University of Helsinki

Master's thesis

A Pretentious Approach to Estimating Character Sums

Author:
Jesse JÄÄSAARI

Supervisor:
Dr. Anne-Maria ERNVALL-HYTÖNEN

November 26, 2013

HELSINGIN YLIOPISTO — HELSINGFORS UNIVERSITET — UNIVERSITY OF HELSINKI

Tiedekunta/Osasto — Fakultet/Sektion — Faculty	Laitos — Institution — Department
Matemaattis-luonnontieteellinen	Matematiikan ja tilastotieteen laitos

Tekijä — Författare — Author

Jesse Jääsaari

Työn nimi — Arbetets titel — Title

A Pretentious Approach to Estimating Character Sums

Oppiaine — Läroämne — Subject

Matematiikka

Työn laji — Arbetets art — Level	Aika — Datum — Month and year	Sivumäärä — Sidoantal — Number of pages
Pro gradu -tutkielma	Marraskuu 2013	85 s.

Tiivistelmä — Referat — Abstract

Vuonna 1837 Peter Dirichlet todisti suuren alkulukuja koskevan tuloksen, jonka mukaan jokainen aritmeettinen jono $\{an+d\}_{n=1}^{\infty}$, missä $(a,d)=1$, sisältää äärettömän monta alkulukua. Todistuksessa hän määritteli ns. Dirichlet'n karakterit joille löydettiin myöhemmin paljon käyttöä lukuteoriassa. Dirichlet'n karakteri $\chi \pmod q$ on jaksollinen (jakson pituutena q), täysin multiplikatiivinen aritmeettinen funktio, jolla on seuraava ominaisuus: $\chi(n)=0$ kun $(n,q)>1$ ja $\chi(n)\neq 0$ kun $(n,q)=1$. Tässä Pro Gradu-tutkielmassa tutkitaan karakterisumman

$$\mathcal{S}_\chi(t)=\sum_{n\le t}\chi(n)$$

kokoa, missä t on positiivinen reaaliluku ja $\chi \pmod q$ on ei-prinsipaali Dirichlet'n karakteri. Triviaalisti jaksollisuudesta seuraa, että $|\mathcal{S}_\chi(t)|\le \min(t,q)$. Ensimmäinen epätriviaali arvio on vuodelta 1918, jolloin George Pólya ja Ivan Vinogradov todistivat, toisistaan riippumatta, että $|\mathcal{S}_\chi(t)|\ll\sqrt{q}\log q$ uniformisti t:n suhteen. Tämä tunnetaan Pólya–Vinogradovin epäyhtälönä. Olettamalla yleistetyn Riemannin hypoteesin, Hugh Montgomery ja Robert Vaughan todistivat, että $|\mathcal{S}_\chi(t)|\ll\sqrt{q}\log\log q$ vuonna 1977. Vuonna 2005 Andrew Granville ja Kannan Soundararajan osoittivat, että jos $\chi \pmod q$ on paritonta rajoitettua kertalukua g oleva primitiivinen karakteri, niin

$$|\mathcal{S}_\chi(t)|\ll_g\sqrt{q}(\log Q)^{1-\frac{\delta_g}{2}+o(1)},\tag{1}$$

missä δ_g on g:stä riippuva vakio ja Q on q tai $(\log q)^{12}$ riippuen siitä oletetaanko yleistetty Riemannin hypoteesi. Todistus perustui teknisiin aputuloksiin, jotka saatiin muotoiltua teeskentelevyyskäsitteen avulla. Granville ja Soundararajan määrittelivät kahden multiplikatiivisen funktion, joiden arvot ovat yksikkökiekossa, välisen etäisyyden kaavalla

$$\mathbb{D}(f,g;x)=\sqrt{\sum_{p\le x}\frac{1-\Re(f(p)\overline{g}(p))}{p}},$$

ja sanoivat, että f on g-teeskentelevä jos $\mathbb{D}(f,g;\infty)$ on äärellinen. Tällä etäisyydellä on paljon hyödyllisiä ominaisuuksia, ja niihin perustuvia menetelmiä kutsutaan teeskentelevyys-menetelmiksi. Johdannon jälkeen luvussa 2 esitetään määritelmiä ja perustuloksia. Luvun 3 tarkoitus on johtaa luvussa 6 tarvittavia aputuloksia. Luvussa 4 määritellään teeskentelevyys, todistetaan etäisyysfunktion $\mathbb{D}(f,g;x)$ ominaisuuksia ja esitetään joitakin sovelluksia. Luvussa 5 johdetaan jälleen teknisiä aputuloksia, jotka seuraavat Montgomery–Vaughanin arviosta. Luvussa 6 tarkastellaan karakterisummia. Aloitamme todistamalla Pólya–Vinogradovin epäyhtälön ja Montgomery–Vaughanin vahvennoksen tälle. Päätuloksena johdamme arvion (1), jossa $\frac{1}{2}\delta_g$ on korvattu vakiolla δ_g. Tämän todisti alunperin Leo Goldmakher. Lopuksi käytämme teeskentelevyys-menetelmiä osoittamaan, että Pólya–Vinogradovin epäyhtälöä voi vahventaa jos karaktereista tehdään erilaisia oletuksia.

Avainsanat — Nyckelord — Keywords

Dirichlet'n karakterit, Eksponenttisummat, Multiplikatiiviset funktiot, Teeskentelevyys

Säilytyspaikka — Förvaringsställe — Where deposited

Kumpulan tiedekirjasto

Muita tietoja — Övriga uppgifter — Additional information

Contents

1 Introduction

In the late 1830s P. Dirichlet made a huge contribution to number theory by proving that every arithmetic progression $\{an + d\}_{n=1}^{\infty}$, with $(a, d) = 1$, contains infinitely many prime numbers. In the proof he introduced ideas that affect modern research even today. One of his ideas was to separate different residue classes using the so-called Dirichlet characters. In this thesis we concentrate, in particular, on their sums. A Dirichlet character $\chi \pmod q$ is a periodic, completely multiplicative arithmetic function that attains the value zero at any integer not coprime to the length of the period[1] q. As we will see, the definition implies that characters have absolute value at most one and so it follows that the absolute value of character sum up to t,

$$\mathcal{S}_\chi(t) := \sum_{n \le t} \chi(n),$$

is trivially bounded from above by $\min(t, q)$. This is known as the trivial estimate.

Character sums arise naturally in many problems in analytic number theory, for example when estimating the size of the least quadratic non-residue modulo p or bounding L-functions. When it comes to sizes of character sums, it is conjectured that for every non-principal Dirichlet character it holds for every $\varepsilon > 0$ that

$$|\mathcal{S}_\chi(t)| \ll_\varepsilon \sqrt{t} \cdot q^\varepsilon. \tag{1}$$

This is a consequence of the Generalised Riemann Hypothesis (the GRH) and currently it is known to be true when $t \gg q^{1-\varepsilon}$, see [3]. For now the bound (1) seems to be very far away but much work has been done to improve the trivial estimate. The first non-trivial result was proved in 1918 independently by G. Pólya [63] and I.M. Vinogradov. Their result is called the Pólya–Vinogradov inequality and it asserts that for any non-principal Dirichlet character $\chi \pmod q$,

$$|\mathcal{S}_\chi(t)| \ll \sqrt{q} \log q,$$

uniformly on t. Pólya's key idea was to expand $\mathcal{S}_\chi(t)$ as a Fourier series which is easy to analyse. After Pólya's result became known, I. Schur gave a different proof for the inequality [68]. Vinogradov found these two proofs independently in reverse order [78], [79].

Apart from reducing the implicit constant in the inequality, no major developments for the uniform bound occurred in the next several decades. Things started to move forward in 1977 when H.L. Montgomery and R. Vaughan [56] proved that, assuming the GRH, we have

$$|\mathcal{S}_\chi(t)| \ll \sqrt{q} \log\log q,$$

for non-principal characters uniformly on t. This is the best possible estimate, since in 1932 Paley [62] showed that there is an infinite family of quadratic characters whose sums are $\gg \sqrt{q} \log\log q$. It should also be mentioned that Paley's construction does not require the assumption that the GRH holds.

There are still some difficulties with these estimates. When $t \le \sqrt{q}$, the trivial bound is stronger than the bound given by the Pólya–Vinogradov inequality. In many applications we need non-trivial estimates for small t, and the trivial bound is far too loose. In 1957 D.A. Burgess [5] showed that $\mathcal{S}_\chi(t) = o(t)$ for quadratic characters for all $t > q^{\frac{1}{4}+o(1)}$, and later A. Hildebrand [42] proved that Burgess's estimates hold in fact when $t > q^{\frac{1}{4}-o(1)}$. Here $o(1) \to 0$ as $q \to \infty$. Burgess later refined his methods [6] – [11] to obtain a little better estimate in that range. Still, for $t > q^{\frac{5}{8}+o(1)}$ Burgess's result is

[1]This period q is called the modulus of the character χ.

weaker than the Pólya–Vinogradov inequality. When $t \leq q^{\frac{1}{4}-o(1)}$ there are no known nontrivial bounds, except in a few special cases.

H. Iwaniec [48] obtained a non-trivial bound, in the range $t > q^{\varepsilon}$, $\varepsilon > 0$, for $\mathcal{S}_\chi(t)$ when $\chi \pmod q$ is primitive and q has only "large" prime factors. This happened in 1974 and in 1990 S.W. Graham and C.J. Ringrose [20] bounded $\mathcal{S}_\chi(t)$ by q and t producing a new bound when q has "small" prime factors. After these improvements, no major contributions were made until the mid-1990s.

Things took an unexpected turn when A. Granville and K. Soundararajan were able to characterise the case when $\mathcal{S}_\chi(t)$ can be large [26]. Analysing Pólya's Fourier expansion it is evident that in order to estimate it, one needs to gain information about the size of the exponential sum

$$\sum_{1\leq |n| \leq N} \frac{\overline{\chi}(n)}{n} e(n\alpha),$$

where $n \in \mathbb{N}$ and $\alpha \in [0,1]$. Montgomery and Vaughan [56] had proved earlier a non-trivial estimate for this assuming that α lies on a minor arc (α admits a rational approximation with a "large" denominator). Granville and Soundararajan looked at the major arc case, and reduced the problem of estimating $\mathcal{S}_\chi(t)$ to bounding the sum

$$\sum_{1\leq |n| \leq N} \frac{\overline{\chi}(n)}{n} e\left(\frac{b}{r}\cdot n\right),$$

where $b \neq 0$ and $(b,r) = 1$. Their analysis showed that, for a small r, this can be large only when there exists a special Dirichlet character ψ for which

$$\sum_{\substack{n \leq \frac{N}{d} \\ n \in S(y)}} \frac{\overline{\chi}(n)\psi(n)}{n}$$

is large, where y is a positive integer. Intuitively this happens at least when $\chi(a) \approx \psi(a)$ for small values of a. Since characters are completely multiplicative functions, we would like to have $\chi(p) = \psi(p)$ for many small primes p. To quantify all this, Granville and Soundararajan introduced a distance between two multiplicative functions, f and g, that have values in the unit disc, defined as

$$\mathbb{D}(f,g;x) = \left(\sum_{p\leq x} \frac{1-\Re(\overline{f}(p)g(p))}{p}\right)^{\frac{1}{2}}.$$

Methods relying on the properties of $\mathbb{D}(f,g;x)$ are called the *pretentious methods.* They are useful also in other contexts outside characters, see e.g. the work of Koukoulopoulos [52], [53]. Besides that, Granville and Soundararajan tried to recover all the basic results from [4] and [14] by pretentious methods. They succeeded to some extent [22].

With some work Granville and Soundararajan [26] managed to improve the Pólya–Vinogradov inequality, both conditionally and unconditionally, for primitive characters of an odd bounded order. Most recently Soundararajan's student L. Goldmakher [33] has improved some of their results. He has also obtained [34] new upper bounds for characters with a specific modulus (similar to the work done by Iwaniec, Graham and Ringrose). Despite all these efforts, the Pólya–Vinogradov inequality remains as the strongest universal estimate for $\mathcal{S}_\chi(t)$ and the bound (1) is still out of reach.

The structure of the thesis is following. In the second chapter we define Dirichlet characters and discuss some of their properties as well as recall some facts from analytic number theory. The most important result found in that chapter is the deduction of Pólya's Fourier expansion.

In Chapter 3 we study more general theory, mean values of multiplicative functions with values inside the unit disc. We will obtain some important corollaries[2] of Halász's Theorem that are a vital ingredient in the proof of the Goldmakher–Granville–Soundararajan estimate in Chapter 6. For the sake of completeness, we also briefly discuss some historical aspects and mean value results in general.

In Chapter 4 we study the distance $\mathbb{D}(f, g; x)$, prove some of its properties and show how we can obtain nontrivial results concerning multiplicative functions with these tools. We conclude the chapter by giving a pretentious proof for the Prime Number Theorem. The main result of this chapter is Theorem 4.9 which gives a lower bound for the distance $\mathbb{D}(\chi(n), \xi(n)n^{it}; y)$. This is needed in Chapter 6 to finish the proof of the Goldmakher–Granville–Soundararajan estimate.

Chapters 5 and 6 contain the most essential part of this thesis. In Chapter 5 we study the exponential sums whose coefficients are multiplicative functions. In particular, we provide a proof for the classical Montgomery–Vaughan bound and obtain some intriguing corollaries that will come in handy later. In the beginning of the section there is a short recap on rational approximations.

Chapter 6 is all about character sums. We start by proving the Pólya–Vinogradov inequality, by adapting ideas from both Pólya [63] and Schur [68]. We also obtain a strengthening of the Pólya–Vinogradov inequality as was done in [56]. Our main goal is to reproduce the Goldmakher–Granville–Soundararajan Estimate[3]: if $\chi \pmod q$ is a primitive character of an odd order g, then

$$|\mathcal{S}_\chi(t)| \ll_g \sqrt{q}(\log Q)^{1-\delta_g+o(1)}$$

where δ_g is a constant depending on g, $o(1) \to 0$ as $g \to \infty$ and Q is $(\log q)^{12}$ when the GRH is assumed and otherwise it is just q.

We conclude by studying the special cases where the best known bounds for character sums are attained. We take a look at two situations. In the first case, we look at the situation in which the modulus q is smooth, i.e., it has only "small" prime factors. In the other case, we consider those characters that have a powerful modulus (the product of all the prime factors is small).

This thesis is not self-contained so we have to assume some prerequisites from the reader. We expect a background in both analytic and elementary number theory. Books [1] and [50] contain all the expected material, respectively. Also a good knowledge on Complex and Fourier analysis is highly recommended. All the required information can be found in [73] and [74]. However, some of the most frequently used results are recalled in Chapter 2, but its content is not sufficient alone.

The aim of this thesis is to present the pretentious methods and study their applications to character sums. Bounding the size of character sums is an interesting problem and we hope to offer a clear presentation on the recent pretentious approach to this subject. We will follow the literature quite closely, but various things are expanded, reformulated and reordered.

---

[2]Actually, the main results do not follow from Halász's Theorems, but rather the methods originated from their proofs.

[3]There is no such name in the literature. This name is from the author.

2 Preliminaries

In this section we present all the basic notations needed. We also recall some tools from analytic number theory that are required to understand the content of this thesis. A major part is devoted to studying our main objects: Dirichlet characters.

2.1 Notations

In general we use standard notations. The most frequently used notations are listed below in appropriate categories:

I. Abbreviations

• RH refers to the Riemann Hypothesis.

• GRH refers to the Generalised Riemann Hypothesis.

• PNT refers to the Prime Number Theorem.

II. Constants

• For a fixed integer $g \geq 3$ we define

$$\delta_g = 1 - \frac{g}{\pi} \sin\left(\frac{\pi}{g}\right).$$

• For a fixed $x \geq 3$ we define

$$s_x = 1 + \frac{1}{\log x}.$$

III. Estimates

• Let f and g be functions. The notation $f = o(g)$ means that for every $\varepsilon > 0$ there exists a constant N_ε such that $|f(n)| \leq \varepsilon\,|g(n)|$ for all $n \geq N_\varepsilon$.

• The notation $f = \mathcal{O}(g)$ means that $|f(z)| \leq C\,|g(z)|$ for some constant C for all values of the argument. Note that the domain of the definition of these functions can be $\mathbb{C}$. If the implicit constant depends on some parameter, say ε, we write $f = \mathcal{O}_\varepsilon(g)$.

• In many cases we will use Vinogradov's notation $f \ll g$ which is equivalent to the previous notation. Again, if the implicit constant depends on some parameter, say ε, we write $f \ll_\varepsilon g$.

• The notation $f \asymp g$ means that $f \ll g$ and $g \ll f$ simultaneously. Note that here the implicit constants usually differ.

• The notation $f \sim g$ means that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

IV. Functions

• The exponential function is $e(x) := e^{2\pi i x}$.

• The distance of $x \in \mathbb{R}$ to the nearest integer is $\|x\|$.

• The logarithm log is always the natural (e-based) logarithm.

• The floor function $\lfloor x \rfloor$ is the largest integer n satisfying $n \leq x$.

• The fractional part is defined as $\{x\} := x - \lfloor x \rfloor$.

• (a, b) and $[a, b]$ are the greatest common divisor and the least common multiple of integers a, b, respectively.

• Conductor of character χ is denoted by $\mathrm{cond}(\chi)$.

• The distance (up to $x \in \mathbb{R}_+$) between two multiplicative functions f and g with values in the unit disc is defined by

$$\mathbb{D}(f, g; x) = \sqrt{\sum_{p \leq x} \frac{1 - \Re(f(p)\overline{g}(p))}{p}}$$

• $M(f; x, T) = \min_{|t| \leq T} \mathbb{D}(f, n^{it}; x)^2$.

• $\varphi(n) = \{\text{Number of } 1 \leq k \leq n \text{ for which } (n, k) = 1\}$ is Euler's totient function.

• The number of positive divisors of a natural number n is denoted by $d(n)$.

• The radical of a positive integer q is

$$\mathrm{rad}(q) = \prod_{p|q} p.$$

• The prime counting functions are defined as $\pi(x) = \{\text{Number of primes } p \leq x\}$ and $\pi(x; q, a) = \{\text{Number of primes } p \equiv a \pmod{q} \text{ with } p \leq x\}$.

• Von Mangoldt's function Λ is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

• Chebyshev's θ-function is

$$\theta(x) = \sum_{p \leq x} \log p.$$

• Chebyshev's ψ-function is

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

• The indicator function of x is

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise,} \end{cases}$$

• Möbius function μ is defined as

$$\mu(n) = \begin{cases} (-1)^k & \text{if } \alpha_j = 1 \text{ for all } i = 1, ..., k, \\ 0 & \text{otherwise,} \end{cases}$$

where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$.

• Liouville function is

$$\lambda(n) = (-1)^{\Omega(n)},$$

where $\Omega(n)$ is the number of prime factors of n counted with multiplicities.

• The Legendre symbol is

$$\left(\frac{n}{p}\right) = \begin{cases} 1 & \text{if } n \text{ is a quadratic residue } \pmod p \text{ and } p \nmid n, \\ -1 & \text{if } n \text{ is a quadratic non-residue } \pmod p \text{ and } p \nmid n, \\ 0 & \text{if } p \mid n. \end{cases}$$

• The Kronecker symbol is defined as follows. Let $n = up_1^{\alpha_1} \cdots p_k^{\alpha_k}$ where $u = \pm 1$. Then

$$\left(\frac{a}{n}\right) = \left(\frac{a}{u}\right) \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{\alpha_i},$$

where $\left(\frac{a}{p_i}\right)$ is an usual Legendre symbol for $p_i > 2$,

$$\left(\frac{a}{2}\right) = \begin{cases} 0 & \text{if } 2|a \\ 1 & \text{if } a \equiv \pm 1 \pmod 8 \\ -1 & \text{if } a \equiv \pm 3 \pmod 8 \end{cases}$$

and finally

$$\left(\frac{a}{1}\right) = 1 \text{ and } \left(\frac{a}{-1}\right) = \begin{cases} -1 & \text{if } a < 0 \\ 1 & \text{if } a \geq 0 \end{cases}$$

• For a given $f : \mathbb{Z} \to \mathbb{C}$ and $y \in \mathbb{R}$ we define the y-smoothed function[4] as follows

$$f_y(n) = \begin{cases} f(n) & \text{if } n \in S(y) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that if f is completely multiplicative, then f_y is also.

V. Indices

• As is usual in analytic number theory, complex variables are denoted $s = \sigma + it$. Here $\sigma = \Re s$ and $t = \Im s$ are the real and imaginary parts of s, respectively.

• $\sum\limits_{n \leq N}$ means that $\sum\limits_{n=1}^{N}$.

• When p appears as an index (for example in $\sum\limits_{p \leq \ell}$ or $\prod\limits_{p \leq \ell}$), the sum or the product is taken over all the primes in the given range.

• The notation $\sum_{\chi \pmod q}$ means the sum over all characters modulo q.

• The sum over reduced residue classes modulo ℓ is denoted by $\sum_{n \pmod \ell}$.

[4] See y-smooth numbers below, VI. Sets.

VI. Sets

- The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ have their usual meanings.

- The closed unit disc will be denoted by $\mathbb{U}$.

- $\mathcal{F} := \{f : \mathbb{Z} \to \mathbb{U} \mid f(mn) = f(m)f(n) \text{ for all } m, n\}$ is the set of all completely multiplicative functions with values in $\mathbb{U}$. The set $\widetilde{\mathcal{F}}$ is the same set expect instead of complete multiplicity we only require f to be multiplicative.

- $\mathcal{S}(y) := \{n \in \mathbb{N} \mid p \leq y \text{ for every prime } p \mid n\}$ is the set of all natural numbers n with no prime divisors exceeding y. Elements belonging to this set are called *y-smooth* numbers.

- $\mu_\ell = \{e\left(\frac{n}{\ell}\right)\}_{n=0}^{\ell-1}$ is the set of the ℓ^{th} roots of unities.

2.2 Dirichlet Characters

In this subsection we define the Dirichlet characters and study some of their basic properties. We define them at first group theoretically and then give an equivalent number theoretic characterization. The latter one is used for the rest of the thesis. This Chapter is mainly based on the books [1] and [14]. We only mention the basic things concerning Dirichlet characters, more information on them can be found, for example, in [1], [14], [30] and [75].

Definition 2.1. A *Dirichlet character* $\chi \pmod q$ is a group homomorphism[5] χ: $(\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$.

The *principal character* modulo q, denoted by $\chi_0 \pmod q$, is the trivial homomorphism whose kernel is the entire group $(\mathbb{Z}/q\mathbb{Z})^*$. The induced modulus of a character is also an important concept:

Given a character $\chi \pmod q$, then $d \in \mathbb{N}$ is an *induced modulus* for χ if the projection map

$$\begin{aligned} \pi : (\mathbb{Z}/q\mathbb{Z})^* &\mapsto (\mathbb{Z}/d\mathbb{Z})^* \\ x + q\mathbb{Z} &\mapsto x + d\mathbb{Z} \end{aligned}$$

is well-defined and there exists a character $\chi' : (\mathbb{Z}/d\mathbb{Z})^* \to \mathbb{C}^*$ for which the following diagram commutes:

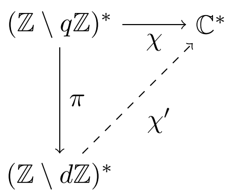

Figure 1. Diagram for the definition of an induced modulus

Now, by the properties of homomorphism, the definitions above can be written number theoretically.

Definition 2.2. A *Dirichlet character* $\chi \pmod q$ is a function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying the following properties:

1. The function χ is q-periodic: $\chi(n + q) = \chi(n)$ for all integers n.
2. If $(n, q) > 1$, then $\chi(n) = 0$. Otherwise $\chi(n) \neq 0$.
3. χ is a completely multiplicative function: $\chi(mn) = \chi(m)\chi(n)$ for all integers m and n.

From now on we simply refer to Dirichlet characters as *characters*. The integer q in the first property is called the *modulus* of the character χ; we will write $\chi \pmod q$. Several properties can be deduced straight from the definition. Property 3 tells that $\chi(1) = \chi(1)\chi(1)$. As we have $(1, n) = 1$ for all $n \in \mathbb{N}$ and by property 2 $\chi(1) \neq 0$, we

[5]Here $(\mathbb{Z}/q\mathbb{Z})^*$ is the group of coprime residue classes modulo q and $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ is the multiplicative group of $\mathbb{C}$.

conclude that $\chi(1) = 1$. We remark that sometimes it is useful to consider property 1 in the form: if $a \equiv b \pmod q$ then $\chi(a) = \chi(b)$.

If (a, q)= 1, then by Euler's Theorem we have $a^{\varphi(q)} \equiv 1 \pmod q$. Thus, by the previously established properties, we have $\chi(a^{\varphi(q)}) = \chi(1) = 1$ and on the other hand by multiplicativity $\chi(a^{\varphi(q)}) = \chi(a)^{\varphi(q)}$. In particular, this means that if n is relatively prime to q, then $\chi(n)$ is $\varphi(q)^{th}$ root of unity. Furthermore, $|\chi(n)|$ is always either 0 or 1. It can also be shown that there are $\varphi(q)$ characters modulo q. For the proof see Theorem 6.15. in [14].

Example 2.3. There are $\varphi(7) = 6$ different characters modulo 7. If we denote $\omega = \exp\left(\frac{\pi i}{3}\right)$ they are

χ/n	0	1	2	3	4	5	6
$\chi_0(n)$	0	1	1	1	1	1	1
$\chi_1(n)$	0	1	ω^2	ω	$-\omega$	$-\omega^2$	-1
$\chi_2(n)$	0	1	$-\omega$	ω^2	ω^2	$-\omega$	1
$\chi_3(n)$	0	1	1	-1	1	-1	-1
$\chi_4(n)$	0	1	ω^2	$-\omega$	$-\omega$	ω^2	1
$\chi_5(n)$	0	1	$-\omega$	$-\omega^2$	ω^2	ω	-1

Table 1. Characters modulo 7. Note that there are two real characters, χ_0 and χ_3.

Next we discuss different types of characters. The *trivial character* is the unique character with period one. By definition this character attains the value one for all the arguments and so it has modulus one. When we speak about characters, the trivial character is excluded unless stated otherwise. A character that assumes the value one at all arguments coprime to the modulus and zero otherwise is called the *principal character*, $\chi_0 \pmod q$.

Now we present a couple of more definitions. The concept of *sign* of the character is important in Chapters 2 and 6. A character χ is called *even* if $\chi(-1) = 1$. Similarly, χ is an *odd character* if $\chi(-1) = -1$. Considering characters modulo 7, Example 2.3. tells that characters χ_0, χ_2, χ_4 are even and χ_1, χ_3, χ_5 are odd. The *order* of the character χ is the smallest positive integer ν for which $\chi^\nu = \chi_0$. For instance, the principal character χ_0 is of order one and clearly the character $\chi_3 \pmod 7$ in the Example 2.3. is of order two. A character is *real* if it assumes only real values. A nice example of a real character is the Kronecker symbol. The proof for this can be found in [30].

The previous notation of an induced modulus translates as follows: Let $\chi \pmod q$ be a character and d any positive divisor of q. The number d is called an *induced modulus* for χ if $\chi(a) = 1$ whenever $(a, q) = 1$ and $a \equiv 1 \pmod d$. The smallest induced modulus d for χ is called the *conductor* of χ. The character is *primitive* if its conductor is q. An example of a primitive character modulo an odd prime is the Legendre symbol, which is also called the quadratic character. Now we study the concept of induced modulus a little closer.

Lemma 2.4. Let $\chi \pmod q$ be a character and assume that $d \mid q$. Then d is an induced modulus for χ if and only if

$$\chi(a) = \chi(b), \text{ whenever } (a, q) = (b, q) = 1 \text{ and } a \equiv b \pmod d.$$

Proof. "If"-part follows by choosing $b = 1$ and using the definition of induced modulus. To prove the "only if"-part we choose a and b such that $(a, q) = (b, q) = 1$ and $a \equiv b \pmod d$. By the choice of a there exists a reciprocal of a, denoted a', such that $aa' \equiv 1$

(mod q). Since d divides q we have $aa' \equiv 1 \pmod{d}$. Since d is an induced modulus we have $\chi(aa') = 1$. On the other hand, $a \equiv b \pmod{d}$ gives that $aa' \equiv ba' \equiv 1 \pmod{d}$ meaning that $\chi(aa') = \chi(ba')$. Thus we have

$$\chi(a)\chi(a') = \chi(b)\chi(a').$$

Since $\chi(a') \neq 0$ we get the claim. □

The following theorem gives a useful characterization for induced modulus:

Theorem 2.5. Let $\chi \pmod{q}$ be a character and assume that $d \mid q$. Then the following statements are equivalent:

1. Number d is an induced modulus for χ.

2. There is a character $\psi \pmod{d}$ such that $\chi(n) = \psi(n)\chi_0(n)$ for all n, where χ_0 is the principal character modulo q.

Proof. Assume that 2. is true. Choose a natural number n such that $(n, q) = 1$ and $n \equiv 1 \pmod{d}$. Then $\chi_0(n) = \psi(n) = 1$ as $\psi(1) = 1$. Thus $\chi(n) = 1$ and hence d is an induced modulus. So 1. follows.
Assume that 1. is true. We define the character ψ as follows: if $(n, d) > 1$, set $\psi(n) = 0$. If $(n, d) = 1$ then, by Dirichlet's Theorem on primes in arithmetic progressions (see Theorem 2.14. below), there exists an integer m such that $m \equiv n \pmod{d}$ and $(m, q) = 1$. This m is unique modulo d and we can define $\psi(n) = \chi(m)$. Next we notice that ψ is a well-defined character modulo d. All the properties of Definition 2.2 are straightforwardly verified by using Lemma 2.4. Now we check that the equation $\chi(n) = \psi(n)\chi_0(n)$ holds for all n.

If $(n, q) = 1$, then $(n, d) = 1$ so $\psi(n) = \chi(m)$ for some $m \equiv n \pmod{d}$. Hence, by Lemma 2.4,

$$\chi(n) = \chi(m) = \psi(n) = \psi(n)\chi_0(n), \tag{2}$$

since $\chi_0(n) = 1$. When $(n, q) > 1$, we have $\chi(n) = \chi_0(n) = 0$ and the equation (2) still holds. This finishes the proof. □

As a consequence (for the detailed argument see [1], Theorem 8.18) we get that every character $\chi \pmod{q}$ can be expressed as a product

$$\chi(n) = \psi(n)\chi_0(n) \text{ for all } n,$$

where χ_0 is the principal character modulo q and ψ is a primitive character modulo the conductor of χ.

Characters also possess the following four identities, which are known as the *orthogonality relations*:

$$\frac{1}{\varphi(q)} \sum_{a=1}^{q} \chi(a) = \begin{cases} 1 & \text{if } \chi = \chi_0 \pmod{q}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \chi(a) = \begin{cases} 1 & \text{if } a \equiv 1 \pmod{q}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\frac{1}{\varphi(q)} \sum_{a=1}^{q} \chi(a)\overline{\psi}(a) = \begin{cases} 1 & \text{if } \chi = \psi, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\frac{1}{\varphi(q)}\sum_{\chi \pmod q}\chi(a)\overline{\chi}(b)=\begin{cases}1 & \text{if } a\equiv b \pmod q \text{ and } (a,q)=1,\\ 0 & \text{otherwise.}\end{cases} \qquad (3)$$

The proofs for all of these are straightforward manipulations. For the details, see [75].

2.3 Dirichlet L-series

L-functions associated to characters are closely connected to the theory of character sums. Here we recall few facts about them. We start by defining the L-function:

Definition 2.6. Let χ be any Dirichlet character. For any $s \in \mathbb{C}$ with $\Re s > 1$, the Dirichlet *L-series* associated to χ is

$$L(s,\chi) := \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

This can be extended to a meromorphic function on the whole complex plane by analytic continuation. In this case $L(s,\chi)$ is called a Dirichlet *L-function.* A basic fact concerning $L(s,\chi)$ is that it can be written as an Euler product

$$L(s,\chi)=\prod_p \frac{1}{1-\chi(p)p^{-s}}.$$

When dealing with character sums, we will notice that several estimates can be made much sharper by assuming the Generalised Riemann hypothesis (The GRH). The classical Riemann hypothesis (The RH) claims that all the non-trivial zeroes of the Riemann zeta-function lie on the critical line $\{\sigma = \frac{1}{2}\}$. The GRH states the following:

Hypothesis 2.7. (The GRH) Let χ be a Dirichlet character. Consider the L-series $L(s,\chi)$ for $\Re s > 1$ and extend it to a meromorphic function to the whole complex plane. Assume that $L(\chi, s) = 0$ with $0 < \sigma < 1$. Then $\sigma = \frac{1}{2}$.

Note that this is more general than the RH, which is just a special case where χ is the trivial character.

How sums of characters are related to L-functions is seen in the following calculation. Recall that the character sum up to $t \in \mathbb{R}$ is defined as

$$\mathcal{S}_\chi(t)=\sum_{n\le t}\chi(n).$$

Note that if $s > 1$, partial summation (Theorem 2.10.) gives

$$\begin{aligned} L(s,\chi) &= \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}\\ &= \int_{1^-}^{\infty}\frac{1}{t^s}dS_\chi(t)\\ &= \Big/_{1^-}^{\infty}\frac{S_\chi(t)}{t^s}+s\int_1^{\infty}\frac{S_\chi(t)}{t^{s+1}}dt\\ &= s\int_1^{\infty}\frac{S_\chi(t)}{t^{s+1}}dt. \end{aligned}$$

Later we obtain a version of Pólya's quantitative Fourier expansion containing an L-function and we use this to derive a bound for $\mathcal{S}_\chi(t)$ under the GRH.

In his original work [17], Dirichlet proved the deep fact that the value $L(1,\chi)$ is non-zero. This plays a central part in the proof of Dirichlet's Theorem on primes in arithmetic progressions. However, we do not need this result in this thesis. For this reason we leave it to the reader to check the proof from the reference mentioned above.

Finally we mention the celebrated result of J.E. Littlewood from 1918 that gives a good estimate for an L-function at argument one under the assumption of the GRH. This theorem will be used in Subsection 6.1 to give an improvement of the Pólya–Vinogradov inequality under the GRH. For the proof see the original article [55].

Theorem 2.8. (Littlewood bound) Let $\chi \pmod q$ be a character and assume that the GRH holds. Then

$$L(1,\chi) \ll \log\log q.$$

Apart from the exact value of the constant, this is the best possible result since S.D. Chowla [12] showed in 1947 that there exist arbitrarily large numbers q and characters $\chi \pmod q$ for which $L(1,\chi) \gg \log\log q$.

2.4 Pólya's Fourier Expansion

Now we obtain a Fourier expansion for a primitive character modulo q as Pólya did in [63]. But let us first introduce the useful concept of a Gauss sum.

The Gauss sum associated to a character $\chi \pmod q$ is defined as

$$\tau(\chi) := \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right).$$

Let us remark that Gauss sum enjoys a nice property that it can be written as

$$\tau(\psi) = \begin{cases} \psi'\left(\frac{q}{q'}\right)\mu\left(\frac{q}{q'}\right)\tau(\psi') & \text{if } \left(\frac{q}{q'}, q'\right) = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where $\psi' \pmod{q'}$ is a character inducing $\psi \pmod q$. This is proved in [14], p.67.

