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On the Existence of an Optimum Savings Program

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1. The purpose of this note is to analyze critically the nature of the solutions which have been offered to the problem of how much a nation should optimally save. Since savings in this connection is the only alternative to consumption, this is equivalent to the problem of how much should a nation optimally consume. An optimal consumption program is one which makes a certain stipulated functional in utilities $[f(U(c(t))]$ as high as possible, subject to certain restrictions on the class of admissible utility and production functions. U here is an indicator of utility and $c(t)$ is consumption at point t.

Two approaches have been adopted in finding this optimal program: a) to define the functions $c(t)$ on a finite time interval $(o \leq t \leq T)$ which corresponds to a finite planning horizon. This makes the domain of the functions closed and bounded. Together with the assumption of continuity of the various functions, this is enough to solve the problem of the optimal program over the relevant time horizon. But the solution is crucially dependent on the length of time period T and the valuation attached to the terminal capital equipment. The latter is not a meaningful concept unless we try to take into account what happens beyond T. This reveals the problem as essentially extending indefinitely over T. This leads to an alternative formulation: b) the functions $c(t)$ are now defined for $t \geq o$, e.g., the time horizon is infinite. The domain is no longer compact. The functions are now defined in an infinite dimensional space, or, in discrete case, in the space of sequences extending to infinity. To choose the optimal consumption program in this case, it is necessary to formulate the problem in such a way that an ordering is introduced on the policy space. This is obvious since, unless there is an order, there is no way of determining the best program. It is the contention of this paper that the attempts in literature up to date either do not pay sufficient attention to introducing this order and thus fail in properly formulating the problem, or they do so in an arbitrary way, which ensures mathematical tractability but little more.* The demonstration of this point emphasizes the necessity of more closely studying these features of infinite programs which enable us to discriminate between them without being totally dependent on one or the other arbitrary assumption.

2. The 'locus classicus' of this problem is to be found in the 1928 paper of Ramsey on the mathematical theory of savings.[1] Subsequently, the following have been the noteworthy contributions to the problem: a) the work of Samuelson and Solow[2] in extending Ramsey's analysis to a world involving multiple capital goods, b) the recent papers on Tinbergen[3] which do not adopt the Ramsey set-up of the problem but are essentially concerned with discovering the policy implications of a one-commodity capital model by using econometrically tested utility and production functions, and c) the papers by Stone[4] and Meade[5] who made strictly Ramsey-type assumptions but tried to uncover its policy implications for aggregative, but more specific situations, characterized by explicit production and utility functions. In the last thirty years, these have been the major contributions to a subject whose fascination is matched only by its difficulties.** In our discussion we shall be primarily concerned with the results of Ramsey and Tinbergen, because the nature of the difficulties involved in selecting a best consumption program over an infinite time comes out very clearly

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*Mathematically, the root of the difficulty in Case b) is in the absence of compactness of the policy space. For the real line, compactness is the same as closedness and boundedness.

**Mr. R. F. Harrod in his interesting recent article in the Economic Journal has a finite time period in mind, since he assumes the world to come to an end as a result of "atomic explosion" at the end of an arbitrarily large but finite time period "n". Thus, as he writes "The Explosion would take the place of Bliss in governing the correct rate of saving." (Second Essay in Dynamic Theory, Economic Journal, June, 1960.)

in these one-commodity models. In a disaggregated model, the same difficulties would persist, fortified by additional complications.

Somewhat different in spirit but bearing essentially on the same problem, we have the remarkable contributions of Malinvaud[6] and Koopmans,[7] on which we shall have occasion to comment briefly.

3. While the contributions mentioned above have been primarily positive, at least in intention, certain critical voices have been heard which have sought to dismiss the problem as being an exercise with little significance for theory and none for policy-making whatsoever.[8] I feel that there are two reasons why such criticism may not be very well taken.

