CONTEXT-DEPENDENT UTILITIES
A SOLUTION TO THE PROBLEM OF CONSTANT ACTS IN SAVAGE

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ABSTRACT. Savage's framework of subjective preference among acts provides a paradigmatic derivation of rational subjective probabilities within a more general theory of rational decisions. The system is based on a set of possible states of the world, and on acts, which are functions that assign to each state a consequence. The representation theorem states that the given preference between acts is determined by their expected utilities, based on uniquely determined probabilities (assigned to sets of states), and numeric utilities assigned to consequences. Savage's derivation, however, is based on a highly problematic well-known assumption not included among his postulates: for any consequence of an act in some state, there is a "constant act" which has that consequence in all states. This ability to transfer consequences from state to state is, in many cases, miraculous – including simple scenarios suggested by Savage as natural cases for applying his theory. We propose a simplification of the system, which yields the representation theorem without the constant act assumption. We need only postulates P1-P6. This is done at the cost of reducing the set of acts included in the setup. The reduction excludes certain theoretical infinitary scenarios, but includes the scenarios that should be handled by a system that models human decisions.

1. INTRODUCTION

In his classic *The Foundations of Statistics*[1] Savage sets up a foundational system within which he derives both subjective probabilities and utilities from the preferences of a rational agent, provided that the preferences satisfy certain plausible postulates. The upshot is that the expected utilities come out as a measure that defines the agent's given preferences. The derivation relies however on additional implicit assumptions, one of which, the CAA discussed below, is quite problematic. Let us first recall the basic structure of the Savage system. It is based on the following four components:

1. A set S of *states* (or states of the world),
2. A set C of *consequences*, which are the consequences of the agent's acts,
3. A set $\mathcal{A}$ of *acts*, where each act is a function, f, which associates with every state, s, the consequence $f(s)$ of performing f in a world that is in state s,
4. The (rational) agent's *preference relation,* $\succcurlyeq$, defined over acts, which is a total preorder. Here, as is customary in current mathematics, "preorder" means a reflexive and transitive relation. A preorder is *total* or *complete* if for any f, g either $f \succcurlyeq g$ or $g \succcurlyeq f$.

Key words and phrases. subjective expected utility, Savage's postulates, constant acts, context-dependent decision making.

[1]The first edition Savage (1954) of Savage's book was published in 1954, all citations made in this paper refer to the second and revised edition Savage (1972) published in 1972.

The intended meaning of $f \succcurlyeq g$ is: f is weakly preferable to g, i.e., is at least as good as g; it is also written $g \preccurlyeq f$. If both $f \succcurlyeq g$ and $g \succcurlyeq f$, then we denote it by $f \equiv g$. Obviously this is an equivalence relation, it means that f and g are equi-preferable: the agent considers them to be equally good. We define: $f \succ g =_{\mathrm{Df}} f \succcurlyeq g$ and $g \not\succcurlyeq f$. This means that f is strictly preferred to g. Note that our notation and terminology differ from Savage's and this can be more than a technicality. For instance, after defining "constant acts" he does not use this term and one has to infer that certain acts are constant only from the notation; that notation, however, is sometimes ambiguous.[2]

Other elements are introduced in Savage's presentation at later stages, as the system is being developed in the book. Thus, there are *events*, which are sets of states that form, under the usual set-theoretic operations, a Boolean algebra, $\mathcal{B}$, in which S is the universal set. And there is the notion of *conditional preference*, that is: $f \succcurlyeq g$ *given* E where E is an event, which is defined using P2 (the sure-thing postulate) and which is supposed to express what the agent prefers under the assumption that $s \in E$. Furthermore, for any $f, g \in \mathcal{A}$, the *combination* of f and g with respect to an event E, in symbols $f|E + g|\overline{E}$, is defined as: $f(s)$ if $s \in E$, $g(s)$ if $s \in \overline{E}$, where $\overline{E} = S - E$ is the compliment of E with respect to S.[3] We sometimes refer to this operation as "cut-and-paste". This notation can be easily generalized to define combinations of n many acts: $f_1|P_1 + \cdots + f_n|P_n$ is the act h such that $h(s) = f_i(s)$ for $s \in P_i$ ($i = 1, \ldots, n$), and this is used under the assumption that $P_1, \ldots, P_n$ is a partition of the set of all states.

