An Epistemic Interpretation of Paraconsistent Weak Kleene Logic

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Abstract

This paper extends Fitting's epistemic interpretation of some Kleene logics, to also account for Paraconsistent Weak Kleene logic. To achieve this goal, a dualization of Fitting's "cut-down" operator is discussed, rendering a "track-down" operator later used to represent the idea that no consistent opinion can arise from a set including an inconsistent opinion. It is shown that, if some reasonable assumptions are made, the truth-functions of Paraconsistent Weak Kleene coincide with certain operations defined in this track-down fashion. Finally, further reflections on conjunction and disjunction in the weak Kleene logics accompany this paper, particularly concerning their relation with containment logics. These considerations motivate a special approach to defining sound and complete Gentzen-style sequent calculi for some of their four-valued generalizations.

Keywords: weak Kleene logic; infectious logic; containment logic; sequent calculus

1 Introduction

Paraconsistent Weak Kleene logic (PWK, for short) is the three-valued logic that arises from the *weak* truth-tables due to Kleene [33], when the intermediate value (which we will, provisionally, call **u**) is taken to be a *designated* value. These truth-tables—represented in Figure 1 below—are peculiarly referred to as weak, because they exhibit a sort of "infectious" behavior of the intermediate value: in fact, this value is assigned to a complex formula whenever one of its components is assigned so.

Moreover, notice that the Paracomplete Weak Kleene logic (K_3^w, for short) is another three-valued logic that arises from these truth-tables, when the intermediate value is not taken to be designated. Naturally, it is because these logics are defined making essential use of the weak Kleene truth-tables that, e.g. in [27], Fitting refers to them (and by extension to their eventual four-valued generalizations, on which more below) as *weak Kleene logics.*[1]

	$\neg$
t	**f**
u	**u**
f	**t**

$\wedge$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**u**
f	**f**	**u**	**f**

$\vee$	**t**	**u**	**f**
t	**t**	**u**	**t**
u	**u**	**u**	**u**
f	**t**	**u**	**f**

Figure 1: The weak Kleene truth-tables

These systems can be compared, primarily, with the three-valued logics defined in terms of the otherwise *strong* Kleene truth-tables—represented in Figure 2 below. Thus, when the intermediate value featured in this strong truth-tables is taken to be non-designated the induced system is usually referred to as Strong Kleene logic (K_3, for short), while the system induced by taking the intermediate value to be designated is usually referred to as Priest's Logic of Paradox (LP, for short). Analogously to the previous remarks, then, it is because these logics are defined making essential use of the strong Kleene truth-tables that, e.g. in [26], Fitting refers to them (and by extension to their eventual four-valued generalizations, on which more below) as *strong Kleene logics.*

[1] It should be noted that PWK and K_3^w are usually identified as the classical, "internal" or $\{\neg, \wedge, \vee\}$-fragments of Halldén's and Bochvar's logics of nonsense presented, respectively, in [7] and [29].

	$\neg$
t	**f**
u	**u**
f	**t**

$\wedge$	**t**	**u**	**f**
t	**t**	**u**	**f**
u	**u**	**u**	**f**
f	**f**	**f**	**f**

$\vee$	**t**	**u**	**f**
t	**t**	**t**	**t**
u	**t**	**u**	**u**
f	**t**	**u**	**f**

Figure 2: The strong Kleene truth-tables

Among the weak Kleene logics, an increasing amount of recent work has been focused on Paraconsistent Weak Kleene (see e.g. [12], [11], [10], [35], [9], [47], [36], [40]), with the salient absence of a cogent philosophical interpretation for it. The aim of this paper is to try to overcome this lack, by offering an *epistemic* interpretation for PWK, that is, an epistemic understanding of its truth-values, its consequence relation and its characteristic truth-tables.

To achieve this goal, we will benefit from Fitting's epistemic interpretation of some of the Kleene logics in works such as [25], [26] and [27]. Interestingly, Fitting himself showed how his epistemic interpretation is flexible enough not only to endow the *strong* Kleene logic K_3 and one of its four-valued generalizations, the logic FDE (studied in [16] and [6]) with an epistemic interpretation, but also to provide such a reading for the paracomplete *weak* Kleene logic K_3^w and one of its four-valued generalizations, the logic $\mathsf{S}_{\mathtt{fde}}$ (on which more below). Quite surprisingly, an attempt to broaden the range of application of this reading to cover the case of Paraconsistent Weak Kleene has not been proposed so far. This is, specifically, what we intend to do in this paper: to show how Fitting's epistemic interpretation of the Kleene logics can also account for the case of PWK. We are, so to say, after the missing piece of the puzzle.

To this extent, the paper is structured as follows. In Section 2 we give a few formal preliminaries and discuss a little bit more rigorously some aspects of the weak Kleene logics and their four-valued generalizations. Section 3 is the main section of the paper, where Fitting's epistemic interpretation of the Kleene logics is reviewed, and our novel reading of Paraconsistent Weak Kleene is presented in full detail. Section 4 has two parts, and in each of them additional formal results are presented. First, some new results are provided, concerning the relation between subsystems of the weak Kleene logics and containment logics—i.e. systems whose valid inference comply with certain set-theoretic containment principles relating the set of propositional variables appearing in the premises and the set of propositional variables appearing in the conclusion. Secondly, making essential use of these new results, sound and complete Gentzen-style sequent calcui for some four-valued generalization of the weak Kleene logics are introduced. Finally, Section 5 outlines some concluding remarks.

2 Preliminaries

Let $\mathcal{L}$ be a propositional language and let Var be a set of propositional variables, assumed to be countably infinite. By $\mathbf{FOR}(\mathcal{L})$ we denote the absolutely free algebra (of formulae), freely generated by Var, with universe $FOR(\mathcal{L})$. In all of the cases considered in this paper the propositional language will be fixed to be the set $\{\neg, \wedge, \vee\}$. In what follows, capital Greek letters Γ, Δ, etc. will denote sets of formulae, and lowercase Greek letters φ, ψ, etc. will denote arbitrary formulae. As usual, a logic L is a pair $\langle FOR(\mathcal{L}), \vdash_{\mathsf{L}}\rangle$, where $\vdash_{\mathsf{L}} \subseteq \wp(FOR(\mathcal{L})) \times FOR(\mathcal{L})$ is a substitution-invariant consequence relation.

For $\mathcal{L}$ a propositional language, an $\mathcal{L}$-matrix (a matrix, for short) is a structure $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$, where $\langle \mathcal{V}, \mathcal{O}\rangle$ is an algebra of the same similarity type as $\mathcal{L}$, with universe $\mathcal{V}$ and a set of operations $\mathcal{O}$, and $\mathcal{D} \subset \mathcal{V}$.[2] Given a matrix, a valuation v is an homomorphism from $FOR(\mathcal{L})$ to $\mathcal{V}$, for which we denote by $v[\Gamma]$ the set $\{v(\gamma) \mid \gamma \in \Gamma\}$, i.e. the image of v under Γ. Finally, by a matrix logic L we understand a pair $\langle FOR(\mathcal{L}), \vDash_{\mathcal{M}}\rangle$ where $\vDash_{\mathcal{M}} \subseteq \wp(FOR(\mathcal{L})) \times FOR(\mathcal{L})$ is a substitution-invariant

[2]Notice that $\mathcal{O}$ is a set that includes for every n-ary operator $\diamond$ in the language $\mathcal{L}$, a corresponding n-ary truth-function $f^{\diamond}_{\mathcal{M}} : \mathcal{V}^n \longrightarrow \mathcal{V}$.

consequence relation defined by letting

$$\Gamma \vDash_{\mathcal{M}} \varphi \iff \text{ for every valuation } v, \text{ if } v[\Gamma] \subseteq \mathcal{D}, \text{ then } v(\varphi) \in \mathcal{D}$$

Moreover, when $\mathsf{L} = \langle FOR(\mathcal{L}), \vDash_{\mathcal{M}} \rangle$ we may alternatively denote $\vDash_{\mathcal{M}}$ as $\vDash_{\mathsf{L}}$.

Moving on to some specifics of our investigation, when analyzing logical systems below, it will be useful to consider *infectious* matrix logics, as defined next.[3]

Definition 2.1. A matrix logic $\mathsf{L} = \langle FOR(\mathcal{L}), \vDash_{\mathcal{M}} \rangle$ is *infectious* if and only if there is an element $\mathbf{x} \in \mathcal{V}$ such that for every n-ary $f^{\diamond}_{\mathcal{M}} \in \mathcal{O}$ it holds that

$$\textit{if } \mathbf{x} \in \{\mathbf{v}_1, \dots, \mathbf{v}_n\}, \textit{ then } f^{\diamond}_{\mathcal{M}}(\mathbf{v}_1, \dots, \mathbf{v}_n) = \mathbf{x}$$

As is easy to see, both Paraconsistent Weak Kleene and Paracomplete Weak Kleene can be faithfully described as three-valued *infectious* logics, with the subtle difference that in the former the infectious value is taken to be designated, whereas in the latter it is not. Furthermore, of much interest to us and of much use in finding an epistemic interpretation for PWK, is to look at infectious *subsystems*—most particularly *four-valued* subsystems—of these three-valued matrix logics. Without loss of generality, in what follows we will be assuming that semantics for these systems count with the classical truth-values $\mathbf{t}$ and $\mathbf{f}$, and two additional truth-values: $\top$ and $\bot$.

Four-valued subsystems of the Kleene logics in general, and some subsystems of PWK in particular, have been studied in recent works, e.g. [48], [47] and [41]. Along this line, *normal* four-valued generalizations of the Kleene logics are taken to be those in which the truth-functions for the connectives coincide with those of Classical Logic, when restricted to the set $\{\mathbf{t}, \mathbf{f}\}$. Among the normal generalizations, however, there are two further salient families which have caught the attention of scholars: the family of regular and the family of monotonic generalizations.

Regular systems are defined in [48, p. 226] to be such that all of its truth-functions comply with the criterion—quoted from [41, p. 4] with notation adjusted to fit ours—that a given column (row) of the truth-table contains $\mathbf{t}$ in the $\top$ or $\bot$ row (column), only if the column (row) has $\mathbf{t}$ in all of its cells; and likewise for $\mathbf{f}$. *Monotonic* systems are presented, as is usual, in terms of all of the truth-functions of the underlying matrix preserving some previously defined order over the truth-values. As detailed in [41, p. 4], the four truth-values of Kleene's four-valued generalizations are ordered in [48] by letting: $\bot \leq \mathbf{f}$, $\bot \leq \mathbf{t}$, $\mathbf{f} \leq \top$, $\mathbf{t} \leq \top$, allowing $\mathbf{t}$ and $\mathbf{f}$ to be incompatible—this is, precisely, the "information" order of the lattice A4 detailed in [6].

As reported in [41], it was proved in [48] that there are 81 four-valued monotonic logics of which only 6 are regular. To this extent, it must be highlighted that the four-valued generalizations of the weak Kleene logics that we are going to discuss next are not regular, although they are indeed monotonic—as is routine to check.

In this vein, it is worth looking at the four-valued logic $\mathsf{S}_{\mathtt{fde}}$ presented below, in Definition 2.2. This system, introduced by Harry Deutsch in [14], has since then been discussed several times in the literature, with different purposes, e.g. in [26], [34], [44], [21], [47], [42].

Definition 2.2. $\mathsf{S}_{\mathtt{fde}}$ is the four-valued logic induced by the matrix $\langle \mathcal{V}_{\mathsf{S}_{\mathtt{fde}}}, \mathcal{D}_{\mathsf{S}_{\mathtt{fde}}}, \mathcal{O}_{\mathsf{S}_{\mathtt{fde}}} \rangle$, where $\mathcal{V}_{\mathsf{S}_{\mathtt{fde}}} = \{\mathbf{t}, \top, \bot, \mathbf{f}\}$, $\mathcal{D}_{\mathsf{S}_{\mathtt{fde}}} = \{\mathbf{t}, \top\}$, $\mathcal{O}_{\mathsf{S}_{\mathtt{fde}}} = \{f^{\neg}_{\mathsf{S}_{\mathtt{fde}}}, f^{\wedge}_{\mathsf{S}_{\mathtt{fde}}}, f^{\vee}_{\mathsf{S}_{\mathtt{fde}}}\}$ and these truth-functions are defined by the truth-tables in Figure 3.

The most important thing about $\mathsf{S}_{\mathtt{fde}}$ is that Fitting, in [27], has taken this logic to be *a four-valued generalization* of K^w_3. Why so? On the one hand, it is legitimate to say it is a *generalization* of Paracomplete Weak Kleene, first, because its semantics include an undesignated infectious value, just like the semantics for K^w_3. Secondly, because this fact—together with the information that it is normal, in the above technical sense—secures that when valuations are restricted to the set $\{\mathbf{t}, \bot, \mathbf{f}\}$ we obtain nothing more than the semantics for K^w_3. On the other hand, it is a *four-valued* generalization,

[3] The following definition is inspired by [31], although a similar definition might be found in [19]. For a generalization of this notion that also applies to non-deterministic matrices (as defined in [3]) see [47].

	$f^{\neg}_{\mathsf{S_{fde}}}$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$
$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$

$f^{\wedge}_{\mathsf{S_{fde}}}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\top$	$\top$	$\top$	$\bot$	$\mathbf{f}$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\bot$	$\mathbf{f}$

$f^{\vee}_{\mathsf{S_{fde}}}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\bot$	$\mathbf{t}$
$\top$	$\mathbf{t}$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$

Figure 3: Truth-tables for the logic $\mathsf{S_{fde}}$ (the four-valued generalization of K_3^w)

because its semantics include an additional non-classical truth-value, namely $\top$, such that both it and its negation are designated—whence e.g. Explosion, the inference $\varphi \wedge \neg\varphi \vDash \psi$, is invalid in $\mathsf{S_{fde}}$. This informs us, moreover, that we are in front of a *paraconsistent* subsystem of K_3^w.

The importance of considering four-valued generalizations of Kleene logics does not rely, however, just on their technical interest. In fact, as we will see later in Section 3, it is only after offering an epistemic interpretation for Belnap-Dunn logic FDE—whose truth-functions are discussed in Section 3.2 below—and looking at Strong Kleene logic K_3 through the eyes of such a reading, that Fitting provided an epistemic interpretation of the latter. Similarly, it was only after offering an epistemic interpretation for $\mathsf{S_{fde}}$ and looking at Paracomplete Weak Kleene with the tools provided by such a reading, that Fitting arrived at an *epistemic interpretation* for K_3^w.

Thus, it will be through a similar path that we will arrive at an epistemic interpretation for PWK. We will, then, find a suitable four-valued generalization of Paraconsistent Weak Kleene, which we will later endow with an epistemic interpretation. Hence, it will be only after looking at PWK through this interpretation, that we will be able to provide an epistemic understanding for it. Our target four-valued system, which we will call $\mathsf{dS_{fde}}$, is presented below in Definition 2.3. This system was first introduced in [47], although with a different name.[4] Here, we prefer to call it $\mathsf{dS_{fde}}$ for it clearly is the dual of Deutsch's $\mathsf{S_{fde}}$.[5]

Definition 2.3. The logic $\mathsf{dS_{fde}}$ is induced by the matrix $\langle \mathcal{V}_{\mathsf{dS_{fde}}}, \mathcal{D}_{\mathsf{dS_{fde}}}, \mathcal{O}_{\mathsf{dS_{fde}}} \rangle$, where $\mathcal{V}_{\mathsf{dS_{fde}}} = \{\mathbf{t}, \top, \bot, \mathbf{f}\}$, $\mathcal{D}_{\mathsf{dS_{fde}}} = \{\mathbf{t}, \top\}$, $\mathcal{O}_{\mathsf{dS_{fde}}} = \{f^{\neg}_{\mathsf{dS_{fde}}}, f^{\wedge}_{\mathsf{dS_{fde}}}, f^{\vee}_{\mathsf{dS_{fde}}}\}$ and these truth-functions are defined by the truth-tables in Figure 4.

