Generalized equilibria in an economy without the survival assumption *

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Econometric Institute Report EI 2002–49

Abstract

It is well known that an equilibrium in the Arrow-Debreu model may fail to exist if a very restrictive condition called the survival assumption is not satisfied. We study two approaches that allow for the relaxation of this condition. Danilov and Sotskov (1990), and Florig (2001) developed a concept of a generalized equilibrium based on a notion of hierarchic prices. Marakulin (1990) proposed a concept of an equilibrium with non-standard prices. In this paper, we establish the equivalence between non-standard and hierarchic equilibria. Furthermore, we show that for any specified system of dividends the set of such equilibria is generically finite. We also provide a generic characterization of hierarchic equilibria and give an easy proof of the core equivalence result.

Key words: competitive equilibrium, hierarchic price, non-standard analysis, satiation, constrained equilibrium.

JEL Classification: D50

1 Introduction

It is well known that an equilibrium in the Arrow-Debreu model may fail to exist if a very restrictive condition called the survival assumption is not satisfied. Its most widely used and widely criticized version requires every consumer to have a positive initial endowment of every good existing in the economy.

To illustrate the problem consider an example (cf. Gale (1976)) of a market with two traders and two commodities: apples and oranges. The first trader

*The earlier version of this paper was published as a CentER Discussion Paper. Whilst the basic results we report are the same in character (except for Theorem 6.1), a number of inaccuracies and minor errors in the proofs have been corrected.

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owns apples and oranges, but has a positive utility only for apples, the second trader cares for both, but owns only oranges. If the price of oranges is positive then the first agent sells his oranges in order to buy more apples, but he already has all the apples. If prices of oranges is zero then the second agent demands an infinite amount of oranges. Thus, no equilibrium results. The reason for this is that the second trader's budget correspondence is not lower hemicontinuous. As the price for oranges falls to zero, the budget set and the demand "explode".

An idea that suggests itself is to redefine a budget correspondence by appropriate refining the notion of prices in order to get an equilibrium that always exists. In particular, an equilibrium in Gale's example would be restored if one manages to define prices for oranges so small that no apples can be bought for any amount of oranges, but still non-zero.

Two realizations of this idea were proposed so far. Gay (1978), Danilov and Sotskov (1990), Mertens (1996), and Florig (1998, 2001) developed an approach based on a notion of a hierarchic price. At equilibrium, all commodities (or commodity bundles treated as separate goods) are divided into several disjoint classes and traded against commodities of the same class according to prices which are an element of some set called a hierarchic price. Moreover, the set of such classes is ordered, superior class commodities cost infinitely much compared to the inferior class ones.

Marakulin (1990) uses non-standard prices in the sense of Robinson's infinitesimal analysis (Robinson (1966)). A similar hierarchic structure of submarkets arises.

An idea to use non-standard numbers to measure prices may look odd at first sight. But a second thought shows that it is not a much bigger abstraction than the use of real numbers for this purpose. Hardly anyone ever paid to anyone else a price of $\sqrt{2}$. Besides, non-standard prices are even natural, since they reflect the fact that costs and values (which are no more than mere numbers) are usually "more divisible" than quantities of consumption goods such as cars, houses, pieces of clothing, etc. The only disadvantage of this approach seems to be that there are still relatively few working economists trained in non-standard analysis. On the other hand, it clearly exceeds standard ways in elegance of proofs and generality of results.

This paper is a continuation of the study of generalized equilibria in a model without the survival assumption. Its first contribution is the reconciliation of standard and non-standard approaches. Starting with non-standard equilibrium, we derive a unique representation of non-standard prices by a hierarchic price, which allows us to characterize non-standard budget sets in pure standard terms. The equivalence between non-standard equilibria and Florig's hierarchic equilibria follows. By use of these results, we prove that it is the possible to represent the set of all equilibrium hierarchic prices as a union of manifolds of dimension less than the number of goods in an economy. This fact allows us to show generic finiteness of hierarchic equilibria for any specified system of dividends. We also provide an existence theorem for this class of equilibria that generalizes the existence result of Marakulin (1990). Furthermore, we show that the set of such equilibria is generically a subset of the union of competitive equilibria and Drèze-Müller coupons equilibria. Finally, we give an easy proof of the equivalence between non-standard (and, therefore, hierarchic) equilibria and the fuzzy rejective core of an economy, a refinement of the weak core introduced in Konovalov (1998).

Section 2 provides the reader with the definition of an equilibrium with non-standard prices and an example which motivates the use of this concept. Section 3 contains a number of auxiliary results that allow us to describe the set of non-standard equilibria in pure standard terms and establish the equivalence between non-standard and hierarchic equilibria. In Section 4 the structure of the set of hierarchic equilibrium prices is investigated. In Section 5 we prove the existence of non-standard dividend equilibria for any specified system of dividends and show the generic finiteness of such equilibria. In Section 6 a generic characterization of non-standard dividend equilibria in terms of constrained equilibria is given. Section 7 contains a core equivalence result.

2 Equilibrium with non-standard prices

We work with an exchange economy $\mathcal{E}$ defined by
$L = \{1, \ldots, l\}$ — the set of commodities;
$Q \subseteq \mathbb{R}^l$ — the set of admissible prices;
$N = \{1, \ldots, n\}$ — the set of agents, where each agent $i \in N$ is characterized by his consumption set $X_i \subset \mathbb{R}^l$, initial endowments $w^i \in X_i$, and preferences given by a correspondence $\mathcal{P}_i : X_i \to 2^{X_i}$, where $\mathcal{P}_i(x^i)$ denotes the set of consumption bundles *strictly* preferred to x^i.

Denote the Cartesian product of individual consumption sets $\prod_{i \in N} X_i$ by X and let $B_i(p) = \{x \in X_i \mid px \leq pw^i\}$ be the budget set of an agent i.

Definition 2.1
An allocation $\bar{x} \in X$ is a *Walrasian equilibrium* if there exists $p \in Q$ such that the following conditions hold:

(i) attainability: $\bar{x}^i \in B_i(p), \quad i \in N,$

(ii) individual rationality: $\mathcal{P}_i(\bar{x}^i) \cap B_i(p) = \emptyset, \quad i \in N,$

(iii) market clearing: $\sum_{i \in N} \bar{x}^i = \sum_{i \in N} w^i.$

Consider the $*$-image ${}^*Q$ of the set Q as the set of all admissible non-standard prices and define by analogy with the standard case non-standard budget sets of consumers:

$$ {}^*B_i(p) = \{x \in {}^*X_i \mid px \leq pw^i\}, \quad p \in {}^*Q, \quad i \in N. $$

By definition, these sets consist of non-standard consumption plans. Consider their standard parts $\bar{B}_i(p)$:

$$ \bar{B}_i(p) = st\, {}^*B_i(p) = \{x \in X_i \mid \exists \tilde{x} \in {}^*B_i(p) : \tilde{x} \approx x\}, $$

where $\tilde{x} \approx x$ denotes infinitesimality of the difference $\tilde{x} - x$: $\|\tilde{x} - x\| \approx 0$. An equilibrium with non-standard prices is formally defined by substitution of the set of possible prices and budget sets in the notion of Walrasian equilibrium for ${}^*Q$ and $\bar{B}_i(p)$, respectively.

Definition 2.2
An allocation $\bar{x} \in X$ is an *equilibrium with non-standard prices* if there exists $p \in {}^*Q$ such that the following conditions hold:

(i) attainability: $\bar{x}^i \in \bar{B}_i(p), \quad i \in N,$

(ii) individual rationality: $\mathcal{P}_i(\bar{x}^i) \cap \bar{B}_i(p) = \emptyset, \quad i \in N,$

(iii) market clearing: $\sum_{i \in N} \bar{x}^i = \sum_{i \in N} w^i.$

It readily follows from the definition above that each Walrasian equilibrium is also an equilibrium with non-standard prices.

Proposition 2.3
Suppose that $\bar{x}$ is a Walrasian equilibrium and $\bar{p} \in Q$ is a corresponding standard vector of equilibrium prices. Then $\bar{x}$ is an equilibrium with non-standard prices.

Proof. Take $\bar{p} \in Q \subset {}^*Q$ as a non-standard equilibrium price vector. Consider an arbitrary individual i. The Walrasian budget set $B_i(\bar{p})$ is by definition a subset of $\bar{B}_i(\bar{p})$ (since $X_i \subset {}^*X_i$). To prove the proposition, we need to show that $\bar{B}_i(\bar{p}) \subseteq B_i(\bar{p})$. Let $x \in \bar{B}_i(\bar{p})$, then there exists $\tilde{x} \in {}^*B_i(\bar{p})$ infinitely near to x. Since $B_i(\bar{p})$ is closed in X_i, $x \in B_i(\bar{p})$ by the non-standard condition for closedness (see, for instance, Anderson (1991), Proposition 2.2.2). Therefore, $B_i(\bar{p}) = \bar{B}_i(\bar{p})$ for every i, which implies that $\bar{x}$ is a Walrasian equilibrium.

□

Reversely, an equilibrium with non-standard prices p is a Walrasian equilibrium if the survival condition is satisfied for each agent i at prices $\bar{p} = {}^{\circ}(p/\|p\|)$. Recall that ${}^{\circ}a$ for $a \in {}^*\mathbb{R}^l$ denotes a *standard part* of a, that is an element of $\mathbb{R}^l$ such that ${}^{\circ}a \approx a$.

Proposition 2.4
Suppose that $\bar{x}$ is a non-standard equilibrium with non-standard prices $p \in {}^*Q$. Let $\bar{p} = {}^{\circ}(p/\|p\|)$. If

$$\inf\ \bar{p}X_i < \bar{p}w^i, \quad i \in N,$$

then $\bar{B}_i(p) = B_i(\bar{p})$ for every $i \in N$, and $\bar{x}$ is a Walrasian equilibrium sustained by the price system $\bar{p}$.

The proof of this fact is relegated to an appendix since it uses Proposition 3.1 that appears later in the paper.

Replacing (*iii*) in the Definition 2.2 by

$$\sum_{i \in N} \bar{x}^i \leq \sum_{i \in N} w^i,$$

gives a definition of a *semi-equilibrium with non-standard prices.*

A satiation effect and dividends

Unfortunately, non-standard equilibria typically do not exist due to a satiation effect caused by measuring prices in non-standard numbers. Recall that an agent is said to be satiated if his demand does not belong to the boundary of his budget set. If there is at least one such an agent in an economy, Walras' law is violated and no competitive equilibrium exists. In our model, even if agent i's preferences are locally non-satiated, his demand belongs to the boundary of the standard set $\bar{B}_i(p)$ but not necessarily to the boundary of the set ${}^*B_i(p)$, which almost entirely consists of non-standard points. In other words, infinitesimal budget excess may be created, which in its turn may result in non-standard equilibrium existence failure.

