Simulation of medium-frequency response in elastic shells using VTCR and PGD methods

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Résumé :

Ce document propose une extension de la Théorie Variationnelle des Rayons Complexes [4, 14] (TVRC) pour calculer la réponse en moyenne fréquence dans des coques élastiques en utilisant la technique dénommée Proper Generalized Decomposition [12, 13] (PGD). Le TVRC est une approche de type Trefftz pour le calcul des vibrations de structures élastiques légèrement amorties dans la gamme de moyenne fréquence. Elle a été entièrement développée pour les problèmes de vibrations acoustiques et structurels à une fréquence fixe. La méthode PGD [12, 13] est une technique de réduction de modèle qui repose sur la construction a priori *d'une représentation de la solution avec des variables séparées sur le domaine fréquentiel. L'approche PGD a montré de bons résultats sur d'autres problèmes multi-paramétriques. Ce travail montre son efficacité sur les exemples considérés qui concernent les chocs pyrotechniques et les problèmes de bande de fréquence de vibration.*

Abstract :

This paper proposes an extension of the Variational Theory of Complex Rays [4, 14] (VTCR) to calculate the medium-frequency bandwidth response in elastic shells using the Proper Generalized Decomposition [12, 13] (PGD) technique. The VTCR is a Trefftz-type approach for calculating vibrations of slightly damped elastic structures in the medium-frequency range. It has been fully developed for acoustic and structural vibration problems at a fixed frequency. The PGD method [12, 13] is a model reduction technique which relies on the a priori *construction of the separated variables representation of the solution over the frequency-space domain. The PGD approach has shown good results on other multi-parametric problems. This work will show its efficiency on the considered examples which concern pyrotechnic shocks and frequency band vibration problems.*

Keywords : Shells ; Variational Theory of Complex Rays ; Proper Generalized Decomposition

1 Introduction

The study of the vibrational response of elastic structures is a key point of the modern structural design process. The low-frequency range nowadays poses no threat to FEM or BEM solvers, even for complex structures [1]. On the other side, the high-frequency range is well studied by the statistical energy analysis (SEA) method which does not takes into account the spatial aspect of the problem [3, 11].

However, the study of the medium-frequency range continue to present problems. The difficulty for the low-frequency methods lies in the length of variation of the phenomena being considered, which is very small if compared to characteristic dimension of the structure. In fact the number of degrees of freedom (DoF) required for such calculations is prohibitive. Nevertheless, much work is currently in progress to extend the frequency range of the SEA-based techniques.

Problems arise also if one tries to apply the SEA method to the medium frequency range. In fact a spatial description of the problem is still needed. The theory depicted in [10, 15] is built upon the ideas of effective energy density and effective vibrational energy. Despite it is extremely attractive, it still encounters some obstacles [9].

The alternative approach developed here is called the Variational Theory of Complex Rays (VTCR). It is a native medium-frequency approach and it has been introduced in [4]. The vibrational response at a fixed frequency is computed using a new variational formulation. It has been developed in order to allow *a priori* independent approximations within the substructures. The transmission conditions are incorporated in the variational formulation. This method hes been successfully applied to bars, beams, plates and shells [4, 6, 14]. This technique computes the vibrational response at a specified frequency.

The Proper Generalized Decomposition (PGD) is an *a priori* model reduction technique. It relies on the *a priori* construction of separated variables representations of the solution. It can be interpreted as an extension of Proper Orthogonal Decomposition (POD) for the *a priori* construction of such separated representation.

The VTCR has already been applied on the medium-frequency band using a Taylor approximation approach [5, 7, 8] and on fluids using the PGD technique [2]. The objective of this article is to present the extension of VTCR theory for shallow shells to a medium-frequency band using the PGD technique.