	$f^{\neg}_{\mathsf{dS_{fde}}}$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$
$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$

$f^{\wedge}_{\mathsf{dS_{fde}}}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{f}$	$\top$	$\mathbf{f}$	$\mathbf{f}$

$f^{\vee}_{\mathsf{dS_{fde}}}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\mathbf{t}$	$\top$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$

Figure 4: Truth-tables for the logic $\mathsf{dS_{fde}}$ (the four-valued generalization of PWK)

The most important thing to say about $\mathsf{dS_{fde}}$ is that this logic can, indeed, be regarded as *a four-valued generalization* of PWK. Why so? On the one hand, it is a *generalization* of Paraconsistent Weak Kleene, first, because its semantics include a designated infectious value, just like the semantics for PWK. Secondly, because this fact—together with the information that it is normal, in the above technical sense—secures that when valuations are restricted to the set $\{\mathbf{t}, \top, \mathbf{f}\}$ we obtain nothing more than the semantics for PWK. On the other hand, it is a *four-valued* generalization, because its semantics include an additional non-classical truth-value, namely $\bot$, such that both it and its negation

[4] In the context of [47], the logic $\mathsf{dS_{fde}}$ is referred to as the system $\mathsf{L}_{\mathbf{nb}'}$.

[5] Meaning that, letting $\Sigma^{\neg} = \{\neg\sigma \mid \sigma \in \Sigma\}$ for every $\Sigma \subseteq FOR(\mathcal{L})$, we can prove

$$\Gamma \vDash_{\mathsf{S_{fde}}} \Delta \iff \Delta^{\neg} \vDash_{\mathsf{dS_{fde}}} \Gamma^{\neg}$$

which we leave to the reader as an exercise. Moreover, for the purpose of giving a deeper meaning to referring to this four-valued generalization of PWK with the name $\mathsf{dS_{fde}}$, it would be interesting to present an intensional system dS which dualizes Deutsch's S and, then, prove that $\mathsf{dS_{fde}}$ is in fact its first-degree fragment. We conjecture this intensional system can be designed by substituting the content-inclusion clause $\gamma_{w'}(B) \leq_{w'} \gamma_{w'}(A)$ featured in Ferguson's semantics for conditionals in the system S (detailed in [20, p. 77-79]), with the alternative clause $\gamma_{w'}(A) \leq_{w'} \gamma_{w'}(B)$.

are undesignated—whence e.g. Implosion, the inference $\psi \vDash \varphi \vee \neg\varphi$, is invalid in $\mathsf{dS_{fde}}$. This informs us, moreover, that we are in front of a *paracomplete* subsystem of PWK.

Before moving on to discuss the target epistemic interpretations of these logics, let us take a brief pause to notice that the weak Kleene logics and their four-valued generalizations are equipped with conjunctions and disjunctions that are utterly peculiar, least to say. In the case of Paracomplete Weak Kleene, a remarkable feature of it is that it invalidates the inference called *Addition*, i.e. $\varphi \vDash \varphi \vee \psi$. Whereas, in the case of Paraconsistent Weak Kleene, a remarkable feature of it is that it invalidates the inference called *Simplification*, i.e. $\varphi \wedge \psi \vDash \psi$. It is reasonable to ask, then, how can the failure of inferences so basic can be made sense of?

First, regarding the failure of Addition, various explanations have been given, which appeal to the meaninglessness [7] or the off-topic character [4] of the issues represented by the newly added disjuncts, to free-choices in deontic logic [50], computational failures [22], [18], analytic connections between premises and conclusions [23] and many other things. Of all these, we will come back to the explanation of the failure of Addition in terms of the absence of analytic connections, when we discuss analytic entailments as modeled by containment logics, in Section 4. We will, obviously, also come back the epistemic explanation of the failure of Addition when we spell out Fitting's epistemic interpretation of Paracomplete Weak Kleene, in Section 3.3.

Notwithstanding the importance of each of these particular explanations, we can refer to a unifying account of all these treatments of disjunctions which do not satisfy Addition, proposed by Thomas Ferguson in the recent paper [18]. There, Ferguson follows Zimmerman's reflections in [50], noting that in many of these accounts the failure of Addition is explained by disjunction having a *conjunctive* flavor to it—or otherwise being nothing more than a conjunction in disguise. The conjunction in question being formed by the explicitly stated disjunction and the implicit requirement that *both* disjuncts satisfy a certain enabling condition, to be further specified in each interpretation. The failure of Addition is accounted for, in this way, by noticing that the fact that one of the disjuncts holds does not guarantee the satisfaction of all the required constraints. In fact, were some of the disjuncts not to satisfy the required enabling condition, then the (apparent) disjunction will not be satisfied—for more on this, see [18, p. 344-349].

Secondly, regarding the failure of Simplification, also various explanations have been given, which appeal to the meaninglessness of one of the conjuncts [29], to a causal, explanatory or otherwise grounding-like connection between premises and conclusions [37], to regressive analytic connections between premises and conclusions [37], and many other things. Of all these, we will come back to the explanation of the failure of Simplification in terms of the absence of regressive analytic connections, when we discuss regressive analytic entailments as modeled by containment logics, in Section 4. We will, again, also come back the epistemic explanation of the failure of Simplification, when we give a our novel epistemic interpretation of Paraconsistent Weak Kleene, in Section 3.4

Until now, no unifying account of all these treatments of conjunctions which do not satisfy Simplification has been proposed. Whether or not it is actually possible to do so, is something we do not know and, in fact, an issue whose discussion will take us probably too far afield. Nevertheless, we can still point out that our epistemic explanation of the failure of Simplification will share some features with the account—implicitly—proposed by Ciuni in [10] to understand the failure of Simplification in Paraconsistent Weak Kleene, when this system is understood as a logic devised to handle paradoxes and semantic pathologies of the like.

There, Ciuni proposed to explain the failure of Simplification in Paraconsistent Weak Kleene by pointing out that, when it is employed as a logic to handle paradoxes, conjunction has a *disjunctive* flavor to it—or otherwise is nothing more than a disjunction in disguise. The disjunction in question being formed by the explicitly stated conjunction and the possibility that *either* of the conjuncts satisfies a certain overriding condition: in the particular case he is discussing, that of being a pathological proposition. In this way, were some of the conjuncts to satisfy this overriding condition, the (apparent) conjunction will be satisfied. We shall highlight that, as the reader will notice in the sequel, our own epistemic interpretation of PWK and therefore of the failure of Simplification in it, will exhibit a similar pattern—although the ingredients will be completely different. We will come back to this similarity below, at the end of Section 3.4.

Having said this, let us now turn to the epistemic interpretation of the Kleene logics.

3 The Epistemic Interpretation of Kleene logics

In this section we will, first, review Fitting's epistemic interpretation of the strong Kleene logic $\mathsf{K_3}$ and its four-valued generalization FDE. After that, we will look at Fitting's epistemic interpretation of the Paracomplete Weak Kleene logic K_3^w and its four-valued generalization $\mathsf{S}_{\mathtt{fde}}$. We will, finally, advance an epistemic interpretation of Paraconsistent Weak Kleene logic and its four-valued generalization $\mathsf{dS}_{\mathtt{fde}}$, which is novel to this work.

3.1 What is an Epistemic Interpretation?

Before jumping to the interpretations themselves, we should explain what are we trying to do in providing an epistemic interpretation for the Kleene logics, i.e. what Fitting did and what will we, consequently, try to do. To briefly answer this question we shall say our aim is to provide an epistemic reading of the truth-values featured in the corresponding Kleene logics, their notion of logical consequence, and the truth-functions characteristic of these systems.

We will devote the specific Sections 3.2 to 3.4 below to discuss the epistemic interpretation of the distinctive truth-functions of each of the strong and weak Kleene logics. Here, we shall talk about what is shared by the epistemic interpretations of each of these systems, namely, the reading of the truth-values and the consequence relation at play.

In relation to these, in e.g. [26] Fitting suggests us to consider the next situation. Suppose we have a certain group of experts $\mathcal{E}$ whose opinion we value and who we are consulting on certain matters, in the form of a series of yes/no questions. When asking these experts about a certain proposition φ some will say it is true, some will say it is false, some may be willing to decline expressing and opinion and some may have reasons for calling it both true and false. Fitting suggests that in these cases we, correspondingly, assign φ a sort of *generalized truth-value*

$$v(\varphi) = \langle P, N \rangle$$

where P is the set of experts who say φ is true, and N is the set of experts who say φ is false [27, p. 57]. Thus, it is possible that $P \cup N \neq \mathcal{E}$ and it is also possible that $P \cap N \neq \emptyset$.[6]

Given this picture, let us now focus on the epistemic reading of the four truth-values $\mathbf{t}, \top, \bot, \mathbf{f}$. These values are usually interpreted—that is, outside of the epistemic interpretation, e.g. in [43]—as, respectively, true only, both-true-and-false, neither-true-nor-false and false only. However, in the context of the epistemic reading we are currently discussing, the assignment of the non-classical values $\bot$ and $\top$ to a certain formula corresponds, respectively, to the case where no expert expresses an opinion towards the formula in question, i.e. to the generalized truth-value $\langle \emptyset, \emptyset \rangle$, and the case where all experts say that the formula in question is true, and at the same time they say it is false, i.e. to the generalized truth-value $\langle \mathcal{E}, \mathcal{E} \rangle$. In the former case, we might say they have an *indeterminate* opinion, and in the latter case we might say they have an *inconsistent* opinion.

Similarly, the assignment of the classical values $\mathbf{t}$ and $\mathbf{f}$ correspond, respectively, to the case where all experts say the formula in question is true and no expert says it is false, i.e. to the generalized truth-value $\langle \mathcal{E}, \emptyset \rangle$, and the case where no expert says that the formula is true and all experts say that the formula is false, i.e. to the generalized truth-value $\langle \emptyset, \mathcal{E} \rangle$. Moreover, with regard to the full set of the four generalized truth-values, we will take the terminological liberty—in alignment with the previous remarks—of calling $\mathbf{t}, \top$ and $\mathbf{f}$ the *determinate* values, while calling $\mathbf{t}, \bot$ and $\mathbf{f}$ the *consistent*

[6] As Fitting highlights in many places, considerations along these lines already suggest we are going to end up, down the road, with a lattice-theoretic structure (called a bilattice) which can be put to very good use for logical investigations. But we will try not to go into the formal details of the connection between this investigations and those concerning bilattices, leaving this discussions for another time.

values. Thus, to account for the strong and weak Kleene logics presented above, it is necessary to think that every time the experts are consulted on a certain proposition φ the resulting general opinion can be represented by one of the four truth-values $\mathbf{t}, \top, \bot, \mathbf{f}$.

This epistemic interpretation of the truth-values $\mathbf{t}, \top, \bot, \mathbf{f}$, certainly puts things under a different light, but it still does not account for an epistemic understanding of a logic. For that purpose we need to give an epistemic understanding of the underlying truth-functions of the given logic (whether its one of the strong or the weak Kleene ones) and of the accompanying definition of *logical consequence*. The latter issue is easier to settle. Being relatively conservative, in what follows we will always be taking logical consequence to be somehow related to *truth-preservation*. More specifically, by this we mean that an argument will be valid if and only if whenever the premises are taken to be true by all experts, so is the conclusion.

The task of giving an *epistemic understanding of the strong and the weak Kleene truth-tables* and of their four-valued generalizations will demand a little bit more work. Fitting achieved this by taking these truth-functions to embody different approaches to determine what experts think of certain complex formulae such as $\varphi \wedge \psi$ and $\varphi \vee \psi$—i.e. whether they think they are true or false—given how these experts stand concerning their components. In a nutshell, we can say that Fitting took the truth-functions characteristic of each of the Kleene logics discussed by him, to incarnate different policies applicable when pooling the opinion of the consulted experts. Thus, in what remains of this section we will be discussing what are these different policies in the case of the strong Kleene logics and of the Paracomplete Weak Kleene logic, showing at last how it is possible to extend this account to provide an epistemic interpretation of the truth-functions of Paraconsistent Weak Kleene.

3.2 The Epistemic Interpretation of strong Kleene logics

Given the above remarks, the last thing required to provide an epistemic interpretation of the strong Kleene logics K_3 and its four-valued generalization FDE is to clarify which policies for pooling the opinion of the consulted experts are characteristic of these logics. The unsurprising answer will be: the most intuitive and straightforward ones.

In fact, concerning a conjunction $\varphi \wedge \psi$, Fitting says that we should calculate its generalized truth-value—the pair comprising, first, the set of experts which think it is true and, second, the set of experts which think it is false—as follows. On the one hand, it seems intuitive to say that the experts which believe $\varphi \wedge \psi$ is true are those who believe both φ *and* ψ are true. That is to say, the set of experts which believe $\varphi \wedge \psi$ is true can be calculated by taking the *intersection* of two sets: the set of experts who think φ is true, and the set of experts who think ψ is true. On the other hand, it also seems intuitive to say that the experts which believe $\varphi \wedge \psi$ is false are those who believe either φ *or* ψ are false. That is to say, the set of experts which believe $\varphi \wedge \psi$ is false can be calculated by taking the *union* of two sets: the set of experts who think φ is false, and the set of experts who think ψ is false. Fitting proposes to formally represent this, given two propositions φ and ψ whose generalized truth-values are $v(\varphi) = \langle P_1, N_1 \rangle$ and $v(\psi) = \langle P_2, N_2 \rangle$, by defining an operation $\sqcap$ between them as

$$\varphi \sqcap \psi = \langle P_1 \cap P_2, N_1 \cup N_2 \rangle$$

Analogous reasoning establishes that for the case of a disjunction $\varphi \vee \psi$, its generalized truth-value should be calculated as follows. On the one hand, it seems intuitive to say that the experts which believe $\varphi \vee \psi$ is true are those who believe φ *or* ψ are true. That is to say, the set of experts which believe $\varphi \vee \psi$ is true can be calculated by taking the *union* of two sets: the set of experts who think φ is true, and the set of experts who think ψ is true. On the other hand, it also seems intuitive to say that the experts which believe $\varphi \vee \psi$ is false are those and only those who believe both φ *and* ψ are false. That is to say, the set of experts which believe $\varphi \vee \psi$ is false can be calculated by taking the *intersection* of two sets: the set of experts who think φ is false, and the set of experts who think ψ is false. Fitting proposes to formally represent this, given two propositions φ and ψ whose generalized truth-values are $v(\varphi) = \langle P_1, N_1 \rangle$ and $v(\psi) = \langle P_2, N_2 \rangle$, by defining an operation $\sqcup$ between them as[7]

$$\varphi \sqcup \psi = \langle P_1 \cup P_2, N_1 \cap N_2 \rangle$$

Finally, for the case of a negation $\neg\varphi$, its generalized truth-value should be calculated by switching the role of the set of experts saying φ is true and the set of experts saying φ is false. That is, those experts who say that φ is true, should be counted as saying that $\neg\varphi$ is false, and those experts saying that φ is false should be counted as saying that $\neg\varphi$ is true. Formally, for a given proposition φ whose generalized truth-value is $v(\varphi) = \langle P_1, N_1 \rangle$, Fitting defines the operation $\neg$ as

$$\neg\varphi = \langle N_1, P_1 \rangle$$

Let us, now, consider the case where every time the experts are consulted on a certain proposition φ, the resulting general opinion can be represented by one of the four truth-values $\mathbf{t}, \top, \bot, \mathbf{f}$. If, in this context, we were to graphically summarize the outcome of the previous pooling directives concerning negation, conjunction and disjunction, we will arrive at the following "truth-tables"

	$\neg$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$
$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$

$\sqcap$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\top$	$\top$	$\top$	$\mathbf{f}$	$\mathbf{f}$
$\bot$	$\bot$	$\mathbf{f}$	$\bot$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$

$\sqcup$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\mathbf{t}$	$\top$	$\mathbf{t}$	$\top$
$\bot$	$\mathbf{t}$	$\mathbf{t}$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$

which are, respectively, those of the truth-functions $f^{\neg}_{\mathsf{FDE}}$, $f^{\wedge}_{\mathsf{FDE}}$ and $f^{\vee}_{\mathsf{FDE}}$ of Belnap-Dunn four-valued logic FDE, as discussed e.g. in [16] and [6]. This suggests that the above remarks amount to an epistemic interpretation of Belnap-Dunn four-valued logic.