The problem is alike the one caused by indivisibilities. Suppose that a smallest available quantity of the good that I need is ε and its price is p_1. If the amount δ of free value at my disposal is less than $p_1\varepsilon$, I can not use it to increase my utility. All the more, if δ is infinitesimal while p_1 is not, no standard quantity of such a good is achievable. In any case, the consequence of this infinitely small value being unused is an inequality $\sum_{i\in N} p\bar{x}^i < \sum_{i\in N} pw^i$, where $\bar{x}^i$ are individual demands, which is inconsistent with the equilibrium market clearing condition[1].

Semi-equilibria with non-standard prices exist if all consumption sets are positive orthants (Marakulin (1990)). In a more general case, where X_i are convex closed bounded from below sets, there exist dividend equilibria with non-standard prices. A notion of dividend equilibrium was proposed by several authors (Makarov (1981), Aumann and Drèze (1986), Mas-Colell (1992)) in order to analyse economies which allowed for possibly satiated preferences. In a dividend equilibrium, each agent i's budget constraint is relaxed by some slack variable d_i in order to allow for redistribution of a budget excess created by satiated agents among non-satiated ones. Such a slack variable can be interpreted as an agent's endowment of coupons (as in Drèze and Müller (1980)) or paper money (as in Kajii (1996)).

For prices $p \in {}^*Q$ and a *system of dividends* $d \in {}^*\mathbb{R}^n_+$ consider the sets

$$\bar{B}_i(p, d_i) = st\,\{x \in {}^*X_i \mid px \le pw^i + d_i\}, \quad i \in N.$$

Definition 2.5
An allocation $\bar{x} \in X$ is a *non-standard dividend equilibrium*, if there exist non-standard vectors $d = (d_1, \ldots, d_n) \in {}^*\mathbb{R}^n_+$ and $p \in {}^*Q$, such that the following conditions hold:

(i) attainability: $\bar{x}^i \in \bar{B}_i(p, d_i), \quad i \in N,$

(ii) individual rationality: $\mathcal{P}_i(\bar{x}^i) \cap \bar{B}_i(p, d_i) = \emptyset, \quad i \in N.$

(iii) market clearing: $\sum\limits_{i\in N} \bar{x}^i = \sum\limits_{i\in N} w^i.$

It is well-known that if agents' preferences are locally non-satiated, then all dividend terms at a dividend equilibrium have to be equal to zero, which makes it an ordinary Walrasian equilibrium. Similarly, one can assert that if $\bar{x}$ is a non-standard dividend equilibrium and local non-satiation of preferences holds, then every d_i is infinitesimal.

[1] See Florig (2000) and Florig and Rivera (2001) to find out more on the connection between equilibrium in a model with indivisible goods and hierarchic equilibrium. In particular, it is shown that a hierarchic equilibrium is a limit of dividend equilibria of an economy where all commodities are assumed to be indivisible with a level of indivisibility going to zero. Thus, a hierarchic equilibrium (and, therefore, a non-standard dividend equilibrium) can be interpreted as a dividend equilibrium of an economy with small indivisibilities.

Proposition 2.6

Suppose that $\bar{x}$ is a non-standard dividend equilibrium such that an equilibrium price vector p is near-standard. Assume that for every $\varepsilon \in \mathbb{R}_{++}$, for all $i \in N$, and every $x^i \in X_i$ there exists $y^i \in \mathcal{P}_i(x^i)$ such that $\|y^i - x^i\| < \varepsilon$. Then $d_i \approx 0$ for every $i \in N$.

Proof. Consider the standard part ${}^\circ p$ of an equilibrium price vector p and show that

$$ {}^\circ p\bar{x}^i = {}^\circ pw^i + {}^\circ d_i, \quad i \in N. \tag{1}$$

The inequality ${}^\circ p\bar{x}^i > {}^\circ pw^i + {}^\circ d_i$ would imply that $\bar{x}^i$ is not attainable for i. Suppose that

$$ {}^\circ p\bar{x}^i < \tau < {}^\circ pw^i + {}^\circ d_i,$$

for some $\tau \in \mathbb{R}$. Then there exists $\varepsilon \in \mathbb{R}_{++}$ such that ${}^\circ py^i < \tau < {}^\circ pw^i + {}^\circ d_i$ for every $y^i \in \{z^i \in X_i \mid \|x^i - z^i\| < \varepsilon\}$. By continuity of scalar multiplication, $p\tilde{y}^i < \tau < pw^i + d_i$ for every $\tilde{y}^i \approx y^i$. Therefore, every element of some standard neighborhood of $\bar{x}^i$ is also an element of the budget set $\bar{B}_i(p, d_i)$. By the local non-satiation condition, the intersection $\bar{B}_i(p, d_i) \cap \mathcal{P}_i(\bar{x}^i)$ is not empty, which is a contradiction with $\bar{x}^i$ being the maximal element on $\bar{B}_i(p, d_i)$. Therefore (1) is true. To complete the proof, one needs to observe that all ${}^\circ d_i$ have to be equal to zero if (1) is to be consistent with the equilibrium market clearing condition.

□

The need to use non-standard prices and dividends is illustrated by the following example (cf. Florig (2001)). In this example not only Walrasian equilibria, but also equilibria and semi-equilibria with non-standard prices fail to exist. However, a non-standard dividend equilibrium exists.

Consider a market with two agents and two goods: white bread and brown bread. Each agent requires a minimum of four slices of bread a day to survive and has an initial endowment of three slices of white bread and one slice of brown bread. The first consumer likes only white bread, and the second consumer likes only brown bread. Formally, consumption sets and initial endowments of agents are given by $X_1 = X_2 = \{x \in \mathbb{R}^2_+ : x_W + x_B \geq 4\}$, $w^1 = w^2 = (3, 1)$, agents' preferences are represented by utility functions $u_1(x) = x_W, u_2(x) = x_B$.

An allocation $\bar{x}^1 = (4, 0)$, $\bar{x}^2 = (2, 2)$ looks especially attractive, in fact, it is the only Pareto optimum. Nevertheless, it can not be obtained through the market mechanism unless non-standard prices and dividends are employed. If prices are different from $(1, 1)$, then one of two agents demands more than 4 slices of bread, which would push the other agent outside of his consumption set. Therefore, neither equilibrium nor semi-equilibrium is possible. If $p = (1, 1)$

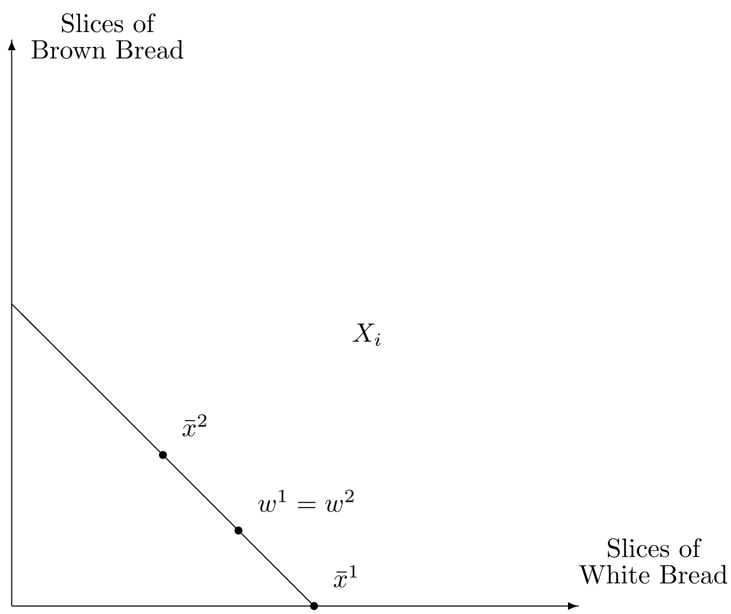

Figure 1: Non-existence of equilibria without non-standard prices and dividends.

then excess demand for brown bread is greater than zero, the same is true for non-standard prices $p = (1+\varepsilon, 1-\varepsilon)$, where ε is some positive infinitesimal. For prices $p = (1-\varepsilon, 1+\varepsilon)$ the excess demand for white bread exceeds zero. By Proposition 2.4, the case ${}^{\circ}p = (1,1)$ is the only one where non-standard prices matter. Thus, the set of Walrasian equilibria, as well as the sets of equilibria and semi-equilibria with non-standard prices, are empty for this economy. However, $\bar{x}$ is a non-standard dividend equilibrium if prices are $p = (1-\varepsilon, 1+\varepsilon)$ and dividends are $d = (0, 2\varepsilon)$. What is happening is that in addition to the standard price system $(1,1)$, a commodity bundle $(-1,1)$ is priced by an infinitesimal value, encouraging in such a way the sale of brown bread and the purchase of white bread. One could say that infinitesimal prices and dividends determine an additional constraint, which plays a role of a rationing scheme. Namely, suppose that d_i represents an agent i's endowment of coupons. Then the second agent spends his amount of coupons 2ε buying one slice of brown bread and selling one slice of white bread, and this is as much as he can achieve. The first agent does not have to spend any coupons at all.

3 Hierarchic prices and non-standard budget sets

In this section we study non-standard prices and budget sets. Among the results displayed are the representation of non-standard prices by an orthonormal set of standard vectors and characterization of non-standard dividend budget sets in pure standard terms. Throughout this section, we use simplified notations, in what follows $X \subset \mathbb{R}^l$ denotes a consumption set of some individual. Without loss of generality we assume that his initial endowments are zero (the general situation can easily be reduced to this case by a shift of the consumption set). In the beginning of the section we deal with the following budget sets

$$B(p) = \{x \in X \ : \ px \leq 0\},$$

— a standard budget set for $p \in \mathbb{R}^l$;

$$^*B(p) = \{x \in {}^*X \ : \ px \leq 0\},$$

— a non-standard budget set for $p \in {}^*\mathbb{R}^l$;

$$\bar{B}(p) = st\ ^*B(p) = \{x \in X \ : \ \exists\, \tilde{x} \in {}^*X \ : \ \tilde{x} \approx x \text{ and } p\tilde{x} \leq 0\},$$

— a standardization of a non-standard budget set.

