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Abstract	This essay examines the philosophical significance of Ω-logic in Zermelo-Fraenkel set theory with choice (ZFC). The dual isomorphism between algebra and coalgebra permits Boolean-valued algebraic models of ZFC to be interpreted as coalgebras. The modal profile of Ω-logical validity can then be countenanced within a coalgebraic logic, and Ω-logical validity can be defined via deterministic automata. I argue that the philosophical significance of the foregoing is two-fold. First, because the epistemic and modal profiles of Ω-logical validity correspond to those of second-order logical consequence, Ω-logical validity is genuinely logical, and thus vindicates a neo-logicist conception of mathematical truth in the set-theoretic multiverse. Second, the foregoing provides a modal-computational account of the interpretation of mathematical vocabulary, adducing in favor of a realist conception of the cumulative hierarchy of sets.	
Keywords (separated by "-")	Modal Ω-logic - Ω-logical Validity - Modal Coalgebraic Automata - Neo-Logicism - Set-theoretic Realism	

Chapter 4   1
Modal Ω-Logic: Automata,   2
Neo-Logicism, and Set-Theoretic Realism   3

Hasen Khudairi   4

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in favor of a realist conception of the cumulative hierarchy of sets.   16

4.1 Introduction   19

This essay examines the philosophical significance of the consequence relation 20
defined in the Ω-logic for set-theoretic languages. I argue that, as with second- 21
order logic, the modal profile of validity in Ω-Logic enables the property to be 22
epistemically tractable. Because of the dual isomorphism between algebras and 23
coalgebras, Boolean-valued models of set theory can be interpreted as coalgebras. 24

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In Sect. 4.2, I demonstrate how the modal profile of Ω-logical validity can be countenanced within a coalgebraic logic, and how Ω-logical validity can further be defined via automata. In Sect. 4.3, I examine how models of epistemic modal algebras to which modal coalgebraic automata are dually isomorphic are availed of in the computational theory of mind. Finally, in Sect. 4.4, the philosophical significance of the characterization of the modal profile of Ω-logical validity for the philosophy of mathematics is examined. I argue (i) that it vindicates a type of neo-logicism with regard to mathematical truth in the set-theoretic multiverse, and (ii) that it provides a modal and computational account of formal grasp of the concept of 'set', adducing in favor of a realist conception of the cumulative hierarchy of sets. Section 4.5 provides concluding remarks.

4.2 Definitions

In this section, I define the axioms of Zermelo-Fraenkel set theory with choice. I define the mathematical properties of the large cardinal axioms to which ZFC can be adjoined, and I provide a detailed characterization of the properties of Ω-logic for ZFC. Because Boolean-valued algebraic models of Ω-logic are dually isomorphic to coalgebras, a category of coalgebraic logic is then characterized which models both modal logic and deterministic automata. Modal coalgebraic models of automata are then argued to provide a precise characterization of the modal and computational profiles of Ω-logical validity.

AQ1

4.2.1 Axioms[1]

- Empty set:
  $\exists x \forall u (u \notin x)$
- Extensionality:
  $x = y \iff \forall u(u \in x \iff u \in y)$
- Pairing:
  $\exists x \forall u(u \in x \iff u = a \lor u = b)$
- Union:
  $\exists x \forall u[u \in x \iff \exists v(u \in v \land v \in a)]$
- Separation:
  $\exists x \forall u[u \in x \iff u \in a \land \phi(u)]$
- Power Set:
  $\exists x \forall u(u \in x \iff u \subseteq a)$

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[1]For a standard presentation, see Jech (2003). For detailed, historical discussion, see Maddy (1988a).

- Infinity: 58
    $\exists x \varnothing \in x \wedge \forall u(u \in x \rightarrow \{u\} \in x)$ 59
- Replacement: 60
    $\forall u \exists! v \psi(u, v) \rightarrow \forall x \exists y (\forall u \in x)(\exists v \in y)\psi(u, v)$ 61
- Choice: 62
    $\forall u[u \in a \rightarrow \exists v(v \in u)] \wedge \forall u, x[u \in a \wedge x \in a \rightarrow \exists v(v \in u \iff v \in$ 63
    $x) \vee \neg v(v \in u \wedge v \in x)] \rightarrow \exists x \forall u[u \in a \rightarrow \exists! v(v \in u \wedge u \in x)]$ 64

4.2.2 Large Cardinals   65

Borel sets of reals are subsets of ω^ω or $\mathbb{R}$, closed under countable intersections and   66
unions.[2] For all ordinals, a, such that $0 < a < \omega_1$, and $b < a$, Σ_a^0 denotes the open   67
subsets of ω^ω formed under countable unions of sets in Π_b^0, and Π_a^0 denotes the   68
closed subsets of ω^ω formed under countable intersections of Σ_b^0.   69

Projective sets of reals are subsets of ω^ω, formed by complementations ($\omega^\omega - u$,   70
for $u \subseteq \omega^\omega$) and projections $[p(u) = \{\langle x_1, \ldots, x_n \rangle \in \omega^\omega \mid \exists y \langle x_1, \ldots, x_n, y \rangle \in u\}]$.   71
For all ordinals a, such that $0 < a < \omega$, Π_0^1 denotes closed subsets of ω^ω; Π_a^1 is   72
formed by taking complements of the open subsets of ω^ω, Σ_a^1; and Σ_{a+1}^1 is formed   73
by taking projections of sets in Π_a^1.   74

AQ2   The full power set operation defines the cumulative hierarchy of sets, V, such that   75
$V_0 = \varnothing$; $V_{a+1} = P(V_0)$; and $V_\lambda = \bigcup_{a < \lambda} V_a$.   76