The question remains, however, of how to use the above considerations to provide an epistemic interpretation for Strong Kleene logic K_3. If we imagine a situation in which all experts are consulted and, for no proposition φ all experts express an inconsistent opinion, this will amount to restricting the FDE valuations to the "consistent" values: namely, $\mathbf{t}, \bot, \mathbf{f}$. The three-valued logic induced by this restriction is K_3 and it is, thus, through this reflections that Fitting arrived at an epistemic interpretation for Strong Kleene logic.

Interestingly, this can be further applied to provide an epistemic interpretation for Priest's Logic of Paradox LP. In fact, if we imagine a situation where, for all propositions on which the experts are asked about, no expert refrains from expressing an opinion, this will amount to restricting the FDE valuations to the "determinate" values: namely, $\mathbf{t}, \top, \mathbf{f}$. The three-valued logic rendered by this constraints is, thus, LP—which, additionally, provides an epistemic interpretation for this interesting Kleene logic.

Let us, then, see how a modification of these remarks may lead to an epistemic interpretation of the *weak* Kleene logics.

3.3 The Epistemic Interpretation of Paracomplete Weak Kleene

How does Fitting arrive at the desired interpretation of Paracomplete Weak Kleene? It is only after taking the weak Kleene truth-tables to summarize a quite distinctive approach to pooling the opinion of the consulted experts. An approach, that is, which must—in a very sensible way—diverge from that of the strong Kleene logics reviewed in the previous subsection.

In fact, in [27] Fitting is quite clear about this, noting that sometimes we may want to collect and ponder the opinion of the consulted experts in special ways. Recall that the framework allows experts to be silent about certain matters when they are asked about their opinions. Thus, e.g. when evaluating a conjunction $\varphi \wedge \psi$ or a disjunction $\varphi \vee \psi$ we may

[7] Fitting actually denotes the operations $\sqcap$ and $\sqcup$ with the symbols $\wedge$ and $\vee$, respectively. However, in an effort to minimize confusion as much as possible, we decided to change these in order to differentiate them from the connectives usually employed to represent conjunction and disjunction. However, this is only a stylistic choice, and none of the formal results depends on this.

> want to 'cut this down' by considering people who have actually expressed an opinion on both propositions [φ] and [ψ] [27, p. 66-67]

Whence, we shall call the resulting alternative conjunctions and disjunctions—following Ferguson in [20] and [21]—the "cut-down variants" of these logical operations. Rendering that the "cut-down" way in which we calculate e.g. the set of experts who believe the conjunction $\varphi \wedge \psi$ is true (alternatively, false), requires taking the *intersection* of set of experts *that we previously classified* as saying $\varphi \wedge \psi$ is true (false), with the set of experts who have actually expressed an opinion towards both φ and ψ. Similarly, for a disjunction $\varphi \vee \psi$ and a negation $\neg\varphi$. In these cases, we may say these variants operate within Fitting's epistemic interpretation of the Kleene logics, following the motto

NO DETERMINATE OPINION CAN ARISE FROM A SET THAT INCLUDES AN INDETERMINATE OPINION

Let us, now, have a closer look at Fitting's proposal to formally model this cut-down approach.

The first step is to define a unary operator—eloquently called a "cut-down operator", by Ferguson—which, for a given proposition φ outputs the set of experts who have expressed any determinate opinion whatsoever (either positive or negative) towards φ. This is done by taking the *union* of the experts who said it is true and the set of experts who said it is false.

This can be rigorously represented with the help of a further operation on generalized truth-values, called the gullability or "accept anything" operation $\oplus$ (see, e.g. [27, p. 56]). Given two propositions φ and ψ, the gullability operation between them gives as a result a generalized truth-value where, on the one hand, all who think φ or ψ are true are brought together and, on the other hand, all who think φ or ψ are false are also brought together.

That is to say, the result of calculating the gullability operation between φ and ψ is a pair, obtained as follows. As a the first coordinate, we have the *union* of the set of experts who think φ is true with the set of experts who think ψ is true. As the second coordinate, we have the *union* of the set of experts who think φ is false with the set of experts who think ψ is false. Speaking more formally, consider two propositions φ and ψ such that their generalized truth-values are, respectively, $v(\varphi) = \langle P_1, N_1 \rangle$ and $v(\psi) = \langle P_2, N_2 \rangle$. The gullability operation applied to them is defined such that

$$\varphi \oplus \psi = \langle P_1 \cup P_2, N_1 \cup N_2 \rangle$$

It is, then, with the aid of this operation that the cut-down $[\![\varphi]\!]$ of a proposition φ, whose generalized truth-value is $v(\varphi) = \langle P_1, N_1 \rangle$, can be defined as

$$[\![\varphi]\!] = \varphi \oplus \neg\varphi = \langle P_1 \cup N_1, P_1 \cup N_1 \rangle$$

noting, furthermore, that the only case where $[\![\varphi]\!] = \langle \emptyset, \emptyset \rangle$ is the case where no expert expressed a determinate opinion towards φ—corresponding to the assignment of the truth-value $\perp$ to φ.

The second step to formally model Paracomplete Weak Kleene's operations in this epistemic setting, is to design, with the help of these tools, e.g. "cut-down" conjunctions and disjunctions. To accomplish this we can use the help of yet another operation on generalized truth-values, called the consensus or "agreement" operation $\otimes$ (see, e.g. [27, p. 56]). Given two propositions φ and ψ, the consensus operation between them gives as a result a generalized truth-value where, on the one hand, all those who agree that φ and ψ are true are brought together and, on the other hand, all those who agree that φ and ψ are false are also brought together.

That is to say, the result of calculating the consensus between φ and ψ is a pair, obtained as follows. As a the first coordinate, we have the *intersection* of the set of experts who think φ is true with the set of experts who think ψ is true. As the second coordinate, we have the *intersection* of the set of experts who think φ is false with the set of experts who think ψ is false. Speaking more formally, consider two propositions φ and ψ such that their generalized truth-values are, respectively, $v(\varphi) = \langle P_1, N_1 \rangle$ and $v(\psi) = \langle P_2, N_2 \rangle$. The consensus operation applied to them is defined such that

$$\varphi \otimes \psi = \langle P_1 \cap P_2, N_1 \cap N_2 \rangle$$

It is, then, with the aid of these formal instruments that we are able to define the target cut-down variants of conjunction and disjunction. Let us focus, for instance, in the case of conjunction. Fitting indicates that we ought to take the generalized truth-value of $\varphi \wedge \psi$ and cut it down to the set of people who have actually expressed an opinion towards both φ and ψ. In other words, the generalized truth-value of this cut-down conjunction should be obtained as follows.

On the one hand, we should cut down the set of experts who believe both φ and ψ are true. This can be done by taking the *intersection* of set of experts who think both φ and ψ are true, with the set of experts who have actually expressed a determinate opinion towards both propositions. On the other hand, we should cut down the set of experts who believe either φ or ψ are false. This can be done by taking the *intersection* of set of experts who think either φ or ψ are false, with the set of experts who have actually expressed a determinate opinion towards both propositions.

Both these moves, together, amount to nothing other than taking the consensus $\otimes$ between the set of experts *that we previously classified* as saying $\varphi \wedge \psi$ is true (or false)—i.e. $\varphi \sqcap \psi$—and the set of experts who have actually expressed an opinion towards both φ and ψ—i.e. the cut-downs of φ and ψ, namely $[\![\varphi]\!]$ and $[\![\psi]\!]$. Similar reasoning leads to similar results for disjunction and negation. More formally, Fitting defines a cut-down conjunction $\vartriangle$ and a cut-down disjunction $\triangledown$ as follows, noting that negation is not altered by these modifications.[8]

$$\varphi \vartriangle \psi = (\varphi \sqcap \psi) \otimes [\![\varphi]\!] \otimes [\![\psi]\!] \qquad \varphi \triangledown \psi = (\varphi \sqcup \psi) \otimes [\![\varphi]\!] \otimes [\![\psi]\!]$$

Finally, if the four values $\mathbf{t}, \top, \bot, \mathbf{f}$ are taken into account, the "truth-tables" for the operations of conjunction, disjunction and negation—understood in this "cut-down" fashion—would be the following, as is easy to check.

	$\neg$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$
$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$

$\vartriangle$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\top$	$\top$	$\top$	$\bot$	$\mathbf{f}$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\bot$	$\mathbf{f}$

$\triangledown$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\bot$	$\mathbf{t}$
$\top$	$\mathbf{t}$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$

These are, respectively, the truth-functions $f^{\neg}_{\mathsf{S_{fde}}}$, $f^{\wedge}_{\mathsf{S_{fde}}}$ and $f^{\vee}_{\mathsf{S_{fde}}}$ of the four-valued logic $\mathsf{S_{fde}}$, depicted in Figure 3 above. Hence, the previous account can be taken to represent nothing more than an epistemic interpretation of this four-valued logic.

Of most importance to us, though, is that our discussion of $\mathsf{S_{fde}}$ as a four-valued generalization of Paracomplete Weak Kleene in Section 2 already suggests how are we going to provide an epistemic interpretation for K^w_3. Indeed, if we imagine a situation in which experts are consulted, and for no proposition φ all the experts think φ is true and all experts think φ is also false, i.e. if all of them have consistent opinions on absolutely all propositions, this will formally amount to restricting the valuations of $\mathsf{S_{fde}}$ to the "consistent" values: namely, the truth-values $\mathbf{t}, \bot, \mathbf{f}$. The three-valued logic induced by this restriction will be no other than K^w_3 and it is, therefore, through these remarks that Fitting arrived at an epistemic interpretation for Paracomplete Weak Kleene.

Let us now illustrate how the *failure of Addition* is, thus, properly understood in this framework. Consider, for example, the case where all experts think φ is true and no expert thinks it is false and, simultaneously, all experts think ψ is neither true nor false—i.e. $v(\varphi) = \langle \mathcal{E}, \emptyset \rangle$ and $v(\psi) = \langle \emptyset, \emptyset \rangle$. Given our previous considerations, we can equivalently say that φ is assigned the truth-value $\mathbf{t}$, whereas ψ is assigned the truth-value $\bot$. As a result of this, the value of the 'cut-down' disjunction $\varphi \triangledown \psi$ will be $\langle \emptyset, \emptyset \rangle$—that is, the epistemic counterpart of $\bot$. Thus, we have a disjunction that is not true which has, nevertheless, one true disjunct. In other words, the failure of Addition is understood in this interpretation by taking disjunction to be a *cut-down disjunction*. Under such a reading, it is clear how from e.g. the fact that all experts think φ is true it does *not* follow that all experts think $\varphi \vee \psi$ is true—the reason being that all experts may have no opinion whatsoever with regard to ψ. Furthermore, allowing us to establish that in the context of this epistemic interpretation the fact that

[8]See Fitting [26, p. 67] and Ferguson [20, p. 24], [21, p. 3].

all experts have an opinion on the given disjuncts works as the aforementioned "enabling condition" (cf. Section 2) for a disjunction to be true.

In what follows we will be proceeding similarly to arrive at our desired epistemic interpretation for Paraconsistent Weak Kleene.

3.4 The Epistemic Interpretation of Paraconsistent Weak Kleene

Our aim, now, is to provide a novel understanding of Paraconsistent Weak Kleene's truth-functions by taking them to summarize a distinctive approach to pooling the opinion of the experts being consulted. This approach will have to, yet again, diverge quite sensibly not only from the pooling policies incarnated by the strong Kleene logics—but also from the one incarnated by Paracomplete Weak Kleene, discussed in the previous section.

We will motivate our approach noting that there might be further special ways in which we might want to collect and ponder the opinion of the experts. Let us recall, for example, that the general framework outlined by Fitting allows experts to have inconsistent opinions about certain matters, i.e. some experts may have reasons for calling a proposition φ both true and false.

Thus, e.g. when considering a conjunction $\varphi \wedge \psi$ or a disjunction $\varphi \vee \psi$ we may want—in a way that is perfectly dual to Fitting's suggestions above—to "track down" people who have expressed an inconsistent opinion towards *either* φ or ψ. Whence, we shall call the resulting alternative conjunctions and disjunctions, the "track-down" variants of these famous logical operations. In these cases, we may say these variant operate within Fitting's epistemic interpretation, following the motto

NO CONSISTENT OPINION CAN ARISE FROM A SET THAT INCLUDES AN INCONSISTENT OPINION

Let us, now, have a closer look at our proposal to formally model this track-down approach.

The first step to technically represent these track-down variants, is to define a unary operator—to be called a "track-down" operator—which, for a given proposition φ outputs the set of experts who have expressed an inconsistent opinion towards φ. This is done by taking the *intersection* of the set of experts who said it is true and the set of experts who said it is false.

This can be formally represented with the help of the consensus operation, letting the track-down $\|\varphi\|$ of a proposition φ, whose generalized truth-value is $v(\varphi) = \langle P_1, N_1 \rangle$, be defined as

$$\|\varphi\| = \varphi \otimes \neg\varphi = \langle P_1 \cap N_1, P_1 \cap N_1 \rangle$$

noting, furthermore, that the only case where $\|\varphi\| = \langle \mathcal{E}, \mathcal{E} \rangle$ is the case where all experts expressed an inconsistent opinion towards φ—corresponding to the assignment of the truth-value $\top$ to φ.

The second step to formally model Paraconsistent Weak Kleene's operations in this epistemic setting, is to design, with the help of these tools, e.g. "track-down" conjunctions and disjunctions. Let us focus, for instance, in the case of conjunction. Our previous motivations indicate we ought to take the generalized truth-value of $\varphi \wedge \psi$ and "track down" the set of people who have expressed an inconsistent opinion towards either φ or ψ. In other words, the generalized truth-value of this track-down conjunction should be obtained as follows.