An ordered orthonormal set $\{q_1, \ldots, q_k\}$ of non-zero vectors in $\mathbb{R}^l$ is called a *hierarchic price*. The first proposition projects the set of non-zero non-standard prices ${}^*\mathbb{R}^l \backslash \{0\}$ onto the set of all possible hierarchic prices $\cup_{k=1}^{l} V_{l,k}$, where $V_{l,k}$ is a Stiefel manifold.

Proposition 3.1
For each $p \in {}^*\mathbb{R}^l \backslash \{0\}$, *there exists an orthonormal set of standard vectors* $\{q_1, q_2, \ldots, q_k\} \in \mathbb{R}^{lk}$ *such that* p *has a unique representation*

$$p = \lambda_1 q_1 + \cdots + \lambda_k q_k, \tag{2}$$

with positive coefficients $\lambda_j \in {}^*\mathbb{R}_{++}$ *satisfying*

$$\frac{\lambda_{j+1}}{\lambda_j} \approx 0, \quad j \in \{1, \ldots, k-1\}. \tag{3}$$

Proof. The proof of the proposition is by induction in the dimension l of the space containing vector $p \in {}^*\mathbb{R}^l$. The proposition is trivially verified for $l = 1$. Suppose that it is true for $l \leq m$ for some natural m and show that it holds for $l = m + 1$.

Assuming $p \neq 0$, let $p' = p / {}^*\|p\|$. Since p' is near-standard (due to compactness of the unit ball in a finite dimensional space and the non-standard criterion of compactness), one may find $q_1 = {}^\circ(p')$ and $\lambda_1 = pq_1$. Let $p'' = p - \lambda_1 q_1$ and consider a subspace $\mathcal{L} = \{x \in \mathbb{R}^l \,|\, xq_1 = 0\}$. Note that by the Transfer Principle (see Anderson (1991), p. 2158), ${}^*\mathcal{L}$ has a similar structure as $\mathcal{L}$:

$$ {}^*\mathcal{L} = \{x \in {}^*\mathbb{R}^l \mid xq_1 = 0\}. $$

By the definition of p'',

$$ p''q_1 = (p, q_1) - (p, q_1)(q_1, q_1) = 0, $$

(since $(q_1, q_1) = 1$), which implies that $p'' \in {}^*\mathcal{L}$. However, $\dim \mathcal{L} \le l$, hence, by the induction agreement, there exists an orthonormal system of standard vectors $\{q_2, \dots, q_k\}$, such that

$$ p'' = \lambda_2 q_2 + \cdots + \lambda_k q_k, $$

where positive coefficients λ satisfy $\lambda_{j+1}/\lambda_j \approx 0$, $j = 2, \dots, k-1$. Consequently,

$$ p = \lambda_1 q_1 + p'' = \lambda_1 q_1 + \lambda_2 q_2 + \cdots + \lambda_k q_k. $$

It is clear that the system $\{q_1, \dots, q_k\}$ is orthonormal. Let us show that $\lambda_2/\lambda_1 \approx 0$. To this end, put $\delta = \|p' - q_1\|^2 \approx 0$. Then

$$ {}^*\|p' - (p', q_1)q_1\|^2 = (p', p') - 2(p', q_1)^2 + (p', q_1)^2(q_1, q_1) = $$
$$ = 1 - (p', q_1)^2 = 1 - (1 - \frac{\delta}{2})^2 = \delta(1 - \frac{\delta}{4}), $$

which implies

$$ {}^*\|p''\| = {}^*\|p - \lambda_1 q_1\| = {}^*\|p\|({}^*\|p' - (p', q_1)q_1\|) = \varepsilon\, {}^*\|p\|, $$

where $\varepsilon = \sqrt{\delta(1 - \frac{\delta}{4})} \approx 0$. On the other hand,

$$ {}^*\|p''\| = {}^*\|p - \lambda_1 q_1\| = {}^*\|\lambda_2 q_2 + \cdots + \lambda_k q_k\| = \sqrt{\lambda_2^2 + \cdots + \lambda_k^2 q_k}. $$

Taking into account $(p', q_1) \approx 1$, one obtains

$$ 0 \le \frac{\lambda_2}{\lambda_1} \le \frac{\sqrt{\lambda_2^2 + \cdots + \lambda_k^2}}{\lambda_1} = \frac{{}^*\|p''\|}{\lambda_1} = \frac{\varepsilon\, {}^*\|p\|}{(p, q_1)} = \frac{\varepsilon}{(p', q_1)} \approx 0. $$

Suppose now that $\lambda_1 q_1 + \cdots + \lambda_k q_k$ and $\lambda'_1 q'_1 + \cdots + \lambda_s q'_s$ are two different representations of the same non-standard vector p. Then

$$ 0 = \sum_{j=1}^{k} \lambda_j q_j - \sum_{j=1}^{s} \lambda'_j q'_j. \tag{4} $$

Assume without loss of generality that $s < k$. Let λ_1, for example, be the largest element in the set $\{\lambda_1, \dots, \lambda_k, \lambda'_1, \dots, \lambda'_s\}$. Since all $\lambda_j, \lambda'_j, j > 1$ are infinitely smaller than λ_1, it must be $\lambda_1 q_1 = \lambda'_1 q'_1$, which implies (recall $\|q_1\| = \|q'_1\| = 1,\ \lambda_1, \lambda'_1 > 0$) that $\lambda_1 = \lambda'_1,\ q_1 = q'_1$. Similarly, taking the largest element of the set $\{\lambda_2, \dots, \lambda_k, \lambda'_2, \dots, \lambda'_s\}$ one will come to $\lambda_2 = \lambda'_2, q_2 = q'_2$, and so on, until $\lambda_s = \lambda'_s, q_s = q'_s$. But then (4) implies $s = k$.

□

Example: $(1+\varepsilon-\varepsilon^2, \varepsilon-2\varepsilon^2, -\varepsilon^2) = (1+\varepsilon-\varepsilon^2)(1,0,0) + (\varepsilon-2\varepsilon^2)(0,1,0) + \varepsilon^2(0,0,-1)$.

If the relations (2)–(3) are true, we say that a hierarchic price $(q_1, \dots, q_k)$ *represents* a non-standard price vector p and denote it by q^p. Conversely, if $q = (q_1, \dots, q_k)$ is a hierarchic price, one may consider for some $\varepsilon \approx 0, \varepsilon > 0$ a vector of non-standard prices

$$p(q,\varepsilon) = q_1 + \varepsilon q_2 + \dots + \varepsilon^{k-1} q_k$$

such that $q^{p(q,\varepsilon)} = q$. Evidently, there is much more than one non-standard vector that satisfies this property, in particular $q^{p(q,\varepsilon)} = q^{p(q,\varepsilon^2)}$. Furthermore, if $p = 0$, then put $q^p = \{0\}$.

The next proposition gives a characterization of the set $\bar{B}(p)$ and is crucial for our analysis. It asserts that if X is a polyhedral set then there is a number $m \in \{1, \dots, k\}$ such that the set $\bar{B}(p)$ consists of elements x such that the vector $(q_j x)_{j=1,\dots,m}$ is lexicographically less than zero.

Remind that a subset $P \subset \mathbb{R}^l$ is called a *polyhedral set* or a *polyhedron* if it is the intersection of a finite number of closed half-spaces. Each polyhedral set is closed and convex but not necessarily compact.

Denote the set $\{x \in X \mid q_1 x = 0, \dots, q_m x = 0\}$ by $X(q_1, \dots, q_m)$ and put $X(\emptyset) = X$. Consider the sets

$$B_m(p) = \{x \in X(q_1, \dots, q_{m-1}) \mid q_m x \le 0\}, \quad m \le k, \tag{5}$$

and

$$B_{k+1}(p) = X(q_1, \dots, q_k). \tag{6}$$

Proposition 3.2
Suppose that $p \in {}^*\mathbb{R}^l$, and at least one of the following alternatives is true:

(a) The set X is a polyhedron;

(b) The set X is closed, star-shaped with respect to 0 and locally polyhedral at 0, that is for some small positive $\gamma \in \mathbb{R}_{++}$ the set $P = \{x \in X : |x_j| \le \gamma,\ j \in L\}$ is a polyhedron.

Then there exists a natural number $m \in \{1, \ldots, k+1\}$ *such that*

$$\bar{B}(p) = B_m(p).$$

Moreover, for $m \leq k$ *there exists* $y \in \bar{B}(p)$ *such that* $q_m y < 0$.

The proof is based on the following lemma.

Lemma 3.3
Suppose that $X \in \mathbb{R}^l$ *is a polyhedral set and* $p \in {}^*\mathbb{R}^l$. *Then* $pX \geq 0$ *implies* $p({}^*X) \geq 0$.

Note first that the conclusion of the lemma does not hold if X is not polyhedral. Take for instance $X = \{x \in \mathbb{R}^2_+ | x_2 \geq (x_1)^2\}$ and $p = (-\varepsilon, 1)$, $\varepsilon \approx 0, \varepsilon > 0$. Then for each $x \in X$ $px = -\varepsilon x_1 + x_2 \geq 0$. But once an element $\tilde{x} = (\varepsilon/2, \varepsilon^2/4) \in {}^*X$ is taken, $p\tilde{x} = -\varepsilon^2/2 + \varepsilon^2/4 < 0$.

Proof. The set X consists of all vectors in $\mathbb{R}^l$ that satisfy some system of linear inequalities:

$$X = \{x \in \mathbb{R}^l \ : \ d_\alpha x \leq g_\alpha, \ \alpha \in A\},$$

where $d_\alpha \in \mathbb{R}^l \backslash \{0\}$, $g_\alpha \in \mathbb{R}$, and A is finite. It follows from the Fundamental Theorem of Linear Inequalities (see, for example, Schrijver (1986), Theorem 7.16) that X can be represented as a sum of a compact polyhedron Y and a convex cone Z with a finite number of generators. In other words, $X = Y + Z$, and for some finite sets $B \subset \mathbb{R}^m$, $C \subset \mathbb{R}^m$

$$Y = \text{ conv } B = \{\sum_{b \in B} \beta_b b \,|\, \beta_b \in \mathbb{R}_+ \ \& \ \sum_{b \in B} \beta_b = 1\, \},$$

and

$$Z = \text{ con } C = \{\sum_{c \in C} \gamma_c c \,|\, \gamma_c \in \mathbb{R}_+\}.$$

Then by definition

$$pX \geq 0 \quad \Longleftrightarrow \quad pb \geq 0 \quad \& \quad pc \geq 0 \quad \forall b \in B, \quad \forall c \in C.$$

By the Transfer Principle, ${}^*X = {}^*Y + {}^*Z$, where ${}^*Y$ and ${}^*Z$ are defined by substitution $\mathbb{R}_+$ for ${}^*\mathbb{R}_+$ in the definitions of Y and Z, respectively. Therefore,

$$pb \geq 0 \quad \& \quad pc \geq 0 \quad \forall b \in B, \quad \forall c \in C \quad \Longleftrightarrow \quad p\,({}^*X) \geq 0.$$

□

Corollary 3.4
Suppose that $X \in \mathbb{R}^l$ is star-shaped with respect to 0 *and locally polyhedral at* 0, $0 \in X$ *and* $p \in {}^*\mathbb{R}^l$. *Then* $pX \geq 0$ *implies* $p({}^*X) \geq 0$.