In the inner model program (cf. Woodin 2001, 2010, 2011; Kanamori 2012a,b),   77
the definable power set operation defines the constructible universe, $L(\mathbb{R})$, in the   78
universe of sets V, where the sets are transitive such that $a \in C \iff a \subseteq C$;   79
$L(\mathbb{R}) = V_{\omega+1}$; $L_{a+1}(\mathbb{R}) = \text{Def}(L_a(\mathbb{R}))$; and $L_\lambda(\mathbb{R}) = \bigcup_{a < \lambda}(L_a(\mathbb{R}))$.   80
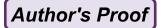
Via inner models, Gödel (1940) proves the consistency of the generalized   81
continuum hypothesis, $\aleph_a^{\aleph_a} = \aleph_{a+1}$, as well as the axiom of choice, relative to the   82
axioms of ZFC. However, for a countable transitive set of ordinals, M, in a model   83
of ZF without choice, one can define a generic set, G, such that, for all formulas, ϕ,   84
either ϕ or $\neg\phi$ is forced by a condition, f, in G. Let $M[G] = \bigcup_{a < \kappa} M_a[G]$, such that   85
$M_0[G] = \{G\}$; with $\lambda < \kappa$, $M_\lambda[G] = \bigcup_{a < \lambda} M_a[G]$; and $M_{a+1}[G] = V_a \cap M_a[G]$.[3]   86
G is a Cohen real over M, and comprises a set-forcing extension of M. The relation   87
of set-forcing, $\Vdash$, can then be defined in the ground model, M, such that the forcing   88
condition, f, is a function from a finite subset of ω into $\{0,1\}$, and $f \Vdash u \in G$ if   89
$f(u) = 1$ and $f \Vdash u \notin G$ if $f(u) = 0$. The cardinalities of an open dense ground   90
model, M, and a generic extension, G, are identical, only if the countable chain   91
condition (c.c.c.) is satisfied, such that, given a chain – i.e., a linearly ordered subset   92
of a partially ordered (reflexive, antisymmetric, transitive) set – there is a countable,   93

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[2]See Koellner (2013), for the presentation, and for further discussion, of the definitions in this and the subsequent paragraph.

[3]See Kanamori (2012a: 2.1; 2012b: 4.1), for further discussion.

maximal antichain consisting of pairwise incompatible forcing conditions. Via set-forcing extensions, Cohen (1963, 1964) constructs a model of ZF which negates the generalized continuum hypothesis, and thus proves the independence thereof relative to the axioms of ZF.[4]

Gödel (1946/1990: 1–2) proposes that the value of Orey sentences such as the GCH might yet be decidable, if one avails of stronger theories to which new axioms of infinity – i.e., large cardinal axioms – are adjoined.[5] He writes that: 'In set theory, e.g., the successive extensions can be represented by stronger and stronger axioms of infinity. It is certainly impossible to give a combinatorial and decidable characterization of what an axiom of infinity is; but there might exist, e.g., a characterization of the following sort: An axiom of infinity is a proposition which has a certain (decidable) formal structure and which in addition is true. Such a concept of demonstrability might have the required closure property, i.e. the following could be true: Any proof for a set-theoretic theorem in the next higher system above set theory . . . is replaceable by a proof from such an axiom of infinity. It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from present axioms plus some true assertion about the largeness of the universe of sets'.

For cardinals, x,a,C, $C \subseteq a$ is closed unbounded in a, if it is closed [if $x < C$ and $\bigcup(C \cap a) = a$, then $a \in C$] and unbounded ($\bigcup C = a$) (Kanamori, op. cit.: 360). A cardinal, S, is stationary in a, if, for any closed unbounded $C \subseteq a$, $C \cap S \neq \emptyset$ (op. cit.). An ideal is a subset of a set closed under countable unions, whereas filters are subsets closed under countable intersections (361). A cardinal κ is regular if the cofinality of κ – comprised of the unions of sets with cardinality less than κ – is identical to κ. Uncountable regular limit cardinals are weakly inaccessible (op. cit.). A strongly inaccessible cardinal is regular and has a strong limit, such that if $\lambda < \kappa$, then $2^\lambda < \kappa$ (op. cit.).

Large cardinal axioms are defined by elementary embeddings.[6] Elementary embeddings can be defined thus. For models A,B, and conditions ϕ, j: A $\rightarrow$ B, $\phi\langle a_1, \ldots, a_n \rangle$ in A if and only if $\phi\langle j(a_1), \ldots, j(a_n) \rangle$ in B (363). A measurable cardinal is defined as the ordinal denoted by the critical point of j, crit(j) (Koellner and Woodin 2010: 7). Measurable cardinals are inaccessible (Kanamori, op. cit.).

Let κ be a cardinal, and $\eta > \kappa$ an ordinal. κ is then η-strong, if there is a transitive class M and an elementary embedding, j: V $\rightarrow$ M, such that crit(j) $= \kappa$, $j(\kappa) > \eta$, and $V_\eta \subseteq M$ (Koellner and Woodin, op. cit.).

κ is strong if and only if, for all η, it is η-strong (op. cit.).

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[4]See Kanamori (2008), for further discussion.

[5]See Kanamori (2007), for further discussion. Kanamori (op. cit.: 154) notes that Gödel (1931/1986: fn48a) makes a similar appeal to higher-order languages, in his proofs of the incompleteness theorems. The incompleteness theorems are examined in further detail, in Sect. 4.4.2, below.

[6]The definitions in the remainder of this subsection follow the presentations in Koellner and Woodin (2010) and Woodin (2010, 2011).

If A is a class, κ is η-A-strong, if there is a j : V $\rightarrow$ M, such that κ is η-strong and j(A $\cap$ V$_\kappa$) $\cap$ V$_\eta$ = A $\cap$ V$_\eta$ (op. cit.).

κ is a Woodin cardinal, if κ is strongly inaccessible, and for all A $\subseteq$ V$_\kappa$, there is a cardinal $\kappa_A < \kappa$, such that κ_A is η-A-strong, for all η such that κ_η, $\eta < \kappa$ (Koellner and Woodin, op. cit.: 8).

κ is superstrong, if j : V $\rightarrow$ M, such that crit(j) = κ and V$_{j(\kappa)}$ $\subseteq$ M, which entails that there are arbitrarily large Woodin cardinals below κ (op. cit.).

Large cardinal axioms can then be defined as follows.