On the one hand, we should collect the set of experts who believe either φ or ψ are false, together with the set of people who have expressed an inconsistent opinion towards either φ or ψ. This can be done by taking the *union* of set of experts who think both φ and ψ are true, with the set of experts who have expressed an inconsistent opinion towards either propositions. On the other hand, we should collect the set of experts who believe either φ or ψ are false, together with the set of people who have expressed an inconsistent opinion towards either φ or ψ. This can be done by taking the *union* of set of experts who think either φ or ψ are false, with the set of experts who have expressed an inconsistent opinion towards either propositions.

Both these moves amount to nothing other than taking the result of the $\oplus$ operation between the set of experts *that we previously classified* as saying $\varphi \wedge \psi$ is true (or false)—i.e. $\varphi \sqcap \psi$—and the set of experts who have expressed an inconsistent opinion towards either φ or ψ—i.e. the track-downs of

φ and ψ, namely $\|\varphi\|$ and $\|\psi\|$. Similar reasoning leads to similar results for disjunction and negation. More formally, we can define a track-down conjunction $\blacktriangle$ and a track-down disjunction $\blacktriangledown$ as follows, noticing that negation is not altered by these modifications.

$$\varphi \blacktriangle \psi = (\varphi \sqcap \psi) \oplus \|\varphi\| \oplus \|\psi\| \qquad \varphi \blacktriangledown \psi = (\varphi \sqcup \psi) \otimes \|\varphi\| \otimes \|\psi\|$$

Thus, just like in the cut-down case where the set of experts being considered is *shrunk* to cover all those experts expressing a determinate opinion on *all* the relevant propositions, in the track-down case something similar happens. In fact, in the track-down case the set of experts being considered is *enlarged* to cover all those experts expressing an inconsistent opinion on *some* of the relevant propositions, whence it is asked of the pooling procedure not to forget that some people do not have a consistent opinion towards the issues in question.

Finally, if the four values $\mathbf{t}, \top, \bot, \mathbf{f}$ are taken into account, the "truth-tables" for the operations of conjunction, disjunction and negation—understood in this "track-down" fashion—would be the following, as is easy to check.

	$\neg$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$
$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$

$\blacktriangle$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{f}$	$\top$	$\mathbf{f}$	$\mathbf{f}$

$\blacktriangledown$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\top$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\mathbf{t}$	$\top$	$\bot$	$\bot$
$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$	$\mathbf{f}$

As advertised, these are, respectively, the truth-functions $f^{\neg}_{\mathsf{dS}_{\mathtt{fde}}}$, $f^{\wedge}_{\mathsf{dS}_{\mathtt{fde}}}$ and $f^{\vee}_{\mathsf{dS}_{\mathtt{fde}}}$ of the four-valued logic $\mathsf{dS}_{\mathtt{fde}}$, whose truth tables we depicted in Figure 4 above. Hence, the previous account can be taken to constitute an epistemic interpretation of this four-valued logic.

Of most importance to us, however, is that our discussion of $\mathsf{dS}_{\mathtt{fde}}$ as a four-valued generalization of Paraconsistent Weak Kleene in Section 2 allows us to transition from the above remarks to an epistemic interpretation for PWK. In fact, if we consider a situation in which experts are consulted, and for no proposition φ all experts refrain from expressing an opinion about it, i.e. if all of them have determinate opinions on absolutely all propositions, this will formally amount to restricting the valuations of $\mathsf{dS}_{\mathtt{fde}}$ to the "determinate" values: namely $\mathbf{t}, \top, \mathbf{f}$. The three-valued logic induced by this restriction is our target logic PWK. It is, therefore, through these considerations that we arrive at our desired *epistemic interpretation for Paraconsistent Weak Kleene*—thus fulfilling the main goal of this paper.

Let us now illustrate, as expected, how the *failure of Simplification* is, thus, properly understood in this framework. Consider, for example, the case where all experts think φ is true and at the same time all experts think it is false, while also all experts think ψ is false and no expert thinks it is true—i.e. $v(\varphi) = \langle \mathcal{E}, \mathcal{E} \rangle$ and $v(\psi) = \langle \emptyset, \mathcal{E} \rangle$. Given our previous considerations, we can equivalently say that φ is assigned the truth-value $\top$, whereas ψ is assigned the truth-value $\mathbf{f}$. As a result of this, the value of the "track-down" conjunction $\varphi \blacktriangle \psi$ will be $\langle \mathcal{E}, \mathcal{E} \rangle$—that is, the epistemic counterpart of $\top$. In a nutshell, the failure of Simplification is understood in this interpretation by taking conjunction to be a *track-down conjunction*. Under such a reading, it is clear how from e.g. the fact that all experts think $\varphi \wedge \psi$ is true it does *not* follow that all experts think that ψ is true—the reason being the truth of the conjunction might be caused by all experts having an inconsistent opinion with regard to φ.

We can, furthermore, connect this to our purported understanding of the failure of Simplification in terms of conjunction being a "disguised disjunction", formed by the conjuncts in question and two more disjuncts representing the possibility that either of the conjuncted propositions triggers a certain overriding condition. It is clear from the above that the epistemic interpretation of Paraconsistent Weak Kleene outlined allows for this reading. In fact, in the context of such an epistemic interpretation, the fact that all experts have an inconsistent opinion on one of the given conjuncts works as the aforementioned "overriding condition" (cf. Section 2) for a conjunction to be true.

The previous reflections put PWK and its four-valued generalization under a new light with regard to philosophical and logical investigations. But, speaking of a formal logic, we believe some philosophical achievements can be made of a broader significance if we can supplement them with attractive formal results. This is why in the next section, we devote ourselves to offer some of these.

In Section 4 we present sound and complete two-sided sequent calculi for the four-valued generalizations of the weak Kleene logics. We motivate the particular calculi below, by pointing out some results on the relation between weak Kleene logics and containment logics.

4 Sequent Calculi

In what follows we will endow the previously discussed four-valued generalizations of the weak Kleene logics with suitable Gentzen-style sequent calculi.

Before moving on, let us notice that *natural deduction* calculi have been recently offered for the weak Kleene logics and some of their four-valued generalizations. Indeed, these were introduced for K_3^w and PWK in [40], for $\mathsf{S}_{\mathtt{fde}}$ in [42], and for some subsystems thereof in [41].[9] Although a similar presentation can be carried out for $\mathsf{dS}_{\mathtt{fde}}$, for matters of space we focus here on Gentzen-style sequent calculi for this logic, leaving the investigation of a natural deduction calculus for it for another occasion.

To arrive at our desired sequent calculi, we will draw inspiration from the techniques introduced by Coniglio and Corbalán in [12] to provide calculi of the like for the systems PWK and K_3^w. The main feature of such proof systems is that they are obtained by taking an appropriate sequent calculus for Classical Logic, and applying different restrictions to the operational rules featured in it. More particularly, these restrictions pertain to some inclusion requirements between the set of propositional variables of the active formulae of the corresponding rules, and the set of propositional variables appearing in some of the side formulae.

The main reason for requiring the rules to comply with these provisos is that weak Kleene logics happen to be closely connected with a family of systems whose valid inferences enjoy certain variable inclusion features. This family of systems, denominated *containment logics* e.g. in [45], gathers logics where an inference holds *only if* certain set-theoretic containment principle holds between the set of propositional variables appearing in the premises and the set of propositional variables appearing in the conclusion.[10]

Thus, Coniglio and Corbalán arrived at sequent calculi for PWK and K_3^w by noticing that these weak Kleene logics can be described as being pretty close—in a sense to be precised below—to containment subsystems of Classical Logic. By following a similar path, we will arrive at sequent calculi for $\mathsf{dS}_{\mathtt{fde}}$ and $\mathsf{S}_{\mathtt{fde}}$, by noticing that these logics can be rightfully described as proper containment subsystems of some other Kleene logics. To accomplish this, we will benefit from connecting weak Kleene logics, in general, with containment logics, later looking at these four-valued systems as an instance of a more general phenomenon.

4.1 Connecting weak Kleene logics and containment logics

Let $var(\Gamma)$ represent the set of propositional variables appearing in the set of formulae Γ, allowing us to refer e.g. to $var(\{\varphi\})$ and $var(\{\varphi, \psi\})$ as $var(\varphi)$ and $var(\varphi, \psi)$, respectively.

A very well-known famlily of containment logics—to which we will refer as Parry logics—are such that all its valid inferences enjoy a property we may call the $\vDash$-Parry Principle, i.e. the property that

$$\Gamma \vDash \psi \quad \textit{only if} \quad var(\varphi) \subseteq var(\Gamma)$$

indeed, a species of Parry's Proscriptive Principle discussed in [39], applied to entailment.[11] Thus, logics satisfying the $\vDash$-Parry Principle saliently inavalidate Addition—i.e. $\varphi \vDash \varphi \vee \psi$—for it may well happen that the propositional variables appearing in ψ are *not* included among those appearing in φ.

[9] In particular, [41] presents natural deduction calculi for the logics $\mathsf{K}_{4\mathsf{b}}^w$ and $\mathsf{K}_{4\mathsf{n}}^w$ referred in [47], respectively, as $\mathsf{L}_{\mathbf{eb'}}$ and $\mathsf{L}_{\mathbf{b'e}}$.

[10] Some important works revolving around these systems are e.g. [39], [46], [1], [37], [38], [19], [24], [11], among others.

[11] In fact, [19] calls it $\vDash$-Proscriptive Principle. There is nothing substantial in the choice of denomination here, though.

Systems of this sort have been studied, discussed and advanced by logicians such as Angell [1], Fine [24], Paoli [37], Epstein [17], Correia [13] and Ferguson [22]—alongside, of course, Parry [39] himself—with the specific aim of modelling *analytic* entailments. An analytic connection between premises and conclusion holds, these authors claim, when the content of the conclusion is included in the content of the premises. Indeed, e.g. [1] and [30] understand a logical behavior of this sort as extending Kant's notion of *analyticity*, according to which the predicate is included in the subject, to also apply to arguments. When the content of a given complex formula is obtained by collecting the content of the propositional variables appearing in it, it is straightforward to see how the requirement that an entailment is analytic directly implies the failure of Addition—for, in this sense, the content of $\varphi \vee \psi$ is usually not taken to be included in the content of φ.

However, notwithstanding the fact that $\vDash$-Parry logics invalidate Addition, it is *not* true that all systems where Addition is invalid are $\vDash$-Parry logics. In fact, Paracomplete Weak Kleene logic is a witness of this case. For Explosion—i.e. the inference $\varphi, \neg\varphi \vDash \psi$—is valid in it, but does not enjoy the $\vDash$-Parry property. This can be generalized, as the following characterization of logical consequence in K_3^w shows.

Observation 4.1 ([49]). *For all sets of formulae* $\Gamma \cup \{\varphi\}$*,*

$$\Gamma \vDash_{\mathsf{K}_3^w} \varphi \Longleftrightarrow \begin{cases} \Gamma \vDash_{\mathsf{CL}} \varphi \text{ and } var(\varphi) \subseteq var(\Gamma), \text{ or} \\ \Gamma \vDash_{\mathsf{CL}} \emptyset \end{cases}$$

Thus, we can say that K_3^w is close to a containment subsystem of Classical Logic. In fact, letting the $\vDash$-Parry fragment of a logic L, denoted $\mathsf{L}_{\mathsf{PP}^{\vDash}}$, be defined such that

$$\Gamma \vDash_{\mathsf{L}_{\mathsf{PP}^{\vDash}}} \varphi \Longleftrightarrow \Gamma \vDash_{\mathsf{L}} \varphi \text{ and } var(\varphi) \subseteq var(\Gamma)$$

it can be observed that the only thing standing between the $\vDash$-Parry fragment of Classical Logic (i.e. $\mathsf{CL}_{\mathsf{PP}^{\vDash}}$) and Paracomplete Weak Kleene are the K_3^w-valid inferences involving inconsistent premises.[12]

Interestingly, in [19] it is pointed out how to obtain $\vDash$-Parry logics, having K_3^w as a starting point. Let us say that a logic $\mathsf{L} = \langle FOR(\mathcal{L}), \vDash_{\mathsf{L}} \rangle$ has *anti-theorems* if there is some $\psi \in FOR(\mathcal{L})$ such that $\psi \vDash_{\mathsf{L}} \emptyset$. Then, a connection between weak Kleene logics and $\vDash$-Parry logics can be established by the following general result.

Proposition 4.2 ([19]). *Let $\mathcal{L}$ be a language and let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an $\mathcal{L}$-matrix such that $\mathcal{V} \setminus \mathcal{D}$ contains an infectious value. If $\mathsf{L} = \langle FOR(\mathcal{L}), \vDash_{\mathcal{M}} \rangle$ has no anti-theorems, then L is a $\vDash$-Parry logic.*

Notice, first, that this explicitly appeals to subsystems of Paracomplete Weak Kleene, as every logic induced by a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ such that $\mathcal{V} \setminus \mathcal{D}$ contains an infectious value is a *subsystem* of K_3^w. Note, moreover, that in light of this observation it is *sufficient* to consider certain paraconsistent subsystems of Paracomplete Weak Kleene to arrive at a $\vDash$-Parry logic. It is in this sense that, in [19] and [18] the four-valued logic $\mathsf{S}_{\mathtt{fde}}$—which is both a paraconsistent logic and a subsystem of K_3^w—is described as a the $\vDash$-Parry fragment of Priest's Logic of Paradox.

Observation 4.3 ([19]). *For all sets of formulae* $\Gamma \cup \{\varphi\}$*,*

$$\Gamma \vDash_{\mathsf{S}_{\mathtt{fde}}} \varphi \Longleftrightarrow \Gamma \vDash_{\mathsf{LP}} \varphi \text{ and } var(\varphi) \subseteq var(\Gamma)$$

Now, moving on to the relation that Paraconsistent Weak Kleene and subsystems thereof have with containment logics, we will highlight that just like we connected containment logics to systems where Addition fails, we can do the same with systems where Simplification fails. Let us consider another family of containment logics—to which we will refer as *Dual* Parry logics—such that all its valid inferences enjoy a property we may call the $\vDash$-Dual Parry Principle, i.e. the property that

$$\Gamma \vDash_{\mathsf{L}} \varphi \quad \textit{only if} \quad \exists \Gamma' \subseteq \Gamma, \Gamma' \neq \emptyset, var(\Gamma') \subseteq var(\varphi)$$

[12] For an extensive discussion of $\mathsf{CL}_{\mathsf{PP}^{\vDash}}$, its relation to Parry logics and weak Kleene logics, see [19].

which is a clear dualization of the $\vDash$-Parry Principle, arrived at by reversing the direction of the famous containment principle discussed before.[13] Thus, logics satisfying the $\vDash$-Dual Parry Principle saliently invalidate Simplification—i.e. $\varphi \wedge \psi \vDash \psi$—for it may well happen that the propositional variables appearing in φ are *not* included among those appearing in ψ.