Firstly, the main analytical interest of these models consists in their attempt to introduce an ordering of consumption programs in an infinite-dimensional policy space that would enable us to choose the best program. As mentioned before, in a world devoid of uncertainty the logical essence of any problem in dynamic programming is inescapably infinite-dimensional. Thus, the Ramsey-Tinbergen approach, while dealing with a one-commodity world, is trying to tackle issues which are at the heart of the theory of capital. Merely to say that the problem is settled in practice by a political decision is not to say that the problem does not exist or that a procedure is indicated as to how to resolve the conceptual difficulties. It only amounts to a confession of failure, without trying to analyze what the failure is due to.

Secondly, from the policy point of view, the importance of the problem should not be underrated. Assume for the time being that we have one central decision maker who is interested in drawing up a savings plan over time. The preference ordering of such a decision maker need not be represented by a cardinal utility function, but it would simplify discussion if we assume cardinal utility. Then, the Ramsey-Tinbergen problem is placed in its proper setting and whenever this setting applies even in a rough way, the resulting analysis will apply.

The discussion on this question becomes all the more important if, following Tinbergen, one assumes that there is a consecutiveness in policy decisions as to permit a splitting up of the problem of resource allocation into different stages. Mathematically, it only implies that the structure of a decision problem is approximately blocktriangular. Then, a decision on how much the nation should optimally save is followed by the optimal distribution of the savings between the different sectors of the economy. Thus, broad macro-economic considerations can be arrived at without entering into details to start with.

4. Let us consider Tinbergen's papers first. His first paper had a somewhat restricted scope in view; he was concerned with finding a savings ratio which would be optimal for all future years, given the utility functions, production functions, and the initial endowment of capital. He also assumed a subjective time preference, independent of diminishing utility or uncertainty. His problem, then, was to maximize the integral of discounted utility over time with respect to a parameter, e.g., the savings ratio. The restrictiveness of this approach is somewhat severe and Tinbergen himself realized it. The source of this restriction does not lie in assuming an arbitrary subjective rate of time preference. For even granting this arbitrariness, the choice of an optimal savings ratio, while, by definition, the best among all programs having fixed savings ratios, is not the same thing as the optimal policy from the point of view of maximizing the integral of discounted utilities. His recent paper deals

with a wider problem, e.g., to choose the profile of savings over time which would render a certain stipulated functional, in his case the integral of utility, this time undiscounted, from zero to infinity, a maximum. Thus, the problem is one in variational calculus.

The problem is as follows: Max $\int_0^\infty U[C(t)]\,dt$ subject to $C(t) = b\,K(t) - \dot{K}(t)$ where b is output-capital ratio, here assumed to be constant, $U(C) > 0$, $U'(C) > 0$, $U''(C) < 0$, and $K(0)$ is given. It is seen that the functional in Tinbergen's case is an improper integral. Thus the question of the best choice of a savings program can only arise in this form if a preliminary property is verified, namely that the stipulated functional has a maximum. Since the functional in this case is an infinite integral, the question boils down to one about its convergence on our previous specifications of the utility and production functions.

If the convergence conditions on the above integral are not satisfied, not only is there no solution to the variational problem in the mathematical sense, but what is even more important, the usefulness of formulating the economic problem in the above manner is a doubtful one. The economic significance of formulating the choice problem as one of maximizing $\int_0^\infty U\,dt$ arises from the possible ordering that the functional imposes on alternative infinite programs. But if the total utility associated with any feasible infinite program is infinitely large, because $\int_0^\infty U\,dt$ does not converge, then, there is no possibility of introducing any order on the policy space through such mappings from the policy space to the utility space. Thus, the above choice of the functional instead of being a 'natural one' turns out to be economically improper, because any proper formulation of a choice problem implies that alternatives should be capable of being discriminated.