Proof. Let P be a polyhedron obtained as an intersection of X with a sufficiently small closed cube, whose interior contains 0. Suppose that there exists $\tilde{x} \in {}^*X$ such that $p\tilde{x} < 0$. Then one can find $\tilde{x}' = \varepsilon\tilde{x} \in {}^*P$ such that $p\tilde{x}' < 0$, for some sufficiently small $\varepsilon \in {}^*\mathbb{R}_{++}$. But $pP \geq 0$, which by Lemma 3.3 implies $p({}^*P) \geq 0$, a contradiction.

□

Proof of Proposition 3.2. Let $q^p = (q_1, \ldots, q_k)$ for some $k \leq l$. If $p = 0$ then $q_1 = 0$, $k = 1$ and $\bar{B}(p) = X = B_1(p)$. Suppose that there exist a number $m \in \{1, \ldots, k\}$ and an element $y \in X(q_1, \ldots, q_{m-1})$ such that $q_m y < 0$. Moreover, assume that m is the smallest such a number, which guarantees that

$$q_j X(q_1, \ldots, q_{j-1}) \geq 0, \quad j \in \{1, \ldots, m-1\}. \tag{7}$$

Note that by construction $y \in B_m(p)$. If $m = 1$ then $\bar{B}(p) = B(q_1) = B_1(p)$ by virtue of Proposition 2.4. Assume $m \in \{2, \ldots, k\}$ and show that $\bar{B}(p) = B_m(p)$.

First, we shall prove that $\bar{B}(p) \subseteq B_m(p)$. Take some arbitrary $x \in \bar{B}(p)$ and suppose that $x \notin B_m(p)$. If so, then by the choice of m and the system of inequalities (7) there exists a number $j \in \{1, \ldots, m\}$ such that

$$q_j x > 0, \quad q_t x = 0, \quad t = 1, \ldots, j-1. \tag{8}$$

Consider some arbitrary non-standard element $\tilde{x} \in {}^*X$ such that $\tilde{x} \approx x$. Then $q_j x > 0$ implies that $q_j \tilde{x}$ is greater than some strictly positive real number.

Take now any standard $x' \in X$ and consider the first non-zero element in the ordered set $\{q_1 x', \ldots, q_{j-1} x'\}$. Since $j \leq m$, it follows again from (7) and the choice of m that such an element, if it exists, is strictly positive. Therefore, a non-standard linear functional $\lambda_1 q_1 + \cdots + \lambda_{j-1} q_{j-1}$ takes only positive values on X :

$$\Big(\lambda_1 q_1 + \cdots + \lambda_{j-1} q_{j-1}\Big) X \geq 0.$$

Then by Lemma 3.3

$$\Big(\lambda_1 q_1 + \cdots + \lambda_{j-1} q_{j-1}\Big)\, {}^*X \geq 0. \tag{9}$$

Consider

$$\frac{1}{\lambda_j} p\tilde{x} = \frac{1}{\lambda_j}[\lambda_1 q_1 + \cdots + \lambda_{j-1} q_{j-1}]\tilde{x} + q_j \tilde{x} + \frac{1}{\lambda_j}[\lambda_{j+1} q_{j+1} + \cdots + \lambda_k q_k]\tilde{x}.$$

The first component of the sum in the right-hand side is positive (it vanishes if $j = 1$), the second component exceeds 0 by a non-infinitesimal amount and the third component is infinitesimal. Therefore, $(1/\lambda_j)p\tilde{x}$ is strictly positive, so that $p\tilde{x} > 0$ for all $\tilde{x} \in {}^*X$ such that $\tilde{x} \approx x$. This contradicts $x \in \bar{B}(p)$. We have shown that $\bar{B}(p) \subseteq B_m(p)$.

Let $x \in B_m(p)$, $y \in X(q_1, \ldots, q_{m-1})$, $q_m y < 0$. Consider a sequence

$$x_n = \frac{1}{n}y + (1 - \frac{1}{n})x.$$

By convexity, $x_n \in {}^*X$ for any $n \in {}^*\mathbb{N}$. Moreover, $px_n < 0$ for all $n \in \mathbb{N}$. Show that $px_{\tilde{n}} < 0$ for some hyperfinite $\tilde{n}$. Suppose that the set

$$A = \{n \in {}^*\mathbb{N} \,:\, px_n \geq 0\}$$

is non-empty. This set is internal as a definable subset of an internal set ${}^*\mathbb{N}$ (cf. Davis (1977), Theorem 1-8.1). Therefore it has a least element $\nu \in {}^*\mathbb{N} \backslash \mathbb{N}$. Take $\tilde{n} = \nu - 1$, then $x_{\tilde{n}} \approx x$ and $px_{\tilde{n}} < 0$, which proves that $x \in \bar{B}(p)$.

To complete the proof it suffices to show that

$$\forall m \in \{1, \ldots, k\} \quad q_m X(q_1, \ldots, q_{m-1}) \geq 0$$

implies $\bar{B}(p) = B_{k+1}(p)$. If $x \in B_{k+1}(p)$ then $px = 0$, so $x \in \bar{B}(p)$. The proof of the inclusion $\bar{B}(p) \subseteq B_{k+1}(p)$ goes along the same lines as in the case $m \leq k$.

□

Next, we turn our attention to the characterization of non-standard dividend budget sets. For $\gamma \in {}^*\mathbb{R}_{++}$ and $p \in {}^*\mathbb{R}^l$ consider the set

$${}^*B(p, \gamma) = \{x \in {}^*X \,:\, px \leq \gamma\}$$

and denote by $\bar{B}(p, \gamma)$ its standardization:

$$\bar{B}(p, \gamma) = st\ {}^*B(p, \gamma).$$

The following auxiliary lemma is useful. It says that small changes in prices and dividends do not alter a standardized dividend budget set.

Lemma 3.5

Let X be a closed convex set, $0 \in X$. Suppose that $p, p' \in {}^*\mathbb{R}^l$ and that the non-standard numbers $\gamma > 0, \gamma' > 0$ satisfy

$$|p - p'| \,/\, \gamma \approx 0 \quad \textit{and} \quad \gamma/\gamma' \approx 1,$$

then $\bar{B}(p, \gamma) = \bar{B}(p', \gamma')$.

Proof. Show first that $\bar{B}(p,\gamma) = \bar{B}(p,\gamma')$. To this end assume $\gamma' > \gamma$ and show that the inclusion

$$\bar{B}(p,\gamma') \subseteq \bar{B}(p,\gamma) \tag{10}$$

holds. Let $x \in \bar{B}(p,\gamma')$. Then one can find $\tilde{x} \approx x, \tilde{x} \in {}^*X$ such that $p\tilde{x} \leq \gamma'$. Suppose that $p\tilde{x} > \gamma$ (otherwise there is nothing to prove), and consider $y = (1-\varepsilon)\tilde{x}$ where $\varepsilon \approx 0$ satisfies $\gamma' = (1+\varepsilon)\gamma$. It is clear that $y \approx \tilde{x} \approx x$, $y \in {}^*X$ by convexity, and

$$py = p\tilde{x} - \varepsilon p\tilde{x} \leq \gamma' - \varepsilon\gamma = \gamma.$$

Thus inclusion (10) follows.

Next we shall establish

$$\bar{B}(p,\gamma') = \bar{B}(p',\gamma').$$

Let $p'' = p - p'$. Since $|p''| \,/\, \gamma' \approx 0$, one can find $\varepsilon \approx 0$, $\varepsilon > 0$ such that

$$|p''| \,/\, \varepsilon\gamma' \approx 0.$$

Then for every near-standard $y \in {}^*X$

$$p'y - \varepsilon\gamma' \leq py \leq p'y + \varepsilon\gamma'.$$

Therefore $py \leq \gamma'$ implies $p'y \leq \gamma' + \varepsilon\gamma'$ and

$$\bar{B}(p,\gamma') \subseteq \bar{B}(p',\gamma' + \varepsilon\gamma') = \bar{B}(p',\gamma').$$

Similarly, $\bar{B}(p',\gamma') \subseteq \bar{B}(p,\gamma')$.

□

Using representation (2) for $p \in {}^*\mathbb{R}^l$, assign to each non-standard $\gamma > 0$ its *infinitesimality level*, that is a number $j = j(p,\gamma) \in \{1,\ldots,k+1\}$ such that

$$j(p,\gamma) = \begin{cases} \min\{m \mid \gamma/\lambda_m \not\approx 0\}, & \text{if } \gamma/\lambda_k \not\approx 0, \\ k+1 & \text{otherwise.} \end{cases}$$

For $j \leq k$, denote by $\mu = \mu(p,\gamma)$ a standard part of the ratio γ/λ_j :

$$\mu(p,\gamma) = {}^\circ(\gamma/\lambda_j).$$

Thus μ is an element of $\mathbb{R}_{++} \cup \{+\infty\}$. Put $\mu = +\infty$ if $j = k+1$.

The next statement gives a complete characterization of a non-standard dividend budget set for a polyhedral set X.

Proposition 3.6
Let X be polyhedral, $p \in {}^*\mathbb{R}^l$, $\gamma \in {}^*\mathbb{R}_{++}$. Assume that $0 \in X$. Then one of the following alternatives is true:

(i) $\bar{B}(p,\gamma) = X(q_1,\ldots,q_{j-1})$ *and* $\mu = +\infty$*;*

(ii) $\bar{B}(p,\gamma) = \{x \in X(q_1,\ldots,q_{j-1}) \mid q_j x \le \mu\}$ *and* $\mu < +\infty$;

(iii) $\bar{B}(p,\gamma) = B_m(p)$ *for some* $m < j$ *and there exists* $y \in \bar{B}(p,\gamma)$ *such that* $q_m y < 0$.