Systems of this sort have been considered by e.g. Epstein [17] and Paoli [37] with the specific aim of modeling, what the latter calls, *regressive analytic* entailments. A regressive analytic connection between premises and conclusions is holds when we

> proceed from simple ingredients (simple ideas as primitive concepts, simple propositions as axioms), down to more complex ones; by analyzing a derived concept or a theorem, we can overturn the procedure and regress to the basic components [37, p. 2]

Whence, this seemingly gives regressive analytic entailments a sort of *explanatory* flavor, the symptom of which appears to be the complexity increase (or stability) from premises to conclusions [37, p. 2]. Yet again, if we apply these ideas to the *content* of premises and conclusions, and then obtain the content of complex expressions by collecting that of the propositional variables appearing in it, it is straightforward to see how the requirement that an inference is regressive analytic directly implies the failure of Simplification—for, in this sense, the content of $\varphi \wedge \psi$ is usually not taken to be included in the content of ψ.[14]

Again, notwithstanding the fact that $\vDash$-Dual Parry systems invalidate Simplification, it is *not* true that all logics that invalidate Simplification are $\vDash$-Dual Parry systems. Indeed, Paraconsistent Weak Kleene logic is a witness of this case. For Implosion—i.e. the inference $\psi \vDash \varphi \vee \neg\varphi$—is valid in it, although this inference does not enjoy the $\vDash$-Dual Parry property. This can be generalized, as the following characterization of logical consequence in PWK shows.

Observation 4.4 ([11])**.** *For all sets of formulae* $\Gamma \cup \{\varphi\}$*,*

$$\Gamma \vDash_{\mathsf{PWK}} \varphi \Longleftrightarrow \begin{cases} \Gamma \vDash_{\mathsf{CL}} \varphi \text{ and } \exists \Gamma' \subseteq \Gamma, \Gamma' \neq \emptyset, var(\Gamma') \subseteq var(\varphi), \text{ or} \\ \emptyset \vDash_{\mathsf{CL}} \varphi \end{cases}$$

Thus, we can say that PWK is close enough to a containment subsystem of Classical Logic. In fact, letting the $\vDash$-Dual Parry fragment of a logic L, denoted $\mathsf{L}_{\mathsf{DPP}^{\vDash}}$, be defined such that

$$\Gamma \vDash_{\mathsf{L}_{\mathsf{DPP}^{\vDash}}} \varphi \Longleftrightarrow \Gamma \vDash_{\mathsf{L}} \varphi \text{ and } \exists \Gamma', \emptyset \neq \Gamma' \subseteq \Gamma, var(\Gamma') \subseteq var(\varphi)$$

it can be observed that the only thing standing between the $\vDash$-Dual Parry fragment of Classical Logic (i.e. $\mathsf{CL}_{\mathsf{DPP}^{\vDash}}$) and Paraconsistent Weak Kleene are the PWK-valid inferences involving tautlogical conclusions.

Interestingly, by a dualization of [19, Observation 1] advanced in [47, p. 297], it can be pointed out how to obtain $\vDash$-Dual Parry logics, having PWK as a starting point. Let us say that a logic $\mathsf{L} = \langle FOR(\mathcal{L}), \vDash_{\mathsf{L}} \rangle$ has *theorems* if there is some $\psi \in FOR(\mathcal{L})$ such that $\emptyset \vDash_{\mathsf{L}} \psi$. Then, a connection between weak Kleene logics and $\vDash$-Dual Parry logics can be established by the following general observation.

Proposition 4.5. *Let $\mathcal{L}$ be a language and let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an $\mathcal{L}$-matrix such that $\mathcal{D}$ contains an infectious value. If $\mathsf{L} = \langle FOR(\mathcal{L}), \vDash_{\mathcal{M}} \rangle$ has no theorems, then L is a $\vDash$-Dual Parry logic.*

Proof. Assume all of the antecedent conditions hold and suppose, for *reductio*, that L is not a $\vDash$-Dual Parry logic. This implies there is an inference $\Gamma \vDash_{\mathcal{M}} \varphi$ such that it is *not true* that $\exists \Gamma' \subseteq \Gamma, \Gamma' \neq \emptyset, var(\Gamma') \subseteq var(\varphi)$. This implies that for all $\gamma \in \Gamma$, $var(\gamma) \nsubseteq var(\varphi)$.

[13] In [15], [17] and [37], similar properties have been called, respectively, Converse Parry Property, Dual Dependence and Regressive Analiticity. Yet again, there is nothing substantial going on in the choice of denomination here—but for a criticism of the terminology employed in [15], see [32, p. 176, fn. 20].

[14] Moreover—as is also argued in [37]—consequence relations enjoying this feature have been motivated by those who favor the idea that the entailments have some causal or grounding flavor to it, as e.g. in [8], which would explain that simple constituents entail some of the compounds they constitute, but not the other way around.

Let $\Sigma \setminus \Delta$ be the result of subtracting from Σ all the elements that are in Δ. Since L has no theorems, moreover, we can assume that there is a valuation v such that $v(\varphi) \notin \mathcal{D}$. Let us refer to the infectious value contained in $\mathcal{D}$ as $\mathbf{x}$. We can construct a valuation v^* such that

$$v^*(p) = \begin{cases} \mathbf{x} & \text{if } p \in var(\Gamma) \setminus var(\varphi) \\ v(p) & \text{otherwise} \end{cases}$$

Since, by the above, we are justified to assume that for all $\gamma \in \Gamma$, $var(\gamma) \setminus var(\varphi) \neq \emptyset$, we know that for all $\gamma \in \Gamma$, there is a $q \in var(\gamma) \setminus var(\varphi)$ such that $v^*(q) = \mathbf{x}$. Whence, for all $\gamma \in \Gamma$, $v^*(\gamma) = \mathbf{x} \in \mathcal{D}$, further implying that $v^*[\Gamma] \subseteq \mathcal{D}$, while at the same time $v^*(\varphi) \notin \mathcal{D}$. Then, v^* witnesses that $\Gamma \nvDash_{\mathcal{M}} \varphi$, which contradicts our initial assumption. Therefore, L is a $\vDash$-Dual Parry logic. □

Notice, first, that this explicitly appeals to subsystems of Paraconsistent Weak Kleene, as every matrix logic induced by a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ such that $\mathcal{D}$ contains an infectious value is a subsystem of PWK. Note, moreover, that in light of this observation it is *sufficient* to consider certain paracomplete subsystems of Paraconsistent Weak Kleene to arrive at a $\vDash$-Dual Parry logic. Thus, as a consequence of these remarks and those made in [47], the four-valued logic $\mathsf{dS_{fde}}$—which is both a paracomplete logic and a subsystem of PWK—can be regarded as the $\vDash$-Dual Parry fragment of Strong Kleene logic.

Observation 4.6. *For all sets of formulae* $\Gamma \cup \{\varphi\}$*,*

$$\Gamma \vDash_{\mathsf{dS_{fde}}} \varphi \Longleftrightarrow \Gamma \vDash_{\mathsf{K_3}} \varphi \text{ and } \exists \Gamma', \emptyset \neq \Gamma' \subseteq \Gamma, var(\Gamma') \subseteq var(\varphi)$$

Proof. That $\Gamma \vDash_{\mathsf{dS_{fde}}} \varphi$ implies $\Gamma \vDash_{\mathsf{K_3}} \varphi$ is established by the fact that $\mathsf{dS_{fde}}$ is a subsystem of $\mathsf{K_3}$, established in [47] and easy to check by looking at their matrices. That $\Gamma \vDash_{\mathsf{dS_{fde}}} \varphi$ implies $\exists \Gamma', \emptyset \neq \Gamma' \subseteq \Gamma, var(\Gamma') \subseteq var(\varphi)$ follows from Proposition 4.5 above. □

Finally, having looked at the systems $\mathsf{dS_{fde}}$ and $\mathsf{S_{fde}}$ as *containment subsystems* of Strong Kleene logic and Priest's Logic of Paradox, respectively, we will now move on to present their corresponding sequent calculi. As we advertised, these will be obtained by imposing certain appropriate *containment provisos* to the operational rules of appropriate Gentzen-style sequent calculi for $\mathsf{K_3}$ and LP.

4.2 Definitions

Definition 4.7. By a sequent $\Gamma \succ \Delta$ we mean an ordered pair $\langle \Gamma, \Delta \rangle$ of (non-simultaneously empty) finite sets of formulae of $FOR(\mathcal{L})$.[15]

Definition 4.8. Let L be a matrix logic $\mathsf{L} = \langle FOR(\mathcal{L}), \vDash_{\mathcal{M}} \rangle$ such that $\mathcal{M} = \langle \mathcal{D}, \mathcal{V}, \mathcal{O} \rangle$. An $\mathcal{M}$ valuation v satisfies a sequent $\Gamma \succ \Delta$ (symbolized $v \vDash_{\mathcal{M}} \Gamma \succ \Delta$) if and only if $v(\gamma) \in \mathcal{D}$ for all $\gamma \in \Gamma$, then $v(\delta) \in \mathcal{D}$ for some $\delta \in \Delta$. A sequent $\Gamma \succ \Delta$ is valid (symbolized $\vDash_{\mathcal{M}} \Gamma \succ \Delta$) if for every $\mathcal{M}$ valuation v, $v \vDash \Gamma \succ \Delta$

Thus, we may interchangeably refer to an inference or sequent $\Gamma \succ \Delta$ which is valid in the logic $\mathsf{L} = \langle FOR(\mathcal{L}), \vDash_{\mathcal{M}} \rangle$ as $\Gamma \vDash_{\mathcal{M}} A$ or $\vDash_{\mathcal{M}} \Gamma \succ \Delta$. Recall, also, that in such cases we may alternatively denote $\vDash_{\mathcal{M}}$ as $\vDash_{\mathsf{L}}$.[16]

Definition 4.9. A sequent rule $\mathfrak{R}$ preserves validity in $\mathcal{M}$ if for every instance $\dfrac{\mathfrak{r}}{\Gamma \succ \Delta}$ of $\mathfrak{R}$ and for every $\mathcal{M}$ valuation v, if $v \vDash_{\mathcal{M}} \Sigma \succ \Pi$ for every $\Sigma \succ \Pi \in \mathfrak{r}$, then $v \vDash_{\mathcal{M}} \Gamma \succ \Delta$

[15]Note that, since we are working with sequents built from *sets*, the Contraction and Exchange rules are going to be built into the system, and no explicit mention of them is going to be necessary.

[16]Notice, that in dealing with sequent calculi we are moving from consequence relations relating sets of premises with a single conclusion, to consequence relations relating sets of premises with multiple conclusions. All our discussion was carried out in the former setting, but can be understood in terms of the latter, whence there is nothing worrisome in this.

Definition 4.10 ([12])**.** The sequent calculus $\mathcal{G}\mathsf{CL}$ contains the following rules.[17]

$$\frac{}{\varphi \succ \varphi}\ [\mathit{Id}]$$

$$\frac{\Gamma \succ \Delta}{\Gamma, \varphi \succ \Delta}\ [\mathit{WL}] \qquad \frac{\Gamma \succ \Delta}{\Gamma \succ \varphi, \Delta}\ [\mathit{WR}] \qquad \frac{\Gamma, \varphi \succ \Delta \quad \Gamma \succ \varphi, \Delta}{\Gamma \succ \Delta}\ [\mathit{Cut}]$$

$$\frac{\Gamma \succ \varphi, \Delta}{\Gamma, \neg\varphi \succ \Delta}\ [\neg L] \qquad \frac{\Gamma, \varphi \succ \Delta}{\Gamma \succ \neg\varphi, \Delta}\ [\neg R]$$

$$\frac{\Gamma, \varphi, \psi \succ \Delta}{\Gamma, \varphi \wedge \psi \succ \Delta}\ [\wedge L] \qquad \frac{\Gamma \succ \varphi, \Delta \quad \Gamma \succ \psi, \Delta}{\Gamma \succ \varphi \wedge \psi, \Delta}\ [\wedge R]$$

$$\frac{\Gamma, \varphi \succ \Delta \quad \Gamma, \psi \succ \Delta}{\Gamma, \varphi \vee \psi \succ \Delta}\ [\vee L] \qquad \frac{\Gamma \succ \varphi, \psi, \Delta}{\Gamma \succ \varphi \vee \psi, \Delta}\ [\vee R]$$

Proposition 4.11 ([12])**.** *Let* $\Gamma \cup \Delta$ *be a finite non-empty set of formulae of* $\mathcal{L}$*.* $\Gamma \succ \Delta$ *is provable in* $\mathcal{G}\mathsf{CL}$ *if and only if* $\vDash_{\mathsf{CL}} \Gamma \succ \Delta$*.*

Definition 4.12. Let us refer to the rules below as the De Morgan rules.

$$\frac{\Gamma, \varphi \succ \Delta}{\Gamma, \neg\neg\varphi \succ \Delta}\ [\neg\neg L] \qquad \frac{\Gamma \succ \varphi, \Delta}{\Gamma \succ \neg\neg\varphi, \Delta}\ [\neg\neg R]$$

$$\frac{\Gamma, \neg\varphi \succ \Delta \quad \Gamma, \neg\psi \succ \Delta}{\Gamma, \neg(\varphi \wedge \psi) \succ \Delta}\ [\neg\wedge L] \qquad \frac{\Gamma \succ \neg\varphi, \neg\psi, \Delta}{\Gamma \succ \neg(\varphi \wedge \psi), \Delta}\ [\neg\wedge R]$$

$$\frac{\Gamma, \neg\varphi, \neg\psi \succ \Delta}{\Gamma, \neg(\varphi \vee \psi) \succ \Delta}\ [\neg\vee L] \qquad \frac{\Gamma \succ \neg\varphi, \Delta \quad \Gamma \succ \neg\psi, \Delta}{\Gamma \succ \neg(\varphi \vee \psi), \Delta}\ [\neg\vee R]$$

Observation 4.13. *The rules* $[\neg\neg L], [\neg\wedge L], [\neg\vee L], [\neg\neg R], [\neg\wedge R], [\neg\vee R]$ *are admissible in* $\mathcal{G}\mathsf{CL}$*.*

Definition 4.14 ([2])**.** Let the calculus $\mathcal{G}\mathsf{K}_3$ be the result of subtracting the rule $[\neg R]$ and adding the De Morgan rules to $\mathcal{G}\mathsf{CL}$. Let the calculus $\mathcal{G}\mathsf{LP}$ bet he result of subtracting the rule $[\neg L]$ and adding the De Morgan rules to $\mathcal{G}\mathsf{CL}$.[18]

[17] Coniglio and Corbalán call this system C, but for matters of uniformity we will adopt the name $\mathcal{G}\mathsf{CL}$, since it gives a more suggestive idea that we are working with a Gentzen-style sequent calculus for CL.