Let $\chi \pmod q$ be primitive for a while. For every n such that $(n,q)=1$, let $\overline{n}$ be the multiplicative inverse of n modulo q i.e., integer $\overline{n}$ is chosen so that $n\overline{n} \equiv 1 \pmod q$. Then we have

$$\chi(n)\tau(\overline{\chi}) = \chi(n)\sum_{m=1}^{q} \overline{\chi}(m) e\left(\frac{m}{q}\right) = \sum_{m=1}^{q} \overline{\chi}(\overline{n}m) e\left(\frac{m}{q}\right) = \sum_{h=1}^{q} \overline{\chi}(h) e\left(\frac{hn}{q}\right). \tag{5}$$

Dividing by $\tau(\overline{\chi})$ we get that

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{h=1}^{q} \overline{\chi}(h) e\left(\frac{hn}{q}\right) \tag{6}$$

Division by $\tau(\overline{\chi})$ is justified since for primitive $\chi \pmod q$, we have $|\tau(\chi)| = \sqrt{q}$. This follows from a simple calculation: at first we note that identity (5) implies that

$$|\chi(n)|^2 |\tau(\overline{\chi})|^2 = \sum_{h=1}^{q} \sum_{h'=1}^{q} \overline{\chi}(h)\chi(h') e\left(\frac{n(h-h')}{q}\right).$$

Now summing over $n = 1, ..., q$, using the fact that the sum of $|\chi(n)|^2$ over these numbers is $\varphi(q)$, and that the sum of the exponentials is 0 unless $h \equiv h' \pmod q$, we get

$$\varphi(q)|\tau(\overline{\chi})|^2 = q\sum_{h=1}^{q}\overline{\chi}(h)\chi(h) = q\varphi(q),$$

which is what we wanted.

Let us then show that equation (6) holds also when $d := (n, q) > 1$. In this case $\chi(n) = 0$ and thus we need to prove that

$$\sum_{h=1}^{q}\overline{\chi}(h)e\left(\frac{hn}{q}\right) = 0.$$

Set $q = q'd$. Let us consider the numbers $h = q' \cdot a + b$, where $a = 0, 1, ..., d-1$ and $b = 0, 1, ..., q'-1$. When a and b run over all possible combinations, h runs over the set $\{0, 1..., q-1\}$. Hence

$$\begin{aligned}\sum_{h=1}^{q}\overline{\chi}(h)e\left(\frac{hn}{q}\right) &= \sum_{b=0}^{q'-1}\sum_{a=0}^{d-1}\overline{\chi}(q'a+b)e\left(\frac{(q'a+b)n}{q}\right)\\ &= \sum_{b=0}^{q'-1}e\left(\frac{bn}{q}\right)\sum_{a=0}^{d-1}\overline{\chi}(q'a+b),\end{aligned}$$

which holds since $\frac{q'an}{q}$ is an integer.

We are done if we show that for fixed q' the identity

$$\sum_{a=0}^{d-1}\overline{\chi}(q'a+b) = 0, \tag{7}$$

holds for any integer b. Considering the left-hand side of (7) as a function of b we see that it is q'-periodic. This is seen as follows. When b is replaced with $b + q'$, then the situation is the same as the range of a changes from $0 \le a \le d-1$ to $1 \le a \le d$. But this does not change anything, since $\overline{\chi}(b) = \overline{\chi}(q'd+b)$. Now, we choose a number c such that $(c, q) = 1$ and $c \equiv 1 \pmod{q'}$. Then, by using the q'-periodicity,

$$\overline{\chi}(c)\sum_{a=0}^{d-1}\overline{\chi}(q'a+b) = \sum_{a=0}^{d-1}\overline{\chi}(cq'a+cb) = \sum_{a=0}^{d-1}\overline{\chi}(aq'+cb) = \sum_{a=0}^{d-1}\overline{\chi}(q'a+b). \tag{8}$$

In Chapter 5 of [14] it is proved that for a primitive $\chi \pmod q$, the function $\chi(n)$ is not q'-periodic for any proper divisor q' of q whenever $(n, q) > 1$. This implies that there exists integers c_1 and c_2 such that $(c_1, q) = (c_2, q) = 1$, $c_1 \equiv c_2 \pmod{q'}$ and $\chi(c_1) \neq \chi(c_2)$. Hence there exists an integer $c \equiv c_1c_2^{-1} \pmod{q'}$ such that $\overline{\chi}(c) \neq 1$ and that has properties $c \equiv 1 \pmod{q'}$, $(c, q) = 1$. Also, clearly $\overline{\chi}(c) \neq 0$. Choosing this c in the equation (8) gives the identity (7).

This shows that the equation (6) is also valid when $(n, q) > 1$. Equation (6) is known as a *finite Fourier expansion* of the character χ.

Let us then proceed to obtaining the quantitative Fourier Expansion for $\mathcal{S}_\chi(t)$. Let t be an integer such that $1 \le t \le q-1$. Write

$$\frac{1}{2}\chi(1) + \chi(2) + \cdots + \chi(t-1) + \frac{1}{2}\chi(t) = \sum_{j=1}^{q}\Phi\left(\frac{2\pi j}{q}\right)\chi(j),$$

where the function $\Phi : [0, 2\pi] \to \mathbb{R}$ is defined as

$$\Phi(x) = \begin{cases} 1 & \text{if } \frac{2\pi}{q} < x < \frac{2\pi t}{q} \\ \frac{1}{2} & \text{if } x = \frac{2\pi}{q},\ x = \frac{2\pi t}{q} \\ 0 & \text{if } 0 \le x < \frac{2\pi}{q},\ \frac{2\pi t}{q} < x \le 2\pi \end{cases}$$

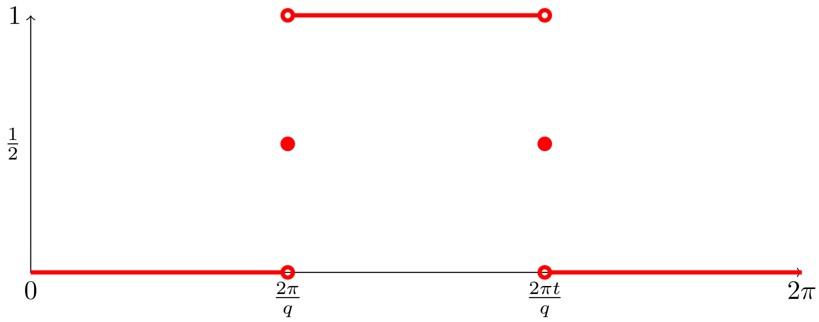

Figure 2. Graph of the function $\Phi(x)$.

Since Φ can be extended to a 2π-periodic function to the whole real line by using periodicity, it can be represented as a Fourier series

$$a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx).$$

A straightforward calculation shows that the Fourier coefficients are

$$a_0 = \frac{t-1}{q}, \qquad a_m = \frac{\sin \frac{2\pi m t}{q} - \sin \frac{2\pi m}{q}}{\pi m} \text{ for } m \ge 1$$

and

$$b_m = -\frac{\cos \frac{2\pi m t}{q} - \cos \frac{2\pi m}{q}}{\pi m} \text{ for all } m \in \mathbb{N}.$$

Therefore, using the formulas

$$\sin\theta \cos\phi = \frac{\sin(\theta+\phi) + \sin(\theta-\phi)}{2}$$

and $\sin(-\theta) = -\sin\theta$, we may write Φ as a Fourier series

$$\begin{aligned} \Phi(x) &= \frac{t-1}{q} + \sum_{m=1}^{\infty} \left(\frac{\sin \frac{2\pi m t}{q} - \sin \frac{2\pi m}{q}}{\pi m} \cos mx - \frac{\cos \frac{2\pi m t}{q} - \cos \frac{2\pi m}{q}}{\pi m} \sin mx \right) \\ &= \frac{t-1}{q} + \sum_{m=1}^{\infty} \frac{1}{\pi m} \left(\sin\left(m \cdot \left(x - \frac{2\pi}{q} \right) \right) - \sin\left(m \cdot \left(x - \frac{2\pi t}{q} \right) \right) \right) \\ &= \frac{t-1}{q} + \frac{1}{\pi} T\left(x - \frac{2\pi}{q} \right) - \frac{1}{\pi} T\left(x - \frac{2\pi t}{q} \right), \end{aligned} \tag{9}$$

where

$$T(y) = \sum_{n=1}^{\infty} \frac{\sin ny}{n}$$

Notice that for a fixed $k \in \mathbb{N}$ we can write the partial sum $T_k(y)$ as

$$\begin{aligned} T_k(y) := \sum_{n=1}^{k} \frac{\sin ny}{n} &= -\int_y^{\pi} \left(\sum_{n=1}^{k} \cos ns \right) ds \\ &= \frac{\pi - y}{2} - \frac{1}{2} \Re \int_y^{\pi} \left(\sum_{n=-k}^{k} e^{ins} \right) ds \\ &= \frac{\pi - y}{2} - \frac{1}{2} \Re \int_y^{\pi} \left(\frac{e^{-i\frac{s}{2}}}{e^{-i\frac{s}{2}}} \cdot \frac{e^{-iks}(e^{(2k+1)is} - 1)}{e^{is} - 1} \right) ds \\ &= \frac{\pi - y}{2} - \int_y^{\pi} \frac{\sin \frac{2k+1}{2} s}{2 \sin \frac{s}{2}} ds \end{aligned}$$

when $y \in]0, \pi]$. Applying the Second Mean Value Theorem for Integration we find $\xi \in [y, \pi]$ so that

$$\begin{aligned} \left| \int_y^{\pi} \frac{\sin \frac{2k+1}{2} s}{2 \sin \frac{s}{2}} ds \right| &= \left| \frac{1}{2 \sin \frac{y}{2}} \int_y^{\xi} \sin \frac{2k+1}{2} s ds + \frac{1}{2 \sin \frac{\pi}{2}} \int_{\xi}^{\pi} \sin \frac{2k+1}{2} s ds \right| \\ &= \left| \frac{1}{2 \sin \frac{y}{2}} \left(\Big/_y^{\xi} \frac{2 \cos \left(\left(k + \frac{1}{2}\right) s \right)}{2k+1} + \Big/_{\xi}^{\pi} \frac{2 \cos \left(\left(k + \frac{1}{2}\right) s \right)}{2k+1} \right) \right| \\ &\leq \frac{4}{2k+1} \cdot \frac{1}{2 \sin \frac{y}{2}} \\ &= \frac{4}{(2k+1)y} \cdot \frac{y}{2 \sin \frac{y}{2}} \\ &< \frac{\pi}{ky}. \end{aligned}$$

In the last estimate we used the well-known inequality $\sin w \geq \frac{2w}{\pi}$, which holds for $w \in [0, \frac{\pi}{2}]$.

Let us denote

$$R_k(y) = \sum_{m=1}^{\infty} \frac{\sin((k+m)y)}{k+m}$$

In this notation $T(y) = T_k(y) + R_k(y)$ for every $k \in \mathbb{N}$. The above calculation shows that, for a fixed $k \in \mathbb{N}$, $|R_k(y)| < \frac{\pi}{ky}$ for all $0 < y \leq \pi$. Furthermore, we have $R_k(0) = R_k(\pi) = 0$ and $R_k(2\pi - y) = -R_k(y)$. From these properties it follows that

$$\sum_{n=1}^{q} \left| R_k \left(\frac{2\pi n}{q} \right) \right| = 2 \sum_{1 \leq n < \frac{q}{2}} \left| R_k \left(\frac{2\pi n}{q} \right) \right| < 2 \sum_{1 \leq n < \frac{q}{2}} \frac{\pi}{k} \cdot \frac{q}{2\pi n} < \frac{q \log q}{k} \tag{10}$$

for every $q \geq 3$. From (6) we deduce that if $\chi(-1) = 1$,

$$\begin{aligned} \sum_{j=1}^{q} \chi(j) \left(\cos \frac{2\pi mj}{q} + i \sin \frac{2\pi mj}{q} \right) &= \overline{\chi}(m) \tau(\chi) \\ &= \overline{\chi}(-m) \tau(\chi) = \sum_{j=1}^{q} \chi(j) \left(\cos \frac{2\pi mj}{q} - i \sin \frac{2\pi mj}{q} \right) \end{aligned}$$

so

$$\sum_{j=1}^{q}\chi(j)\sin\frac{2\pi mj}{q}=0 \quad \text{and} \quad \sum_{j=1}^{q}\chi(j)\cos\frac{2\pi mj}{q} = \overline{\chi}(m)\tau(\chi)-\sum_{j=1}^{q}\chi(j)\sin\frac{2\pi mj}{q} = \overline{\chi}(m)\tau(\chi).$$

Similarly, if $\chi(-1)=-1$,

$$\begin{aligned}\sum_{j=1}^{q}\chi(j)\left(\cos\frac{2\pi mj}{q}+i\sin\frac{2\pi mj}{q}\right) &= \overline{\chi}(m)\tau(\chi)\\ &= -\overline{\chi}(-m)\tau(\chi)\\ &= \sum_{j=1}^{q}\chi(j)\left(-\cos\frac{2\pi mj}{q}+i\sin\frac{2\pi mj}{q}\right)\end{aligned}$$

so

$$\sum_{j=1}^{q}\chi(j)\cos\frac{2\pi mj}{q}=0 \quad \text{and} \quad \sum_{j=1}^{q}\chi(j)\sin\frac{2\pi mj}{q} = \frac{1}{i}\left(\overline{\chi}(m)\tau(\chi)-\sum_{j=1}^{q}\chi(j)\cos\frac{2\pi mj}{q}\right) = \frac{\overline{\chi}(m)\tau(\chi)}{i}.$$

Using these identities we get

$$\begin{aligned}&\frac{1}{2}\chi(1)+\chi(2)+...+\chi(t-1)+\frac{1}{2}\chi(t)\\ =\;& \sum_{j=1}^{q}\left(a_0+\left(\sum_{m=1}^{\infty}a_m\cos\frac{2\pi mj}{q}+b_m\sin\frac{2\pi mj}{q}\right)\right)\chi(j)\\ =\;& a_0\underbrace{\sum_{j=1}^{q}\chi(j)}_{=0}+\sum_{m=1}^{\infty}\left(\sum_{j=1}^{q}\left(a_m\cos\frac{2\pi mj}{q}+b_m\sin\frac{2\pi mj}{q}\right)\chi(j)\right)\\ =\;& \begin{cases}\tau(\chi)\sum_{m=1}^{\infty}a_m\overline{\chi}(m) & \text{if } \chi(-1)=1\\ \frac{\tau(\chi)}{i}\sum_{m=1}^{\infty}b_m\overline{\chi}(m) & \text{if } \chi(-1)=-1\end{cases}\end{aligned}$$

Combining this with (9) we deduce immediately that for every fixed $k\in\mathbb{N}$, the representation

$$\begin{aligned}\mathcal{S}_\chi(t) &= \frac{1}{2}(\chi(1)+\chi(t))+\sum_{j=1}^{q}\left(a_0+\frac{1}{\pi}\left(T\left(\frac{2\pi(j-1)}{q}\right)-T\left(\frac{2\pi(j-t)}{q}\right)\right)\right)\chi(j)\\ &= \frac{1}{2}(\chi(1)+\chi(t))+\sum_{m=1}^{k}\left(\sum_{j=1}^{q}\left(a_m\cos\frac{2\pi mj}{q}+b_m\sin\frac{2\pi mj}{q}\right)\chi(j)\right)\\ &+ \frac{1}{\pi}\sum_{j=1}^{q}\left(R_k\left(\frac{2\pi(j-1)}{q}\right)-R_k\left(\frac{2\pi(j-t)}{q}\right)\right)\chi(j)\\ &= \frac{1}{2}(\chi(1)+\chi(t))+\tau(\chi)\sum_{m=1}^{k}\ell_m\overline{\chi}(m)\\ &+ \frac{1}{\pi}\sum_{j=1}^{q}\left(R_k\left(\frac{2\pi(j-1)}{q}\right)-R_k\left(\frac{2\pi(j-t)}{q}\right)\right)\chi(j), \qquad (11)\end{aligned}$$

where

$$\ell_m = \begin{cases} a_m & \text{if } \chi(-1) = 1 \\ -ib_m & \text{if } \chi(-1) = -1, \end{cases}$$

holds. The first term is $\ll 1$ and the last term is $\ll \frac{q \log q}{k}$ for every natural number k because of (10).

If $\chi(-1) = 1$, then $\overline{\chi}(-m) = \chi(-1)\overline{\chi}(m) = \overline{\chi}(m)$ and we can calculate

$$\mathcal{S}_\chi(t) = \frac{\tau(\chi)}{2\pi i} \sum_{m=1}^{k} \frac{2\pi i}{\pi m} \left(\frac{e\left(\frac{mt}{q}\right) - e\left(-\frac{mt}{q}\right) - e\left(\frac{m}{q}\right) + e\left(-\frac{m}{q}\right)}{2i} \right) \overline{\chi}(m) + \mathcal{O}\left(1 + \frac{q \log q}{k}\right) \tag{12}$$

The contribution coming from the terms $e\left(\pm\frac{m}{q}\right)$ can be included in the error term. The proof is omitted, since the author could not verify it. Apparently this is implicitly contained in [63]. On the page 12 in his PhD-Thesis, Goldmakher claims that this is done in [58]. However, the author did not find it there. So, we will take the above fact concerning the terms $e\left(\pm\frac{m}{q}\right)$ for granted. Many researchers state (see [26], [44] and [56] for instance) that Polyá derived the quantitative Fourier expansion (13) in [63], although only the expansion (12) is obtained there.

Moving back to the proof, we can continue our calculation from (12):

$$\begin{aligned} |\mathcal{S}_\chi(t)| &= \frac{\tau(\chi)}{2\pi i} \sum_{m=1}^{k} \frac{\overline{\chi}(m)}{m} \left(\left(1 - e\left(-\frac{mt}{q}\right)\right) - \left(1 - e\left(\frac{mt}{q}\right)\right)\right) + \mathcal{O}\left(\frac{q \log q}{k}\right) \\ &= \frac{\tau(\chi)}{2\pi i} \sum_{1 \le |m| \le k} \frac{\overline{\chi}(m)}{m} \left(1 - e\left(-\frac{mt}{q}\right)\right) + \mathcal{O}\left(1 + \frac{q \log q}{k}\right). \end{aligned}$$

Similarly, if $\chi(-1) = -1$ we get

$$\begin{aligned} \mathcal{S}_\chi(t) &= \frac{\tau(\chi)}{2\pi i} \sum_{m=1}^{k} \frac{2\pi i^2}{\pi m} \left(\frac{e\left(\frac{mt}{q}\right) + e\left(-\frac{mt}{q}\right) - e\left(\frac{m}{q}\right) - e\left(-\frac{m}{q}\right)}{2} \right) \overline{\chi}(m) + \mathcal{O}\left(1 + \frac{q \log q}{k}\right) \\ &= \frac{\tau(\chi)}{2\pi i} \sum_{1 \le |m| \le k} \frac{\overline{\chi}(m)}{m} \left(1 - e\left(-\frac{mt}{q}\right)\right) + \mathcal{O}\left(1 + \frac{q \log q}{k}\right). \end{aligned}$$

So we have obtained the following result:

Theorem 2.9. Let $\chi \pmod{q}$ be a primitive character, $1 \le t \le q$ and k be a natural number. Then

$$\mathcal{S}_\chi(t) = \frac{\tau(\chi)}{2\pi i} \sum_{\substack{m \in \mathbb{Z} \\ 1 \le |m| \le k}} \frac{\overline{\chi}(m)}{m} \left(1 - e\left(-\frac{mt}{q}\right)\right) + \mathcal{O}\left(1 + \frac{q \log q}{k}\right). \tag{13}$$

This is known as *Pólya's quantitative Fourier expansion*.

2.5 Results from Analytic Number Theory

Next we present some important methods and results from analytic number theory. Although we expect the reader to have a strong knowledge of this field, we recall some theory that is considered vital for the most of the proofs in this thesis. Our first result is the partial summation formula which is one of the most fundamental tools in this field. The proof can be found in any book dealing with basic analytic number theory, e.g. [1].

Theorem 2.10. (Partial summation) Let $0 \leq \lambda_1 \leq \lambda_2 \leq ...$ be a divergent sequence of real numbers. Let n be a natural number and $x > 1$ a real number. Then for any continuous piecewise continuously differentiable function f, and a sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$ one has

$$\sum_{\lambda_n \leq x} a_n f(\lambda_n) = A(x)f(x) - \int_{\lambda_1}^{x} A(u)f'(u)du,$$

where $A(x) := \sum_{\lambda_n \leq x} a_n$.

The next theorem is very well-known:

Theorem 2.11. (The Prime Number Theorem) Let $\pi(x)$ be the number of primes p which are less or equal to x. Then

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

We postpone the proof of this till the end of Chapter 4. Next we will move to theorems concerning the distribution of primes in arithmetic progressions. For the proofs of these classical theorems, see [14, 40].

Theorem 2.12. (Brun–Titchmarsh) Let $\pi(x;q,a)$ be the number of primes $p \equiv a \pmod q$ with $p \leq x$. Then we have

$$\pi(x;q,a) \leq \frac{x}{\varphi(q)\log\sqrt{\frac{x}{q}}} \text{ for all } q \leq x.$$

Besides that, we will use the corollary stating that the number of primes $\equiv a \pmod q$ lying on the interval $[x, x+y]$ is

$$\pi(x+y;q,a) - \pi(x;q,a) \ll \frac{y}{\varphi(q)\log\sqrt{\frac{y}{q}}}$$

when $q \leq y$

We will also need another similar theorem:

Theorem 2.13. (Siegel–Walfisz) Let us define

$$\psi(x;q,a) := \sum_{\substack{n\leq x \\ n\equiv a (\mathrm{mod}\ q)}} \Lambda(n).$$

Then for any real number N there exists a constant C_N such that

$$\psi(x;q,a) = \frac{x}{\varphi(q)} + \mathcal{O}\left(x\exp\left(-C_N(\log x)^{\frac{1}{2}}\right)\right),$$

when $(a, q) = 1$ and $q \leq (\log x)^N$.

We remark that sometimes this theorem is used in the weaker form (see e.g. [49])

$$\psi(x; q, a) = \frac{x}{\varphi(x)} + \mathcal{O}(x(\log x)^{-A}),$$

which holds for any $q \geq 1$, $(a, q) = 1$, $x \geq 2$ and $A \geq 0$.

We will also need a consequence stating that for fixed $\varepsilon > 0$ and $A > 0$, and for all $x \geq \exp(m^\varepsilon)$, we have

$$\theta(x; m, a) := \sum_{\substack{p \leq x \\ p \equiv a \pmod m}} \log p = \frac{x}{\varphi(m)} \left(1 + \mathcal{O}\left(\frac{1}{(\log x)^A}\right)\right). \tag{14}$$

This is stated in [32].

Another consequence which we will use is an analogue of Mertens' Theorem for primes in arithmetic progressions:

$$\sum_{\substack{m \leq p \leq x \\ p \equiv a \pmod m}} \frac{1}{p} = (1 + o(1)) \cdot \frac{1}{\varphi(m)} \log \log x. \tag{15}$$

This is proved in [45].

For the sake of completeness we state Dirichlet's Theorem whose proof first introduced characters in 1837.

Theorem 2.14. (Dirichlet) Let a and d be positive integers such that $(a, d) = 1$. Then the arithmetic progression $\{an + d\}_{n=1}^{\infty}$ contains infinitely many prime numbers.

Dirichlet's original proof can be found in [17]. There are also other proofs, an elementary one by Selberg [70], and one by Serre [71] which uses density arguments.

Next we state couple of estimates which are useful in the final chapter:

Lemma 2.15. Let n be a natural number. Then

$$\frac{n}{\varphi(n)} \ll \log \log n, \tag{16}$$

$$\log d(n) \ll \frac{\log n}{\log \log n} \tag{17}$$

and

$$\prod_{p|n} \left(\frac{p+1}{p-1}\right) \ll (\log \log(n+2))^2. \tag{18}$$

Proof. The first one is Theorem 16 in [65] and the second one is exercise 1.3.3. in [60]. The last estimate is Satz I.5.1 in [64]. □

Finally we present the result due to Mertens, which is often used in this thesis. The proof is short, so we will include it. We refer to this Lemma as *Mertens' Theorem.*

Lemma 2.16. (Mertens) There exists a constant $c > 0$ such that

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + c + \mathcal{O}\left(\frac{1}{\log x}\right).$$

Proof. Write

$$R(x) := \sum_{p \le x} \frac{\log p}{p} - \log x.$$

It is a well-known consequence of partial summation that $R(x) \ll 1$ (see Theorem 4.10 in [1]). Then note that the partial summation gives

$$\begin{aligned}
\sum_{p \le x} \frac{1}{p} &= \frac{1}{\log x} \sum_{p \le x} \frac{\log p}{p} + \int_2^x \frac{1}{t \log^2 t} \sum_{p \le t} \frac{\log p}{p} \text{ dt} \\
&= \int_2^x \frac{dt}{t \log t} + \frac{R(x)}{\log x} + \int_2^x \frac{R(t)}{t \log^2 t} dt \\
&= \log \log x - \log \log 2 + \mathcal{O}\left(\frac{1}{\log x} \right) + \int_2^x \frac{R(t)}{t \log^2 t} dt \\
&= \log \log x + \left(\int_2^\infty \frac{R(t)}{t \log^2 t} dt - \log \log 2 \right) + \mathcal{O}\left(\frac{1}{\log x} \right),
\end{aligned}$$

as desired. □

3 Mean Values of Multiplicative Functions

The aim of this thesis is to use the so-called pretentious methods to bound the sizes of character sums. In the previous chapter we observed that characters are multiplicative functions whose modulus is at most one. Thus they belong to a class of multiplicative functions which take values in the unit disc $\mathbb{U}$. The goal of this chapter is to obtain some consequences of methods used to prove Halász's Theorem and some of its strengthenings. These consequences are vital ingredients for the proof of the Goldmakher–Granville–Soundararajan estimate. We also briefly discuss methods which imply that it is possible to obtain a similar estimate when we set certain requirements for our multiplicative function. In subsection 3.1 we mainly follow Granville's exposition [21].

We begin by recalling what is the difference between a multiplicative function and a completely multiplicative function. An arithmetic function f is *multiplicative* if $f(mn) = f(m)f(n)$ when $(m,n) = 1$. Function f is *completely multiplicative* if the equation $f(mn) = f(m)f(n)$ holds for all pairs (m,n) of natural numbers and $f(1) = 1$.

3.1 Halász's Theorems

Let f be a multiplicative function with values in the unit disc $\mathbb{U}$. Throughout this thesis we write

$$\theta(f,x) := \prod_{p\leq x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+...\right)\left(1-\frac{1}{p}\right)$$

and

$$M(f;x,T) := \min_{|y|\leq T}\sum_{p\leq x}\frac{1-\Re(f(p)p^{-iy})}{p}.$$

In multiplicative number theory, we are often interested in mean values of multiplicative functions

$$\frac{1}{x}\sum_{n\leq x}f(n). \tag{19}$$

If f is real-valued, it turns out that

$$\frac{1}{x}\sum_{n\leq x}f(n)\to\theta(f,\infty) \text{ as } x\to\infty$$

This was proved in two parts. Wintner [82] proved this in the case $\theta(f,\infty)\neq 0$ in 1944. The harder case $\theta(f,\infty) = 0$ was settled by Wirsing [83] in 1967. This verified an old conjecture by Erdös and Wintner [18] that every real-valued multiplicative function f with $-1\leq f(n)\leq 1$, for all n, has a mean value i.e. limit of (19) exists as $x\to\infty$. On the other hand, this no longer holds if f is complex valued. Consider for example the function $f(n) = n^{it}$ for real $t\neq 0$. The mean value does not exist, since by the standard integral comparison we find

$$\frac{1}{x}\sum_{n\leq x}n^{it}\sim\frac{x^{it}}{1+it}.$$

We are particularly interested in when the mean value (19) exists and is non-zero, as $x\to\infty$. First results concerning such situation are from the early 60s when H. Delange [15, 16] showed that if (19) has a non-zero limit then the series

$$\sum_p\frac{f(p)-1}{p}$$

is convergent. Above we concluded that the mean-value does not exist for the function $f(n) = n^{it}$. Example of a function that does have a non-zero mean value is the constant function $f(n) = 1$. Apart from these, other examples of such functions are not easy to find. It turned out that no other such functions exist. This surprising result was proved by G. Halász in 1968:

Theorem 3.1. (Halász's First Theorem) Let f be a multiplicative function with $|f(n)| \leq 1$.

(1) If

$$\sum_{p} \frac{1 - \Re(f(p)p^{-it})}{p}$$

diverges for all $t \in \mathbb{R}$, then $\frac{1}{x} \sum_{n \leq x} f(n) \to 0$ as $x \to \infty$.

(2) If there exists a real number t for which

$$\sum_{p} \frac{1 - \Re(f(p)p^{-it})}{p}$$

converges, then

$$\frac{1}{x} \sum_{n \leq x} f(n) \sim \frac{x^{it}}{1+it} \theta(f^{(t)}, x),$$

where $f^{(t)}(n) = f(n)n^{-it}$.

The proof of this is long and so we leave it out. The right method is to integrate along carefully chosen contours and estimate the integrals by the Cauchy–Schwarz inequality. For the details we refer to [37] and [38].

Halász also established how fast the mean value converges in the case (1) of Theorem 3.1. This is known as Halász's Second Theorem. The idea of the proof is a straight modification of the proof for his first Theorem; see [37] and [39].

Theorem 3.2. (Halász's Second Theorem) Let f be a multiplicative function with $|f(n)| \leq 1$. Then

$$\frac{1}{x} \sum_{n \leq x} f(n) \ll e^{-\frac{M(f;x,\log x)}{16}}.$$

By Halász's Second Theorem, we have an estimate for sums of multiplicative functions with values in the unit disc. This raises the question whether something stronger holds under tighter assumptions, i.e., when the values of the multiplicative function lie on a certain subset of the unit disc. In 1995 R.R. Hall [41] showed that a version of Halász's Second Theorem holds for multiplicative functions with values in the unit disc and values at prime arguments on a certain fixed subset of $\mathbb{U}$.

In other words, Hall's aim was to study the validity of the estimate

$$\sum_{n \leq x} f(n) \ll x \exp\left(-\kappa \sum_{p \leq x} \frac{1 - \Re f(p)}{p}\right) \tag{20}$$

under the conditions $f(p) \in \mathscr{D}$ for all primes p where $\mathscr{D}$ is a subset of the unit disc and κ is a real constant depending on $\mathscr{D}$. He proved that if $\mathscr{D}$ is a closed convex subset of

$\mathbb{U}$, with $1 \in \mathscr{D}$, then (20) holds. For $\alpha \in [0,1]$ we define

$$\hbar(\alpha) := \frac{1}{2\pi} \int_0^{2\pi} \max_{\delta \in \mathscr{D}} \Re((1-\delta)(\alpha - e^{-i\theta}))d\theta.$$

The constant $\kappa = \kappa(\mathscr{D})$ is chosen to be the largest $\alpha \in [0,1]$ such that $\hbar(\alpha) \geq 1$ (this exists since $\hbar(0) \leq 1$). More on this matter can be found from the reference mentioned above [41] and from Tenenbaum's book [76]. Related to this, it should also be mentioned that the spectrum of mean values of functions $f \in \mathcal{F}$ is studied in [28].

By modifying Halász's method to prove Theorem 3.2. Montgomery [59] proved a refinement for his result. Recently Granville and Soundararajan [24] improved Montgomery's result to

Theorem 3.3. Let $f \in \mathcal{F}$, $x \geq 3$, $T \geq 1$ and $M = M(f; x, T)$. If f is completely multiplicative, then

$$\frac{1}{x}\left|\sum_{n \leq x} f(n)\right| \leq \left(M + \frac{12}{7}\right) e^{\gamma - M} + \mathcal{O}\left(\frac{1}{T} + \frac{\log\log x}{\log x}\right)$$

If f is multiplicative, then

$$\frac{1}{x}\left|\sum_{n \leq x} f(n)\right| \leq \prod_p \left(1 + \frac{2}{p(p-1)}\right)\left(M + \frac{4}{7}\right) e^{\gamma - M} + \mathcal{O}\left(\frac{1}{T} + \frac{\log\log x}{\log x}\right)$$

where γ is the Euler–Mascheroni constant.

In [76] Tenenbaum slightly refined Montgomery's method and obtained the following corollary, which is known as the *Halász–Montgomery–Tenenbaum Theorem*:

Theorem 3.4. Let $f \in \mathcal{F}, x \geq 3$, $T \geq 1$ and $M(f; x, T)$ be as before. Then

$$\frac{1}{x}\sum_{n \leq x} f(n) \ll (1 + M(f; x, T))e^{-M(f;x,T)} + \frac{1}{\sqrt{T}}.$$

With the methods Tenenbaum used, Goldmakher proved a slightly similar estimate for a different sum, which enabled him to refine methods presented in article [26] to obtain an improvement of the Pólya–Vinogradov inequality for characters of an odd order. This result is obtained in the next subsection.

3.2 Halász-type Results

For a given $f \in \mathcal{F}$, let us denote

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

which converges in the half plane $\Re s > 1$. To prove the main results of this subsection, we need two lemmas. The proofs of these lemmas are long and complicated, for which reason they are omitted. The first lemma is taken from the article [57].

Lemma 3.5. For any $f \in \mathcal{F}$ and $x \geq 3$, we have

$$\sum_{n \leq x} \frac{f(n)}{n} \ll \frac{1}{\log x} \int_{\frac{1}{\log x}}^{1} \frac{1}{\alpha} H(\alpha) d\alpha,$$

where

$$H(\alpha) := \left(\sum_{k \in \mathbb{Z}} \max_{s \in \mathcal{B}_k(\alpha)} \left| \frac{F(s)}{s-1} \right|^2 \right)^{\frac{1}{2}}$$

and $\mathcal{B}_k(\alpha)$ is the rectangular region of the complex plane defined as

$$\mathcal{B}_k(\alpha) := \left\{ s \in \mathbb{C} \mid 1+\alpha \le \sigma \le 2 \text{ and } |t-k| \le \frac{1}{2} \right\}.$$

When estimating the quantity $F(s)$ we will use the following result due to Tenenbaum [76]:

Lemma 3.6. If $f \in \mathcal{F}$ and $x \geq 3$, we have

$$F(1+\alpha+it) \ll \begin{cases} (\log x) e^{-M(f;x,T)} & \text{if } |t| \le T \\ \frac{1}{\alpha} & \text{if } |t| > T \end{cases}$$

uniformly for $\alpha \in \left[\frac{1}{\log x}, 1\right]$.

Now we are ready to obtain Goldmakher's Halász-type results. The following proofs are borrowed from [33].

Theorem 3.7. For $f \in \mathcal{F}$, $x \geq 2$ and $T \geq 1$ we have

$$\sum_{n \le x} \frac{f(n)}{n} \ll (\log x) e^{-M(f;x,T)} + \frac{1}{\sqrt{T}}.$$

Proof. Let $s \in \mathcal{B}_k(\alpha)$. A simple use of the Pythagorean Theorem and an easy manipulation shows that $|s-1|^2 \geq \alpha^2 + \left(k - \frac{1}{2}\right)^2 \geq \frac{1}{4}(\alpha^2 + k^2)$ when $k > 0$ and $|s-1|^2 \geq \alpha^2 + \left(k+\frac{1}{2}\right)^2 \geq \frac{1}{4}(\alpha^2+k^2)$ when $k < 0$. Also, $|s-1|^2 \geq \alpha^2$ for $k = 0$ (see Figure 3).

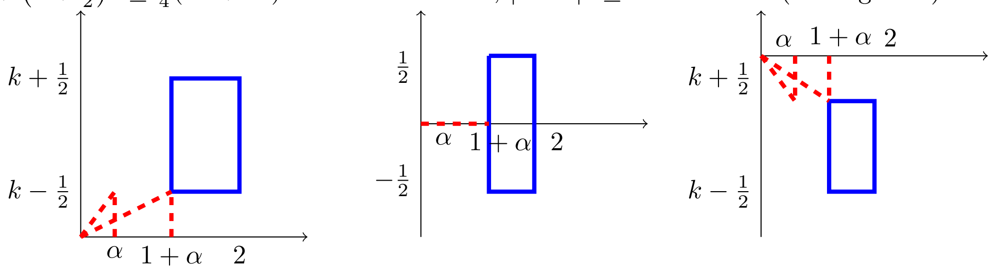

Figure 3. Use of the Pythagorean Theorem in cases $k > 0$, $k = 0$ and $k < 0$.