1.1. **The Problem of the Constant-act Assumption.** One crucial element of the system is the notion of *constant acts* or, in Savage's phrasing, "acts that are constant" (p. 25). The idea is that a constant act has the same consequence in all states. To be precise, being a constant act is not a property of a single act, but is subject to an axiom that applies to a bunch of acts: the preference between two constant acts, given some event, does not depend on the event. The fifth postulate (P5) posits the existence of two non-equivalent constant acts.

Savage's representation theorem claims that a preference relation that satisfies the postulates determines a unique (finitely additive) probability on $\mathcal{B}$ and a utility function (unique up to a linear transformation) which assigns numeric utilities to consequences, such that $f \succcurlyeq g$ iff the expected utility of f is greater or equal to that of g. The derivation of a probability and a utility is carried out in two stages. In the first stage a finitely additive probability is derived from a preference relation, which satisfies the postulates P1–P6. As far as constant acts are concerned, this derivation does not require more than P5 (the existence of two non-equivalent constant acts is sufficient). But in the second stage—the derivation of a utility in chapter 5—Savage tacitly assumes the following:

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[2] Savage's "simple ordering" is, in our terminology, a total preorder. He uses 'F' for the set of consequences and he characterizes total preorders as "simple orderings". In particular, he uses boldface letters $\mathbf{f}, \mathbf{g}, \ldots$ for acts and italics $f, g, \ldots$ for values of "acts that are constant", writing $\mathbf{f} \equiv g$ when $\mathbf{f}(s) = g$ for all states s. He also uses 'f' for constant act whose value is f. Furthermore, he sometimes switches to italicized notation even when the function is not constant, as he does in the statement of P4 on p. 31, where he writes $f_A(s)$ instead of $\mathbf{f}_A(s)$, or in Theorem 1 on page 70, where he writes $f(s) = f_i$ instead of $\mathbf{f}(s) = f_i$ as he should.

[3] Some writers use '$f \oplus_E g$' or 'fEg' or '$[f$ on E, g on $\overline{E}]$' for combined acts.

TABLE 1. Savage omelet example.

Act	State	
	Good	Rotten
break into bowl	six-egg omelet	no omelet and all five eggs destroyed
break into saucer	six-egg omelet and a saucer to wash	five-egg omelet and a saucer to wash
throw away	five-egg omelet and one good egg wasted	five-egg omelet

CAA (constant-acts assumption): For every consequence $a \in C$ there exists a constant act $\mathfrak{c}_a$, such that $\mathfrak{c}_a(s) = a$, for all $s \in S$.

Note that after introducing "acts that are constant" Savage hardly uses the term anymore and one has to infer that such and such acts are constant only from the notation, which is not always consistent (see Footnote 2). Fishburn (Fishburn (1970)) who observed that CAA is required for the proof of the representation theorem, has also pointed out the problematic nature of CAA (cf. Footnote 4 below). Among others who have also emphasized the need for CAA in Savage's system are Pratt (1974); Seidenfeld and Schervish (1983); Shafer (1986). This assumption, we shall argue, does not sit well with certain simple scenarios of decision making, which Savage considers as the kind of situations that his system is supposed to handle.

The difficulty is the fact that the very possibility of some consequence may depend on the world being in a certain state: *the consequence could not exist in a different state of the world.* At the beginning of his book ((Savage, 1972, p. 14)) Savage proposes the following omelet-making problem to illustrate the way his system works. The agent, call him John (in the book it is 'you'), has to finish making an omelet, which was begun by his wife. She broke into a bowl five good eggs and John finds a sixth egg, which can be added to the bowl or thrown away (we assume that there is no option of keeping it for future use). John does not know if the egg is good or rotten and has to decide between three acts: (1) Break it into the bowl (2) break it into a saucer to see if it is good or rotten (3) throw it away. There are two possible states of the world *good* and *rotten*, which are determined by the state of the sixth egg. The consequences of each act are given in Table 1, as it appears in the book.