$\exists$xΦ is a large cardinal axiom, because:

(i) Φx is a Σ_2-formula;
(ii) if κ is a cardinal, such that V $\models$ $\Phi(\kappa)$, then κ is strongly inaccessible; and
(iii) for all generic partial orders $\mathbb{P}$ $\in$ V$_\kappa$, V$^{\mathbb{P}}$ $\models$ $\Phi(\kappa)$; I$_{NS}$ is a non-stationary ideal; A^G is the canonical representation of reals in L($\mathbb{R}$), i.e. the interpretation of A in M[G]; H(κ) is comprised of all of the sets whose transitive closure is $< \kappa$ (cf. Rittberg 2015); and L($\mathbb{R}$)$^{\mathbb{P}max}$ $\models$ $\langle$H(ω_2), $\in$, I$_{NS}$, A$^G\rangle$ $\models$ 'ϕ'. $\mathbb{P}$ is a homogeneous partial order in L($\mathbb{R}$), such that the generic extension of L($\mathbb{R}$)$^{\mathbb{P}}$ inherits the generic invariance, i.e., the absoluteness, of L($\mathbb{R}$). Thus, L($\mathbb{R}$)$^{\mathbb{P}max}$ is (i) effectively complete, i.e. invariant under set-forcing extensions; and (ii) maximal, i.e. satisfies all Π_2-sentences and is thus consistent by set-forcing over ground models (Woodin (ms): 28).

Assume ZFC and that there is a proper class of Woodin cardinals; A $\in$ $\mathbb{P}(\mathbb{R})$ $\cap$ L($\mathbb{R}$); ϕ is a Π_2-sentence; and V(G), s.t. $\langle$H(ω_2), $\in$, I$_{NS}$, A$^G\rangle$ $\models$ 'ϕ': Then, it can be proven that L($\mathbb{R}$)$^{\mathbb{P}max}$ $\models$ $\langle$H(ω_2), $\in$, I$_{NS}$, A$^G\rangle$ $\models$ 'ϕ', where 'ϕ' := $\exists$A $\in$ $\Gamma^\infty\langle$H(ω_1), $\in$, A$\rangle$ $\models$ ψ.

The axiom of determinacy (AD) states that every set of reals, a $\subseteq$ ω^ω is determined, where κ is determined if it is decidable.

Woodin's (1999) Axiom (*) can be thus countenanced:

AD$^{L(\mathbb{R})}$ and L[($\mathbb{P}\omega_1$)] is a $\mathbb{P}$max-generic extension of L($\mathbb{R}$),

from which it can be derived that $2^{\aleph_0} = \aleph_2$. Thus, ¬CH; and so CH is absolutely decidable.

4.2.3   Ω-Logic

For partial orders, $\mathbb{P}$, let V$^{\mathbb{P}}$ = V$^{\mathbb{B}}$, where $\mathbb{B}$ is the regular open completion of ($\mathbb{P}$).[7] M$_a$ = (V$_a$)^M and M$_a^{\mathbb{B}}$ = (V$_a^{\mathbb{B}}$)^M = (V$_a^{M^{\mathbb{B}}}$). *Sent* denotes a set of sentences in a first-order language of set theory. T $\cup$ {ϕ} is a set of sentences extending ZFC. *c.t.m* abbreviates the notion of a countable transitive $\in$-model. *c.B.a.* abbreviates the notion of a complete Boolean algebra.

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[7]The definitions in this section follow the presentation in Bagaria et al. (2006).

Define a $c.B.a.$ in V, such that $V^{\mathbb{B}}$. Let $V_0^{\mathbb{B}} = \emptyset$; $V_\lambda^{\mathbb{B}} = \bigcup_{b<\lambda} V_b^{\mathbb{B}}$, with λ a limit ordinal; $V_{a+1}^{\mathbb{B}} = \{f : X \to \mathbb{B} \,|\, X \subseteq V_a^{\mathbb{B}}\}$; and $V^{\mathbb{B}} = \bigcup_{a \in On} V_a^{\mathbb{B}}$. 167 168

ϕ is true in $V^{\mathbb{B}}$, if its Boolean-value is $1^{\mathbb{B}}$, if and only if 169

$V^{\mathbb{B}} \models \phi$ iff $[\![\phi]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$. 170

Thus, for all ordinals, a, and every $c.B.a.$ $\mathbb{B}$, $V_a^{\mathbb{B}} \equiv (V_a)^{V^{\mathbb{B}}}$ iff for all $x \in V^{\mathbb{B}}$, 171 $\exists y \in V^{\mathbb{B}} [\![x = y]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$ iff $[\![x \in V^{\mathbb{B}}]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$. 172

Then, $V_a^{\mathbb{B}} \models \phi$ iff $V^{\mathbb{B}} \models {}$'$V_a \models \phi$'. 173

Ω-logical validity can then be defined as follows: 174

For $T \cup \{\phi\} \subseteq Sent$, 175

$T \models_\Omega \phi$, if for all ordinals, a, and $c.B.a.$ $\mathbb{B}$, if $V_a^{\mathbb{B}} \models T$, then $V_a^{\mathbb{B}} \models \phi$. 176

Supposing that there exists a proper class of Woodin cardinals and if $T \cup \{\phi\} \subseteq$ 177 $Sent$, then for all set-forcing conditions, $\mathbb{P}$: 178

$T \models_\Omega \phi$ iff $V^T \models {}$'$T \models_\Omega \phi$', 179

where $T \models_\Omega \phi \equiv \emptyset \models {}$'$T \models_\Omega \phi$'. 180

The Ω-Conjecture states that $V \models_\Omega \phi$ iff $V^{\mathbb{B}} \models_\Omega \phi$ (Woodin ms). Thus, Ω- 181 logical validity is invariant in all set-forcing extensions of ground models in the 182 set-theoretic multiverse. 183

The soundness of Ω-Logic is defined by universally Baire sets of reals. For a 184 cardinal, e, let a set A be e-universally Baire, if for all partial orders $\mathbb{P}$ of cardinality 185 e, there exist trees, S and T on ω X λ, such that $A = p[T]$ and if $G \subseteq \mathbb{P}$ is 186 generic, then $p[T]^G = \mathbb{R}^G - p[S]^G$ (Koellner 2013). A is universally Baire, if it 187 is e-universally Baire for all e (op. cit.). 188

Ω-Logic is sound, such that $V \vdash_\Omega \phi \to V \models_\Omega \phi$. However, the completeness 189 of Ω-Logic has yet to be resolved. 190

Finally, in category theory, a category C is comprised of a class Ob(C) of objects 191 a family of arrows for each pair of objects C(A,B) (Venema 2007: 421). A functor 192 from a category C to a category D, $\mathbf{E}: C \to D$, is an operation mapping objects and 193 arrows of C to objects and arrows of D (422). An endofunctor on C is a functor, $\mathbf{E}:$ 194 $C \to C$ (op. cit.). 195

