text{cl}(p_i))$ are just the Voronoi cells $\langle p_i \rangle$. More precisely, $x \in E$ is contained in the interior of a cell $\text{int}(\text{cl}(p))$ iff $m(x) = \{p\}$. ♦

By this recipe, the cells of the resulting topological tessellation of E coincide with the cells of the Voronoi tessellation of E . Moreover, the geometrically defined boundary of the Voronoi tessellation coincides with the topological boundary. In sum, every geometrically defined Voronoi tessellation based on a finite set of prototypical points gives rise to a uniquely defined topological tessellation defined by (X, m, P) that is extensionally equivalent.

Compared with the geometrical construction of a Voronoi tessellation of the plane a topological tessellation requires much less structural presuppositions. This is a conceptual advantage insofar, as certain problems caused by the presence of representational artifacts disappear. For example, for Euclidean spaces there are many different metrical structures that define the same underlying topological structure.⁷ With respect to these different metrics, one and the same set P of prototypical points may

⁷ A prominent case is provided by the family of Minkowski-metrics $d_i(x, y)$ for $1 \leq i \leq \infty$. This problem is briefly discussed for $i = 1$ (taxi cab or Manhattan metric) and $i = 2$ (ordinary Euclidean metric in Gärdenfors (2000, chapter 3.9). The Euclidean metric d_2 has the structural advantage that the cells of its Voronoi tessellations turn out to be convex.

give rise to different Voronoi tessellations. Which one should be chosen as the right one? Another problem that may be attributed to the too specific mathematical apparatus used for the definition of a Voronoi tessellation T of a conceptual space concerns the boundary $bd(T)$ of T . It has been dubbed the “thickness problem” (cf. Douven et al. 2013).

The “thickness problem” may be explicated as follows. By the very construction of Voronoi tessellations, the boundaries of cells are “thin” compared with the interiors of the cells since they are lines consisting of points that have equal distances to two (or more) prototypical points. Douven et al. (2013) rightly point out that this assumption for most conceptual spaces is not very plausible. For instance, for the conceptual space of the color spectrum the boundary, say, between “red” and “orange” is defined by the points that have exactly the same distance from the prototypical points of “red” and “orange”. Empirically, this does not make much sense. What should it mean that a certain shade of color has the same distance from a prototypical “red” and a prototypical “orange”? Moreover, in the general case, there is no reason to assume that boundaries are “thin” compared with the regular open cells of the Voronoi tessellation. Douven et al. (2013) propose to overcome this shortcoming by the introduction of “collated Voronoi diagrams” that arise as the result of projecting similar ordinary Voronoi diagrams onto each other such that their boundaries define a blurred and more or less “thick” area thereby taking into account the vagueness of concepts and their boundaries.

For topological tessellations no “thickness” problem arises, since they do not distinguish between “thick” and “thin” as geometrical tessellations do (in an artificial way). The following example shows that the topological approach easily deals with tessellations with cells whose boundaries are “thicker” than the cells themselves:

(4.6) Example. Let X be the set $\{\alpha, \mathbf{N}, \omega\}$, $\mathbf{N} = \{1, 2, \dots\}$ the set of natural numbers and α and ω two objects different from all elements of \mathbf{N} and from each other. Take $P = \{\alpha, \omega\}$ and define a pole distribution (X, m, P) by

$$m(i) = \{\alpha, \omega\} \text{ for } i \in \mathbf{N}, \quad m(\alpha) = \{\alpha\}, \quad m(\omega) = \{\omega\}$$

The corresponding topological structure (X, OX) to this pole distribution is given by

$$cl(\alpha) = \{\alpha\} \cup \mathbf{N}, \quad cl(i) = \{i\}, \quad cl(\omega) = \{\omega\} \cup \mathbf{N}$$

$$int(\alpha) = \{\alpha\}, \quad int(\mathbf{N}) = \emptyset, \quad int(\omega) = \{\omega\}$$

$$intcl(\alpha) = \{\alpha\}, \quad int(cl(\omega)) = \{\omega\}, \quad bd(\omega) = bd(\alpha) = \mathbf{N}.$$

The cardinality of the boundary of the regular open cells $\{\alpha\}$ and $\{\omega\}$ is infinite and thus much larger than the cardinality of the regular open cells $intcl(\alpha) = \{\alpha\}$ and $int(cl(\omega)) = \{\omega\}$. ♦

The example (4.6) shows that the topological approach has no difficulty to accommodate the “thickness of boundaries”. Topological tessellations are flexible enough to accept cells with boundaries that are intuitively much “thicker” than the cells they are boundaries of.