John's ranking of the acts (that is, his preference relation, $\succcurlyeq$) reflects both his probabilistic estimates regarding the likeliness of each state, as well as the utility values of the consequences; for example, if he is sufficiently confident that the egg is good and if washing the saucer is, for him, of considerable nuisance, he will prefer "break into bowl" to "break into saucer". His preferences for these three acts cannot, of course, determine the probabilities and utilities, but if the set of acts over which the preference relation is defined is sufficiently rich (where "sufficiently rich" is determined by the postulates), then we get probabilities and utilities. Obviously the consequence "six-egg omelet" means an omelet made of the six eggs of the story, in the case where the sixth egg is good. Yet CAA requires that there should be a constant act that yields that consequence also in the state in which the sixth

egg is rotten. It would involve a miraculous production of a good six-egg omelet out of five good eggs and a rotten one.[4]

The problem arises also in the second scenario, which Savage proposes for the very purpose of clarifying what is implied by a constant act (*ibid.* p. 25). A person, call her Jane, plans to go with friends on a picnic, and she has to choose between buying a tennis racquet and buying a bathing suit (assume that buying both is ruled out for financial reasons). The bathing suit would be handier if the picnic is held near water where one can swim; the racquet would be better, if the picnic is not held near water but near a tennis court. One might consider the possession of a bathing suit and the possession of a tennis racquet as constant, state-independent consequences. But Savage makes it clear that this would not do, since the preference order of possessing a racquet and possessing a bathing suit depends on the state of the world, where the state of the world includes the picnic-location. Savage argues that the payoffs should be entities such as: "a refreshing swim with friends, or sitting on a shadeless beach twiddling a brand-new tennis racquet while one's friends swim". That, however, does not make the constant-acts problem easier. To get a constant act, we have to appeal to the theoretical possibility that while Jane sits on a shadeless beach twiddling a brand new tennis racket, she has somehow the enjoyment of a refreshing swim with her friends.

Perhaps the constant-acts problem is not so difficult if we consider getting sums of money, or some other quantitative goods, as being of equivalent value to the consequences in question. In the omelet scenario, John may consider getting k as being equivalent to a six-egg omelet and this can serve also as a payoff in the state "rotten". But it is not clear what the equivalence of k with a six-egg omelet means in the given context where John has to finish making the omelet. We may consider replacing Table 1 by the following table, in which the entries are dollar amounts; this would turn the problem into a problem of choosing between gambles. (Obviously, k is assumed to be the largest payoff, l is the smallest, $m > n$ and $q > n$.)

Act	State	
	Good	Rotten
Gamble 1	k	l
Gamble 2	m	n
Gamble 3	p	q

And we may consider offering John the choice of not completing the task – throwing out all eggs – and getting in return to choose a gamble from the table above. But this artificial dubious device undermines the big attraction of Savage's system: its ability to evaluate consequences that do not consist in winning or loosing sums of money or goods. If all consequences are to be replaced by dollar sums before the system is applied, the main point of the system is lost.

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[4] In passing, Fishburn ((Fishburn, 1970, p. 166-7)) also voiced this unsatisfactory feature of CAA. He pointed out that, for any states $s, s' \in S$, if $W(s)$ and $W(s')$ are respectively the sets of consequences that may occur under s and s', then it might well be that $W(s) \neq W(s')$ (or even that $W(s) \cap W(s') = \varnothing$), in which case the CAA fails. He remarked that he is not aware of any axiomatic system that does not make the assumption that $W(s) = W(s') = C$ for all $s, s' \in S$, and he left this line of research as an open question (see also (Fishburn, 1981, p. 162)).

One objective of this paper is to show that CAA is not required for applying Savage's system to any finitistic problem, that is to say, *a problem that is stated in terms of finitely many evants, finitely many acts and finitely many possible consequences.* All that we need is the existence of two distinguished constant acts.