The crucial importance of convergence conditions may be understood intuitively, when one considers that the problem of choosing the best consumption path is defined as determing which program of consumption makes the integral of utility as large as possible. Now, only if the integral is finite do we make sense in talking about one path being better than the other because it gives us a higher sum total of satisfaction. If the sum total can be made infinite for some consumption profiles, then usually one cannot discriminate between the alternatives. In those very special cases, where there is one path dominating all the others in the sense that it can provide more utility at any instant of time than any other, the problem of choice is a trivial one. The question of choosing between alternative infinite programs arises because such domination does not generally exist.*

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*Malinvaud has suggested an interesting procedure to compare alternative infinite programs, each of which gives infinite satisfaction. Thus, in Orwellian language, all infinite programs are equal, but some are more equal than the others. So far as I can see, Malinvaud's procedure consists in dividing the class of feasible infinite programs into two subclasses, a) programs which differ only for a number of periods T, where T is arbitrary and finite, but are identical afterwards; b) all the other programs which are feasible. Now for a given T, the programs belonging to a) are comparable amongst themselves and the one giving the highest value of the functional up to T should be chosen. By making T arbitrarily large, we widen the set of admissible programs and so long as T remains finite, however large, a best program exists among programs admitted.

This procedure while extremely interesting, hinges on the crucial assumption that letting $T \to \infty$, the various best programs (best for each choice of T) will tend to a 'the best program.' If this does not hold, then the procedure suggested by Malinvaud does not work, but it must still be granted that Malinvaud's procedure provides us with the necessary conditions for determining the best program in the infinite dimensional policy space.

Returning to Tinbergen's problem, it may be useful to divide the class of admissible utility functions into two subclasses: a) the functions which do not admit a bliss point; b) the functions admitting a bliss point. What the existence of bliss implies is that as consumption grows longer and larger over time, the utility associated with this consumption has a finite upper bound. It may, however, be profitable to distinguish further under b) three sub-cases:

i. There is a finite level of consumption corresponding to this upper bound on the utility function;

ii. There is no finite level of consumption which corresponds to the upper bound on the utility function, but there is an asymptotic approach towards the upper bound with consumption increasing. An example of (ii) is given by the hyperbolic utility function given in Tinbergen: (3, 1956) $U(C) = U_o\,(1 - C^o/C)$. Here the upper bound U_o is never attained but only approached as $C \to \infty$. The function is only defined for $C^o \neq o$.

iii. In this case, not only is there a finite consumption which attains finite bliss, but to push consumption beyond this point lowers total utility. This is the saturation case mentioned by Tinbergen (3, 1960) towards the end of his paper. This case is improper for our present discussion, because it implies $U'(C) < 0$, while we have assumed on general qualitative grounds that $U'(C) > 0$.

On assumption a) which is Tinbergen's first example, U (C) grows indefinitely with C increasing. One necessary condition for $\int_o^\infty U[C(t)]\,dt$ to converge is that $U(t) \to 0$ as $t \to \infty$, but with only $U'(C) \to 0$, U (t) does not tend to zero even if C is increasing very fast. Thus the necessary condition for the convergence of the integral is not satisfied. Hence, the problem is not adequately formulated even from the economic point of view, not to speak of its mathematical impropriety.

On assumptions b) i, ii, which admit bliss, but rule out negative marginal utility for any level of consumption, $U[C(t)]$ tends towards the bliss level as $t \to \infty$. Thus, if $U_0 = B$ (bliss), then, once again, the integral $\int_0^\infty U[C(t)]\,dt$ does not converge for what tends to 0 is $[B-U(t)]$ and not $U(t)$.

Thus, on either set of assumptions, the functional is divergent and, therefore, no order can be introduced among the alternative paths through using functional criterion $\int_0^\infty U\,dt$.

This is borne out by an analysis of Tinbergen's results. Tinbergen employs two sets of considerations to obtain the best program of savings. One of them is the obvious balance equation that says that consumption plus savings (= investment) equals income. The operative principle, however, is that at any point of time consumption is determined by equating its marginal utility to the marginal utility of savings, since savings is the only alternative to consuming. Marginal utility of savings is calculated as the integral of additional satisfaction obtained in perpetuity derived from saving an extra unit today, where future utility is subject to no discounting.