2 The reference problem

Just for the sake of clarity, let us formulate the problem for an assembly of two substructures. The method can easily be generalized to the case of n substructures. The two reference surfaces of the isotropic and homogeneous sub-domains of the shallow shells are Ω_1 and Ω_2. $\partial\Omega_1$ and $\partial\Omega_2$ denote the boundaries of the surfaces Ω_1 and Ω_2 respectively. It is required to study the harmonic vibration of the structures on a frequency band $\omega \in [\omega_1, \omega_2]$. In order to expand the VTCR method let us discretize the domain in a vector of fixed frequencies $\underline{\omega}$ and apply the VTCR technique to every fixed frequency step. All the quantities of the system can be defined in the complex domain : an amplitude $S(\underline{x})$ corresponds to $S(\underline{x})e^{j\omega_i t}$. For each shell, the generic displacement $\underline{u}^z = [\underline{v}^z, w^z]'$ (tangential displacement $\underline{v}^z = [u^z, v^z]'$, normal displacement w^z), the moment and the resultant (associated with operators $\underline{\underline{M}}$ and $\underline{\underline{N}}$ respectively) are taken into account. The structures are assumed to be slightly curved. The effect of the environment on Ω_1 is represented in Figure 1 and consists of a displacement field $\underline{u}_{1d}$ on $\partial_{\underline{u}_{1d}}\Omega_1$, a force density $\underline{F}_{1d}$ on $\partial_{\underline{F}_{1d}}\Omega_1$ and a surface load $\underline{f}_{1d}$ on Ω_1. Similar quantities are defined for Ω_2. The common boundary is Γ.

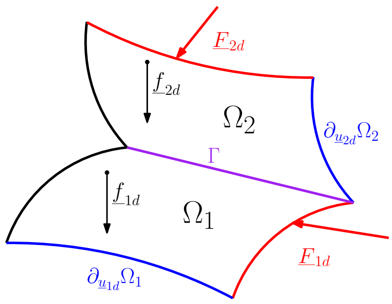

FIGURE 1 – Reference example described in section 2.

The shell theory used here is the standard theory of shallow shells which is a particular case of the Donnel-Mushtari-Vlasov theory of thin shells (see [16]). At first the geometry of the shallow shell can be approximated its projection on the local x, y plane. The displacement class is restricted to (Kirchhoff's kinematic assumption) :

$$\underline{u}^z = \underline{u} - z\underline{\phi} \tag{1}$$

$$\underline{\phi} = \underline{grad}(w) - \underline{\underline{R}}\underline{v} \tag{2}$$

$$\underline{u} = [\underline{v}, w]' \tag{3}$$

where $\underline{u}$, $\underline{v}$ and w are the total, in-plane and off-plane displacements of a point of the middle surface. The displacement troughtout the thickness is $\underline{u}^z$. The symbol $\square'$ is the transpose operator.

$$\underline{\underline{R}} = \begin{bmatrix} \frac{1}{Rx} & 0 \\ 0 & \frac{1}{Ry} \end{bmatrix} \tag{4}$$

is the curvature matrix. The transverse deformation energy is neglected. Define for the generic subdomain Ω the field $\mathcal{D} = \left\{\underline{u}, \underline{\underline{N}}, \underline{\underline{M}}\right\}$ such that

$$\underline{u}^z \in \mathcal{U}^z \quad \text{finite energy displacement set,} \tag{5}$$
$$\left\{\underline{\underline{N}}, \underline{\underline{M}}\right\} \in \mathcal{S} \quad \text{finite energy generalized stress set,} \tag{6}$$
$$div\left(\underline{div}\left(\underline{\underline{M}}\right)\right) + Tr\left(\underline{\underline{RN}}\right) + f_{dz} + \varrho h \omega_i^2 w = 0 \quad \text{on} \quad \Omega, \tag{7}$$
$$\underline{div}\left(\underline{\underline{N}}\right) + \underline{f}_{ds} + \varrho h \omega_i^2 \underline{v} = 0 \quad \text{on} \quad \Omega, \tag{8}$$
$$\underline{f}_d = \left[f_{dx}, f_{dy}, f_{dz}\right]', \tag{9}$$
$$\underline{f}_{ds} = \left[f_{dx}, f_{dy}\right]', \tag{10}$$
$$\underline{\underline{M}} = -\mathbf{K}_{cp} : \underline{\underline{grad}}\left(\underline{grad}\left(w\right)\right), \tag{11}$$
$$\underline{\underline{N}} = \frac{12}{h^2}\mathbf{K}_{cp} : \left(\underline{\underline{\varepsilon}} - \underline{\underline{R}}w\right), \tag{12}$$
$$\underline{\underline{\varepsilon}} = \underline{\underline{grad}}\left(\underline{v}\right)_{sym} = \frac{1}{2}\left(\underline{\underline{grad}}\left(\underline{v}\right) + \underline{\underline{grad}}\left(\underline{v}\right)^H\right), \tag{13}$$
$$D = \frac{Eh^3}{12\left(1-\nu^2\right)}, \tag{14}$$
$$E = E_0\left(1 + i\eta\right). \tag{15}$$