1.2. **The Significance of the Set of Acts and the Boolean Algebra.** The weaker the postulates and the presuppositions which are needed to get the representation theorem, the stronger the theorem is. The basic presupposition of Savages system is that the preference relation is defined over some very rich set of acts. In some places Savage even considers every function from states to consequences to be an act, in situations in which the set of states, as well as the set of consequences, has the cardinality of the continuum. This is exorbitant. Of course the set of acts should be sufficient for handling the kind of problems that the system is designed for. As a rule, these problems are stated in terms of finitely many simple acts, where a simple act is an act, f, which has finitely many values, such that, $f^{-1}(x)$ is an event (a member of the Boolean algebra $\mathcal{B}$) for each consequence x that is a value of f. Such acts are called by Savage gambles. It is easily seen that a simple act, f, can be written in the form $f = f|P_1 + \ldots + f|P_n$, where $P_1, \ldots, P_n$ is a partition of S, $P_i = f^{-1}(x_i)$ and the x_i are consequences.

In the initial scenario the agent is supposed to decide between given options that belong to some finite set of simple acts. P6 implies however that the preference is to be defined over richer sets that involve more refined events (cf. Theorem 2.3 below). But, as we shall show, we never need more than simple acts. (In Section 3, we comment on how our model can be generalized to treat certain infinitary cases.)

Now the richness of the set of acts is also determined by the richness of the Boolean algebra $\mathcal{B}$ of events, namely the collection of subsets that constitute events. As noted, Savage considers possibilities in which this Boolean algebra consists of all subsets of real numbers. But his proof of the representation theorem requires only that it be a σ-algebra, that is, closed under unions of countable many sets. Our results can be now stated as follows:

i. While we assume that the Boolean algebra is a σ-algebra, we can derive the representation theorem if we consider only a preference defined over simple acts, which include two non-equivalent constant ones.

ii. Moreover, we can also give up the assumption that the algebra is a σ-algebra and get the representation theorem, nonetheless. In fact, we need only a countable Boolean algebra so that the simple acts defined over it satisfy P6.

(i) is proved by using Savage's derivation of probabilities from two constant acts. We deviate from him in the derivation of expected utilities for simple acts (where the set of consequences is arbitrary). In the next section, we lay out the basic ideas behind our construction, the full technical details will be left to the full paper. (ii) is a more difficult result that is based on a more difficult derivation of probabilities. We do not have the space for getting into it here.

2. CONTEXT-DEPENDENT DECISION-MAKING

2.1. **Subjective Probability.** To derive subjective probability from preferences, Savage uses P1-P6. The construction starts with a derivation of qualitative probabilities.

Definition 2.1. For any events E, F, say that E is *weakly more probable* than F, written $E \succeq F$, if, for any constant acts $\mathfrak{c}_a$ and $\mathfrak{c}_b$ such that $\mathfrak{c}_a \succcurlyeq \mathfrak{c}_b$,

$$\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E} \succcurlyeq \mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}. \tag{2.1}$$

Savage's P4 guarantees that (2.1) does not depend on the choice of the pair of constant acts. It is also not difficult to show that $\succcurlyeq$ is a qualitative probability. The task is to show that this qualitative probability admits a numerical *representation*: there exists a real-valued probability measure μ defined on an algebra of events satisfying:

$$E \succeq F \iff \mu(E) \geq \mu(F). \tag{2.2}$$

Savage's proof of the existence of a quantitative probability that satisfies (2.2) requires the assumption that the algebra of events is closed under countable unions, i.e., it is a σ-algebra. (That one can do without this assumption is, as noted above, the content of our second result.) So far only two non-equivalent constant acts are required.[5]

Theorem 2.2 (Savage)**.** Let $\succcurlyeq$ be a preference relation among acts. Suppose that $\succcurlyeq$ satisfies P1-6 and that the Boolean algebra $\mathcal{B}$ of events is a σ-algebra, then there exists a unique (finitely additive) probability measure μ for which (2.2) holds.

The proof of the theorem establishes also the following theorem, which holds under the assumption that the algebra of events is a σ-algebra.

Theorem 2.3. Given the probability measure μ obtained above, for any event E and any $0 \leq \rho \leq 1$, there exists some $F \subseteq E$ such that $\mu(F) = \rho\mu(E)$.

Note that, unlike Theorem 2.2, Theorem 2.3 fails if the assumption that the Boolean algebra is a σ-algebra is omitted. A weaker version of it holds: The set of all ρ for which the equality holds is dense in $(0, 1)$.

2.2. **Utility for All Acts.** The following are some simple properties of the two distinguished constant acts, which are immediate from the definitions above and Theorem 2.2.

