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Paradoxes of Demonstrability

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1. Introduction

In this paper I consider two paradoxes that arise in connection with the concept of demonstrability, or absolute provability. I assume—for the sake of the argument—that there is an intuitive notion of *demonstrability*, which should not be conflated with the concept of *formal deducibility* in a (formal) system or the relativized concept of *provability* from certain axioms. Demonstrability is an epistemic concept: the rough idea is that a sentence is demonstrable if it is provable from knowable basic ("self-evident") premises by means of simple logical steps. A statement that is demonstrable is also knowable and a statement that is actually demonstrated is known to be true.

By casting doubt upon apparently central principles governing the concept of demonstrability, the paradoxes of demonstrability presented here tend to undermine the concept itself—or at least our understanding of it. As long as we cannot find a diagnosis and a cure for the paradoxes, it seems that the coherence of the concepts of demonstrability and demonstrable knowledge are put in question. There are of course ways of putting the paradoxes in quarantine, for example by imposing a hierarchy of languages à la Tarski, or a ramified hierarchy of propositions and propositional functions à la Russell.[1] These measures, however, helpful as they may be in avoiding contradictions, do not seem to solve the underlying conceptual problems.

I offer these paradoxes to Howard Sobel on his 80th birthday in the hope that he will try his hand at their resolution—or at least that they may give him some pleasure. For many years now Howard has pursued a non-proposition strategy to solve various versions of the liar paradox.[2] So far Howard has applied his strategy to semantical paradoxes, for which he claims that the strategy works generally:

> "To resolve putative liar paradoxes it is sufficient to attend to the distinction between liar-sentences and the propositions they would express, and to exer-

[1] Cf. Church (1976).
[2] Sobel (1992), (2008), (2009a-c).

> cise the option of turning would be deductions of paradox (or contradictions) into reductions of the existence of these propositions."
>
> (Sobel 1992, p. 51, See also Sobel 2009b)

Although structurally similar to the semantical paradoxes, the paradoxes discussed in this paper involve epistemic notions: "demonstrability", "knowability", "knowledge"... These notions are "factive" (e.g., if A is demonstrable, then A is true), but similar paradoxes arise in connection with "non-factive" notions like "believes", "says", "asserts".[3] There is no consensus in the literature concerning the analysis of the notions involved—often referred to as "propositional attitudes"—or concerning the treatment of the paradoxes they give rise to.

So, here is the question I want to put to you, Howard: Will your propositional strategy take care of the epistemic paradoxes? And how will it work? I am looking forward to many lively and enjoyable discussions with you on these matters.

2. An Elementary Calculus of Demonstrability

Although the concept Dem(x) of demonstrability is far from precise, it appears that it must satisfy the following principles:

(D0) $\vdash$ A, for any theorem A of minimal (propositional) logic
(D1) $\vdash \mathrm{Dem}(\langle A\rangle) \to A$
(R1) If $\vdash A$ and $\vdash A \to B$, then $\vdash B$
(R2) If $\vdash A$, then $\vdash \mathrm{Dem}(\langle A\rangle)$

Here $\vdash A$ means that A is a thesis (theorem) of our calculus of demonstrability. For any sentence A, $\langle A\rangle$ is a standard name of A.

We also assume that there is a sentence D for which it holds:

(D2) $\vdash D \leftrightarrow (\mathrm{Dem}(\langle D\rangle) \to G)$

That is, we assume that there is a sentence D which is provably equivalent to the sentence $\mathrm{Dem}(\langle D\rangle) \to G$, where G is any sentence (e.g., $\bot$, "Santa Claus exist",...). By means of Gödel numbering and a weak theory of arithmetic as part of our background theory we could prove a diagonal lemma, which would have (D2) as a special case.

[3] Cf. Burge (1978), (1984).

3. A Curry-Type Paradox of Demonstrability

Suppose now:

{1}	(1) Dem(<D>)	assumption
{}	(2) Dem(<D>) → D	by (D1)
{1}	(3) D	(1), (2) → elimination
{1}	(4) Dem(<D>) → G	(3), (D2) minimal logic
{1}	(5) G	(1), (4) → elimination
{}	(6) Dem(<D>) → G	(1) – (5) → introduction
{}	(7) D	(6), (D2) minimal logic
{}	(8) Dem(<D>)	from (7) using (R2)
{}	(9) G	(6), (8) → elimination

So we get ⊢ G, for any sentence G, which is absurd. In particular,

⊢ Dem(<G>), for any G.

Informally the argument goes as follows. Suppose that the sentence D is demonstrable. What is demonstrable is true, so we conclude that D is true. But D is equivalent to Dem(<D>) → G, so the latter sentence is true as well. From Dem(<D>) and Dem(<D>) → G, we conclude G. Thus, we derived G from the assumption that D is demonstrable. Hence, we have shown that Dem(<D>) → G holds, under no assumption at all. From this we infer D using (D2). So by (R2), Dem(<D>). Finally we use modus ponens to conclude G. But G is an arbitrary sentence. That is, our assumptions concerning demonstrability have lead to an absurdity.