A $\mathbf{E}$-coalgebra is a pair $\mathbb{A} = (A, \mu)$, with A an object of C referred to as the 196 carrier of $\mathbb{A}$, and $\mu: A \to \mathbf{E}(A)$ is an arrow in C, referred to as the transition map of 197 $\mathbb{A}$ (390). 198

$\mathbb{A} = \langle A, \mu: A \to \mathbf{E}(A) \rangle$ is dually isomorphic to the category of algebras over the 199 functor μ (417–418). If μ is a functor on categories of sets, then Boolean-algebraic 200 models of Ω-logical validity are isomorphic to coalgebraic models. 201

The significance of the foregoing is that coalgebraic models may themselves be 202 availed of in order to define modal logic and automata theory. Coalgebras provide 203 therefore a setting in which the Boolean-valued models of set theory, the modal 204 profile of Ω-logical validity, and automata can be interdefined. In what follows, $\mathbb{A}$ 205 will comprise the coalgebraic model – dually isomorphic to the complete Boolean- 206 valued algebras defined in the Ω-Logic of ZFC – in which modal similarity types 207 and automata are definable. As a coalgebraic model of modal logic, $\mathbb{A}$ can be defined 208 as follows (407): 209

For a set of formulas, Φ, let $\nabla\Phi := \Box \bigvee \Phi \wedge \bigwedge \diamond\Phi$, where $\diamond\Phi$ denotes the set 210
$\{\diamond\phi \mid \phi \in \Phi$ (op. cit.). Then, 211

$\diamond\phi \equiv \nabla\{\phi, \mathrm{T}\},$ 212

$\Box\phi \equiv \nabla\varnothing \vee \nabla\phi$ (op. cit.). 213

Let an **E**-coalgebraic modal model, $\mathbb{A} = \langle S, \lambda, R[.]\rangle$, such that $\mathbb{S}, s \Vdash \nabla\Phi$ if and 214
only if, for all (some) successors σ of s∈S, $[\Phi, \sigma(\mathrm{s}) \in \mathbf{E}(\Vdash_\mathbb{A})]$ (op. cit.). 215

A coalgebraic model of deterministic automata can be thus defined (391). An 216
automaton is a tuple, $\mathbb{A} = \langle A, a_I, C, \delta, F\rangle$, such that A is the state space of 217
the automaton $\mathbb{A}$; $a_I \in A$ is the automaton's initial state; C is the coding for the 218
automaton's alphabet, mapping numerals to properties of the natural numbers; δ: 219
A X C $\rightarrow$ A is a transition function, and F⊆ A is the collection of admissible 220
states, where F maps A to {1,0}, such that F: A $\rightarrow$ 1 if a∈F and A $\rightarrow$ 0 if a∉F 221
(op. cit.). The determinacy of coalgebraic automata, the category of which is dually 222
isomorphic to the Set category satisfying Ω-logical consequence, is secured by the 223
existence of Woodin cardinals: Assuming ZFC, that λ is a limit of Woodin cardinals, 224
that there is a generic, set-forcing extension G $\subseteq$ the collapse of $\omega < \lambda$, and that 225
$\mathbb{R}^* = \bigcup\{\mathbb{R}^G[a] \mid a < \lambda\}$, then $\mathbb{R}^* \models$ the axiom of determinacy (AD) (Koellner and 226
Woodin, op. cit.: 10). 227

Finally, $\mathbb{A} = \langle A, \alpha: A \rightarrow \mathbf{E}(A)\rangle$ is dually isomorphic to the category of algebras 228
over the functor α (417–418). For a category C, object A, and endofunctor **E**, 229
define a new arrow, α, s.t. $\alpha: \mathbf{E}A \rightarrow A$. A homomorphism, f, can further be 230
defined between algebras $\langle A, \alpha\rangle$, and $\langle B, \beta\rangle$. Then, for the category of algebras, 231
the following commutative square can be defined: (i) $\mathbf{E}A \rightarrow \mathbf{E}B$ ($\mathbf{E}f$); (ii) $\mathbf{E}A \rightarrow$ 232
A (α); (iii) $\mathbf{E}B \rightarrow B$ (β); and (iv) A $\rightarrow$ B (f) (cf. Hughes, 2001: 7–8). The same 233
commutative square holds for the category of coalgebras, such that the latter are 234
defined by inverting the direction of the morphisms in both (ii) [A $\rightarrow$ **E**A (α)], and 235
(iii) [B $\rightarrow$ **E**B (β)] (op. cit.). 236

Thus, $\mathbb{A}$ is the coalgebraic category for modal, deterministic automata, dually 237
isomorphic to the complete Boolean-valued algebraic models of Ω-logical validity, 238
as defined in the category of sets. 239

4.3   Epistemic Modal Algebras and the Computational   240
Theory of Mind   241

Beyond the remit of Boolean-valued models of set-theoretic languages, models of 242
epistemic modal algebras are availed of by a number of paradigms in contemporary 243
empirical theorizing, including the computational theory of mind and the theory 244
of quantum computability. In Epistemic Modal Algebra, the topological boolean 245
algebra, A, can be formed by taking the powerset of the topological space, X, defined 246
above; i.e., $A = P(X)$. The domain of A is comprised of formula-terms – eliding 247
propositions with names – assigned to elements of P(X), where the proposition- 248
letters are interpreted as encoding states of information. The top element of the 249

algebra is denoted '1' and the bottom element is denoted '0'. We interpret modal 250
operators, f(x), – i.e., intensional functions in the algebra – as both concerning 251
topological interiority, as well as reflecting *epistemic possibilities*. An Epistemic 252
Modal-valued Algebraic structure has the form, $F = \langle A, D_{P(X)}, \rho \rangle$, where ρ is 253
a mapping from points in the topological space to elements or regions of the 254
algebraic structure; i.e., $\rho : D_{P(X)} \times D_{P(X)} \rightarrow A$. A model over the Epistemic- 255
Modal Topological Boolean Algebraic structure has the form $M = \langle F, V \rangle$, where 256
$V(a) \leq \rho(a)$ and $V(a,b) \wedge \rho(a, b) \leq V(b)$.[8] For all $x_{x/a,\phi}, y \in A$: 257