Therefore, by Lemma 3.6, we estimate

$$\begin{aligned} H(\alpha) &= \left(\sum_{k \in \mathbb{Z}} \max_{s \in \mathcal{B}_k(\alpha)} \left| \frac{F(s)}{s-1} \right|^2 \right)^{\frac{1}{2}} \ll \left(\sum_{k \in \mathbb{Z}} \frac{1}{k^2+\alpha^2} \max_{s \in \mathcal{B}_k(\alpha)} |F(s)|^2 \right)^{\frac{1}{2}} \\ &\ll (\log x) e^{-M(f;x,T)} \left(\sum_{|k| \le T - \frac{1}{2}} \frac{1}{k^2+\alpha^2} \right)^{\frac{1}{2}} + \frac{1}{\alpha} \left(\sum_{|k| > T - \frac{1}{2}} \frac{1}{k^2+\alpha^2} \right)^{\frac{1}{2}} \\ &\ll \frac{1}{\alpha} (\log x) e^{-M(f;x,T)} + (\log x) e^{-M(f;x,t)} \left(\sum_{k \le T} \frac{1}{k^2} \right)^{\frac{1}{2}} + \frac{1}{\alpha} \left(\sum_{k > T - \frac{1}{2}} \frac{1}{k^2} \right)^{\frac{1}{2}} \\ &\ll \frac{1}{\alpha} (\log x) e^{-M(f;x,T)} + \frac{1}{\alpha\sqrt{T}}. \end{aligned}$$

Plugging this estimate to the one in Lemma 3.5. yields

$$\begin{aligned}\sum_{n\le x}\frac{f(n)}{n} &\ll \int_{\frac{1}{\log x}}^{1}\frac{1}{\alpha^2}\left(e^{-M(f;x,t)}+\frac{1}{\log x\sqrt{T}}\right)d\alpha\\ &= \left(e^{-M(f;x,t)}+\frac{1}{\log x\sqrt{T}}\right)(\log x-1)\\ &\ll (\log x)e^{-M(f;x,T)}+\frac{1}{\sqrt{T}},\end{aligned}$$

as desired. □

As an easy corollary we get a result we use in concrete estimates:

Corollary 3.8. For $f\in\mathcal{F}$, $x\ge 2$, $y\ge 2$ and $T\ge 1$,

$$\sum_{\substack{n\le x\\ n\in S(y)}}\frac{f(n)}{n}\ll(\log y)e^{-M(f;y,T)}+\frac{1}{\sqrt{T}}.$$

Proof. If $x\le y$, the condition $n\in S(y)$ is superfluous and so Theorem 3.7. gives the claim. Assume then that $y<x$ holds. Since $f\in\mathcal{F}$ we have also $f_y\in\mathcal{F}$. Thus by Theorem 3.7. we have

$$\sum_{\substack{n\le x\\ n\in S(y)}}\frac{f(n)}{n}=\sum_{n\le x}\frac{f_y(n)}{n}\ll(\log x)e^{-M(f_y;x,T)}+\frac{1}{\sqrt{T}}.$$

By Mertens' Theorem we have

$$\begin{aligned}M(f_y;x,T) &= \min_{|t|\le T}\sum_{p\le x}\frac{1-\Re f_y(p)p^{-it}}{p}=\min_{|t|\le T}\left(\sum_{p\le y}\frac{1-\Re f(p)p^{-it}}{p}+\sum_{y<p\le x}\frac{1}{p}\right)\\ &= M(f;y,T)+\log\left(\frac{\log x}{\log y}\right)+\mathcal{O}(1).\end{aligned}$$

Combining the above two estimates gives the desired result:

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in S(y)}}\frac{f(n)}{n} &\ll (\log x)e^{-M(f;y,T)+\log\left(\frac{\log y}{\log x}\right)}+\frac{1}{\sqrt{T}}\\ &= (\log x)\left(\frac{\log y}{\log x}\right)e^{-M(f;y,T)}+\frac{1}{\sqrt{T}}\\ &= (\log y)e^{-M(f;y,T)}+\frac{1}{\sqrt{T}}.\end{aligned}$$

□

4 Pretentiousness

In this section we define a natural distance between two multiplicative functions, with values in the closed unit disc $\mathbb{U}$, and the concept of pretentiousness. We study their basic properties, derive estimates for certain distances and in the end we take a look at some applications. This section is mostly based on the works by Granville and Soundararajan [21, 23, 25].

4.1 The Multiplicative Mimicry Metric

We begin by defining the distance between two multiplicative functions f and g that have values in $\mathbb{U}$. One obvious candidate to quantify how close f can be to g is

$$\sum_{p^k \leq x} \frac{|f(p^k) - g(p^k)|}{p},$$

but this is not sufficient for our purposes. The quantity that proves to be good enough is defined as follows.

Definition 4.1. Let f and g be multiplicative functions with values in the unit disc i.e. $|f(n)| \leq 1$ and $|g(n)| \leq 1$ for all natural numbers n. Then we can define their *distance* up to a real number x as

$$\mathbb{D}(f, g; x)^2 := \sum_{p \leq x} \frac{1 - \Re(f(p)\overline{g}(p))}{p}$$

This distance does not have an established name but there is many different variations in the literature, for example the *Multiplicative Mimicry Metric* and the *Granville–Soundararajan Distance*. However, in this thesis we just refer to it as the *distance*.

Although $\mathbb{D}(f, g; x)$ is called a metric, it is not a metric in the general case. It is possible that the distance from function f to itself might be non-zero. This happens for example when $f(p) = 0$ for some prime $p \leq x$.

We remark that if h is a multiplicative function with $|h(p)| = 1$ for all primes p, then we have $\mathbb{D}(f, g; x) = \mathbb{D}(fh, gh; x)$ for all $f, g \in \mathcal{F}$ and $x \in \mathbb{R}_+$. This *invariance property* is useful in some situations. It is also worthwhile to observe that

$$|\Re(f(p)\overline{g}(p))| \leq \sqrt{\Re^2(f(p)\overline{g}(p)) + \Im^2(f(p)\overline{g}(p))} = |f(p)\overline{g}(p)| \leq 1,$$

and so, by using Mertens' Theorem, the estimate

$$0 \leq \mathbb{D}(f, g; x) \leq \sqrt{\sum_{p \leq x} \frac{2}{p}} = \sqrt{2 \log\log x + c + \mathcal{O}\left(\frac{1}{\log x}\right)} = (1 + o(1))\sqrt{2 \log\log x} \quad (21)$$

holds.

Note that this measure has the property $\mathbb{D}(f, g; x) = 0$ if and only if $f(p) = g(p)$ and $|f(p)| = 1$ for all primes $p \leq x$. Moreover, it follows easily that the distance satisfies the following inequality

Theorem 4.2. (Pretentious Triangle Inequality[6]): Let f, g, F and G be multiplicative functions with values in the unit disc. Then

$$\mathbb{D}(f, g; x) + \mathbb{D}(F, G; x) \geq \mathbb{D}(fF, gG; x)$$

[6]This is not a triangle inequality in the traditional meaning.

Proof. Since

$$\mathbb{D}(f,g;x)^2 = \sum_{p\le x} \frac{1-\Re(f(p)\overline{g}(p))}{p} = \sum_{p\le x} \frac{1-\Re(1\cdot \overline{\overline{f}(p)g(p)})}{p} = \mathbb{D}(1,\overline{f}g;x)^2$$

we can without loss of generality assume that f and F are identically one. We calculate

$$\begin{aligned}
(\mathbb{D}(1,g;x)+\mathbb{D}(1,G;x))^2 &= \sum_{p\le x}\left(\frac{1-\Re g(p)}{p}+\frac{1-\Re G(p)}{p}\right)+2\mathbb{D}(1,g;x)\mathbb{D}(1,G;x)\\
&= \sum_{p\le x}\frac{1-\Re g(p)+1-\Re G(p)+2\sqrt{1-\Re g(p)}\sqrt{1-\Re G(p)}}{p}\\
&\overset{(*)}{\ge} \sum_{p\le x}\frac{1-\Re g(p)+1-\Re G(p)+\Im g(p)\Im G(p)}{p}\\
&\overset{(**)}{\ge} \sum_{p\le x}\frac{1-\Re(g(p)G(p))}{p} = \mathbb{D}(1,gG;x)^2,
\end{aligned}$$

as desired. The step $(*)$ follows since

$$2\Re g(p)+\Im^2 g(p)\le 2\Re g(p)+1-\Re^2 g(p) = 2-(\Re g(p)-1)^2\le 2,$$

which implies that $2(1-\Re g(p))\ge \Im^2 g(p)$. The estimate $(**)$ follows by writing $g(p)=a+bi$, $G(p)=c+di$ and noting that it is enough to prove that $1+ac\ge a+c$. This is true since it is equivalent to $(1-a)(1-c)\ge 0$, which in turn follows as $a,c\le 1$. $\square$

There is also another version of the distance. Indeed, let $\eta_j:\mathbb{U}\times\mathbb{U}\to\mathbb{R}_{\ge 0}$ be a sequence of functions satisfying the ordinary triangle inequality: $\eta_j(z_1,z_3)\le \eta_j(z_1,z_2)+\eta_j(z_2,z_3)$ for every $j\in\mathbb{N}$ and for all $z_1,z_2,z_3\in\mathbb{U}$. Then we can define a metric in $\mathbb{U}^{\mathbb{N}}$ by setting

$$d(z,w)=\left(\sum_{j=1}^{\infty}\eta_j(z_j,w_j)^2\right)^{\frac{1}{2}},$$

where $z=(z_1,z_2,...)$ and $w=(w_1,w_2,...)$, assuming that the sum converges. A straightforward use of the Cauchy–Schwarz inequality (see [29], Lemma 4.1) shows that the triangle inequality holds:

$$d(z,w)\le d(z,y)+d(y,w). \tag{22}$$

An important class of such functions is

$$\eta_j(z_j,w_j)^2 = a_j(1-\Re(z_j\overline{w}_j)), \tag{23}$$

where constants a_j are non-negative. With some work (see [23], Lemma 4.1.1) one can show that these functions satisfy the ordinary triangle inequality for a fixed j. Let $p_1<p_2<...$ be all the prime numbers. Choosing $a_j=\frac{1}{p_j}$ for all j such that $p_j\le x$ and 0 for larger values of j, d-function gives the distance $\mathbb{D}(f,g;x)$ when we set $z_j=f(p_j)$ and $w_j=g(p_j)$. An immediate consequence from this and (22) is that the distance satisfies also the ordinary triangle inequality: $\mathbb{D}(f,g;x)\le\mathbb{D}(f,h;x)+\mathbb{D}(h,g;x)$.

We can also choose $a_j=1/p^\alpha$, with $\alpha>1$ in the function η above, $z_j=f(p_j)$ and $w_j=g(p_j)$ to obtain an α-scaled distance

$$\mathbb{D}_\alpha(f,g;\infty)^2 := \sum_p \frac{1-\Re(f(p)\overline{g}(p))}{p^\alpha}$$

Using similar arguments as in the case $\alpha = 1$, it is possible to show that this distance also satisfies the pretentious triangle inequality. Furthermore, for a fixed $x \in \mathbb{R}$, choosing $\alpha = s_x = 1 + \frac{1}{\log x}$, we get that these two distances are closely related:

$$\mathbb{D}(f, g; x)^2 = \mathbb{D}_{s_x}(f, g; \infty)^2 + \mathcal{O}(1).$$

To see this, just consider the difference

$$\left|\mathbb{D}(f, g; x)^2 - \mathbb{D}_{s_x}(f, g; \infty)^2\right| \leq 2 \sum_{p \leq x} \left|\frac{1}{p} - \frac{1}{p^{s_x}}\right| + 2 \sum_{p > x} \frac{1}{p^{s_x}} \ll 1.$$

For the boundedness of the two summands, see the proof of Lemma 3.2 in [53].

Now we can give the definition of pretentiousness. Below are two different definitions of this concept. The second definition is not used in this thesis.

Definition 4.3. Let f and g be multiplicative functions with values in the unit disc $\mathbb{U}$. We say that function f is *g-pretentious*, if the distance $\mathbb{D}(f, g, \infty)$ is finite i.e.

$$\sum_p \frac{1 - \Re(f(p)\overline{g}(p))}{p} < \infty.$$

The another meaning is the following. Fixing $\delta > 0$, we say that a function f is $(g, x; \delta)-$*pretentious* if

$$\sum_{p \leq x} \frac{1 - \Re(f(p)\overline{g}(p))}{p} \leq \delta \log \log x,$$

for every $x \in \mathbb{R}$.

4.2 Simple Examples

We give a few examples of pretentiousness:

• It is possible that a multiplicative function f is not f-pretentious. This is seen, for instance, by taking f to be a completely multiplicative function with $f(p) = \frac{1}{p}$ for all primes p. Namely,

$$\mathbb{D}(f, f; \infty)^2 = \sum_p \frac{1 - |f(p)|^2}{p} \geq \frac{3}{4} \sum_p \frac{1}{p} = \infty.$$

• For every function $f \in \mathcal{F}$ with $|f(p)| = 1$ we evidently have $\mathbb{D}(f, f; \infty) = 0$. Examples of such functions include the constant function $f(n) = 1$ and the Liouville function $\lambda(n)$.

• To obtain examples of completely multiplicative functions that are not n^{it}-pretentious for any $t \neq 0$ we need the following result:

Lemma 4.4. Let $f \in \mathcal{F}$ and suppose that there exists $k \geq 1$ such that $f(p)^k = 1$ for every prime p. Then $\mathbb{D}(f(n), n^{it}; \infty) = \infty$ for every non-zero t.

Proof. Let us begin with Dirichlet series. We first prove that:

Claim. For $x \geq 2$ we have

$$\exp\left(\sum_{p \leq x} \frac{f(p)}{p^{1+it}}\right) \asymp \sum_{n \geq 1} \frac{f(n)}{n^{1+\frac{1}{\log x}+it}}.$$

Set $s = 1 + \frac{1}{\log x} + it$ throughout the proof. To prove the claim, we start by recalling that $\sum_{p \le x} \frac{\log p}{p} = \log x + \mathcal{O}(1)$ (see the proof of Lemma 2.16). Using this we deduce

$$\left| \sum_{p > x} \frac{f(p)}{p^s} \right| \le \sum_{p > x} \frac{1}{p^{1+\frac{1}{\log x}}} \ll \int_x^\infty \frac{1}{t^{s_x} \log t} d\theta(t) \ll 1,$$

and

$$\left| \sum_{p \le x} \left(\frac{f(p)}{p^s} - \frac{f(p)}{p^{1+it}} \right) \right| \le \sum_{p \le x} \frac{1 - p^{-\frac{1}{\log x}}}{p} \ll \frac{1}{\log x} \sum_{p \le x} \frac{\log p}{p} \ll 1.$$

These together yield

$$\sum_{p \le x} \frac{f(p)}{p^{1+it}} = \sum_p \frac{f(p)}{p^s} + \mathcal{O}(1). \tag{24}$$

Writing the Dirichlet series to Euler product we obtain

$$\begin{aligned}
\log \sum_{n=1}^\infty \frac{f(n)}{n^s} &= \log \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots \right) \\
&\overset{(*)}{=} \log \prod_p \frac{1}{1 - \frac{f(p)}{p^s}} \\
&= -\sum_p \log \left(1 - \frac{f(p)}{p^s} \right) \\
&\overset{(**)}{=} \sum_p \sum_{n=1}^\infty \frac{f(p)^n}{np^{sn}},
\end{aligned}$$

where $(*)$ follows since f is completely multiplicative and in the step $(**)$ we used the Taylor series of $\log(1-z)$.

Now, clearly the sum

$$\sum_p \sum_{n=2}^\infty \frac{f(p)^n}{np^{ns}}$$

converges as $|f(p)| \le 1$ for every p. Thus (24) tells that

$$\sum_{p \le x} \frac{f(p)}{p^{1+it}} = \log \sum_{n=1}^\infty \frac{f(n)}{n^s} + \mathcal{O}(1).$$

We take the exponential from both sides to get the claim. □

Now let us move to the proof of Lemma 4.4. Suppose that the statement is false: there exists a real $t \ne 0$ such that $\mathbb{D}(f(n), n^{it}; \infty) < \infty$. If $k = 1$ we clearly have $\mathbb{D}(1, n^{it}, \infty) = \mathbb{D}(f(n), n^{it}; \infty)$. If $k \ge 2$, a repeated application of the pretentious triangle inequality yields

$$\begin{aligned}
\mathbb{D}(1, n^{ikt}, \infty) &= \mathbb{D}(f(n)^k, n^{ikt}; \infty) \\
&\le \mathbb{D}(f(n)^{k-1}, n^{i(k-1)t}; \infty) + \mathbb{D}(f(n), n^{it}; \infty) \\
&\le \mathbb{D}(f(n)^{k-2}, n^{i(k-2)t}; \infty) + 2\mathbb{D}(f(n), n^{it}; \infty) \\
&\ \ \vdots \\
&\le \mathbb{D}(f(n)^2, n^{2it}; \infty) + (k-2)\mathbb{D}(f(n), n^{it}; \infty) \\
&\le 2\mathbb{D}(f(n), n^{it}; \infty) + (k-2)\mathbb{D}(f(n), n^{it}; \infty) \\
&= k\mathbb{D}(f(n), n^{it}; \infty) \\
&< \infty.
\end{aligned}$$

Hence $\mathbb{D}(1, n^{ikt}; \infty)$ is finite for every integer $k \geq 1$.

Let $s' = 1 + \frac{1}{\log x} + ikt$. Then, by the claim above

$$\log \zeta(s') = \sum_{p \leq x} \frac{1}{p^{1+ikt}} + \mathcal{O}(1).$$

This implies that

$$\begin{aligned} \log |\zeta(s')| &= \Re(\log \zeta(s')) = \sum_{p \leq x} \frac{\Re(p^{-ikt})}{p} + \mathcal{O}(1) \\ &= \sum_{p \leq x} \frac{1 - (1 - \Re(\overline{p^{ikt}}))}{p} + \mathcal{O}(1) \\ &= \sum_{p \leq x} \frac{1}{p} - \mathbb{D}(1, n^{ikt}; x) + \mathcal{O}(1) \\ &= \log \log x + \mathcal{O}_t(1). \end{aligned}$$

Thus $|\zeta(s')| \gg \log x$ $(*)$. However, by partial summation we have

$$\zeta(s') = \frac{1}{s'-1} - s' \int_1^\infty \frac{\{u\}}{u^{s'+1}} du \qquad \text{for all } \Re s' > 0,$$

(see exercise 2.1.6 in [60]) and so

$$\zeta(s') = \frac{1}{s'-1} + \mathcal{O}(1 + |t|) = \frac{1}{it} + \mathcal{O}\left(1 + |t| + \frac{1}{|t|^2 \log x}\right).$$

This contradicts with $(*)$. The proof is completed. □

• As an immediate corollary, Möbius function μ, character χ and the function $\mu\chi$ are not n^{it}-pretentious for any $t \neq 0$.

• Finally we mention that a multiplicative function cannot pretend two characters simultaneously very well. We do not present the proofs for the next two theorems, since they rely on the Pólya–Vinogradov inequality which is not yet proven. Also these results are not essential for the rest of the thesis. Detailed proofs are presented in [25].

Theorem 4.5. Let χ be a primitive character mod q. Then there exist a constant $c > 0$ such that

$$\mathbb{D}(1, \chi; x)^2 \geq \frac{1}{2} \log\left(\frac{c \log x}{\log q}\right).$$

Moreover, if f is a multiplicative function, χ and ψ are characters with conductors below Q, then for all $x \geq Q$

$$\mathbb{D}(f, \chi; x)^2 + \mathbb{D}(f, \psi; x)^2 \geq \frac{1}{8} \log\left(\frac{c \log x}{2 \log Q}\right).$$

Theorem 4.6. Let $\chi \pmod q$ be a primitive character and $t \in \mathbb{R}$. Then there exists an absolute constant $c > 0$ such that for all $x \geq q$

$$\mathbb{D}(1, \chi(n) n^{it}; x)^2 \geq \frac{1}{2} \log\left(\frac{c \log x}{\log(q(1 + |t|))}\right).$$

Moreover, if f is a multiplicative function, χ and ψ are two distinct primitive characters with conductor below Q, then for all $x \geq Q$ we have

$$\mathbb{D}(f, \chi(n)n^{it}; x)^2 + \mathbb{D}(f, \psi(n)n^{iu}; x)^2 \geq \frac{1}{8} \log \left(\frac{c \log x}{2 \log(Q(1 + |t-u|))} \right).$$

4.3 An Enlightening Example

In this subsection we show how the distance function $d(f, g)$ can be used while studying multiplicative functions. We derive some formulas for the ζ-function as Granville and Soundararajan did in their paper [25].

We consider an arbitrary multiplicative function f with values in the unit disc. Let $q_1 < q_2 < ...$ be all the prime powers. Choosing $a_j = \Lambda(q_j)/(q_j^\sigma \log q_j)$ for $\sigma > 1$, in (23) and taking g to be identically one, the d-function takes the form

$$\begin{aligned} d(f,1)^2 &= \sum_{j=1}^{\infty} \frac{\Lambda(q_j)}{q_j^\sigma \log q_j}(1 - \Re f(q_j)) = \sum_p \sum_{k=1}^{\infty} \frac{1}{k} p^{-k\sigma} - \sum_p \sum_{k=1}^{\infty} \frac{1}{k} p^{-k\sigma} \Re f(p^k) \\ &= \log \zeta(\sigma) - \log |F(\sigma)| = \log \frac{\zeta(\sigma)}{|F(\sigma)|}, \end{aligned}$$

where $F(\sigma) := \sum_{n=1}^{\infty} f(n) n^{-s}$. Now it is easy to obtain the following relation for the zeta-function:

Theorem 4.7. Let f and g be completely multiplicative functions with $|f(n)| \leq 1$ and $|g(n)| \leq 1$. Let s be a complex number with $\Re s > 1$, and set $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$, $G(s) = \sum_{n=1}^{\infty} g(n) n^{-s}$, $F \bigotimes G(s) = \sum_{n=1}^{\infty} f(n) g(n) n^{-s}$. Then for $\sigma > 1$,

$$\sqrt{\log \frac{\zeta(\sigma)}{|F(\sigma)|}} + \sqrt{\log \frac{\zeta(\sigma)}{|G(\sigma)|}} \geq \sqrt{\log \frac{\zeta(\sigma)}{|F \bigotimes G(\sigma)|}}.$$

Proof. This follows directly from the pretentious triangle inequality

$$d(f, 1) + d(g, 1) \geq d(fg, 1).$$

□

By choosing $f(n) = n^{-it}$ and $g(n) = n^{-it'}$, the previous identity transforms to the form

Corollary 4.8. The following identity holds

$$\sqrt{\log \frac{\zeta(\sigma)}{|\zeta(\sigma + it)|}} + \sqrt{\log \frac{\zeta(\sigma)}{|\zeta(\sigma + it')|}} \geq \sqrt{\log \frac{\zeta(\sigma)}{|\zeta(\sigma + it + it')|}}.$$

Similar identities can be found for other functions e.g. the Dirichlet L-function. Indeed, by taking $f(n) = \chi(n) n^{-it}$ and $g(n) = \psi(n) n^{-it'}$ in Theorem 4.7, we obtain

$$\sqrt{\log \frac{\zeta(\sigma)}{|L(\sigma + it, \chi)|}} + \sqrt{\log \frac{\zeta(\sigma)}{|L(\sigma + it', \psi)|}} \geq \sqrt{\log \frac{\zeta(\sigma)}{|L(\sigma + it + it', \chi\psi)|}}.$$

4.4 The Size of $\mathbb{D}(\chi(n), \xi(n)n^{it}; y)$

The aim of this section is to prove a theorem which gives a lower bound for the distance between a primitive character χ and $\xi(n)n^{it}$, where ξ is a character with a small conductor. This will be useful in Chapter 6. Our proof is a combination of [26] and [33]. We remark that here the summation $\sum_{n \pmod{\ell}}$ means that we sum over all the reduced residue classes modulo ℓ.

Theorem 4.9. Let $\chi \pmod q$ be a primitive character of odd order g. Suppose ξ (mod m) is a primitive character such that $\chi(-1)\xi(-1) = -1$. If $m < (\log y)^A$, for some fixed $A > 0$, then for all $|t| \leq \log^2 y$, we have

$$\mathbb{D}(\chi(n), \xi(n)n^{it}; y)^2 \geq \left(1 - \frac{g}{\pi}\sin\frac{\pi}{g} + o(1)\right)\log\log y.$$

Proof. Let us first consider the case $t = 0$. Since χ has odd order we have $\chi(-1) = 1$ and thus $\xi(-1) = -1$. This in turn implies that ξ has an even order, say $\ell \geq 2$. Disregarding the arithmetic properties of χ and viewing it as an element of the set $\mu_g \cup \{0\}$ we get

$$\begin{aligned}
\mathbb{D}(\chi, \xi; y)^2 &= \sum_{p \leq y} \frac{1 - \Re(\chi(p)\overline{\xi}(p))}{p} = \underbrace{\sum_{\substack{p \leq y \\ \xi(p)=0}} \frac{1 - \Re(\chi(p)\overline{\xi}(p))}{p}}_{\geq 0} + \sum_{\substack{p \leq y \\ \xi(p)\neq 0}} \frac{1 - \Re(\chi(p)\overline{\xi}(p))}{p} \\
&\geq \sum_{-\frac{\ell}{2} < n \leq \frac{\ell}{2}} \left(\sum_{\substack{p \leq y \\ \xi(p)=e\left(\frac{n}{\ell}\right)}} \frac{1 - \Re(\chi(p)\overline{\xi}(p))}{p} \right) \\
&\geq \sum_{-\frac{\ell}{2} < n \leq \frac{\ell}{2}} \left(\sum_{\substack{p \leq y \\ \xi(p)=e\left(\frac{n}{\ell}\right)}} \frac{1}{p} \right) \min_{z \in \mu_g \cup \{0\}} \left(1 - \Re\left(ze\left(-\frac{n}{\ell}\right)\right)\right).
\end{aligned}$$

Now observe that

$$\begin{aligned}
\min_{z \in \mu_g \cup \{0\}} \left(1 - \Re\left(ze\left(-\frac{n}{\ell}\right)\right)\right) &= \min_{r \in \{0,1,\ldots,g-1\}} \left(1 - \Re\left(e\left(\frac{r}{g} - \frac{n}{\ell}\right)\right)\right) \\
&= \min_{r \in \{0,1,\ldots,g-1\}} \left(1 - \cos\left(2\pi\left(\frac{r}{g} - \frac{n}{\ell}\right)\right)\right) \\
&= \min_{r \in \{0,1,\ldots,g-1\}} \left(1 - \cos\left(\frac{2\pi}{g}\left(r - \frac{ng}{\ell}\right)\right)\right) \\
&= 1 - \cos\left(\frac{2\pi}{g}\left\|\frac{ng}{\ell}\right\|\right).
\end{aligned}$$

Let us now record a simple result which is very useful:

Claim. It holds that

$$\sum_{\substack{a \pmod m \\ \xi(a)=e\left(\frac{n}{\ell}\right)}} 1 = \frac{\varphi(m)}{\ell}. \tag{25}$$

Proof. First of all, the number of elements $a \pmod m$ is the same as the number of elements of the set $(\mathbb{Z}/m\mathbb{Z})^*$. Since ξ has order ℓ, there exists $b \in (\mathbb{Z}/m\mathbb{Z})^*$ such that the numbers $1, \xi(b), \xi(b)^2, ..., \xi(b)^{\ell-1}$ are all distinct (otherwise there exists an integer $0 < k < \ell$ for which ξ^k gives the principal character). On the other hand, they are also ℓ^{th} roots of unity and therefore we find a number $g \in (\mathbb{Z}/m\mathbb{Z})^*$ such that $\xi(g) = e\left(\frac{1}{\ell}\right)$. Let H be the kernel of ξ i.e., the set of elements $a \in (\mathbb{Z}/m\mathbb{Z})^*$ for which $\xi(a) = 1$. Then from the basics of abstract algebra, H is a normal subgroup of $(\mathbb{Z}/m\mathbb{Z})^*$ and $g^nH = \{a \in (\mathbb{Z}/m\mathbb{Z})^* | \xi(a) = e\left(\frac{n}{\ell}\right)\}$. Thus $(\mathbb{Z}/m\mathbb{Z})^*$ can be decomposed as a disjoint union of the ℓ cosets g^nH with $0 \leq n \leq \ell - 1$. Since $|g^nH| = |H|$ for each n and $|(\mathbb{Z}/m\mathbb{Z})^*| = \varphi(m)$, the claim follows. □

Using this and the estimate (15) we obtain

$$\begin{aligned}
\sum_{\substack{p \leq y \\ \xi(p)=e\left(\frac{n}{\ell}\right)}} \frac{1}{p} &= \sum_{\substack{a \ (\mathrm{mod}\ m) \\ \xi(a)=e\left(\frac{n}{\ell}\right)}} \sum_{\substack{p \leq y \\ p \equiv a \ (\mathrm{mod}\ m)}} \frac{1}{p} \\
&\geq \frac{\varphi(m)}{\ell}(1+o(1)) \cdot \frac{1}{\varphi(m)} \cdot \log\log y \\
&= (1+o(1))\frac{1}{\ell} \cdot \log\log y.
\end{aligned}$$

Writing $\frac{g}{\ell} = \frac{g'}{\ell'}$ with (g', ℓ')=1 and observing that

$$\begin{aligned}
\cos\left(\frac{2\pi}{g}\left\|\frac{ng}{\ell}\right\|\right) &= \cos\left(\frac{2\pi}{g}\left(k - \frac{ng'}{\ell'}\right)\right) \\
&\overset{g\cdot\ell' = g'\cdot\ell}{=} \cos\left(\frac{2\pi}{g}\cdot k - \frac{2\pi n}{\ell}\right) \\
&= \cos\left(\frac{2\pi(\ell'\cdot k - g'\cdot n)}{g\ell'}\right)
\end{aligned}$$

for some integer k, we combine the above results to

$$\begin{aligned}
\mathbb{D}(\chi, \xi; y)^2 &\geq (1+o(1))\frac{1}{\ell}\log\log y \cdot \frac{\ell}{\ell'} \sum_{-\frac{\ell'}{2} < n \leq \frac{\ell'}{2}} \left(1 - \cos\left(\frac{2\pi n}{g\ell'}\right)\right) \\
&\sim \left(1 - \frac{\sin\left(\frac{\pi}{g}\right)}{\ell' \tan\left(\frac{\pi}{g\ell'}\right)}\right)\log\log y. \qquad (26)
\end{aligned}$$

The last step follows from the calculation:

$$\begin{aligned}
\sum_{-\frac{\ell'}{2} < n \leq \frac{\ell'}{2}} \cos\left(\frac{2\pi n}{g\ell'}\right) &= \Re\left(\sum_{-\frac{\ell'}{2} < n \leq \frac{\ell'}{2}} e\left(\frac{n}{g\ell'}\right)\right) \\
&= \Re\left(\frac{e^{-\frac{\pi}{g}i + \frac{2\pi}{g\ell'}i}(1 - e^{\frac{2\pi}{g}i})}{1 - e^{\frac{2\pi}{g\ell'}}}\right) \\
&= \Re\left(\frac{1}{2}\cdot\frac{\left(e^{\frac{\pi}{g}i} - e^{-\frac{\pi}{g}i}\right)\left(e^{\frac{\pi}{g\ell'}i} + e^{-\frac{\pi}{g\ell'}i}\right)}{e^{\frac{\pi}{g\ell'}i} - e^{-\frac{\pi}{g\ell'}i}}\right) \\
&= \frac{\sin\left(\frac{\pi}{g}\right)}{\tan\left(\frac{\pi}{g\ell'}\right)}
\end{aligned}$$

Noting that by a simple differentiation we deduce $\ell' \tan\left(\frac{\pi}{g\ell'}\right) > \frac{\pi}{g}$, thus finishing the case $t = 0$.

Assume then that $t \neq 0$. If $t = o\left(\frac{\log\log y}{\log y}\right)$, the claim follows from the case $t = 0$:

$$\begin{aligned}
\mathbb{D}(\chi(n), \xi(n)n^{it}; y)^2 &= \sum_{p \le y} \frac{1 - \Re\left(\chi(p)\overline{\xi}(p)e^{-it\log p}\right)}{p} \\
&= \sum_{p \le y} \frac{1 - \Re\left(\chi(p)\overline{\xi}(p)(1 + \mathcal{O}(|t|\log p))\right)}{p} \\
&= \mathbb{D}(\chi(n), \psi(n); y)^2 + \mathcal{O}\left(|t| \sum_{p \le y} \frac{\log p}{p}\right) \\
&= \mathbb{D}(\chi(n), \psi(n); y)^2 + o(\log\log y) \\
&\ge (\delta_g + o(1))\log\log y.
\end{aligned}$$

In the second to last step of the argument we, yet again, used the fact $\sum_{p \le y} \frac{\log p}{p} = \log y + \mathcal{O}(1)$ and the assumption on the size of t.