5. Weakly Scattered Alexandroff Spaces. Alexandroff topologies (X, OX) bring to the fore the order-theoretic features of topology by relating the topology OX to the specialization order (X, \leq) . The essential feature that distinguishes Alexandroff topological spaces (X, OX) from ordinary topological spaces is the existence of an “open hull” $V(A) \in OX$ for every $A \in PX$.

In this section we recall the necessary elements of the theory of Alexandroff topological spaces in order to show that weakly scattered Alexandroff spaces provide a natural generalization of polar spaces.

For quite a long time Alexandroff spaces did not find much attention in mathematics or elsewhere. This changed, when it became clear that topology may have a bearing on theoretical computer science. As it turned out, Alexandroff topological spaces and related structures became important, see Gierz & alii (2003), Goubault-Larrecq (2013).

Recall that an Alexandroff space X is defined as a topological space such that arbitrary intersections of open sets are open (and not only finite ones) (cf. (2.6)). Clearly, every topological space X having only finitely many elements is an Alexandroff space. In other words, Alexandroff topology becomes an interesting concept only for spaces of infinite cardinality. Thus, the Alexandroff topology of the color circle and similar conceptual spaces defined by prototype distributions (X, m, P) may qualify as an interesting Alexandroff topology (cf. Rumfitt (2015)).

Let (X, OX) be a polar space defined by (X, m, P) . The specialization order (X, \leq) is defined as $x < y$ iff y is a prototype of x , i.e., $y \in m(x)$. The smallest open subset that contains x is the set $V(x) = \{x, m(x)\}$. Thus, the polar topology OX is just Alexandroff topology defined by the specialization order (X, \leq) . In particular, $\uparrow x := \{x, m(x)\} = \{y; x \leq y\}$. Thus, for general Alexandroff spaces (X, OX) with specialization order (X, \leq) one has:

$$(5.1) \quad x \in \text{int}(A) := \uparrow x = \{y; x \leq y\} \subseteq A$$

Thus, Rumfitt's definition of the interior operator int for polar spaces is just a special case of the standard definition of int for general Alexandroff spaces in that $m(x)$ is replaced by $\uparrow x$. This corresponds to the standard definition for general topological spaces. In the case of an Alexandroff the existence of an open neighborhood $U(x)$ contained in $\text{int}(A)$ can be expressed more specifically that $\text{int}(A)$ must contain $\uparrow x$. Thus, Rumfitt's definition of int for polar spaces is just a special case of the general definition

of the interior operator in of Alexandroff spaces. Correspondingly, for a general Alexandroff (X, OX) space one obtains for its closure operator:

$$(5.2) \quad cl(A) = \{x; \downarrow x \cap A \neq \emptyset\} \quad \downarrow x := \{y; y \leq x\}$$

The simplest class of weakly scattered Alexandroff spaces are polar spaces defined by pole distributions (X, m, P) . More general weakly scattered Alexandroff spaces share many properties with polar spaces. Thus it seems justified to assert that weakly scattered Alexandroff spaces may be considered as the “right” generalization of polar spaces.

(5.3) Proposition. Let (X, OX) be a weakly scattered Alexandroff space with $ISO(X)$ the set of isolated points. Then $ISO(X)$ is the set of maximal points of the specialization order (X, \leq) . Moreover, $ISO(X)$ generates the filter DX of dense sets of X .

Proof. Let p be a maximal element of (X, \leq) . By definition of the Alexandroff topology the singleton $\{p\}$ is open, i.e., $\uparrow p = \{p\}$. Hence all maximal elements are isolated, and, vice versa, all isolated elements are maximal. Since (X, OX) is weakly scattered and Alexandroff one has $X = \bigcup cl(ISO(X)) = \bigcup_{p \in ISO(X)} cl(p)$. Thus, by definition of the Alexandroff topology all $x \in X$ there is at least one p such that $x \in cl(p) = \downarrow p = \{x; x \leq p\}$.