$f(1) = 1;$ 258
$f(x) \leq x;$ 259
$f(x \wedge y) = f(x) \wedge f(y);$ 260
$f[f(x)] = f(x);$ 261
$V(a, a) > 0;$ 262
$V(a, a) = 1;$ 263
$V(a, b) = V(b, a);$ 264
$V(a, b) \wedge V(b, c) \leq V(a, c);$ 265
$V(a = a) = \rho(a, a);$ 266
$V(a, b) \leq f[V(a, b)];$ 267
$V(\neg\phi) = \rho(\neg\phi) - f(\phi);$ 268
$V(\diamond\phi) = \rho\phi - f[-V(\phi)];$ 269
$V(\square\phi) = f[V(\phi)]$ (cf. Lando, op. cit.).[9]   270

Marcus (2001) argues that mental representations can be treated as algebraic 271
rules characterizing the computation of operations on variables, where the values 272
of a target domain for the variables are universally quantified over and the function 273
is one-one, mapping a number of inputs to an equivalent number of outputs (35–36). 274
Models of the above algebraic rules can be defined in both classical and weighted, 275
connectionist systems: Both a single and multiple nodes can serve to represent the 276
variables for a target domain (42–45). Temporal synchrony or dynamic variable- 277
bindings are stored in short-term working memory (56–57), while information 278
relevant to long-term variable-bindings are stored in registers (54–56). Examples 279
of the foregoing algebraic rules on variable-binding include both the syntactic 280
concatenation of morphemes and noun phrase reduplication in linguistics (37–39, 281
70–72), as well as learning algorithms (45–48). Conditions on variable-binding 282
are further examined, including treating the binding relation between variables 283
and values as tensor products – i.e., an application of a multiplicative axiom 284
for variables and their values treated as vectors (53–54, 105–106). In order to 285
account for recursively formed, complex representations, which he refers to as 286
structured propositions, Marcus argues instead that the syntax and semantics of such 287
representations can be modeled via an ordered set of registers, which he refers to as 288
'treelets' (108). 289

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[8]See Lando (2015), McKinsey (1944) and Rasiowa (1963), for further details.

[9]Note that, in cases of Boolean-valued epistemic topological algebras, models of corresponding coalgebras will be topological (cf. Takeuchi 1985 for further discussion).

A strengthened version of the algebraic rules on variable-binding can be accom- 290
modated in models of epistemic modal algebras, when the latter are augmented 291
by cylindrifications, i.e., operators on the algebra simulating the treatment of 292
quantification, and diagonal elements.[10] By contrast to Boolean Algebras with 293
Operators, which are propositional, cylindric algebras define first-order logics. 294
Intuitively, valuation assignments for first-order variables are, in cylindric modal 295
logics, treated as possible worlds of the model, while existential and universal 296
quantifiers are replaced by, respectively, possibility and necessity operators ($\diamond$ and 297
$\square$) (Venema 2013: 249). For first-order variables, $\{v_i \mid i < \alpha\}$ with α an arbitrary, 298
fixed ordinal, $v_i = v_j$ is replaced by a modal constant $\mathbf{d}_{i,j}$ (op. cit: 250). The 299
following clauses are valid, then, for a model, M, of cylindric modal logic, with 300
$E_{i,j}$ a monadic predicate and T_i for i, j $< \alpha$ a dyadic predicate: 301

M, w $\Vdash$ p $\iff$ w $\in$ V(p); 302
M, w $\Vdash$ $\mathbf{d}_{i,j}$ $\iff$ w $\in$ $E_{i,j}$; 303
M, w $\Vdash$ $\diamond_i \psi$ $\iff$ there is a v with $wT_i v$ and M, v $\vdash$ ψ (252).[11] 304

Finally, a cylindric modal algebra of dimension α is an algebra, $\mathbb{A} = \langle A, +, \bullet,$ 305
$-, 0, 1, \diamond_i, \mathbf{d}_{ij} \rangle_{i,j < \alpha}$, where $\diamond_i$ is a unary operator which is normal ($\diamond_i 0 = 0$) and 306
additive $[\diamond_i(x + y) = \diamond_i x + \diamond_i y)]$ (257). 307

The philosophical interest of cylindric modal algebras to Marcus' cognitive 308
models of algebraic variable-binding is that variable substitution is treated in the 309
modal algebras as a modal relation, while universal quantification is interpreted as 310
necessitation. The interest of translating universal generalization into operations of 311
epistemic necessitation is, finally, that – by identifying epistemic necessity with 312
apriority – both the algebraic rules for variable-binding and the recursive formation 313
of structured propositions can be seen as operations, the implicit knowledge of 314
which is apriori. 315

In quantum information theory, let a constructor be a computation defined over 316
physical systems. Constructors entrain nomologically possible transformations from 317
admissible input states to output states (cf. Deutsch 2013). On this approach, 318

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[10]See Henkin et al (op. cit.: 162–163) for the introduction of cylindric algebras, and for the axioms governing the cylindrification operators.

[11]Cylindric frames need further to satisfy the following axioms (op. cit.: 254):

1. p $\to$ $\diamond_i$p
2. p $\to$ $\square_i \diamond_i$ p
3. $\diamond_i \diamond_i$ p $\to$ $\diamond_i$p
4. $\diamond_i \diamond_j$ p $\to$ $\diamond_j \diamond_i$ p
5. $\mathbf{d}_{i,i}$
6. $\diamond_i(\mathbf{d}_{i,j} \wedge p) \to \square_i(\mathbf{d}_{i,j} \to p)$
    [Translating the diagonal element and cylindric (modal) operator into, respectively, monadic and dyadic predicates and universal quantification: $\forall xyz[(T_i xy \wedge E_{i,j}y \wedge T_i xz \wedge E_{i,j}z) \to y = z]$ (op. cit.)]
7. $\mathbf{d}_{i,j}$ $\iff$ $\diamond_k(\mathbf{d}_{i,k}], \wedge \mathbf{d}_{k,j})$.