where $\mathbf{K}_{cp}$ is Hooke's plane stress operator, ϱ the density, η the damping coefficient, h the thickness of the shell, E_0 the Young modulus, ν the Poisson's ratio, $\square^H$ is the hermitian operator, $\underline{\underline{N}}$ and $\underline{\underline{M}}$ are the usual stress resultants and stress moment resultants tensors respectively. The subspaces $\mathcal{D}_1$ and $\mathcal{D}_2$ associated with the homogenized conditions ($\underline{f}_{1d} = \underline{f}_{2d} = 0$) are denoted $\mathcal{D}_{0\,1}$ and $\mathcal{D}_{0\,2}$. The boundary conditions are :

$$\underline{u}_1 = \underline{u}_{1d} \quad \text{on} \quad \partial_{\underline{u}_{1d}}\Omega_1, \qquad \underline{u}_2 = \underline{u}_{2d} \quad \text{on} \quad \partial_{\underline{u}_{1d}}\Omega_1, \tag{16}$$
$$w_{1,n_1} = w_{1d,n_1} \quad \text{on} \quad \partial_{w_{1d,n_1}}\Omega_1, \qquad w_{2,n_2} = w_{2d,n_2} \quad \text{on} \quad \partial_{w_{2d,n_2}}\Omega_2, \tag{17}$$
$$\left(\underline{\underline{N}}_1 - \underline{\underline{R}}_1\underline{\underline{M}}_1\right)\hat{\underline{n}}_1 = \underline{\mu}_1 = \underline{\mu}_{1d} \quad \text{on} \quad \partial_{\underline{\mu}_{1d}}\Omega_1, \qquad \left(\underline{\underline{N}}_2 - \underline{\underline{R}}_2\underline{\underline{M}}_2\right)\hat{\underline{n}}_2 = \underline{\mu}_2 = \underline{\mu}_{2d} \quad \text{on} \quad \partial_{\underline{\mu}_{2d}}\Omega_2, \tag{18}$$
$$\left(\underline{div}\left(\underline{\underline{M}}_1\right)\right)\hat{\underline{n}}_1 + \left(\hat{\underline{t}}_1'\underline{\underline{M}}_1\hat{\underline{n}}_1\right), t_1 = Q_1 = Q_{1d}, \quad \text{on} \quad \partial_{Q_{1d}}\Omega_1 \qquad \left(\underline{div}\left(\underline{\underline{M}}_2\right)\right)\hat{\underline{n}}_2 + \left(\hat{\underline{t}}_2'\underline{\underline{M}}_2\hat{\underline{n}}_1\right), t_2 = Q_2 = Q_{2d}, \quad \text{on} \quad \partial_{Q_{2d}}\Omega_2 \tag{19}$$
$$\hat{\underline{n}}_1'\underline{\underline{M}}_1\hat{\underline{n}}_1 = M_1 = M_{1d} \quad \text{on} \quad \partial_{M_{1d}}\Omega_1, \qquad \hat{\underline{n}}_2'\underline{\underline{M}}_2\hat{\underline{n}}_2 = M_2 = M_{2d} \quad \text{on} \quad \partial_{M_{2d}}\Omega_2, \tag{20}$$
$$\tag{21}$$

