Medium Frequency Range Analysis in non-homogeneous slender structures

P. Gibert[a], E. Gay[a]

a. *MEcanique et DYnamique des SYStemes 8bis bd Dubreuil 91400 Orsay*

Résumé :

Des phénomènes de propagation d'ondes dans des milieux homogènes ou non interviennent dans de nombreux domaines de la physique. En mécanique on utilise notamment le principe des ondes guidées lors de l'inspection de panneaux composites par Contrôle Non Destructif. Nous proposons une méthode numérique simplifiée permettant de déterminer les chemins de propagation d'ondes dans des structures élancées, non homogènes, dans le cas où la longueur d'onde est de l'ordre de grandeur de la dimension transverse, et petite par rapport à une dimension longitudinale macroscopique du système. Pour cela on utilise une méthode asymptotique de type W.K.B.J., dont la résolution à l'ordre un se ramène à un système Hamiltonien construit à partir des propriétés propagatives locales et dont les trajectoires fournissent les chemins de propagation. Les lois de conservation de l'énergie obtenues à l'ordre deux permettent de déduire la localisation des zones de concentration d'énergie engendrées par l'hétérogénéité.

Abstract :

A lot of physics fields involve wave propagation within non-homogeneous materials. Numerous methods use guided waves in slender bodies such as composite laminates to verify their integrity, especially for the Non Destructive Inspection of aircraft structures, or to identify their mechanical properties. A simplified numerical approach of the wave propagation within non-homogeneous slender structures is proposed here. In our conditions, the wavelength is significantly smaller than the characteristic macroscopic length of the structure but it is in the same range as the transverse size. The Finite Element approach is impossible in this case. A specific asymptotic approach based on W.K.B.J. method is then used. A first-order non linear partial derivative equation is set up using the material local propagative properties and is solved using an Hamiltonian system. The energy propagation is thus precisely described. The propagation of energy could be deviated depending on the material heterogeneities and some areas are submitted to energy concentration.

Mots clefs : **guided waves ; non-homogeneous slender structures ; composite materials**

1 Introduction

Numerous physical fields involve waves propagation, either in three-dimensional material, or in one or two-dimensional waves-guides. The detection and identification of defects and damaged areas are provided through the measurements of the propagative properties of guided waves [1, 4]. Numerous papers deal with theoretical or experimental applications, in civil engineering - inspection of aging civil structures, for example after earthquakes, aircraft engineering - Non Destructive Testing of composite components. These studies concern a special kind of guided wave : the Lamb waves. The detection of various kinds of defect has been investigated : porosity, moisture, thermic deterioration or delamination of composite material can be detected using this method.

The finite elements approach of the high frequency analysis raises the question of compatibility between the mesh size and the wavelength or material heterogeneity. For complex materials such as

composites laminates with a periodic microstructure, the phenomena at different scales could interfere with each other. The global vibrational behavior depends on the wavelength and several frequency ranges may coexist as well as several models of vibrational behaviors. In some complex cases, it is even possible to observe three scales : the macroscopic dimension or thickness of a plate, the period of the representative elementary model, and the wavelength. A comparison between these scales is required to define simplified models. On the other hand, the numerical determination of the propagative properties of waveguide with constant longitudinal properties have already been reported, for example for the propagation of Lamb waves in multi-layered composites [2].
The wave propagation is here studied within non-homogeneous materials with a distributed heterogeneity. This heterogeneity could have been induced by a mechanical deformation, it could be a distributed defect generated during the manufacturing process or in use. This defect is either physical (for example a variation of the thickness or a local curvature of the a plate) or mechanical (non-constant constitutive behaviour law). This paper aims to demonstrate that it is possible to completely predict the propagation of energy, the local energy concentration and the extinction of modes along virtual boundaries, using a simplified numerical approach. This analysis and related numerical tools could support experimentation to detect and identify the distributed defects.
The well-known W.K.B.J. asymptotic method is used and adjusted by introducing a small scale parameter that determines the wavelength depending on a macroscopic characteristic length. The wavelength is supposed to have a similar size than the transverse dimension of the waveguides. After the description of the mechanical process of the method in section 2, the first and second order approximations are studied in sections 3 and 4. In section 5, the method is illustrated using examples and the multi-purpose MFRA.Waves software is introduced.