information is defined in terms of constructors, i.e., intensional computational 319
properties. The foregoing transformations, as induced by constructors, are referred 320
to as tasks. Because constructors encode the counterfactual to the effect that, were 321
an initial state to be computed over, then the output state would result, modal 322
notions are thus constitutive of the definition of the tasks at issue. There are, further, 323
both topological and algebraic aspects of the foregoing modal approach to quantum 324
computation.[12] The composition of tasks is formed by taking their union, where the 325
union of tasks can be satisfiable while its component tasks might not be. Suppose, 326
e.g., that the information states at issue concern the spin of a particle. A spin-state 327
vector will be the sum of the probabilities that the particle is spinning either upward 328
or downward. Suppose that there are two particles which can be spinning either 329
upward or downward. Both particles can be spinning upward; spinning downward; 330
particle-1 can be spinning upward while particle-2 spins downward; and vice versa. 331
The state vector, V which records the foregoing possibilities – i.e., the superposition 332
of the states – will be equal to the product of the spin-state of particle-1 and the 333
spin-state of particle-2. If the particles are both spinning upward or both spinning 334
downward, then V will be .5. However – relative to the value of each particle 335
vector, referred to as its eigenvalue – the probability that particle-1 will be spinning 336
upward is .5 and the probability that particle-2 will be spinning downward is .5, 337
such that the probability that both will be spinning upward or downward = $.5 \times .5$ 338
= .25. Considered as the superposition of the two states, V will thus be unequal to 339
the product of their eigenvalues, and is said to be entangled. If the indeterminacy 340
evinced by entangled states is interpreted as inconsistency, then the computational 341
properties at issue might further have to be defined on a distribution of epistemic 342
possibilities which permit of hyperintensional distinctions.[13] 343

4.4 Modal Coalgebraic Automata and the Philosophy of Mathematics 344 345

This section examines the philosophical significance of the Boolean-valued models 346
of set-theoretic languages and the modal coalgebraic automata to which they are 347
dually isomorphic. I argue that, similarly to second-order logical consequence, 348
(i) the 'mathematical entanglement' of Ω-logical validity does not undermine its 349
status as a relation of pure logic; and (ii) both the modal profile and model- 350
theoretic characterization of Ω-logical consequence provide a guide to its epistemic 351

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[12]For an examination of the interaction between topos theory and an S4 modal axiomatization of computable functions, see Awodey et al. (2000).

[13]The nature of the indeterminacy in question is examined in Saunders and Wallace (2008), Deutsch (2010), Hawthorne (2010), Wilson (2011), Wallace (2012: 287–289), Lewis (2016: 277–278), and Khudairi (ms). For a thorough examination of approaches to the ontology of quantum mechanics, see Arntzenius (2012: ch. 3).

tractability.[14] I argue, then, that there are several considerations adducing in favor of   352
the claim that the interpretation of the concept of set constitutively involves modal   353
notions. The role of the category of modal coalegebraic deterministic automata   354
in (i) characterizing the modal profile of Ω-logical consequence, and (ii) being   355
constitutive of the formal understanding-conditions for the concept of set, provides,   356
then, support for a realist conception of the cumulative hierarchy.   357

4.4.1   Neo-Logicism   358

Frege's (1884/1980; 1893/2013) proposal – that cardinal numbers can be explained   359
by specifying an equivalence relation, expressible in the signature of second-order   360
logic and identity, on lower-order representatives for higher-order entities – is the   361
first attempt to provide a foundation for mathematics on the basis of logical axioms   362
rather than rational or empirical intuition. In Frege (1884/1980. cit.: 68) and Wright   363
(1983: 104–105), the number of the concept, **A**, is argued to be identical to the   364
number of the concept, **B**, if and only if there is a one-to-one correspondence   365
between **A** and **B**, i.e., there is a bijective mapping, R, from **A** to **B**. With Nx: a   366
numerical term-forming operator,   367

- ∀**A**∀**B**∃R[[Nx: **A** = Nx: **B** ≡ ∃R[∀x[**A**x → ∃y(**B**y ∧ Rxy ∧ ∀z(**B**z ∧ Rxz → y   368
  = z))] ∧ ∀y[**B**y → ∃x(**A**x ∧ Rxy ∧ ∀z(**A**z ∧ Rzy → x = z))]]]].   369

Frege's Theorem states that the Dedekind-Peano axioms for the language of   370
arithmetic can be derived from the foregoing abstraction principle, as augmented   371
to the signature of second-order logic and identity.[15] Thus, if second-order logic   372
may be counted as pure logic, despite that domains of second-order models are   373
definable via power set operations, then one aspect of the philosophical significance   374
of the abstractionist program consists in its provision of a foundation for classical   375
mathematics on the basis of pure logic as augmented with non-logical implicit   376
definitions expressed by abstraction principles.   377

There are at least three reasons for which a logic defined in ZFC might not   378
undermine the status of its consequence relation as being logical. The first reason for   379
which the mathematical entanglement of Ω-logical validity might be innocuous is   380
that, as Shapiro (1991: 5.1.4) notes, many mathematical properties cannot be defined   381
within first-order logic, and instead require the expressive resources of second-order   382
logic. For example, the notion of well-foundedness cannot be expressed in a first-   383
order framework, as evinced by considerations of compactness. Let E be a binary   384
relation. Let m be a well-founded model, if there is no infinite sequence, $a_0, \ldots,$ 385

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[14]The phrase, 'mathematical entanglement', is owing to Koellner (2010: 2).

[15]Cf. Dedekend (1888/1963) and Peano (1889/1967). See Wright (1983: 154–169) for a proof sketch of Frege's theorem; Boolos (1987) for the formal proof thereof; and Parsons (1964) for an incipient conjecture of the theorem's validity.

a_i, such that $Ea_0, \ldots, Ea_{i+1}$ are all true. If m is well-founded, then there are no infinite-descending E-chains. Suppose that T is a first-order theory containing m, and that, for all natural numbers, n, there is a T with $n + 1$ elements, $a_0, \ldots, a_n$, such that $\langle a_0, a_1 \rangle, \ldots, \langle a_n, a_{n-1} \rangle$ are in the extension of E. By compactness, there is an infinite sequence such that $a_0 \ldots a_i$, s.t. $Ea_0, \ldots, Ea_{i+1}$ are all true. So, m is not well-founded.

By contrast, however, well-foundedness can be expressed in a second-order framework:

$\forall X[\exists x Xx \rightarrow \exists x[Xx \wedge \forall y(Xy \rightarrow \neg Eyx)]]$, such that m is well-founded iff every non-empty subset X has an element x, s.t. nothing in X bears E to x.

