On the Paradox of the Adder

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Nuel Belnap formulated his paradox in the following manner:

> Let δ symbolize the denotation function, and let us baptize the symbol "'$\delta a + 1$'", using the letter "a" for its name; that is, we declare that a='$\delta a + 1$'. So by applying the denotation-function, δ, to both sides, we readily obtain $\delta a = \delta$('$\delta a + 1$'). But δ('$\delta a + 1$')=$\delta a + 1$ (because δ is denotation), so $\delta a = \delta a + 1$. As a result, there is a paradox. Ordinary Peano arithmetic says that the adding-one function has no fixed point: $n + 1 \neq n$, all n. But this stands in contradiction to what we have just shown, that $\delta a = \delta a + 1$, i.e., that δa is a fixed point of adding one.

Nuel Belnap (2006: *Prosentence, Revision, Truth, and Paradox*, http://www.pitt.edu/ belnap/143prosentence.pdf, Philosophy and Phenomenological Research Vol. LXXIII, No. 3, November, pp. 705–712.)

It is clear that in the formula '$\delta a + 1$', the term 'δa' denotes a number and not a numeral, because in this context the addition is a mathematical operation that is defined on numbers and not signs, thus the reasoning is grammatically correct. From a semantic point of view, Belnap emphasized the similarity between his paradox and another circular paradox:

> We may also "solve" the Adder in analogy to the three-valued "solution" to the Liar paradox. That is, we may let u be "the ungrounded number," following Kleene arithmetic by declaring that u is a fixed point for the adding-one function: $u + 1 = u$. Then we can let the denotation of a be u, which "solves" the paradox.

> ... the Adder is exactly the same as the Liar, *mutatis mutandis*: Given the Liar and no hierarchy, either negation has a fixed point, in which case we are not doing (ordinary two-valued) semantics, or else, if negation has no fixed point, we have a contradiction in our semantic theory.

Let me shed some light on the connection to the Liar paradox following Belnap's argumentation in another way. Let '$\sim$' be Kleene's three-valued negation, where 's' is a sentence: 1=True, 0=False, 2=not True and not False, and $|s|$ is the logical value of s sentence.

The truth table:

s	$\sim s$
1	0
0	1
2	2

This means:

(A1) If $1 = |s|$ then $0 = |\sim s|$
(A2) If $0 = |s|$ then $1 = |\sim s|$
(A3) If $2 = |s|$ then $2 = |\sim s|$
(A4) $0 \neq 1$ and $1 \neq 2$

Let L be the Liar sentence formulated by (1):

*(1)	$L \leftrightarrow_{df} \sim L$	(This is a circular definition.)
*(2)	$L \leftrightarrow \sim L$	(1)
*(3)	$2 = \|L\| \leftrightarrow 2 = \|\sim L\|$	(2) (A3)
(4)	If $L \leftrightarrow_{df} \sim L$ then $2 = \|L\|$	(3)

In other words, the Liar sentence does not express a proposition, because it is neither true nor false. Consider, then, the following inference of the Adder paradox:

(A1)	For all x, if x is a finite number then $x \neq x+1$	
(A2)	$\forall x$, $x = \delta(\lceil x \rceil)$	
*(1)	$a =$ '$\delta a + 1$'	
*(2)	$\delta(a) = \delta($'$\delta a + 1$'$)$	(1) (A2)
*(3)	$\delta($'$\delta a + 1$'$) = \delta a + 1$	(A2)
*(4)	$\delta a = \delta a + 1$	(1) (3)
(5)	If $a =$ '$\delta a + 1$'. Then $\delta a = \delta a + 1$	(1)(4)
(6)	If δa is a finite number then $\delta a \neq \delta a + 1$	(A1)
(7)	δa is not a finite number but something else, e.g. a infinite cardinal number.	(1)(5)(6)

In both cases there is a fixed point.

The above solution does not work if we insist that the Liar sentence is true or false or that δa is a finite number. Solving the Liar paradox in the framework of classical logic or semantic and definition theory, we must use language levels and different truth predicates. The following brief outline of Tarski's solution is based on an interpretation of the Axiom of Specification: To every set of sentence names P and to every one-to-one map from sentence names to sentences $S(x)$ there corresponds a set of sentence names T whose elements are exactly those elements x of P for which $S(x)$ holds.