2 The mechanical context - The W.K.B.J. method

The shell is defined as a tridimensional structure using its mid-surface Γ and its constant thickness $h^\epsilon = \frac{L}{\epsilon}$, where ϵ is a scale parameter considered small and L is the given reference macroscopic length. D^ϵ is the physical domain defined by the relation (1) :

$$D^\epsilon = \{M = \begin{pmatrix} m(\tau_1,\tau_2) \\ z \end{pmatrix}, m \in \Gamma, -\frac{h^\epsilon}{2} < z < \frac{h^\epsilon}{2}\} \tag{1}$$

where τ_1, τ_2 are the curvilinear coordinates of m on Γ and D is the stretched and fixed domain :

$$D = \{\begin{pmatrix} m(\tau_1,\tau_2) \\ Z \end{pmatrix}, m \in \Gamma, -\frac{L}{2} < Z < \frac{L}{2}\} \tag{2}$$

The unitary tangent vectors $a_{,\alpha}$ are defined so that $m_{,\alpha} = A_\alpha.a_{,\alpha}$, $\alpha = 1,2$ and $a_\alpha, \alpha = 1,2$ are supposed orthogonal as a simplifying assumption.

The elastic waves propagate along the mid-surface Γ with a wavelength in the same range as the shell thickness. The frequencies are thus given as a function of $\frac{1}{\epsilon}$. As a consequence, the terminology Medium Frequency Range is used.

The W.K.B.J. method [3, 7] is used to evaluate the propagative tridimensional solutions. Then for a pulsation $\omega^\epsilon = \frac{\Omega}{\epsilon}$ at a given Ω, tridimensional solutions are considered as :

$$u(\tau_1,\tau_2,z,t) = e^{i\omega^\epsilon t - i\frac{S(\tau_1,\tau_2)}{\epsilon}}.U^\epsilon(\tau_1,\tau_2,Z = \frac{z}{\epsilon}) \tag{3}$$

U^ϵ is then gradually determined using the asymptotic expansion :

$$U^\epsilon = U_0 + \epsilon\, U_1 + \epsilon^2\, U_2 + \dots \tag{4}$$

Then the new unknown parameters are the phase $S(\tau_1,\tau_2)$, and the induced displacements $U_0(\tau_1,\tau_2)$, $U_1(\tau_1,\tau_2)$, etc

The solution is locally related to a waveguide mode for an infinite plate :

$$u \approx C^{cst} e^{i(\Omega t^{\epsilon} - k_{loc1} X_{loc1} - k_{loc2} X_{loc2})} U_0(0,0,Z) \tag{5}$$

where $k_{loc\alpha} = \frac{1}{A_\alpha(0,0)} S_{,\alpha}(0,0), \alpha = 1,2$ with $S_{,\alpha} = \frac{\partial S}{\tau_{\partial\alpha}}$ and C^{cst} is a constant, $X_{locj}, j = 1,2$ are the local cartesian coordinates and $t^{\epsilon} = \frac{t}{\epsilon}$ the fast time.

(5) is a waveguide solution relation for the tangent infinite plate at $(0,0)$, where the mechanical properties are the same as those at $(0,0)$. The general solution (3) is thus locally a waveguide eigenmode at frequency Ω and S is closely related to the local wave number and then to the wavelength.

S, U_0, U_1 are now determined step by step using the asymptotic expansions method in a global variational formulation of a dynamical problem, on tridimensional space of variables τ_1, τ_2, Z.