These two considerations enable him to deduce the time-path of consumption over time, which turns out to be a constant magnitude, equal to the subsistence level of consumption. Now, this apparently surprising result is due to the fact that when we are considering as the functional the undiscounted sum of utilities over an <u>infinite</u> period, with Tinbergen's assumptions it can never pay to consume anything more than the minimum, because what we lose by not consuming today will be made up by what we gain from the satisfaction due to extra product over an infinite time. Thus, if there were no restriction on minimum consumption, which merely reflects an arbitrary choice of origin, the rule would have been to save 100 per cent, inasmuch as however much one reduces consumption, marginal utility of consumption never increases as much to equal the marginal utility of savings for any positive level of consumption. Thus, savings is always more profitable, till one reaches the corner situation where everything is saved.*

This result is the reflection of the property that the infinite integral is divergent.

Another way of looking at this problem would be to maximize the integral of utility over a finite time, T, and then letting T vary to infinity and see what turns out to be the maximal program over an infinite period. Thus, the problem is now to maximize $\int_0^T U\,[C(t)]\,dt$. This is a well-behaved problem subject to our specifications on the utility and production functions. Now, assume that $C_1\,(t)$ is the function that maximizes this integral. Change the upper bound from T to T + 1, we have then function $C_2\,(t)$ that answers our problem. The problem $\int_0^\infty U[C(t)]\,dt$ is soluble if and only if the functions $C_1\,(t)$, $C_2\,(t)$, tend towards a limit function C(t) as $t \rightarrow \infty$, which maximizes the above integral.

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*This need not be the case in general, but is a consequence of Tinbergen's assumptions of a constant marginal productivity of capital b) as well as a very slow decline in marginal utility resulting from the shape of the utility function. In this case, the path with no consumption dominates all paths having non-zero consumption.

In Tinbergen's case, what happens is that a limit function exists but does not maximize the functional. Because, by means of letting T vary in the manner described above, what he gets is an indefinite postponement of current consumption above the subsistence minimum. Thus, the limiting best path of consumption when $T \to \infty$ is a constant magnitude equal to the subsistence. That this does not maximize the integral is borne out when one realizes that it gives only the minimum utility over time, because at any single point of time only the minimum utility is being attained.

5. Ramsey, however, had a different functional in mind, namely, min. $\int_0^\infty [B - U(c)]\, dt$. In economic terms the question no longer is what is the program of savings that maximizes utility over time, but what is the best way to get to bliss. In this case the utility function has always an asymptote. This problem has a meaningful answer for all cases where the bound on utility function is attained for a finite level of consumption. In fairness to Ramsey, one must say that this was the case which he explicitly considered. The reason that a meaningful solution exists in this case is that the difference between current utility and bliss summed over time is a finite magnitude. Mathematically, the expression $[B - U(c)] \to 0$ as $t \to \infty$. Thus, the necessary condition for the convergence of the integral is satisfied. Furthermore, the specification that bliss is actually <u>attained</u> for a finite level of consumption provides one possible sufficient condition. Then $B - U(c) = 0$ from certain time t onwards. For these cases, Ramsey's procedure is the converse of the functional $\int U\, dt$ and one that enables us to discriminate between alternative paths. If these conditions are not satisfied, e.g., a finite bliss exists but no finite upperbound on consumption, then much depends on whether consumption is growing sufficiently rapidly to speed up the rate of convergence of utility levels to bliss. This depends on the choice of production and utility functions.

The following example illustrates this point. Take the case of hyperbolic utility, $U(C) = U_0 \left(1 - \frac{C^o}{C}\right)$, where $U_0 = B$

Then, we have
$$\int_0^\infty \left[U_o - U_o \left(1 - \frac{C^o}{C(t)}\right)\right] dt = \int_0^\infty \frac{U_o C_o}{C(t)}\, dt$$

Now, for $C(t) = \bar{\bar{C}}$, this integral does not converge. Thus, a constant consumption program is ruled out. This is trivial from the economic point of view. But suppose that $C(t) = C_o + \alpha t$, then, also, the integral does not converge. Then consumption is growing linearly but $\int_1^\infty \frac{1}{t}\, dt$ is divergent. Our qualitative specifications on the production function do not, however, rule out the case of systems growing asymptotically at linear rates and that is interesting.