**Lemma 2.4.** For any events E, F,

(1) $\mu(E) > \mu(F)$ iff $\mathfrak{c}_1|E + \mathfrak{c}_0|\overline{E} \succ \mathfrak{c}_1|F + \mathfrak{c}_0|\overline{F}$,
(2) $\mu(E) = \mu(F)$ iff $\mathfrak{c}_1|E + \mathfrak{c}_0|\overline{E} \equiv \mathfrak{c}_1|F + \mathfrak{c}_0|\overline{F}$.

We show that, under P1-6 and the assumption that there exist two constant acts $\mathfrak{c}_0$ and $\mathfrak{c}_1$, the agent's preferences can be represented by a utility function in Savage's system without appealing to CAA. To this end, we first observe that to each act $f \in \mathcal{A}$ satisfying $\mathfrak{c}_1 \succcurlyeq f \succcurlyeq \mathfrak{c}_0$ there corresponds a combined act using the two distinguished constant acts which is indifferent to f under $\succcurlyeq$.

Lemma 2.5. For and $f \in \mathcal{A}$, if $\mathfrak{c}_1 \succcurlyeq f \succcurlyeq \mathfrak{c}_0$, there exists an event E_f such that

$$\mathfrak{c}_1|E_f + \mathfrak{c}_0|\overline{E_f} \equiv f. \tag{2.3}$$

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[5]This observation is also noted in (Fishburn, 1981, p. 161) where the author remarked that "[as far as obtaining a unique probability measure is concerned] Savage's $\mathscr{C}$ [i.e., the set of consequences] can contain as few as two consequences." See (Fishburn, 1970, §14.1-3) for a clean exposition of Savage's proof of (2.2), and see especially §14.3 for an illustration of the role of P1-6 played in deriving numerical probability.
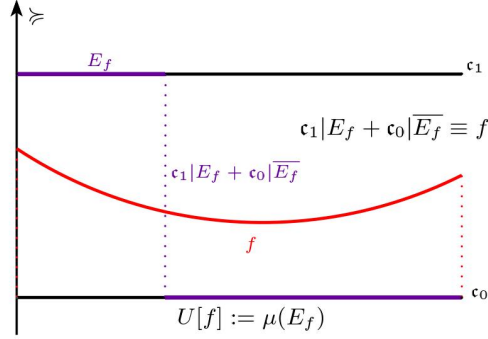
FIGURE 1. The case where $\mathfrak{c}_1 \succcurlyeq f \succcurlyeq \mathfrak{c}_0$

In proving this lemma, we make full use of the derived personal probability μ from Theorem 2.2, the proof given here is somewhat standard in utility theory. Figure 1 provides an illustration of the general method involved in the proof, where $\mathfrak{c}_1|E_f + \mathfrak{c}_0|\overline{E_f}$ is the act that yields $\mathfrak{c}_1$ if E_f occurs, status quo otherwise. The aim is to find the appropriate E_f so that the given event f is indifferent to this combined act.

Proof of Lemma 2.5. Let us consider the following two sets of events.

$$B := \left\{ E \ \middle| \ \mathfrak{c}_1|E + \mathfrak{c}_0|\overline{E} \succcurlyeq f \right\};$$
$$C := \left\{ E \ \middle| \ \mathfrak{c}_1|E + \mathfrak{c}_0|\overline{E} \preccurlyeq f \right\}. \tag{2.4}$$

It is easily seen that B and C are nonempty, for at least we have $S \in B$ and $\emptyset \in C$. Let μ be the probability measure derived from Theorem 2.2, Next, consider the following sets defined in terms of B, C and μ:

$$B_\mu := \left\{ \mu(E) \ \middle| \ E \in B \right\};$$
$$C_\mu := \left\{ \mu(E) \ \middle| \ E \in C \right\}. \tag{2.5}$$