[18] Let us clarify a number of things. First, the sequent calculus for LP is presented by [5] without a proper name, whence we call it $\mathcal{G}\mathsf{LP}$ here. The sequent calculus for K_3 is not presented in [5], but it is pointed out that it should be constructed this way, which is done in e.g. [28]—although in [28] the axioms only feature literals, i.e. propositional variables or their negations, which is again inessential given the rest of the rules. Secondly, in [5] and [28], these calculi are presented with both the left and right Weakening rules being *absorbed* into the axioms. There is nothing substantial to this, given in the context of a calculus satisfying $[\mathit{Cut}]$, both sets of rules are interderivable. Thirdly, both in the context of [5] and [28], the calculi for LP and K_3 are taken as the result of subtracting from $\mathcal{G}\mathsf{CL}$ both negation rules $[\neg L]$ and $[\neg R]$ and adding, alongside with the De Morgan rules, the axioms (which can be traced back to [2]) we call $[\mathit{Exhaustion}]$ and $[\mathit{Exclusion}]$, respectively

$$\frac{}{\Gamma \succ \varphi, \neg\varphi, \Delta}\ [\mathit{Exhaustion}] \qquad \frac{}{\Gamma, \varphi, \neg\varphi \succ \Delta}\ [\mathit{Exclusion}]$$

However, this difference in presentation is inessential. For, in the context of a calculus satisfying $[\mathit{Id}]$, $[\mathit{WL}]$, $[\mathit{WR}]$ and $[\mathit{Cut}]$, on the one hand $[\mathit{Exhaustion}]$ and $[\neg R]$ are interderivable and, on the other, $[\mathit{Exclusion}]$ and $[\neg L]$ are interderivable. This can be witnessed by the following derivations

$$\frac{\Gamma, \varphi \succ \Delta \quad \dfrac{}{\Gamma \succ \varphi, \neg\varphi, \Delta}\ [\mathit{Exhaustion}]}{\Gamma \succ \neg\varphi, \Delta}\ [\mathit{Cut}] \qquad \frac{\dfrac{\dfrac{\dfrac{}{\varphi \succ \varphi}\ [\mathit{Id}]}{\Gamma, \varphi \succ \varphi}\ [\mathit{WL}]}{\Gamma, \varphi \succ \varphi, \Delta}\ [\mathit{WR}]}{\Gamma \succ \varphi, \neg\varphi, \Delta}\ [\neg R] \qquad \frac{\Gamma \succ \varphi, \Delta \quad \dfrac{}{\Gamma, \varphi, \neg\varphi \succ \Delta}\ [\mathit{Exclusion}]}{\Gamma, \neg\varphi \succ \Delta}\ [\mathit{Cut}] \qquad \frac{\dfrac{\dfrac{\dfrac{}{\varphi \succ \varphi}\ [\mathit{Id}]}{\Gamma, \varphi \succ \varphi}\ [\mathit{WL}]}{\Gamma, \varphi \succ \varphi, \Delta}\ [\mathit{WR}]}{\Gamma, \varphi, \neg\varphi \succ \Delta}\ [\neg L]$$

Theorem 4.15 ([2])**.** *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{K}_3$ if and only if $\vDash_{\mathsf{K}_3} \Gamma \succ \Delta$.*

Theorem 4.16 ([2])**.** *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{LP}$ if and only if $\vDash_{\mathsf{LP}} \Gamma \succ \Delta$.*

Theorem 4.17 ([2])**.** *Let $\Gamma \cup \Delta$ be finite non-empty set of formulae of $\mathcal{L}$. The sequent $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{K}_3$, then there is a Cut-free derivation of it. Similarly for $\mathcal{G}\mathsf{LP}$.*

Definition 4.18 ([12])**.** Let the calculus $\mathcal{G}\mathsf{PWK}$ result from $\mathcal{G}\mathsf{CL}$ minus the rules $[\wedge R]$ and $[\wedge L]$, and the additional restriction that the rule $[\neg L]$ must comply with the proviso that $var(\varphi) \subseteq var(\Delta)$—in which case, we will call this rule $[\neg^H L]$.[19]

Definition 4.19 ([12])**.** Let the calculus $\mathcal{G}\mathsf{K}_3^w$ result from $\mathcal{G}\mathsf{CL}$ minus the rules $[\vee R]$ and $[\vee L]$, and the additional restriction that the rule $[\neg R]$ must comply with the proviso that $var(\varphi) \subseteq var(\Gamma)$—in which case, we will call this rule $[\neg^B R]$.[20]

Theorem 4.20 ([12])**.** *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{PWK}$ if and only if $\vDash_{\mathsf{PWK}} \Gamma \succ \Delta$.*

Theorem 4.21 ([12])**.** *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{K}_3^w$ if and only if $\vDash_{\mathsf{K}_3^w} \Gamma \succ \Delta$.*

Theorem 4.22 ([12])**.** *Let $\Gamma \cup \Delta$ be finite non-empty set of formulae of $\mathcal{L}$. If the sequent $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{PWK}$, then there is a Cut-free derivation of it. Similarly for $\mathcal{G}\mathsf{K}_3^w$.*

Let us now turn to the calculi $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ and $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ for the four-valued generalizations of PWK and K_3^w, i.e. $\mathsf{dS}_{\mathtt{fde}}$ and $\mathsf{S}_{\mathtt{fde}}$. Their presentation is heavily inspired in the above discussed calculi presented in [12] for PWK and K_3^w—where they are properly discussed as the $\{\neg, \wedge, \vee\}$-fragment of Halldén's and Bochvar's logics of nonsense, respectively.

Definition 4.23. Let the calculus $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ result from $\mathcal{G}\mathsf{K}_3$, adding the restrictions that the rule $[\neg L]$ must comply with the proviso that $var(\varphi) \subseteq var(\Delta)$, and the rules $[\wedge L]$ and $[\neg\vee L]$ must comply with the proviso that $var(\varphi, \psi) \subseteq var(\Delta)$—in which case, we will call these rules $[\neg^H L]$, $[\wedge^H L]$ and $[\neg\vee^H L]$.

Definition 4.24. Let the calculus $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ result from $\mathcal{G}\mathsf{LP}$, adding the restrictions that the rule $[\neg R]$ must comply with the proviso that $var(\varphi) \subseteq var(\Gamma)$, and the rules $[\vee R]$ and $[\neg\wedge R]$ must comply with the proviso that $var(\varphi, \psi) \subseteq var(\Gamma)$—in which case, we will call these rules $[\neg^B R]$, $[\vee^B R]$ and $[\neg\wedge^B R]$.

4.3 Soundness and Completeness for $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$

In what follows proceed to prove the soundness and completeness results for the sequent calculus $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. For soundness, the proof is standard, by the usual means.

Lemma 4.25. *Every sequent rule of the calculus $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ preserves $\mathsf{dS}_{\mathtt{fde}}$-validity.*

Proof. Obviously the axiom and the structural rules preserve validity. We prove the case for the restricted operational rules and leave the rest as an exercise to the reader.

$[\neg^H L]$ Let v be a $\mathsf{dS}_{\mathtt{fde}}$ valuation such that $v \vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \varphi, \Delta$ and assume $var(\varphi) \subseteq var(\Delta)$. Suppose $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma \cup \{\neg\varphi\}$. Then, by hypothesis, $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$, or $v(\varphi) \in \{\mathbf{t}, \top\}$. Since $v(\neg\varphi) \in \{\mathbf{t}, \top\}$, then $v(\varphi) \in \{\mathbf{f}, \top\}$. If $v(\varphi) = \mathbf{f}$, then $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$. If $v(\varphi) = \top$, then $v(p) = \top$ for some $p \in var(\varphi)$. Since $var(\varphi) \subseteq var(\Delta)$, there is a $\delta \in \Delta$ such that $q \in var(\delta)$ and $v(q) = \top$, whence $v(\delta) = \top$ for some $\delta \in \Delta$. In both cases it follows that $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$, establishing that $v \vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma, \neg\varphi \succ \Delta$.

[19] Coniglio and Corbalán call this system H_3, but for matters of uniformity we will adopt the name $\mathcal{G}\mathsf{PWK}$, since it gives a more suggestive idea that we are working with a Gentzen-style sequent calculus for PWK.

[20] Coniglio and Corbalán call this system B_3 but, yet again, for matters of uniformity we will adopt the name $\mathcal{G}\mathsf{K}_3^w$, since it gives a more suggestive idea that we are working with a Gentzen-style sequent calculus for K_3^w.

$[\wedge^H L]$ Let v be a $\mathsf{dS}_{\mathtt{fde}}$ valuation such that $v \vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma, \varphi, \psi \succ \Delta$ and assume $var(\varphi, \psi) \subseteq var(\Delta)$. Suppose $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma \cup \{\varphi \wedge \psi\}$. If $v(\varphi \wedge \psi) = \mathbf{t}$, then given the $\mathsf{dS}_{\mathtt{fde}}$ truth-function for conjunction, we can establish that $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{t}$, whence by hypothesis $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$. If $v(\varphi \wedge \psi) = \top$, then either there is a $p \in var(\varphi)$ such that $v(p) = \top$, or there is a $q \in var(\psi)$ such that $v(q) = \top$. Either way, since $var(\varphi, \psi) \subseteq var(\Delta)$ we know that there is a $\delta \in \Delta$ such that there is an $r \in var(\delta)$ for which $v(r) = \top$, whence $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$. In both cases it follows that $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$, establishing $v \vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma, \varphi \wedge \psi \succ \Delta$.

$[\neg\vee^H L]$ Let v be a $\mathsf{dS}_{\mathtt{fde}}$ valuation such that $v \vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma, \neg\varphi, \neg\psi \succ \Delta$ and assume $var(\varphi, \psi) \subseteq var(\Delta)$. Suppose $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma \cup \{\neg(\varphi \vee \psi)\}$. If $v(\neg(\varphi \vee \psi)) = \mathbf{t}$, then given the $\mathsf{dS}_{\mathtt{fde}}$ truth-functions for negation and disjunction, we can establish that $v(\neg\varphi) = \mathbf{t}$ and $v(\neg\psi) = \mathbf{t}$, and hence by hypothesis $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$. If $v(\neg(\varphi \vee \psi)) = \top$, then either there is a $p \in var(\varphi)$ such that $v(p) = \top$, or there is a $q \in var(\psi)$ such that $v(q) = \top$. Either way, since $var(\varphi, \psi) \subseteq var(\Delta)$ we know that there is a $\delta \in \Delta$ such that there is an $r \in var(\delta)$ for which $v(r) = \top$, whence $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$. In both cases it follows that $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta$, establishing $v \vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma, \neg(\varphi \vee \psi) \succ \Delta$.

This concludes the proof. □

Theorem 4.26 (Soundness of $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$)**.** *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$, then $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$.*

Proof. If $\Gamma \succ \Delta$ is an axiom, then it is valid in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. By induction on the depth of a derivation of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ it follows, by the above Lemma 4.25, that $\Gamma \succ \Delta$ is valid in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. □

Proposition 4.27 (Non-triviality of $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$)**.** *Let Γ be a finite non-empty set of formulae of $\mathcal{L}$. The sequent $\Gamma \succ \emptyset$ is not provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$.*

Proof. Let v be a $\mathsf{dS}_{\mathtt{fde}}$-valuation such that $v(p) = \top$ for every $p \in var(\Gamma)$. It follows that $v \nvDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \emptyset$ and thus $\nvDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \emptyset$. By contraposition of Soundness, we can conclude that the sequent $\Gamma \succ \emptyset$ is not provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. □

We now turn to completeness.

Proposition 4.28. *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$, then it is provable in $\mathcal{G}\mathsf{K}_3$.*

Proof. Straightforward, since $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ is a restriction of $\mathcal{G}\mathsf{K}_3$. □

Lemma 4.29. *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{K}_3$ and $var(\Gamma) \subseteq var(\Delta)$, then $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ without using the Cut rule.*

Proof. Remember that proofs in sequent calculi are rooted binary trees such that the root is the sequent being proved and the leafs of the tree are instances of $[Id]$, in other words, sequents of the form $\varphi \succ \varphi$.

Now, assume that Π is a Cut-free derivation of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{K}_3$ such that $var(\Gamma) \subseteq var(\Delta)$. If Π is a Cut-free derivation in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$, then the result is established. If Π is not a Cut-free derivation in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$, then there must be in Π applications of the rules $[\neg L]$, $[\wedge L]$ and $[\neg\vee L]$ where the required provisos are not satisfied

$$\frac{\Gamma^* \succ \Delta^*, \varphi}{\Gamma^*, \neg\varphi \succ \Delta^*}\ [\neg L] \qquad \frac{\Gamma^*, \varphi, \psi \succ \Delta^*}{\Gamma^*, \varphi \wedge \psi \succ \Delta^*}\ [\wedge L] \qquad \frac{\Gamma^*, \neg\varphi, \neg\psi \succ \Delta^*}{\Gamma^*, \neg(\varphi \vee \psi) \succ \Delta^*}\ [\neg\vee L]$$

Now, since Π is a Cut-free proof, we are guaranteed that the root sequent $\Gamma \succ \Delta$ contains *all* the propositional variables appearing in Π. Since, by hypothesis, we know that $var(\Gamma) \subseteq var(\Delta)$, we can affirm that $var(\Pi) = var(\Delta)$.

What is left is, then, to design a procedure to transform Π into a Cut-free proof of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. We do this in two steps. First, we enlarge every node of Π by adding Δ to its right-hand side. By doing this, we obtain a rooted binary tree Π', whose leafs are sequents of the form $\varphi \succ \varphi, \Delta$. Second, we extend each leaf with a branch starting in an instance of $[Id]$, that is, a sequent of the form $\varphi \succ \varphi$, followed by any number of necessary iterated applications of the right Weakening rule $[WR]$, so that the sequent $\varphi \succ \varphi, \Delta$ is obtained.