In order to show that $ISO(X)$ generates DX as a filter one argues as follows. Let A be a dense subset of X . Then $X = cl(A) = \bigcup_{a \in A} cl(a) = \bigcup_{a \in A} \downarrow a$. In order that $ISO(X)$ is contained in $cl(A)$ one must have for all $p \in ISO(X) \subseteq A \cup_{a \in A} \downarrow a$, since p is maximal in (X, \leq) . Hence, $ISO(X)$ is a minimal generator of DX as a filter. ♦

(5.4) Corollary. Weakly scattered Alexandroff spaces (X, OX) satisfy the McKinsey axiom, i.e., $bd(bd(A)) = bd(A)$ for all $A \subseteq X$. ♦

Due to (5.4), for weakly scattered Alexandroff spaces no problem of higher-order vagueness arises: Higher-order boundaries coincide with ordinary boundaries, i.e., $bd(A) = bd^n(A)$, for $n \geq 1$. As is well known, the validity of the McKinsey axiom is equivalent to the fact that the logic of weakly scattered Alexandroff spaces is S4.1.

With respect to tessellations general weakly scattered Alexandroff spaces behave very much like polar spaces: Every weakly scattered Alexandroff space (X, OX) has a canonical atomic regular open tessellation defined by the atoms of O^*X :

(5.5) Proposition. Let (X, OX) be a weakly scattered Alexandroff space with specialization order (X, \leq) , and $ISO(X)$ the set of maximal elements. Then O^*X is an atomic Boolean algebra with atoms $int(cl(p))$, $p \in ISO(X)$, as generators. Thereby one obtains a regular open atomic tessellation of X by $X = \bigvee_{p \in ISO(X)} int(cl(p))$.

Proof. Let $ISO(X)$ be the set of maximal elements of the specialization order (X, \leq) . By definition, for each $x \in X$ there is at least one $p \in ISO(X)$ such that $x \leq p$. Since the Alexandroff topology is the upper topology of the specialization order (X, \leq) the singletons $\{p\}$ are open, and the closures $cl(p)$ of $\{p\}$ are the down sets $\downarrow p := \{x; x \leq p\}$. The sets $int(cl(p)) := \{x; \uparrow x \subseteq \downarrow p\}$ for $p \in ISO(X)$ are atoms of O^*X : For different p, p^* the sets $intcl(p)$ and $intcl(p^*)$ are disjoint and regular open, since $int(cl(p)) \cap int(cl(p^*)) = intcl(\{p\} \cap \{p^*\}) = \emptyset$ due to the fact that the operator $intcl$ is a nucleus, i.e., it distributes over finite intersections.

That $intcl(p)$ is an atom in O^*X is seen as follows: Assume $x \in A = intcl(A) \subseteq intcl(p)$. Since $intcl(p)$ is open one has $\uparrow x \subseteq cl(p)$. This entails $x \leq p$ and therefore $p \in \uparrow x \subseteq A$. Hence $intcl(p) \subseteq intcl(A) = A$. Hence $A = intcl(p)$.

Now we prove that any regular open $A = \bigvee_{n \in M'} intcl(n)$ for some $M' \subseteq M$. Let $M_A := \{p; p \in A \cap M\}$. Clearly, $\bigvee_{n \in M_A} intcl(p) \subseteq A$. If we can show that $A \subseteq \bigvee_{n \in M_A} intcl(p)$ we are done.

Assume $x \in A$ and define $M_x := \{p; x \leq p \text{ and } p \in M\}$. Clearly $M_x \subseteq M_A$. Hence $x \in$

$\text{cl}(A_M)$. Assume $y \in \uparrow x$. This is the case iff $x \leq y$ and this entails $M_y \subseteq M_x$. So we obtain $\uparrow x \subseteq \text{cl}(A_M)$. This means $x \in \text{intcl}(A_M) = \text{intcl}(\bigcup_{p \in A_M} \{p\}) = \bigvee_{p \in A_M} \text{intcl}(p)$. This is to say $A \subseteq \bigvee_{p \in A_M} \text{intcl}(p)$. ♦

Being Alexandroff and weakly scattered is a sufficient reason for a topological space (X, OX) to ensure that O^*X defines a regular open atomic tessellation of X . These two conditions are not necessary, however. It is easy to find (non-weakly-scattered) Alexandroff spaces or (non-Alexandroff) weakly scattered spaces that have regular open atomic tessellations. In order to succinctly summarize the results of this paper the following diagram may be helpful:

G-Spaces \Rightarrow Polar Spaces \Rightarrow MK-Alexandroff Space \Rightarrow O*-Atomic Spaces

The terms of this diagram have the following meaning:

G-Spaces	=	Conceptual spaces with tessellations <u>geometrically</u> defined by prototypes.
Polar Spaces	=	Conceptual spaces with tessellations <u>topologically</u> defined by pole distributions.
MK A-Spaces	=	Alexandroff Spaces that satisfy the MK-axiom.
O*-Atomic Spaces	=	Spaces such that O^*X is atomic.