Let $\alpha_* = \inf B_\mu$ and $\alpha^* = \sup C_\mu$. Note that, for any $a > \alpha_*$, there must exist some $a' \in B_\mu$ such that $a > a' \geq \alpha_*$ (for, otherwise, a is a lower bound of B_μ strictly greater than α_*, which contradicts the assumption $\alpha_* = \inf B_\mu$). Since $a' \in B_\mu$ then, by the definition of B_μ in (2.5), there is some event, say, $F' \in B$ such that $\mu(F') = a'$. Further, let F be an event such that $\mu(F) = a$ (the existence of F is guaranteed by Theorem 2.3). Then, by Lemma 2.4, $\mu(F) = a > \mu(F') = a' \geq \alpha_*$ implies $\mathfrak{c}_1|F + \mathfrak{c}_0|\overline{F} \succ \mathfrak{c}_1|F' + \mathfrak{c}_0|\overline{F'} \succcurlyeq f$. It follows, via P1, that, for any F,

$$\mu(F) > \alpha_* \implies F \notin C. \tag{2.6}$$

The contrapositive of (2.6) says that, for any F, $F \in C$ implies that $\mu(F) \leq \alpha_*$. In other words, α_* is an upper bound of C_μ, and hence $\alpha^* = \sup C_\mu \leq \alpha_*$. Using a symmetric argument one can show that $\alpha^* \geq \alpha_*$. Hence $\alpha^* = \alpha_*$.

Next, let E_f be such that $\mu(E_f) = \alpha^* = \alpha_*$ (again, the existence of E_f is guaranteed by Theorem 2.3). The proof is completed if we can show that $E_f \in B \cap C$.

Suppose, to the contrary, $E_f \notin B$, then, by P1, $f \succ \mathfrak{c}_1 | E_f + \mathfrak{c}_0 | \overline{E_f}$. The latter implies, via P6, there exists a partition $\{P_i\}_{i=1}^n$ such that,

$$\mathfrak{c}_1 \Big| P_i + \left(\mathfrak{c}_1 | E_f + \mathfrak{c}_0 | \overline{E_f} \right) \Big| \overline{P_i} \quad \text{for all } i = 1, \ldots, n, \tag{2.7}$$

that is,

$$f \succ \mathfrak{c}_1 \Big| E_f \cup P_i + \mathfrak{c}_0 \Big| \overline{E_f \cup P_i} \qquad \text{for all } i = 1, \ldots, n. \tag{2.8}$$

Then, it follows that $E_f \cup P_i \in C$ for *all* $i = 1, \ldots, n$. On the other hand, note that P_i's form a partition of S, we consider two cases:

(1) If for *some* P_j in the partition we have $\mu(E_f \cup P_j) > \mu(E_f) = \alpha_*$, then, by (2.6), $E_f \cup P_j \notin C$, a contradiction.
(2) If $\mu(E_f \cup P_j) \leq \mu(E_f) = \alpha_*$ for *all* $j = 1, \ldots, n$, then it is easily seen that $\mu(E_f) = 1$. By Lemma 2.4(2), it follows that $\mathfrak{c}_1 | E_f + \mathfrak{c}_0 | \overline{E_f} \equiv \mathfrak{c}_1 | S + \mathfrak{c}_0 | \overline{S} = \mathfrak{c}_1$, and hence $E_f \in B$, but this contradicts the hypothesis $E_f \notin B$.

Hence, E_f must be in B. Similarly, it can be shown that $E_f \in C$. Then we have $E_f \in B \cap C$. This completes the proof of the lemma. $\qquad\square$

Remark 2.6. 1. In light of the lemma, for any $f \in \mathcal{A}$ satisfying $\mathfrak{c}_1 \succcurlyeq f \succcurlyeq \mathfrak{c}_0$, let E_f be such that (2.3) holds, we define the *utility* of f to be