One aspect of the philosophical significance of well-foundedness is that it provides a distinctively second-order constraint on when the membership relation in a given model is intended. This contrasts with Putnam's (1980) claim, that first-order models *mod* can be intended, if every set *s* of reals in *mod* is such that an ω-model in *mod* contains *s* and is constructible, such that – given the Downward Lowenheim-Skolem theorem[16] – if *mod* is non-constructible but has a submodel satisfying '*s* is constructible', then the model is non-well-founded and yet must be intended. The claim depends on the assumption that general understanding-conditions and conditions on intendedness must be co-extensive, to which I will return in Sect. 4.4.2

A second reason for which Ω-logic's mathematical entanglement might not be pernicious, such that the consequence relation specified in the Ω-logic might be genuinely logical, may again be appreciated by its comparison with second-order logic. Shapiro (1998) defines the model-theoretic characterization of logical consequence as follows:
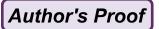
'(10) Φ is a logical consequence of [a model] Γ if Φ holds in all possibilities under every interpretation of the nonlogical terminology which holds in Γ' (148).

A condition on the foregoing is referred to as the 'isomorphism property', according to which 'if two models M, M' are isomorphic vis-a-vis the nonlogical items in a formula Φ, then M satisfies Φ if and only if M' satisfies Φ' (151).

Shapiro argues, then, that the consequence relation specified using second-order resources is logical, because of its modal and epistemic profiles. The epistemic tractability of second-order validity consists in 'typical soundness theorems, where one shows that a given deductive system is 'truth-preserving' (154). He writes that: '[I]f we know that a model is a good mathematical model of logical consequence (10), then we know that we won't go wrong using a sound deductive system. Also, we can know that an argument is a logical consequence . . . via a set-theoretic proof in the metatheory' (154–155).

The modal profile of second-order validity provides a second means of accounting for the property's epistemic tractability. Shapiro argues, e.g., that: 'If the isomorphism property holds, then in evaluating sentences and arguments, the only 'possibility' we need to 'vary' is the size of the universe. If enough sizes are

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[16] For any first-order model M, M has a submodel M' whose domain is at most denumerably infinite, s.t. for all assignments s on, and formulas $\phi(x)$ in, M', $M,s \Vdash \phi(x) \iff M',s \Vdash \phi(x)$.

represented in the universe of models, then the modal nature of logical consequence   427
will be registered ... [T]he only 'modality' we keep is 'possible size', which is   428
relegated to the set-theoretic metatheory' (152).   429

Shapiro's remarks about the considerations adducing in favor of the logicality   430
of non-effective, second-order validity generalize to Ω-logical validity. In the   431
previous section, the modal profile of Ω-logical validity was codified by the dual   432
isomorphism between complete Boolean-valued algebraic models of Ω-logic and   433
the category, 𝔸, of coalgebraic modal logics. As with Shapiro's definition of logical   434
consequence, where Φ holds in all possibilities in the universe of models and   435
the possibilities concern the 'possible size' in the set-theoretic metatheory, the Ω-   436
Conjecture states that $V \models_\Omega \phi$ iff $V^{\mathbb{B}} \models_\Omega \phi$, such that Ω-logical validity is invariant   437
in all set-forcing extensions of ground models in the set-theoretic multiverse.   438

Finally, the epistemic tractability of Ω-logical validity is secured, both – as on   439
Shapiro's account of second-order logical consequence – by its soundness, but also   440
by its isomorphism to the coalgebraic category of deterministic automata, where the   441
determinacy thereof is again secured by the existence of Woodin cardinals.   442

4.4.2 Set-Theoretic Realism   443

In this section, I argue, finally, that the modal profile of Ω-logic can be availed of   444
in order to account for the understanding-conditions of the concept of set, and thus   445
crucially serve as part of the argument for set-theoretic realism.   446

Putnam (op. cit.: 473–474) argues that defining models of first-order theories is   447
sufficient for both understanding and specifying an intended interpretation of the   448
latter. Wright (1985: 124–125) argues, by contrast, that understanding-conditions   449
for mathematical concepts cannot be exhausted by the axioms for the theories   450
thereof, even on the intended interpretations of the theories. He suggests, e.g., that: 451

'[I]f there really were uncountable sets, their existence would surely have to flow   452
from the concept of set, as intuitively satisfactorily explained. Here, there is, as it   453
seems to me, no assumption that the content of the ZF-axioms cannot exceed what is   454
invariant under all their classical models. [Benacerraf] writes, e.g., that: 'It is granted   455
that they are to have their 'intended interpretation': 'e' is to mean set-membership.   456
Even so, and conceived as encoding the intuitive concept of set, they fail to entail   457
the existence of uncountable sets. So how can it be true that there are such sets?   458
Benacerraf's reply is that the ZF-axioms are indeed faithful to the relevant informal   459
notions only if, in addition to ensuring that 'E' means set-membership, we interpret   460
them so as to observe the constraint that 'the universal quantifier has to mean all or at   461
least all sets' (p. 103). It follows, of course, that if the concept of set does determine   462
a background against which Cantor's theorem, under its intended interpretation, is   463
sound, there is more to the concept of set that can be explained by communication of   464
the intended sense of 'e' and the stipulation that the ZF-axioms are to hold. And the   465
residue is contained, presumably, in the informal explanations to which, Benacerraf   466
reminds us, Zermelo intended his formalization to answer. At least, this must be so if   467
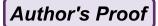
the 'intuitive concept of set' is capable of being explained at all. Yet it is notable that
Benacerraf nowhere ventures to supply the missing informal explanation – the story
which will pack enough into the extension of 'all sets' to yield Cantor's theorem,
under its intended interpretation, as a highly non-trivial corollary'(op. cit).

In order to provide the foregoing explanation in virtue of which the concept of set
can be shown to be associated with a realistic notion of the cumulative hierarchy, I
will argue that there are several points in the model theory and epistemology of set-
theoretic languages at which the interpretation of the concept of set constitutively
involves modal notions. The aim of the section will thus be to provide a modal
foundation for mathematical platonism.

One point is in the coding of the signature of the theory, T, in which Gödel's
incompleteness theorems are proved (cf. Halbach and Visser 2014). Relative to,

(i) a choice of coding for an ω-complete, recursively axiomatizable language, L,
of T – i.e. a mapping between properties of numbers and properties of terms
and formulas in L;

(ii) a predicate, phi; and

(iii) a fixed-point construction:

Let phi express the property of 'being provable', and define (iii) such that,
for all consistent theories T of L, there are sentences, p_{phi}, corresponding to
each formula, phi(x), in T, s.t. for 'm' := p_{phi},

$\vdash_T p_{phi}$ iff phi(m).

One can then construct a sentence, 'm' := ¬phi(m), such that L is incom-
plete (the first incompleteness theorem).