The arrows are to be read as follows: G-spaces define in a canonical way polar spaces due to the fact that their metric and the set of prototypes define a polar distribution. Polar spaces are a proper subclass of weakly scattered Alexandroff spaces. These spaces

have been proved to possess an atomic regular open tessellation, i.e., they are O^* -atomic spaces.⁸

6. Concluding Remarks. In this paper some general reasons have been put forward for the claim that conceptual spaces should be conceived of as being endowed with the topological structure of Alexandroff spaces. One reason is that Alexandroff topologies are an expedient conceptual device to take care of the important role that prototypes and paradigmatic elements play in human categorization. In the simplest case, this is already evidenced by polar spaces (X, OX) that are defined by pole distributions (X, m, P) . Weakly scattered Alexandroff spaces may be considered as a convenient generalization of this case since they preserve most of the useful properties exhibited by polar spaces, and on the other hand allow to get rid of the overly narrow framework that permits only one level of prototypes. Instead, in the case of weakly scattered Alexandroff spaces a partially ordered hierarchy of prototypes may occur. This enables us to deal with properties of different paradigmatic levels. If paradigms and prototypes play an important role in human categorization, and this seems to be the case, polar spaces, and, more generally, weakly scattered Alexandroff spaces provide a natural.

As already said in the introduction, the basic assumption of the conceptual spaces approach in cognitive science is that concepts can be usefully represented as well-formed subsets of a conceptual space that is structured in one way or other. The basic task of this approach is to find appropriate structures that allow to single out empirically useful concepts as structurally well formed. Topological structures are generally recognized as basic spatial structure for all kinds of spaces that show up in all realms of knowledge. Thus, it does not appear unreasonable to expect that topological structures

⁸ It should be noted, however, that there are Alexandroff spaces that are not weakly scattered but have nevertheless atomic O^*X , and there are weakly scattered spaces that are not Alexandroff but nevertheless are O^* -atomic spaces.

may also play a role in the theory of conceptual spaces. Topological concepts are sufficiently flexible to be adapted to various empirical necessities. It is a matter of empirical research to find out which topological structures for which type of conceptual spaces may be helpful.

Gärdenfors and others have argued that convex structures of Euclidean spaces are appropriate for singling out meaningful concepts. The resulting Voronoi tessellations depend on a rather special geometrical structure. This has the danger of generating structural artifacts that may lack empirical content. For instance, it may be difficult to justify empirically that a given conceptual space should be endowed with a specific metric and not with another one. For instance, the convexity of the Voronoi cells of a certain discretization of a conceptual space depends on a specific metric, the choice of a different metric would yield different cells. As another example of a problem generated by the too specific structure imposed on conceptual spaces is that problem that Douven et al. (2013) have called the problem of “thickness”: Assuming a Euclidean structure of conceptual spaces entails a very special structure of the boundaries of their concepts that is difficult to justify empirically. More precisely, for Euclidean conceptual spaces the boundaries of concepts suffer from “extensional anorexia”, so to speak, in that they are overly thin (cf. Douven et al. (2013)). Topological tessellations do not suffer from this illness, since discretizations of conceptual spaces based on rather austere topological means generate less representational artifacts than those that are based on a much richer geometrical structure.

Be this as it may topological structures are fundamental spatial structures, arguably even the most fundamental ones. Thus, if we agree with Gärdenfors’ thesis (Gärdenfors (2000, 262) that “to understand the structure of our thoughts ... we should aim at unveiling our conceptual spaces”, we should invest some efforts to understand the

topological structures of our conceptual spaces. The topology of Alexandroff spaces may be a useful tool for this purpose.

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Zur Kritik des zweiten Gutachters:

1. Die Bemerkung über “heriditarily ...” ist ebenfalls nur von mathematischer Bedeutung.
2. “Topological tessellation” kann etwas genauer expliziert werden. Es geht mir um die Analogie.