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Mathematical Models of Abstract Systems: Knowing abstract geometric forms

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Abstract. — Scientists use models to know the world. It is usually assumed that mathematicians doing pure mathematics do not. Mathematicians doing pure mathematics prove theorems about mathematical entities like sets, numbers, geometric figures, spaces, etc., they compute various functions and solve equations. In this paper, I want to exhibit models build by mathematicians to study the fundamental components of spaces and, more generally, of mathematical forms. I focus on one area of mathematics where models occupy a central role, namely homotopy theory. I argue that mathematicians introduce genuine models and I offer a rough classification of these models.

Résumé. — Les scientifiques construisent des modèles pour connaître le monde. On suppose, en général, que les mathématiciens qui font des mathématiques pures n'ont pas recours à de tels modèles. En mathématiques pures, on prouve des théorèmes au sujet d'entités mathématiques comme les ensembles, les nombres, les figures géométriques, etc., on calcule des fonctions et on résout des équations. Dans cet article, je présente certains modèles construits par des mathématiciens qui permettent d'étudier les composantes fondamentales des espaces et, plus généralement, des formes mathématiques. Cet article explore principalement la théorie de l'homotopie, secteur des mathématiques où les modèles occupent une place centrale. Je soutiens que les mathématiciens introduisent des modèles au sens courant du terme et je présente une première classification de ces modèles.

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Contents

1. Knowledge of pure mathematics

The goal of what is customarily called "pure mathematics" seems to be clear and straightforward: to acquire knowledge about mathematical objects by developing theories about them. These theories contain definitions of the objects with some of their properties and mathematicians *prove* theorems about them. In some cases, pure mathematicians have to compute certain formulas to obtain the information about a specific situation, e.g. a certain cohomology group. From a philosophical point of view, this picture is handy and precious: the justification of mathematical knowledge follows a foundationalist pattern thoroughly familiar to philosophers. This is certainly fine to a large extent. But is that all? Some mathematicians have opened the door to other facets of the practice. To wit:

> What do we view as our chief goal when we 'do' mathematics? It is customary, at least among pure mathematicians, to say that we seek to prove theorems. Theorems and nothing else are the currency of the field: they buy you a thesis, invitations to deliver colloquia and especially a job. [...]
>
> Opposed to this, however, is the idea of a model. Models are most prominent in applied mathematics where they express the

> essential point at which chaos of experiment gets converted into a well-defined mathematical problem. But pure mathematics is full of models too: one area, let's say, has uncovered a complex set of examples and is stuck making a direct attack on them. Often the best approach is to isolate part of the structure, in effect defining a model which is easier to attack. ([68])

In this passage, the mathematician David Mumford uses the term 'model' in a way that is not unlike the way scientists and philosophers of science use it[1]. The idea as presented by Mumford is straightforward: one 'abstracts' or 'isolates' data from a complex situation. If this abstraction is done properly, then one obtains a model of the original complex and messy examples. By studying the model and its properties, one can explain various phenomena, predict certain effects and, which is probably more relevant here, understand crucial aspects of the situation[2]. As Mumford says himself, this is not usually thought as being a part of the practice of the pure mathematician. But it is — Mumford does not have one doubt about it — and, in fact, it is nowadays crucial and does reflect important epistemological aspects of contemporary mathematics.

At first sight, the idea that pure mathematicians need to build models in a way that is analogous to scientists seems to be preposterous. Indeed, models usually stand between empirical data and theories. They are either built from the data themselves, the so-called phenomenological models, or they are built from the theoretical principles, the so-called theoretical models. They constitute a bridge between a collection of specific, singular data and universal, general laws, usually presented in the language of mathematics. They are used to represent certain aspects of a situation, devised so that mathematical methods can be applied to the situation and thus, certain inferences can be performed rigorously. Models turn complex, singular situations into tractable, comprehensible, understandable, explainable and, in the best cases, predictable situations.

What on earth would pure mathematicians model in the first place? As *pure* mathematicians, they do not model empirical situations or systems. So what is it that they model? The only possible answer is that they want to model mathematical objects. If one accepts that the latter makes sense, then one comes upon the second bizarrerie: Why would pure mathematicians have to model mathematical objects? What is it about these objects that require mathematicians to model them? Why would anyone model math-

(1) And *not* the way logicians use it, I should hastily add. Some standard references on the use of models in the factual sciences are [67, 84].

(2) Whether this constitutes an adequate description of the nature and use of models in the factual sciences is an issue all to itself and is certainly not directly relevant here.

ematical objects by other mathematical objects? And why with models? Why are the latter called 'models' in the first place? Are models introduced for reasons similar to those introduced for real physical systems? Does it make sense to model a type of mathematical objects in order to apply mathematical methods to them? Well, it does. Fortunately for us, Mumford gives an example in the foregoing quote:

> This is how algebraic topology got going in the 50's: the category of homotopy types of spaces was defined and the field exploded once this 'model' for topological spaces was made explicit. This type of model is based on throwing away part of the structure so as to concentrate on specific aspects which work as self-consistent non-trivial structure in their own right. ([68])

We now have specific data to work with: the objects modeled are topological spaces. In the philosophical literature, topological spaces would be called the "target system"([38]). The model is given by the category of homotopy types of spaces and these could be called the "model system". We are also told that the field of algebraic topology exploded once a model was introduced and used. The model apparently opens up avenues and provides a rich harvest of results. Furthermore, it is a model since, according to Mumford, one abstracts from topological spaces and thus obtains a new type of mathematical object that can be studied in its own right and yield relevant information about the original objects.

Why do mathematicians need models of topological spaces? What is it about them that requires that mathematicians find models? And what kind of mathematical objects will these models be? There is, after all, a whole general theory of topological spaces, as anyone can learn from [13, 14, 55, 69]. For applications, one can then turn to specific classes of topological spaces, e.g. metric spaces, L^p-spaces, etc. I will, in the next section, propose a partial answer to these questions.

Is Mumford's use of the term 'model' legitimate? Or is it only an analogy with the case of models in the factual sciences? Even if it is only an analogy, why does Mumford believe that it is an appropriate analogy? I will try to argue that it is a legitimate usage of the term. I will also try to show that there are in fact different senses of the term involved. The purpose of this paper is to explore the various uses of models in algebraic topology. I will also try to make some preliminary steps into the epistemological consequences of this dimension of contemporary mathematical practice. I hasten to add that I present this paper as a first step only, that my goal is merely to open the door and set up the table for further studies. Let us now briefly turn to the mathematical background required to understand the technical details involved in the proposed claims.

2. Space and its models

The concept of space evolved radically during the 19th century. What was one concept, Euclidian space, with a fixed, clear reference, namely the real physical space, became a plethora of different concepts with no clear physical referent. At the most general level, the culminating point of this development was the concept of topological space, as introduced by Hausdorff in 1914. The accompanying notion, that of continuity, was finally made clear and rigorous by Hausdorff's definition. But the methods used to study topological spaces were split in two: there were the combinatorial methods of the so-called *analysis situs* and there were the methods based on the new set theory introduced by Cantor and applied by Hausdorff himself in his book. Once the notion was accepted, clarified, simplified and extensively used and once the notion of homeomorphism[3], that is the criterion of identity for topological spaces, was given clearly by Kuratowski, the problem of classifying topological spaces emerged naturally.

Here is how Seifert and Threlfall presented what they called the principal problem of topology in the introduction of their textbook originally published in German in 1934:

> The principal problem of topology is to decide whether two given figures are homeomorphic and, when possible, to enumerate all classes of nonhomeomorphic figures. Although extensive theories exist which treat arbitrary subsets of Euclidean space, we will not deal with the concept of a figure in that generality. To do so would entangle us in set theoretic difficulties. The concept of a complex, as introduced by L.E.J. Brouwer, and further narrowed down during the course of our investigations to that of a manifold, will be sufficiently restrictive so that it bypasses the set theoretic difficulties but will also be broad enough to include almost all figures of interest. The topology which we treat here is not, then, set theoretic topology but is a topology of complexes and manifolds. ([79, p. 4])

The goal for the pure mathematician in this case is straightforward: enumerate all classes of non homeomorphic figures. This is a typical classification problem. That is easy to say but another matter to complete it.

(3) Let X and Y be topological spaces. A *homeomorphism* between X and Y is a continuous map $f : X \to Y$ such that $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$, where f^{-1} is the inverse map $f^{-1} : Y \to X$ and is continuous.

To show that two spaces are homeomorphic, one has to construct a homeomorphism between them. It is not because such a homeomorphism has not been found that there cannot be one. At some point, doubts sink in and one tries to show that two given spaces cannot be homeomorphic. In this case, one has to show there there is *no* possible way to define a homeomorphism between two given figures. How does one do that? A different mathematical strategy is required. Much of the machinery of algebraic topology was conceived with precisely this goal in mind: to find ways to establish classes of non homeomorphic spaces. One possible strategy consists in finding ways to solve the problem to classes of spaces and then try to extend these methods to the other classes. This is where models of certain spaces might turn out to be useful. If they can simplify the context, provide various handles for computations and clearly capture the relevant properties for the classification problem, then it is certainly a good idea to develop and study these models.

Other obstructions to the classification problem emerged along the way, obstructions that seem to be circumstantials. The first one mentioned in the foregoing quote is that arbitrary subsets of Euclidean space include numerous pathological spaces that arise simply because set theoretical tools are employed to define them. This situation is echoed by another important mathematician of the 1930s:

> Nowadays we tend, almost automatically, to identify physical space with the space of three variables and to interpret physical continuity in the classical function theoretical manner. But the space of three real variables is not the only possible model of physical space, nor is it a satisfactory model for dealing with certain types of problems. Whenever we attack a topological problem by analytic methods it almost invariably happens that to the intrinsic difficulties of the problem, which we can hardly hope to avoid, there are added certain extraneous difficulties in no way connected with the problem itself, but apparently associated with the particular type of machinery in dealing with it[4]. (Alexander, quoted in [45, p. 564].)

The 'particular type of machinery' that Alexander has in mind here is the set-theoretical machinery. We can immediately draw an epistemological moral: there are numerous models of (physical) space and they rest on various mathematical frameworks. A model, in the natural sciences and in

(4) Notice that the usual definition of a topological space as a set of points satisfying the usual axioms of a topology is seen here as a model of physical space. But this definition is, in a sense, too broad for it allows for models of spaces that are "pathological" from the geometric point of view.

applied mathematics, is devised with a specific function in mind, with a purpose, a problem to be solved. In that respect, some models are better than others and a target system might have different, incompatible model systems. Is this the case in pure mathematics too? As the foregoing quotes suggest, there was a sentiment that set-theoretical tools were inadequate to model various aspects of geometric spaces. This was already clear to some mathematicians as early as the beginning of the 1930s. One way around these unwanted difficulties is to develop alternative approaches and, in this specific case, it rested on the notion of a complex.