The strain (6,1) vector is then derived for a displacement u using relation (3) as a function of the amplitude U in the local coordinate system (time t is not taken into account) :

$$\boldsymbol{\epsilon(u)} = \frac{1}{\epsilon}\{-i\boldsymbol{L_{S'}(U)} + \boldsymbol{\epsilon_g(U)}\} + \{\boldsymbol{\epsilon_m(U)} + \boldsymbol{A.U} - iZ\boldsymbol{M_{S'}(U)}\} + \epsilon\{...\} \tag{6}$$

with :

$$\boldsymbol{L_{S'}(U)}^{tr} = \begin{pmatrix} S_1'U_1 & S_2'U_2 & 0 & S_2'U_Z & S_1'U_Z & S_1'U_2 + S_2'U_1 \end{pmatrix} \tag{7}$$

$$\boldsymbol{\epsilon_g(U)}^{tr} = \begin{pmatrix} 0 & 0 & \frac{\partial U_Z}{\partial Z} & \frac{\partial U_2}{\partial Z} & \frac{\partial U_1}{\partial Z} & 0 \end{pmatrix} \tag{8}$$

$$\boldsymbol{\epsilon_m(U)}^{tr} = \begin{pmatrix} \frac{1}{A_1}\frac{\partial U_1}{\partial \tau_1} & \frac{1}{A_2}\frac{\partial U_2}{\partial \tau_2} & 0 & \frac{1}{A_2}\frac{\partial U_Z}{\partial \tau_2} & \frac{1}{A_1}\frac{\partial U_Z}{\partial \tau_1} & \frac{1}{A_1}\frac{\partial U_2}{\partial \tau_1} + \frac{1}{A_2}\frac{\partial U_1}{\partial \tau_2} \end{pmatrix} \tag{9}$$

$$\boldsymbol{M_{S'}(U)}^{tr} = \begin{pmatrix} S_1'\frac{U_1}{R_1} & S_2'\frac{U_2}{R_2} & 0 & S_2'\frac{U_Z}{R_2} & S_1'\frac{U_Z}{R_1} & S_1'\frac{U_2}{R_1} + S_2'\frac{U_1}{R_2} \end{pmatrix} \tag{10}$$

and A is a (6,3) matrix that only depends on the local geometry of the mid-surface Γ.

A virtual displacement is chosen with a similar relation : $\boldsymbol{v}(\tau_1, \tau_2, z, t) = e^{i\omega^{\epsilon}t - i\frac{S(\tau_1,\tau_2)}{\epsilon}}.\boldsymbol{V}(\tau_1, \tau_2, Z = \frac{z}{\epsilon})$ with compact support in D, the dynamical equations are derived using a variational formulation given by :

$$\left\{\begin{array}{l} \int_D\{\frac{1}{\epsilon}(-i\boldsymbol{L_{S'}(U^{\epsilon})} + \boldsymbol{\epsilon_g(U^{\epsilon})}) + (\boldsymbol{\epsilon_m(U^{\epsilon})} + \boldsymbol{A.U^{\epsilon}} - iZ\boldsymbol{M_{S'}(U^{\epsilon})}) + ...\}^{tr}.C. \\ \{\frac{1}{\epsilon}(i\boldsymbol{L_{S'}(V^*)} + \boldsymbol{\epsilon_g(V^*)}) + (\boldsymbol{\epsilon_m(V^*)} + \boldsymbol{A.V^*} + iZ\boldsymbol{M_{S'}(V^*)}) + ...\}\epsilon dD = \\ = \frac{\Omega^2}{\epsilon^2}\int_D \rho \boldsymbol{U}^{\epsilon tr}.\boldsymbol{V}^*\epsilon dD \quad \forall \boldsymbol{V} : D \to \Re^3 \end{array}\right. \tag{11}$$

where $dD = A_1A_2 d\tau_1 d\tau_2 dZ$, and V has a compact support in $\Re^3$.

The asymptotic expansion relation (4) is enclosed in (11) to identify the successive orders of ϵ.