For the larger values of t, the idea is as follows: we partition the interval $[2, y)$ into subintervals of the form $]x, (1+\delta)x]$ with $\delta \asymp (\log x)^{-3}$. Then for each prime in such a interval, p^{it} can be approximated by x^{it}. Indeed, by using the Intermediate Value Theorem we get

$$\begin{aligned}
|p^{it} - x^{it}| &= |e^{-it\log p} - e^{-it\log x}| \\
&= 2\left|\sin\left(\frac{t}{2}(\log p - \log x)\right)\right| \\
&\le |t(\log p - \log x)| \\
&= |t| \cdot |\log((1+\delta)x) - \log x| \\
&= |t| \cdot |\log(1+\delta)| \le \delta|t|.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\sum_{x < p \le (1+\delta)x} \frac{1 - \Re\left(\chi(p)\overline{\xi}(p)p^{-it}\right)}{p} &= \sum_{x < p \le (1+\delta)x} \frac{1 - \Re\left(\chi(p)\overline{\xi}(p)x^{-it}\right)}{p} + \mathcal{O}\left(\delta|t| \sum_{x < p \le (1+\delta)x} \frac{1}{p}\right) \\
&= \sum_{x < p \le (1+\delta)x} \frac{1 - \Re\left(\chi(p)\overline{\xi}(p)e(\theta_x)\right)}{p} + \mathcal{O}\left(\frac{\delta^2\log^2 y}{\log x}\right) \qquad (27)
\end{aligned}$$

where $\theta_x = -\frac{t}{2\pi}\log x$. The last equality follows from Mertens' Theorem and the estimates $|t| \le \log^2 y$, $\log(1+y) \ll y$:

$$\delta|t| \sum_{x < p \le (1+\delta)x} \frac{1}{p} \ll \delta \log^2 y \log\left(\frac{\log(1+\delta) + \log x}{\log x}\right) \ll \delta\log^2 y \cdot \frac{\log(1+\delta)}{\log x} \ll \frac{\delta^2\log^2 y}{\log x}.$$

As before, viewing the character χ as an element of the set $\mu_g \cup \{0\}$, we have that the main term is

$$\begin{aligned}
\sum_{x < p \le (1+\delta)x} \frac{1 - \Re(\chi(p)\xi(p))p^{-it}}{p} &= \sum_{-\frac{\ell}{2} \le n < \frac{\ell}{2}} \sum_{\substack{x < p \le (1+\delta)x \\ \xi(p) = e\left(\frac{n}{\ell}\right)}} \frac{1 - \Re\left(\chi(p)e\left(-\frac{n}{\ell}\right)e(\theta_x)\right)}{p} \\
&\ge \sum_{-\frac{\ell}{2} \le n < \frac{\ell}{2}} \sum_{\substack{x < p \le (1+\delta)x \\ \xi(p) = e\left(\frac{n}{\ell}\right)}} \frac{1}{p} \min_{z \in \mu_g \cup \{0\}} \left(1 - \Re\left(z \cdot e\left(\theta_x - \frac{n}{\ell}\right)\right)\right).
\end{aligned}$$

We need two lemmas to estimate the sums appearing on the right-hand side. The first one states that

Lemma 4.10. Let $\varepsilon > 0$ be fixed, $\xi \pmod m$ be a non-principal character of order ℓ and $y < (\log m)^A$ for given $A > 0$. Then for $x \geq \exp((\log y)^\varepsilon)$ we have

$$\sum_{\substack{x<p\leq(1+\delta)x \\ \xi(p)=e\left(\frac{n}{\ell}\right)}} \frac{1}{p} = \frac{\delta}{\ell \log x}(1+o(1)).$$

Proof. The idea behind the proof is to use the Siegel–Walfizs Theorem. In order to do so, the sum must be expressed as a sum where the primes are in an arithmetic progression. This is done easily:

$$\sum_{\substack{x<p\leq(1+\delta)x \\ \xi(p)=e\left(\frac{n}{\ell}\right)}} \frac{1}{p} = \sum_{\substack{a \pmod m \\ \xi(p)=e\left(\frac{n}{\ell}\right)}} \sum_{\substack{x<p\leq(1+\delta)x \\ p\equiv a (\mathrm{mod}\ m)}} \frac{1}{p}.$$

Observe that from the restriction $x < p \leq (1+\delta)x$ it follows that

$$1 < \frac{x\log x}{p \log p} < \frac{(1+\delta)x\log x}{x\log x} = 1+\delta$$

and so

$$\frac{x\log x}{p\log p} = 1+\mathcal{O}(\delta)$$

Now, a straightforward application of the estimate (14) gives that the inner sum equals

$$\begin{aligned}
\sum_{\substack{x<p\leq(1+\delta)x \\ p\equiv a (\mathrm{mod}\ m)}} \frac{1}{p} &= \sum_{\substack{x<p\leq(1+\delta)x \\ p\equiv a (\mathrm{mod}\ m)}} \left(\frac{1}{x\log x}\cdot\frac{x\log x}{p\log p}\cdot \log p\right) \\
&= \frac{1+\mathcal{O}(\delta)}{x\log x} \sum_{\substack{x<p\leq(1+\delta)x \\ p\equiv a (\mathrm{mod}\ m)}} \log p \\
&= \frac{1+\mathcal{O}(\delta)}{x\log x} \left(\theta((1+\delta)x;m,a) - \theta(x;m,a)\right) \\
&= \frac{1+\mathcal{O}(\delta)}{x\log x} \left(\frac{((1+\delta)-1)x}{\varphi(m)} + \mathcal{O}\left(\frac{(1+\delta)x}{\varphi(m)(\log(1+\delta)x)^A} - \frac{x}{\varphi(m)(\log x)^A}\right)\right) \\
&= \frac{\delta}{\varphi(m)\log x} + \mathcal{O}\left(\frac{\delta^2}{\varphi(m)\log x}\right) + \mathcal{O}_\varepsilon\left(\frac{1}{\varphi(m)(\log x)^{1+\frac{4}{\varepsilon}}}\right) \\
&= \frac{\delta}{\varphi(m)\log x}\left(1+\mathcal{O}_\varepsilon\left(\frac{1}{\log y}\right)\right).
\end{aligned}$$

The last equality followed from the assumptions concerning the sizes of x and δ. An application of (25) finishes the proof. □

The following lemma provides an easy way to calculate the rest of the right-hand side:

Lemma 4.11. Given $g \geq 3$ odd, $\ell \geq 2$ even, $\theta \in]-\frac{1}{2},\frac{1}{2}]$, and set $\ell' = \frac{\ell}{(\ell,g)}$. Then

$$\frac{1}{\ell}\sum_{-\frac{\ell}{2}<n\leq\frac{\ell}{2}} \min_{z\in\mu_g\cup\{0\}} \left(1-\Re\left(z\cdot e\left(\theta-\frac{n}{\ell}\right)\right)\right) = 1 - \frac{\sin\frac{\pi}{g}}{\ell'\tan\frac{\pi}{g\ell'}} F_{g\ell'}(-g\ell'\theta),$$

where

$$F_N(x) := \cos\frac{2\pi\{x\}}{N} + \left(\tan\frac{\pi}{N}\right)\sin\frac{2\pi\{x\}}{N}.$$

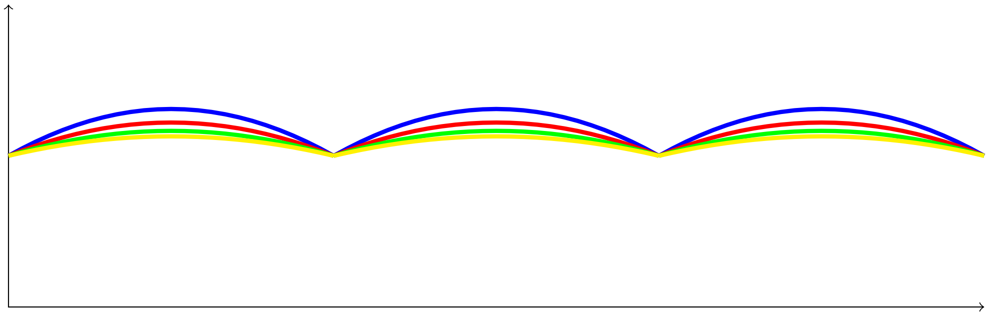

Figure 4. Illustration of functions $F_6(x)$ (blue), $F_7(x)$ (red), $F_8(x)$ (green) and $F_9(x)$ (yellow).

Proof. We make a couple of apparent remarks of the function $F_N(x)$. First of all it is 1-periodic, and so we can always assume that $x \in [0,1[$. It is also clear that it is symmetric about the line $x = \frac{1}{2}$ and concave in the interval $x \in [0,1[$. Finally, the average value of the function $F_N(x)$ in the interval $[0,1[$ is

$$\overline{F}_N = \int_0^1 F_N(x)dx = \int_0^1 \left(\cos\frac{2\pi x}{N} + \left(\tan\frac{\pi}{N}\right)\sin\frac{2\pi x}{N}\right)dx = \frac{N}{\pi}\tan\frac{\pi}{N}.$$

Then set $d = (g, \ell)$, $\ell^{'} = \frac{\ell}{d}$ and $g^{'} = \frac{g}{d}$. To prove the lemma we have to show that

$$\sum_{-\frac{\ell}{2}<n\leq\frac{\ell}{2}} \max_{z\in\mu_g\cup\{0\}} \Re\left(z\cdot e\left(\theta - \frac{n}{\ell}\right)\right) = d\cdot\frac{\sin\frac{\pi}{g}}{\tan\frac{\pi}{g\ell'}} F_{g\ell'}(-g\ell^{'}\theta).$$

Let $\mathscr{B}_0 = \left\{e(t)| - \frac{1}{2g} < t \leq \frac{1}{2g}\right\}$ and $\mathscr{B}_m = e\left(\frac{m}{g}\right)\mathscr{B}_0$ for $m \geq 1$. Notice that the disjoint union of $\mathscr{B}_m$, for $m = 0, 1, ..., g-1$, forms the unit circle. For the sake of convenience, we define $\mathscr{B}_{-t} = \mathscr{B}_{g-t}$ for all natural numbers t. Thus for any $n \in \mathbb{Z}$ there exists a unique $m_n \in \left]-\frac{g}{2}, \frac{g}{2}\right]$ such that $e\left(\theta - \frac{n}{\ell}\right) \in \mathscr{B}_{m_n}$, and by definition $e\left(-\frac{m_n}{g}\right)e\left(\theta - \frac{n}{\ell}\right) \in \mathscr{B}_0$. Since, for all the other n on this interval, $e\left(-\frac{m_n}{g}\right)e\left(\theta - \frac{n}{\ell}\right) \notin \mathscr{B}_0$, we obtain

$$\max_{z\in\mu_g\cup\{0\}} \Re\left(z\cdot e\left(\theta - \frac{n}{\ell}\right)\right) = \Re\left(e\left(-\frac{m_n}{g}\right)e\left(\theta - \frac{n}{\ell}\right)\right) = \Re\left(e(\theta)e\left(\frac{f(n)}{g\ell}\right)\right)$$

where the function $f : \mathbb{Z} \to \mathbb{Z}$ is defined to be $f(n) = -(gn + \ell m_n)$. Hence

$$\sum_{-\frac{\ell}{2}<n\leq\frac{\ell}{2}} \max_{z\in\mu_g\cup\{0\}} \Re\left(z\cdot e\left(\theta - \frac{n}{\ell}\right)\right) = \Re\left(e(\theta)\sum_{-\frac{\ell}{2}\leq n<\frac{\ell}{2}} e\left(\frac{f(n)}{g\ell}\right)\right). \tag{28}$$

The next step is to express the left-hand side as a sum of a geometric series. By elementary means we can establish

Lemma 4.12. The following claims hold:

1. If $n_1 \equiv n_2 \pmod{\ell'}$ then $f(n_1) \equiv f(n_2) \pmod{g\ell}$.

2. The restricted map $f|[-\frac{\ell'}{2}+\ell'\theta, \frac{\ell'}{2}+\ell'\theta[\cap\mathbb{Z}$ is an injection into

$$\left]-\frac{\ell}{2}-g\ell\theta, \frac{\ell}{2}-g\ell\theta\right]\cap\mathbb{Z}.$$

Proof. 1. The condition implies that $\ell|g(n_1-n_2)$. So there exists an integer m such that

$$-\frac{n_1}{\ell}=\frac{m}{g}-\frac{n_2}{\ell}. \tag{29}$$

As in the previous page, we have that $e\left(-\frac{m_{n_1}}{g}\right)e\left(\theta-\frac{n_1}{\ell}\right)\in\mathscr{B}_0$. Combining this with (29) yields $e\left(-\frac{m_{n_1}}{g}\right)e\left(\theta-\frac{m}{g}-\frac{n_2}{\ell}\right)\in\mathscr{B}_0$ which further implies that $e\left(\frac{m-m_{n_1}}{g}\right)\in e\left(\frac{n_2}{\ell}-\theta\right)\mathscr{B}_0$. Clearly $e\left(-\frac{m_{n_2}}{g}\right)$ belongs also to the set $e\left(\frac{n_2}{\ell}-\theta\right)\mathscr{B}_0$. This implies that $m_{n_1}\equiv m+m_{n_2} \pmod{g}$, from which we calculate

$$\begin{aligned} f(n_1)-f(n_2) &= -(gn_1+\ell m_{n_1})+(gn_2+\ell m_{n_2}) \\ &= g(n_2-n_1)+\ell(m_{n_2}-m_{n_1}) \\ &\equiv 0-\ell\cdot\frac{g(n_1-n_2)}{\ell} \\ &\equiv 0 \pmod{g\ell'}. \end{aligned}$$

This finishes the proof.

2. For injectivity we prove a little stronger result: $n_1\equiv n_2 \pmod{\ell'}$ if and only if $f(n_1)\equiv f(n_2)\pmod{\ell}$. The proof is short: If $f(n_1)\equiv f(n_2)\pmod{\ell}$, then we have $g(n_2-n_1)\equiv 0\pmod{\ell}$. Dividing by (g,ℓ) gives $n_1-n_2\equiv 0\pmod{\ell'}$. On the other hand, if $n_1\equiv n_2\pmod{\ell'}$ we have $f(n_1)-f(n_2)\equiv g(n_2-n_1)\equiv 0\pmod{\ell}$. The last congruence follows from the fact $(g,\ell)\cdot\ell'=\ell$. The other direction follows from part 1. This concludes the proof and the injectivity follows.

Now, choose an integer $n\in\left[-\frac{\ell'}{2}+\ell'\theta, \frac{\ell'}{2}+\ell'\theta\right[$ or equivalently $\theta-\frac{n}{\ell'}\in\left]-\frac{1}{2},\frac{1}{2}\right]$. Remember that $e\left(\theta-\frac{n}{\ell'}\right)\in\mathscr{B}_{m_n}$. Furthermore,

$$\theta-\frac{n}{\ell'}\in\left]A+\frac{2m_n-1}{2g}, A+\frac{2m_n+1}{2g}\right]$$

for some integer A. Actually, since $-\frac{g-1}{2}\le m_n\le\frac{g-1}{2}$ it follows that $A-\frac{1}{2}<\theta-\frac{n}{\ell'}\le A+\frac{1}{2}$ and so, in view of the choice of n, $A=0$. Hence

$$\theta-\frac{n}{\ell'}\in\left]\frac{2m_n-1}{2g}, \frac{2m_n+1}{2g}\right]$$

from which it is easy to verify that $f(n)\in\left]-\frac{\ell}{2}-g\ell\theta, \frac{\ell}{2}+g\ell\theta\right]$. This completes the proof. $\square$

The first claim tells that we actually have

$$\sum_{-\frac{\ell}{2}\le n<\frac{\ell}{2}} e\left(\frac{f(n)}{g\ell}\right)=d\cdot\sum_{-\frac{\ell'}{2}\le n'<\frac{\ell'}{2}} e\left(\frac{f(n')}{g\ell}\right) \tag{30}$$

Noting that $d|f(n)$ for all n and using the second claim of the Lemma 4.12. we get that the set

$$\left\{f(n')| -\frac{\ell'}{2}+\ell'\theta \le n' \le \frac{\ell'}{2}+\ell'\theta\right\}$$

contains ℓ' distinct multiplies of d, all contained in $\left]-\frac{\ell}{2}-g\ell\theta, \frac{\ell}{2}+g\ell\theta\right]$. On the other hand, this set contains exactly ℓ' multiples of d. Therefore we can calculate,

$$\begin{aligned}
\sum_{-\frac{\ell'}{2}\le n'<\frac{\ell'}{2}} e\left(\frac{f(n')}{g\ell}\right) &= \sum_{-\frac{\ell'}{2}+\ell'\theta\le n'<\frac{\ell'}{2}+\ell'\theta} e\left(\frac{f(n')}{g\ell}\right) \\
&= \sum_{\frac{1}{d}\left(-\frac{\ell}{2}-g\ell\theta\right)<s\le\frac{1}{d}\left(\frac{\ell}{2}-g\ell\theta\right)} e\left(\frac{sd}{g\ell}\right) \\
&= \sum_{-\frac{\ell'}{2}-g\ell'\theta<s\le\frac{\ell'}{2}-g\ell'\theta} e\left(\frac{s}{g\ell'}\right) \\
&\overset{(*)}{=} \frac{e\left(\frac{1}{g\ell'}\lfloor\frac{\ell'}{2}-g\ell'\theta\rfloor\right)\left(1-e\left(-\frac{1}{g\ell'}\right)^{\ell'}\right)}{1-e\left(-\frac{1}{g\ell'}\right)} \\
&= \frac{e\left(-\theta+\frac{1-2\{-g\ell'\theta\}}{2g\ell'}\right)\left(e\left(\frac{1}{2g}\right)-e\left(-\frac{1}{2g}\right)\right)}{e\left(\frac{1}{2g\ell'}\right)-e\left(-\frac{1}{2g\ell'}\right)} \\
&= \frac{e\left(-\theta+\frac{1-2\{-g\ell'\theta\}}{2g\ell'}\right)\sin\frac{\pi}{g}}{\sin\frac{\pi}{g\ell'}}. \qquad (31)
\end{aligned}$$

In the step $(*)$ we used the summation formula of the geometric progression. Collecting (28), (30) and (31) together gives

$$\begin{aligned}
\sum_{-\frac{\ell}{2}<n\le\frac{\ell}{2}} \max_{Z\in\mu_g\cup\{0\}} \Re\left(z\cdot e\left(\theta-\frac{n}{\ell}\right)\right) &= d\cdot\frac{\sin\frac{\pi}{g}}{\sin\frac{\pi}{g\ell'}}\cdot\Re\left(e(\theta)\cdot e\left(-\theta+\frac{1-2\{-g\ell'\theta\}}{2g\ell'}\right)\right) \\
&= d\cdot\frac{\sin\frac{\pi}{g}}{\sin\frac{\pi}{g\ell'}}\cdot\cos\left(\frac{\pi(1-2\{-g\ell'\theta\})}{g\ell'}\right) \\
&= d\cdot\frac{\sin\frac{\pi}{g}}{\sin\frac{\pi}{g\ell'}}\cdot F_{g\ell'}(-g\ell'\theta).
\end{aligned}$$

This finishes the proof of Lemma 4.11. □

Using Lemmas 4.10 and 4.11, the main term can be estimated as

$$\begin{aligned}
\sum_{x<p\le(1+\delta)x}\frac{1-\Re(\chi(p)\overline{\xi}(p)p^{-it})}{p} &\ge \sum_{-\frac{\ell}{2}\le n<\frac{\ell}{2}}\left(\sum_{\substack{x<p\le(1+\delta)x\\ \xi(p)=e\left(\frac{n}{\ell}\right)}}\frac{1}{p}\right)\min_{z\in\mu_g\cup\{0\}}\left(1-\Re ze\left(\theta_x-\frac{n}{\ell}\right)\right) \\
&= \frac{\delta(1+o(1))}{\log x}\left(1-\frac{\sin\frac{\pi}{g}}{\ell'\tan\frac{\pi}{g\ell'}}F_{g\ell'}(-g\ell'\theta_x)\right).
\end{aligned}$$

Let us define a new function

$$G(s) := 1 - \frac{\sin\frac{\pi}{g}}{\ell' \tan\frac{\pi}{g\ell'}} F_{g\ell'}\left(\frac{tg\ell'}{2\pi} \cdot s\right).$$

We notice that G is minimized for those s for which $F_{g\ell'}$ is maximized i.e., when $\frac{tg\ell'}{2\pi} \cdot s$ is a half-integer. Thus

$$G(s) \geq 1 - \frac{\sin\frac{\pi}{g}}{\ell' \tan\frac{\pi}{g\ell'}} > 0,$$

where the last inequality sign is justified by a straightforward differentiation.

When this is combined with our earlier bound (27) we get

$$\begin{aligned}
\sum_{x<p\leq(1+\delta)x} \frac{1-\Re(\chi(p)\overline{\xi}(p)p^{-it})}{p} &= \frac{(1+o(1))\delta}{\log x} G\left(-\frac{2\pi}{t}\cdot\theta_x\right) + \mathcal{O}\left(\frac{\delta^2\log^2 y}{\log x}\right) \\
&= \frac{(1+o(1))\delta}{\log x} G(\log x) + \mathcal{O}\left(\frac{\delta^2\log^2 y}{\log x}\right) \\
&= \frac{(1+o(1))\delta}{\log x} G(\log x), \qquad (32)
\end{aligned}$$

where $o(1) \to 0$ as $y \to \infty$. The last step follows since $\mathcal{O}\left(\frac{\delta^2\log^2 y}{\log x}\right) = \mathcal{O}\left(\frac{1}{\log^4 y \log x}\right) = o(1)$, which in turn follows from the fact $\delta \asymp (\log y)^{-3}$.

Let $x_0 = \exp((\log y)^{\varepsilon})$ and $x_r = x_0(1+\delta)^r$ for $r \geq 1$. With the aid of (32) we deduce

$$\begin{aligned}
\mathbb{D}(\chi(n), \xi(n)n^{it}; y)^2 &\geq \sum_{x_0<p\leq y} \frac{1-\Re(\chi(p)\overline{\xi}(p)p^{-it})}{p} \\
&\geq \sum_{\substack{r\geq 0\\ x_{r+1}\leq y}} \sum_{x_r<p\leq x_{r+1}} \frac{1-\Re(\chi(p)\overline{\xi}(p)p^{-it})}{p} \\
&\geq \sum_{\substack{r\geq 0\\ x_{r+1}\leq y}} \frac{(1+o(1))\delta}{\log x_r} G(\log x_r) \\
&\geq (1+o(1))\log(1+\delta) \sum_{\substack{r\geq 0\\ x_{r+1}\leq y}} \frac{G(\log x_r)}{\log x_r}. \qquad (33)
\end{aligned}$$

The sum on the right is a left Riemann sum for the integral $\int_{\log x_0}^{\log x_m} \frac{G(z)}{z} dz$, where m is an integer for which $x_m \leq y < x_{m+1}$ and the length of the subinterval is $\Delta = \log(1+\delta)$. For all $s \geq \log x_0$ we estimate trivially

$$\left|\frac{d}{ds}\frac{G(s)}{s}\right| \leq \left|\frac{G'(s)}{s}\right| + \left|\frac{G(s)}{s^2}\right| \leq \frac{\frac{\sin\frac{\pi}{g}}{\ell'\tan\frac{\pi}{g\ell'}} \cdot \frac{tg\ell'}{2\pi} F_{g\ell'}(0)}{\log x_0} + \frac{2}{(\log x_0)^2} \ll 1. \qquad (34)$$

Now, we would like to use the error formula for the left Riemann sum, which states that for a continuously differentiable function f it holds that

$$\left|\int_a^b f(x)dx - R\right| \leq \frac{b-a}{2} \cdot \Delta \cdot \max_{x\in[a,b]} f'(x), \qquad (35)$$

where R is the value of a left Riemann sum of the function f in the interval $[a, b]$.

We face a problem, since the formula requires $\frac{G(s)}{s}$ to be continuously differentiable which it is not (it has cusps at integer points). The problem can be avoided by choosing δ such that 1 is an integer multiple of $\log(1+\delta)$ and x_0 such that $\log x_0$ is an integer; see Figure 5.

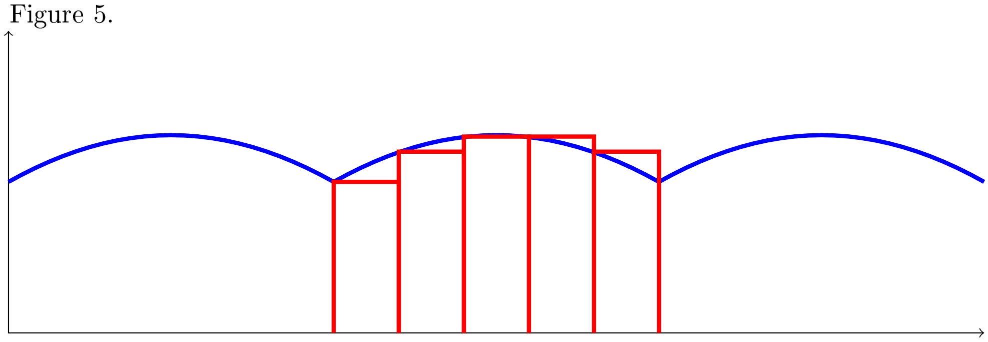

Figure 5. The left Riemann sum for $\int_{\log x_0}^{\log x_m} \frac{G(z)}{z} dz$ for above choices of δ and x_0.

From (34) and (35) we get

$$\begin{aligned}\left|\log(1+\delta)\sum_{\substack{r\geq 0\\ x_{r+1}\leq y}} \frac{G(\log x_r)}{\log x_r} - \int_{\log x_0}^{\log y} \frac{G(s)}{s} ds\right| &\leq \left|\log(1+\delta)\sum_{\substack{r\geq 0\\ x_{r+1}\leq y}} \frac{G(\log x_r)}{\log x_r} - \int_{\log x_0}^{\log x_m} \frac{G(s)}{s} ds\right| \\ &+ \left|\int_{\log x_m}^{\log y} \frac{G(s)}{s} ds\right| \\ &\leq \log(1+\delta)\cdot \frac{\log x_m - \log x_0}{2} \cdot \max_{s\in[\log x_0, \log x_m]} \left|\frac{d}{ds}\frac{G(s)}{s}\right| \\ &+ \left|\int_{\log x_m}^{\log y} \frac{G(s)}{s} ds\right| \\ &\ll \log(1+\delta)\cdot \log y + \frac{1}{\log^2 y} \ll \frac{1}{\log^2 y} \ll 1,\end{aligned}$$

where the first estimate on the last line follows since $\delta \asymp (\log y)^{-3}$ and $\log(1+\delta) \ll \delta$. Thus the estimate (33) gives

$$\mathbb{D}(\chi(n), \xi(n)n^{it}; y)^2 \geq (1+o(1)) \int_{\log x_0}^{\log y} \frac{G(s)}{s} ds + \mathcal{O}(1).$$

We are done if we manage to prove that

$$\int_{\log x_0}^{\log y} \frac{G(s)}{s} ds \geq (\delta_g + o(1)) \log\log y,$$

where $\delta_g = 1 - \frac{g}{\pi}\sin\frac{\pi}{g}$.

Setting $N = g\ell'$ and making a change of variable $\frac{Nts}{2\pi} \mapsto s$ we see that it is enough to prove that

$$\int_{\log x_0}^{\log y} \frac{1}{s} ds - \int_{\log x_0}^{\log y} \frac{\sin\frac{\pi}{g}}{s\cdot \frac{N}{g}\tan\frac{\pi}{N}} F_N\left(\frac{Nt}{2\pi}\cdot s\right) ds \geq (\delta_g + o(1)) \log\log y.$$

On the other hand, we have

$$\begin{aligned}\int_{\log x_0}^{\log y} \frac{\sin\frac{\pi}{g}}{s\cdot\frac{N}{g}\tan\frac{\pi}{N}} F_N\left(\frac{Nt}{2\pi}\cdot s\right) ds &= \int_{\log x_0}^{\log y} \frac{F_N\left(\frac{Nt}{2\pi}\right)}{\frac{2\pi}{Nt}\cdot\frac{Nt}{2\pi}\cdot s}\cdot\frac{\sin\frac{\pi}{g}}{\frac{N}{g}\tan\frac{\pi}{N}} ds \\ &= \int_{\frac{Nt}{2\pi}\log x_0}^{\frac{Nt}{2\pi}\log y} \frac{Nt}{2\pi}\cdot\frac{\sin\frac{\pi}{g}}{\frac{N}{g}\tan\frac{\pi}{N}}\cdot\frac{2\pi}{Nt}\cdot\frac{F_N(s)}{s} ds\end{aligned}$$

by using the formula[7]

$$\int_{\phi(a)}^{\phi(b)} f(x)dx = \int_a^b f(\phi(t))\phi'(t)dt,$$

with a choice $\phi(s) = \frac{Nt}{2\pi}\cdot s$.

Hence it suffices to prove that

$$\int_{\log x_0}^{\log y} \frac{1}{s} ds - \int_{\frac{Nt}{2\pi}\log x_0}^{\frac{Nt}{2\pi}\log y} \frac{\sin\frac{\pi}{g}}{\frac{N}{g}\tan\frac{\pi}{N}}\cdot\frac{F_N(s)}{s} ds \geq (\delta_g + o(1))\log\log y$$

which is equivalent to

$$\begin{aligned}\int_{\frac{Nt}{2\pi}\log x_0}^{\frac{Nt}{2\pi}\log y} \frac{1}{s}F_N(s)ds &\leq \left(\int_{\log x_0}^{\log y}\frac{1}{s}ds - (\delta_g + o(1))\log\log y\right)\cdot\frac{\frac{\pi}{g}\cdot\frac{N}{\pi}\tan\frac{\pi}{N}}{\sin\frac{\pi}{g}} \\ &= \left(\frac{\frac{\pi}{g}}{\sin\frac{\pi}{g}}(\log\log y - \log\log x_0) - \left(\frac{\frac{\pi}{g}}{\sin\frac{\pi}{g}} - 1 + o(1)\right)\log\log y\right)\frac{N}{\pi}\tan\frac{\pi}{N} \\ &\overset{(*)}{=} (1 - \varepsilon + o(1))\overline{F}_N\log\log y \\ &= (\overline{F}_N + o(1))\log\log y\end{aligned}$$

where $\frac{\log\log y}{\log y} \ll |t| \leq \log^2 y$ and $\overline{F}_N$ is the average value of the function F_N over the unit interval. The step $(*)$ is based on the definition $x_0 = \exp((\log y)^{\varepsilon})$.

Now we split the consideration into two cases. For clarity, we set $a(y) = \frac{Nt}{2\pi}\log x_0$ and $b(y) = \frac{Nt}{2\pi}\log y$.

If $a(y) \geq 1$ we have split the interval $[1, x]$ to unit intervals, with at most one exception, and bound the term $\frac{1}{s}$ trivially on each of these intervals. This gives

$$\int_{a(y)}^{b(y)} \frac{1}{s}\cdot F_N(s)ds = \overline{F_N}\cdot\log\frac{b(y)}{a(y)} + \mathcal{O}(1) \leq (\overline{F_N} + o(1))\log\log y,$$

and the claim follows.

If $a(y) < 1$ we can assume that $b(y) \geq 1$ by choosing implicit constant in the estimate $|t| \gg \frac{\log\log y}{\log y}$ large enough. Now, splitting the interval into two parts $[a(y), 1]$ and $[1, b(y)]$, making a change of variable $s \mapsto \frac{1}{s}$ in the first integral and doing the same calculation as above we deduce

$$\begin{aligned}\int_{a(y)}^{b(y)} \frac{1}{s}\cdot F_N(s)ds &= \int_{a(y)}^{1}\frac{1}{s}F_N(s)ds + \int_1^{b(y)}\frac{1}{s}F_N(s)ds \\ &= \int_1^{\frac{1}{a(y)}}\frac{1}{s}F_N\left(\frac{1}{s}\right)ds + \overline{F_N}\cdot\log b(y) + \mathcal{O}(1).\end{aligned}$$

[7]The formula holds for a continuously differentiable function $\phi : [a, b] \to I$ and a continuous function $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval.

Hence it suffices to prove that

$$\int_1^x \frac{1}{s} F_N\left(\frac{1}{s}\right) ds \leq \overline{F_N} \cdot \log x + \mathcal{O}(1).$$

But this follows from the fact that $F_N\left(\frac{1}{x}\right) \leq \overline{F_N}$ for sufficiently large x (since F_N is concave in the unit interval). Thus the proof is completed. □

4.5 A Pretentious Proof for the Prime Number Theorem

As an application of pretentious methods we will show how the Prime Number Theorem can be obtained by the pretentious triangle inequality and Halász–Montgomery–Tenenbaum Theorem. The main applications of pretentious methods in this thesis are focused on character sums and are studied in Chapter 6. The third example of the power of these methods is the proof of the Distribution Theorem that concerns the equidistribution of functions with $|f(n)| = 1$ on the unit circle. This is discussed in [23, 29]. Still, the most astonishing result obtained using pretentious methods is the recent proof of the Arithmetic Quantum Unique Ergodicity Conjecture originally due to Rudnik and Sarnak. It was proved by Soundararajan and Holowinsky. This matter is not treated here, but we encourage the reader to take a look at the papers [46, 47, 72].

Maybe the most significant work concerning the distribution of primes was Riemann's memoir [66], which introduced the idea of applying complex analytic methods to study the function $\pi(x)$. Inspired by these ideas Vallée–Poussin [77] and Hadamard [35], [36] managed, independently, prove the PNT in 1896. The central idea in both of these proofs was to show that the ζ-function is non-vanishing on the line $\{\sigma = 1\}$.

Many other proofs have been found since. These include the elementary proofs of Erdös [19] and Selberg [69] in 1946. Both of these proofs were based on the "fundamental inequality" due to Selberg. The question whether to write a joint paper on the matter lead to a bitter dispute between these two mathematicians, see [31]. Still, probably the simplest proof is due to Newman [61] which dates back to 1980. The proof is based on a clever use of contour integration.

In this section we give a pretentious proof for the PNT following the manuscript [29]. There are also other pretentious proofs for this matter which make use of the Brun–Titchmarsh Theorem [21]. Our starting point is that the Prime Number Theorem is true if

$$\left|\sum_{n\leq x} \mu(n)\right| = o(x)$$

This implication is proved, for example, in [67] where it is Theorem 4.3.

We continue with the following result:

Lemma 4.13. Let f be a real-valued multiplicative function with $-1 \leq f(n) \leq 1$ and $|\alpha| \leq (\log x)^{10}$, $x \geq 1$. Then we have

$$\mathbb{D}(f, p^{i\alpha}; x) \geq \min\left(\frac{1}{2}\sqrt{\log\log x} + \mathcal{O}(1), \frac{1}{3}\mathbb{D}(1, f; x) + \mathcal{O}(1)\right).$$

Proof. Since $\mathbb{D}(f, p^{i\alpha}; x) = \mathbb{D}(f, p^{-i\alpha}; x)$ the triangle inequality gives

$$\mathbb{D}(1, p^{2i\alpha}; x) = \mathbb{D}(p^{i\alpha}, p^{-i\alpha}; x) \leq 2\mathbb{D}(f, p^{i\alpha}; x). \tag{36}$$

Notice that we have

$$\begin{aligned}
\mathbb{D}(1,p^{i\alpha};x)^2 &= \sum_{p\le x}\frac{1-\Re(p^{-i\alpha})}{p}\\
&= \sum_{p\le x}\frac{1}{p}-\sum_{p\le x}\frac{\Re(p^{-i\alpha})}{p}\\
&= \log\log x-\log\left|\zeta\left(1+\frac{1}{\log x}+i\alpha\right)\right|+\mathcal{O}(1).
\end{aligned}$$

If $\frac{1}{100}\le|\alpha|\le(\log x)^{10}$ we use the fact $|\zeta(s)|\ll\log(2+|s|)$, which holds when $|s-1|\gg 1$, to obtain

$$\begin{aligned}
\mathbb{D}(1,p^{2i\alpha};x)^2 &\ge \log\log x-\frac{\log\left|\zeta\left(1+\frac{1}{\log x}+2i\alpha\right)\right|}{\log\log x}\cdot\log\log x+\mathcal{O}(1)\\
&\ge \log\log x-\frac{\log\log(2+|1+\frac{1}{\log x}+2i\alpha|)}{\log\log x}\cdot\log\log x+\mathcal{O}(1)\\
&\ge (1-\varepsilon)\log\log x+\mathcal{O}(1)
\end{aligned}$$

for some fixed $0<\varepsilon<1$.

This follows since

$$0<\frac{\log\log(2+|1+\frac{1}{\log x}+2i\alpha|)}{\log\log x}\le\frac{\log\log\left(2+\sqrt{\left(1+\frac{1}{\log x}\right)^2+4(\log x)^{20}}\right)}{\log\log x}<1$$

when x is large enough.

Now,

$$\mathbb{D}(f,p^{i\alpha};x)^2\ge\frac{1}{4}\mathbb{D}(1,p^{2i\alpha};x)^2\ge\frac{1}{4}(1-\varepsilon)\log\log x+\mathcal{O}(1),$$

which gives the desired bound. On the other hand, if $|\alpha|\le\frac{1}{100}$ we have

$$\mathbb{D}(1,p^{2i\alpha};x)=\mathbb{D}(1,p^{i\alpha};x)+\mathcal{O}(1). \tag{37}$$

This follows from the identity $\frac{|s|}{|s-1|}-|s|\le|\zeta(s)|\le\frac{|s|}{|s-1|}+|s|$, which holds for $s>1$, as we have[8]

$$\begin{aligned}
&\log\left|\zeta\left(1+\frac{1}{\log x}+i\alpha\right)\right|-\log\left|\zeta\left(1+\frac{1}{\log x}+2i\alpha\right)\right|\\
&= \log\left(\left|\frac{\zeta\left(1+\frac{1}{\log x}+i\alpha\right)}{\zeta\left(1+\frac{1}{\log x}+i\alpha\right)}\right|\right)\\
&\le \log\left(\frac{\frac{|1+\frac{1}{\log x}+i\alpha|}{|\frac{1}{\log x}+i\alpha|}+|1+\frac{1}{\log x}+i\alpha|}{\frac{|1+\frac{1}{\log x}+2i\alpha|}{|\frac{1}{\log x}+2i\alpha|}-|1+\frac{1}{\log x}+2i\alpha|}\right)\\
&\ll 1.
\end{aligned}$$

[8]The last estimate is hard to obtain by a direct calculation. However, it is easily checked by using a computer.