> The property which distinguishes a complex from an arbitrary point set of a space is its triangulability: a complex is a point set consisting of finitely many or countably infinitely many not-necessarily straight-line intervals, triangles, tetrahedra, or corresponding higher dimensional building stones, assembled together as a structure. (...) As a consequence of the triangulability property, most so-called pathological point sets will be excluded from our considerations. A close connection with objects of geometric interest is then achieved, ... Examples of complexes are: all Riemann surfaces; Euclidean space of arbitrary dimension; open subsets and algebraic curves and surfaces lying within that space; the projective plane and projective 3-space; all Euclidean and non-Euclidean space forms, regions of discontinuity of metric groups of motions and, finally, position and phase spaces of mechanical systems. [79, p. 5]

Complexes were defined to model geometric spaces of interest from a combinatorial point of view. They are of interest because they can be used to compute various properties of the spaces they model. Between roughly 1910 and the 1940s, various types of complexes were introduced for different purposes. We will here focus on complexes that play an important role in the study of homotopy types. But before we do so, we have to make a pause to discuss some of the basic philosophical facets of models in the sciences.

3. Models in pure mathematics

As far as I know, there is no commonly accepted typology of models in philosophy of science[5]. There are obvious examples of types of models in science and technology: scale models, idealized models, analogical models, phe-

(5) In the article on models in science of the Stanford Encyclopedia of Philosophy, one finds an interesting discussion of various kinds of models and issues related to them, but there is no general classification of models. For a daring and unorthodox attempt at such a typology, see [77].

nomenological models, mathematical models, theoretical models, bayesian models to mention but the most obvious. One also finds animal models or, more generally, in vivo models and in vitro models in the biological sciences. This list is not exhaustive nor is it exclusive. It is easy to give examples of each of these types, e.g. Bohr's model of the atom. Particular models can be physical objects, equations, descriptions, some sort of idealized or fictitious systems, not to say that it can be a combination of some of these elements.

Some philosophers have argued that the best we can do is to provide a *functional* characterization of scientific models. In other words, what scientific models would have in common is their *function* and the latter would roughly be to represent some system in the real world[6]. Other philosophers have lately attempted an ontological characterization, trying to capture the common *nature* of scientific models. For instance, Gabriele Contessa has recently offered a provisional taxonomy of scientific models based on ontological properties[7]. According to him, there are three types of models: material models, mathematical models and fictional models. Material models include, at the very least, scale models, physical models and biological models. For instance, that particular well-known construction made up of wooden or plastic balls and rods in many biology classes standing for the DNA helix is a material model and so is the scale model of a car built up by engineers to test air resistance. Mathematical models usually boil down to a differential equation or a system of differential equations, as in the Lokta-Volterra model in population biology. They constitute the bread and butter of dynamical systems theory. Contessa leaves their ontological status to philosophers of mathematics. I will get back to this issue in the next paragraph. Finally, we have fictional models. These include the proverbial ideal pendulum in classical mechanics, various models of the atom or its nucleus and a myriad of other well-known cases. What intrigues philosophers about these models is that they are not concrete objects nor are they purely abstract objects since they stand for concrete systems in a crucial manner. According to Contessa, these fictional models belong to the ontological genus of imaginary objects.

As the foregoing quote by Mumford already indicates, the functional aspect inherent to scientific models is also a key part in the usage of the terminology in the mathematical case. However, I claim that there is an ontological component in this too. Although it might seem that our forthcoming examples of models in mathematics cannot fall under the ontological classification offered so far, I venture to make some preliminary and rough remarks in this respect.

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(6) See, for instance, [39].
(7) See [28]. See also [38, 87].

Of course, the models we are talking about are not material models[8]. They are trivially mathematical models, since they are mathematical objects themselves. What is peculiar in our case is that the target system is itself a mathematical system. One *could* claim that our models are also fictional objects on the grounds that *all* mathematical objects are fictional objects[9]. But I believe that this claim would completely miss the point, in particular the *ontological point* of the story. The ontological component has to do, in a nutshell, with the fact that we are dealing, in a sense to be clarified fully elsewhere, with conceptual *artifacts*[10]. Mathematical models are not merely mathematical objects that satisfy certain properties. They are objects that are given in a way that they have such and such properties that are wanted for a certain purpose. The latter are guaranteed by the artifactual nature of the objects. They are, in a very loose sense, designed and constructed with these properties in mind. It is a clear case where form and function go together, where form is designed with a specific function in mind[11]. In other words, we are dealing with a case of conceptual design, that is designing conceptual objects[12]. In our cases, either the models are constructed systematically according to an explicit plan or they are *carved out* of a given system of objects by certain *choices* that then determine the whole structure of the models. Needless to say, the objects are not *literally* constructed, that is physically or materially, and the models are not *literally* carved out by a series of choices. We are talking about conceptual or abstract artifacts after all.

Thus, from the ontological point of view, we are trying to identify *within* mathematical objects, a category of objects that were *designed* by mathematicians with a specific purpose in mind, namely to stand in for other mathematical objects. It is important to note that there can be and, in fact, there often is, a whole theory of these objects. We are not dwelling here with the other categories of mathematical objects, but we are submit-

(8) But, as is well known, there are material models of geometrical objects, for instance polytopes and they do serve as models. Some of the cases we are interested in are related to these. Obviously, the material models of geometrical objects are not used by mathematicians in the way that a scale model can be used by engineers. They can be prosthetic to visual thinking or to thinking in general.

(9) This latter claim goes back at least to Hans Vaihinger's philosophy of the *as if*. (See [88].) It is gaining in popularity in philosophy of mathematics. See, for instance, [25, 59]. I am not, however, endorsing a form of fictionalism here. See [85, 86, 65] for some arguments against this view.

(10) Artifacts are often presented as having a dual nature: as material objects and embodying intentional functions. See the whole issue of [57] and [56]. I would argue that mathematical objects also have a crucial material facet in the form of notational systems and devices.

(11) Just to be entirely clear: I am not using the term "function" in its mathematical sense in this context.

(12) Since I am talking about design, this would be a natural place to introduce beauty in the picture.

ting the idea that there are mathematical objects that should be thought of as being conceptual artifacts. This ontological category is of course coupled with an epistemological function and both the ontological and the epistemological facets have to be taken together for our claim to be comprehensible and plausible. The upshot is that I claim that we are introducing a legitimate notion of model since: 1. models in the factual sciences and in pure mathematics share basically the same epistemological function and 2. models in the factual sciences and in pure mathematics are fundamentally artifacts.

This being said, we can be more specific about the types of models that one can find in pure mathematics. We will encounter three different types of mathematical models. I hasten to add that there are probably more. The first two are akin to a scale model, but that comparison might be more misleading than anything else! The underlying idea of the first type is to replace a mathematical object, for instance a topological space, by another topological space which is *constructed* in a certain way such that the constructed space has all the same topological properties as the original space and, furthermore, it has nice computational properties, this latter term understood in a loose sense. Furthermore, the construction can be performed systematically if not for all topological spaces, at least for topological spaces that are deemed of interest, usually qualified as being "nice" topological spaces. Since the model building revolves around a recipe for constructing objects, we will call these models, *object models*.

We say that X is a *object model* of Y if:

1. X is constructed systematically according to an explicit plan or rule;
2. X stands for Y, that is, X possesses all the relevant properties of Y;
3. X can be manipulated systematically, that is one can compute with X or one can compare features of X with other entities of the same kind;
4. X allows one to explain, understand and even predict certain aspects of Y.

This characterization is very general, but it can *not* be applied to concrete models and other models in the factual sciences. Of course, in the case of mathematics, the objects X and Y are *mathematical* objects and, in this case and as we have already indicated, it is perhaps the nature of the Y's that is more curious. But although this already constitute a crucial difference between our models and concrete models, the key difference lies in the fact that object models possess *all* the relevant properties of the target system. This is of course impossible in the case of models of concrete systems and is attributable to the fact that all our objects are mathematical objects.

The second notion is a variation on the first, but it introduces a level of abstraction in the picture. In this case, the model is not of the same kind as the target object and it is not (necessarily) constructed from an explicit rule. However it is a model of the target system since it is possible to construct an object like the target object from the model and the latter possesses or captures the essential properties of the target object.

We say that X is a *codified model* of Y if:

1. X contains all the information to construct an object equivalent to Y;
2. X itself codifies fundamental properties of Y;
3. X can be manipulated systematically, that is one can compute with X or one can compare features of X with other entities of the same kind;
4. X allows one to explain, understand and even predict certain aspects of Y.

The third notion is more general, more abstract and is what Mumford has in mind. In this case, the target system is a given category – in the mathematical sense of the term – and one constructs a new category and a functor between the categories. Of course, it is a very specific construction with very specific properties. The constructed category is the model in this case[13].

We say that $\widehat{\mathbf{T}}$ is a *systemic model* of $\mathbf{T}$ if:

1. $\widehat{\mathbf{T}}$ and $\mathbf{T}$ are categories;
2. The category $\mathbf{T}$ is what is called a *homotopical category*, that is it has a distinguished set W of maps, called the *weak equivalences* which contains all the identities of $\mathbf{T}$, and thus all the objects of the latter, and it satisfies the two out of six property[14];
3. The category $\widehat{\mathbf{T}}$ is the *homotopy category* of $\mathbf{T}$; it is constructed by formally inverting the weak equivalences of $\mathbf{T}$ and there is a canonical quotient functor $I : \mathbf{T} \to \widehat{\mathbf{T}}$;

(13) What is usually called ‘pure category theory’ might be seen in this light in general, that is as providing abstract models of various fields of mathematics, e.g. abelian categories, cartesian categories, cartesian closed categories, etc. We will not enter this discussion in this paper.

(14) This condition says that, given any three morphisms f, g and h in $\mathbf{T}$ for which two composition gf and hg exist and are in W, the four morphisms f, g, h and hgf are also in W. See [31, p. 23].

4. $\widehat{\mathbf{T}}$ allows one to explain, understand and even predict certain aspects of $\mathbf{T}$.

The terminology will become clear as we process in the discussion. We are aware that the last definition is at the same time highly technical and, here, imprecise. We won't give the precise technical definition in this paper. What is important to note for our purpose is that the homotopy category $\widehat{\mathbf{T}}$ has the same objects as $\mathbf{T}$. The morphisms are different, for the morphisms in $\widehat{\mathbf{T}}$ are equivalence classes of morphisms of $\mathbf{T}$. What this means, fundamentally, is that we are looking at the same objects but with a different and more abstract, criterion of identity. It is also important to emphasize immediately that this purely formal definition encompasses a very large class of significantly different examples.

What is fascinating in the case of topological spaces is that these three types of models interact constantly in contemporary algebraic topology and their interaction played a key role in the development of what is now called "modern classical homotopy theory".

4. Algebraic topology and homotopy types

As its name indicates, algebraic topology is the study of topological spaces with the means of algebraic methods. For this to be possible, two essential steps have to be carried out. First, for the purpose of computability, spaces have to be such that they can be manipulated easily by combinatorial means, that is, spaces themselves have to be thought of as being made up, not of points as in point-set topology, but of parts, or in the jargon of the field, of cells and these cells are fitted together according to certain rules to form the spaces. Thus, at its very roots, algebraic topology offers various ways of building spaces out of cells and these are already models of spaces. Second, these models have to be manipulated to construct algebraically tractable systems of data. More specifically, one wants to associate to a space X a series of invariants that would characterize the space X in the following sense: if X and Y are different spaces, then they should have at least one different (algebraic) invariant. Ideally, one should also be able to show that if two spaces X and Y have the same invariants, then they should be identical as spaces. This is of course very vague and it took quite some time for mathematicians to settle for a proper definition of topological spaces, a proper criterion of identity for topological spaces, a proper understanding of the invariants involved, that is from numerical invariants to algebraic invariants, the realization that there was an autonomous abstract geometric form underlying all these invariants, the development of a theory to study these geometric forms, their invariants, etc. I will not follow the historical steps here. Although the historical process is worth looking at from a philosophical point of view, for our purposes here it would simply be too long and intricate. (See [29] for an overview of the history of algebraic topology and [45] for a short description of the main actors and developments.) Let us now consider some basic definitions to fix ideas and the notation.