Order ϵ^{-1}

$$\int_D (-i\boldsymbol{L_{S'}(U_0)} + \boldsymbol{\epsilon_g(U_0)}).C.(i\boldsymbol{L_{S'}(V^*)} + \boldsymbol{\epsilon_g(V^*)})dD - \Omega^2 \int_D \rho \boldsymbol{U_0^{tr}}.\boldsymbol{V}^* dD = 0 \tag{12}$$

Order ϵ^0

$$\begin{aligned} &\int_D (-i\boldsymbol{L_{S'}(U_0)} + \boldsymbol{\epsilon_g(U_0)}).C.(\boldsymbol{\epsilon_m(V^*)} + \boldsymbol{A.V^*} + iZ\boldsymbol{M_{S'}(V^*)})dD \\ &+ \int_D (\boldsymbol{\epsilon_m(U_0)} + \boldsymbol{A.U_0} - iZ\boldsymbol{M_{S'}(U_0)}).C.(i\boldsymbol{L_{S'}(V^*)} + \boldsymbol{\epsilon_g(V^*)})dD \\ &+ \int_D (-i\boldsymbol{L_{S'}(U_1)} + \boldsymbol{\epsilon_g(U_1)}).C.(i\boldsymbol{L_{S'}(V^*)} + \boldsymbol{\epsilon_g(V^*)})dD \\ &- \Omega^2 \int_D \rho \boldsymbol{U_1^{tr}}.\boldsymbol{V}^* dD = 0 \end{aligned} \tag{13}$$

... and so on ; these relations give the ability to define successively S, $\boldsymbol{U}_0$, $\boldsymbol{U}_1$, ...

3 The eikonale equation

In the relation (12) the point (τ_1, τ_2) is considered as a parameter ; it can be rewritten as the variational equation along the thickness of a waveguide problem given at the point (τ_1, τ_2) and using its mechanical characteristics :

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} (\boldsymbol{\epsilon_g}(\boldsymbol{U}_0) - i.\boldsymbol{L_k}(\boldsymbol{U}_0))^{tr}.\boldsymbol{C}(\tau_1,\tau_2,Z).(\boldsymbol{\epsilon_g}(\boldsymbol{V}^*) + i.\boldsymbol{L_k}(\boldsymbol{V}^*))\,dZ = \Omega^2 \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho(\tau_1,\tau_2)\,\boldsymbol{U}_0^{tr}.\boldsymbol{V}^*\,dZ$$
$$\boldsymbol{U}_0(\tau_1,\tau_2,.)\ :]-\frac{L}{2},\frac{L}{2}[\to \Re^3 \qquad \forall \boldsymbol{V}\ :]-\frac{L}{2},\frac{L}{2}[\to \Re^3 \tag{14}$$

where $\boldsymbol{k} = \boldsymbol{S'}$ is the local wave propagation vector. The equation (14) is the variational expression of Lamb waves for an infinite plate whose material properties are given at $\boldsymbol{m}$ [5]. Then Ω is related to $\boldsymbol{k}$ using the dispersion relation of the waveguide at (τ_1, τ_2). The waveguide eigenmodes are sinusoidal waves which propagate at the frequency Ω such that :

$$\boldsymbol{U}(\tau_1,\tau_2,Z,t) = e^{i(\Omega t - k_1\tau_1 - k_2\tau_2)}\boldsymbol{U}(Z) \tag{15}$$

If $(\Omega_j^{\boldsymbol{k},\boldsymbol{m}}, \boldsymbol{U}^{\boldsymbol{k},\boldsymbol{m}}{}_j(z)),\ j = 1,2,...$ are respectively the j-th eigenpulsation and its related normalized eigenmode at a point m for a wave-vector k, the Hamilton function is defined by :

$$H_j(\boldsymbol{k},\boldsymbol{m}) = \Omega_j(\boldsymbol{k},\boldsymbol{m}) - \Omega \tag{16}$$

Then for a given index j :

$$H_j(\boldsymbol{S'},\boldsymbol{m}) = 0 \tag{17}$$

and :

$$\boldsymbol{U}_0(\boldsymbol{m},Z) = \phi(\boldsymbol{m}).\boldsymbol{U}_j^{\boldsymbol{k},\boldsymbol{m}} \tag{18}$$

where ϕ is an undetermined function of point $\boldsymbol{m}(\tau_1,\tau_2)$.