Thus the triangle inequality and estimates (36), (37) give

$$\begin{aligned}\mathbb{D}(f,p^{i\alpha};x) &\geq \mathbb{D}(1,f;x)-\mathbb{D}(1,p^{i\alpha};x)\\ &\geq \mathbb{D}(1,f;x)-\mathbb{D}(1,p^{2i\alpha};x)+\mathcal{O}(1)\\ &\geq \mathbb{D}(1,f;x)-2\mathbb{D}(f,p^{i\alpha};x)+\mathcal{O}(1)\end{aligned}$$

from which it follows

$$\mathbb{D}(f,p^{i\alpha};x)\geq \frac{1}{3}\mathbb{D}(1,f;x)+\mathcal{O}(1),$$

as desired. $\square$

Combining the previous lemma with the Halász–Montgomery–Tenenbaum Theorem (Theorem 3.4 with a choice $T=(\log x)^{10}$) yields

$$\begin{aligned}\frac{1}{x}\sum_{n\leq x} f(n) &\ll \exp\left(-\frac{1}{4}\log\log x+\mathcal{O}(1)\right)+\mathbb{D}(1,f;x)^2\exp\left(-\frac{1}{9}\mathbb{D}(1,f;x)^2\right)+\frac{1}{(\log x)^5}.\\ &\ll \mathbb{D}(1,f;x)^2\exp\left(-\frac{1}{9}\mathbb{D}(1,f;x)^2\right)+\frac{1}{(\log x)^{\frac{1}{4}+o(1)}}.\end{aligned}$$

Choosing $f=\mu$ in the above formula and using Mertens' Theorem gives

$$\begin{aligned}\left|\sum_{n\leq x}\mu(n)\right| &\ll x\cdot(\log\log x)\exp\left(-\frac{2}{9}\log\log x\right)+\frac{x}{(\log x)^{\frac{1}{4}+o(1)}}\\ &\ll \frac{x}{(\log x)^{\frac{2}{9}+o(1)}} \qquad (38)\\ &= o(x).\end{aligned}$$

This completes the proof. $\square$

We remark that the PNT is also true if

$$\psi(x)=x+o(x).$$

This is proved in [73] where it is Proposition 2.1 in Chapter 7. In the view of this, if we want a version with an error estimate, it follows from (38) that

$$\psi(x)=x+\mathcal{O}\left(\frac{x}{(\log x)^{\frac{2}{9}+o(1)}}\right)$$

(see the remark on p. 34 in [29]).

This method yields a worse error term than the classical proofs. The best known error term is given by

$$\psi(x)=x+\mathcal{O}\left(x\exp\left(-(\log x)^{\frac{3}{5}+o(1)}\right)\right).$$

This was obtained by Korobov [51] and Vinogradov [80] in 1958.

5 Exponential Sums with Multiplicative Coefficients

In this chapter our goal is to prove the Montgomery–Vaughan bound which estimates the exponential sums whose coefficients are multiplicative functions. The bound itself is not used later in this thesis, but the corollaries are. At first we, however, recall some things from the theory of rational approximations, which are used frequently in Chapter 6.

5.1 On Rational Approximations

At first we prove Dirichlet's Approximation Theorem:

Theorem 5.1. For any real number α and natural number N, there exists integers b and r such that $1 \leq r \leq N$ and

$$\left|\alpha - \frac{b}{r}\right| \leq \frac{1}{rN}.$$

Proof. We consider the fractional parts $\{0 \cdot \alpha\}, \{\alpha\}, ..., \{N \cdot \alpha\}$ which belong to the half open interval $[0, 1[$. This interval can be written as a union of N subintervals

$$[0, 1[= \bigcup_{m=0}^{N-1} \left[\frac{m}{N}, \frac{m+1}{N}\right[.$$

Now we have $N + 1$ fractional parts and N subintervals, so by the pigeonhole principle there exists $0 \leq k < \ell \leq N$ such that $\{k \cdot \alpha\}$ and $\{\ell \cdot \alpha\}$ belong to the same subinterval. In particular, there exists an integer b such that $|(\ell - k)\alpha - b| < \frac{1}{N}$. Choosing $r = \ell - k$ finishes the proof. □

Now we can define a classical concept related to the circle method:

Definition 5.2. Let $\alpha \in [0, 1]$. We say that α lies on a *minor arc* if it has rational approximation with a large denominator. Otherwise, if there is no such rational approximation, we say that α lies on a *major arc*. Here, the concepts "small" and "large" depend on the context. These names come from the phenomenon that points lying on the major arc contribute to the main term in the estimate and points lying on the minor arc contribute to the error term. In this thesis the situation is roughly the following: Let M be a fixed natural number and c some positive constant. By Dirichlet's Approximation Theorem there exists a reduced fraction $\frac{b}{r}$ with $1 \leq r \leq M$ such that

$$\left|\alpha - \frac{b}{r}\right| \leq \frac{1}{rM}.$$

Notice that there may be many such fractions. If there exists a reduced fraction with $r > c$, we say that α lies on a minor arc. If we always have $r \leq c$, then α lies on a major arc.

5.2 The Montgomery–Vaughan Bound

Now we will move on to studying exponential sums of the form

$$\sum_{n \leq x} f(n)e(n\alpha) \qquad \text{and} \qquad \sum_{n \leq x} \frac{f(n)}{n} e(n\alpha), \tag{39}$$

where $f \in \widetilde{\mathcal{F}}$ and $\alpha \in \mathbb{R}$. In [56] Montgomery and Vaughan showed that there is cancellation in (39) if α belongs to the minor arc. This refined an old result due to Daboussi

[13]. In this section we are going to prove this. First we evaluate (39) at rational points. The following proof closely follows the original proof [56].

Theorem 5.3. Suppose that $q \leq N$ and $(a,q) = 1$. Then

$$\sum_{n \leq N} f(n) e\left(n \cdot \frac{a}{q}\right) \ll N \left(\frac{1}{\log 2N} + \frac{1}{\sqrt{\varphi(q)}} + \sqrt{\frac{q}{N}} \left(\log \frac{2N}{q} \right)^{\frac{3}{2}} \right)$$

uniformly for $f \in \widetilde{\mathcal{F}}$.

Since the proof is very involved, we will first lay out the structure behind it. Our outline has 6 steps:

1. First we easily reduce the problem to bounding the exponential sum which has coefficients of the form $f(mn)\Lambda(m)$. Then we observe that $f(mn)$ can be replaced with $f(m)f(n)$ with a suitable error.

2. Since Von Mangoldt's function vanishes unless m is a power of a prime, it suffices to bound the sum over all pairs $(m,n) = (p^k, n)$ with $mn \leq N$. It is almost immediate that those pairs with $k \geq 2$ contribute the term which is $\ll N$, and therefore it does not give us any trouble.

3. Estimating the sum over pairs (p,n) is a more subtle problem. To do this we split the area bordered by the coordinate axis and the curve $xy = N$ into rectangles of the form $]P', P''] \times]N', N'']$, with the side lengths having certain properties described in the actual proof. The reason for such division is explained in step 5.

4. However, our partition has a weakness: there exists points that do not lie on any rectangles constructed in the previous step. Luckily, the contribution of such exceptional points can be estimated quite easily with the Cauchy–Schwarz inequality and some well-known sieve estimates.

5. Then we move back to the set up in step 3. The fundamental idea of the proof is to estimate the contribution of the points lying in the rectangles one rectangle at a time. In other words we seek a non-trivial bound for the sum

$$\sum_{(p,n) \in \mathscr{R}} f(p) f(n) e\left(pn \cdot \frac{a}{q}\right) \log p$$

where $\mathscr{R}$ is one of the rectangles we have constructed. This is done again with the help of the Cauchy–Schwarz inequality.

6. The final step of the proof is to apply the estimate obtained in the previous step to each rectangle constructed in step 3. Then the theorem is obtained by summing all these bounds and the estimate for the contribution of the exceptional points derived in the step 4.

This concludes the outline, and so we are ready to begin the proof.

Proof. Since $f \in \widetilde{\mathcal{F}}$, the Cauchy–Schwarz inequality yields

$$\sum_{n \leq N} f(n) e\left(n \cdot \frac{a}{q}\right) \log \frac{N}{n} \ll \left(\sum_{n \leq N} \left(\log \frac{N}{n} \right)^2 \right)^{\frac{1}{2}} \left(\sum_{n \leq N} |f(n)|^2 \right)^{\frac{1}{2}} \ll N.$$

Thus

$$\sum_{n \leq N} f(n) e\left(n \cdot \frac{a}{q}\right) \log N \ll N + \left| \sum_{n \leq N} f(n) e\left(n \cdot \frac{a}{q}\right) \log n \right|.$$

Because of the well-known formula $\log n = \sum_{d|n} \Lambda(d)$, it is enough to show that

$$\sum_{mn \le N} f(mn)\Lambda(m)e\left(mn \cdot \frac{a}{q}\right) \ll N + \frac{N \log N}{\sqrt{\varphi(q)}} + \sqrt{Nq}\left(\log \frac{2N}{q}\right)^{\frac{3}{2}} \log N. \tag{40}$$

Next we would like to use the fact that f is multiplicative, that is $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$ such that $(m, n) = 1$. This leads us to consider the sum

$$\tau := \sum_{mn \le N} \Lambda(m) \left| f(mn) - f(m)f(n) \right|.$$

Now, since f is multiplicative, we actually have

$$\tau = \sum_{\substack{mn \le N \\ (m,n)>1}} \Lambda(m)|f(mn) - f(m)f(n)| \le \underbrace{\sum_{\substack{mn \le N \\ (m,n)>1}} \Lambda(m)|f(mn)|}_{=:\tau_1} + \underbrace{\sum_{\substack{mn \le N \\ (m,n)>1}} \Lambda(m)|f(m)f(n)|}_{=:\tau_2}.$$

Since $\Lambda(m) = 0$ unless $m = p^k$, in which case it equals $\log p$, we can write

$$\tau_1 = \sum_{p,k \ge 1} \sum_{\substack{n \le Np^{-k} \\ p|n}} \left| f(p^k n) \right| \log p$$

and

$$\begin{aligned}
\tau_2 &= \sum_{p,k \ge 1} (\log p)|f(p^k)| \sum_{m \le Np^{-k}} |f(m)| = \sum_{p,k \ge 1} (\log p)|f(p^k)| \sum_{\substack{\ell \le Np^{-k-j} \\ p \nmid \ell}} |f(p^j \ell)| \\
&= \sum_{p,k \ge 1} (\log p)|f(p^k)| \sum_{j \ge 1} |f(p^j)| \sum_{\substack{\ell \le Np^{-k-j} \\ p \nmid \ell}} |f(\ell)|.
\end{aligned}$$

Next we estimate the sums τ_1 and τ_2 separately. For τ_1 we collect the terms for which $p^\ell || p^k n$, and observe that the estimate $|f(n)| \le 1$ together with the Cauchy–Schwarz inequality gives

$$\begin{aligned}
\tau_1 &= \sum_{p,k \ge 1} (\log p) \sum_{\ell \ge 1} \left(\sum_{\substack{n \le Np^{-k} \\ p^\ell || n}} |f(p^k n)| \right) = \sum_{p,k \ge 1} (\log p) \sum_{\ell \ge 1} \left(\sum_{\substack{m \le Np^{-k} \\ p^{k+\ell} || n}} |f(p^{k+\ell})| \cdot |f(m)| \right) \\
&\overset{k+\ell=j}{\le} \sum_{p,j \ge 2} (\log p) \left| f(p^j) \right| (j-1) \sum_{m \le Np^{-j}} |f(m)| \\
&\ll N \sum_{p,j \ge 2} jp^{-j} \left| f(p^j) \right| \log p = N \sum_{p,j \ge 2} jp^{-\frac{3}{8}j} p^{-\frac{5}{8}j} |f(p^j)| \log p \\
&\ll N \left(\sum_{p,j \ge 2} j^2 p^{-\frac{3}{4}j} \log^2 p \right)^{\frac{1}{2}} \left(\sum_{p,j \ge 2} p^{-\frac{5}{4}j} \left| f(p^j) \right|^2 \right)^{\frac{1}{2}} \\
&\ll N \left(\sum_{p,j \ge 2} j^2 p^{-\frac{3}{4}j} \log^2 p \right)^{\frac{1}{2}} \left(\sum_{n} n^{-\frac{5}{4}} \left| f(n) \right|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The sums on the right are clearly convergent, and thus $\tau_1 \ll N$. Similarly, for τ_2 we have

$$\begin{aligned}
\tau_2 &\ll N \sum_{p,j,k\geq 1} \left|f(p^j)f(p^k)\right| p^{-\frac{1}{3}j-\frac{1}{3}k} p^{-\frac{2}{3}j-\frac{2}{3}k} \log p \\
&\ll N \sum_{p,j,k\geq 1} \left(|f(p^j)p^{-\frac{j}{3}}|^2 + |f(p^k)p^{-\frac{k}{3}}|^2\right) p^{-\frac{2j}{3}-\frac{2k}{3}} \log p \\
&\ll N \sum_{p,j\geq 1} \left|f(p^j)\right|^2 p^{-\frac{4j}{3}} (\log p) \sum_{k\geq 1} p^{-\frac{2k}{3}} \\
&\ll N \left(\sum_n |f(n)|^2 n^{-\frac{4}{3}} \log n\right) \\
&\ll N.
\end{aligned}$$

Combining these bounds we get $\tau \ll N$. Now we have reduced our problem to bounding the expression

$$\sum_{mn\leq N} f(m)f(n)\Lambda(m)e\left(mn\cdot\frac{a}{q}\right).$$

The pairs $(m,n) = (p^k, n)$, $k \geq 2$, contribute a term which is

$$\ll \sum_{p,k\geq 2} |f(p^k)| \log p \sum_{n\leq Np^{-k}} |f(n)| \ll N \sum_{p,k\geq 2} \left|f(p^k)\right| p^{-k} \log p \ll N.$$

Thus, in order to prove (40), it is enough to show that

$$\sum_{pn\leq N} f(p)f(n)e\left(pn\cdot\frac{a}{q}\right)\log p \ll N + \frac{N\log N}{\sqrt{\varphi(q)}} + \sqrt{Nq}\left(\log\frac{2N}{q}\right)^{\frac{3}{2}} \log N. \tag{41}$$

Now we start partitioning the region $\{(x,y) \in \mathbb{R}^2 \mid xy \leq N\}$ into rectangles. For all $0 \leq i \leq \log_2 N$ we define the rectangles

$$\mathscr{R}_i :=]0, 2^i] \times \underbrace{]N2^{-i-1}, N2^{-i}]}_{=:\mathscr{N}_i}.$$

We also set

$$\mathscr{C}_i := \min\left(i+1, \lfloor\log_2 N\rfloor - i + 1, \left\lfloor\frac{1}{2}\log_2\left(\frac{64N}{q}\right)\right\rfloor\right). \tag{42}$$

We are left with covering regions

$$\mathscr{D}_i := \left\{(x,y) \in \mathbb{R}^2 \mid xy \leq N, x > 2^i, \frac{N}{2^{i+1}} < y \leq \frac{N}{2^i}\right\}.$$

This can be done in the following manner: for a fixed i, we place rectangles $\mathscr{R}_{ijk}$ into the region $\mathscr{D}_i$, where $j = 1, 2, ..., \mathscr{C}_i$ and for each such j we choose those k which satisfy $2^{j-1} < k \leq 2^j$. The rectangles $\mathscr{R}_{ijk}$ are defined recursively: let

$$\mathscr{R}_{i12} := \left]2^i, \frac{4}{3}2^i\right] \times \left]\frac{1}{2}N2^{-i}, \frac{3}{4}N2^{-i}\right]. \tag{43}$$

The sides of such a rectangle divide the region $\mathscr{D}_i$ into two regions, with the same height, which is a half of the height of $\mathscr{D}_i$. Let these regions be $\mathscr{D}_i'$ and $\mathscr{D}_i''$. For both of these regions we choose rectangles similar to (43) but heights are half of what they used to be. In this case they are $\mathscr{R}_{i23}$ and $\mathscr{R}_{i24}$, respectively. Rectangle $\mathscr{R}_{i23}$ divides the region $\mathscr{D}_i'$ into two parts $\mathscr{D}_i^{'*}$ and $\mathscr{D}_i^{'**}$. Similarly, rectangle $\mathscr{R}_{i24}$ divides region $\mathscr{D}_i''$ into two parts $\mathscr{D}_i^{''*}$ and $\mathscr{D}_i^{''**}$ (see Figure 6.). Then we apply the same procedure until $j = \mathscr{C}_i$. So, in the j^{th} step we place the 2^{j-1} rectangles $\mathscr{R}_{ijk}$ of the form

$$\mathscr{R}_{ijk} = \left]\frac{2^{i+j}}{k}, \frac{2^{i+j+1}}{2k-1}\right] \times \left]\frac{(k-1)N}{2^{i+j}}, \frac{(2k-1)N}{2^{i+j+1}}\right]. \tag{44}$$

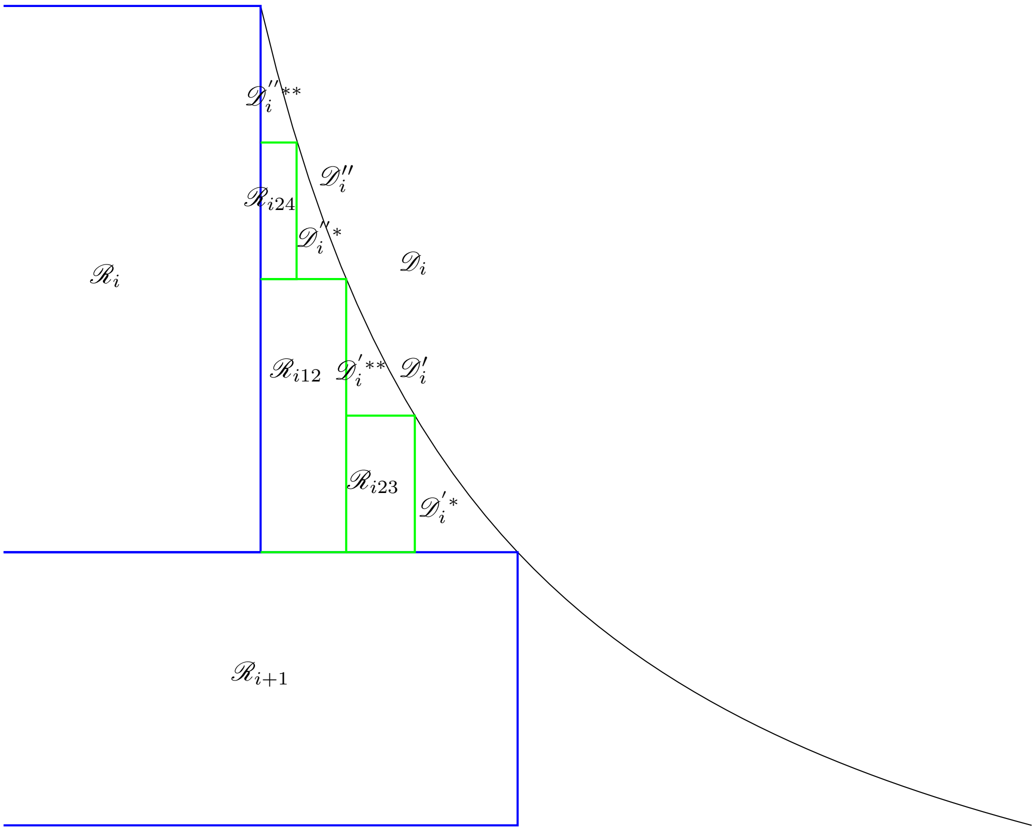

Figure 6. Illustration of placement of rectangles $\mathscr{R}_{ijk}$ to the region $\mathscr{D}_i$.

By the choice of $\mathscr{C}_i$, it is straightforward to verify that all rectangles $\mathscr{R}_{ijk}$ are of the form $]p, p'] \times]n, n']$, where

$$p' - p = \frac{2^{i+j+1}}{2k-1} - \frac{2^{i+j}}{k} = \frac{2^{i+j}}{(2k-1)k} \geq \frac{2^i}{2^j - 1} \geq \frac{2^i}{2^{\mathscr{C}_i} - 1} \geq \frac{2^i}{2^{i+1} - 1} \geq \frac{1}{4},$$

$$n' - n = \frac{(2k-1)N}{2^{i+j+1}} - \frac{(k-1)N}{2^{i+j}} = \frac{N}{2^{i+j+1}} \geq \frac{N}{2^{i+\mathscr{C}_i+1}} \geq \frac{N}{2^{\lfloor \log_2 N \rfloor} + 1} \geq \frac{1}{4},$$

and

$$(p'-p)(n'-n) \geq \frac{2^i}{2^{\mathscr{C}_i} - 1} \cdot \frac{N}{2^{i+\mathscr{C}_i+1}} > \frac{N}{2^{2\mathscr{C}_i+1}} \geq \frac{N}{\frac{64}{q} \cdot 2} \geq \frac{q}{128} \gg q.$$

However, such a partition does not cover all the points (p, n) for which $pn \leq N$. Let the set of such points be $\mathscr{E}$. Let us define

$$\mathscr{H}_i := \{(p, n) \in \mathscr{E} \mid n \in \mathscr{N}_i\}.$$

Now we define the sets $\mathscr{E}_1, \mathscr{E}_2$ and $\mathscr{E}_3$ in the following way

$$\mathscr{E}_1 := \bigcup_{\mathscr{C}_i = i+1} \mathscr{H}_i$$
$$\mathscr{E}_2 := \bigcup_{\mathscr{C}_i = \lfloor \log_2 N \rfloor - i + 1} \mathscr{H}_i$$
$$\mathscr{E}_3 := \bigcup_{\mathscr{C}_i = \left\lfloor \frac{1}{2} \log_2 \left(\frac{64N}{q} \right) \right\rfloor} \mathscr{H}_i.$$

Clearly $\mathscr{E} = \mathscr{E}_1 \cup \mathscr{E}_2 \cup \mathscr{E}_3$.

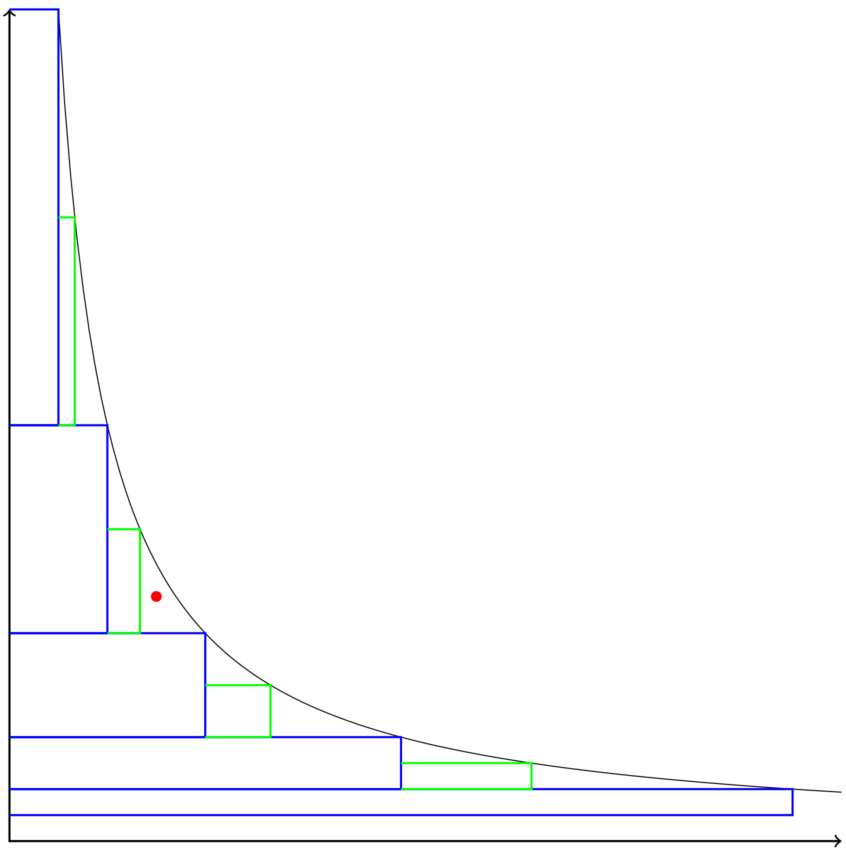

Figure 7. Illustration of the case $N = 17$ and $q = 7$. Note that in this particular case, the set $\mathscr{E}$ contains only one point, namely $(3, 5)$ (marked red in the picture). Rectangles $\mathscr{R}_i$ are coloured blue and rectangles $\mathscr{R}_{ijk}$ are coloured green[9].

The next step is to estimate the contribution of the points $(p, n) \in \mathscr{E}$ to the left-hand side of (41). Let us make few geometric observations. For a fixed p with $2^i < p < 2^{i+1}$ the pairs (p, n) which belong to the set $\mathscr{E}$ lie in an interval of length

$$\leq \frac{N}{2^{i+1}} \cdot \frac{1}{2^{\mathscr{C}_i}} \tag{45}$$

[9]Note that in the case $(N, q) = (17, 7)$ the only possibly triplets (i, j, k) are $(1, 1, 2), (2, 1, 2), (3, 1, 2)$ and $(4, 1, 2)$ as one can easily see from the restrictions of parameters i, j, k and the equation (42).

Moreover, for a fixed n there exists an integer i such that $n \in \left[\frac{N}{2^{i+1}}, \frac{N}{2^i}\right]$ and after that we choose an integer ℓ such that

$$n \in \left[\frac{N}{2^{i+1}} + \frac{N\ell}{2^{\mathscr{C}_i+i+1}}, \frac{N}{2^{i+1}} + \frac{N(\ell+1)}{2^{\mathscr{C}_i+i+1}}\right].$$

Then all the primes p such that $(p, n) \in \mathscr{E}$ lie on an interval of length

$$\frac{N}{n} - \frac{2^{\mathscr{C}_i+i+1}}{2^{\mathscr{C}_i} + 2\lfloor\frac{\ell}{2}\rfloor + 1} \tag{46}$$

Deriving the facts (45) and (46) is laborious, technical and does not require any non-standard ideas, for which reasons we skip the details. The geometrical interpretation of (46) is presented in Figure 8.

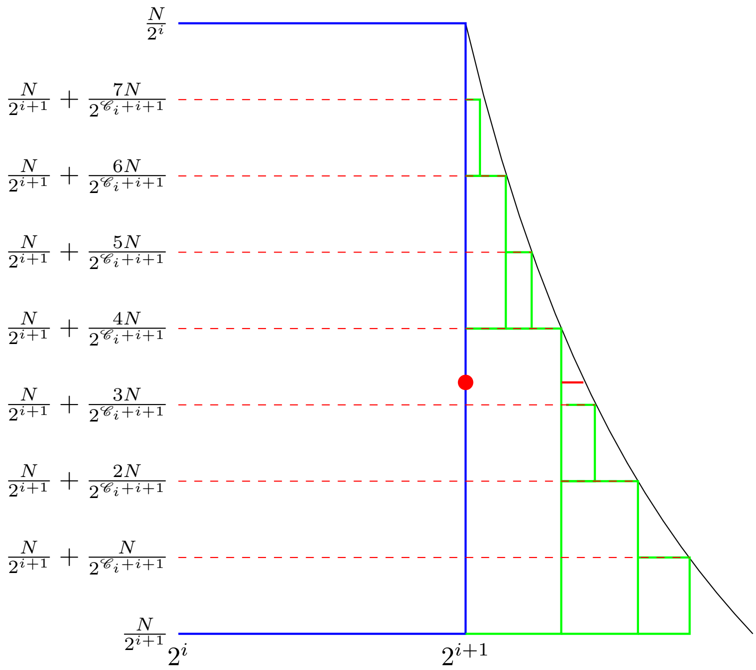

Figure 8. The case where $\mathscr{C}_i = 3$. If we fix $n \in \left[\frac{N}{2^{i+1}} + \frac{3N}{2^{\mathscr{C}_i+i+1}}, \frac{N}{2^{i+1}} + \frac{4N}{2^{\mathscr{C}_i+i+1}}\right]$, then all the pairs $(p, n) \in \mathscr{E}$ lie on the horizontal line segment which is coloured red in the picture.

For $\mathscr{E}_1$ observe that for a fixed p, the number of n such that $(p, n) \in \mathscr{E}_1$ is $\ll Np^{-2}$ by (45) as $\mathscr{C}_i = i + 1$. For a fixed n, (46) tells that primes p such that $(p, n) \in \mathscr{E}_1$ lie on an interval of length $\ll 1$ and consequently their number is also $\ll 1$. These, together with the Cauchy–Schwarz inequality, imply

$$\begin{aligned}\sum_{\mathscr{E}_1} |f(p)f(n)| \log p &\ll \left(\sum_{\mathscr{E}_1} |f(n)|^2\right)^{\frac{1}{2}} \left(\sum_{\mathscr{E}_1} (\log p)^2\right)^{\frac{1}{2}} \\ &\ll \left(\sum_{n \le N} |f(n)|^2\right)^{\frac{1}{2}} \left(\sum_{p \le N} Np^{-2} (\log p)^2\right)^{\frac{1}{2}} \\ &\ll N.\end{aligned}$$

When $(p,n) \in \mathscr{E}_2$ we have $\lfloor \log_2 N \rfloor \le 2i$ and so

$$n \le \frac{N}{2^{\frac{1}{2}\lfloor \log_2 N \rfloor}} \le \frac{N}{2^{\frac{1}{2}(\log_2 N - 1)}} = \sqrt{2N}.$$

Notice that for a fixed n, primes p such that $(p,n) \in \mathscr{E}_2$, lie on an interval of length Nn^{-2} due to (46). So, by the Brun–Titchmarsh Theorem, the number of such primes p is

$$\le \pi(x + Nn^{-2}; 2, 1) - \pi(x; 2, 1) \ll \pi(x + 4Nn^{-2}; 2, 1) - \pi(x; 2, 1) \ll \frac{Nn^{-2}}{\log 4Nn^{-2}}.$$

Moreover, for a fixed p the numbers n such that $(p,n) \in \mathscr{E}_2$ lie in an interval of length

$$\le \frac{N}{2^{\mathscr{C}_i + i + 1}} = \frac{N}{2^{\lfloor \log_2 N \rfloor + 2}} \ll 1,$$

due to (45). So the number of such pairs is $\ll 1$. Then, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
\sum_{\mathscr{E}_2} |f(p)f(n)| \log p &\ll \left(\sum_{\mathscr{E}_2} |f(n)|^2 \right)^{\frac{1}{2}} \left(\sum_{\mathscr{E}_2} (\log p)^2 \right)^{\frac{1}{2}} \\
&\ll \left(\sum_{n \le \sqrt{2N}} |f(n)|^2 \cdot \frac{Nn^{-2}}{\log 4Nn^{-2}} \right)^{\frac{1}{2}} \left(\sum_{p \le N} Np^{-2} (\log p)^2 \right)^{\frac{1}{2}} \\
&\ll N.
\end{aligned}$$

If $(p,n) \in \mathscr{E}_3$ we have

$$\left\lfloor \frac{1}{2} \log_2 \left(\frac{64N}{q} \right) \right\rfloor - 1 \le i \le \lfloor \log_2 N \rfloor - \left\lfloor \frac{1}{2} \log_2 \left(\frac{64N}{q} \right) \right\rfloor + 1.$$

Hence,

$$p \le \frac{N}{n} \le \frac{N}{2^{\lfloor \frac{1}{2} \log_2 \left(\frac{64N}{q} \right) \rfloor}} \le \frac{N}{2^{\frac{1}{2} \log_2 \left(\frac{64N}{q} \right) - 1}} = \frac{\sqrt{Nq}}{4} \le \sqrt{Nq}$$

and

$$p \ge 2^{\lfloor \frac{1}{2} \log_2 \left(\frac{64N}{q} \right) \rfloor - 1} \ge 2^{\frac{1}{2} \log_2 \left(\frac{64N}{q} \right) - 2} = 2\sqrt{\frac{N}{q}} \ge \sqrt{\frac{N}{q}}.$$

For each of those primes p, (45) tells that the number of n such that $(p,n) \in \mathscr{E}_3$ is $\ll \sqrt{Nq}p^{-1}$. For a fixed n, all primes p lie on an interval of length $\ll \sqrt{Nq}n^{-1}$ because of (46). Thus, by the Brun–Titchmarsh Theorem, the number of such primes is

$$\le \pi(x + \sqrt{Nq}n^{-1}; 2, 1) - \pi(x; 2, 1) \ll \pi(x + 2\sqrt{Nq}n^{-1}; 2, 1) - \pi(x; 2, 1) \ll \frac{\sqrt{Nq}n^{-1}}{\log 2Nqn^{-2}}.$$

Combining these with the Cauchy–Schwarz inequality yields

$$\begin{aligned}
\sum_{\mathscr{E}_3} |f(p)f(n)| \log p &\ll \left(\sum_{\mathscr{E}_3} |f(n)|^2 \log \frac{2N}{n} \right)^{\frac{1}{2}} \left(\sum_{\mathscr{E}_3} \log p \right)^{\frac{1}{2}} \\
&\ll \sqrt{Nq} \left(\sum_{n \in [\sqrt{\frac{N}{q}}, \sqrt{Nq}]} |f(n)|^2 \frac{\log(2Nn^{-1})}{n \log(2Nqn^{-2})} \right)^{\frac{1}{2}} \left(\sum_{p \in [\sqrt{\frac{N}{q}}, \sqrt{Nq}]} \frac{\log p}{p} \right)^{\frac{1}{2}}.
\end{aligned}$$

Since

$$\frac{\log(2Nn^{-1})}{\log(2Nqn^{-2})} \leq \log\left(\frac{4N}{q}\right)$$

we can continue our calculation to obtain

$$\sum_{\mathscr{E}_3} |f(p)f(n)| \log p \ll \sqrt{Nq \log \frac{2N}{q}} \log q.$$

Combining the above estimates we get

$$\sum_{\mathscr{E}} f(p)f(n)e\left(pn \cdot \frac{a}{q}\right) \log p \ll N + \sqrt{Nq \log \frac{2N}{q}} \log N. \tag{47}$$

Now, let us prove a technical lemma:

Lemma 5.4. Fix a natural number K. For $1 \leq k \leq K$ let $\mathscr{R}(k) = \mathscr{I}(k) \times \mathscr{J}(k)$ be a rectangle, where $\mathscr{I}(k) :=]I(k), I'(k)]$ and $\mathscr{J}(k) :=]J(k), J'(k)]$. For the set of such rectangles, we assume that the following properties hold:

- Line segments $\mathscr{I}(k)$ are disjoint.
- $\mathscr{I}(k) \subset]0, A]$ for some real A.
- $I'(k) - I(k) \leq B$ for some real B, uniformly on k.
- Line segments $\mathscr{J}(k)$ are disjoint.
- $\mathscr{J}(k) \subset]0, C]$ for some real C.
- $J'(k) - J(k) \leq D$ for some real D, uniformly on k.
- $J'(k) \leq 2J(k)$.

Let

$$\mathscr{T} := \sum_{k=1}^{K} \sum_{(p,n)\in\mathscr{R}(k)} f(p)f(n)e\left(pn \cdot \frac{a}{q}\right) \log p.$$

Then, for $(a, q) = 1$ and $q \leq BD$, we have

$$\mathscr{T} \ll \sqrt{CAD \log 2A + \frac{CABD}{\varphi(q)} + CAB + CAq \log\left(\frac{2BD}{q}\right)}. \tag{48}$$

Proof. We look at one rectangle at a time. Let $\mathscr{R} = \mathscr{I} \times \mathscr{J}$ be one of the rectangles $\mathscr{R}(k)$. The Cauchy-Schwarz inequality gives

$$\begin{aligned}\left|\sum_{(p,n)\in\mathscr{R}} f(p)f(n)e\left(pn \cdot \frac{a}{q}\right) \log p\right|^2 \quad &\leq \quad \left(\sum_{n\in\mathscr{J}} |f(n)|^2\right) \\ &\cdot \left(\sum_{n\in\mathscr{J}} \left|\sum_{p\in\mathscr{I}} f(p)e\left(pn \cdot \frac{a}{q}\right) \log p\right|^2\right). \end{aligned} \tag{49}$$

Let us define a new function

$$\mathscr{W}(n) := \max\left(0, 2 - \frac{|2n - 2J - D|}{D}\right).$$

Notice that $-D = 2J - 2J - D \leq 2n - 2J - D \leq 2(J + D) - 2J - D = D$, and so we have $\mathscr{W}(n) \geq 1$ for all $n \in \mathscr{J}$. Using the fact $|z|^2 = z\overline{z}$ we have that the second factor on the right-hand side of (49) is

$$\begin{aligned}
&\ll \sum_n \mathscr{W}(n) \left| \sum_{p \in \mathscr{I}} f(p) e\left(pn \cdot \frac{a}{q}\right) \log p \right|^2 \\
&= \sum_n \mathscr{W}(n) \left(\sum_{p \in \mathscr{I}} f(p) e\left(pn \cdot \frac{a}{q}\right) \log p \right) \left(\sum_{p \in \mathscr{I}} \overline{f}(p) e\left(-pn \cdot \frac{a}{q}\right) \log p \right) \\
&= \sum_{p, p' \in \mathscr{I}} f(p) \overline{f}(p') (\log p)(\log p') \sum_n \mathscr{W}(n) e\left((p - p') n \cdot \frac{a}{q}\right) \\
&\ll (\log A)^2 \sum_{p, p' \in \mathscr{I}} \min\left(D, \frac{1}{D \left\|(p - p') a q^{-1}\right\|^2}\right).
\end{aligned}$$

The last step follows by observing that[10]

$$\sum_{L - M \leq m \leq L + M} e(m\alpha) \left(1 - \frac{|m - L|}{M}\right) \ll \frac{1}{M} \left(\frac{\sin(M\pi\alpha)}{\sin \pi\alpha}\right)^2 \ll \frac{1}{M\|\alpha\|^2}$$

and that $\mathscr{W}(n) > 0$ if and only if $-\frac{D}{2} + J < n < \frac{3}{2}D + J$.