Here $q = (q_1,\ldots,q_k)$ *is a hierarchic price representing* p, $j = j(p,\gamma)$ *is the infinitesimality level of* γ.

Proof. Consider an $(l+1)$-dimensional set $X^* = X \times \{1\}$, and a "budget" set

$$\{x \in {}^*X^* | p^* x \le 0\}, \tag{11}$$

where the price vector $p^* \in \mathbb{R}^{l+1}$ is given by

$$p^* = \begin{cases} (p', -\gamma), & \text{if } \gamma/\lambda_j \approx +\infty, \quad p' = p - \sum\limits_{t \ge j} \lambda_t q_t, \\ (p', -\lambda_j \mu), & \text{if } \gamma/\lambda_j < +\infty, \quad p' = \sum\limits_{t \le j} \lambda_t q_t. \end{cases}$$

By construction $|p - p'|/\gamma \approx 0$ and $\lambda_j \mu/\gamma \approx 1$. Since $0 \in X$, Lemma 3.5 implies that the projection of the standardization of the set defined in (11) onto the first l components of the set X^* coincides with $\bar{B}(p,\gamma)$. At the same time,

$$p^* = \sum_{t<j} \lambda_t q_t^* + \lambda_j^* q_j^*,$$

where $q_t^* = (q_t, 0)$ if $t < j$,

$$q_j^* = \begin{cases} (0,-1), & \text{if } \gamma/\lambda_j \approx +\infty, \\ (q_j, -\mu), & \text{if } \gamma/\lambda_j < +\infty, \end{cases}$$

and

$$\lambda_j^* = \begin{cases} \gamma, & \text{if } \gamma/\lambda_j \approx +\infty, \\ \lambda_j, & \text{if } \gamma/\lambda_j < +\infty. \end{cases}$$

Since a hierarchic price $\{q_t^*\}_{t \le j}$ represents p^* (strictly speaking, the vector q_j^* should be normalized), Proposition 3.2 is applicable. The alternatives $(i)-(iii)$ follow immediately. The case $m < j$ corresponds to alternative (iii). If $m = j$ and $\gamma/\lambda_j < +\infty$, then $\mu < +\infty$, and the budget restriction in the definition of $B_m(p^*)$ has the form

$$\Big((x,1),(q_j,-\mu)\Big) \le 0 \Rightarrow q_j x \le \mu,$$

so alternative (ii) follows. If $m = j$ and $\gamma/\lambda_j \approx +\infty$, then alternative (i) is true. The case $m = j+1$ will not occur because the assumption $0 \in X$ guarantees that $0 \in \bar{B}(p,\gamma)$.

□

This proposition makes it a mere formality to show that non-standard dividend equilibria coincide with Florig's hierarchic equilibria. One only needs to observe that non-standard prices and dividends on the one hand, and hierarchic prices and revenues on the other, provide an individual with the same budget opportunities. Consider a hierarchic price $q = (q_1, \dots, q_k)$ and define the *q-value* of a consumption bundle x as a vector

$$qx = (q_1 x, \dots, q_k x, +\infty, \dots, +\infty) \in (\mathbb{R} \cup \{+\infty\})^l.$$

A *hierarchic revenue* r is an element of $(\mathbb{R} \cup \{+\infty\})^l$. For a hierarchic price q, revenue r and $i \in N$ consider

$$s_i(q,r) = \min \{s \in \{1, \dots, l\} | \exists x \in X_i : (q_1 x, \dots, q_s x) \prec (r_1, \dots, r_s)\},$$

where $\prec$ denotes a lexicographic ordering. In principle, s_i is the first level at which a consumer i is not at minimum wealth. Given $s_i(q,r)$, consider an augmented revenue vector

$$\rho_i(q,r) = (r_1, \dots, r^{s_i(q,r)}, +\infty, \dots, +\infty) \in (\mathbb{R} \cup \{+\infty\})^l$$

and the budget set of consumer i

$$B_i(q,r) = \{x \in X_i \mid qx \preceq \rho_i(q,r)\}.$$

This construction guarantees closedness of the budget set $B_i(q,r)$.

Definition 3.7
A list of net trades $x \in \prod_{i \in N}(X_i - w^i)$ is a *hierarchic equilibrium* of the economy $\mathcal{E}$ if there exist a hierarchic price q and positive hierarchic revenues $r^i \in \mathbb{R}^l_+$, $i \in N$ such that:

(*i*) $x^i \in B_i(q, r^i)$ and $\mathcal{P}_i(x) \cap B_i(q, r^i) = \emptyset, \quad i \in N$;

(*ii*) $\sum_{i \in N} x^i = 0$.

Theorem 3.8
Suppose that consumption sets X_i of all individuals are polyhedral. Then the set of hierarchic equilibria of an economy $\mathcal{E}$ coincides with the set of non-standard dividend equilibria.

Proof. Suppose that x is a hierarchic equilibrium, and that q and r are the corresponding hierarchic price and revenue, respectively. Then each consumer maximizes his preferences on the set

$$B_i(q, r^i) = \{x \in X_i(q_1, \dots, q_{s_i - 1}) \mid q_{s_i} x \leq r_{s_i}\},$$

which immediately implies that x is a non-standard dividend equilibrium at prices $p = q_1 + \varepsilon q_2 + \cdots \varepsilon^{k-1} q_k$ for some $\varepsilon > 0$, $\varepsilon \approx 0$, and dividends

$$d_i = \varepsilon^{s_i - 1} r_{s_i}, \quad i \in N.$$

Conversely, if x is a non-standard dividend equilibrium at $p \in {}^*\mathbb{R}^l$ and $d \in {}^*\mathbb{R}^n_+$, then the representation q^p will be a hierarchic equilibrium price, and for each $i \in N$ the components of a hierarchic equilibrium revenue r^i will be

$$r^i_t = \begin{cases} 0 & \text{if } t < j(p, d_i) \\ {}^\circ(d_i / \lambda_{j(p,d_i)}) & \text{if } t = j(p, d_i) \\ +\infty & \text{if } t > j(p, d_i). \end{cases}$$

□

Propositions 3.2 and 3.6 allow us to set a useful relation between the standardization of the non-standard budget set ${}^*B(p, \gamma)$ and its *standard part*, i.e. the set of all standard elements satisfying the constraint $px \leq \gamma$:

$${}^\circ B(p, \gamma) = \{x \in X \ : \ px \leq \gamma\}$$

Proposition 3.9
If X is a compact polyhedral set, $0 \in X$, $p \in {}^*\mathbb{R}^l$, and $\gamma \in {}^*\mathbb{R}_+$, then

$$\bar{B}(p, \gamma) = cl \, {}^\circ B(p, \gamma). \tag{12}$$

Proof. Since ${}^\circ B(p, \gamma) \subseteq \bar{B}(p, \gamma)$ and since the latter set is closed, the claim is proved if for an arbitrary $x \in \bar{B}(p, \gamma)$ we point out a standard sequence $(x_\alpha)_{\alpha \in \mathbb{N}} \subset X$ converging to x such that

$$px_\alpha \leq \gamma \tag{13}$$

for every $\alpha \in \mathbb{N}$.

Suppose that $\bar{B}(p, \gamma) = B_m(p)$ for some $m < k+1$. Then there exists $y \in \bar{B}(p, \gamma)$ such that $q_m y < 0$. We can choose for x_α a convex combination

$$x_\alpha = (1/\alpha) y + (1 - 1/\alpha) x, \quad \alpha \in \mathbb{N}.$$

Actually, since

$$px_\alpha = (1/\alpha) \lambda_m q_m y + (1 - 1/\alpha) \lambda_m q_m x + \lambda_{m+1} [\ldots] < 0,$$

where the value in the square brackets is near-standard (recall that x and y are near-standard), it is easy to see that (13) holds for every $\gamma \in \mathbb{R}_+$.

If the alternative (ii) of Proposition 3.6 is true, we can put

$$x_\alpha = (1 - 1/\alpha)x, \quad \alpha \in \mathbb{N}.$$

All other cases do not require special inspection because for every $x \in \bar{B}(p, \gamma)$ we have $px \leq \gamma$, so it is possible to put $x_\alpha = x$.

□

In general, the relation (12) does not hold if X is not a polyhedron (a counter-example is given in Marakulin (2001), page 84).

4 Manifolds of hierarchic prices

As was mentioned before, a hierarchic price $q = (q_1, \ldots, q_k)$ is, by definition, an element of a Stiefel manifold $V_{l,k}$, which is a surface of dimension $kl - k(k+1)/2$ (see Dubrovin et al. (1985)). This is intuitively clear, as q is parametrized by kl coordinates $(q_{mt})_{m=1,\ldots,k}^{t=1,\ldots,l}$ related by $k(k+1)/2$ equations:

$$\sum_{t=1,\ldots,l} q_{mt} q_{m't} = \begin{cases} 1, & \text{if } m = m', \\ 0, & \text{if } m \neq m'. \end{cases}$$

The set of all possible hierarchic prices Θ is then a union of k Stiefel manifolds and a set containing a zero vector:

$$\Theta = (\bigcup_{k=1}^{l} V_{l,k}) \cup \{0\},$$

where components of Θ of the highest dimension are $V_{l,l}$ and $V_{l,l-1}$,

$$dim\, V_{l,l} = dim\, V_{l,l-1} = l(l-1)/2.$$

In this section we prove that if all consumption sets in an economy are polyhedral, then the set of all relevant hierarchic equilibrium prices $\bar{\Theta}$ can be described as a finite union of manifolds whose dimension does not exceed $l - 1$. The term "relevant" means here that for each non-standard dividend equilibrium one can choose a hierarchic equilibrium price from $\bar{\Theta}$. Thus, the situation is similar to what we have in the purely standard case, where a price vector belongs to a sphere or an $(l-1)$-simplex in $\mathbb{R}^l$. This is surprising, since it means that introducing non-standard or hierarchic prices does not change the dimension of the set of all possible equilibrium prices.

Besides, it means that if a non-standard equilibrium is described as a solution of a system of equations $\psi(q) = 0$, one can use an $(l-1)$-dimensional set as the domain of the correspondence ψ. This will be useful, when we study generic properties of the set of non-standard equilibria.