$$U[f] := \mu(E_f), \tag{2.9}$$

where μ is obtained through Theorem 2.2 and E_f is from (2.3).
2. Notice that, if there exists another event E'_f for which (2.3) holds, then we have $\mathfrak{c}_1 | E_f + \mathfrak{c}_0 | \overline{E_f} \equiv \mathfrak{c}_1 | E'_f + \mathfrak{c}_0 | \overline{E'_f}$. It follows, via Lemma 2.4(2), that $\mu(E'_f) = \mu(E_f)$, hence $U[f]$ is well defined.
3. For the two distinguished constant acts $\mathfrak{c}_1$ and $\mathfrak{c}_0$, trivially we have $E_{\mathfrak{c}_1} = S$ and $E_{\mathfrak{c}_0} = \varnothing$, then (2.9) yields that $U[\mathfrak{c}_1] = 1$ and $U[\mathfrak{c}_0] = 0$.
4. It is plain that U does not need to be uniquely defined by (2.9): if h is any monotonically increasing function on the reals (or any order preserving function), then U can also be defined by $h \circ \mu$.
5. If $f \succ \mathfrak{c}_1$ (or $\mathfrak{c}_0 \succ f$), it is easy to see that Lemma 2.5 can be adjusted to show that there exists some E_f such that $f | E_f + \mathfrak{c}_0 | \overline{E_f} \equiv \mathfrak{c}_1$ (or $\mathfrak{c}_1 | E_f + f | \overline{E_f} \equiv \mathfrak{c}_0$), in which case U can be defined standardly as in (2.11) below.

Theorem 2.7. *Let $\succcurlyeq$ be a preference relation over acts, if $\succcurlyeq$ satisfies P1-6, then there exists a real-valued function U on $\mathcal{A}$ satisfying, for all $f, g \in \mathcal{A}$,*

$$f \succcurlyeq g \iff U[f] \geq U[g], \tag{2.10}$$

where

$$U[f] := \begin{cases} \frac{1}{\mu(E_f)} & \text{if } f \succ \mathfrak{c}_1, \\ \mu(E_f) & \text{if } \mathfrak{c}_1 \succcurlyeq f \succcurlyeq \mathfrak{c}_0, \\ \frac{\mu(E_f)}{\mu(E_f) - 1} & \text{if } \mathfrak{c}_0 \succ f. \end{cases} \tag{2.11}$$

2.3. **Context-dependent Expected Utility for Simple Acts.** We now proceed to show that, assuming P1-6, the utility of a simple act can be further expressed as its expected utility of its consequences. Let us denote the set of all simple acts by $\mathcal{A}_0$. Recall that a simple act $f \in \mathcal{A}_0$ is one that has a finite number of consequences, say, $x_1, \ldots, x_n$, and let $P_1, \ldots, P_n$ be the corresponding sets of states under which they obtain. It is easily seen that $\{P_i\}_{i=1}^n$ forms a partition of S:

$$
\begin{aligned}
& P_i = f^{-1}(x_i) \quad (i = 1, \ldots, n), \\
& P_i \cap P_j = \varnothing \quad (i \neq j) \quad \text{and} \quad \bigcup_{i=1}^n P_i = S.
\end{aligned}
\tag{2.12}
$$

We seek to define a *context-dependent* utility function u over consequences such that the utility of a simple act $U[f]$ can be represented by its expected utility:

$$
U[f] = \sum_{i=1}^n \mu(P_i) u(P_i, x_i),
\tag{2.13}
$$

where $u(P_i, x_i)$ is the utility of consequences x_i *given* P_i. As it will be shortly shown, in all cases in which $\mu(P_i) > 0$ this value depends only on the consequence x_i. And this value is the same across different acts. We thus can speak of context-dependent utilities. We can assign utilities to consequences, but these utilities can be used for the purpose of calculating expected utilities as long as the consequence is obtained as a value of states that constitute a set of probability greater than 0.

We adopt the following notation:

$$
\mathfrak{c}_x^*(s) := \begin{cases} x & \text{if } s \in E, \\ 0 & \text{if } s \notin E, \end{cases} \quad \text{for some } E \in \mathcal{B}.
\tag{2.14}
$$

We refer to $\mathfrak{c}_x^*$ as a *locally constant act* which yields x in all states in E, 0 (status quo) otherwise. It is obvious that $\mathfrak{c}_x^*$ is a generalization of Savage's notion of constant act. Now with (2.14), a simple act f satisfying (2.12) can be expressed by the combination of a series of locally constant acts as follows

$$
f = \mathfrak{c}_{x_1}^* | P_1 + \cdots + \mathfrak{c}_{x_n}^* | P_n.
\tag{2.15}
$$