The point of the above discussion is that even in this modified Ramsey case, if there is no finite level of consumption corresponding to bliss, the usual restrictions on the utility function and production function do not rule out the possibility that no meaningful solution exists. Thus, even Ramsey's choice of the functional is not always free from difficulty. It may be that empirical restrictions on parameters are such that the linear growth case is always ruled out in practice. But then the problem is an empirical one and not a purely logical one, a possibility not envisaged by Ramsey himself. Moreover, the importance of this possibility is stronger when one realizes that Ramsey thought the optimum savings to be independent of the choice of production function, not to speak of numerical restrictions on the parameters.

There is a further economic reason why the Ramsey choice is not economically very meaningful in these cases. This is because the <u>existence</u> of an optimum solution is not invariant with respect to the choice of the utility indicator involved. Suppose instead of taking U, we take U^2 or U^3, then even with a linear growth of consumption, the problem admits of an answer. Since in the absence of uncertainty, or separability in the non-stochastic case, we get utility indicator unique only up to monotonic transformation, such lack of invariance only underlines the failure of this functional to impose a natural ordering on the infinite policy space, when finite bliss at finite consumption levels is ruled out. For the finite case, only the quantitative result (the actual savings program) depend on the cardinal measurability of utility. But in the more general situation, even the qualitative answer changes with the change in the assumed utility function.

6. Another answer, which has been attempted to this question is to introduce a subjective rate of time preference and then our functional is not the integral of utility over time but the integral of discounted utility of consumption over time, where the rate of discount is a pure rate of time preference.* Tinbergen himself had used these specifications in his first paper, with the added restriction that the savings rate be a constant. Now, in the variational case we have $\text{Max}\int_0^\infty U(C)\, e^{-\rho t}\, dt$ subject to $C(t) = bK(t) - \dot{K}(t)$ or $C(t) = K^P - K(t)$, if we assume Cobb-Douglas production function.

Even here, the functional is bounded above provided the combined effects of diminishing marginal utility and time preference relative to the rate of growth of consumption are such as to satisfy the convergence conditions. To assume that such conditions will always be satisfied, is

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*The "pure rate of time preference" should be distinguished from the notion of a 'social rate of time preference,' which may be logically implied in any consideration relating to equity between the generations.

very largely to prejudge the whole set of issues; since one does not know _a priori_ what the rate of growth would be, even if the rate of pure time preference is numerically ascertainable. However, if this condition is assumed satisfied, then we do not need any further restriction on the class of utility functions, other than their usual curvature properties. Thus, introducing this time premium enables us to talk of a best infinite program of savings.

Alternatively, if a best infinite consumption program exists, this under certain circumstances may be interpreted as implying the existence of a time premium. This presumably explains the result which Koopmans obtained in his recent paper[7] about impatience being logically implied in an infinite consumption program. The justification for the above statement is that while setting up his problem, Koopmans assumes axiomatically that a best program exists in the policy space. In this case, given Koopmans other assumptions relating to stationarity, etc., it was logical to derive time preference as a _consequence_, rather than as an _empirically testable_ hypothesis, as has been the general tradition in the literature on capital theory. It should, however, be recognized that Koopmans' result is not confined only to the above choice of the functional but includes a wider class in which the above formulation would be necessarily included.

7. The upshot of the above discussion has been to indicate that the various optimal infinite programs that have been discussed in the literature suffer from either an improper formulation which makes the solutions economically irrelevant, or from the restrictiveness that arises from crucial dependences on certain arbitrary assumptions. A functional criterion

such as Max $\int U\,dt$ does not impose any ordering on the policy space, because with the usual qualitative specifications on the admissible utility and production functions, the integral does not converge, and thus the results obtained from this procedure are economically meaningless. On the other hand, functional criteria such as Min $\int (B-U)\,dt$ where B is reached for a finite consumption or Max $\int e^{-\rho t} U\,dt$ impose some ordering on the utility space but they do it in completely arbitrary fashions. In a sense, such arbitrary preference orderings are operationally meaningless statements so long as we do not have any method of refuting them. Since our interest lies primarily in the meaningfulness of the order introduced and not only in the mathematical requirements of introducing such order, it is difficult to help but feel that such formulations have very little significance apart from ensuring solubility of the mathematical problem.

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