the conditions on Γ are :

$$\underline{u}_1 = \underline{u}_2, \tag{22}$$

$$w_{1,n_{w_{1d,n_1}}} = -w_{2,n_{w_{2d,n_2}}}, \tag{23}$$

$$\left(\underline{\underline{N}}_1 - \underline{\underline{R}}_1\underline{\underline{M}}_1\right)\underline{\hat{n}}_1 = \left(\underline{\underline{N}}_2 - \underline{\underline{R}}_2\underline{\underline{M}}_2\right)\underline{\hat{n}}_2, \tag{24}$$

$$\left(\underline{div}\left(\underline{\underline{M}}_1\right)\right)\underline{\hat{n}}_1 + \left(\underline{\hat{t}}'\underline{\underline{M}}_1\underline{\hat{n}}_1\right)_{,\underline{\hat{t}}_1} = \left(\underline{div}\left(\underline{\underline{M}}_2\right)\right)\underline{\hat{n}}_2 + \left(\underline{\hat{t}}'\underline{\underline{M}}_2\underline{\hat{n}}_2\right)_{,\underline{\hat{t}}_2}, \tag{25}$$

$$\underline{\hat{n}}_1'\underline{\underline{M}}_1\underline{\hat{n}}_1 = \underline{\hat{n}}_2'\underline{\underline{M}}_2\underline{\hat{n}}_2, \tag{26}$$

the condition of the generic corner m is :

$$2\sum_{i=1}^{2}\underline{\hat{n}}_i{}'\underline{\underline{M}}_i\underline{\hat{t}}_i = S_{md} \tag{27}$$

where $\underline{\hat{n}}_1$ is the versor normal to the boundary directed outside the element Ω_1 and $\underline{\hat{t}}_1$ is the tangent versor of that boundary. Similar quantities are defined for the boundaries of element Ω_2. In these equations, for the sake fo brevity, we have defined some useful quantities like $\underline{\mu}_1$ while stating the corresponding boundary constraint $\underline{\mu}_{1d}$.

3 The weak variational formulation of VTCR

The VTCR is a weak variational formulation of the whole boundary conditions. The theory uses independent approximations within substructures. It looks for the solutions $\left\{\underline{u}_1, \underline{\underline{N}}_1, \underline{\underline{M}}_1\right\} \in \mathcal{D}_1$ and $\left\{\underline{u}_2, \underline{\underline{N}}_2, \underline{\underline{M}}_2\right\} \in \mathcal{D}_2$ such that