Let X and Y be topological spaces, I the usual unit interval $[0, 1]$ with the standard topology, and let $f, g : X \to Y$ be two continuous maps.

Definition 4.1. — *A homotopy α from f to g, denoted by $\alpha : f \Rightarrow g$, is a continuous map $\alpha : X \times I \to Y$ such that for all $x \in X$,*

$$\alpha(x, 0) = f(x),$$
$$\alpha(x, 1) = g(x).$$

In words, a homotopy is a continuous deformation of the image of f into the image of g. In the vernacular language, one could say that a homotopy is a morphing of f into g. When there is a homotopy between f and g, we say that they are *homotopic*. It can be shown that 'being homotopic' is an equivalence relation between continuous maps $X \to Y$.

Although the notion of homotopy was introduced informally by Poincaré and defined rigorously by Brouwer in 1912, the notion of homotopy equivalence and homotopy type of spaces was introduced by Hurewicz only in 1935. (See [15] and [1].) It took another fifteen years for mathematicians to understand that homotopy types were at the core of algebraic topology and another fifteen to twenty years to see that they underly all geometric structures. (See [5], [7] for more details.)

Let us introduce the definition of homotopy equivalence of spaces.

Definition 4.2. — *Two spaces X and Y are said to be homotopy equivalent if there are continuous maps $f : X \to Y$ and $g : Y \to X$ and homotopies $\alpha : f \circ g \Rightarrow \mathrm{id}_Y$ and $\beta : g \circ f \Rightarrow \mathrm{id}_X$, that is the composites $f \circ g$ and $g \circ f$ are* homotopic *to the identities id_Y and id_X respectively. When two spaces X and Y are homotopy equivalent, we say that they are of the same homotopy type or that they are tokens of the same homotopy type.*

Here are a few standard examples. Let 1 denote the one-point space. Then it is easy to show that 1 and the real line $\mathbb{R}$ are homotopy equivalent! (The proof follows from the fact that all continuous maps $\mathbb{R} \to \mathbb{R}$ are homotopic.) A space homotopy equivalent to 1 is said to be *contractible*. For instance, any open interval (a, b) is contractible and so is any closed interval

$[a, b]$. A space with a single point and the real line $\mathbb{R}$ are tokens of the same homotopy *type*. And so is the real plane $\mathbb{R}^2$ and even the n-dimensional space $\mathbb{R}^n$.

It can be shown that 'being homotopy equivalent' is an equivalence relation between spaces. It is tempting at this stage to quotient the class of all topological spaces under this equivalence relation and declare that an equivalence class is a homotopy type. But I claim that this would be a serious mistake. For mathematical purposes, this is not the equivalence relation that matters. The crucial equivalence relation is the relation between maps of spaces, that is being homotopic. We will come back to this point in a short while.

From a philosophical point of view, homotopy types constitute a challenge. Here are some of the most obvious questions they raise: what *are* homotopy types? Should it be considered to be an object in its own right? If not, what is it? How is a homotopy type given? What is it to know a homotopy type? One of the main points of this paper is precisely that homotopy types are known via models. But this claim has to be clarified.

For instance, it is tempting to interpret Mumford as claiming that a homotopy type is a model of a topological space. I believe that it can be said that homotopy types are abstracted from topological spaces. However, I do not want to say that a homotopy type is a model of a topological space. For homotopy types capture fundamental properties of topological spaces. They constitute their underlying forms. Furthermore, in the foregoing quote, Mumford talks about the *category* of homotopy types as being models of spaces. We will get back to the category of homotopy types in section 6. To get a better grasp of what this is suppose to mean, let us recall the notion of homeomorphic spaces or what is sometimes called 'topological type'. This will allow us to contrast and compare the two notions and see how homotopy types are abstract geometric forms of topological spaces.

A homeomorphism is an isomorphism in the category **Top** of topological spaces. As it is easy to see, a homeomorphism is necessarily a bijection. We say that two spaces X and Y are *homeomorphic*, written $X \simeq_{\mathbf{Top}} Y$, if there is a homeomorphism between them. Notice that "being homeomorphic" is an equivalence relation between topological spaces. This is where homotopy types come in as models of topological spaces. The classification of homotopy types is coarser than the one provided by homeomorphic types.

Notice that whereas a homeomorphism is an identity between composite maps and identity maps, a homotopy equivalence is a continuous transformation between composite maps and identity maps. Furthermore, the homotopies involved in the homotopy equivalence need not be bijections. Note also that there can be more than one homotopy replacing the identities of homeomorphisms. In this sense, the identity between two spaces is not, from the homotopical point of view, an all or nothing affair.

Homotopy *theory* was more or less launched in the 1930's by Heinz Hopf and Witold Hurewicz (and closely followed by many mathematicians, one of which was J.H.C. Whitehead, A.N. Whitehead's nephew). The dream here is to have what is called a complete set of invariants, that is two spaces would be homotopically equivalent if and only if they had the same set of invariants. As I write, such sets have been found for specific cases, but the general problem seems to be intractable. (See [8] for the case of simply-connected 4-manifolds, for instance, and [9] for recent progress.) The theory uses various tools, e.g. homotopy groups, homology and cohomology groups, fibrations, loop spaces, suspensions, spectral sequences, Postnikov towers, etc., in order to establish the existence of homotopies between various maps or to prove that two spaces are not homotopy equivalent.

I will not look explicitly at the specific tools and machines of homotopy theory in this paper, e.g. homotopy groups, etc. I want to concentrate on one aspect of homotopy theory: models of topological spaces and their relations to homotopy types. In the 1940s, mathematicians realized that they could construct models of topological spaces that were tailored for the needs of homotopy theory[15]. In the process, a particularly adequate notion of cell was also introduced, namely CW-cells and their associated CW-complexes. As we will see, there is, already at this stage, a very close connection between the algebraic data, the homotopy types and their geometric realization, i.e. the spaces.

5. Cells and CW-complexes

Any child knows that various surfaces and volumes can be constructed from simple pieces by following a plan. In fact, all our buildings are constructed this way: from simple pieces and 'gluing' rules, we assemble elaborate spaces. Furthermore, children also learn how to determine certain properties of a space, e.g. a line segment, a surface or a solid, by combining properties of its parts, when, for instance, they have to compute the volume of a solid. These simple facts ought to be applicable to abstract spaces too and, not surprisingly, they are. One has to start with simple blocks of different dimensions, called 'cells' or 'n-cells' in this case and combinatorial rules to put these cells to form a space. Combinatorial topology, as it was

(15) These were introduced by J.H.C. Whitehead and R. H. Fox. See [93], [94], [36], [35].

used to be called before it became algebraic topology, starts from this principle: a space can be constructed from simple, knowable pieces according to simple and clear construction rules. The first such constructions were simplicial complexes. A more general and flexible notion was introduced by J.H.C. Whitehead for the purpose of homotopy theory in 1949. Very roughly speaking, the construction set is made up, in this case, of spheres of various dimensions that can be glued together in complicated ways at various points. Before we can give the formal definitions, we need to recall some basic conventions.

The n-dimensional disk, or n-disk D^n is the subspace of $\mathbb{R}^n$ defined by

$$D^n = \{x \in \mathbb{R}^n : |x| \leqslant 1\},$$

with the induced topology and where the function $|\cdot| : \mathbb{R}^n \to [0, \infty[$ is the standard norm on $\mathbb{R}^n$. The open n-disk, D^n, is, of course, the interior of D^n, hence

$$D^n = \{x \in \mathbb{R}^n : |x| < 1\}$$

and the *boundary* of D^n in $\mathbb{R}^n$ is the standard $(n-1)$-sphere

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

We can now define the cells.

DEFINITION 5.1. — *An n-cell is a space homeomorphic to the open n-disk D^n. A cell is a space which is an n-cell for some $n \geqslant 0$.*

Let us unpack this somewhat. Since D^0 is the same as $\mathbb{R}^0$, which is the same as $\{0\}$ by definition, and $D^0 = D^0$, a 0-cell is simply a point. A 1-cell is clearly essentially a line segment. A 2-cell is a disk and a 3-cell is a sphere. Although we won't give any argument for this, it is fairly easy to convince oneself that a n-cell is of dimension n.

Roughly speaking, a CW-complex is a way to glue cells together — possibly infinitely many of them — according to certain reasonable patterns. For a space X to be a CW-complex, it has to be constructed by the following procedure. One starts with a collection X^0 of 0-cells, that is a collection of points. We then proceed by induction by attaching n-cells to already constructed $(n-1)$-cells and we glue them appropriately. For instance, we can construct a sphere S^2 by taking one 0-cell, a point, and one 2-cell, an open disk and by gluing the boundary of the open disk to the point. This is more or less a balloon. Compare this construction with a basketball or a soccer ball: in each of these, there are more cells, namely the different patches glued together to form the ball. Each one of these correspond to different ways of constructing a sphere. It is easy to see that any n-sphere can be constructed by taking a 0-cell and an n-cell and gluing the boundary of the n-cell to the point. Thus, any n-sphere can be decomposed into two cells, a very simple decomposition. The operations of gluing along a circle and identifying points have been formalized by mathematicians along the way and it is more convenient to use the language of category theory at this stage. Gluing can best be described by the operation of taking the push-out of two maps. Let A be a subspace of X, which we represent by a monomorphism $i : A \rightarrowtail X$ and $f : A \to Y$ be an arbitrary continuous map. Gluing X to Y along the image of A in Y amounts to constructing $X \amalg_f Y$ in the following push-out:

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \longrightarrow & X \amalg_f Y \end{array}$$

Definition 5.2. — *A CW-complex is a space X together with a filtration of subspaces*

$$X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \ldots \subseteq X^n \subseteq \ldots \subseteq X$$

such that

1. *the n-skeleton X^n is constructed from X^{n-1} by attaching n-cells e^n_α with the maps $\phi_\alpha : S^{n-1} \to X^{n-1}$;*
2. *$X = \bigcup_{n \geqslant 1} X^n$;*
3. *A subset $A \subseteq X$ is closed if and only if $A \cap X^n$ is closed in X^n for all $n \geqslant 0$. This says that the space X carries the weak topology with respect to the spaces $\{X^n\}_{n \geqslant 0}$.*

Almost all interesting geometric spaces are CW-complexes. We have already mentioned that all n-spheres can be trivially decomposed into CW-complexes. A 1-dimensional CW-complex is a graph (seen as a topological space). The real projective n-space $\mathbb{R}\mathrm{P}^n$, the complex projective n-space $\mathbb{C}\mathrm{P}^n$ can all be constructed as CW-complexes.

CW-complexes are “nice” spaces. They are necessarily Hausdorff, locally contractible and every compact manifold is homotopy equivalent to a CW-complex. (See [41, pp. 519 - 529] or [78, pp. 199-200] for proofs.)