Theorem 4.1
Suppose that X_i is a polyhedral set for every $i \in N$. Then there exists a set $\bar{\Theta} \subset \Theta$ such that

(i) for each non-standard dividend equilibrium $\bar{x}$ there exists $q \in \bar{\Theta}$ such that q is an equilibrium hierarchic price for $\bar{x}$ (or, which is equivalent, such that q represents non-standard equilibrium prices p corresponding to the equilibrium $\bar{x}$).

(ii) $\bar{\Theta}$ is a union of manifolds of dimension less than or equal to $l-1$.

Proof. We continue to use the convention $w^i = 0$ for every $i \in N$. Suppose that $\bar{x}$ is a non-standard dividend equilibrium and $p \in {}^*\mathbb{R}^l$ and $d \in {}^*\mathbb{R}^n_+$ are corresponding prices and dividends. It is possible to classify consumers according to the structure of their budget sets $\bar{B}(p, d_i)$. Proposition 3.6 implies that each consumer i faces (for some $m = m(i) \in \{1, \ldots, k+1\}$) $m-1$ budget constraints in the form of the equalities:

$$q_t x = 0, \quad t \in \{1, \ldots, m-1\},$$

and (possibly) one budget constraint in the form of the inequality: $q_m x \leq 0$ or $q_m x \leq \mu_j$, where $\mu_i = {}^\circ(d_i/\lambda_{j(p,d_i)})$. Let $N_m \subseteq N$ be the set of all agents for whom the last budget restriction involves the vector q_m. For all such agents we have

$$\bar{B}_i(p - \sum_{t>m} \lambda_t q_t, d_i) = \bar{B}_i(p, d_i).$$

Put $i \in N_1$ if agent i faces no restrictions at all, that is if $\bar{B}_i(p, d_i) = X_i$. Thus, given non-standard prices p and dividends d_i we partition the set of agents N into k subsets $N_1, \ldots, N_k$.

Consider an arbitrary agent i; $i \in N_m$ for some m. His budget set is a subset of the set

$$X_i(q_1, \ldots, q_{m-1}) = \{x \in X_i \mid q_1 x = 0, \ldots, q_{m-1} x = 0\}.$$

Each vector q_t, $t = 1, \ldots, m-1$, supports $X_i(q_1, \ldots, q_{t-1})$, (though it does not have to support X_i), which implies that the sets

$$X_i \supset X_i(q_1) \supset \ldots \supset X_i(q_1, \ldots, q_{m-1})$$

form a finite sequence of faces of X_i contained in each other. Denote the face $X_i(q_1, \ldots, q_{t-1})$ by $F_i^t(p)$ (note that the superscript t specifies that there are $t-1$ equalities).

Let us construct a set Θ_p which contains a hierarchic price $q^p = (q_1, \ldots, q_k)$ representing p. It will be done in k steps, and Θ_p will finally be obtained as a product of k spheres in some linear space.

Consider the set N_k. Without loss of generality it is not empty (otherwise one can throw away the last component of q^p and consider a new equilibrium price $p' = p - \lambda_k q_k$). Let L_k be the linear hull of faces F_i^k, $i \in N_k$,

$$L_k = span\left(\bigcup_{i \in N_k} F_i^k(p)\right).$$

It is clear that the vector q_k must belong to this subspace (if necessary, q_k can be replaced by its projection on L_k). Take the unit sphere S_k in L_k as the last component of Θ_p,

$$S_k = \{x \in L_k \mid \|x\| = 1\}.$$

Secondly, consider $F_i^{k-1}(p)$ — superfaces of $F_i^k(p)$ for $i \in N_k$ — that is the sets

$$\{x \in X_i \quad | \quad q_t x = 0, \quad t \leq k-2\},$$

and faces $F_i^{k-1}(p)$ for $i \in N_{k-1}$. Taking a linear hull of the union of the sets $F_i^{k-1}(p)$ for all i from N_k and N_{k-1}, we obtain a linear space M_{k-1} that contains L_k. Denote by L_{k-1} the orthogonal complement to L_k in the space M_{k-1},

$$L_{k-1} = M_{k-1} \cap (L_k)^{\perp},$$

and take the unit sphere in L_{k-1} as the next component of Θ_p,

$$S_{k-1} = \{x \in L_{k-1} \mid \|x\| = 1\}.$$

(Note that since the vectors q_t are mutually orthonormal, the vector q_{k-1} always has a non-zero projection on L_{k-1}).

The procedure described above is reiterated $k-1$ times. As a result, a system of mutually orthogonal subspaces $L_1, \ldots, L_k$ is constructed, where

$$L_m = M_m \cap (L_{m+1})^{\perp},$$

$$M_m = span\left(\bigcup_{i \in N_t, t \geq m} F_i^m(p)\right).$$

By construction, $M_1 = \mathbb{R}^l$, $M_1 = L_1 \oplus \cdots \oplus L_k$. The set Θ_p is defined as the product of spheres in L_m, $m = 1, \ldots, k$,

$$\Theta_p = S_1 \times \cdots \times S_k,$$

so its dimension is equal to $l - k$.

Look now at the set

$$\Upsilon = \{\Theta_p \mid p \in {}^*\mathbb{R}^l \text{ is an equilibrium price of an economy } \mathcal{E}\}.$$

This set is finite because each X_i has only a finite number of faces. Finally, let

$$\bar{\Theta} = \bigcup_{\Theta_p \in \Upsilon} \Theta_p.$$

By construction, for every non-standard dividend equilibrium $\bar{x}$ there exists a corresponding non-standard equilibrium price vector p, such that its hierarchic representation q^p belongs to $\bar{\Theta}$.

□

5 Existence and finiteness of non-standard equilibria

Gerard Debreu (1970), one of the founders of equilibrium analysis, was the first to establish the finiteness of equilibria for "almost all" exchange economies. His approach was based on a variation of initial endowments while all other parameters of the model were fixed. Using Sard's theorem applied to the aggregate excess demand function, Debreu obtained finiteness of equilibria for an open set containing almost all – in the sense of Lebesgue measure – allocations of initial resources. In subsequent contributions to the issue, Smale (1974), Dubey (1980) and others used also variations of utility functions, which required the use of Thom's theorems of openness and density of transversal intersections.

In the present paper we follow the latter approach, i.e., only utility functions vary, while initial endowments are kept fixed. This is done because of the fact that for almost all initial allocations of resources, the survival condition is satisfied, in which case non-standard equilibria coincide with usual Walrasian equilibria. Their local uniqueness follow then by the Debreu's (1970) theorem.

In this section, we assume that preferences of agents are given by utility functions defined over the set $X = \prod_{i \in N} X_i$

$$u_i : \prod_{i \in N} X_i \to \mathbb{R}, \quad i \in N.$$

However, the existence result of this section can easily be generalized to the case of non-complete and non-transitive preferences. We introduce a new concept of non-standard dividend equilibrium similar to the one given in Marakulin (1990) but with a specified system of dividends. If the survival assumption is satisfied in the model, then this concept boils down to a notion of equilibrium with individual slacks proposed by Kajii (1996).

Fix a *standard* strictly positive vector $\delta \in \mathbb{R}^n_{++}$. For a given non-standard number $\varepsilon \in {}^*\mathbb{R}_+$ consider the dividend budget sets

$$\bar{B}_i(p, \varepsilon\delta_i) = st\ \{x \in X_i \ \mid \ px \leq pw^i + \varepsilon\delta_i\}, \quad i \in N.$$

Definition 5.1

An allocation $\bar{x}$ is a δ-*equilibrium* of an economy $\mathcal{E}$, if there exist $\varepsilon \in {}^*\mathbb{R}_+$ and $p \in {}^*Q$ such that the following conditions hold:

(i) attainability:

$$\bar{x}^i \in \bar{B}_i(p, \varepsilon\delta_i), \quad i \in N,$$

(ii) individual rationality:

$$u_i(\bar{x}) = \max_{x^i \in \bar{B}_i(p, \varepsilon\delta_i)} u_i(\bar{x}|x^i), \quad i \in N,$$

(iii) market clearing:

$$\sum_{i \in N} \bar{x}^i = \sum_{i \in N} w^i.$$

The specifics of δ-equilibria are twofold. First, the ratio of individual dividends is assumed to be given a priori and fixed, as in the case of Kajii's equilibria with individual slacks, or, for instance, Mas-Colell's (1992) equilibria with slack, where the uniform dividend scheme was applied. Second, dividends of all consumers have the same "order of smallness" ε. Therefore, income is redistributed at most at one infinitesimality level, which may not generally be the case for non-standard dividend equilibria. Interpretation of the components δ_i depends on the further specification of the model. For instance, they may represent initial stocks of coupons or paper money or express market shares of individuals.

We continue with an example which illustrates that the system of dividends has to be specific for the number of non-standard equilibria to be finite. Let $X_1 = X_2 = X_3 = \{(x_1, x_2) : 0 \leq x_j \leq 10, j = 1, 2\}, Q = \{p \in \mathbb{R}^l : \ \|p\| \leq 2\}$, $w^1 = (2, 1)$, $w^2 = w^3 = (2, 0), u_1(x) = 5 - (x_1 - 1)^2 - (x_2 - 2)^2$, $u_2(x) = u_3(x) = x_1$. Allocations $\bar{x}^1 = (1, 1)$, $\bar{x}^2 = (2 + \lambda, 0)$, $\bar{x}^3 = (3 - \lambda, 0)$, $0 \leq \lambda \leq 1$ constitute a continuum of non-standard dividend equilibria for $p = (\varepsilon, 1)$, $\varepsilon \approx 0$, $\varepsilon > 0$, and dividends $d = (0, \lambda\varepsilon, (1 - \lambda)\varepsilon)$. This example is robust against sufficiently small perturbations of utility functions. Observe that if variations of initial endowments are considered, then the number of non-standard equilibria is generically finite. Indeed, the survival assumption ($w^i \gg 0$, $i = 1, 2, 3$) is satisfied for almost all perturbations of initial endowments, in which case non-standard dividend equilibria coincide with ordinary Walrasian equilibria (see Proposition 2.4).