4. Related Paradoxes: Sundholm's Paradox of Knowability and Kaplan-Montague's Paradox of the Knower

It is instructive to compare the above paradox with Göran Sundholm's *paradox of knowability* (Sundholm 2008).

We consider a sentence α such that:

(1)	α = 'α is not knowable'	
(2)	α is knowable	Assumption
(3)	α is knowable → α is true	what is knowable is true
(4)	α is true	(2), (3) modus ponens
(5)	'α is not knowable' is true	(1), (4) identity substitution
(6)	α is not knowable	from (5) using the T-schema
(7)	$\perp$	from (2), (6)

(8)	α is not knowable	Demonstrated on no assumption
(9)	'α is not knowable' is knowable	What is demonstrated is knowable
(10)	α is knowable	(1), (9) identity substitution
(11)	$\perp$	from (6) and (10)

The two paradoxes are closely related. The paradox in Section 3 was obtained by analyzing Sundholm's paradox and making the underlying assumptions explicit. However, there are some differences between the two paradoxes. Consider an ordinary Liar sentence:

α = 'α is not true'.

One obtains Sundholm's paradoxical sentence by replacing 'true' by 'knowable'. We get:

β = 'β is not knowable'.

As was shown by Geach (1955) and Löb (1955), independently of each other, Liar sentences can also be constructed from implication alone, without the use of negation.[4] Hence, we get Liar sentences of the following kind:

γ = 'If γ is true, then G'.

Replacing 'true' here by 'demonstrable', we get:

δ = 'If δ is demonstrable, then G'

Using this sentence, we get a paradox using essentially the same reasoning as in Section 3.

In this connection one should also mention the *Knower paradox* of Kaplan and Montague, which starts out from a sentence D such that:

(1) $\vdash \mathrm{D} \leftrightarrow \mathrm{K}(<\neg\mathrm{D}>)$,

where K(x) means that the sentence x is known to be true. Intuitively, the sentence D says that its own negation is known to be true.[5]

Suppose:

(2)	D	Assumption
(3)	$\mathrm{K}(<\neg\mathrm{D}>)$	(1), (2)
(4)	$\mathrm{K}(<\neg\mathrm{D}>) \rightarrow \neg\mathrm{D}$	What is known must be true

[4] The general idea of negation-free paradoxes goes back to the Curry paradox (1942).
[5] See Kaplan and Montague (1960) and Montague (1963) for the original paradox and Anderson (1983) for an excellent discussion thereof.

(5)	$\neg D$	(2), (4) $\rightarrow$ elimination
(6)	$\perp$	from (2), (5)
(7)	$\neg D$	(2)-(6), $\neg D$ is demonstrated on no assumption
(8)	$K(<\neg D>)$	what is demonstrated is known
(9)	D	from (1), (8)
(10)	$\perp$	from (7), (9)

5. A Yabloesque Curry-Type Paradox of Demonstrability

Stephen Yablo has constructed an elegant version of the Liar paradox that does not, in any obvious way, involve self-reference or circular reference.[6] Here I am going to modify Yablo's construction in such a way that it connects to the Curry type-paradox of demonstrability presented above. The aim is to construct a Paradox of Demonstrability that neither involves negation nor circular reference, i.e., a Yabloesque Curry-type paradox of Demonstrability.

Consider an infinite sequence of sentences S_0, S_1,...., S_n, ... where:

S_0 = <for all i > 0: if Dem(S_i), then G>.
S_1 = <for all i > 1: if Dem(S_i), then G>.
....
....
S_n = <for all i > n: if Dem(S_i), then G>
etc.

G is here an arbitrary sentence, for example "Santa exists".

Axioms:

(a) 'Dem(<A>) $\rightarrow$ A
(b) if $\vdash$ S, then $\vdash$ Dem(S).
By his principle we are justified in inferring Dem(S), when we have constructed a demonstration of S
(c) for all n,
Dem(S_n) $\leftrightarrow$ Dem(<for all i > n, if Dem(S_i), then G>).
(d) If $\vdash$ A $\rightarrow$ B, then $\vdash$ Dem(<A>) $\rightarrow$ Dem(<B>).

This is the way the argument goes. We prove three lemmas:

Lemma 1. For all n, if Dem(S_n), then Dem(S_{n+1}).

[6] Cf. Yablo (1993).

Lemma 2. For all n, if Dem(S_n), then G.
Lemma 3. For all n, Dem(S_n).

Once we have proved these three lemmas, we argue as follows:

(1)	Dem(S_0)	by lemma 3
(2)	Dem(S_0) $\rightarrow$ G	by lemma 2
(3)	G	from (1) and (2) by modus ponens.

Let us now proceed to prove the three lemmas.