Our claim, not surprisingly, is that CW-complex are *object models* of (nice) spaces: they clearly satisfy the four conditions listed at the end of section 2. CW-complex constitute a clear example of models of topological spaces designed specifically for the purposes of homotopy theory. Indeed, one and the same space X can be decomposed in different CW-complexes and some decompositions are chosen because they simplify computations. Furthermore, one can use the building pattern of CW-complexes to develop arguments that proceed combinatorially cell-by-cell. Various constructions can be performed on CW-complexes, that is products, quotients, suspension, wedge and smash products, allowing one to build more intricate complexes to model various spaces naturally. In fact, these constructions are better seen as being performed in the category of CW-complexes with continuous mappings between them.

In fact, CW-complexes come very close to solve the principal problem of topology. In order to see this, we need to recall a different version of homotopy equivalence, namely the so-called *weak homotopy equivalence*[16].

Definition 5.3. — *A map $f : X \to Y$ is a* weak homotopy equivalence *if for each base point $x \in X$, the induced map $f_* : \pi_n(X, x) \to \pi_n(Y, f(y))$ is a bijection of sets for $n = 0$ and an isomorphism of groups for $n \geqslant 1$*[17].

Two spaces, X and Y are said to be *weakly homotopy equivalent* if there is a weak homotopy equivalence between them. The latter definition is deceptive. The terminology suggests that it is also an equivalence relation. The main problem here is that the continuous map $f : X \to Y$ has to be given to start with and although it can easily be verified that the definition is reflexive and transitive, it need not be symmetric. Indeed, there is no guarantee that the map f has an inverse. The difference between being weakly homotopy equivalent and being homotopy equivalent can be illustrated by the following analogy. In the case of two spaces being homotopy equivalent, we have rules that allow us to continuously transform the spaces into each other (and we are allowed to identify points on the way). Being weakly homotopy equivalent means that we have a way of mapping one space into the other and that as far as our indicators can tell, that is the homotopy groups, the spaces are essentially indistinguishable. Notice also that the underlying sets of two weakly homotopy equivalent spaces have the same cardinality.

CW-complexes were designed for homotopy theory precisely because the relations of being homotopy equivalent and being weakly homotopy equivalent coincide in the following precise sense:

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(16) At this stage, I have to assume that the reader is familiar with homotopy groups. See [41, chapter 4] or [78, chapter 11]

(17) It is of course possible to give examples of weak homotopy equivalences between spaces that are not homotopy equivalent. See [41, chapter 4].

THEOREM 5.4 (Whitehead [41, chapter 4]). — *If X and Y are* connected *CW-complexes and if $f : X \to Y$ is a continuous map such that $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism for all n, then f is a homotopy equivalence.*

Thus, whenever two connected CW-complexes X and Y are weakly homotopy equivalent, they are homotopy equivalent, that is X and Y are tokens of the same homotopy type[18]. Thus, in the case of connected CW-complexes, the required information is included in the maps that induce the isomorphisms of groups. In this sense, the homotopy groups and the maps inducing the isomorphisms give us all the invariants we need. From that perspective, CW-complexes are very nice models of topological spaces indeed.

But they are not quite what we dreamed of. At this point, two options are open: simplify the data in one way or another or try a more abstract approach. In fact, it is when these two approaches are combined together that one obtains a very powerful machinery.

6. Algebraic models of homotopy types of spaces

Let us turn to homotopy n-types, introduced by R.H. Fox in 1941. Informally and very roughly, a homotopy n-type is a type of spaces that are homotopy equivalent in dimension n. For us, the interesting aspect of homotopy n-types is that they constitute 'toy' models, in the sense that one controls systematically the complexity involved right from the start. Thus, once one has the general notion of homotopy types and then the notion of weak homotopy types, it is a simplification to move to homotopy n-types and try to understand them. The hope would then be to have a better picture of the general notion from this more simple situation.

A homotopy n-type is a homotopy type such that all its homotopy groups higher than n are trivial, i.e. for all $j > n$, $\pi_j = 0$. Usually, tokens of homotopy n-types are taken to be CW-complex (this is how we know that these spaces exist). Thus, a homotopy 1-type is exhibited by a CW-complex such that for all $j > 1$, $\pi_j = 0$.

Let us start with homotopy 0-types. The latter is exhibited by any space whose homotopy groups π_j are all trivial for $j > 0$. In particular, π_1 is

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(18) In fact, a CW-complex is always homotopy equivalent to a polyhedron and is, in that sense, a "nice" space. One has to be careful here: as Hatcher rightly emphasizes, Whitehead's theorem does *not* say that two CW complexes with isomorphic homotopy groups are homotopy equivalent. It says that if there is a map that induces isomorphisms between homotopy groups, then that map is a homotopy equivalence. Hatcher [41, p. 348] gives examples of spaces that have isomorphic homotopy groups but that are *not* homotopy equivalent.

trivial, thus any homotopy 0-type is contractible. One can in fact think of the path-components given by π_0 as being contractible also, and thus as being homotopically equivalent to geometric points. These are the simplest and they are modeled by *sets*. In other words, a set is a *codified model* of a discrete homotopy 0-type. In this framework, one can think of sets as made up of geometric points, the later being a contractible space (and presumably one has in mind its contracted image) and thus, all the elements of these sets are urelements and not sets themselves.

Consider now homotopy 1-types. For simplicity, we could first consider *connected* homotopy 1-types. Using some basic algebraic topology, one can show that they correspond to groups! Indeed, let G be a group. One constructs the Eilenberg-Mac Lane space $K(G, 1)$[19]. It is by construction a path-connected space and it can be shown that the homotopy type of (a CW-complex) $K(G, 1)$ is uniquely determined by G[20]. In other words, there is a functor

$$B : \mathbf{Grp} \to \mathbf{CWcomp}$$

that associates to a group G its classifying space BG, which in this case is the space $K(G, 1)$, such that $\pi_1(BG) \cong G$ and $\pi_i(BG) = 0$ for $i > 1$. This statements says precisely how groups constitute models of connected homotopy 1-types.

Some important remarks have to be made at this point. Of course, in contrast with CW-complexes, groups were not invented for the purpose of homotopy theory. Furthermore, groups are connected to CW-complexes, which are themselves models of topological spaces. Finally, groups are of an entirely different nature from spaces and this is a crucial element. Whereas a CW-complex is a (constructed) space, a group is not a space[21]. A group is a model of a very different kind indeed, but it is a model nonetheless. Although a given group G itself is not constructed (but it might be), there is a definite construction from G to its corresponding CW-complex which is necessary to see that it stands for a space in a certain manner. The important fact is that G contains the information that leads to the construction of the space BG. In our terminology, groups constitute codified models of spaces as all the following extensions in this section.

If we remove the restriction and consider arbitrary homotopy 1-types, then it can be shown that they correspond to *groupoids*. Recall that a

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(19) In fact, there are many different possible constructions. See, for instance, [41, sections 1.B and 4.2].

(20) "Having a unique homotopy type of $K(G, 1)$'s associated to each group G means that algebraic invariants of spaces that depend only on homotopy type, such as homology and cohomology groups, become invariants of groups." [41, p. 90].

(21) Of course, theyre are topological groups, but we are talking of groups here.

groupoid is a category in which every morphism is invertible, i.e. in which every morphism is an isomorphism.

DEFINITION 6.1. — *A* groupoid *is a category in which every morphism is an isomorphism, that is for every* $f : X \to Y$*, there is an* $f^{-1} : Y \to X$ *such that* $f \circ f^{-1} = \mathrm{id}_Y$ *and* $f^{-1} \circ f = \mathrm{id}_X$*.*

Thus any group G is a groupoid, since it can be seen as a one object category satisfying the property stipulated in the foregoing definition. (For a brief history of the concept of groupoid, see [18].)

Now, let X be a topological space. It is possible to define a category, denoted by $\Pi_1(X)$, directly from the homotopy properties of paths in X. By doing so, we are here thinking of a space as an algebra of paths. The objects of $\Pi_1(X)$ are its points x, y, ... A morphism $x \to y$ of $\Pi_1(X)$ is a *path class* from x to y, that is an homotopy class of paths from x to y. It can be shown that this in indeed a category and in fact a groupoid. It is called the *fundamental groupoid* of X. (See [21, section 6.2] for a proof and some of its properties or [40, chap. 3] for a presentation in the simplicial context.) Of course, $\Pi_1(-)$ is a functor from a category of topological spaces to the category of groupoids. It sends a space X to a groupoid $\Pi_1(X)$, a continuous map of spaces $f : X \to Y$ to a functor $\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y)$ and a homotopy class of homotopies to natural transformations. (It is thus what is called a 2-functor.) This is a very natural construction and the result seems to unlock a key to the nature of homotopy types. This construction from topological spaces to groupoids is perhaps closer to what Mumford had in mind: in some sense, we are abstracting from topological spaces to construct the corresponding groupoids and the latter reflect only specific properties of topological spaces. In this sense, they constitute models of topological spaces. Grothendieck's conjectured already in the early 1980's that the higher-dimensional component inherent to homotopy types can be captured by an appropriate notion of higher-dimensional groupoid.

Groupoids model arbitrary homotopy 1-types. To see this, one starts with groupoids and construct a homotopy 1-type from it in such a way that maps of groupoids, e.g. functors, become continuous maps of spaces and natural transformations are translated into homotopy classes of homotopies. It is possible to construct, for any groupoid G, its Eilenberg-Mac Lane space $|G|$ and prove that the latter is a CW-complex and, in fact, a homotopy 1-type, i.e. such that the homotopy type of the constructed space is uniquely determined by the groupoid G. It is then possible to show that the (2-)category of homotopy 1-types is equivalent (in a precise mathematical sense) to the (2-)category of groupoids.

The algebraic objects corresponding to homotopy 2-types, called crossed-modules, were introduced already by Whitehead. A crossed-module is a pair of groups G_1, G_0 and a right action of G_0 on G_1 together with a group homomorphism $\phi : G_1 \to G_0$ respecting the action. (For the algebraic expression of these conditions, see [60, p. 285] or [70, pp. 79-81].) In the context of homotopy theory, G_1 is the homotopy group π_2 and the group G_0 is π_1. Whitehead together with Mac Lane have shown in 1950 that there is functor B from the category of crossed modules to the category of CW-spaces showing how crossed-modules can be taken as algebraic models of homotopy 2-types. (Beware that what Mac Lane and Whitehead call a 3-type is now called a 2-type. See [61], [23], [20].)

We have moved from groups, to groupoids and to crossed-modules. It is impossible not to look for some kind of uniformity on the algebraic side that corresponds to the uniformity on the topological side. Is there such a thing as a 2-groupoid such that there would be a correspondence between these and homotopy 2-types? Recall that a groupoid can be defined as being a category in which all morphisms are isomorphisms or invertible. Can we define a 2-groupoid as a 2-category in which all morphisms are isomorphisms? It is indeed possible. First, there is a definition of a 2-category.

In the technical jargon, a 2-category is a category enriched over **Cat**, that category of (small) categories.

More specifically, a 2-category is given by the following data:

1. A set of objects $a, b, c, ...$;
2. A function which assigns to each ordered pair of objects (a, b) a category $T(a, b)$;
3. For each ordered triple $< a, b, c >$ of objects a functor

   $$K_{a,b,c} : T(b, c) \times T(a, b) \to T(a, c)$$

   called composition;
4. For each object a, a functor $U_a : 1 \to T(a, a)$;

such that the composition satisfy the obvious associative law and U_a is a left and right identity for this composition.