The Hamiltonian method is used to solve the nonlinear first order equation (17). To do this, the nonlinear system of the ordinary Hamilton equations is used (index j is omitted from now on, the dot notation is used for the time derivative) :

$$\dot{\boldsymbol{k}} = -\frac{\partial H}{\partial \boldsymbol{m}} \tag{19}$$
$$\dot{\boldsymbol{m}} = \frac{\partial H}{\partial \boldsymbol{k}} \tag{20}$$

and then S is determined by :

$$\dot{S} = \boldsymbol{k}.\dot{\boldsymbol{m}} \tag{21}$$

The initial conditions can be defined at a given point m_0. A k_0 vector has then to be chosen such that $H(k_0, m_0) = 0$ or along a given source line γ where $S = 0$. A wave vector k_0 orthogonal to γ has also to be selected at each m_0 in γ. Then a trajectory is defined using a geometrical parameter ζ, which is either an orientation angle for $\boldsymbol{k}$ for a source point or a curvilinear coordinate for a source line.

The projection of the relations (19) and (20) in the local coordinate system (a_1, a_2) provides a nonlinear system of 4 first order differential equations given by :

$$\chi = \psi_H(\chi) \tag{22}$$

where the components of χ are τ_1, τ_2 and the local coordinates of $\boldsymbol{k}$.

It is possible to demonstrate that S only depends on $\boldsymbol{m}$ and $\boldsymbol{S'} = \boldsymbol{k}$. S is then defined as a function of the local coordinates (t, ζ). Singularities of transformation $(\tau_1, \tau_2) \to (t, \chi)$ may occur, particularly close to the caustic curves (envelopes of the trajectories, see Figure 1). As explained in the next section, these areas are submitted to a concentration of energy and asymptotic expansion (4). The use of the exponential function is not relevant anymore because the kind of solution has changed. A local analysis has to be carried, using specific Airy functions and a multiple scale asymptotic expansion [3].

The trajectories and then the energy are deviated and attracted near the singular points χ_0 for equation (22) such that $\psi(\chi_0) = \psi'(\chi_0) = 0$. For example, a point with a minimal cut-off frequency is singular. As the adequate eigenmode is chosen depending on the kind of damage, such points are linked to the maximal damage. Other phenomena may occur for eigenmodes with a null group velocity [6].

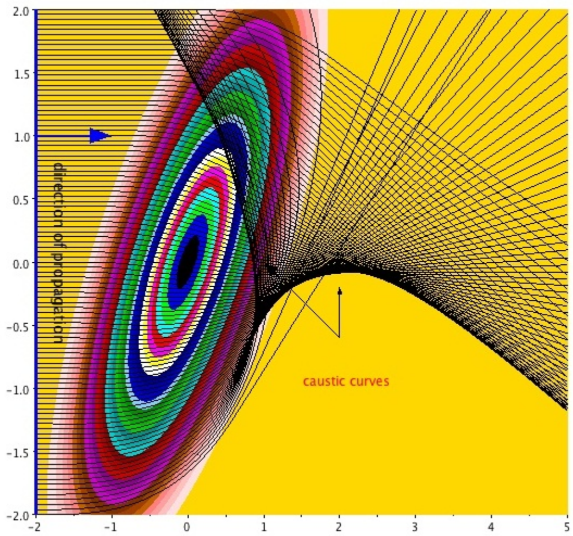

FIGURE 1 – Wave propagation within a composite plate with a degradation of the behaviour law. A caustic curve is generated after this damaged area.

4 The transport equation

The equation (13) can be considered as a forced response equation for an unknown displacement U_1 induced by a loading depending on ϕ. The Fredholm condition shows the existence of a solution and a new equation is obtained to determine ϕ, and then the first order solution U_0 is completed. If only the modulus of ϕ is taken into account, depending on the local energy of the wave, this equation can be given by :

$$div(|\phi|^2 \frac{\partial H}{\partial \boldsymbol{k}}(\boldsymbol{S'}, \boldsymbol{x})) = 0 \tag{23}$$

Then the square of the wave amplitude times the group velocity of the wave is constant along a trajectory. The wave paths are directly linked to the way energy propagates. As explained earlier, some complex situations may occur, leading to a concentration of energy. This phenomena could be localized after a defect where the wave is refracted, or ahead the boundaries of a forbidden frequency zone.