$$\begin{aligned}
&\int_{\partial_{\underline{v}_{1d}}\Omega_1} \delta\underline{\mu}_1^H\left(\underline{v}_1 - \underline{v}_{1d}\right)ds + \int_{\partial_{\underline{v}_{2d}}\Omega_2} \delta\underline{\mu}_2^H\left(\underline{v}_2 - \underline{v}_{2d}\right)ds + \sum_{m \text{ corners}}\left(\sum_{\Omega_i \text{ on corners}} w_{mi}^H\left(2\underline{\hat{n}}_{mi}'\underline{\underline{M}}_{mi}\underline{\hat{t}}_{mi} - S_{md}\right)\right)\\
&+\int_{\partial_{w_{1d}}\Omega_1} \delta Q_1^H\left(w_1 - w_{1d}\right)ds + \int_{\partial_{w_{2d}}\Omega_2} \delta Q_2^H\left(w_2 - w_{2d}\right)ds\\
&-\int_{\partial_{w_{1d,n}}\Omega_1} \delta M_1^H\left(w_{1,n} - w_{1d,n}\right)ds - \int_{\partial_{w_{2d,n}}\Omega_2} \delta M_2^H\left(w_{2,n} - w_{2d,n}\right)ds\\
&+\int_{\partial_{\underline{\mu}_{1d}}\Omega_1} \delta\underline{v}_1^H\left(\underline{\mu}_1 - \underline{\mu}_{1d}\right)ds + \int_{\partial_{\underline{\mu}_{2d}}\Omega_2} \delta\underline{v}_2^H\left(\underline{\mu}_2 - \underline{\mu}_{2d}\right)ds\\
&\quad+\int_{\partial K_{1d}\Omega_2} \delta w_1^H\left(K_1 - K_{1d}\right)ds + \int_{\partial K_{2d}\Omega_1} \delta w_2^H\left(K_2 - K_{2d}\right)ds\\
&\quad-\int_{\partial M_{1d}\Omega_1} \delta w_{1,n}^H\left(M_1 - M_{1d}\right)ds - \int_{\partial M_{2d}\Omega_2} \delta w_{2,n}^H\left(M_2 - M_{2d}\right)ds\\
&\quad+\int_\Gamma \frac{1}{2}\left(\left(\delta\underline{\mu}_1 - \delta\underline{\mu}_2\right)^H\left(\underline{v}_1 - \underline{v}_2\right) + \left(\delta Q_1 - \delta Q_2\right)^H\left(w_1 - w_2\right) - \left(\delta M_1 + \delta M_2\right)^H\left(w_{1,n} + w_{2,n}\right)\right)ds\\
&\quad+\int_\Gamma \frac{1}{2}\left(\left(\delta\underline{v}_1 + \delta\underline{v}_2\right)^H\left(\underline{\mu}_1 + \underline{\mu}_2\right) + \left(\delta w_1 + \delta w_2\right)^H\left(Q_1 + Q_2\right) - \left(\delta w_{1,n} - \delta w_{2,n}\right)^H\left(M_1 - M_2\right)\right)ds = 0\\
&\quad\forall\left\{\underline{u}_1, \underline{\underline{N}}_1, \underline{\underline{M}}_1\right\} \in \mathcal{D}_1 \vee \forall\left\{\underline{u}_2, \underline{\underline{N}}_2, \underline{\underline{M}}_2\right\} \in \mathcal{D}_2 \vee \forall\left\{\delta\underline{u}_1, \delta\underline{\underline{N}}_1, \delta\underline{\underline{M}}_1\right\} \in \mathcal{D}_{0\,1} \vee \forall\left\{\delta\underline{u}_2, \delta\underline{\underline{N}}_2, \delta\underline{\underline{M}}_2\right\} \in \mathcal{D}_{0\,2}
\end{aligned} \tag{28}$$

where the sum on the corners consider just the elements that share the same corner. The solution is researched in the form of a sum of shape functions

$$\underline{u} \approx \underline{u}_{VTCR}\left(\underline{x}_{rel}\right) = \sum_{i=1}^{m} a_i\underline{\hat{c}}_i e^{j\underline{k}_i'\underline{x}_{rel}} \tag{29}$$

where a_i are unknown parameters, $\underline{x}_{rel}$ is the position vector with respect of a certain point of the element (usually the geometric center) and $\underline{\hat{c}}_i$ and $\underline{k}_i$ are chosen so that equations (7) and (8) are identically satisfied. The substitution of (29) into the variational formulation using the same set of shape functions as test functions brings us to solve the set of linear equations

$$\underline{\underline{B}}\,\underline{a} = \underline{l} \tag{30}$$

where $\underline{\underline{B}} = \underline{\underline{B}}(\omega_i)$ is the square matrix associated to the bilinear form, $\underline{l} = \underline{l}(\omega_i)$ is linear form and $\underline{a} = \underline{a}(\omega_i)$ are the unknown parameters. We need to underline that terms in (30) must be calculated for every frequency step. In order to solve this set of set of equations the PGD technique is considered. The final solution is a matrix where the columns are the solution of (30). It is a rectangular matrix $\underline{\underline{A}} = \underline{\underline{A}}\left(\underline{\hat{k}}, \underline{\omega}\right) = [\underline{a}(\omega_1), \underline{a}(\omega_2), \ldots, \underline{a}(\omega_n)]$ where $\underline{\hat{k}}_i$ is the versor of the ray $\underline{k}_i$. In the same way the general linear form is $\underline{\underline{l}}$ and the bilinear form $\underline{\underline{\underline{B}}}$. We need to stress out that only the norm of $\underline{k}_i$ depends on the frequency ω_i. Therefore its versor is independent and the solution is uncoupled. For shallow shells membrane modes and bending ones are weakly coupled. Therefore they can be studied separately with just a corrective coupling term.