The distinctive feature of 2-categories is that they have two composition operations, a "vertical" composition that is part of the vertical categories $T(-, -)$ and an "horizontal" composition given by the functor $K_{(-,-,-)}$ and these two operations agree with each other. There is a algebraic presentation of 2-categories which makes these compositions and their relationships more explicit. (See [60, chap. XII] for a detailed presentation and other relevant notions.) A strict 2-groupoid is then defined to be a 2-category in which all 1-cells and 2-cells are isomorphisms.

Can we simply go on? Can we define 3-groupoids as 3-categories in which all morphisms are isomorphisms? It is precisely when we get to this step that the situation becomes surprising. It is of course possible to define a 3-category and then derive from the latter definition the notion of a 3-groupoid. However, the latter notion does *not* capture homotopy 3-types. As usual, moving up a dimension yields considerable more leeway.

Before we look at homotopy 3-types and their algebraic models in more details, it is probably a good idea to stop and discuss a few surprising results relating groups, categories and crossed-modules. Suppose we work in a category of sets S. In fact, all we need is a category with sufficiently good properties, e.g. finite limits[22]. We now define an internal category in S: it consists of two objects C_0 and C_1 together with four morphisms in S:

$$C_1 \times_{C_0} C_1 \xrightarrow{\ c\ } C_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} C_0, \qquad C_0 \xrightarrow{\ i\ } C_1$$

where the morphism c is defined on the following pullback:

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{\ \pi_2\ } & C_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ C_1 & \xrightarrow[\ d_0\ ]{} & C_0 \end{array}$$

Informally, C_0 is the object or set of objects, C_1 is the set of morphisms, the morphism i sends to each object of C_0, its identity morphism, d_0 sends a morphism to its domain, d_1 sends a morphism to its codomain and c is the composition of morphisms.

These maps are naturally constrained to satisfy four commutativity conditions, each of which corresponding to an axiom of category theory. (See [60, p. 268].) If we are working in a category S of sets, then this yields a small category. But it can also be done in any category with the right

(22) This is typical of one of the ways one can work with categories as a foundational framework: simply stipulate the properties and the structure needed for the specific piece of mathematics to be developed. As far as practicing mathematicians are concerned, this is perfectly fine. Things become more murky when we are dealing with categories themselves...

properties. Thus, we can consider a internal category in the category **Grp** of groups and group homomorphisms and it can be verified that in this case that an internal category in **Grp** is the same as an internal group in the category **Cat** of small categories. (See [60, p. 269].)

It is certainly a surprising fact that the category of internal categories in **Grp** is equivalent, that is essentially the same, as the category of crossed-modules. (See [60, pp. 285-287] for a proof. Even Mac Lane, who introduced categories and worked with crossed-modules with Whitehead qualified this result as "striking"!) In fact, it is possible to be even more precise, as follows. In the same way that we have defined an internal category in a category S, it is possible to define an internal group G in a category S. In particular, it is possible to define a group object in the category **Grpoid** of groupoids. This is what is called a **2-group** $\mathfrak{G}$. Again, this is in fact just a special case of the above, a 2-group can also be defined as a groupoid in the category **Grp** of groups. Finally, a 2-group can also be desbribed as a 2-category with one object in which all 1-morphisms and 2-morphisms are invertible.

What we see here is that already at the level of homotopy 2-types, there are many different ways to generalize the notion of crossed-modules and each of these provides algebraic models of 2-types. It turns out that these notions *can* be generalized to n-types and that they indeed model *all* homotopy n-types. (See [71], [72].)

However, homotopy 2-types can be modeled by strict 2-groupoids, but this fails when we move to 3-types. It is the strictness condition that fails when we move to (connected) 3-types: there are homotopy 3-types that cannot be modeled by strict 3-groupoids. More precisely, it can be shown that there is no strict 3-groupoid $\mathfrak{G}$ such that its geometric realization is weakly equivalent to the 3-type of the sphere S^2! (See [23, 24, 19, 20], [10], [81, chap. 4].) A simply connected 3-type is modeled by a braided categorical group. (See [49] for details.)

Thus, when we move to homotopy n-types, for $n > 2$, we are forced to consider what are called *weak* n-categories, in particular *weak* n-groupoids. In fact, it is taken as a desiderata that any formalization of the latter notion should automatically model homotopy types as a special case[23].

Let us pause and come back to models. We have shown that CW-complexes can be thought of as object models of topological spaces. We have now looked at codified models of homotopy types. The picture is somewhat more complicated and, in fact, more interesting. Whereas object models of

(23) Here are some references on the topic: [3], [12], [22], [27], [47], [54], [82], [89, 90].

topological spaces were introduced to provide a handle on the classification problem, codified models of homotopy types are introduced not only because the classification problem is just as hard, but also because homotopy types are inherently more abstract than topological spaces. An algebraic gadget is a model of a homotopy type when it is possible to systematically construct, from the algebraic gadget, a space such that its homotopy invariants are included, in some appropriate sense, in the algebraic gadget one started with. In this sense, the models capture all the essential information of the spaces. The algebraic models are clearly different from the topological models that were introduced. Notice that the two types of models are related: from the codified algebraic model, one constructs a topological model, e.g. a CW complex, and then establishes that the first is, in this specific sense, a model of the homotopy type of a space. The shift illustrates one of our points: the fact that mathematicians are building various kinds of models of spaces. The variety is important, for it provides a way to test how stable a certain notion is. The convergence is another important question: when one is able to show that apparently different models are equivalent, in an adequate sense of equivalence, then one has another indication that the various models capture important ingredients of the systems studied.

Category theory provides something more: the search for the "best" possible model, that is, a universal model. But for this to be possible, one has to climb the ladder of abstraction.

7. The abstract framework: categorical contexts

Once categories had been introduced by Eilenberg & Mac Lane in the early 1940's, Eilenberg & Steenrod quickly realized that they could use the language of categories, functors and natural transformations to provide a clear and systematic foundations for algebraic topology[24]. The object of study is the category **Top** of topological spaces and continuous mappings[25]. Eilenberg & Steenrod suggested that the algebraic invariants associated to topological spaces and continuous maps should be functors from the category **Top** to various categories of algebraic structures, like the category **Grp** of groups and group homomorphisms. Thus, in their book, Eilenberg & Steenrod gives axiom of homology theories and these axioms stipulates that a homology theory is a *functor* from a category of topological spaces to an algebraic category, e.g. groups, abelian groups, modules, vector spaces,

(24) See [58] and [64] for more on these aspects of the history of category theory.
(25) From the point of view of what used to be called "combinatorial topology", it took quite some time to show that combinatorial methods could be applied to an *arbitrary* topological space. One has to keep in mind that in algebraic topology, one studies spaces that are usually geometrically motivated and ignores more pathological cases that follow under the axiomatic characterization of the concept.

rings, etc. One of the points of Eilenberg & Steenrod's book is to set up the theory so that one can compare various homology theories systematically, which means via natural transformations between functors.

Both Eilenberg & Mac Lane ([33]) as well as Eilenberg & Steenrod ([34]) made suggestions as to how to formulate the basic ingredients of homotopy theory into the language of category theory. Again, the idea is that the links between the topological data and the algebraic data should be functorial. Mathematicians first tried to modify the axioms given by Eilenberg & Steenrod for homology (and cohomology) groups to characterize homotopy groups. Thus, in the case of homotopy theory, we have a series of functors:

1. $\pi_0 : \mathbf{Top} \to \mathbf{Set}$
2. $\pi_1 : \mathbf{Top} \to \mathbf{Grp}$
3. $\pi_2 : \mathbf{Top} \to \mathbf{AbGrp}$

   $\vdots$

n. $\pi_n : \mathbf{Top} \to \mathbf{AbGrp}$

   $\vdots$

The first functor yields, to each topological space X, the set of connected components $\pi_0(X)$. The second functor is the fundamental group π_1 of a space X, a construction introduced by Poincaré in 1895, although, of course, not as a functor. The other π_i's were all introduced at once by Hurewicz and they are known as the higher homotopy groups.

Most topological invariants *are* defined up to homotopy. For instance, homology theories and cohomology theories are functors from a category **Top** of topological spaces and an algebraic category, e.g. the category **AbGrp** of abelian groups. But these functors are *all* homotopy invariant: they preserve homotopy equivalences which are not the equivalences of **Top**[26]. (See, for precise statements and proofs, [41, pages 110-111 and page 201].)

This is where the systemic models come in. The natural thing to do is to construct a quotient category of **Top**, denoted by $\mathbf{ho}(\mathbf{Top})$, as follows: the objects of $\mathbf{ho}(\mathbf{Top})$ are the objects of **Top** but a morphism in $\mathbf{ho}(\mathbf{Top})$ is a homotopy *class* $[f]$ of maps. Thus the family of morphisms between two spaces X and Y is $[X, Y]$. Homology and cohomology theories, as well as the

(26) The specialist will have noticed that we restrict our presentation to the category **Top** of topological spaces, whereas one often considers the category $\mathbf{Top}_*$ of *pointed* spaces. Although this is technically important, it does not affect our discussion in any relevant manner.

homotopy groups, are best described as being functors $H : \mathbf{ho}(\mathbf{Top}) \to C$, for some algebraic category C. This appears to be innocuous and conceptually sound.

Indeed, in $\mathbf{ho}(\mathbf{Top})$ an isomorphism *is* an homotopy equivalence. Thus, whereas in the category **Top**, isomorphisms are homeomorphisms, in the category $\mathbf{ho}(\mathbf{Top})$, isomorphisms are homotopy equivalences. By moving to equivalence classes of morphisms between spaces, we have restored the *equality* sign between certain morphisms: instead of having $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$, we have $[f \circ g] = [\mathrm{id}_Y]$ and $[g \circ f] = [\mathrm{id}_X]$. Surprisingly, perhaps, the passage from the category **Top** to the category $\mathbf{ho}(\mathbf{Top})$ does not modify the objects of the category, but the criterion of identity for its morphisms.

However, once more, things are not so simple. As Peter Freyd has remarked already in the late sixties, the category $\mathbf{ho}(\mathbf{Top})$ "has always been the best example of an abstract category, historically and philosophically." ([37, page 1]) What Freyd means by this is the fact, proved by him, that there cannot be a *faithful* embedding $F : \mathbf{ho}(\mathbf{Top}) \to \mathbf{Set}$. (Recall that a functor $F : C \to D$ is *faithful* when for every pair of parallel morphisms $f, f' : X \to Y$ in C, if $F(f) = F(f') : F(X) \to F(Y)$ in D, then $f = f'$ was already the case in C. I urge the reader to look at Freyd's *proof*, for it is quite interesting in itself.) This has to be compared with more familiar cases. For instance, there are faithful embeddings $F : \mathbf{Grp} \to \mathbf{Set}$ — the forgetful functor will do. Thus, in this sense, the category **Grp** of groups is concrete. And so are most of the categories of structures one usually think of. In Freyd's terminology, the homotopy category $\mathbf{ho}(\mathbf{Top})$ is abstract. One interpretation of Freyd's result is that, from a functorial point of view, there is no way to interpret the morphisms of $\mathbf{ho}(\mathbf{Top})$ as set-theoretical functions. It is in this precise sense that the category $\mathbf{ho}(\mathbf{Top})$ is abstract — a concrete category C being a category for which the morphisms of C can be thought of as genuine functions between the objects of C[27]. I, for one, interpret this results as revealing the presence of a conceptual fault between the universe of homotopy types and the universe of (extensional) sets. But the conceptual quake set off by Freyd's result has not attracted the attention of philosophers of mathematics.