The previous estimate, with the Cauchy–Schwarz inequality, yields

$$\begin{aligned}
\mathscr{T} &\ll (\log A) \left(\sum_k \sum_{n \in \mathscr{J}(k)} |f(n)|^2 \right)^{\frac{1}{2}} \left(\sum_k \sum_{p, p' \in \mathscr{I}(k)} \min\left(D, \frac{1}{D \left\|(p - p') a q^{-1}\right\|^2}\right) \right)^{\frac{1}{2}} \\
&\ll (\log A) \sqrt{C} \left(\frac{DA}{\log 2A} + \sum_{0 \leq h \leq B} \sum_{\substack{p \leq A \\ p + h = p'}} \min\left(D, \frac{1}{D \left\|h a q^{-1}\right\|^2}\right) \right)^{\frac{1}{2}}.
\end{aligned}$$

By a well-known result from sieve theory (see Theorem 3.11. in [40] or Exercise 9.4.6 in [60]) the number of primes $p \leq A$, such that $p + h$ is also a prime, is

$$\ll \frac{A}{(\log 2A)^2} \prod_{p | h} \left(1 + \frac{1}{p}\right) \ll \frac{hA}{(\log 2A)^2} \cdot \frac{1}{\varphi(h)}.$$

Hence

$$\mathscr{T} \ll (\log A) \sqrt{\frac{CAD}{\log 2A} + \frac{CAE}{(\log 2A)^2}} \ll \sqrt{CAD \log A + CAE}, \tag{50}$$

where

$$E := \sum_{0 \leq k \leq B} \frac{h}{\varphi(h)} \min\left(D, \frac{1}{D \left\|h a q^{-1}\right\|^2}\right).$$

[10]The first estimate is justified in the same way as the closed form representation of Fejer's kernel is derived. See [74].

Estimate (50) implies that in order to prove (48), it is enough to show that

$$E \ll \frac{BD}{\varphi(q)} + B + D\log 2B + q\log\left(\frac{2BD}{q}\right). \tag{51}$$

Indeed, when this is established, we can calculate

$$\begin{aligned}
\mathscr{T} &\ll \sqrt{CAD\log A + CA\left(\frac{BD}{\varphi(q)} + B + D\log 2B + q\log\left(\frac{2BD}{q}\right)\right)} \\
&\ll \sqrt{CAD\log 2AB + \frac{CABD}{\varphi(q)} + CAB + CAq\log\left(\frac{2BD}{q}\right)} \\
&\ll \sqrt{CAD\log 2A + \frac{CABD}{\varphi(q)} + CAB + CAq\log\left(\frac{2BD}{q}\right)},
\end{aligned}$$

which is (48). The last estimate is based on the observation $B \le A$ and so

$$\frac{\log 2AB}{\log 2A} = \frac{\log 2A + \log B}{\log 2A} \ll 1.$$

Now we proceed to proving (51). We have

$$\frac{h}{\varphi(h)} \ll \sum_{m|h} \frac{1}{m},$$

so

$$E \ll \sum_{m\le B} \frac{1}{m} \sum_{n \le \frac{B}{m}} \min\left(D, \frac{1}{D\left\|mnaq^{-1}\right\|^2}\right).$$

Now we observe that the inner sum on the right-hand side of the previous estimate is of the form

$$\mathscr{F} := \sum_{n\le f} \min\left(D, \frac{1}{D\left\|bnr^{-1}\right\|^2}\right),$$

with $r = \frac{q}{(m,q)}$ and $(b,r)=1$. It holds that[11]

$$\mathscr{F} \ll \min\left(Df, \frac{(D+r)(f+r)}{r}\right). \tag{52}$$

Using this we get

$$\begin{aligned}
E &\ll \sum_{\substack{m\le B \\ BD \le \frac{mq}{(m,q)}}} \frac{BD}{m^2} + \sum_{\substack{m\le B \\ BD > \frac{mq}{(m,q)}}} \frac{1}{m}\left(\frac{BD}{mq}(m,q) + \frac{B}{m} + D + \frac{q}{(m,q)}\right) \\
&\ll \sum_{r|q} \sum_{s > \frac{BD}{q}} \frac{BD}{r^2 s^2} + \sum_{r|q}\sum_{s} \frac{BD}{rs^2 q} + B + D\log 2B + \sum_{r|q} \sum_{s < \frac{BDr}{q}} \frac{q}{r^2 s}.
\end{aligned}$$

Clearly this is dominated by the right-hand side of (51), and so we are done. □

[11] The estimate (52) is taken for granted in the original paper [56]. The author could not verify this fact, but apparently the method used in the proof of Weyl's old estimate (see [84]) $\sum_{n\le N} \min\left(M, \frac{1}{\|n\alpha+\gamma\|}\right) \ll \left(q + M + N + \frac{MN}{q}\right)\log\left(\frac{2MN}{q}\right)$ applies with some changes. Discussion on pp. 101 – 102 in [29] supports this intuition.

Now we are ready to complete the proof. For each rectangle $\mathscr{R}_i$, apply Lemma 5.4. with the choices $K = 1$, $A = B = 2^i$ and $C = D = N2^{-i}$:

$$\sum_{(p,n)\in\mathscr{R}_i} f(p)f(n)e\left(pn\cdot\frac{a}{q}\right)\log p$$
$$\ll \sqrt{\frac{N^2}{2^i}\log 2^{i+1} + \frac{N^2}{\varphi(q)} + N2^i + Nq\log\frac{2N}{q}}$$
$$\ll N\sqrt{(i+1)2^{-i}} + \frac{N}{\sqrt{\varphi(q)}} + \sqrt{N2^i} + \sqrt{Nq\log\frac{2N}{q}}. \tag{53}$$

Now, for all pairs (i,j) with $1 \le j \le \mathscr{C}_i$ we apply Lemma 5.4. with the 2^{j-1} element set of rectangles $\mathscr{R}_{ijk}$, where $2^{j-1} < k \le 2^j$. Thus, by (44), we can take $K = 2^{j-1}$, $C = N2^{-i}$, $A = 2^{i+1}$, $B = 2^{i-j+1}$ and $D = 32N2^{-i-j}$. With these choices the conditions for (48) are satisfied, and hence we have

$$\sum_{2^{j-1}<k\le 2^j}\ \sum_{(p,n)\in\mathscr{R}_{ijk}} f(p)f(n)e\left(pn\cdot\frac{a}{q}\right)\log p$$
$$\ll \sqrt{64N^2 2^{-i-j}\log 2^{i+2} + \frac{128N^2 2^{-2j}}{\varphi(q)} + N2^{i-j+2} + 2Nq\log\left(\frac{64N2^{-2j+1}}{q}\right)}$$
$$\ll N\sqrt{(i+1)2^{-i-j}} + \frac{N2^{-j}}{\sqrt{\varphi(q)}} + \sqrt{N2^{i-j}} + \sqrt{Nq\log\frac{2N}{q}}.$$

By the choice of $\mathscr{C}_j$ we have $\mathscr{C}_j \ll \log(2N/q)$. By summing over all j such that $1 \le j \le \mathscr{C}_i$, we get

$$\sum_{1\le j\le\mathscr{C}_i}\ \sum_{2^{j-1}<k\le 2^j}\ \sum_{(p,n)\in\mathscr{R}_{ijk}} f(p)f(n)e\left(pn\cdot\frac{a}{q}\right)\log p$$
$$\ll N\sqrt{(i+1)2^{-i}} + \frac{N}{\varphi(q)} + \sqrt{N2^i} + \sqrt{Nq}\left(\log\frac{2N}{q}\right)^{\frac{3}{2}}.$$

Applying (53) over all i such that $0 \le i \le \log_2 N$ we obtain

$$\sum_{\substack{pn\le N\\ (p,n)\notin\mathscr{E}}} f(p)f(n)e\left(pn\cdot\frac{a}{q}\right)\log p \ll N + \frac{N}{\sqrt{\varphi(q)}}\log N + \sqrt{Nq}\left(\log\frac{2N}{q}\right)^{\frac{3}{2}}\log N.$$

This combined with (47) gives (41), which consequently completes the proof. □

As a consequence we obtain the minor arc estimate used in applications.

Corollary 5.5. Assume that $f \in \widetilde{\mathcal{F}}$, $\left|\alpha - \frac{a}{q}\right| \le \frac{1}{q^2}$ with $(a,q) = 1$. Then for every $R \in [2,r]$ and $N \ge Rr$ we have

$$\sum_{n\le N} f(n)e(n\alpha) \ll \frac{N}{\log N} + \frac{N}{\sqrt{R}}(\log R)^{\frac{3}{2}}.$$

Moreover, under the same assumptions

$$\sum_{Rr\le n\le N} \frac{f(n)}{n}e(n\alpha) \ll \log\log N + \frac{(\log R)^{\frac{3}{2}}}{\sqrt{R}}\log N.$$

Proof. Let $\frac{b}{r}$ be a reduced fraction with $r \le N$. We start by noting that partial summation gives

$$\begin{aligned}\sum_{n=1}^{N} f(n)e(n\alpha) &= e\left(\left(\alpha - \frac{b}{r}\right)N\right)\sum_{n\le N} f(n)e\left(n\cdot\frac{b}{r}\right)\\ &- 2\pi i\left(\alpha-\frac{b}{r}\right)\int_1^N\left(\sum_{n\le u} f(n)e\left(n\cdot\frac{b}{r}\right)e\left(\left(\alpha-\frac{b}{r}\right)u\right)\right)du.\end{aligned}$$

Then, by using the trivial bound $|f(n)| \le 1$ when $u \le r$ and the Montgomery–Vaughan bound for $u > r$, we have by a simple integration

$$\begin{aligned}\sum_{n=1}^{N} f(n)e(n\alpha) &\ll \frac{N}{\log 2N} + \frac{N}{\sqrt{\varphi(r)}} + \sqrt{Nr}\left(\log\frac{2N}{r}\right)^{\frac{3}{2}}\\ &+ \left|\alpha-\frac{b}{r}\right|\int_1^r u\,du + \left|\alpha-\frac{b}{r}\right|\int_r^N\left(\frac{u}{\log 2u}+\frac{u}{\sqrt{\varphi(r)}}+\sqrt{ur}\left(\log\frac{2u}{r}\right)^{\frac{3}{2}}\right)du\\ &\ll \left(\frac{N}{\log N}+\frac{N}{\sqrt{\varphi(r)}}+\sqrt{Nr}\left(\log\frac{2N}{r}\right)^{\frac{3}{2}}\right)\left(1+N\left|\alpha-\frac{b}{r}\right|\right). \qquad (54)\end{aligned}$$

If $q > \sqrt{N}$, the corollary follows by choosing $b = a$ and $r = q$ in (54) using the assumption $\left|\alpha - \frac{a}{q}\right| \le \frac{1}{q^2}$. Suppose that $q \le \sqrt{N}$. Then, by Dirichlet's Approximation Theorem, there exists b, r such that $(b, r) = 1$, $r \le \frac{2N}{q}$ and $\left|\alpha - \frac{b}{r}\right| \le \frac{q}{2rN}$. Thus either $r = q$ or

$$1 \le |ar - bq| = rq\left|\left(\alpha-\frac{b}{r}\right)-\left(\alpha-\frac{a}{q}\right)\right| \le \frac{q^2}{2N}+\frac{r}{q} \le \frac{1}{2}+\frac{r}{q}.$$

So in any case $r \ge \frac{1}{2}q$, meaning that $\left|\alpha - \frac{b}{r}\right| \le \frac{1}{N}$. Thus (54) gives

$$\sum_{n=1}^{N} f(n)e(n\alpha) \ll \frac{N}{\log N} + N\sqrt{q}(\log q)^{\frac{3}{2}},$$

as desired. The second statement follows from the first by partial summation. □

Goldmakher [33] derived an estimate for y-smooth integers from the previous corollary.

Corollary 5.6. Let $f \in \mathcal{F}$ $\alpha \ne 0$ and b, r positive integers with $(b, r) = 1$ such that $\left|\alpha - \frac{b}{r}\right| \le \frac{1}{r^2}$. Then for $x \ge 2$ and $y \ge 16$ we have

$$\sum_{\substack{n\le x\\ n\in S(y)}} \frac{f(n)}{n}e(n\alpha) \ll \log r + \frac{1+(\log r)^{\frac{5}{2}}}{\sqrt{r}}\log y + \log\log y.$$

Proof. If $x \le r^2$ the claim is trivial by the estimate $|f(n)| \le 1$. Hence we assume that $x > r^2$. If $x \le y^{\log r}$, the claim follows from Corollary 5.5:

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in S(y)}} \frac{f(n)}{n}e(n\alpha) &= \sum_{n\le x}\frac{f_y(n)}{n}e(n\alpha) = \sum_{n<r^2}\frac{f_y(n)}{n}e(n\alpha) + \sum_{r^2\le n\le x}\frac{f_y(n)}{n}e(n\alpha)\\ &\ll \log r + \frac{(\log r)^{3/2}}{\sqrt{r}}\log x + \log\log x \ll \log r + \frac{(\log r)^{5/2}}{\sqrt{r}}\log y + \log\log y,\end{aligned}$$

as desired. Therefore it is enough to bound

$$\sum_{\substack{y^{\log r}<n\leq x\\ n\in S(y)}} \frac{f(n)}{n}e(n\alpha).$$

When $n > y^{\log r}$ we have $n > r \cdot n^{1-\frac{1}{\log y}}$. Thus

$$\sum_{\substack{y^{\log r}<n\leq x\\ n\in S(y)}} \frac{f(n)}{n}e(n\alpha) \ll \frac{1}{r} \sum_{\substack{y^{\log r}<n\leq x\\ n\in S(y)}} \frac{1}{n^{1-\frac{1}{\log y}}} \leq \frac{1}{r}\prod_{p\leq y}\left(1-\frac{1}{p^{1-\frac{1}{\log y}}}\right)^{-1}. \tag{55}$$

Furthermore, we have

$$\log \prod_{p\leq y}\left(1-\frac{1}{p^{1-\frac{1}{\log y}}}\right)^{-1} = \sum_{p\leq y}\frac{1}{p^{1-\frac{1}{\log y}}} + \mathcal{O}(1) \overset{(*)}{=} \log\log y + \mathcal{O}(1).$$

Using this in (55) yields

$$\sum_{\substack{y^{\log r}<n\leq x\\ n\in S(y)}} \frac{f(n)}{n}e(n\alpha) \ll \frac{1}{r}\log y,$$

and so we are done.

In the step (*) we used the fact that for $y \geq 10$, $n \geq 2$,

$$\sum_{p\leq y}\frac{1}{p^{1-\frac{1}{\log N}}} = \log\log y + \mathcal{O}\left(\frac{\exp\left(\frac{\log y}{\log N}\right)}{1+\frac{\log y}{\log N}}\right) \tag{56}$$

The proof of this is straightforward, but very messy. The reader may consult the details from [32] where it is presented as Lemma 2.6. □

6 Large Character Sums

Now we turn our attention to estimating character sums

$$\mathcal{S}_\chi(t) := \sum_{n \le t} \chi(n), \tag{57}$$

for a non-principal Dirichlet character χ modulo q. In this chapter we prove classical upper bounds for this sum. After that we discuss the recent application of pretentious methods to studying $\mathcal{S}_\chi(t)$, obtaining improvements to the old results in some situations.

6.1 The Pólya–Vinogradov Inequality

From the periodicity of the characters we know that the sum (57) is $\ll q$. However, this can be sharpened significantly as was proven independently by Pólya and Vinogradov in 1918. This result is called the Pólya–Vinogradov inequality which we are going to prove next.

Theorem 6.1. (Pólya–Vinogradov Inequality) Let χ be a non-principal character mod q. Then

$$|\mathcal{S}_\chi(t)| \ll \sqrt{q} \log q.$$

In his original work, Pólya deduced Theorem 6.1 for primitive characters straight from his Fourier expansion. Indeed, inequalities (10), (11), the triangle inequality and the fact that $|\tau(\chi)| = \sqrt{q}$ imply

$$|\mathcal{S}_\chi(t)| < 1 + \sqrt{q}\left(\sum_{m=1}^{n} \frac{2}{\pi m}\right) + \frac{2}{\pi} \cdot \frac{q \log q}{n} < 1 + \frac{2}{\pi}\sqrt{q}\,(1 + \log n) + \frac{2}{\pi} \cdot \frac{q \log q}{n},$$

for all natural numbers n. In particular, choosing $n = \lfloor q^{\frac{1}{2}+\varepsilon} \rfloor$, with $\varepsilon > 0$, yields Theorem 6.1. in this particular case. Now, we present another proof due to I. Schur [14], [68].

Proof. Let us first assume that χ is a primitive character. In Chapter 2 we saw that it can be represented in the form

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{h=1}^{q} \overline{\chi}(h) e\left(\frac{hn}{q}\right).$$

By summing over all $n \le t$ we get

$$\mathcal{S}_\chi(t) = \frac{1}{\tau(\overline{\chi})} \sum_{h=1}^{q-1} \overline{\chi}(h) \sum_{n \le t} e\left(\frac{hn}{q}\right),$$

since $\chi(q) = 0$. Taking absolute values, multiplying by $\sqrt{q}$ and using the fact $|\tau(\chi)| = \sqrt{q}$ yields

$$\sqrt{q} \cdot |\mathcal{S}_\chi(t)| \le \sum_{h=1}^{q-1} \left| \sum_{n \le t} e\left(\frac{hn}{q}\right) \right|. \tag{58}$$

Let us denote

$$f(h) := \sum_{n \le t} e\left(\frac{hn}{q}\right).$$

Notice that also

$$f(q-h)=\sum_{n\le t} e\left(\frac{n(q-h)}{q}\right)=\sum_{n\le t} e\left(-\frac{hn}{q}\right)=f(-h)=\overline{f}(h)$$

and consequently $|f(q-h)|=|\overline{f}(h)|=|f(h)|$. Hence (58) can be written as

$$\sqrt{q}\cdot|\mathcal{S}_\chi(t)|\le 2\sum_{h<q/2}|f(h)|+\frac{(-1)^q+1}{2}\left|f\left(\frac{q}{2}\right)\right|. \tag{59}$$

Evidently $f(h)$ is a geometric sum

$$f(h)=\sum_{m=1}^{\lfloor t\rfloor} e\left(\frac{hm}{q}\right).$$

Thus

$$|f(h)|=\left|e\left(\frac{h\lfloor t+1\rfloor}{2q}\right)\frac{e\left(-\frac{\lfloor t\rfloor h}{2q}\right)-e\left(\frac{\lfloor t\rfloor h}{2q}\right)}{e\left(-\frac{h}{2q}\right)-e\left(\frac{h}{2q}\right)}\right|=\left|\frac{\sin\frac{\pi\lfloor t\rfloor h}{q}}{\sin\frac{\pi h}{q}}\right|\le\frac{1}{\sin\frac{\pi h}{q}}.$$

Now, by using the inequality $\sin t\ge\frac{2t}{\pi}$ (which holds for all $t\in[0,\pi/2]$) for $t=\frac{\pi h}{q}$, we obtain

$$|f(h)|\le\frac{q}{2h}.$$

when $h\le\frac{q}{2}$. If q is odd, (59) gives

$$\sqrt{q}\cdot|\mathcal{S}_\chi(t)|\le q\sum_{m<\frac{q}{2}}\frac{1}{m}<q\log q.$$

If q is even, $|f(q/2)|\le 1$ and so (59) gives

$$\sqrt{q}\cdot|\mathcal{S}_\chi(t)|\le q\left(\sum_{h<\frac{q}{2}}\frac{1}{h}+\frac{1}{q}\right)<q\log q.$$

In both cases

$$|\mathcal{S}_\chi(t)|<\sqrt{q}\log q,$$

as desired. Now assume that χ is a non-primitive character modulo q and let c denote its conductor. From Chapter 2 we know that if $c\mid q$, $c<q$, then

$$\chi(m)=\psi(m)\chi_0(m),$$

where χ_0 is the principal character modulo q and ψ is a primitive character modulo c. Using this we can calculate

$$\begin{aligned}\mathcal{S}_\chi(t) &= \sum_{\substack{n\le t\\(n,q)=1}}\psi(n)=\sum_{n\le t}\psi(n)\sum_{d\mid(n,q)}\mu(d)=\sum_{n\le t}\sum_{\substack{d\mid q\\ d\mid n}}\mu(d)\psi(n)\\ &= \sum_{d\mid q}\mu(d)\sum_{q\le\frac{t}{d}}\psi(qd)=\sum_{d\mid q}\mu(d)\psi(d)\sum_{x\le\frac{t}{d}}\psi(x).\end{aligned}$$

Since the Pólya–Vinogradov inequality holds for a primitive $\psi \pmod c$, we have

$$|\mathcal{S}_\chi(t)| \leq \sum_{d|q} |\mu(d)\psi(d)| \left| \sum_{x \leq \frac{t}{d}} \psi(x) \right| < \sqrt{c} \log c \left| \sum_{d|q} \mu(d)\psi(d) \right|. \tag{60}$$

Next, we observe that $|\mu(d)\psi(d)|$ is either 0 or 1. It equals one if and only if $\mu(d) = 1$ and $|\psi(d)| = 1$. That is exactly when d is squarefree, say $d = p_1 p_2 \cdots p_l$ and $(d, c) = 1$. This implies that no prime factor p_i divides c. Hence each p_i divides $\frac{q}{c}$ and thus d divides $\frac{q}{c}$. Therefore

$$\sum_{d|q} |\mu(d)\psi(d)| \leq \sum_{d|\frac{q}{c}} 1 = d\left(\frac{q}{c}\right) \ll \left(\frac{q}{c}\right)^{\delta},$$

for every $\delta > 0$. In the last estimate we used the well-known fact that $d(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$. For the proof, see Theorem 13.12 in [1].

In particular $d\left(\frac{q}{c}\right) \ll \sqrt{\frac{q}{c}}$, and so (60) implies that

$$|\mathcal{S}_\chi(t)| \ll \sqrt{\frac{q}{c}} \cdot \sqrt{c} \log c \ll \sqrt{q} \log c \ll \sqrt{q} \log q,$$

as desired. □

This is quite close to being sharp. Applying partial summation to the Gauss sum $\tau(\chi)$ for primitive $\chi \pmod q$, we deduce

$$\sqrt{q} = |\tau(\chi)| \leq 1 + \frac{2\pi}{q} \int_1^q |\mathcal{S}_\chi(t)| dt \leq 1 + 2\pi \max_{t \leq q} |\mathcal{S}_\chi(t)|$$

which gives $\max_{t \leq q} |\mathcal{S}_\chi(t)| \gg \sqrt{q}$.

Next we prove an improvement for the Pólya–Vinogradov inequality by assuming that the GRH is true. This result was proved by Montgomery and Vaughan [56] as an application of the Montgomery–Vaughan bound which we produced in Chapter 5. Our proof is in line with their original paper.

Theorem 6.2. Assume the GRH. Then for any non-principal character $\chi \pmod q$

$$|\mathcal{S}_\chi(t)| \ll \sqrt{q} \log \log q.$$

Let $\psi \pmod r$ be a primitive character which induces character χ. Then $r|q$ and as before

$$\sum_{n \leq t} \chi(n) = \sum_{\substack{n \leq t \\ \left(n, \frac{q}{r}\right) = 1}} \psi(n) = \sum_{d|\frac{q}{r}} \mu(d)\psi(d) \sum_{m \leq \frac{t}{d}} \psi(m).$$

Hence,

$$\left| \sum_{n \leq t} \chi(n) \right| \leq 2^{\omega\left(\frac{q}{r}\right)} \max_t \left| \sum_{n \leq t} \psi(n) \right|.$$

As we clearly have[12] $2^{\omega\left(\frac{q}{r}\right)} \ll \sqrt{\frac{q}{r}}$, it suffices to prove Theorem 6.2. only for primitive characters. For the proof we need the following lemma, whose proof can be found in [56].

[12] Here $\omega(n)$ is the number of distinct prime divisors of n.

Lemma 6.3. Let χ be a non-principal character modulo q, and suppose that $L(s,\chi)\neq 0$ for $\Re s>1/2$. Suppose also that $(\log q)^4\le y\le q$ and $x\le q$. Then

$$\sum_{n\le x}\chi(n)=\sum_{\substack{n\le x,\\ n\in S(y)}}\chi(n)+\mathcal{O}(xy^{-\frac{1}{2}}(\log q)^4),$$

We observe that for primitive $\chi \pmod q$ the trivial bound $|\mathcal{S}_\chi(t)|\le q$ gives

$$\begin{aligned}
\frac{\tau(\chi)}{2\pi i}\sum_{1\le|n|\le k}\frac{\overline{\chi}(n)}{n} &= \frac{\tau(\chi)}{2\pi i}(1-\chi(-1))\sum_{n=1}^{k}\frac{\overline{\chi}(n)}{n}\\
&= \frac{\tau(\chi)}{2\pi i}(1-\chi(-1))L(1,\chi)-\frac{\tau(\chi)}{2\pi i}(1-\chi(-1))\sum_{n=k}^{\infty}\frac{\overline{\chi}(n)}{n}\\
&\ll \frac{\tau(\chi)}{2\pi i}(1-\chi(-1))L(1,\chi)+\mathcal{O}\left(\sqrt{q}\cdot\frac{q}{k}\right)
\end{aligned}$$

for every natural number k. Choosing $k=q$ and combining this bound with Pólya's Fourier expansion gives

$$\begin{aligned}
\mathcal{S}_\chi(t) \ll\ & \frac{\tau(\chi)}{2\pi i}(1-\chi(-1))L(1,\chi)\\
+\ & \frac{\tau(\chi)}{2\pi i}\left(\sum_{n=1}^{q}\frac{\overline{\chi}(n)}{n}e\left(-\frac{nt}{q}\right)+\chi(-1)\sum_{n=1}^{q}\frac{\overline{\chi}(n)}{n}e\left(\frac{nt}{q}\right)\right)+\mathcal{O}(1+\sqrt{q}+\log q).
\end{aligned}$$

Notice that the error term is $\ll\sqrt{q}\log\log q$. Taking into account Littlewood's estimate under the GRH, $L(1,\chi)\ll\log\log q$ (Theorem 2.8) and the fact $|\tau(\chi)|=\sqrt{q}$, we note that in order to deduce Theorem 6.2 it is enough to show the following:

Lemma 6.4. Assume the GRH. Let χ be a primitive character modulo $q>1$. Then we have

$$\sum_{n=1}^{q}\frac{\chi(n)}{n}e(n\alpha)\ll\log\log q,$$

uniformly in α.

Prior proving this we present a result which is used repeatedly throughout rest of this thesis:

Lemma 6.5. (The Granville–Soundararajan Identity) Let $\chi \pmod q$ be a primitive character. Assume that we are given integers b and r such that $(b,r)=1$ with $b\neq 0$ and $r\ge 1$. Then for all $N\ge 2$ and $y\ge 2$ we have

$$\sum_{\substack{n\le N\\ n\in S(y)}}\frac{\overline{\chi}(n)}{n}e\left(n\cdot\frac{b}{r}\right)=\sum_{\substack{d|r\\ d\in S(y)}}\frac{\overline{\chi}(d)}{d}\cdot\frac{1}{\varphi\left(\frac{r}{d}\right)}\sum_{\psi \pmod{\frac{r}{d}}}\tau(\overline{\psi})\psi(b)\sum_{\substack{m\le\frac{N}{d}\\ m\in S(y)}}\frac{(\overline{\chi}\psi)(m)}{m}$$

Proof. Summing over all the greatest possible common divisors of n and r we find

$$\sum_{\substack{n\le N\\ n\in S(y)}}\frac{\overline{\chi}(n)}{n}e\left(n\cdot\frac{b}{r}\right)=\sum_{\substack{d|r\\ d\in S(y)}}\frac{\overline{\chi}(d)}{d}\sum_{\substack{m\le\frac{N}{d}\\ (m,\frac{r}{d})=1\\ m\in S(y)}}\frac{\overline{\chi}(m)}{m}e\left(\frac{mb}{\frac{r}{d}}\right). \tag{61}$$

On the other hand, we have

$$e\left(\frac{mb}{\frac{r}{d}}\right) = \sum_{0\le k<\frac{r}{d}} e\left(\frac{k}{\frac{r}{d}}\right)\delta_{mb+\frac{r}{d}\mathbb{Z}}(k).$$

Using the orthogonality relation (3) we can write

$$\delta_{mb}(k) = \frac{1}{\varphi\left(\frac{r}{d}\right)} \sum_{\psi \pmod{\frac{r}{d}}} \overline{\psi}(k)\psi(mb).$$

Putting these to (61) and switching the order of summation gives the claim. □

Now we are ready to prove Lemma 6.4:

Proof of Lemma 6.4. Suppose that q is large and let t be an integer such that $1 \le t \le q$.
If $t \le 100$, we trivially have

$$\sum_{n\le t} \chi(n)e(n\alpha) \ll \frac{t}{\log 2t}. \tag{62}$$

If $t > 100$, we have $t(\log t)^{-3} > 1$, and then by Dirichlet's Approximation Theorem there exists $b, r \in \mathbb{Z}$ such that $(b, r) = 1$, $r \le t(\log t)^{-3}$ and

$$\left|\alpha - \frac{b}{r}\right| \le \frac{(\log t)^3}{rt}.$$

Let $y := (\log q)^{20}$. We have two cases to consider:

1) Assume that $r \ge (\log t)^3$. Then, by Corollary 5.5, the formula (62) still holds:

$$\sum_{n\le t} \chi(n)e(n\alpha) \ll \frac{t}{\log t} + \frac{t}{\sqrt{(\log t)^3}} \cdot (\log t)^{\frac{3}{2}} \ll \frac{t}{\log 2t}.$$

2) Assume that $1 \le r < (\log t)^3$. Then, for all $r \le y$, we use the same argument as in the proof of Lemma 6.5 to obtain

$$\sum_{\substack{n\le t\\ n\notin S(y)}} \chi(n)e\left(\frac{b}{r}\cdot n\right) = \sum_{d|r} \frac{\chi(d)}{\varphi(\frac{r}{d})} \sum_{\psi (\mathrm{mod}\ \frac{r}{d})} \psi(b)\tau(\overline{\psi}) \sum_{\substack{m\le \frac{t}{d}\\ m\notin S(y)}} \psi\chi(m). \tag{63}$$

The character $\psi\chi$ is a non-principal character modulo $\frac{qr}{d}$, where $r \ge d$, and since q is large, the parameter y satisfies the inequality

$$\left(\log\left(\frac{qr}{d}\right)\right)^4 \le y \le \frac{qr}{d}.$$

Thus we can use (63) and apply Lemma 6.3 to deduce

$$\begin{aligned}
\sum_{\substack{n\le t\\ n\notin S(y)}} \chi(n)e\left(\frac{b}{r}\cdot n\right) &\ll \sum_{d|r} \frac{\chi(d)}{\varphi\left(\frac{r}{d}\right)} \sum_{\psi \pmod{\frac{r}{d}}} \psi(b)\tau(\overline{\psi}) \cdot \frac{t}{d} \cdot \sqrt{\frac{1}{(\log q)^{20}}} \left(\log \frac{qr}{d}\right)^4 \\
&\ll \sum_{d|r} \frac{\chi(d)}{\varphi\left(\frac{r}{d}\right)} \cdot \varphi\left(\frac{r}{d}\right) \cdot \sqrt{\frac{r}{d}} \cdot \frac{t}{d} \cdot (\log q)^{-6} \\
&\ll \sum_{d|r} \sqrt{\frac{r}{d}} \cdot \frac{t}{d}(\log q)^{-6} \ll t(\log q)^{-4}.
\end{aligned}$$

Using this we get

$$\sum_{\substack{n\le t\\ n\notin S(y)}} \chi(n)e(n\alpha) \ll t(\log q)^{-4}\left(1+t\left|\alpha-\frac{b}{r}\right|\right) \ll \frac{t}{\log t}.$$

Altogether

$$\sum_{n\le t}\chi(n)e(n\alpha) = \sum_{\substack{n\le t\\ n\in S(y)}} \chi(n)e(n\alpha) + \mathcal{O}\left(\frac{t}{\log t}\right).$$

Therefore, by (62), we always have

$$\sum_{n\le t}\chi(n)e(n\alpha) \ll \frac{t}{\log 2t} + \sum_{\substack{n\le t\\ n\in S(y)}} 1.$$

Using partial summation to the previous estimated yields

$$\begin{aligned}\sum_{n=1}^{q}\frac{\chi(n)}{n}e(n\alpha) &\ll \int_e^q \frac{dt}{t\log t} + \sum_{n\in S(y)}\frac{1}{n}\\ &= \log\log q + \prod_{p\le y}\left(1-\frac{1}{p}\right)^{-1}\\ &\ll \log\log q,\end{aligned}$$

which completes the proof. □

The upper bound in Theorem 6.2 is known as the *Montgomery–Vaughan Estimate*, as opposed to the Montgomery–Vaughan Bound which was proved in Chapter 5.

This is essentially the best possible bound, since Paley showed in [62] that the bound is optimal for a certain infinite class of quadratic characters. More precisely, he used the Chinese Remainder Theorem and the Quadratic Reciprocity Law to construct an infinite family $\mathcal{A}$ of characters such that for every character $\chi \pmod q$ in $\mathcal{A}$ there exists a natural number N_χ for which[13]

- $\chi(p) = \chi_{-4}(p)$ for all $p \le N_\chi$
- $q \le 1 + 4\prod_{p\le N_\chi} p$

While not vital, it is interesting to know how large the implicit constant is in the Pólya–Vinogradov inequality. Hildebrand [43] has studied this in the 1980s, and he was able to obtain improvements to the trivial bounds for primitive characters

$$\frac{|\mathcal{S}_\chi(t)|}{q\log q} \le \begin{cases} \frac{1}{2\pi\sqrt{2}} + o(1) \approx 0.11254 + o(1) & \text{if } \chi(-1)=1\\ \frac{1}{2\pi} + o(1) \approx 0.15915 + o(1) & \text{if } \chi(-1)=-1\end{cases}$$

given by Landau in [54]. Hildebrand's result was that for any non-principal character $\chi \pmod q$

$$\frac{|\mathcal{S}_\chi(t)|}{q\log q} \le \begin{cases} \frac{4}{\sqrt{6}}\cdot\frac{2}{3\pi^2} + o(1) \approx 0.11030 + o(1) & \text{if } \chi(-1)=1\\ \frac{4}{\sqrt{6}}\cdot\frac{1}{3\pi} + o(1) \approx 0.17327 + o(1) & \text{if } \chi(-1)=-1\end{cases}$$

[13] Here χ_{-4} is the non-trivial character modulo 4.