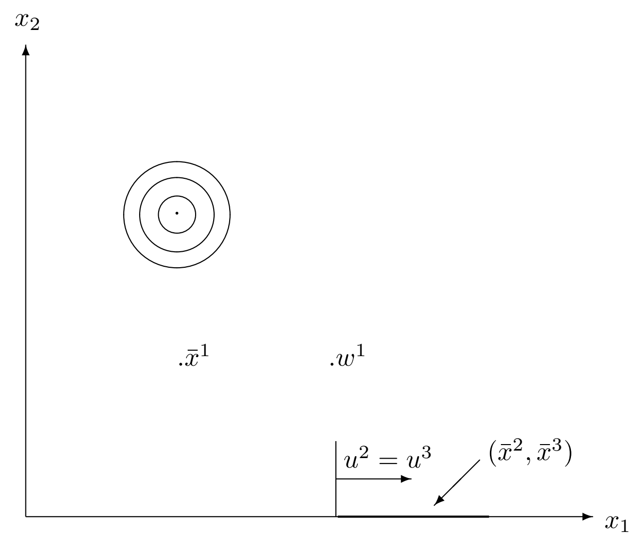

Figure 2: Continuum of non-standard dividend equilibria.

Next, we will show that δ-equilibria exist even if the survival condition or any of its analogues is not satisfied. Note that the existence of non-standard dividend equilibria unlike that of hierarchic equilibria (see Florig (2001)) does not require any conditions on consumption sets aside from convexity and compactness. Moreover, the compactness assumption can always be relaxed and substituted for closedness and boundedness from below.

Theorem 5.2
Let $Q = \{p \in \mathbb{R}^l \ : \ \|p\| \leq 1\}$. Assume that the set X_i is convex and compact, and the utility function u_i is continuous in x and strictly quasi-concave in x^i for every $i \in N$. Then for each $\delta \in \mathbb{R}^n_{++}$ a δ-equilibrium exists.

The idea of the proof is to change slightly (in fact, by an infinitesimal value) agents' income functions pw^i to make them meet the strong survival assumption, and apply a non-standard translation of the standard existence theorem. Kajii's (1996) theorem is appropriate for this purpose. An equilibrium allocation is found as a standard part of a non-standard equilibrium allocation obtained in such a way.

Proof. Let us take an arbitrary standard strictly positive $\gamma > 0$ and extend the economy with a new consumer 0 having a consumption set

$$X_0 = \{x \in \mathbb{R}^l \mid \|x\| \leq \gamma\},$$

initial endowments $w^0 = 0$, and utility function $u_0 = 0$. Revise income functions of individuals: instead of pw^i, consider functions

$$\alpha_0(p) = -\gamma\|p\|/2,$$

$$\alpha_i(p) = pw^i + \frac{\gamma\delta_i\|p\|}{2\sum_{j\in N}\delta_j}, \quad i \in N,$$

as new incomes. At each non-zero prices some amount of value is taken from consumer 0 and redistributed among other agents in proportions given by δ. Put $N_0 = N \cup \{0\}$. We have obtained an exchange economy with $n+1$ consumers and generalized income functions

$$\mathcal{E}^\gamma = < N_0, \{X_i, u_i, \alpha_i, w^i\}_{i\in N_0}, Q > .$$

The functions α_i are continuous, satisfy Walras law:

$$\sum_{i=0}^{n} \alpha_i(p) = -\frac{\gamma\|p\|}{2} + p\sum_{i\in N} w^i + \frac{\gamma\|p\|\sum_{i\in N}\delta_i}{2\sum_{j\in N}\delta_i} = p\sum_{i\in N} w^i,$$

and satisfy the strong survival assumption: for each non-zero $p \in Q$

$$\alpha_0(p) = -\gamma\|p\|/2 > -\gamma\|p\| = \inf\ pX_0,$$

$$\alpha_i(p) > pw^i \geq \inf\ pX_i, \quad i \in N.$$

By Corollary 1 in Kajii (1996), for each $\gamma > 0$ there exists some $\varepsilon_\gamma \geq 0$, a pair $(x_\gamma^0, x_\gamma) \in X_0 \times X$, and a price vector $p_\gamma \in Q$ such that each x_γ^i maximizes u_i on the set

$$B_i(p_\gamma, \alpha_i, \varepsilon_\gamma\delta_i) = \{x \in X_i \mid p_\gamma x \leq \alpha_i(p_\gamma) + \varepsilon_\gamma\delta_i\},$$

and total demand is equal to total supply:

$$x_\gamma^0 + \sum_{i\in N} x_\gamma^i = \sum_{i\in N} w^i.$$

Choose any strictly positive infinitesimal $\gamma \in {}^*\mathbb{R}_{++}$. By the Transfer Principle, we can find $\tilde{\varepsilon} \in {}^*\mathbb{R}_+$, an allocation $(\tilde{x}^0, \tilde{x}) \in {}^*X_0 \times {}^*X$ and prices $\tilde{p} \in {}^*Q$ such that each $\tilde{x}^i$ maximizes u_i on the set

$${}^*B_i(\tilde{p}, \alpha_i, \tilde{\varepsilon}\delta_i) = \{x \in {}^*X_i \mid \tilde{p}x \leq \alpha_i(\tilde{p}) + \tilde{\varepsilon}\delta_i\},$$

and

$$\tilde{x}^0 + \sum_{i\in N} \tilde{x}^i = \sum_{i\in N} w^i. \tag{14}$$

Since $\tilde{x}$ belongs to ${}^*X$ and X is a compact set, $\tilde{x}$ is near-standard. Let $\bar{x} = {}^\circ\tilde{x}$ be a standard part of $\tilde{x}$. We shall prove that $\bar{x}$ is a δ-equilibrium provided prices are $\tilde{p}$ and

$$\varepsilon = \tilde{\varepsilon} + \frac{\gamma\|\tilde{p}\|}{2\sum_{i\in N}\delta_i}.$$

Taking standard parts from both sides of (14), one easily gets market clearing:

$$\sum_{i \in N} \bar{x}_i = \sum_{i \in N} w^i.$$

Moreover, for each i

$$\bar{x}^i \in \bar{B}_i(\tilde{p}, \varepsilon\delta_i) = st\,\{x \in {}^*X_i \mid \tilde{p}x \leq \tilde{p}w^i + \varepsilon\delta_i\}.$$

To complete the proof, we need to show that $\bar{x}^i$ maximizes u_i on the set $\bar{B}_i(\tilde{p}, \varepsilon\delta_i)$ given that $\tilde{x}^i$ is a utility maximum on the set

$${}^*B_i(\tilde{p}, \varepsilon\delta_i) = \{x \in {}^*X_i \mid \tilde{p}x \leq \tilde{p}w^i + \varepsilon\delta_i\}.$$

Suppose that for some $i \in N$ it does not hold. Then there exists a standard $y \in \bar{B}_i(\tilde{p}, \varepsilon\delta_i)$ such that $u_i(y) > u_i(\bar{x}^i)$. By the definition of the set $\bar{B}_i(\tilde{p}, \varepsilon\delta_i)$ there exists $y' \in {}^*B(\tilde{p}, \varepsilon\delta_i)$ such that $y' \approx y$. By continuity of the utility function u_i,

$$u_i(y') > u_i(x')$$

for any $x' \approx \bar{x}^i$. Therefore, $u_i(y') > u_i(\tilde{x}^i)$, a contradiction with $\tilde{x}^i$ being the best element on the set ${}^*B_i(\tilde{p}, \varepsilon\delta_i)$.

□

Obviously, the existence of δ-equilibria implies the existence of non-standard dividend equilibria. Thus we have the existence theorem of Marakulin (1990) as a corollary of Theorem 5.2.

Furthermore, we will focus our attention on a specific class of δ-equilibria, for which there exists at least one agent who consumes an element of the interior of his consumption set.

Definition 5.3
A δ-equilibrium $\bar{x}$ is *proper* if there exists $i_0 \in N$ such that $\bar{x}^{i_0} \in \mathit{int}\, X_{i_0}$.

In the rest of the section, we hold X^i and w^i fixed for every $i \in N$. The finiteness result is established under the following **assumptions:**

A1. For all $i \in N$ the set X_i is a bounded from below polyhedron with a non-empty interior.

A2. Utility functions u^i are defined and twice differentiable on an open neighbourhood $\tilde{X}$ of the set X.

Denote by U the linear space $C^2(\tilde{X}, \mathbb{R}^n)$ and endow it with the standard topology of C^2 uniform convergence on compacts: if $\{f_t\}_{t=1}^{\infty} \subset C^2(\tilde{X}, \mathbb{R}^n)$, then

$f_t \to f_0 \in C^2(\tilde{X}, \mathbb{R}^n)$ if and only if $f_t \mid_K \to f_0$ if $t \to \infty$ in the norm $\| \, \|_{C^2}$ of the vector space $C^2(K, \mathbb{R}^n)$ for every compact set $K \subset \tilde{X}$. The norm $\| \, \|_{C^2}$ is defined by

$$\|g\|_{C^2(K,\mathbf{R}^n)} = \max\{\|g_i\|_{C(K,\mathbf{R}^n)}, \quad \|\tfrac{\partial g_i}{\partial x_j}\|_{C(K,\mathbf{R}^n)},$$
$$\|\tfrac{\partial^2 g_i}{\partial x_j \partial x_s}\|_{C(K,\mathbf{R}^n)}, \quad i \in N, \quad j, s \in N \times L\},$$

where

$$\|g\|_{C(K,\mathbf{R}^n)} = \max\{|g(x)| : x \in K\}.$$

Thus we can think of an economy as of given by an element u of the set U.

Recall that a subset of a topological space is called *residual* if it is the countable intersection of open dense sets. In particular, the Baire Category Theorem asserts that a residual subset of a complete metric space is dense. The main result of this section is the following theorem.

Theorem 5.4
For any strictly positive vector δ, there exists a residual (of the second category and hence dense) set $G \subseteq U$ such that for each $u \in G$ the set of proper δ-equilibria is finite.

The proof is rather cumbersome and will be organized into a number of steps. We give now a brief guide how the finiteness of δ-equilibria will be obtained. First we consider the set of equilibrium hierarchic prices Θ_F that correspond to δ-equilibria from the relative interior of an arbitrary face F of the polyhedron X. After that, we construct a mapping $\Psi_u, \ u \in U$, which is defined on $ri\ F \times \Theta_F$ and takes its values in some finite-dimensional space. This mapping characterizes δ-equilibria from $ri\ F$. Then, in the range of Ψ_u, we find a manifold Δ_F such that $\Psi_u^{-1}(\Delta_F)$ contains all δ-equilibria from $ri\ F$. Thus, to establish finiteness of δ-equilibria, it is sufficient to show that the manifold $\Psi_u^{-1}(\Delta_F)$ has dimension zero (is discrete). We show that it is indeed so if Ψ_u is transversal to Δ_F. Finally, it follows by Thom's theorems of density and openness of transversal sections that the mapping Ψ_u is transversal to Δ_F for a residual set of economies $u \in U$.