The goal is to represent simple acts in the form of (2.15) by expected utilities.[6] Observe that, if $\mu(P_i) = 0$ for some P_i, then the term $\mu(P_i) u(P_i, x_i)$ in (2.13) is 0, in which case consequence x_i can be seen as having no contribution to the total utility calculation. As a rule, one can assign in this situation an arbitrary finite value to the consequence $f(s)$ where $s \in P_i$. If, on the other hand, $\mu(P_i) \neq 0$, consider act $\mathfrak{c}_{x_i}^* | P_i + \mathfrak{c}_0 | \overline{P_i}$. Then in light of Theorem 2.7, define a *context-dependent* utility of x_i in

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[6] Savage ((Savage, 1972, p. 71)) uses $\sum_i \rho_i f_i$ to denote the class of simple acts for which, to use his notations, there exist partitions B_i of S such that $P(B_i) = \rho_i$ and $f(s) = f_i$ for $s \in B_i$. He further remarks that if a simple act $\mathbf{f}$ is such that "the consequences f_i will befall the person in case B_i occurs, then the value of $\mathbf{f}$ is independent of how the partition B_i is chosen." In other words, his utility function, once derived, is *state-independent*. We, on the other hand, take that the value of a consequence depends on the states under which it obtains. Thus, we allow that for two simple acts f, g with different partitions $\{P_i\}_{i=1}^n$ and $\{Q_i\}_{i=1}^n$ for which $\mu(P_i) = \mu(Q_i)$ and $f(s) = g(t)$ for $s \in P_i$ and $t \in Q_i$ $(i = 1, \ldots, n)$, $f \not\equiv g$. That is, we allow Theorem 1 ((Savage, 1972, p. 70)) to fail in our decision model where utilities are *context-dependent*.

P_i in terms of the utility of $\mathfrak{c}_{x_i}^* | P_i + \mathfrak{c}_0 | \overline{P_i}$ as follows

$$u(P_i, x_i) := \begin{cases} c & \text{if } \mu(P_i) = 0, \\ \dfrac{U\left[\mathfrak{c}_{x_i}^* | P_i + \mathfrak{c}_0 | \overline{P_i}\right]}{\mu(P_i)} & \text{if } \mu(P_i) \neq 0, \end{cases} \tag{2.16}$$

where c can be any number in $[0, 1]$. Finally, it remains to verify that $\succcurlyeq$ among simple acts indeed admits an expected utility representation using the probability measure μ and utility function u given above. We put this claim in the form of the following theorem. The rather straightforward proof is omitted.

Theorem 2.8. Let $\succcurlyeq$ be a preference relation over acts, if $\succcurlyeq$ satisfies P1-6, then there exist a probability measure μ on events and a utility function u on the consequences such that, for any $f, g, \in \mathcal{A}_0$,

$$f \succcurlyeq g \iff \sum_{x \in f(S)} \mu[f(s) = x] u(f^{-1}(x), x) \geq \sum_{x \in g(S)} \mu[g(s) = x] u(g^{-1}(x), x).$$

3. INFINITARY CASES

Our method can be generalized to treat certain infinitary case. There are acts, f, in which there are countably many consequences, say $x_1, x_2, \ldots, x_n, \ldots$ such that $f^{-1}(x_n)$ is a non-null set for every n. In other words, we allow the number of cells of the partition in (2.12) to be unbounded. Then (2.16) and Theorem 2.8 also apply to this case, where the expected utility of f can be defined by

$$\sum_{i=1}^{\infty} \mu[f(s) = x_i] u(f^{-1}(x_i), x_i) \tag{3.1}$$

provided that $\sum_{i=1}^{\infty} \mu[f(s) = x_i] \cdot |u(f^{-1}(x_i), x_i)|$ converges. It is defined as the sum of the positive values minus the sum of the negative ones. Note that μ does not need to be countably additive. The expectation in that case is defined for discrete random variables, for which the sum absolutely converges.

Finally, we point out that Savage needed the CAA because he wanted to extend the expectation to continuous random variables, that is, he wanted to define the integral:

$$\int X(s) \, d\mu(s) \tag{3.2}$$

where X is a measurable function, which is interpreted in his system as a general act with potentially uncountably many consequences, and μ is a finitely additive probability. Mathematically this is interesting. But we do not think that it is required for applying his system to decision scenarios which a rational human agent is expected to face.[7]

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[7]Thanks are due to three anonymous reviewers for helpful comments.

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