5 Software development

The authors have developed the MFRA.Waves software (Medium Frequency Range Analysis) to support experimental Non Destructive Inspection. It has the ability to calculate waves propagative properties within multi-layered materials. It also takes into account the interaction of the waves with continuous damage in structures such as beam, plate and shell. Medium Frequency Range waves are considered as a particles flow whose trajectory is related to energy concentration as described in this paper. Figure 2 provides a comparison of wave propagation within an intact plate and a damaged one.

The damage is modeled as a continuous decrease of the Young's modulus, represented in the figure by the white ellipses.

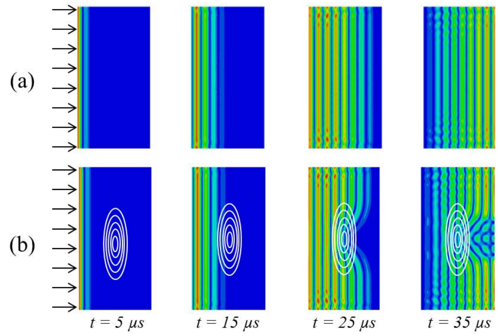

FIGURE 2 – Wave propagation within a composite plate :(a) Intact, and (b) With a continuous damage. Parameters used to model the plate and solicitation are : plate dimension : $3 * 150 * 75$ mm, density $= 1.53$, $E_L = 20$ GPa, $E_T = 8$ GPa, $G_{LT} = 4$ GPa, frequency = 233 kHz, mode S_0

Wave propagation is altered by this degradation. The software MFRA.Waves provides thus a valuable support to interpret the experimental response of the laminate to a wave solicitation.

6 Conclusion

The method provides a numerical solution for medium frequency wave propagation in composite slender structure. It is based on a semi-analytical approximation using a W.K.B.J. asymptotic expansion. It uses the local propagation properties, such as the relations of dispersion, to solve a first-order nonlinear partial derivative equation using a characteristic method. The wave propagation is then described by the particles movements computed by a Hamilton equation solver. The local displacements and the wave vectors as well as the magnitude of the energy are determined using this method. The energy concentration areas are highlighted through the post-processing of the trajectories after solving the Hamilton equations. An heterogeneity induces a perturbation of the distribution of energy : some areas are submitted to a concentration of energy while other ones receive less energy.
Future works will concern the application of the method to other fields of physics such as electromagnetism or optics.

Références

[1] Y. Bar-Cohen, A. Mai, Z. Chang 1998 Defects detection and Characterization using Leaky Lamb wave (LLW) dispersion data, *Proceeding of the ASNT Asia-Pacific Conference on NDT. and* 7^{th} *Annual Research Symposium, Anaheim, CA, 24-26 March.*

[2] M. El Allami, H. Rhimini, M. Sidki 2010 Propagation des ondes de Lamb : Résolution par la methode des éléments finis et post-traitement par la transformée en ondelettes, *Compte-rendu du* 10^{eme} *Congres Francais d'Acoustique, Lyon, 12-16 avril 2010.*

[3] Ali Nayfeh Perturbation methods, *Pure and Applied Mathematics - Wiley-Interscience Publications* (1973)

[4] A. Raghavan, C.E.S. Cesnik 2007 Review of guided-wave structural health monitoring, *The Shock and Vibration Digest*, **39** n^o2, 91-114

[5] D. Royer, E. Dieulesaint, Ondes elastiques dans les solides, Tome 1 : Propagation libre et guidée *Ed. Masson (1996)*

[6] D. Royer, D. Clorennec, C. Prada 2010 Caracterisation de plaques et de tubes par modes de Lamb a vitesse de groupe nulle, *I2M - 10/2010 Méthodes innovantes en CND* 73-94

[7] M.A.Slawinski Waves and Rays in Elastic Continua *World Scientific Publishing Co.Pte.Ltd.* (2007)