From this procedure, we get a rooted binary tree Π'' which is undoubtedly a Cut-free derivation in $\mathcal{G}\mathsf{K}_3$ of the sequent $\Gamma \succ \Delta$, such that the critical instances of the rules $[\neg L]$, $[\vee L]$ and $[\neg\wedge R]$ have in Π'' the form

$$\frac{\Gamma^* \succ \varphi, \Delta^*, \Delta}{\Gamma^*, \neg\varphi \succ \Delta^*, \Delta}\ [\neg L] \qquad \frac{\Gamma^*, \varphi, \psi \succ \Delta^*, \Delta}{\Gamma^*, \varphi \wedge \psi \succ \Delta^*, \Delta}\ [\wedge L] \qquad \frac{\Gamma^*, \neg\varphi, \neg\psi \succ \Delta^*, \Delta}{\Gamma^*, \neg(\varphi \vee \psi) \succ \Delta^*, \Delta}\ [\neg\vee L]$$

and are, thus, admissible in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. Finally, from this we infer that Π'' is a Cut-free derivation in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ of the sequent $\Gamma \succ \Delta$. □

Corollary 4.30. *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$ but $var(\Gamma) \subsetneq var(\Delta)$, then there is a $\Gamma' \subseteq \Gamma$ such that $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma' \succ \Delta$, where $var(\Gamma') \subseteq var(\Delta)$.*

Proof. First, notice that if $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$, then $var(\Gamma) \neq \emptyset \neq var(\Delta)$. Now, assume $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$ but $var(\Gamma) \nsubseteq var(\Delta)$. Hence, define $\Gamma' = \Gamma \setminus \{\gamma \in \Gamma \mid var(\gamma) \nsubseteq var(\Delta)\}$, whence $\Gamma' \subset \Gamma$ and $var(\Gamma') \subseteq var(\Delta)$. Suppose there is a $\mathsf{dS}_{\mathtt{fde}}$ valuation v such that $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma'$. If $v(\gamma) = \top$ for some $\gamma \in \Gamma'$, then $v(p) = \top$ for some $p \in var(\gamma)$ and, therefore, $v(p) = \top$ for some $p \in var(\Gamma')$. Since $var(\Gamma') \subseteq var(\Delta)$, then $v(q) = \top$ for some $q \in var(\Delta)$, whence there is a $\delta \in \Delta$ such that $v(\delta) = \top$. This establishes $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma' \succ \Delta$. If $v(\gamma) = \mathbf{t}$ for all $\gamma \in \Gamma'$, then suppose for *reductio* that $v(\delta) \in \{\bot, \mathbf{f}\}$ for all $\delta \in \Delta$, which implies that $\nvDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma' \succ \Delta$. But then, $v(p) \in \{\mathbf{t}, \bot, \mathbf{f}\}$ for all $p \in var(\Delta)$. And since $var(\Gamma') \subseteq var(\Delta)$, this will also require that $v(q) \in \{\mathbf{t}, \bot, \mathbf{f}\}$ for all $q \in var(\Gamma')$. Consider, now, a $\mathsf{dS}_{\mathtt{fde}}$ valuation v^* such that

$$v^*(p) = \begin{cases} \top & \text{if } p \in var(\Gamma) \setminus var(\Delta) \\ v(p) & \text{if } p \in var(\Delta) \end{cases}$$

Then, by the above this will imply $v^*(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma$, but $v^*(\delta) \in \{\bot, \mathbf{f}\}$ for all $\delta \in \Delta$, whence v^* witnesses $\nvDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$, contrdaciting our initial assumption. Therefore, if $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$ but $var(\Gamma) \subsetneq var(\Delta)$, then there is a $\Gamma' \subseteq \Gamma$ such that $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma' \succ \Delta$, where $var(\Gamma') \subseteq var(\Delta)$. □

Theorem 4.31 (Completeness of $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$)**.** *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$, then $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ without using the Cut rule.*

Proof. Assume $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$. By Observation 4.6, we know that $\vDash_{\mathsf{K}_3} \Gamma \succ \Delta$, and also by Theorem 4.15 we are granted that $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{K}_3$. To finally establish that $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ without using the Cut rule, we consider two cases. First, if $var(\Gamma) \subseteq var(\Delta)$, we know by Lemma 4.29 that this is the case. Second, if $var(\Gamma) \subsetneq var(\Delta)$, we know by Corollary 4.30 that there is a $\Gamma' \subseteq \Gamma$ such that $\vDash_{\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta'$, where $var(\Gamma') \subseteq var(\Delta)$. Now, by Lemma 4.29 we know that $\Gamma' \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ without using the Cut rule, by means of a proof Π_1 (i.e. a rooted binary tree) whose root is $\Gamma' \succ \Delta$ and whose leafs are instances of $[Id]$, of the form $\varphi \succ \varphi$. Finally, we transform Π_1 into a proof Π_1', by extending down the node $\Gamma' \succ \Delta$ by means of the required iterated applications of the left Weakening rule $[WL]$, until we arrive at the sequent $\Gamma \succ \Delta$. But this rooted binary tree Π_1' is now a proof in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ of the sequent $\Gamma \succ \Delta$, without using the Cut rule. □

Corollary 4.32 (Cut-elimination for $\mathsf{dS}_{\mathtt{fde}}$)**.** *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae in $\mathcal{L}$. If the sequent $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$, then there is a Cut-free derivation of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$.*

Proof. Assume that $\Gamma \succ \Delta'$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$. By Theorem 4.34, that is, because the system is sound, we know that $\vDash_{\mathcal{G}\mathsf{dS}_{\mathtt{fde}}} \Gamma \succ \Delta$. But, then, by Theorem 4.39, that is, becuase the system is complete, we know that $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{dS}_{\mathtt{fde}}$ without using the Cut rule. □

4.4 Soundness and Completeness for $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$

Lemma 4.33. *Every sequent rule of the calculus $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ preserves $\mathsf{S}_{\mathtt{fde}}$-validity.*

Proof. Obviously the axiom and the structural rules preserve validity. We prove the case for the restricted operational rules and leave the rest as an exercise to the reader.

$[\neg^B R]$ Let v be a $\mathsf{S}_{\mathtt{fde}}$ valuation such that $v \vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma, \varphi \succ \Delta$ and assume that $var(\varphi) \subseteq var(\Gamma)$. Suppose $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma$. Thus, $v(p) \in \{\mathbf{t}, \top, \mathbf{f}\}$ for all $p \in var(\Gamma)$. Since $var(\varphi) \subseteq var(\Gamma)$, we also know that $v(q) \in \{\mathbf{t}, \top, \mathbf{f}\}$ for all $q \in var(\varphi)$. Hence, $v(\varphi) \in \{\mathbf{t}, \top, \mathbf{f}\}$. If $v(\varphi) \in \{\top, \mathbf{f}\}$, then $v(\neg\varphi) \in \{\mathbf{t}, \top\}$, whence $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta \cup \{\neg\varphi\}$. If $v(\varphi) = \mathbf{t}$, then by hypothesis there is a $\delta \in \Delta$ such that $v(\delta) \in \{\mathbf{t}, \top\}$, whence $v(\delta) \in \{\mathbf{t}, \top\}$ for some $\delta \in \Delta \cup \{\neg\varphi\}$. Either way, this establishes $v \vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \neg\varphi, \Delta$.

$[\vee^B R]$ Let v be a $\mathsf{S}_{\mathtt{fde}}$ valuation such that $v \vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \varphi, \psi, \Delta$ and assume that $var(\varphi, \psi) \subseteq var(\Gamma)$. Suppose $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma$. Hence, $v(p) \in \{\mathbf{t}, \top, \mathbf{f}\}$, for all $p \in var(\Gamma)$. Since $var(\varphi, \psi) \subseteq var(\Gamma)$ we know that $v(q) \in \{\mathbf{t}, \top, \mathbf{f}\}$, for all $q \in var(\varphi, \psi)$ and, moreover, $v(\varphi) \in \{\mathbf{t}, \top, \mathbf{f}\}$ and $v(\psi) \in \{\mathbf{t}, \top, \mathbf{f}\}$. By hypothesis, there is a $\delta \in \Delta \cup \{\varphi, \psi\}$ such that $v(\delta) \in \{\mathbf{t}, \top\}$. Thus, either there is a $\delta \in \Delta$ such that $v(\delta) \in \{\mathbf{t}, \top\}$, or $v(\varphi) \in \{\mathbf{t}, \top\}$, or $v(\psi) \in \{\mathbf{t}, \top\}$. Finally, given $v(\varphi) \in \{\mathbf{t}, \top, \mathbf{f}\}$ and $v(\psi) \in \{\mathbf{t}, \top, \mathbf{f}\}$, and given the $\mathsf{S}_{\mathtt{fde}}$ truth-function for disjunction, we can establish that in all these cases it follows that there is a $\delta \in \Delta \cup \{\varphi \vee \psi\}$ such that $v(\delta) \in \{\mathbf{t}, \top\}$. Therefore, $v \vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \varphi \vee \psi, \Delta$.

$[\neg\wedge^B R]$ Let v be a $\mathsf{S}_{\mathtt{fde}}$ valuation such that $v \vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \neg\varphi, \neg\psi, \Delta$ and assume that $var(\varphi, \psi) \subseteq var(\Gamma)$. Suppose $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma$. Thus, $v(p) \in \{\mathbf{t}, \top, \mathbf{f}\}$, for all $p \in var(\varphi, \psi)$. Since $var(\varphi, \psi) \subseteq var(\Gamma)$ we know that $v(q) \in \{\mathbf{t}, \top, \mathbf{f}\}$, for all $q \in var(\varphi, \psi)$ and, moreover, that $v(\neg\varphi) \in \{\mathbf{t}, \top, \mathbf{f}\}$ and $v(\neg\psi) \in \{\mathbf{t}, \top, \mathbf{f}\}$. By hypothesis, there is a $\delta \in \Delta \cup \{\neg\varphi, \neg\psi\}$ such that $v(\delta) \in \{\mathbf{t}, \top\}$. Thus, either there is a $\delta \in \Delta$ such that $v(\delta) \in \{\mathbf{t}, \top\}$, or $v(\neg\varphi) \in \{\mathbf{t}, \top\}$, or $v(\neg\psi) \in \{\mathbf{t}, \top\}$. Finally, given $v(\neg\varphi) \in \{\mathbf{t}, \top, \mathbf{f}\}$ and $v(\neg\psi) \in \{\mathbf{t}, \top, \mathbf{f}\}$, and given the $\mathsf{S}_{\mathtt{fde}}$ truth-functions for negation and conjunction, we can establish that in all these cases it follows that there is $\delta \in \Delta \cup \{\neg(\varphi \wedge \psi)\}$ such that $v(\delta) \in \{\mathbf{t}, \top\}$. Therefore, $v \vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \neg(\varphi \wedge \psi), \Delta$.

This concludes the proof. □

Theorem 4.34 (Soundness of $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$)**.** *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$, then $\vDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta$.*

Proof. If $\Gamma \succ \Delta$ is an axiom, then it is valid in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$. By induction on the depth of a derivation of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ it follows, by the above Lemma 4.33, that $\Gamma \succ \Delta$ is valid in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$. □

Proposition 4.35 (Non-triviality of $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$)**.** *Let Γ be a finite non-empty set of formulae of $\mathcal{L}$. The sequent $\Gamma \succ \emptyset$ is not provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$.*

Proof. Let v be a $\mathsf{S}_{\mathtt{fde}}$-valuation such that $v(p) = \top$ for every $p \in var(\Gamma)$. It follows that $v \nvDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \emptyset$ and thus $\nvDash_{\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \emptyset$. By contraposition of Soundness, we can conclude that the sequent $\Gamma \succ \emptyset$ is not provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$. □

We now turn to completeness.

Proposition 4.36. *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$, then it is provable in $\mathcal{G}\mathsf{LP}$.*

Proof. Straightforward, since $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ is a restriction of $\mathcal{G}\mathsf{LP}$. □

Lemma 4.37. *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{LP}$ and $var(\Delta) \subseteq var(\Gamma)$, then $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ without using the Cut rule.*

Proof. Remember that proofs in sequent calculi are rooted binary trees such that the root is the sequent being proved and the leafs of the tree are instances of $[Id]$, in other words, sequents of the form $\varphi \succ \varphi$.

Now, assume that Π is a Cut-free derivation of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{LP}$ such that $var(\Delta) \subseteq var(\Gamma)$. If Π is a Cut-free derivation in $\mathcal{G}\mathsf{S}_{\texttt{fde}}$, then the result is established. If Π is not a Cut-free derivation in $\mathcal{G}\mathsf{S}_{\texttt{fde}}$, then there must be in Π applications of the rules $[\neg R]$, $[\vee R]$ or $[\neg\wedge R]$ where the required provisos are not satisfied

$$\frac{\Gamma^*, \varphi \succ \Delta^*}{\Gamma^* \succ \neg\varphi, \Delta^*}\,[\neg R] \qquad \frac{\Gamma^* \succ \varphi, \psi, \Delta^*}{\Gamma^* \succ \varphi \vee \psi, \Delta^*}\,[\vee R] \qquad \frac{\Gamma^* \succ \neg(\varphi \wedge \psi), \Delta^*}{\Gamma^* \succ \neg\varphi, \neg\psi, \Delta^*}\,[\neg\wedge R]$$

Now, since Π is a Cut-free proof, we are guaranteed that the root sequent $\Gamma \succ \Delta$ contains *all* the propositional variables appearing in Π. Since, by hypothesis, we know that $var(\Delta) \subseteq var(\Gamma)$, we can affirm that $var(\Pi) = var(\Gamma)$.

What is left is, then, to design an algorithmic procedure to transform Π into a Cut-free proof of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{S}_{\texttt{fde}}$. We do this in two steps. First, we enlarge every node of Π by adding Γ to its left-hand side. By doing this, we obtain a rooted binary tree Π', whose leafs are sequents of the form $\Gamma, \varphi \succ \varphi$. Second, we extend each leaf with a branch starting in an instance of $[Id]$, that is, a sequent of the form $\varphi \succ \varphi$, followed by any number of necessary iterated applications of the left Weakening rule $[WL]$, so that the sequent $\Gamma, \varphi \succ \varphi$ is obtained.

From this procedure, we get a rooted binary tree Π'' which is undoubtedly a Cut-free derivation in $\mathcal{G}\mathsf{LP}$ of the sequent $\Gamma \succ \Delta$, such that the critical instances of the rules $[\neg R]$, $[\vee L]$ and $[\neg\wedge R]$ have in Π'' the form

$$\frac{\Gamma, \Gamma^*, \varphi \succ \Delta^*}{\Gamma, \Gamma^* \succ \neg\varphi, \Delta^*}\,[\neg R] \qquad \frac{\Gamma, \Gamma^* \succ \varphi, \psi, \Delta^*}{\Gamma, \Gamma^* \succ \varphi \vee \psi, \Delta^*}\,[\vee R] \qquad \frac{\Gamma, \Gamma^* \succ \neg(\varphi \wedge \psi), \Delta^*}{\Gamma, \Gamma^* \succ \neg\varphi, \neg\psi, \Delta^*}\,[\neg\wedge R]$$

and are, thus, admissible in $\mathcal{G}\mathsf{S}_{\texttt{fde}}$. Finally, from this we infer that Π'' is a Cut-free derivation in $\mathcal{G}\mathsf{S}_{\texttt{fde}}$ of the sequent $\Gamma \succ \Delta$. □

Corollary 4.38. *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\vDash_{\mathsf{S}_{\texttt{fde}}} \Gamma \succ \Delta$ but $var(\Delta) \subsetneq var(\Gamma)$, then there is a $\Delta' \subseteq \Delta$ such that $\vDash_{\mathsf{S}_{\texttt{fde}}} \Gamma \succ \Delta'$, where $var(\Delta') \subseteq var(\Gamma)$.*

Proof. First, notice that if $\vDash_{\mathsf{S}_{\texttt{fde}}} \Gamma \succ \Delta$, then $var(\Gamma) \neq \emptyset \neq var(\Delta)$. Now, assume $\vDash_{\mathsf{S}_{\texttt{fde}}} \Gamma \succ \Delta$ but $var(\Delta) \nsubseteq var(\Gamma)$. Hence, define $\Delta' = \Delta \setminus \{\delta \in \Delta \mid var(\delta) \nsubseteq var(\Gamma)\}$, whence $\Delta' \subset \Delta$ and $var(\Delta') \subseteq var(\Gamma)$. Suppose additionally, for *reductio*, that there is a $\mathsf{S}_{\texttt{fde}}$ valuation v such that $v(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma$, but $v(\delta) \in \{\bot, \mathbf{f}\}$ for all $\delta \in \Delta'$, thus implying $\nvDash_{\mathsf{S}_{\texttt{fde}}} \Gamma \succ \Delta'$. Construct now a $\mathsf{S}_{\texttt{fde}}$ valuation v^* such that

$$v^*(p) = \begin{cases} \bot & \text{if } p \in var(\Delta) \setminus var(\Gamma) \\ v(p) & \text{if } p \in var(\Gamma) \end{cases}$$