(27) Of course, Freyd's result does not contradict the fact that any category C can be embedded faithfully in the functor category $\mathbf{Set}^{C^{op}}$ by the so-called Yoneda embedding. For one thing, the latter category is considerably bigger than the category **Set**.

8. Defining abstract homotopy types

> The fundamental theorems of this paper are theorems 3.1 and 3.2. [...] These theorems (3.1 and 3.2) are of considerable interest in themselves. They exhibit a duality which is quite striking and seem to indicate a relatively unexplored region which I might designate as "algebra of mapping classes". In this connection they should be compared with the fundamental theorem of fibre spaces to which they bear an evident analogy. ([35, p. 40])

8.1. Abstract categorical contexts

The homotopy category is the abstract embodiment of the universe of homotopy types, if only because the isomorphisms are homotopy equivalences. But as a matter of fact, when one does homotopy theory, one seldom works only in **ho**(**Top**). Rather, one works in *models of homotopy theory* or, to use a slightly different terminology, specific instances of such a theory and uses functors between these categories and homotopy categories. Mathematicians do not work directly with homotopy types, but always with tokens of those types. This applies to categories as well and, thus, one works with *model categories*[28]. The latter provide the abstract setting in which one can define and do homotopy theory and one and the same category can provide *different* model categories. They are thus the universes of homotopy types. One does not try to define homotopy types as sets with a structure and then define a category with these objects and the appropriate morphisms between them, rather one tries to define directly a category of such entities and fully use categorical properties and constructions to understand them.

Casting homotopical ideas in the language of category theory leads to an abstract understanding of homotopy theory: its basic concepts and constructions turn out to be expressible in terms of the structure and properties of a category, in other words in terms of arrows only. However, as is very often the case, translating these concepts in categorical language introduces subtle differences and opens up vast possibilities[29]. We will provide the details of the lifting process in an appendix and move directly to the issues that are relevant to our enterprise.

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(28) The notion of model category is a technical notion and although the term 'model' is used here, it is clearly not used in the way we are using the term 'model' in this paper.

(29) As is very often the case, translating concepts into the categorical language is at the same time a generalization and an abstraction. This, in turn, yields new methods of definition and proof which are then transferable to other contexts.

In the nineteen fifties and sixties, these notions were lifted — no pun intended — from the topological context to various algebraic contexts. Then Dan Quillen introduced axioms for a purely abstract homotopy theory that allowed him to prove important theorems in rational homotopy theory.[30] (See [75], [76].) These axioms define what is now called a *model category.* It is then possible to construct a homotopy theory for that model category. It turns out that the important notions for these constructions are weak equivalences, fibrations and cofibrations.[31] I will here follow Dwyer and Spalinsky [32] and Hovey [44].

DEFINITION 8.1. — *A* model category *is a category C with three distinguished classes of maps:*

1. *weak equivalences,*
2. *fibrations,*
3. *cofibrations,*

each of which is closed under composition and contains all identity maps and satisfying the following axioms:

MC1 C has all finite limits and colimits.[32]

MC2 If f and g are maps in C such that $g \circ f$ is defined and if two of the three maps $f, g, g \circ f$ are weak equivalences, then so is the third. (The so-called 2-out-of-3 property.)

---

(30) I have to underline the fact that Quillen's development of an abstract homotopy theory was not the first nor is it the last. In a sense, it goes back to the beginning of algebraic topology itself with the notion of simplicial complexes. But as far as I can tell, J. H. C. Whitehead was the first to call explicitly for an algebraic analysis of homotopy theory in the 1940's. Whitehead's approach was based on a combinatorial framework with adequate categorical properties, namely CW-complexes, a notion that he introduced. See [93], [94]. In 1955, Daniel Kan entitled a series of four papers "Abstract Homotopy" in which he defined a homotopy in any category with certain objects satisfying very weak properties. (See [50], [51], [52], [53].) In particular, he showed how to define homotopy groups in the category of simplicial sets and showed that they corresponded to the traditional homotopy groups of spaces under precise correspondences. However, he did not define a type of category appropriate for homotopy theory. This step was more or less initiated by Quillen, inspired by Grothendieck's approach to homological algebra via abelian categories. In fact, Quillen saw homotopical algebra as a generalization of homological algebra, since a specific model category on a category of algebraic structures is identical with what is usually considered to be homological algebra. But there were and still are other approaches, basically depending on one's needs. See [4], [6], [16], [17], [42], [73], [80]. Hence what constitutes an adequate abstract homotopy theory is still being discussed. See [74]. There are still fundamental developments going on in the field revealing pivotal conceptual elements of the theory. See [31].

(31) To be more precise, it seems that weak equivalences are conceptually the basic elements and that fibrations and cofibrations are required to develop the homotopical machinery. See [31].

(32) In some definitions, this condition is relaxed to small limits and colimits.

MC3 *If f is a retract of g and g is a fibration, cofibration, or a weak equivalence, then so is f.*

MC4 *A map f that is both a cofibration and a weak equivalence is called a* trivial cofibration *and a map f is a* trivial fibration *if it is both a fibration and a weak equivalence.*[33] *Given a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[g]{} & Y \end{array}$$

a lift exists in the diagram in either of the following situations: (i) i is a cofibration and p is a trivial fibration, or (ii) i is a trivial cofibration and p is a fibration;

MC5 *Any map f can be factored in two ways: (i) $f = p \circ i$, where i is a cofibration and p is a trivial fibration, and (ii) $f = p \circ i$, where i is a trivial cofibration and p is a fibration.*

The investigations of the logical consequences of these axioms is what Quillen called *homotopical algebra*. It is the theory of the properties of any model category, thus a specific case of general abstract nonsense. To obtain specific results about particular cases, one has to specify a category with a chosen model structure. Here are some examples of model categories.

The first example is totally trivial. Let C be any category with finite limits and colimits. Then by stipulating that a morphism is a weak equivalence if and only if it is an isomorphism and that every morphism is both a cofibration and a fibration, one obtains a model category. That such a category satisfies the properties of a model category is immediately verified.

More interesting examples are offered by the category of topological spaces and the category of pointed topological spaces. In **Top**, one can take as weak equivalences the standard homotopy equivalences, as fibrations Hurewicz fibrations and as cofibrations closed cofibrations. As was shown first by Ström, this yields a model category. (See [83].)

A different and more useful model structure on the same category is provided by the following choices: weak equivalences are the usual *weak* homotopy equivalences, fibrations are the so-called Serre fibrations and cofibrations are retracts of specified maps. (See [32, section 8] or [44, section

(33) Some authors call a trivial fibration an *acyclic fibration* (respectively a trivial cofibration an acyclic cofibration).

2.4] for definitions and details.) These two examples illustrate how one and the same category can bear different model categories.

Quillen's axiomatization of a model category completely changed the face of homotopy theory. Model categories revealed new *classical* homotopy theories and opened the door to completely new and unforseeable homotopy theories.

> It's hard to tell from this vantage point whether or not it was one of the original goals of the theory, but it's major feature of homotopical algebra [sic] that the notion of homotopy itself "explodes" in this context. ([46, p. 653])

Furthermore, homotopy theory became independent of the original topological context.

> As such, the methods and ideas are algebraic and combinatorial and, despite the deep connection with the homotopy theory of topological spaces, exist completely outside any topological context. This point of view was effectively introduced by Kan, and later encoded by Quillen in the notion of a closed model category. Simplicial homotopy theory, and more generally the homotopy theories associated to closed model categories, can then be interpreted as a purely algebraic enterprise, ... The point is that homotopy is more than the standard variational principle from topology and analysis: homotopy theories are everywhere, along with functorial methods of relating them. [40, p. i]

Quillen's objective was to use homotopical methods in order to solve algebraic problems. This was made possible once he was able to show that the categories of algebraic structures he was interested in, namely the category of differential graded Lie algebras, could be turned into model categories. (See [76].) Quillen showed that there were many others: the category of simplical sets, the category of simplicial groups, or more generally categories of simplicial objects, the latter family covering a wide range of cases. Nowadays, these methods extend to other fields, most notably in algebraic geometry with the ground breaking work of Voevodsky. His proof of the Milnor conjecture rests on the transfer of homotopical methods to algebraic geometry and model categories play a pivotal role in this transfer. (See [91], [66].)

8.2. Homotopy categories

Model categories provide the materials to build a homotopy theory. Observe that any model category C, since it has finite limits and colimits, has an initial object, denoted as usual by 0 and a terminal object, denoted by 1. This allows us to define two classes of important objects in any model category.

DEFINITION 8.2. — *Let C be a model category. An object X of C is said to be* cofibrant *if the morphism $0 \to X$ is a cofibration and an object Y is said to be* fibrant *if the morphism $Y \to 1$ is a fibration.*

Given these definitions, it is possible to restate Whitehead's theorem in the language of model categories: any weak equivalence $f : X \to Y$ is a homotopy equivalence if X and Y are both fibrant and cofibrant. One can then show that CW-complexes are fibrant and cofibrant.

It is easy to convince oneself that in any model category C it is possible, for any given objects X and Y of C, to construct directly from the data of the model structure a cylinder on X and a path space on Y and, therefore, the notion of a left homotopy and the notion of a right homotopy. Furthermore, it can be shown that if X is a cofibrant object of C and Y is a fibrant object of C and $f, g : X \to Y$ are morphisms of C, then the left homotopy and the right homotopy relations coincide and are equivalence relations on $\mathrm{Hom}_C(X, Y)$. (See, for instance, [32, section 4] or [44, 9 – 11].) Thus, to make sure that left homotopies coincide with right homotopies, one takes the full subcategory C_{cf} of C with objects the cofibrant and fibrant objects of C. We therefore have a purely abstract notion of homotopy types at our disposal, completely independent of the topological setting we started with.

Although this gives us a completely reasonable definition of homotopy, we still do not have a homotopy theory yet: although homotopies between morphisms in C_{cf} are definable, homotopy equivalences are not necessarily invertible, as was the case with weak equivalences of CW-complexes. The strategy at that stage is the same as in the case of the category **Top** of topological spaces: move to the quotient category $C_{cf}/\sim$. And as in the case of **Top**, a lot of information is lost about the model structure, particularly the higher dimensional information, but the hope is that the homotopically relevant information is kept and is easily manageable. This is as always when one tries to translate a given structure into a different structure in the hope of capturing the essential ingredients of a situation. This is one of the reasons why there are various notions of model structures in the literature: it is a question of fine tuning between what is lost and what is gained by the same token.

This is *one* construction of the homotopy category of the model category C. There is a different construction yielding a different category, denoted by $\mathrm{Ho}(C)$, which is *categorically equivalent* — not isomorphic! — to the category $C_{cf}/\sim$. Furthermore, there are naturally defined functors $C \to C_{cf}/\sim$ and $\gamma : C \to \mathrm{Ho}(C)$. In both cases, the weak equivalences of the model category C become homotopy equivalences. More precisely, a morphism $f : X \to Y$ of the model category C is a weak equivalence if and only its image $\gamma(f)$ is an isomorphism in $\mathrm{Ho}(C)$. We are now fully in the framework of systemic models.