4 The PGD technique

Let us study the "best" p^{th}-order approximation POD-type on the frequency-direction domain $\Psi \times \Theta$ of $\underline{\underline{A}}\left(\underline{\hat{k}}, \underline{\omega}\right)$. The idea of PGD is to define the solution which minimizes the distance to the initial function $\underline{\underline{A}}\left(\underline{\hat{k}}, \underline{\omega}\right)$ with respect to a particular norm $\|\Box\|_{\Psi\times\Theta}$:

$$\underline{\underline{A}}\left(\underline{\hat{k}}, \underline{\omega}\right) = \sum_{i=1}^{p} \underline{\alpha}_i(\underline{\omega})\,\underline{\beta}_i\left(\underline{\hat{k}}\right) = \arg\left\{\min_{\underline{\alpha}_i(\underline{\omega}),\underline{\beta}_i\left(\underline{\hat{k}}\right)}\left\{\left\|\underline{\underline{l}}\left(\underline{\hat{k}}, \underline{\omega}\right) - \underline{\underline{\underline{B}}}\left(\underline{\hat{k}}, \underline{\omega}\right)\sum_{i=1}^{p} \underline{\alpha}_i(\underline{\omega})\,\underline{\beta}_i\left(\underline{\hat{k}}\right)\right\|^2_{\Psi\times\Theta}\right\}\right\}. \tag{31}$$

where $\underline{\alpha}_i(\underline{\omega})$ is an unknown function of the frequency only and $\underline{\beta}_i\left(\underline{\hat{k}}\right)$ is an unknown function of the direction only as in the usual POD technique. In literature there are some ways to deal with the resolution of this problem [2, 12, 13].

5 Numerical example

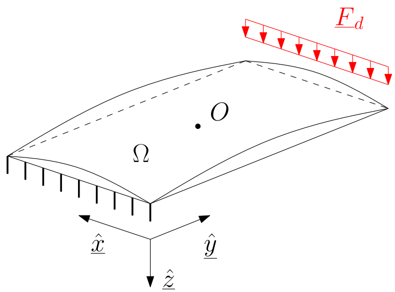

FIGURE 2 – Shallow shell example considered in Section 5.

The generic example that we want to solve is shown in Figure 2. It is a shallow shell subjected to a distributed load $\underline{F}_d$. The mechanical properties of this example are : $E = 75$ GPa, $\eta = 0.0001$, $\nu = 0.33$, $\varrho = 2750$ Kg/m, $[1600, 2000]$ Hz, $\underline{F}_d = [0, 0, 1]'$ N/m. At first the frequency is discretized and the VTCR is applied for each element $\omega_i \in \underline{\omega}$. The shape functions used are of the form

$$\underline{u} \approx \underline{u}_{VTCR}(\underline{x}_{rel}) = \sum_{n=1}^{N_r} a_n \underline{\hat{c}}_n (\omega_i)_n\, e^{jk_n(\omega_i)\underline{\hat{k}}'_n \underline{x}_{rel}} \tag{32}$$

where $\underline{x}_{rel}$ is the distance between a generic point P and a point O chosen in the element (usually it is the geometric center of the element), $\hat{\underline{k}}'_n$ is the versor of the generic wave vector.

After that the solution of the problem in the whole frequency-direction domain is searched using the PGD approach.

6 Conclusions

The proposed approach which is a fusion of the VTCR method with the PGD technique has been introduced to compute the frequency response of elastic shallow shells structures over a medium-frequency band. Since it is a very general approach the method seems to be well suited for solving general vibrational problems such as pyrotechnic shocks in spacecraft.

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