Hence, v^* is such that $v^*(\gamma) \in \{\mathbf{t}, \top\}$ for all $\gamma \in \Gamma$, but $v^*(\delta) \in \{\bot, \mathbf{f}\}$ for all $\delta \in \Delta$, whence v^* witnesses $\nvDash_{\mathsf{S}_{\texttt{fde}}} \Gamma \succ \Delta$, contradicting our initial assumption. Thus, there is a $\Delta' \subset \Delta$ such that $\vDash_{\mathsf{S}_{\texttt{fde}}} \Gamma \succ \Delta'$, where $var(\Delta') \subseteq var(\Gamma)$. Therefore, if $\vDash_{\mathsf{S}_{\texttt{fde}}} \Gamma \succ \Delta$ but $var(\Delta) \subsetneq var(\Gamma)$, then there is a $\Delta' \subseteq \Delta$ such that $\vDash_{\mathsf{S}_{\texttt{fde}}} \Gamma \succ \Delta'$, where $var(\Delta') \subseteq var(\Gamma)$. □

Theorem 4.39 (Completeness of $\mathcal{G}\mathsf{S}_{\texttt{fde}}$)**.** *Let $\Gamma \cup \Delta$ be a finite non-empty set of formulae of $\mathcal{L}$. If $\vDash_{\mathsf{S}_{\texttt{fde}}} \Gamma \succ \Delta$, then $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{S}_{\texttt{fde}}$ without using the Cut rule.*

Proof. Assume $\vDash_{\mathsf{S}_{\texttt{fde}}} \Gamma \succ \Delta$. By Observation 4.3, we know that $\vDash_{\mathsf{LP}} \Gamma \succ \Delta$, and also by Theorem 4.16 we are granted that $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{LP}$. To finally establish that $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{S}_{\texttt{fde}}$ without using the Cut rule, we consider two cases. First, if $var(\Delta) \subseteq var(\Gamma)$, we know by Lemma 4.37 that this is the case. Second, if $var(\Delta) \subsetneq var(\Gamma)$, we know by Corollary 4.38 that there is a $\Delta' \subseteq \Delta$ such that $\vDash_{\mathsf{S}_{\texttt{fde}}} \Gamma \succ \Delta'$, where $var(\Delta') \subseteq var(\Gamma)$. Now, by Lemma 4.37 we know that $\Gamma \succ \Delta'$ is provable in $\mathcal{G}\mathsf{S}_{\texttt{fde}}$ without using the Cut rule, by means of a proof Π_1 (i.e. a rooted binary tree) whose root is $\Gamma \succ \Delta'$ and whose leafs are instances of $[Id]$, of the form $\varphi \succ \varphi$. Finally, we transform Π_1 into a proof

Π_1', by extending down the node $\Gamma \succ \Delta'$ by means of the required iterated applications of the right Weakening rule [WR], until we arrive at the sequent $\Gamma \succ \Delta$. But this rooted binary tree Π_1' is now a proof in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ of the sequent $\Gamma \succ \Delta$, without using the Cut rule. □

Corollary 4.40 (Cut-elimination for $\mathsf{S}_{\mathtt{fde}}$)**.** *Let $\Gamma \cup \Delta$ be a finite nonempty set of formulae in $\mathcal{L}$. If the sequent $\Gamma \succ \Delta$ is provable in $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$, then there is a Cut-free derivation of $\Gamma \succ \Delta$ in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$.*

Proof. Assume that $\Gamma \succ \Delta'$ is provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$. By Theorem 4.34, that is, because the system is sound, we know that $\vDash_{\mathcal{G}\mathsf{S}_{\mathtt{fde}}} \Gamma \succ \Delta$. But, then, by Theorem 4.39, that is, becuase the system is complete, we know that $\Gamma \succ \Delta$ is provable in $\mathcal{G}\mathsf{S}_{\mathtt{fde}}$ without using the Cut rule. □

5 Conclusion

In this paper we showed that, by following Fitting's epistemic interpretation of the strong Kleene logics K_3 and FDE, and the Paracomplete Weak Kleene logic K_3^w, an up to now unnoticed epistemic interpretation of Paraconsistent Weak Kleene logic PWK is available. This interpretation is carried out by focusing on a four-valued generalization of PWK, namely the logic $\mathsf{dS}_{\mathtt{fde}}$, and showing that its truth-functions can be interpreted in terms of what we called track-down operations. These operations, built inspired in the idea that no consistent opinion can arise from a set that includes an inconsistent opinion, coincide with the truth-functions of Paraconsistent Weak Kleene when certain reasonable constraints are assumed.

In addition to providing this novel interpretation of Paraconsistent Weak Kleene, the failure of Conjunctive Simplification in such a system and its sublogics is discussed in terms of track-down conjunctions and, also, in connection with containment logics. Concerning this latter relation, Paraconsistent Weak Kleene is shown to be closely related, and its theoremless subsystems are shown to belong, to a family of systems that respect a containment principle dual to Parry's Proscriptive Principle for entailment. These considerations mirror the previous remarks made in the literature concerning the other three-valued weak Kleene logic, namely K_3^w, whose subsystems counting with no anti-theorems were shown by Ferguson to respect Parry's Proscriptive Principle for entailment.

These observations allowed us to design sound and complete Gentzen-style sequent calculi for this four-valued generalizations of PWK and K_3^w, i.e. the systems we referred to as $\mathsf{dS}_{\mathtt{fde}}$ and $\mathsf{S}_{\mathtt{fde}}$, drawing inspiration from the techniques recently applied by Coniglio and Corbalán to provide calculi of these sort for the three-valued weak Kleene logics. The main feature of these calculi, both for logics of the three- and four-valued kinds, was the presence of linguistic (i.e. variable inclusion) provisos in some of the operational rules for the calculi, pertaining the set of propositional variables of the active formulae of the corresponding rules, and the set of propositional variables appearing in some of the side formulae of such rules.

References

[1] R. B. Angell. Deducibility, entailment and analytic containment. In J. Norman and R. Sylvan, editors, *Directions in Relevant Logic*, pages 119–143. Springer, Dordrecht, 1989. DOI:10.1007/978-94-009-1005-8_8.

[2] A. Avron. Natural 3-valued logics—characterization and proof theory. *The Journal of Symbolic Logic*, 56(1):276–294, 1991. DOI:10.2307/2274919.

[3] A. Avron and A. Zamansky. Non-deterministic semantics for logical systems. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 16, pages 227–304. Springer, Dodrecht, 2011. DOI:10.1007/978-94-007-0479-4_4.

[4] J. Beall. Off-Topic: A New Interpretation of Weak-Kleene Logic. *The Australasian Journal of Logic*, 13(6):136–142, 2016.

[5] J. C. Beall. Multiple-conclusion LP and default classicality. *The Review of Symbolic Logic*, 4(2): 326–336, 2011. DOI:10.1017/S1755020311000074.

[6] N. Belnap. A useful four-valued logic. In M. Dunn and G. Epstein, editors, *Modern uses of Multiple-Valued Logic*, volume 2, pages 5–37. Springer, Dodrecht, 1977. DOI:10.1007/978-94-010-1161-7_2.

[7] D. A. Bochvar. On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus. *History and Philosophy of Logic*, 2(1-2): 87–112, 1981. DOI:10.1080/01445348108837023. Translated by Merrie Bergmann.

[8] B. Bolzano. *Wissenschaftslehre*. Seidel, Sulzbach, 1837.

[9] S. Bonzio, J. Gil-Ferez, F. Paoli, and L. Peruzzi. On Paraconsistent Weak Kleene logic: Axiomatization and Algebraic Analysis. *Studia Logica*, 105(2):253–297, 2017. DOI:10.1007/s11225-016-9689-5.

[10] R. Ciuni. Conjunction in Paraconsistent Weak Kleene Logic. In P. Arazim and M. Dančák, editors, *The Logica Yearbook 2014*, pages 61–76. College Publications, London, 2015.

[11] R. Ciuni and M. Carrara. Characterizing Logical Consequence in Paraconsistent Weak Kleene. In L. Felline, A. Ledda, F. Paoli, and E. Rossanese, editors, *New Developments in Logic and the Philosophy of Science*, pages 165–176. College Publications, London, 2016.

[12] M. Coniglio and M. I. Corbalán. Sequent calculi for the classical fragment of Bochvar and Halldén's Nonsense Logics. In D. Kesner and P. Viana, editors, *Proceedings of the Seventh Workshop on Logical and Semantic Frameworks with Applications (LSFA 2012)*, volume 113 of *EPTCS*, pages 125–136, 2012.

[13] F. Correia. Semantics for analytic containment. *Studia Logica*, 77(1):87–104, 2004. DOI:10.1023/B:STUD.0000034187.37935.24.

[14] H. Deutsch. Relevant analytic entailment. *The Relevance Logic Newsletter*, 2(1):26–44, 1977.

[15] K. Došen. A note on the law of identity and the converse Parry property. *Notre Dame Journal of Formal Logic*, 19(1):174–176, 1978. DOI:10.1305/ndjfl/1093888223.

[16] M. Dunn. Intuitive semantics for first-degree entailments and 'coupled trees'. *Philosophical Studies*, 29(3):149–168, 1976. DOI:10.1007/BF00373152.

[17] R. L. Epstein. *The Semantic Foundations of Logic*, volume I: Propositional Logics. Oxford University Press, New York, 2nd edition, 1995.

[18] T. M. Ferguson. A computational interpretation of conceptivism. *Journal of Applied Non-Classical Logics*, 24(4):333–367, 2014. DOI:10.1080/11663081.2014.980116.

[19] T. M. Ferguson. Logics of nonsense and Parry systems. *Journal of Philosophical Logic*, 44(1): 65–80, 2015. DOI:10.1007/s10992-014-9321-y.

[20] T. M. Ferguson. Cut-Down Operations on Bilattices. In *Proceedings of the 45th International Symposium on Multiple-Valued Logic (ISMVL 2015)*, pages 24–29. IEEE Computer Society, 2015. DOI:10.1109/ISMVL.2015.14.

[21] T. M. Ferguson. Rivals to Belnap–Dunn Logic on Interlaced Trilattices. *Studia Logica*, 2016. DOI:10.1007/s11225-016-9695-7. *Forthcoming*.

[22] T. M. Ferguson. Faulty Belnap computers and subsystems of FDE. *Journal of Logic and Computation*, 26(5):1617–1636, 2016. DOI:10.1093/logcom/exu048.

[23] T. M. Ferguson. *The Proscriptive Principle and Logics of Analytic Implication*. PhD thesis, City University of New York, 2017.

[24] K. Fine. Angellic content. *Journal of Philosophical Logic*, 45(2):199–226, 2016. DOI:10.1007/s10992-015-9371-9.

[25] M. Fitting. Bilattices and the semantics of logic programming. *The Journal of Logic Programming*, 11(2):91–116, 1991. DOI:10.1016/0743-1066(91)90014-G.

[26] M. Fitting. Kleene's three valued logics and their children. *Fundamenta Informaticae*, 20(1, 2, 3):113–131, 1994. DOI:10.3233/FI-1994-201234.

[27] M. Fitting. Bilattices are nice things. In T. Bolander, V. Hendricks, and S. A. Pedersen, editors, *Self-Reference*, pages 53–78. CSLI Publications, 2006.

[28] R. French and S. Standefer. Non-triviality done proof-theoretically. In A. Baltag, J. Seligman, and T. Yamada, editors, *Logic, Rationality, and Interaction: 6th International Workshop (LORI 2017)*, pages 438–450, Berlin, 2017. Springer. DOI:10.1007/978-3-662-55665-8_30.

[29] S. Halldén. *The logic of nonsense*. Uppsala Universitets Arsskrift, Uppsala, 1949.

[30] J. Hintikka. *Logic, Language-Games and Information: Kantian Themes in the Philosophy of Logic*. Clarendon Press, Oxford, 1973.

[31] L. Humberstone. Power Matrices and Dunn–Belnap Semantics: Reflections on a Remark of Graham Priest. *The Australasian Journal of Logic*, 11(1):14–45, 2014.

[32] L. Humberstone and R. Meyer. The relevant equivalence property. *Logic Journal of the IGPL*, 15 (2):165–181, 2007.

[33] S. C. Kleene. *Introduction to Metamathematics*. North-Holland, Amsterdam, 1952.

[34] C. A. Oller. Paraconsistency and analyticity. *Logic and Logical Philosophy*, 7:91–99, 1999. `DOI:10.12775/LLP.1999.008`.

[35] H. Omori. Halldén's Logic of Nonsense and its expansions in view of Logics of Formal Inconsistency. In *Proceedings of the 27th International Workshop on Database and Expert Systems Applications (DEXA 2016)*, pages 129–133. IEEE Computer Society, 2016. `DOI:10.1109/DEXA.2016.039`.

[36] H. Omori and D. Szmuc. Conjunction and disjunction in infectious logics. In A. Baltag, J. Seligman, and T. Yamada, editors, *Logic, Rationality, and Interaction: 6th International Workshop (LORI 2017)*, pages 268–283, Berlin, 2017. Springer. `DOI:10.1007/978-3-662-55665-8_19`.

[37] F. Paoli. Regressive analytical entailments. Technical Report number 33, *Konstanzer Berichte zur Logik und Wissenschaftstheorie*, 1992.

[38] F. Paoli. Tautological entailments and their rivals. In J.-Y. Béziau, W. A. Carnielli, and D. M. Gabbay, editors, *Handbook of Paraconsistency*, pages 153–175. College Publications, 2007.

[39] W. T. Parry. Ein axiomensystem für eine neue art von implikation (analytische implikation). *Ergebnisse eines mathematischen Kolloquiums*, 4:5–6, 1933.

[40] Y. Petrukhin. Natural Deduction for Three-Valued Regular Logics. *Logic and Logical Philosophy*, 26(2):197–206, 2016. `DOI:10.12775/LLP.2016.025`.

[41] Y. Petrukhin. Natural Deduction for Four-Valued both Regular and Monotonic Logics. *Logic and Logical Philosophy*, 2017. `DOI:10.12775/LLP.2017.001`. *Forthcoming*.

[42] Y. Petrukhin. Natural Deduction for Fitting's Four-Valued Generalizations of Kleene's Logics. *Logica Universalis*, 2017. `DOI:10.1007/s11787-017-0169-0`. *Forthcoming*.

[43] G. Priest. *An introduction to non-classical logic: from If to Is*. Cambridge University Press, 2nd edition, 2008.

[44] G. Priest. Plurivalent logics. *The Australasian Journal of Logic*, 11(1):2–13, 2014.

[45] R. Routley, V. Plumwood, R. Meyer, and R. Brady. *Relevant Logics and Their Rivals: The Basic Philosophical and Semantical Theory*. Ridgeview, Atascadero, CA, 1982.

[46] R. Sylvan. Relevant containment logics and certain frame problems of AI. *Logique et Analyse*, 31:11–25, 1988.

[47] D. Szmuc. Defining LFIs and LFUs in extensions of infectious logics. *Journal of Applied Non-Classical Logics*, 26(4):286–314, 2017. `DOI:10.1080/11663081.2017.1290488`.

[48] N. Tomova. About four-valued regular logics. *Logical Investigations*, 15:223–228, 2009.

[49] A. Urquhart. Basic many-valued logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 2, pages 249–295. Springer, Dodrecht, 2001. `DOI:10.1007/978-94-017-0452-6_4`.

[50] T. E. Zimmermann. Free choice disjunction and epistemic possibility. *Natural language semantics*, 8(4):255–290, 2000. `DOI:10.1023/A:1011255819284`.

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