Furthermore, Quillen has given a definition of an equivalence between homotopy categories, thus providing a way to determine when two homotopy categories or theories are essentially the same. (See [75] or [44, section 1.3].)

Here is an interesting and important example of two seemingly different categories that yield Quillen equivalent homotopy categories. First, we need to introduce yet two more important categories, the simplicial category and the category of simplicial sets.

Let $\mathbf{\Delta}$ be the category whose objects are the sets $[n] = \{0, 1, ..., n\}$, for $n \geqslant 0$ a nonnegative integer with the natural order and with morphisms the order preserving maps between these sets, that is functions $f : [n] \to [m]$ such that if $i \leqslant j$ then $f(i) \leqslant f(j)$. This is the simplicial category[34].

The simplicial category has a geometric interpretation which plays a vital part in our story. In the euclidean space $\mathbb{R}^{n+1}$ the *standard geometric n-simplex* Δ_n is the convex closure of the standard basis $e_0 = (1, ..., 0), \ldots, e_n = (0, ..., 1)$, that is the subspace of $\mathbb{R}^{n+1}$ defined by

$$\Delta_n = \{p = \sum_{i=0}^{n} t_i e_i | t_0 \geqslant 0, ..., t_n \geqslant 0, \sum_{i=0}^{n} t_i = 1\}.$$

In the standard basis, Δ_0 is simply the point 1, Δ_1 is "the" unit interval [0, 1], Δ_2 is a triangle with the standard unit points as vertices, Δ_3 is a tetrahedron, etc. In general, Δ_0 is a point, Δ_1 is an interval or an edge, Δ_2 is a triangle or a face, etc. By convention, we can stipulate that Δ_{-1} is the empty set. One can see that the standard geometric n-simplex has $n + 1$ faces. These can be seen as images of the following injective maps, called

(34) As emphasized by Mac Lane in [60], the simplicial category has a protean character. It is the category of finite ordinal numbers and, as such, a full category of the category **Ord** of all linearly ordered sets. It is also a full subcategory of **Cat**, the category of small categories. It is the strict monoidal category containing the universal monoid. Finally, it is a subcategory of **Top**. When one adds categories of simplicial objects, that is functor categories from the simplicial category into a category C, then its roles in mathematics in general become daunting.

the *face maps*, $\delta_i : \Delta_{n-1} \to \Delta_n$,

$$\delta_i(x_0, ..., x_{n-1}) = (x_0, ..., x_{i-1}, 0, x_i, ..., x_{n-1}), \text{ for } i = 0, ..., n.$$

It is also possible to smash an $n+1$-simplex onto an n-simplex. By looking at specific cases, it is easy to see that it is precisely what the surjective maps $\sigma_i : \Delta_{n+1} \to \Delta_n$ do:

$$\sigma_i(x_0, ..., x_{n+1}) = (x_0, ..., x_i + x_{i+1}, ..., x_{n+1}), \text{ for } i = 0, ..., n.$$

These maps are called the *degeneracy maps*. The δ_i's and the σ_i's satisfy various relations that determine completely the maps between the standard geometric n-simplices.[35] Needless to say, the standard geometric n-simplices are object models of topological spaces.

The trick is now to identify the elements of an object $[n]$ of $\mathbf{\Delta}$ with the vertices $e_0, \ldots, e_n$ of Δ_n. We thus get a map from the objects of $\mathbf{\Delta}$ to the standard geometric n-simplexes, which are objects of **Top**. By assigning to maps $[n] \to [m]$ between objects of $\mathbf{\Delta}$, continuous functions $\Delta_n \to \Delta_m$, we obtain a functor $\mathbf{\Delta} \to \mathbf{Top}$. Thus the combinatorial data $[n]$ and their maps are embodied in geometrical objects Δ_n and geometrical transformations.

The category **sSet** of simplicial sets is the category whose objects are contravariant functors $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$ and whose morphisms are natural transformations between them. Notice immediately that there is no topological structure involved. Thus, simplicial sets are purely combinatorial (in contrast with CW-complexes which, although they were tailored for homotopy theory, have a topology). If we unpack this definition, we can see that a simplicial set X can be thought of as a graded set $(X_n)_{n\geqslant 0}$ together with two set of functions $d_i : X_n \to X_{n-1}$ for $i = 0, \ldots, n$ and $s_j : X_n \to X_{n+1}$ for $j = 0, \ldots, n$ satisfying the identities

$$d_i d_j = d_{j-1} d_i \text{ for } i < j; \tag{8.1}$$

$$s_i s_j = s_{j+1} s_i \text{ for } i \leqslant j; \tag{8.2}$$

$$d_i s_j = s_{j-1} d_i \text{ for } i < j; \tag{8.3}$$

$$d_i s_j = \mathrm{id}_{X_n} \text{ for } i = j \text{ or } i = j+1; \tag{8.4}$$

$$d_i s_j = s_j d_{i-1} \text{ for } i > j+1. \tag{8.5}$$

The maps d_i are called *face maps* and the maps s_j are called *degeneracy* maps. The elements of X are called *simplices* of X and the elements of a single X_n are called *n-simplices* of X.

(35) It is the σ's that make the simplicial category so useful in homotopy theory. It is possible to consider a subcategory of $\mathbf{\Delta}$ by restricting the morphisms to *strictly* order preserving maps, that is if $i < j$, then $f(i) < f(j)$. The resulting category, although useful in algebraic topology, does not play a role in homotopy theory.

Once again, there is a geometric content underlying the whole construction. One should informally think of each n-simplex x in X_n as a copy of the standard geometric n-simplex Δ_n. Thus, at each stage n, a simplicial set X has a set of n-simplices. These simplices are related to one another by the face and the degeneracy maps.

This geometric content has a formal realization: a simplicial set X admits a geometric realization $|X|$, a topological space constructed as follows. For each element x in X_n, one takes a one n-simplex and these are glued together according to the rules given by the faces and degeneracies. More specifically, the geometric realization is the quotient of the disjoint union of standard simplices Δ_n over an equivalence relation determined by the face and the degeneracy maps:

$$|X| = \bigsqcup_{i=1}^{n} X_n \times \Delta_n / \sim .$$

It can be shown that the geometric realization of a simplicial set is in fact a CW-complex. Thus, in this precise sense, a simplicial set can be thought of as a model of topological spaces. They thus constitute combinatorial models of spaces.

There is a model structure on the category of **sSet**. This was proved by Quillen in 1967 and the verification is far from trivial. (For a detailed proof, see [48].) The weak equivalences are *geometric homotopy equivalences*, that is a morphism $f : X \to Y$ between simplicial sets such that the induced continuous map $|f| : |X| \to |Y|$ is a homotopy equivalence between the geometric realizations. The cofibrations are monomorphisms and the fibrations are the so-called Kan fibrations. This reinforces the idea that simplicial sets are models of spaces. For the category of simplicial sets is a model category, thus a homotopical category. One can therefore construct a systemic model of the category of simplicial sets.

The punch line is that the homotopy category **ho**(**sSet**) of simplicial sets is Quillen equivalent to the homotopy category **ho**(**Top**) of the category of topological spaces with weak homotopical equivalences as equivalences, Serre fibrations as fibrations and certain retracts as cofibrations. This follows from the fact that the category **ho**(**Top**) is Quillen equivalent to the homotopy category of CW-complexes and it can be shown that the latter is Quillen equivalent to **ho**(**sSet**). In this precise technical sense, **ho**(**sSet**) is a good combinatorial model of homotopy types: it is essentially the same as far as the homotopy theory of homotopy types is concerned. (See [40] for details.) We now see how to construct various systemic models of homotopy types: when a homotopy category of a homotopical category is Quillen equivalent to the homotopy category of CW-complexes. It is important to note that, once again, we abstract a certain amount of information when we move to the homotopy category.

In closing, I have to point to important recent work that extend the foregoing considerations in conceptually interesting directions[36]. They represent a crucial conceptual shift in the way categorical constructions are extended to homotopy types or, more precisely, model categories. I will not give any technical details in this section. As we have seen, given a category C, it is possible to construct different model categories from it and, therefore, different homotopy categories. In a paper published in 2001, Daniel Dugger showed how to construct a universal model category from a given C. (See [30, 26, 11].) The notion of universality used here is the one coming from category theory and it is a precise technical notion. The important fact is that one can look at this universal model category and consider its properties. It is, in a precise sense, the best possible model and, as such, gives us information about what could be considered the "robust" content of any such model category. Since the latter are likely to occupy an important place in the foundations of mathematics (see [2, 92]), its detailed study might be extremely rewarding from a philosophical point of view. It might yield models for the whole universe of mathematics, an interesting prospect if there is one.

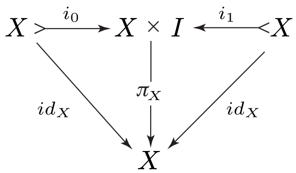
9. Conclusion

I hope I have provided enough information to at least bring the reader to consider seriously that pure mathematician are indeed building models and that the latter activity is fundamental in certain fields. I have tried to indicate how this practice evolved over the last seventy years or so and that the notion of model itself took different form and served different purposes. It is, as always, crucial to keep in mind the epistemological functions of these constructions: they are, right from the start, introduced as devices that capture essential properties of given mathematical objects and that provide definite epistemological gains over the objects one is trying to understand. They are not shadows of real objects, they are the keys to our understanding of fundamental features of basic mathematical ideas.

(36) This is but one direction! There is simply too much for us to cover. The explosion of material is still going on very strong.

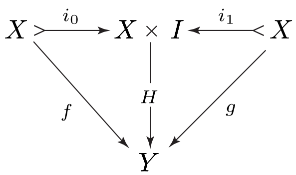
10. Appendix: Lifting homotopical concepts in a categorical framework

To raise homotopy theory to the abstract level, we basically have to start from scratch. We need to consider an equivalent way to define the relation of homotopy between morphisms. To fix ideas, we start with the familiar category of topological spaces and continuous spaces. (I basically follow Kathryn Hess's presentation to start with. See [43].) Given a space X, the *cylinder* on X is the product space $X \times I$ together with the monomorphisms $i_0 : X \rightarrowtail X \times I$ and $i_1 : X \rightarrowtail X \times I$ defined by $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$ such that the following diagram commutes:



The map $\pi_X : X \times I \to X$ is the standard projection, namely $\pi_X(x, t) = x$ for all t. When X is the standard circle S^1, it is clear that we get what is customarily called a cylinder with height equal to a unit. So, the foregoing construction is a simple generalization of the standard concept. The underlying informal process depicted by this construction is that we have a copy of X at the bottom of the cylinder, a copy of X at the top of the cylinder and a continuous "stretch" between the two. We have basically made a "morphing" of X with itself. But this is still not the abstract concept of cylinder. We will get to that shortly.