Proof of Lemma 1.
We want to prove that Dem(S_n) entails Dem(S_{n+1}).

(1)	Dem(S_n)	Assumption
(2)	Dem(<for all i > n, if Dem(S_i), then G>)	(1), Axiom (c)
(3)	(for all i > n, if Dem(S_i), then G) $\rightarrow$ (for all i > n+1, if Dem(S_i), then G)	mathematical induction on n.
(4)	Dem(<for all i > n, if Dem(S_i), then G>) $\rightarrow$ Dem(<for all i > n+1, if Dem(S_i), then G>)	(3) Axiom (d)
(5)	Dem(<for all i > n+1, if Dem(S_i), then G>)	(2), (4) $\rightarrow$ Elim.
(6)	Dem(S_{n+1})	(5), Axiom (c)
(7)	Dem(S_n) $\rightarrow$ Dem(S_{n+1}).	

Proof of Lemma 2.

(1)	Dem(S_n)	Asumption
(2)	Dem(<for all i > n, if Dem(S_i), then G>)	from (1) by Axiom (c)
(3)	Dem(<if Dem(S_{n+1}), then G>)	from (2) by logic and Axiom (d)
(4)	if Dem(S_{n+1}), then G	(3), Axiom (a)
(5)	Dem(S_{n+1})	(1), Lemma 1
(6)	G	(4), (5) $\rightarrow$ Elim.
(7)	Dem(S_n) $\rightarrow$ G	

Proof of Lemma 3.
We want to show that for every n, Dem(S_n).

(1)	for all n, Dem(S_n) $\rightarrow$ G	by Lemma 2

From this we deduce that for every n,

(2) for all k > n, Dem(S_k) $\rightarrow$ G

Hence, for every n

(3) $\mathrm{Dem}(<\text{for all } k > n, \mathrm{Dem}(S_k) \rightarrow G>)$ By Axiom (b)
(4) $\mathrm{Dem}(S_n)$. (3) Axiom (c).

Hence, it seems that the notion of demonstrability cannot satisfy all of the four axioms (a)-(d).

6. A Final Suggestion

Suppose we wish to avoid the paradoxes of demonstrability. One idea is to start out from the idea that a sentence is demonstrable if and only if it has a demonstration. Consider now the equevalence:

(D) $\mathrm{Dem}(S) \leftrightarrow \exists d(d \text{ is a demonstration of } S)$.

However, a demonstration may itself involve the concept of demonstrability. In the course of the demonstration there may occur propositions of the type Dem(S). This introduces a kind of impredicativity (or vicious circularity) in the definition (D). To break the circularity, we need to distinguish between demonstrations of different rank. First there are those demonstrations that are of rank 0. They are the demonstrations that do not involve the concept of demonstrability at all. If $d(A, \alpha)$ is a demonstration of rank α of the proposition A, then we can obtain from it a demonstration $d(\mathrm{Dem}(<A>), \alpha+1)$ of the sentence $\mathrm{Dem}(<A>)$ of rank $\alpha+1$. Hence, we assume that there is a well-founded ordering of all demonstrations, where each demonstration has an ordinal number as its rank. Then, we can also associate a rank with every demonstrable sentence A. The rank of A is the smallest ordinal α such that A has a demonstration of rank α. We write $\vdash_\alpha A$, if and only if A has a demonstration of rank α. Then, we have:

(R_α) If $\vdash_\alpha A$, then $\vdash_{\alpha+1} \mathrm{Dem}(<A>)$.
(i) If A is at all demonstrable, then rank(A) is the smallest α such that $\vdash_\alpha A$,
(ii) If $\alpha < \beta$, and $\vdash_\alpha A$, then $\vdash_\beta A$.
(iii) $\mathrm{rank}(\mathrm{Dem}(<A>)) = \mathrm{rank}(A) + 1$.

Consider now the paradoxical inference in Section 3 above.
Suppose that:

$$\vdash_\alpha D \leftrightarrow (\mathrm{Dem}(<D>) \rightarrow G)$$

Then, we can show:

(1) $\vdash_{\alpha} \mathrm{Dem}(<D>) \rightarrow G$

and hence:

(2) $\vdash_{\alpha} D$

However, we cannot prove:

(3) $\vdash_{\alpha} \mathrm{Dem}(<D>)$

from (2). Instead, we only get

(4) $\vdash_{\alpha+1} \mathrm{Dem}(<D>)$,

and this is not sufficient to get the paradoxical conclusion G.

By distinguishing between demonstrability of different order, the paradoxical conclusion is avoided. In a similar way the paradox in Section 5 is avoided.

By giving up the idea of universal concepts of demonstration and demonstrability in favor of that of a well-founded hierarchy of demonstrations and demonstrability concepts, the paradoxes are avoided. Whether this idea can be developed into a satisfactory philosophical solution of the paradoxes of demonstrability I do not know. But the idea seems worth pursuing.

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