If χ is primitive, then the multiplication with the factor $\frac{4}{\sqrt{6}}$ is unnecessary. In the same year, Hildebrand made a further reduction of the constants, see [44] Corollary 4. If we assume that $\chi \pmod q$ is primitive, it is possible to show that [26]

$$\frac{|\mathcal{S}_\chi(t)|}{q \log q} \leq \begin{cases} \frac{69}{70} \cdot \frac{c+o(1)}{\pi\sqrt{3}} & \text{if } \chi(-1) = 1 \\ \frac{c+o(1)}{\pi} & \text{if } \chi(-1) = -1, \end{cases}$$

where $c = \frac{1}{4}$ if q is cubefree and $c = \frac{1}{3}$ otherwise.

The next question to ask is whether we can improve Theorems 6.1 and 6.2. It turns out that with certain assumptions this is possible. This is what we are trying to achieve in the following sections.

Until the year 2005 there were no significant improvements to the Pólya–Vinogradov inequality in the unconditional case for over ninety years. Things took a turn when Granville and Soundararajan came up with an improvement for characters of an odd order under some additional assumptions [26]. In the next three sections we will discuss the pretentious methods used in work of Goldmakher which, on the other hand, was built upon [26]. In section 6.2. we examine Hildebrand's paper [44]. The paper does not contain any pretentious methods, but it is still an important piece of work. Actually, Theorem 6.7. realizes the possibility that the Pólya–Vinogradov inequality, for the character $\chi \pmod q$, can be improved if there exists a primitive character of a small conductor having certain additional properties. The proof also shows that possible improvements are closely connected to the rational approximation properties of $\frac{t}{q}$.

6.2 The Work of Hildebrand

As said before, the work presented in this subsection had a huge influence on the works of Granville and Soundararajan. Next we are going to examine Hildebrand's main theorem and then discuss the main implications of it. Our treatise follows the original paper [44]. Before stating the theorem, some new definitions are needed. Let χ (mod q) be a primitive character. At first we define the refined character sums

$$\mathcal{T}_\chi(t) = \sum_{n \leq t} \frac{\chi(n)}{n}.$$

and

$$\mathcal{S}'_\chi(t) = \mathcal{S}_\chi(t) - \frac{\tau(\chi)\varepsilon(\chi)}{\pi i} L(1, \chi),$$

where $\tau(\chi)$ is a Gauss sum associated with character χ and $\varepsilon(\chi)$ is defined as

$$\varepsilon(\chi) = \begin{cases} 1 & \text{if } \chi(-1) = -1 \\ 0 & \text{if } \chi(-1) = 1 \end{cases}$$

Let $\psi_0 \pmod{k_0}$ be a primitive character with $1 \leq k_0 \leq (\log q)^{\frac{1}{8}}$ such that

$$\sup_{t \geq 1} |\mathcal{T}_{\chi\psi_0}(t)| = \max_{\psi} \sup_{t \geq 1} |\mathcal{T}_{\chi\psi}(t)|,$$

where the maximum on the right-hand side is taken over all primitive characters ψ (mod k) for which $1 \leq k \leq (\log q)^{\frac{1}{8}}$. For a given $t \geq 1$, the Dirichlet's Approximation Theorem tells that there exists integers $r \geq 0$ and $s \geq 1$ such that

$$\frac{t}{q} = \frac{r}{s} + \alpha, \ (r, s) = 1, \ 1 \leq s \leq (\log q)^{\frac{1}{8}} \text{ and } |\alpha| \leq \frac{1}{s(\log q)^{\frac{1}{8}}}. \tag{64}$$

Denote

$$\ell = \min\left(\frac{1}{\alpha}, q\right).$$

Finally we define a function g as

$$g(n) = (\overline{\chi} * \mu\psi_0)(n) = \sum_{d|n} \overline{\chi}(n)\mu\left(\frac{n}{d}\right)\psi_0\left(\frac{n}{d}\right).$$

An interested reader may consult the original paper [44] for the proof of the following lemma.

Lemma 6.6. Let $f \in \mathcal{F}$ and $x \geq 1$. Then for any positive integer k we have

$$\sum_{\substack{n \leq x \\ (n,k)=1}} \frac{f(n)}{n} = \prod_{p|k}\left(1 - \frac{f(p)}{p}\right)\sum_{n\leq x}\frac{f(n)}{n} + \mathcal{O}((\log\log(k+2))^3),$$

where the constant is absolute. This lemma appeared in Hildebrand's paper [44] and played a role in the proof of Theorem 6.7. We state it here, since we are going to use it later in a different purpose.

Hildebrand's main result in [44] states that

Theorem 6.7. If $k_0 \mid s$ we have

$$\mathcal{S}'_\chi(t) = \frac{-\tau(\chi)\tau(\psi_0)\overline{\psi}_0(-r)g\left(\frac{s}{k_0}\right)}{\pi i \varphi(s)}\varepsilon(\chi\psi_0)\mathcal{T}(\overline{\chi\psi_0}, \ell) + \mathcal{O}\left(\sqrt{q}(\log q)^{\frac{19}{20}}\right).$$

If $k_0 \nmid s$ we simply have

$$\mathcal{S}'_\chi(t) \ll \sqrt{q}(\log q)^{\frac{19}{20}}.$$

Theorem 6.7. clearly shows that $\mathcal{S}_\chi(t)$ depends strongly on the Diophantine approximation properties of $\frac{t}{q}$. It is also evident how the existence of a character ψ_0 matters. In fact, the following consequence is true:

Corollary 6.8. Under the same notations as in Theorem 6.7. we have

$$|\mathcal{S}'_\chi(t)| \ll \sqrt{q}\log q\left(\frac{\log\log(s+2)}{\sqrt{s}} \cdot \frac{\log \ell}{\log q} + (\log q)^{-\frac{1}{20}}\right).$$

Proof. If $k_0 \nmid s$, the statement is clear. If $k_0|s$, we use the following two easy estimates

$$|\mathcal{T}_{\overline{\chi\psi}}(\ell)| \leq \sum_{n\leq \ell}\frac{1}{n} \ll \log \ell$$

and

$$\frac{|\tau(\psi_0)| \cdot \left|g\left(\frac{s}{k_0}\right)\right|}{\varphi(s)} \leq \frac{\sqrt{k_0}}{\varphi(s)}\prod_{p|\frac{s}{k_0}} 2 \ll \frac{\sqrt{s}}{\varphi(s)} \ll \frac{\log\log(s+2)}{\sqrt{s}},$$

where the last estimate follows from Lemma 2.15.

Plugging these into the estimate in Theorem 6.7 gives

$$\begin{aligned}|\mathcal{S}'_\chi(t)| &\ll \sqrt{q}\cdot\frac{\log\log(s+2)}{\sqrt{s}}\cdot\log\ell + \sqrt{q}(\log q)^{\frac{19}{20}} \\ &= \sqrt{q}\log q\left(\frac{\log\log(s+2)}{\sqrt{s}}\cdot\frac{\log\ell}{\log q} + (\log q)^{-\frac{1}{20}}\right),\end{aligned}$$

as desired. $\square$

So, we have an improvement for the Pólya–Vinogradov inequality when s is sufficiently large. Another interesting corollary (see [44], Corollary 2) shows that the Pólya–Vinogradov bound for $\mathcal{S}'_\chi(t)$ is attained very rarely:

Corollary 6.9. Let $\chi \pmod q$ be a primitive character and let $(\log q)^{-\frac{1}{21}} \leq \varepsilon \leq 1$. Then the set of real numbers $\alpha \in [0,1]$, for which

$$|\mathcal{S}'_\chi(\alpha q)| \geq \varepsilon\sqrt{q}(\log q)$$

holds, has Lebesgue measure $\ll q^{-c\varepsilon}$ for some constant c.

6.3 Improvement for the Characters of an Odd Order

In this subsection we adapt the following convention concerning whether the GRH is assumed:

$$Q = \begin{cases} q & \text{unconditionally} \\ (\log q)^{12} & \text{conditionally under the GRH} \end{cases}$$

The goal of this section is to obtain the following improvement to the Pólya–Vinogradov inequality for primitive characters of an odd order.

Goldmakher–Granville–Soundararajan Estimate: Let $\chi \pmod q$ be a primitive character of an odd order g. Then

$$|\mathcal{S}_\chi(t)| \ll_g \sqrt{q}(\log Q)^{1-\delta_g+o(1)},$$

where $\delta_g = 1 - \frac{g}{\pi}\sin\frac{\pi}{g}$.

This was conjectured for the first time by Granville and Soundararajan in [26]. Actually, in that paper they showed that a weaker bound, δ_g replaced with $\frac{\delta_g}{2}$, holds. This was achieved by the following theorem, which characterize when $\mathcal{S}_\chi(t)$ can be large.

Theorem 6.10. Let χ be a character mod q. Let $\psi \pmod m$ be the primitive character among those with conductor below $(\log q)^{\frac{1}{3}}$ for which the distance $\mathbb{D}(\chi,\psi;q)$ is minimal (of course, if there are many of those, pick any one of them). Then we have the following estimate

$$\max_t |\mathcal{S}_\chi(t)| \ll (1-\chi(-1)\psi(-1))\frac{\sqrt{qm}}{\varphi(m)}\log Q \exp\left(-\frac{1}{2}\mathbb{D}(\chi,\psi;q)^2\right) + \sqrt{q}(\log Q)^{\frac{6}{7}}.$$

Basically this says that $\max_t |\mathcal{S}_\chi(t)|$ is small, that is $\ll (\log q)^{\frac{6}{7}}$, unless there exists a primitive character $\psi \pmod m$ with opposite parity and whose distance to χ is small: $\mathbb{D}(\chi,\psi;q)^2 \leq \frac{2}{7}\log\log q$.

For the sake of completeness, we record another theorem from that article, which gives a characterization for large character sums:

Theorem 6.11. Let $\psi \pmod m$ be a primitive character opposite parity than χ. Then we have

$$\max_t |\mathcal{S}_\chi(t)| + \frac{\sqrt{qm}}{\varphi(m)}\log\log Q \gg \frac{\sqrt{qm}}{\varphi(m)}\log Q \exp(-\mathbb{D}(\chi,\psi;q)^2).$$

Proofs of these result require several technical lemmas and are thus omitted. They are found in [26] as Theorems 2.1, 2.2 in the unconditional case, respectively, and Theorems 2.4, 2.5 in the conditional case, respectively.

Now the weaker for of the Goldmakher–Granville–Soundararajan estimate follows easily. Indeed, suppose that $\chi \pmod q$ has an odd order g and let ψ be the character with conductor below $(\log q)^{\frac{1}{3}}$ such that the distance $\mathbb{D}(\chi, \psi; q)$ is minimal. If $\chi(-1)\psi(-1) = 1$ then Theorem 6.10. implies $\mathcal{S}_\chi(t) \ll \sqrt{q}(\log Q)^{\frac{6}{7}}$ and Theorem follows at once as we have $\frac{6}{7} < \frac{1}{2} + \frac{1}{2} \cdot \frac{g}{\pi} \sin\left(\frac{\pi}{g}\right)$ for every integer $g \geq 3$. On the other hand, if $\chi(-1)\psi(-1) = -1$, we use Theorem 4.9, which says that $\mathbb{D}(\chi, \psi; Q)^2 \geq (\delta_g + o(1)) \log\log Q$, and Theorem 6.10 to get the desired result:

$$\begin{aligned}
|\mathcal{S}_\chi(t)| &\ll \frac{\sqrt{qm}}{\varphi(m)} \log Q \cdot \exp\left(\left(-\frac{1}{2}\delta_g + o(1)\right) \log\log Q\right) \\
&\ll \frac{\sqrt{q}m}{\varphi(m)} (\log Q)^{1-\frac{1}{2}\delta_g + o(1)} + \sqrt{q}(\log Q)^{\frac{6}{7}} \\
&\ll \sqrt{q}(\log Q)^{1-\frac{1}{2}\delta_g + o(1)} + \sqrt{q}(\log Q)^{\frac{6}{7}} \\
&\ll \sqrt{q}(\log Q)^{1-\frac{1}{2}\delta_g + o(1)}.
\end{aligned}$$

Now we move back to the Goldmakher–Granville–Soundararajan estimate. Goldmakher settled the unconditional case in his Ph.D.-thesis [32]. Later he improved his methods to also obtaining the conditional case in [33] which we will follow closely throughout this subsection.

As mentioned earlier, Goldmakher's proof of this theorem refines ideas used by Granville and Soundararajan in their work and then introduces a new idea of using methods following Halász's Theorems. First we state the result which shows that restricting to Q-smooth numbers does not cause problems when the GRH is assumed:

Theorem 6.12. Let $\chi \pmod q$ be primitive, α be a real number, $y \geq 1$, and assume that the GRH holds. Then

$$\sum_{n \leq x} \frac{\overline{\chi}(n)}{n} e(n\alpha) = \sum_{\substack{n \leq x \\ n \in S(y)}} \frac{\overline{\chi}(n)}{n} e(n\alpha) + \mathcal{O}\left(\frac{(\log q) \cdot \log x}{y^{\frac{1}{6}}}\right)$$

for $1 \leq x \leq q^{\frac{3}{2}}$.

Proof is omitted due to its complexity, see [26]. We remark that this tells that under the GRH,

$$\sum_{1 \leq |n| \leq q} \frac{\overline{\chi}(n)}{n} e(n\alpha) = \sum_{\substack{1 \leq |n| \leq q \\ n \in S(Q)}} \frac{\overline{\chi}(n)}{n} e(n\alpha) + \mathcal{O}(1), \tag{65}$$

when we choose $x = q$ and $y = Q$.

Let us discuss the main ideas behind the proof of the Goldmakher–Granville–Soundararajan Estimate. Choosing $k = q$ in Pólya's Fourier expansion we see that we must bound

$$\sum_{1 \leq |n| \leq q} \frac{\overline{\chi}(n)}{n} e(n\alpha),$$

with $\alpha = 0$ or $\alpha = -\frac{t}{q}$. We have already seen in (65) that it is possible to restrict ourselves to Q-smooth number with a small error. So, in fact, we are studying the sum

$$\sum_{\substack{1 \leq |n| \leq q \\ n \in S(Q)}} \frac{\overline{\chi}(n)}{n} e(n\alpha).$$

As usual, let $\frac{b}{r}$ be the rational approximation of α with $1 \leq r \leq M$ and $\left|\alpha - \frac{b}{r}\right| \leq \frac{1}{rM}$. Here M is a positive number which is specified later. For those α lying on the minor arc[14], $r > \log Q$, we may use Corollary 5.6. of the Montgomery–Vaughan bound to get the desired result. The major arc case is the one requiring much work. We will prove in Lemma 6.13. that α can be replaced with its rational approximation such that a possible shortening of the summing range is negligible. If α belongs to the major arc, r will be small and thus only the innermost sum on the right-hand side of the Granville–Soundararajan identity can make a significant contribution. In order to understand this kind of a sum, we study the size of a more general sum

$$\sum_{\substack{n \leq x \\ n \in S(y)}} \frac{g(n)}{n}$$

with $g \in \mathcal{F}$. To estimate this, Goldmakher introduced a new idea of applying methods originated from the proofs of strengthenings for Halász's Theorems. The main point was to obtain Corollary 3.8. Now we are ready to handle those α lying on a major arc. It is proved in [2] that the multiplicative function $f \in \mathcal{F}$ cannot mimic two different characters simultaneously very well. Thus if we can identify an "exceptional character" $\xi \pmod{m}$ for which f most nearly mimics, then f is far from mimicking other primitive characters.

If we plug the estimate of Corollary 3.8. to the Granville–Soundararajan identity, we get the upper bound in terms of quantities $M(f\overline{\psi}, \frac{N}{d}, T)$, where T is some parameter and ψ is a character of modulus dividing r. If m does not divide r then none of the characters ψ are induced by the exceptional character ψ. The repulsion principles [2] tell that all $M(f\overline{\psi}; y, \log^2 y)$ are bounded from below and hence the contribution coming from characters ψ is not very large.

On the other hand, if m divides r, it is possible that some of the characters ψ are induced by the exceptional character ξ. These characters will contribute the main term. The estimates, however, are fairly standard. As before, the characters not induced by ψ do not give a major contribution. Putting our estimates to Pólya's Fourier expansion and using Theorem 4.9 finishes the proof.

Now, let us make the above rigorous. As mentioned above, the minor arcs are easy to deal with. Let us choose

$$M = \exp\left(\exp\left(\frac{\log\log Q}{\log\log\log Q}\right)\right). \tag{66}$$

If $r > \log Q$ we have by Corollary 5.6. that

$$\begin{aligned}
\sum_{\substack{n \leq t \\ n \in S(Q)}} \frac{\overline{\chi}(n)}{n} e(n\alpha) &\ll \log r + \frac{1 + (\log r)^{\frac{5}{2}}}{\sqrt{r}} \cdot \log Q + \log\log Q \\
&\ll \exp\left(\frac{\log\log Q}{\log\log\log Q}\right) + \frac{\exp\left(\frac{\frac{5}{2}\cdot\log\log Q}{\log\log\log Q}\right)}{\sqrt{\log Q}} \cdot \log Q + \log\log Q \\
&\ll (\log Q)^{\frac{1}{2}+o(1)}.
\end{aligned} \tag{67}$$

Notice that since χ has an odd order, we must have $\chi(-1) = 1$ and hence

$$\sum_{1 \leq |n| \leq q} \frac{\overline{\chi}(n)}{n} = \sum_{1 \leq n \leq q} \frac{\overline{\chi}(n)}{n} - \overline{\chi}(-1) \sum_{1 \leq n \leq q} \frac{\overline{\chi}(n)}{n} = 0.$$

[14] In his PhD.-thesis, Goldmakher set that α lies on a minor arc if $r > (\log q)^{2\delta_g}$.

Putting this and (67) to Pólya's Fourier expansion and using Theorem 6.12 yields the desired result:

$$|\mathcal{S}_\chi(t)| \ll \sqrt{q} \cdot (\log Q)^{\frac{1}{2}+o(1)} + 1 + \log q \ll \sqrt{q} \cdot (\log Q)^{1-\delta_g+o(1)}.$$

Now we move to the major arc case. We continue by reducing the problem to the case where α is a rational number by proving a slightly more general result.

Theorem 6.13. Let $f \in \mathcal{F}$, $\alpha \in \mathbb{R}$, $x \geq 16$, $y \geq 16$ and $M \geq 2$. By Dirichlet's Approximation Theorem, there exists a reduced fraction $\frac{b}{r}$ with $r \leq M$ such that

$$\left|\alpha - \frac{b}{r}\right| \leq \frac{1}{rM}. \tag{68}$$

Set $N = \min\left\{x, \frac{1}{|r\alpha - b|}\right\}$. Then for all $R \in [2, \frac{N}{2}]$,

$$\sum_{\substack{n\leq x \\ n\in S(y)}} \frac{f(n)}{n} e(n\alpha) = \sum_{\substack{n\leq N \\ n\in S(y)}} \frac{f(n)}{n} e\left(\frac{b}{r}\cdot n\right) + \mathcal{O}\left(\log R + \frac{(\log R)^{\frac{3}{2}}}{\sqrt{R}}(\log y)^2 + \log\log y\right).$$

If $M \geq (\log y)^4 \log\log y$, then the error term is actually $\mathcal{O}(\log\log y)$.

Proof. If $N = x$, then $\left|\alpha - \frac{b}{r}\right| \leq \frac{1}{rx}$ and so

$$\sum_{\substack{n\leq x \\ n\in S(y)}} \frac{f(n)}{n} e\left(e(n\alpha) - e\left(\frac{b}{r}\cdot n\right)\right) \ll \sum_{\substack{n\leq x \\ n\in S(y)}} \frac{1}{n}\cdot n\left|\alpha - \frac{b}{r}\right| \ll 1, \tag{69}$$

which furnishes this case. Hence we assume that $N = \frac{1}{|r\alpha-b|} < x$. It is evident that $N \geq M$ (this being equivalent to (68)) and that

$$\left|\alpha - \frac{b}{r}\right| = \frac{1}{rN}.$$

By Dirichlet's Approximation Theorem, there exists a reduced fraction $\frac{b'}{r'}$ with $r' \leq 2N$ for which

$$\left|\alpha - \frac{b'}{r'}\right| \leq \frac{1}{2Nr'}.$$

We observe that $\frac{b}{r} \neq \frac{b'}{r'}$ holds. If this is not the case, then $r = r'$ (as $\frac{b}{r}$ and $\frac{b'}{r'}$ are reduced fractions) and then

$$\frac{1}{rN} = \left|\alpha - \frac{b}{r}\right| = \left|\alpha - \frac{b'}{r'}\right| \leq \frac{1}{2r'N} < \frac{1}{r'N} = \frac{1}{rN},$$

which is a contradiction.

Therefore

$$\frac{1}{rr'} \leq \left|\frac{b}{r} - \frac{b'}{r'}\right| \leq \frac{1}{2Nr'} + \frac{1}{rN}.$$

This rearranges to $r' \geq N - \frac{r}{2}$. Moreover, since $r \leq M \leq N$, we have $\frac{N}{2} \leq r' \leq 2N$. Let us then split the summing range into four parts: $1 \leq n \leq N$, $N < n \leq Rr'$, $Rr' < n \leq e^{(\log y)^2}$, $e^{(\log y)^2} \leq n \leq x$, where we yet again sum only over the y-smooth integers. We bound these sums separately.

First of all, we trivially have

$$\sum_{\substack{N<n\le Rr' \\ n\in S(y)}} \frac{f(n)}{n} e(n\alpha) \ll \log \frac{Rr'}{N} = \log R + \mathcal{O}(1).$$

By applying Corollary 5.6. to the y-smoothed function f_y we deduce

$$\begin{aligned} \sum_{\substack{Rr'<n\le e^{(\log y)^2} \\ n\in S(y)}} \frac{f(n)}{n} e(n\alpha) &= \sum_{Rr'<n\le e^{(\log y)^2}} \frac{f_y(n)}{n} e(n\alpha) \\ &\ll \log\log y + \frac{(\log R)^{\frac{3}{2}}}{\sqrt{R}} (\log y)^2. \end{aligned}$$

We also have

$$\sum_{\substack{e^{(\log y)^2}<n\le x \\ n\in S(y)}} \frac{f(n)}{n} e(n\alpha) \ll \sum_{\substack{e^{(\log y)^2}<n\le x \\ n\in S(y)}} \frac{1}{n} \ll \frac{1}{y} \sum_{n\in S(y)} \frac{1}{n^{1-\frac{1}{\log y}}} \overset{(*)}{\ll} 1.$$

The estimate $(*)$ follows as in the proof of Corollary 5.6.

Putting these bounds together yields

$$\sum_{\substack{n\le x \\ n\in S(y)}} \frac{f(n)}{n} e(n\alpha) = \sum_{\substack{n\le N \\ n\in S(y)}} \frac{f(n)}{n} e(n\alpha) + \mathcal{O}\left(1 + \log R + \frac{(\log R)^{\frac{3}{2}}}{\sqrt{R}} (\log y)^2 + \log\log y\right).$$

Combining this with the estimate (69) finishes the proof. If $M \ge 2(\log y)^4 \log\log y$, we take $R = (\log y)^4 \log\log y$ to get the desired result. □

Now observe that M chosen in (66) is $\ge 2(\log Q)^4 \log\log Q$ for $Q \ge 16$. So Theorem 6.13 says that there exists $N \in [M, q]$ for which

$$\sum_{\substack{1\le |n|\le q \\ n\in S(Q)}} \frac{\overline{\chi}(n)}{n} e(n\alpha) = \sum_{\substack{1\le |n|\le N \\ n\in S(Q)}} \frac{\overline{\chi}(n)}{n} e\left(\frac{b}{r}\cdot n\right) + \mathcal{O}(\log\log Q).$$

The main term on the right-hand side can be written as

$$\sum_{\substack{1\le |n|\le N \\ n\in S(Q)}} \frac{\overline{\chi}(n)}{n} e\left(\frac{b}{r}\cdot n\right) = \sum_{\substack{n\le N \\ n\in S(Q)}} \frac{\overline{\chi}(n)}{n} e\left(\frac{b}{r}\cdot n\right) - \overline{\chi}(-1) \sum_{\substack{n\le N \\ n\in S(Q)}} \frac{\overline{\chi}(n)}{n} e\left(-\frac{b}{r}\cdot n\right).$$

By using the Granville–Soundararajan identity separately to two sums appearing on the right-hand side we find that

$$\begin{aligned} \sum_{\substack{1\le |n|\le N \\ n\in S(Q)}} \frac{\overline{\chi}(n)}{n} e\left(\frac{b}{r}\cdot n\right) &= \sum_{\substack{d\mid r \\ d\in S(Q)}} \frac{\overline{\chi}(d)}{d} \cdot \frac{1}{\varphi\left(\frac{r}{d}\right)} \cdot \qquad (70) \\ &\cdot \sum_{\psi \pmod{\frac{r}{d}}} (1-\chi(-1)\psi(-1))\tau(\psi)\overline{\psi}(b) \left(\sum_{\substack{n\le \frac{N}{d} \\ n\in S(Q)}} \frac{\overline{\chi\psi}(n)}{n}\right). \end{aligned}$$

We will prove the following:

Theorem 6.14. Let $N \geq 2$, $Q \geq 16$, χ be a character, and $\frac{b}{r}$ be a reduced fraction[15] with $1 \leq r \leq \log Q$. Assume that as ψ ranges over all primitive characters of conductor less than r, $M(\overline{\chi\psi}; Q, \log^2 Q)$ is minimized when $\psi = \xi \pmod{m}$. Then

$$\sum_{\substack{1\leq|n|\leq N \\ n\in S(Q)}} \frac{\overline{\chi}(n)}{n} e\left(\frac{b}{r}\cdot n\right) \ll \frac{1}{\sqrt{r}}(\log Q)^{\frac{2}{3}+o(1)} + \sqrt{r}e^{\mathcal{O}(\sqrt{\log\log Q})} + \begin{cases} (1-\chi(-1)\xi(-1))\frac{\sqrt{m}}{\varphi(m)}(\log Q)e^{-M(\overline{\chi\xi};Q,\log^2 Q)} & \text{if } m|r \\ 0 & \text{otherwise} \end{cases} \tag{71}$$

Proof. Assume first that $b \neq 0$. We consider the set of all primitive characters with conductor below or equal to r. Here we also consider the constant function 1 as a primitive character modulo 1. Let us enumerate these characters as $\psi_k \pmod{m_k}$ such that

$$M(\overline{\chi\psi_1}; Q, \log^2 Q) \leq M(\overline{\chi\psi_2}; Q, \log^2 Q) \leq \cdots$$

First we consider the inner sum on the right-hand side of (70), that is

$$\mathcal{X}(d) := \frac{1}{\varphi\left(\frac{r}{d}\right)} \sum_{\psi \pmod{\frac{r}{d}}} (1-\chi(-1)\psi(-1))\tau(\psi)\overline{\psi}(b) \sum_{\substack{n\leq\frac{N}{d} \\ n\in S(Q)}} \frac{(\overline{\chi\psi})(n)}{n}$$

Since the behaviour of characters $\psi \pmod{\frac{r}{d}}$ is determined by the primitive characters inducing them, we will define a set $\mathcal{K}_d$ as

$$\mathcal{K}_d = \left\{k : m_k | \frac{r}{d}\right\}$$

By the consequence of Theorem 2.5 (see Chapter 2.) we can write $\mathcal{X}(d)$ in terms of ψ_k as

$$\mathcal{X}(d) = \frac{1}{\varphi\left(\frac{r}{d}\right)} \sum_{k\in\mathcal{K}_d} (1-\chi(-1)\psi_k(-1))\tau(\psi_k\chi_0)\overline{\psi_k}(b)\chi_0(b) \left(\sum_{\substack{n\leq\frac{N}{d} \\ n\in S(Q)}} \frac{\overline{\chi\psi_k}\chi_0(n)}{n} \right)$$

Here χ_0 is the principal character modulo the conductor of ψ.

Furthermore, with the help of equation (4) this can be written as

$$\begin{aligned} \mathcal{X}(d) &= \frac{\chi_0(b)}{\varphi\left(\frac{r}{d}\right)} \sum_{k\in\mathcal{K}_d} \mu\left(\frac{r}{dm_k}\right)\psi_k\left(\frac{r}{dm_k}\right) \\ &\cdot (1-\chi(-1)\psi_k(-1))\tau(\psi_k)\overline{\psi}_k(b) \sum_{\substack{n\leq\frac{N}{d} \\ n\in S(Q) \\ \left(n,\frac{r}{d}\right)=1}} \frac{(\overline{\chi\psi_k})(n)}{n}. \end{aligned} \tag{72}$$

[15] If $b = 0$ we will require that $r = 1$

Applying Lemma 6.6. with Q-smoothed version $(\overline{\chi\psi})_Q$ of the function $\overline{\chi\psi}$ in place of f and noting that $d \le r \le Q$, we obtain

$$\begin{aligned}
\sum_{\substack{n \le \frac{N}{d} \\ n \in S(Q) \\ \left(n, \frac{r}{d}\right)=1}} \frac{(\overline{\chi\psi})(n)}{n} &= \sum_{\substack{n \le N \\ \left(n, \frac{r}{d}\right)=1}} \frac{(\overline{\chi\psi})_Q(n)}{n} + \mathcal{O}(\log d) \\
&= \prod_{p \mid \frac{r}{d}} \left(1 - \frac{(\overline{\chi\psi})_Q(p)}{p}\right) \sum_{n \le N} \frac{(\overline{\chi\psi})_Q(n)}{n} + \mathcal{O}(\log r) \\
&= \prod_{p \mid \frac{r}{d}} \left(1 - \frac{(\overline{\chi\psi})(p)}{p}\right) \sum_{\substack{n \le N \\ n \in S(Q)}} \frac{(\overline{\chi\psi})(n)}{n} + \mathcal{O}(\log r).
\end{aligned}$$

Thus, by substituting this back to (72), we see that the main term of $\mathcal{X}(d)$ will be

$$\frac{\chi_0(b)}{\varphi\left(\frac{r}{d}\right)} \sum_{k \in \mathcal{K}_d} \mu\left(\frac{r}{dm_k}\right) \psi_k\left(\frac{r}{dm_k}\right) (1-\chi(-1)\psi_k(-1))\tau(\psi_k)\overline{\psi}_k(b) \prod_{p \mid \frac{r}{d}} \left(1 - \frac{(\overline{\chi\psi_k})(p)}{p}\right) \sum_{\substack{n \le N \\ n \in S(Q)}} \frac{(\overline{\chi\psi_k})(n)}{n}$$

and the error term is of the size

$$\begin{aligned}
&\ll \frac{\chi_0(b)}{\varphi\left(\frac{r}{d}\right)} \sum_{k \in \mathcal{K}_d} \mu\left(\frac{r}{dm_k}\right) \psi_k\left(\frac{r}{dm_k}\right) \tau(\psi_k)\overline{\psi}_k(b) \log r \\
&\ll \frac{1}{\varphi\left(\frac{r}{d}\right)} \sum_{k \in \mathcal{K}_d} \sqrt{m_k} \log r \ll \sqrt{\frac{r}{d}} \log r
\end{aligned}$$

by using the fact that $|\tau(\psi_k)| = \sqrt{m_k}$, that characters have absolute value at most one, $|\mathcal{K}_d| = \varphi\left(\frac{r}{d}\right)$ and $m \mid \frac{r}{d}$. The total contribution of such error terms to the left-hand side of (71) equals

$$\sum_{d \mid r} \frac{\overline{\chi}(d)}{d} \mathcal{X}(d) \ll \sum_{d \mid r} \frac{1}{d} \sqrt{\frac{r}{d}} \log r \ll \sqrt{r} \log r \ll \sqrt{r} \log\log Q,$$

because $r \le \log Q$. This is negligible compared to the bounds claimed in the statement of the theorem. Now we move back to the main term. We will estimate the contribution of characters ψ_k, $k \ge 2$. By using Corollary 3.8 it can be estimated upwards as

$$\ll \frac{1}{\varphi\left(\frac{r}{d}\right)} \sum_{\substack{k \in \mathcal{K}_d \\ k \ge 2}} \sqrt{m_k} \left(\prod_{p \mid \frac{r}{d}} \left(1 + \frac{1}{p}\right)\right) \left((\log Q) e^{-M(\overline{\chi\psi_k}; Q, \log^2 Q)} + \frac{1}{\log Q}\right) \tag{73}$$

simply from the fact that $|\chi(n)| \le 1$ for every character and $|\tau(\psi_k)| = \sqrt{m_k}$. Recall from Chapter 4 that we have, from the estimate (21), that for all $g \in \mathcal{F}$ and $T \ge 0$ and $0 \le M(g; y, T) \le 2 \log\log y + \mathcal{O}(1)$. Therefore

$$(\log Q) e^{-M(\overline{\chi\psi}; Q, \log^2 Q)} \gg \frac{1}{\log Q}.$$

Note that we also have $m_k \le \frac{r}{d}$ for all $k \in \mathcal{K}_d$ and from (18) it follows

$$\prod_{p \mid \frac{r}{d}} \left(1 + \frac{1}{p}\right) \ll \log\log\left(\frac{r}{d} + 2\right).$$

Using these facts we can continue our calculation from (73) to obtain that the total contribution of characters $\psi_k \pmod{m_k}$ with $k \geq 2$ is

$$\ll \frac{1}{\varphi\left(\frac{r}{d}\right)}\sqrt{\frac{r}{d}}\left(\log\log\left(\frac{r}{d}+2\right)\right)(\log Q)\sum_{\substack{k\in\mathcal{K}_d\\ k\geq 2}} e^{-M(\overline{\chi\psi_k};Q,\log^2 Q)}.$$

To make progress we must estimate the quantity $M(\overline{\chi\psi_k}, Q, \log^2 Q)$ from below. For this we use the following results from [2]. Lemma 3.3 from that source states that if $k \geq 2$ then

$$M(\overline{\chi\psi_k}; Q, \log^2 Q) \geq \left(\frac{1}{3}+o(1)\right)\log\log Q. \tag{74}$$

From Lemma 3.1 of [2] we deduce that if $k > \sqrt{\log\log Q}$ we have a sharper estimate

$$\begin{aligned} M(\overline{\chi\psi_k}; Q, \log^2 Q) &= \min_{|t|\leq \log^2 Q} \mathbb{D}(\overline{\chi}, \psi_k(n)n^{it}; Q)^2 \\ &\geq \left(1-\frac{1}{\sqrt{\log\log Q}}\right)\log\log Q + \mathcal{O}(\sqrt{\log\log Q}) \\ &= \log\log Q + \mathcal{O}(\sqrt{\log\log Q}). \end{aligned}$$

These imply that the contribution of all characters ψ_k, $k \geq 2$, $k \in \mathcal{K}_d$ to $\mathcal{X}(d)$ is

$$\begin{aligned} &\ll \frac{1}{\varphi\left(\frac{r}{d}\right)}\sqrt{\frac{r}{d}}\left(\log\log\left(\frac{r}{d}+2\right)\right)(\log Q)\left((\log Q)^{-\frac{1}{3}+o(1)} + e^{\mathcal{O}(\sqrt{\log\log Q})}\cdot\frac{1}{\log Q}\right) \\ &\ll \frac{1}{\varphi\left(\frac{r}{d}\right)}\sqrt{\frac{r}{d}}\left(\log\log\left(\frac{r}{d}+2\right)\right)(\log Q)^{\frac{2}{3}+o(1)} + \sqrt{\frac{r}{d}}\left(\log\log\left(\frac{r}{d}+2\right)\right)e^{\mathcal{O}(\sqrt{\log\log Q})}. \end{aligned}$$

Now the total contribution of such characters to the left-hand side of (71) can be estimated as follows

$$\begin{aligned} &\ll \sum_{d|r}\frac{1}{d}\left(\frac{1}{\varphi\left(\frac{r}{d}\right)}\sqrt{\frac{r}{d}}\left(\log\log\left(\frac{r}{d}+2\right)\right)(\log Q)^{\frac{2}{3}+o(1)} + \sqrt{\frac{r}{d}}\left(\log\log\left(\frac{r}{d}+2\right)\right)e^{\mathcal{O}(\sqrt{\log\log Q})}\right) \\ &\ll \sqrt{r}(\log\log(r+2))\sum_{d|r}\left(\frac{1}{d}\right)^{\frac{3}{2}}\left(\frac{1}{\varphi\left(\frac{r}{d}\right)}(\log Q)^{\frac{2}{3}+o(1)} + e^{\mathcal{O}(\sqrt{\log\log Q})}\right) \\ &\ll \frac{1}{r}(\log\log(r+2))(\log Q)^{\frac{2}{3}+o(1)}\sum_{d|r}\frac{d^{\frac{3}{2}}}{\varphi(d)} + \sqrt{r}(\log\log(r+2))e^{\mathcal{O}(\sqrt{\log\log Q})}, \end{aligned}$$

where we made a change of variable $d \mapsto \frac{r}{d}$ on the third line. Using the estimates (16) and $d(r) \ll (\log Q)^{o(1)}$ (this follows from (17) as $r \leq \log Q$), we obtain that the contribution of all such a terms to the right-hand side of the Granville–Soundararajan identity is

$$\begin{aligned} &\leq \frac{1}{r}(\log\log(r+2))(\log Q)^{\frac{2}{3}+o(1)}d(r)\frac{r^{\frac{3}{2}}}{\varphi(r)} + \sqrt{r}(\log\log(r+2))e^{\mathcal{O}(\sqrt{\log\log Q})} \\ &\ll \frac{1}{\sqrt{r}}(\log\log(r+2))(\log Q)^{\frac{2}{3}+o(1)}\cdot\log\log r\cdot(\log Q)^{o(1)} + \sqrt{r}(\log\log(r+2))e^{\mathcal{O}(\sqrt{\log\log Q})} \\ &\ll \frac{1}{\sqrt{r}}(\log Q)^{\frac{2}{3}+o(1)} + \sqrt{r}e^{\mathcal{O}(\sqrt{\log\log Q})}. \end{aligned}$$

The last estimate follows from the assumption $r \leq \log Q$.