Not surprisingly, the concept of a cylinder provides an equivalent definition of a homotopy between continuous maps. Two continuous maps $f, g : X \to Y$ are said to be *left homotopic* if there is a continuous map $H : X \times I \to Y$ such that the following diagram commutes:

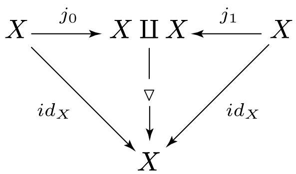


Here the commutativity of the left triangle expresses the fact that f is at the "bottom" of the cylinder and the commutativity of the right triangle means that g is at the top. Since H is a continuous map defined on the whole cylinder, it transforms f into g in a continuous manner, that is, it is a homotopy from f to g. In contemporary terminology, H morphs f into g.

Clearly, the relation of being left homotopic is an equivalence relation between morphisms. In the categories **Top** and $\mathbf{Top}_*$, this is merely an equivalent characterization of the notion of an homotopy of maps.

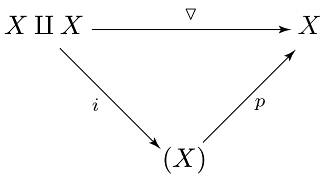
All the previous constructions took place in the category **Top**. It is possible to move to an arbitrary category satisfying certain simple properties, namely the category **C** has to have finite limits and colimits[37]. From now on, we will assume that we work in a category having the right properties. In the latter context, the notion of a cylinder of an object X is *not* defined by taking the product of X with a unit interval I (together with the induced morphisms). It is defined from very basic morphisms available in such a category.

First, given an object X in a category **C**, a coproduct of X with itself is defined as an object, denoted by $X \amalg X$, together with two morphisms $j_0 : X \to X \amalg X$ and $j_1 : X \to X \amalg X$ satisfying the usual universal property. The latter property together with the given morphism $id_X : X \to X$ automatically yield the so-called folding morphism $\nabla : X \amalg X \to X$ which arises in the following diagram:



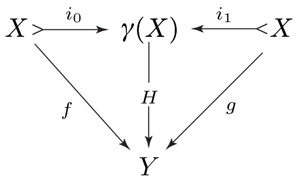
I want to emphasize the fact that the folding morphism comes from the categorical machinery and nothing else. It is a simple example of general abstract nonsense at work. There is an obvious representation of the folding morphism that justifies its name: it is as if one would "fold" the object X over itself and then project these two copies of X which are right above one another down on X by the folding morphism.

Given an object X, a *cylinder* $\gamma(X)$ on X is given by a factorization of the folding morphism thus:

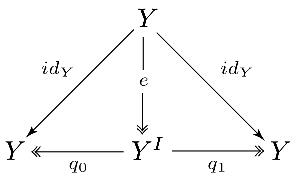


(37) Often, this is relaxed to all *small* limits and colimits. But this is a fine technical point that we will ignore here.

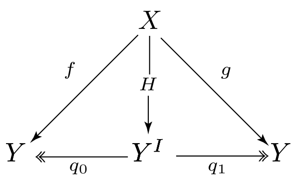
where the morphism $p : \gamma(X) \to X$ has to be a weak equivalence (a property to which we will come back). Defining the morphisms $i_0 = i \circ j_0$ and $i_1 = i \circ j_1$, we get the abstract notion of a left homotopy H between two morphisms $f : X \to Y$ and $g : X \to Y$ in $\mathbf{C}$ by stipulating that f is left homotopic to g if there exists a left homotopy H making the following diagram commutative:



The dual notion of a *right homotopy* is given by the concept of a path space. Classically, a *path space on* Y, denoted by Y^I, is the space $\{\alpha : I \to Y \mid \alpha \text{ is continuous}\}$ with the compact-open topology and the continuous maps $q_0 : Y^I \to Y$, $q_1 : Y^I \to Y$, defined by $q_0(\alpha) = \alpha(0)$ and $q_1(\alpha) = \alpha(1)$, such that the following diagram commutes:

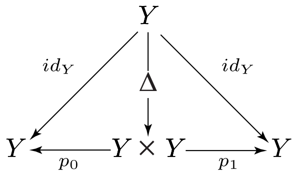


where $e(y)$ is the constant path at y. Two continuous maps $f, g : X \to Y$ are said to be *right homotopic* if there is a continuous map $H : X \to Y^I$ such that the following diagram commutes:

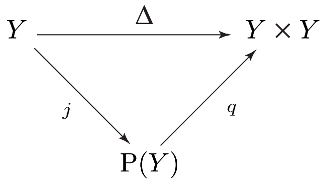


As was the case with left homotopies, being right homotopic is also an equivalence relation.

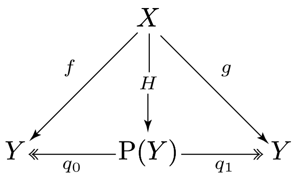
The construction of a path *object* $\mathrm{P}(Y)$ on an object Y of a category $\mathbf{C}$ is based on similar basic categorical properties. First, given a product $Y \times Y$ of Y with itself, the *diagonal* morphism $\Delta : Y \to Y \times Y$ is defined canonically by the universal property of products:



A path object $\mathrm{P}(Y)$ on Y is given by a factorization of the diagonal morphism:



where this time the morphism j is a weak equivalence. Defining the morphisms $q_0 = p_0 \circ q$ and $q_1 = p_1 \circ q$, an abstract right homotopy between two morphisms $f : X \to Y$ and $g : X \to Y$ is given by a morphism $H : X \to \mathrm{P}(Y)$ such that the diagram

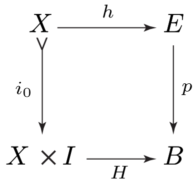


commutes.

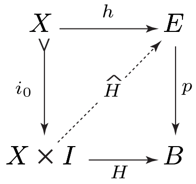
It is well-known that for topological spaces satisfying some mild natural properties, there is a natural bijection between continuous maps of the form $X \times I \to Y$ and $X \to Y^I$ and thus, we can see how the notion of being right homotopic corresponds naturally to the standard definition.[38] In the abstract context, one can specify precisely under what conditions left and right homotopies coincide. We will come back to these conditions in the next section.

The cylinder object and the path object play a crucial role in the definition of fundamental concepts of homotopy theory which are also expressible in terms of diagrams. A continuous map $p : E \to B$ is said to satisfy the *homotopy lifting property* if whenever there is a commutative square

(38) In fact, in the appropriate conditions, this is a basic example of an adjoint situation.

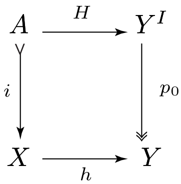


then there is a diagonal map $\widehat{H} : X \times I \to E$

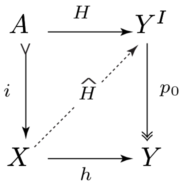


such that the two resulting triangles commute. As the name of the property indicates, the map $\widehat{H} : X \times I \to E$ lifts the homotopy $H : X \times I \to B$ through p and extends h over i_0. A continuous map $p : E \to B$ which satisfy the homotopy lifting property for all such commutative squares is said to be a *Hurewicz fibration*.

The dual notion of a cofibration is immediate. A subspace $i : A \rightarrowtail X$ satisfies the *homotopy extension property* if whenever there is a commutative square



there is a diagonal map $\widehat{H} : X \to Y^I$



such that the two resulting triangles commute. Again, as the name indicates, the map $\widehat{H} : X \to Y^I$ extends the homotopy $H : A \to Y^I$ over i and lifts h through p_0. A continuous map $i : A \to X$ which satisfies the homotopy extension property for all such commutative squares is said to be a *Hurewicz cofibration*.

Although we have presented them in an abstract setting, the homotopy lifting property and the homotopy extension property arose from specific problems and needs in the homotopy theory of topological spaces. (See [29] or [63].) They are two technical properties that play a crucial role in many homotopical contexts involving covering spaces, fiber bundles and other important topological constructions. They are also related to the notion of homotopy equivalence of spaces, thus to homotopy types. It is certainly worth sketching the link if only to give an idea of the role played by these properties.

Let us start with a definition.

Definition 10.1. — *A morphism $f : A \to X$ of a category C is a* retract *of a morphism $g : B \to Y$ if and only if there is commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & A \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \longrightarrow & Y & \longrightarrow & X \end{array}$$

where the horizontal composites are the identities.

This is not the definition one finds in topology textbooks. In the latter, a retract or a retraction is a map $f : X \to A$ having a right inverse, i.e. there is a map $g : A \to X$ such that $f \circ g = \mathrm{id}_A$. It is, however, a special case of the former definition. If $f : X \to A$ is a retraction, then the following diagram is a particular case of the definition:

$$\begin{array}{ccccc} A & \xrightarrow{g} & X & \xrightarrow{f} & A \\ {\scriptstyle \mathrm{id}_A}\downarrow & & {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_A} \\ A & \xrightarrow[g]{} & X & \xrightarrow[f]{} & A \end{array}$$

Thus, if the spaces are represented by their identity maps, we can say that the space A is a retract of the space X. In the categorical definition, the morphisms are taken as objects.

Retractions underlie a concept closely related to homotopy types. A continuous map $r : X \to A$ is a *deformation retraction* if it is a retraction and the composition with the inclusion $i : A \rightarrowtail X$ is homotopic to the identity map id_X. Equivalently, a deformation retraction is a homotopy

$H : X \times I \to X$ such that $\forall x \in X$ and $\forall a \in A$, $H(x, 0) = x$, $H(x, 1) \in A$ and $H(a, 1) = a$. In this case, the subspace A is said to be a deformation retract of X. It is immediate that a deformation retraction is a special case of a homotopy equivalence. A few examples of deformation retracts immediately explain the terminology. In figure 4, the eight figure is a deformation retract of the ambient space with two holes and the arrows illustrate the process of retraction.

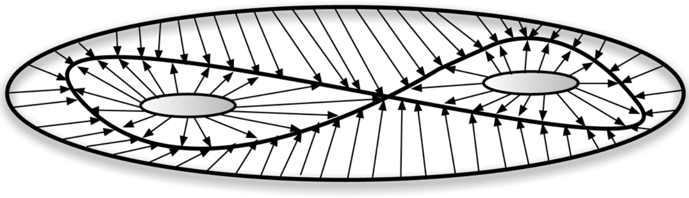


Figure 1. — A deformation retraction

Another example of a deformation retraction is provided by the deformation of the Mőbius band onto its core circle S^1.

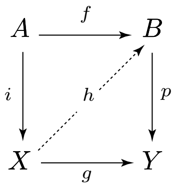
Deformation retractions are linked to the homotopy extension property by two important results. First, if the pair (X, A) satisfies the homotopy extension property and the inclusion $i : A \rightarrowtail X$ is a homotopy equivalence, then A is a deformation retract of X. Second, two spaces X and Y are homotopy equivalent if and only if there is a third space containing both X and Y as deformation retracts. (See [41, pages 14 – 17] for precise statements and proofs.)

As we have already indicated, the key properties at work in homotopy theory are the homotopy lifting property and the homotopy extension property. They can be defined for any category C as follows.

Given a commutative square

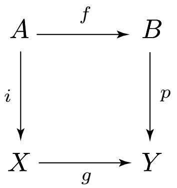
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[g]{} & Y \end{array}$$

a *lift* or a *lifting* in the diagram is a morphism $h : X \to B$ such that the resulting diagram



commutes, i.e. $h \circ i = f$ and $p \circ h = g$.

A morphism $i : A \to X$ is said to have the *left lifting property* (LLP) with respect to another morphism $p : B \to Y$ and p is said to have the *right lifting property* (RLP) with respect to the morphism i if a lift $h : X \to B$ exists for any commutative square



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