"Axiom of specification: To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds." Paul R. Halmos (1960: *Naive Set Theory*, http://www.questia.com/read/10345494, Van Nostrand., Princeton, NJ., p.6)

Let P, P_1 be sets of sentence names; T, T_1, T_2 sets of true sentence names; and S, δ_1 one-to-one maps (bijective functions) from sentence names to sentences.

(1)	$\forall P \forall S \exists T \forall x (x \in T \leftrightarrow (x \in P \,\&\, S(x)))$	Axiom of Specification
(2)	$\forall S \exists T \forall x (x \in T \leftrightarrow (x \in P_1 \,\&\, S(x)))$	(1)
(3)	$\exists T \forall x (x \in T \leftrightarrow (x \in P_1 \,\&\, \delta_1(x)))$	(2) δ_1 where $\delta_1(\lambda) \leftrightarrow_{df} \lambda \notin T_1$
(4)	$\forall x (x \in T_2 \leftrightarrow (x \in P_1 \,\&\, \delta_1(x)))$	(3)T_2
(5)	$\lambda \in T_2 \leftrightarrow (\lambda \in P_1 \,\&\, \delta_1(\lambda))$	(4)
(6)	$\lambda \in T_2 \leftrightarrow (\lambda \in P_1 \,\&\, \lambda \notin T_1)$	(5)
(7)	$\lambda \in P_1 \rightarrow (\lambda \in T_2 \leftrightarrow \lambda \notin T_1)$	(6)

Assuming that we have only a unique set of Truth ($T_2 = T_1$), the consequence is logical fallacy, thus if $\lambda \in P_1$, that is, if λ is a true or false sentence

($x \in P_1 :=x$ expresses a proposition), then we must use different sets of true sentences that are extensions of different truth predicates at different language levels.

Belnap claims:

> We may "solve" the Adder paradox in the Tarski way, by classifying it as due to bad grammar ($a =$'$\delta a + 1$' both uses "a" on the left, and mentions it on the right).

Belnap is right regarding the similarity, but he fails to localize the root of the disease. He should not degrade Tarski's solution of semantic paradoxes by using quotation marks, but should apply Tarski's theory properly. For Tarski, every semantic functor, such as the 'denote' function or 'true' predicate,has a certain level that is connected to a given language. This means that we have a 'denote$_1$' function (δ_1) in language$_1$ and 'denote$_2$' function (δ_2) in language$_2$. Let us suppose that: (D2) $\forall x \in$ terms of language$_2$, $x = \delta_2(\lceil x \rceil) :=$ For every x terms of language$_2$, $\lceil x \rceil$ denotes$_2$ x that is " 'a' " denotes "a", " 'b' " denotes "b", etc. "Julius Caesar" denotes a Roman general but " 'Julius Caesar' " denotes a name of a Roman general. (Where $\lceil\ \rceil$ is the symbol for the citation function or quasi-quotation in terms of Quine.) Note that, by using quotation marks, we mention rather than use the first two letters of the alphabet. Applying the considerations above we arrive at the following formulation of the Adder paradox.

Let δ_1 symbolize the denotation function in language L_1 defined for all the terms of L_1, and let us baptize the symbol " '$\delta_1(a) + 1$' ", using the letter "a" for its name; that is, we declare that $a =$ '$\delta_1(a) + 1$', where "$a =$ '$\delta_1(a) + 1$' " is an object language sentence. Following Tarski, we can talk about this sentence on a metalanguage level. We can talk about truth, or about what terms denote on both sides of identity relations, only at the level of metalanguage. We can claim at the metalanguage level that $\delta_2(a) = \delta_2$('$\delta_1(a) + 1$'), where δ_2 means the 'denote' function in L_2 defined for all the terms of L_2. (For the sake of simplicity we suppose that the metalanguage contains the object language as a part.) It follows from D2 that δ_2('$\delta_1(a) + 1$') $= \delta_1(a) + 1$, hence $\delta_2(a) = \delta_1(a) + 1$, but this sentence of L_2 does not contradict Peano's axioms. In this way the contradiction immediately disappears, thus—not surprisingly—we have eliminated the paradox. Sometimes it is worth using Tarski's solution rather than merely mentioning it.