Moreover, L cannot prove its own consistency:

If:

$\vdash_T$ 'm' iff ¬phi(m),

Then:

$\vdash_T C \rightarrow m$.

Thus, L is consistent only if L is inconsistent (the second incompleteness
theorem).

In the foregoing, the choice of coding bridges the numerals in the language
with the properties of the target numbers. The choice of coding is therefore
intensional, and has been marshalled in order to argue that the very notion of
syntactic computability – via the equivalence class of partial recursive functions,
λ-definable terms, and the transition functions of discrete-state automata such as
Turing machines – is constitutively semantic (cf. Rescorla 2015). Further points
at which intensionality can be witnessed in the phenomenon of self-reference in
arithmetic are introduced by Reinhardt (1986). Reinhardt (op. cit.: 470–472) argues
that the provability predicate can be defined relative to the minds of particular agents
– similarly to Quine's (1968) and Lewis' (1979) suggestion that possible worlds can
be centered by defining them relative to parameters ranging over tuples of spacetime
coordinates or agents and locations – and that a theoretical identity statement can be
established for the concept of the foregoing minds and the concept of a computable
system.

In the previous section, intensional computational properties were defined via 512
modal coalgebraic deterministic automata, where the coalgebraic categories are 513
dually isomorphic to the category of sets in which Ω-logical validity was defined. 514
Coalgebraic modal logic was shown to elucidate the modal profile of Ω-logical con- 515
sequence in the Boolean-valued algebraic models of set theory. The intensionality 516
witnessed by the choice of coding may therefore be further witnessed by the modal 517
automata specified in the foregoing coalgebraic logic. 518

A second point at which understanding-conditions may be shown to be con- 519
stitutively modal can be witnessed by the conditions on the epistemic entitlement 520
to assume that the language in which Gödel's second incompleteness theorem is 521
proved is consistent (cf. Dummett 1963/1978; Wright 1985). Wright (op. cit.: 91, 522
fn.9) suggests that '[T]o treat [a] proof as establishing consistency is implicitly 523
to exclude any doubt . . . about the consistency of first-order number theory'. 524
Wright's elaboration of the notion of epistemic entitlement, appeals to a notion 525
of rational 'trust', which he argues is recorded by the calculation of 'expected 526
epistemic utility' in the setting of decision theory (2004; 2014: 226, 241). Wright 527
notes that the rational trust subserving epistemic entitlement will be pragmatic, 528
and makes the intriguing point that 'pragmatic reasons are not a special genre of 529
reason, to be contrasted with e.g. epistemic, prudential, and moral reasons' (2012: 530
484). Crucially, however, the very idea of expected epistemic utility in the setting 531
of decision theory makes implicit appeal to the notion of possible worlds, where the 532
latter can again be determined by the coalgebraic logic for modal automata. 533

A third consideration adducing in favor of the thought that grasp of the concept 534
of set might constitutively possess a modal profile is that the concept can be defined 535
as an intension – i.e., a function from possible worlds to extensions. The modal 536
similarity types in the coalgebraic modal logic may then be interpreted as dynamic- 537
interpretational modalities, where the dynamic-interpretational modal operator has 538
been argued to entrain the possible reinterpretations both of the domains of the 539
theory's quantifiers (cf. Fine 2005, 2006), as well as of the intensions of non-logical 540
concepts, such as the membership relation (cf. Uzquiano 2015).[17] 541

The fourth consideration avails directly of the modal profile of Ω-logical 542
consequence. While the above dynamic-interpretational modality will suffice for 543

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[17]For an examination of the philosophical significance of modal coalgebraic automata beyond the philosophy of mathematics, see Baltag (2003). Baltag (op. cit.) proffers a colagebraic semantics for dynamic-epistemic logic, where coalgebraic functors are intended to record the informational dynamics of single- and multi-agent systems. For an algebraic characterization of dynamic-epistemic logic, see Kurz and Palmigiano (2013). For further discussion, see Khudairi (ms). The latter proceeds by examining undecidable sentences via the epistemic interpretation of multi-dimensional intensional semantics. See Reinhardt (1974), for a similar epistemic interpretation of set-theoretic languages, in order to examine the reduction of the incompleteness of undecidable sentences on the counterfactual supposition that the language is augmented by stronger axioms of infinity; and Maddy (1988b), for critical discussion. Chihara (2004) argues, as well, that conceptual possibilities can be treated as imaginary situations with regard to the construction of open-sentence tokens, where the latter can then be availed of in order to define nominalistically adequate arithmetic properties.

possible reinterpretations of mathematical terms, the absoluteness and generic 544
invariance of the consequence relation is such that, if the Ω-conjecture is true, then 545
Ω-logical validity is invariant in all possible set-forcing extensions of ground models 546
in the set-theoretic multiverse. The truth of the Ω-conjecture would thereby place 547
an indefeasible necessary condition on a formal understanding of the intension for 548
the concept of set. 549

4.5 Concluding Remarks 550

In this essay I have examined the philosophical significance of the isomorphism 551
between Boolean-valued algebraic models of modal Ω-logic and modal coalgebraic 552
models of automata. I argued that – as with the property of validity in second- 553
order logic – Ω-logical validity is genuinely logical, and thus entails a type 554
of neo-logicism in the foundations of mathematics. I argued, then, that modal 555
coalegebraic deterministic automata, which characterize the modal profile of Ω- 556
logical consequence, are constitutive of the interpretation of mathematical concepts 557
such as the membership relation. The philosophical significance of modal Ω-logic 558
is thus that it can be availed of to vindicate both a neo-logicist foundation for set 559
theory and a realist interpretation of the cumulative hierarchy of sets. 560

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AQ6

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AQ1. We have retain the "Footnote 1" in the section. Please check if okay.
AQ2. Please provide details of "Woodin (2001), Gödel (1940), Cohen (1963, 1964), Hughes (2001) and Parsons (1964)" in reference list.
AQ3. Please provide publisher location for "Boolos (1987), Reinhardt (1974), Frege (1884/1980), Gödel (1931/1986), and Gödel (1946/1990)".
AQ4. Please cite "Henkin et al. (1971), Wright (2004), Wright (2012), and Wright (2014)" in text.
AQ5. Please provide publisher details for "Koellner (2013) and Peano (1889/1967)".
AQ6. Please provide conference location for "Woodin (2010)".