Now we turn to the character $\psi_1 \pmod{m_1}$. Note that the exceptional character $\psi_1 \pmod{m_1}$ appears in the sum $\mathcal{X}(d)$ only when $1 \in \mathcal{K}_d$, i.e. when it induces a character modulo $\frac{r}{d}$. There are two cases to consider. If $m_1 \nmid r$, then $1 \notin \mathcal{K}_d$ for any d, and so the exceptional character does not contribute anything. Hence, the theorem is proved in this case. The situation is much more interesting when $m_1|r$. Then the total contribution of this character is

$$\sum_{d|\frac{r}{m_1}} \frac{\overline{\chi}(d)}{d} \quad \cdot \quad \frac{1}{\varphi\left(\frac{r}{d}\right)}\mu\left(\frac{r}{dm_1}\right)\psi_1\left(\frac{r}{dm_1}\right)(1-\chi(-1)\psi_1(-1))$$
$$\cdot \quad \tau(\psi_1)\overline{\psi}_1(b)\left(\prod_{p|\frac{r}{dm_1}}\left(1-\frac{(\overline{\chi\psi_1})(p)}{p}\right)\right)\sum_{\substack{n\le N\\ n\in S(Q)}}\frac{(\overline{\chi\psi_1})(n)}{n}.$$

By making a change of variables $d \mapsto \frac{r}{dm_1}$ we see that this can be written in the form

$$\frac{m_1}{r}(1-\chi(-1)\psi_1(-1))\tau(\psi_1)\overline{\psi}_1(b)\left(\sum_{\substack{n\le N\\ n\in S(Q)}}\frac{(\overline{\chi\psi_1})(n)}{n}\right)\sum_{d|\frac{r}{m_1}}\overline{\chi}\left(\frac{r}{dm_1}\right)\mathscr{A}(d), \tag{75}$$

where

$$\mathscr{A}(d) := \frac{d}{\varphi(dm_1)}\mu(d)\psi_1(d)\prod_{p|d}\left(1-\frac{(\overline{\chi\psi_1})(p)}{p}\right).$$

Since $\mathscr{A}(d)=0$ when $\mu(d)=0$ or $\psi_1(d)=0$, it is enough to consider those numbers d which are squarefree and satisfy $(d,m_1)\ne 1$. We can calculate

$$\begin{aligned}
\mathscr{A}(d) &= \frac{1}{\varphi(m_1)}\prod_{p|d}\left(\frac{\overline{\chi}(p)-\psi_1(p)\cdot p}{\varphi(p)}\right)\\
&\ll \frac{1}{\varphi(m_1)}\prod_{p|d}\left(\frac{p+1}{p-1}\right)\\
&\overset{(*)}{\ll} \frac{1}{\varphi(m_1)}(\log\log(d+2))^2.
\end{aligned}$$

In the step $(*)$ we used Lemma 2.15. When this is combined with (75) and Corollary 3.8. we get that the total contribution of the character ψ_1 is

$$\begin{aligned}
&\ll (1-\chi(-1)\psi_1(-1))\frac{m_1}{r}\tau(\psi_1)\overline{\psi}_1(b)\left((\log Q)^{-M(\overline{\chi\psi_1};Q,\log^2 Q)}+\frac{1}{\log Q}\right)\sum_{d|\frac{r}{m_1}}\frac{1}{\varphi(m_1)}(\log\log(d+2))^2\\
&\ll (1-\chi(-1)\psi_1(-1))\frac{m_1}{r}\cdot\sqrt{m_1}\cdot(\log Q)^{-M(\overline{\chi\psi_1};Q,\log^2 Q)}\cdot\frac{1}{\varphi(m_1)}\cdot\frac{r}{m_1}\\
&\ll (1-\chi(-1)\psi_1(-1))\frac{\sqrt{m_1}}{\varphi(m_1)}(\log Q)^{-M(\overline{\chi\psi_1};Q,\log^2 Q)}.
\end{aligned} \tag{76}$$

This concludes the case $b\ne 0$. The remaining case $b=0$ is easy to handle. If ψ_1 is the trivial character, then $m_1=1$ and the theorem follows from Corollary 3.8:

$$\begin{aligned}
\sum_{\substack{1\le|n|\le N\\ n\in S(Q)}}\frac{\overline{\chi\psi}(n)}{n} &\ll (1-\chi(-1))(\log Q)e^{-M(\overline{\chi\psi};Q,\log^2 Q)}\\
&= (1-\chi(-1)\psi_1(-1))\frac{\sqrt{m_1}}{\varphi(m_1)}(\log Q)^{-M(\overline{\chi\psi_1};Q,\log^2 Q)},
\end{aligned}$$

as wanted.

If it is not, then we are done by applying Corollary 3.8 and using (74):

$$\sum_{\substack{1\le|n|\le N\\ n\in S(Q)}} \frac{\overline{\chi\psi}(n)}{n} \ll (\log Q)e^{-M(\overline{\chi\psi};Q,\log^2 Q)} \ll (\log Q)\cdot(\log Q)^{-\frac{1}{3}+o(1)} = \frac{1}{\sqrt{r}}(\log Q)^{\frac{2}{3}+o(1)},$$

as desired. The last equality is true from our convention that $r = 1$ when $b = 0$. So we get the bounds claimed in the theorem also in this case. Proof is completed. $\square$

Now we are in the position to prove Goldmakher–Granville–Soundararajan Estimate. Combinig Theorems 6.12, 6.13 (with $f = \overline{\chi}$ and $y = Q$, of course) and 6.14 we immediately deduce that

Theorem 6.15. Let $\chi \pmod q$ be a primitive character and suppose that as ψ ranges over all primitive characters of conductor less than $\log Q$, $M(\chi\overline{\psi}, Q, \log^2 Q)$ is minimized when $\psi = \xi \pmod m$. Then

$$\sum_{1\le|n|\le q} \frac{\overline{\chi}(n)}{n}e(n\alpha) \ll (1-\chi(-1)\xi(-1))\frac{\sqrt{m}}{\varphi(m)}(\log Q)e^{-M(\chi\overline{\xi},Q,\log^2 Q)} + (\log Q)^{\frac{2}{3}+o(1)}$$

for $\alpha \neq 0$.

Combining this with the Pólya's Fourier expansion yields in the case $\alpha \neq 0$ that

$$|\mathcal{S}_\chi(t)| \ll (1-\chi(-1)\xi(-1))\frac{\sqrt{m}}{\varphi(m)}\sqrt{q}(\log Q)e^{-M(\chi\overline{\xi},Q,\log^2 Q)} + \sqrt{q}(\log Q)^{\frac{2}{3}+o(1)}.$$

Using Theorem 4.9 gives the Goldmakher–Granville–Soundararajan estimate

$$\begin{aligned} |\mathcal{S}_\chi(t)| &\ll \sqrt{q}(\log Q)\exp\left(-(\delta_g+o(1))\log\log Q\right) + \sqrt{q}(\log Q)^{\frac{2}{3}+o(1)} \\ &\ll \sqrt{q}(\log Q)\cdot(\log Q)^{\delta_g+o(1)} \\ &= \sqrt{q}(\log Q)^{1-\delta_g+o(1)}. \end{aligned}$$

We deal the case $\alpha = 0$ separately. We have two cases to consider: ψ is either the trivial character or it is not. The arguments are practically the same as in the case $b = 0$ of Theorem 6.14.

Assume first that ξ is trivial. If necessary, we use Theorem 6.12. to restrict the sum

$$\sum_{1\le|n|\le q} \frac{\overline{\chi}(n)}{n} = (1-\chi(-1))\sum_{n\le q}\frac{\overline{\chi}(n)}{n}$$

to Q-smooth arguments. Since ξ is trivial we especially have $\xi(-1) = 1$ and so we can calculate

$$\begin{aligned} (1-\chi(-1))\sum_{\substack{n\le q\\ n\in S(Q)}}\frac{\overline{\chi}(n)}{n} &= (1-\chi(-1)\xi(-1))\sum_{\substack{n\le q\\ n\in S(Q)}}\frac{\overline{\chi}\xi(n)}{n} \\ &= (1-\chi(-1)\xi(-1))\frac{\sqrt{m}}{\varphi(m)}e^{-M(\chi\overline{\xi};Q,\log^2 Q)} \end{aligned}$$

using Corollary 3.8.

On the other hand, if ξ is not trivial, we have by Corollary 3.8 and (74),

$$(1-\chi(-1)\xi(-1))\sum_{\substack{n\le q\\ n\in S(Q)}}\frac{\overline{\chi}(n)}{n} \ll (\log Q)^{\frac{2}{3}+o(1)}.$$

Similarly as before, putting these bounds to Pólya's Fourier expansion and using Theorem 4.9 gives the desired result. So the Goldmakher–Granville–Soundararajan estimate holds also when $\alpha = 0$. This concludes the proof. □

Goldmakher [33] also showed, using the construction inspired by Paley's work [62], that the estimate under the GRH is the best possible. Instead of Quadratic Reciprocity Law, he used a more general version of this due to Vostokov and Orlova [81].

6.4 Character Sums to Smooth Moduli

There are much better estimates for certain special characters that the Pólya–Vinogradov inequality produces. In this chapter we examine a few such cases.

Recall that a positive integer y is called *smooth* if its prime factors are small related to y. If the product of all prime factors of y is small, then y is called *powerful.* We define the radical of q to be

$$\mathrm{rad}(q) = \prod_{p|q} p.$$

In this section we show how the results of Granville and Soundararajan can be applied to give bounds for sums of characters whose modulus is either smooth or powerful. Our main reference of this section is [34] and other sources are mentioned when needed. In order to keep this thesis relatively short, we will skip all the proofs of all the results related to L-functions.

For characters with smooth conductors, Goldmakher proved the following improvement for the Pólya–Vinogradov inequality

Theorem 6.16. Let $\chi \pmod q$ be a primitive character with q squarefree. Then

$$|\mathcal{S}_\chi(t)| \ll \sqrt{q}\log q\left(\left(\frac{\log\log\log q}{\log\log q}\right)^{\frac{1}{2}} + \left(\frac{(\log\log\log q)^2\log(p(q)d(q))}{\log q}\right)\right)^{\frac{1}{4}},$$

where $d(q)$ is the number of divisor of q and $p(q)$ is its largest prime factor.

The key idea is to apply Theorem 6.10. due to Granville and Soundararajan. In order to do so, we must gain understanding of the size of $\mathbb{D}(\chi, \psi; q)$. However, it is easy to see that this actually equals $\mathbb{D}(\chi\overline{\psi}, 1; q)$. and so we are left to study the quantity $\mathbb{D}(\chi, 1; q)$. In [34] Goldmakher showed[16] that this distance is closely connected to the value of the L-function at slightly right on the line $\sigma = 1$:

$$\mathbb{D}(\chi, 1; y)^2 = \log\left|\frac{\log y}{L(s_y, \chi)}\right| + \mathcal{O}(1).$$

Combining this to Theorem 6.10. yields

Theorem 6.17. Let $\chi \pmod q$ be a primitive character and $\psi \pmod m$ be as in Theorem 6.10. Then

$$\begin{aligned} |\mathcal{S}_\chi(t)| &\ll \sqrt{q}\log q\cdot\left(\sqrt{\frac{\log q}{|L(s_q, \chi\overline{\psi})|}}\right)^{-1} + \sqrt{q}(\log q)^{\frac{6}{7}} \\ &= \sqrt{q}\sqrt{(\log q)|L(s_q, \chi\overline{\psi})|} + \sqrt{q}(\log q)^{\frac{6}{7}}. \end{aligned}$$

[16] Lemma 3 there

Thus we have to bound $L(s_q, \chi\overline{\psi})$ non-trivially. Lemma 4 in [34] tells that

Lemma 6.18. Given a primitive character $\chi \pmod q$, let r be any positive number such that for all $p \geq r$, $\text{ord}_p q \leq 1$. Here $\text{ord}_p q$ is the largest non-negative integer ℓ such that $p^\ell | q$. Also let

$$q' = \prod_{p<r} p^{\text{ord}_p q}.$$

Then for all $y > 3$ it holds that

$$|L(s_y, \chi)| \ll \log q' + \frac{\log q}{\log\log q} + \sqrt{(\log q)(\log(p(q)d(q)))}.$$

We wish to apply this lemma to prove Theorem 6.16. We face a problem, since $\chi\overline{\psi}$ might not be primitive. Instead we will apply it to the primitive character which induces $\chi\overline{\psi}$. Thus we need to understand the conductor of $\chi\overline{\psi}$. We start by quoting the well-known result that for any non-principal characters $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$, $\text{cond}(\chi_1\chi_2) | [\text{cond}(\chi_1), \text{cond}(\chi_2)]$. For the proof we refer to [34], Lemma 5.1.

Let $\chi \pmod q$ and $\psi \pmod m$ be primitive characters as before. Denote the primitive character which induces $\chi\overline{\psi}$ by $\xi \pmod r$. Then, choosing $\chi_1 = \chi$ and $\chi_2 = \overline{\psi}$ in the well-known fact mentioned above gives $r | [q, m]$, so $r \leq qm$. On the other hand, by choosing $\chi_1 = \chi\overline{\psi}$ and $\chi_2 = \psi$ we get $q | [r, m]$, which implies $q \leq rm$. These together imply that

$$\frac{q}{m} \leq r \leq qm. \tag{77}$$

We also need the following result which connects the L-functions related to $\chi\overline{\psi}$ and ξ.

Lemma 6.19. Let $\chi \pmod q$, $\psi \pmod m$ and $\xi \pmod r$ be as before. Then, for all $s \in \mathbb{C}$ with $\Re s > 1$,

$$\left| \frac{L(s, \chi\overline{\psi})}{L(s, \xi)} \right| \ll 1 + \log\log m.$$

The proof is omitted, but it can be found from [34] where it is recorded as Lemma 5.2. Now we are ready to prove Theorem 6.16.

Proof of Theorem 6.16. The immediate consequence of the previous lemma and the assumption that the conductor of $\psi \pmod m$ is less than $(\log q)^{\frac{1}{3}}$, is that

$$|L(s_q, \chi\overline{\psi})| \ll |L(s_q, \xi)| \log\log\log q. \tag{78}$$

We also have that $r | [q, m]$ and so for all primes $p > m$ we have

$$\text{ord}_p r \leq \max(\text{ord}_p q, \text{ord}_p m) = \text{ord}_p q \leq 1,$$

since q is squarefree. Now we can apply Lemma 6.18. to the character ξ with choices $y = q$ and

$$q' = \prod_{p \leq m} p^{\text{ord}_p r}$$

to give

$$|L(s_q, \xi)| \ll \log q' + \frac{\log r}{\log\log r} + \sqrt{(\log r)\log(P(r)d(r))}. \tag{79}$$

The remaining thing is to bound the right-hand side in terms of q. For the first term

$$\begin{aligned}\log q' &= \sum_{p\le m}(\mathrm{ord}_p q)\log p\\ &\le \sum_{p\le m}(\mathrm{ord}_p q)\log p+\sum_{p\le m}(\mathrm{ord}_p m)\log p\\ &\le \theta(m)+\log m \ll (\log q)^{\frac{1}{3}},\end{aligned}$$

since the conductor of $\psi \pmod m$ is below $(\log q)^{\frac{1}{3}}$. The second term can be estimated with (77) as follows

$$\frac{\log r}{\log\log r} \ll \frac{\log q}{\log\log q}.$$

For the final term we have

$$d(r) \le d(qm) \le d(q)d(m) \le d(q)(\log q)^{\frac{1}{3}}$$

and

$$P(r) \le \max(P(q), P(m)) \le P(q)P(m) \le P(q)(\log q)^{\frac{1}{3}}.$$

Combining these, and taking the clear estimate $\log r \ll \log q$ into account, estimate (79) yields

$$\begin{aligned}|L(s_q,\xi)| &\ll (\log q)^{\frac{1}{3}}+\frac{\log q}{\log\log q}+\sqrt{(\log q)\log\left(P(q)d(q)(\log q)^{\frac{2}{3}}\right)}\\ &\ll \frac{\log q}{\log\log q}+\sqrt{(\log q)\log(P(q)d(q))},\end{aligned}$$

where the last estimate follows since $(\log q)^{\frac{1}{3}} \ll \frac{\log q}{\log\log q}$ and $(\log q)\log((\log q)^{\frac{2}{3}}) \ll (\log q)^2 \ll (\log q)(\log p(q)d(q))$. Combining this with (78) and Theorem 6.17 completes the proof. □

As a consequence we also have the following estimate

Corollary 6.20. If $\chi \pmod q$ is primitive with q squarefree, then

$$|\mathcal{S}_\chi(t)| \ll \sqrt{q}\log q\left(\frac{(\log\log\log q)^2}{\log\log q}+\frac{(\log\log\log q)^2\log p(q)}{\log q}\right)^{\frac{1}{4}}.$$

This follows at once from Theorem 6.16 as $\log d(q) \ll \frac{\log q}{\log\log q}$ (see Lemma 2.15).

For primitive characters with powerful conductors, we have another estimate under some extra conditions.

Theorem 6.21. Let $\chi \pmod q$ be a primitive character with q large and

$$\mathrm{rad}(q) \le (\exp(\log q)^{\frac{3}{4}}). \tag{80}$$

Then

$$|\mathcal{S}_\chi(t)| \ll_\varepsilon \sqrt{q}(\log q)^{\frac{7}{8}+\varepsilon}$$

For the proof we need the following lemma:

Lemma 6.22. Let $\chi \pmod q$ be a primitive character with q large and

$$\mathrm{rad}(q) \le \exp(2(\log q)^{\frac{3}{4}}).$$

Then for all $y > 3$ we have

$$|L(s_y, \chi)| \ll_\varepsilon (\log q)^{\frac{3}{4}+\varepsilon}.$$

For the proof we refer to [34], Theorem 2. Now it is easy to complete the proof of Theorem 6.21:

Proof of Theorem 6.21. We maintain the notations of this section. We start by noting that $\mathrm{rad}(m) = \prod_{p|m} p \leq \prod_{p \leq m} p = \exp(\theta(m))$. Therefore the PNT, fact that conductor of $\psi \pmod m$ is less than $(\log q)^{\frac{1}{3}}$, and the assumption (80) assert that

$$\begin{aligned}
\mathrm{rad}(r) &\leq \mathrm{rad}(q)\mathrm{rad}(m) \\
&\leq \exp\left((\log q)^{\frac{3}{4}} + \sum_{p|m} \log p\right) \\
&\leq \exp((\log q)^{\frac{3}{4}} + C(\log q)^{\frac{1}{3}}) \leq \exp\left(\frac{4}{3}(\log q)^{\frac{3}{4}}\right) \qquad (81)
\end{aligned}$$

for a sufficiently large q. Here C was just an absolute constant. The estimate (77) tells that for a large enough q,

$$\left(\frac{\log r}{\log q}\right)^{\frac{3}{4}} \geq \left(\frac{\log \frac{q}{m}}{\log q}\right)^{\frac{3}{4}} \geq \left(1 - \frac{\log\log q}{\log q}\right) \geq \frac{2}{3}.$$

Combining this with (81) implies that $\mathrm{rad}(r) \leq \exp(2(\log r)^{\frac{3}{4}})$. This with Lemma 6.22. and (78) produces

$$\begin{aligned}
|L(s_q, \chi\overline{\psi})| &\ll (\log\log\log q)|L(s_q, \xi)| \\
&\ll_\varepsilon (\log\log\log q)(\log r)^{\frac{3}{4}+\varepsilon} \\
&\leq (\log\log\log q)(\log qm)^{\frac{3}{4}+\varepsilon} \\
&\ll_\varepsilon (\log q)^{\frac{3}{4}+\varepsilon}.
\end{aligned}$$

Plugging this into Theorem 6.17. gives the desired result:

$$\begin{aligned}
|\mathcal{S}_\chi(t)| &\ll_\varepsilon \sqrt{q}(\log q)^{\frac{7}{8}+\varepsilon} + \sqrt{q}(\log q)^{\frac{6}{7}} \\
&\ll_\varepsilon \sqrt{q}(\log q)^{\frac{7}{8}+\varepsilon}.
\end{aligned}$$

□

References

[1] Apostol, T.M.: Introduction to Analytic Number Theory. Undergraduate Texts in Mathematics. Springer (1976).

[2] Balog, A., Granville, A., Soundararajan, K.: Multiplicative Functions in Arithmetic Progressions. Annales mathématiques du Québec. Volume 37, no. 1, (2013), pp. 3-30.

[3] Bober, J. & Goldmakher, L.: The distribution of the maximum of character sums. Mathematika, Vol. 59 (02), (2013), pp. 427-442.

[4] Bombieri, E.: Le grand crible dans la theorie analytique des nombres. Asterisque, (18):103, 1987.

[5] Burgess, D.A.: The distribution of quadratic residues and non-residues. Mathematika 4 (1957), pp. 106-112.

[6] Burgess, D.A.: On character sums and primitive roots, Proc. Lond. Math. Soc. (3), 12 (1962), pp. 179-192.

[7] Burgess, D.A.: On character sums and L-series, Proc. Lond. Math. Soc. (3), 12 (1962), pp. 193-206

[8] Burgess, D.A.: On character sums and L-series II, Proc. Lond. Math. Soc. (3), 13 (1963), pp. 524-536.

[9] Burgess, D.A.: Estimation of character sums modulo a small power of a prime. J. Lond. Math. Soc. (2), 30 (1984), no. 3, pp. 385-393.

[10] Burgess, D.A.: Estimation of character sums modulo a power of a prime. Proc. Lond. Math. Soc. (3), 52 (1986), no. 2, pp. 215-235.

[11] Burgess, D.A.: The character sum estimate with $r = 3$. J. Lond. Math. Soc. (2), 33 (1986), no. 2, pp. 219-226.

[12] Chowla, S.D.: On the class number of the corpus $P(\sqrt{-k})$. Proc. Nat. Inst. Sci. India 13 (1947), pp. $197 - 200$.

[13] Daboussi, H.: Fonctions multiplicatives presque périodiques B. D'aprés un travail commun avec Hubert Delange. Journés Arithmt́ique de Bordeaux (Conf. Univ. Bordeaux, 1974), 321-324. Astŕisque, 24-25. Soc. Math France, Paris, 1975.

[14] Davenport, H.: Multiplicative Number Theory. Springer-Verlag, New York (1980).

[15] Delange, H.: Sur les fonctions arithmétiques multiplicatives. Ann. Sci. Ecole Norm. Sup. (3), 78 (1961), pp. $273 - 304$.

[16] Delange, H.: On a class of multiplicative functions. Scripta Math. 26 (1963) pp. $121 - 141$.

[17] Dirichlet, P.: Beweis des Satzes, dafs jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enhält. Memoirs of the Royal Prussian Academy of Sciences in Berlin 1837 (Berlin. Royal Acad of Sciences, 1839).

[18] Erdös, P.: Some unsolved problems. Michigan Math. J., 4 (1957) pp. $91 - 300$.

[19] Erdös, P.: On a new method in elementary number theory which leads to an elementary proof of the prime number theorem. Proc. Nat. Acad. Scis. U.S.A. 35 (1949), pp. 374-384.

[20] Graham, S.W. and Ringrose, C.J.: Lower Bounds for least quadratic nonresidues. Analytic number theory (Allerton Park, IL, 1989), Birkhäuser, 1990, pp. 269-309.

[21] Granville, A.: Sums of multplicative functions. Lecture notes for Summer School in Analytic Number Theory and Diophantine Approximation, University of Ottawa, Ontario Canada (2008). Available at
http://aix1.uottawa.ca/droy/summer-school-2008/granvillelectures.pdf

[22] Granville, A.: A Pretentious Introduction to Analytic Number Theory. Lecture slides, Canadian Number Theory Association XII Meeting University of Lethbridge June 17-22, 2012. Available at
http://www.cs.uleth.ca/ cnta2012/slides-cnta12/Andrew-Granville.pdf

[23] Granville, A. & Soundararajan, K.: The distribution of prime numbers. Lecture notes for the course Pretentious Introduction to Analytic Number Theory (2011). University of Montreal. Available at
http://www.dms.umontreal.ca/ andrew/Courses/MAT6627.H11.html

[24] Granville, A. & Soundararajan, K.: Decay of mean-values of multiplicative functions. Canad. J. Math. 55, no. 6 (2003), pp. 1191-1230.

[25] Granville, A. & Soundararajan, K.: Pretentious multiplicative functions and an inequality for the zeta-function. Anatomy of integers, pp. 191-197, CRM Proc. Lecture Notes (2008), 46, Amer. Math. Soc., Providence, RI.

[26] Granville, A. & Soundararajan, K.: Large character sums: Pretentious characters and the Pólya-Vinogradov theorem. J. Amer. Math. Soc. 20 (2007), no. 2, pp.357-384.

[27] Granville, A. & Soundararajan, K.: Large character sums. J. Amer. Math. Soc. 14 (2001), no. 2, pp. 365-397.

[28] Granville, A. & Soundararajan, K.: The Spectrum of Multiplicative Functions. Ann. of Math. 153 (2001), pp. 407-470.

[29] Granville, A., Soundararajan, K.: Multiplicative number theory: The pretentious approach. To appear. Manuscript available at
http://www.dms.umontreal.ca/ andrew/PDF/Pretentious010611.pdf

[30] Green, B.J.: Proof of Hálasz's theorem. Lecture notes for the course Prime Numbers, University of Cambridge (2012). Available at
https://www.dpmms.cam.ac.uk/ bjg23/primenumbers-2012/notes4.pdf

[31] Goldfeld, D.: The Elementary Proof of the Prime Number Theorem: An Historical Perspective. Available at
http://www.math.columbia.edu/ goldfeld/ErdosSelbergDispute.pdf

[32] Goldmakher, L.: Multiplicative Mimicry and Improvements of the Pólya-Vinogradov Inequality. (2009), Phd-Thesis, University of Michigan.

[33] Goldmakher, L.: Multiplicative mimicry and improvements of the Pólya-Vinogradov inequality. Algebra and Number Theory Vol. 6 (2012), No. 1, pp. 123-163.

[34] Goldmakher, L.: Character sums to smooth moduli are small. Canad. J. Math. 62 (2010), no. 5, pp. 1099-1115.

[35] Hadamard, J.: Etude sur les proprietes des fonctions entieres et en particulier dune fonction consideree par Riemann, J. de Math. Pures Appl. (4) 9 (1893), 171-215; reprinted in Oeuvres de Jacques Hadamard, C.N.R.S., Paris, 1968, vol 1, pp. 103-147.

[36] Hadamard, J.: Sur la distribution des zeros de la fonction $\zeta(s)$ et ses consequences arithmetiques. Bull. Soc. Math. France 24 (1896), pp. 199-220; reprinted in Oeuvres de Jacques Hadamard, C.N.R.S., Paris, 1968, vol 1, pp. 189-210.

[37] Halász, G.: Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen. Acta Math. Acad. Sci. Hungar. 19. (1968) pp. 365-403.

[38] Halász, G.: On the distribution of additive and mean-values of multiplicative functions. Stud. Sci. Math. Hungar 6 (1971), pp. 211-233.

[39] Halász, G.: On the distribution of additive arithmetic functions. Acta Arith. XXVII (1975) pp. 143-152.

[40] Halberstam, H., Richert, H.-E.: Sieve methods. London: Academic Press 1974.

[41] Hall, R. R.: A sharp inequality of Halász type for the mean value of a multiplicative arithmetic function. Mathematika 42 (1995), no. 1, pp. 144-157.

[42] Hildebrand, A.: A note on Burgess's character sum estimate. C.R. Acad. Sci. Roy. Soc. Canada 8 (1986), pp. 35-37.

[43] Hildebrand, A.: On the constant in the Pólya-Vinogradov inequality. Canad. Math. Bull. 31 (1988), no. 3. pp. 347-352.

[44] Hildebrand, A.: Large Values of Character Sums. Journal of Number Theory 29 (1988), pp. 271-296.

[45] Hildebrand, A.: Introduction to Analytic Number Theory. Lecture notes (Fall 2005), University of Illinois. Available at
http://www.math.uiuc.edu/ hildebr/ant/main6.pdf

[46] Holowinsky, R.: Sieving for mass equidistribution. Ann. of Math. (2) 172 (2010), no. 2, pp. 1499-1516.

[47] Holowinsky, R. & Soundararajan, K.: Mass equidistribution for Hecke eigenforms. Ann. of Math. (2) 172 (2010), no. 2, pp.1517-1528.

[48] Iwaniec, H.: On zeros of Dirichlet's L series. Invent. math. 23 (1974), pp. 97-104.

[49] Iwaniec, H. & Kowalski, E.: Analytic Number Theory. Providence: American Mathematical Society (2004).

[50] Jones, G.A. & Jones, J.M.: Elementary Number Theory. Springer Undergraduate Mathematics Series. 1998.

[51] Korobov, N.M.: Estimates of trigonometric sums and their applications. (Russian) Uspehi Mat. Nauk 13 (1958), no. 4 (82), pp. 185-192.

[52] Koukoulopoulos, D.: On multiplicative functions which are small on average. Geometric and Functional Analysis Volume 23, Issue 5 (2013) pp. 1569-1630.

[53] Koukoulopoulos, D.: Pretentious multiplicative functions and the prime number theorem for arithmetic progressions. Compos. Math., 149 (2013), no. 7, pp. 1129-1149.

[54] Landau, E.: Abschätzungen von Charaktersummen. Einheiten und Klassenzahlen, Nachrichten königl. Ges. Wiss. Göttingen (1918), pp. 79-97.

[55] Littlewood, J.E.: On the class number of the corpus $P(\sqrt{-k})$. Proc. London Math. Soc. 27 (1928), pp. 358-372.

[56] Montgomery, H.L. & Vaughan, R.C.: Exponential Sums with Multiplicative Coefficients. Inventiones Mathematicae Vol. 43 (1977) pp. 69-82.

[57] Montgomery, H.L. & Vaughan R.C.: Mean values of multiplicative functions. Period. Math. Hungar. 43, no. $1-2$, (2001) pp. $199-214$.

[58] Montgomery, H.L.: & Vaughan, R.C.: Multiplicative Number Theory I: Classical Theory. Cambridge University Press, 2006.

[59] Montgomery, H.L.: A note on mean values of multiplicative functions. Institute Mittag-Leffler, Report No. 17 (1978).

[60] Murty, R.M.: Problems in Analytic Number Theory, Graduate Texts in Mathematics, vol. 206, Springer-Verlag, New York, 2001, Readings in Mathematics.

[61] Newman, D.J.: Simple analytic proof of the prime number theorem. Amer. Math. Monthly 87 (1980), no. 9, pp. 693-696.

[62] Paley, R.E.A.C.: A theorem on characters. J. London Math. Soc. 7. (1932) pp. 28-32.

[63] Pólya, G.: Über die Verteilung der quadratischen Reste und Nichtreste, Göttinger Nachrichten (1918), pp. 21-29.

[64] Prachar, K.: Primzahlverteilung. Springer–Verlag, Berlin/New York, (1957).

[65] Ramachandra, K.: Theory of Numbers: A textbook. Alpha Science (2007).

[66] Riemann, B.: Über die Anzahl der Primzahlen unter einer gegebenen Grösse. Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin (1859).

[67] Saksman, E.: Analytic Number Theory. Lecture notes, University of Helsinki, Fall 2011.

[68] Schur, I.: Einige Bemerkungen zu der vorstehenden Arbeit des Herrn G. Pólya: Über die Verteilung der quadratischen Reste und Nichtreste. Göttinger nachrichten (1918), pp. 30-36.

[69] Selberg, A.: An elementary proof of the prime-number theorem. Ann. of Math. (2) 50, (1949). pp. 305-313.

[70] Selberg, A.: An elementary proof of Dirichlet's theorem about primes in an arithmetic progression. Annals of Mathematics 50 (2), (1949), pp. 297-304.

[71] Serre, J.P.: A Course in Arithmetic. Graduate Texts in Mathematics. Springer (1973)

[72] Soundararajan, K.: Weak subconvexcity for central values of L-functions. Ann. of Math (2) 172 (2010), no. 2, pp. 1469-1498.

[73] Stein, E. & Shakarchi, R.: Complex Analysis (Princeton Lectures in Analysis, No. 2), 2003.

[74] Stein, E. & Shakarchi, R.: Fourier Analysis: An Introduction (Princeton Lectures in Analysis), 2003.

[75] Strömbergsson, A.: Analytic Number Theory, Lecture notes, Uppsala University, (Fall 2008). Available at:
http://www2.math.uu.se/astrombe/analtalt08/www notes.pdf

[76] Tenenbaum, G.: Introduction to analytic and probabilistic number theory, Cambridge University Press, 1995.

[77] de la Vallée-Poussin, C.J.: Recherches analytiques sur la theorie des nombres premiers. Ann. Soc. Sci. Bruxelles 20 (1896), pp. 183-256.

[78] Vinogradov, I.M.: On an asymptotic equality in the theory of quadratic forms, J. Phys.-Mat. ob-va Permsk Univ. 1 (1918), pp. 18-28.

[79] Vinogradov, I.M.: Sur la distribution des résidus et des nonrésidus des puissances. J. Phys.-Mat. ob-va Permsk Univ. 1 (1918) pp. 94-98.

[80] Vinogradov, I.M.: A new estimate of the function $\zeta(1+it)$. (Russian) Izv. Akad. Nauk SSSR. Ser. Mat. 22 (1958), pp. 161-164

[81] Vostokov, S.V. & Orlova, K.Y.: Generalization and Application of the Eisenstein Reciprocity Law. Vestnik St. Petersburg University Mathematics, Vol. 41 No. 1, (2008), pp. 15-20.

[82] Wintner, A.: Eulerian products and analytic continuation. Duke Math. J. 11, (1944). pp. 277-285.

[83] Wirsing, E.: Das asymptotische verhalten von Summen über multiplikative Funktionen II. Acta Math. Acad. Sci. Hung. 18 (1967), pp. 411-467.

[84] Wolf, J.: Arithmetic structures in difference sets, Part III essay, University of Cambridge (2003). Available at
http://www.juliawolf.org/research/preprints/partIIIweb.pdf