In our argument we heavily rely on Proposition 3.1 that gives a representation of a non-standard price vector by a hierarchic price and Proposition 3.6 that provides us with the characterization of non-standard dividend budget sets. When using Proposition 3.6 to characterize δ-equilibria, one should remember that all dividend terms $\varepsilon\delta_i$ have the same infinitesimality level $j(p, \varepsilon\delta_i) = j(p, \varepsilon)$. Moreover, it follows from Lemma 3.5 that all components q_s of an equilibrium hierarchic price with s higher than j do not matter. Hence, we can assume without loss of generality that for any hierarchic representation $q = \{q_1, \ldots, q_k\}$

of non-standard equilibrium prices p

$$k = j(p, \varepsilon) \quad \text{and} \quad \varepsilon/\lambda_k < +\infty.$$

The case $\varepsilon/\lambda_k \approx +\infty$ is not interesting, since it can be reduced to the prices $p' = \sum_{j<k} \lambda_t q_t$ for which $\varepsilon/\lambda_{j-1} \approx 0$.

Proposition 3.6 implies that each agent i faces $m-1$ budget restrictions in the form of equalities:

$$q_t x = q_t w_i, \quad t < m, \quad x \in X_i, \tag{15}$$

and one restriction in the form of inequality

$$q_m x \leq q_m w^i, \quad x \in X_i,$$

for some natural number $m < k$. Moreover, there always exists $y \in X_i$ for which the last inequality is strict. For $m = k$ the last restriction transforms to

$$q_k x \leq q_k w^i + \delta_i \mu, \quad \mu = {}^\circ(\varepsilon/\lambda_k) \tag{16}$$

and can be realized as an equality if $\mu = 0$. Equations (15) determine a face $F_i^m(p)$ of the polyhedron X_i such that $w^i \in F_i^m(p)$. In other words,

$$\bar{B}_i(p, \varepsilon\delta_i) = \{x \in F_i^m(p) \quad | \quad q_m x \leq q_m w^i\} \tag{17}$$

for $m < k$, and

$$\bar{B}_i(p, \varepsilon\delta_i) = \{x \in F_i^k(p) \quad | \quad q_k x \leq q_k w^i + \delta_i \mu\} \tag{18}$$

otherwise.

From now on we again use the convention $w^i = 0$, $i \in N$ in order to facilitate notations. Consider a partition $\mathcal{N}(p) = \{N_1, \ldots, N_k\}$ of the set N such that each set N_m contains the agents, whose last budget restriction involves q_m.

Fix some face $F \subseteq X$ and consider a δ-equilibrium x that belongs to $ri\ F$ — the relative interior of F. Let p be the corresponding non-standard equilibrium price vector. It is easy to see that

$$ri\ F_i \subseteq F_i^m(p), \quad i \in N,$$

where F_i are faces of X_i that compose the face F :

$$F = \prod_{i \in N} F_i.$$

Faces $F_i^m(p) \subseteq X^i$ are determined by prices p and equalities (15) for the appropriate $m \leq k$. The definition of a δ-equilibrium implies that each x^i maximizes utility $u_i(.)$ on the set

$$\{x \in ri\ F_i \quad | \quad q_m x \leq 0\}$$

if $m < k$, $i \in N_m$ and on the set

$$\{x \in ri\ F_i \quad | \quad q_k x \leq \delta_i \mu\}$$

if $i \in N_k$.

Let Θ_F be the set that contains all hierarchic equilibrium prices that correspond to δ-equilibria from F. The construction used in the proof of Theorem 4.1 implies that this set can be represented as a finite union of manifolds Θ_F^ξ such that for each Θ_F^ξ there exists $k \in \{1, \ldots, l\}$ such that

$$\Theta_F^\xi = \prod_{m=1}^{k} (L_m^\xi \backslash \{0\}),$$

where $L_1^\xi, \ldots, L_k^\xi$ are mutually orthogonal subspaces of $\mathbb{R}^l$. From now on, we fix an arbitrary element Θ of this finite union and denote by Θ^m its components $L_m \backslash \{0\}$, $m = 1, \ldots, k$,

$$\Theta = \Theta^1 \times \cdots \times \Theta^k.$$

We proceed with introducing a mapping Ψ_u that characterizes δ-equilibria from the face F. This mapping will be constructed as a product of the following correspondences.

Mappings $\Psi_i^u : \tilde{X} \to \mathbb{R}^{L \times \{i\}}$, $i \in N$ are defined by fragments of the gradient vectors of agents' utility functions related to their own consumption:

$$(\Psi_i^u)_j(x) = \frac{\partial u^i}{\partial x_j^i}(x), \quad j \in L.$$

Mappings Ψ_i^F, $i \in N$ reflect a condition that a δ-equilibrium allocation belongs to the face $F = \prod_{i \in N} F_i$. Remind that we consider only $x^i \in ri\ F_i$. We replace here this condition with the milder requirement $x^i \in span\ F_i$. Choose vectors $c_t^i \in \mathbb{R}^l$ such that $x^i \in span\ F_i$ iff x^i is a solution of a system of $t(i)$ linear equations

$$c_t^i x^i = 0, \quad t \in T_i, \quad T_i = \{1, \ldots, t(i)\}, \tag{19}$$

and suppose that all rows of a matrix

$$C_i = \begin{pmatrix} c_1^i \\ \vdots \\ c_{t(i)}^i \end{pmatrix}$$

are linearly independent for each $i \in N$. A mapping $\Psi_i^F : \tilde{X} \to \mathbb{R}^{T_i}$ is defined by

$$\Psi_i^F(x) = C_i x^i.$$

Thus, the condition $\Psi_i^F(x) = 0$ is necessary for x^i to belong to $ri\ F_i$.

Let $N_1, \ldots, N_k$ be an arbitrary partition of the set N. A mapping $\Psi^m : \tilde{X} \times \Theta^m \to \mathbb{R}^{N_m \setminus \{i_0\}}$ responds to the budget restriction $q_m x^i = 0$ for $i \in N_m$:

$$(\Psi^m(x, q_m))_i = q_m x^i, \quad i \in N_m \setminus \{i_0\}, \quad m = 1, \ldots, k.$$

For technical reasons, the budget restriction for agent i_0 is removed. This restriction follows from those of other individuals and feasibility of an equilibrium allocation.

A mapping $\Psi^{mc} : \tilde{X} \to \mathbb{R}^{L \times \{n+1\}}$ represents the equilibrium market clearing condition:

$$(\Psi^{mc}(x))_j = \sum_{i \in N} x^i_j, \quad j \in L.$$

Finally, we need the identity mapping $\Psi^q : \Theta \to \Theta$,

$$\Psi^q(q_1, \ldots, q_k) = (q_1, \ldots, q_k).$$

A mapping Ψ_u is defined as a product of the correspondences described above:

$$\Psi_u = \prod_{i \in N} \Psi^u_i \times \Psi^{mc} \times \prod_{i \in N} \Psi^F_i \times \prod_{m=1}^{k} \Psi^m \times \Psi^q.$$

This mapping has domain $Z = \tilde{X} \times \Theta$ and takes its values in a finite-dimensional space $\mathbb{R}^T \times \Theta$, where

$$T = \left(\bigcup_{i=1}^{n+1} L \times \{i\} \right) \cup (N \setminus \{i_0\}) \cup \left(\bigcup_{i \in N} T_i \right).$$

A mapping $\Psi : U \times Z \to \mathbb{R}^T \times \Theta$ is defined by

$$\Psi(u, z) = \Psi_u(z).$$

At the next step, we describe a submanifold Δ_F such that for each proper δ-equilibrium $(x, q_1, \ldots, q_k)$ its value $\Psi_u(x, q_1, \ldots, q_k)$ belongs to Δ_F for some choice of parameters $i_0, \Theta, N_1, \ldots, N_k$.

Partition the set N_m into two subsets N'_m and N''_m, where N'_m consists of those agents for whom the last budget restriction is binding and N''_m contains those agents who are locally satiated on the face F_i. The necessary first-order conditions of a local extremum for the agents of the first type can be formulated as follows: there exist $y \in \mathbb{R}^{T_i}$ and $\lambda^m \in \mathbb{R}_{++}$ such that

$$\Psi^u_i(x) = \frac{\partial u_i}{\partial x^i}(x) = \lambda^m q_m + y C_i.$$

For agents of the second type those conditions have a simpler form: there exists $y \in \mathbb{R}^{T_i}$ such that

$$\Psi^u_i(x) = \frac{\partial u_i}{\partial x^i}(x) = y C_i.$$

When defining a manifold Δ_F, we take into consideration that $\Psi_i^m(x,q)$ is equal to zero at equilibrium for agents from N'_m whenever $m < k$, or to $\mu\delta_i$ if $m = k$; and corresponds to a free variable for $i \in N''_m$, $m = 1, \ldots, k$.

Define a submanifold $\Delta_F \subset \mathbb{R}^T \times \Theta$ as follows:

$$\Delta_F \overset{def}{=} \Big\{(\nu_1, \ldots, \nu_n, \beta_1, \ldots, \beta_k, \sigma, q, \varphi_1, \ldots, \varphi_n) \in \mathbb{R}^T \times \Theta \mid$$

$$\nu_i = \begin{cases} \lambda_i^m q_m + yC_i, \quad y \in \mathbb{R}^{T_i}, & \text{if } i \in N'_m, \quad m = 1, \ldots, k, \\ yC_i, \quad y \in \mathbb{R}^{T_i}, & \text{if } i \in N''_m, \quad m = 1, \ldots, k, \end{cases} \tag{20}$$

$$(\beta_m)_i = \begin{cases} 0 & \text{if } i \in N'_m \backslash \{i_0\}, \quad m = 1, \ldots, k-1, \\ \delta_i \mu & \text{if } i \in N'_k \backslash \{i_0\}, \quad m = k, \end{cases} \tag{21}$$

$$\varphi_i = 0, \quad i \in N, \quad \sigma = 0, \quad \|q_m\| = 1, \quad m = 1, \ldots, k\Big\}.$$

Here μ, λ_i^m, and y are free variables;
ν_i is an l-dimensional vector that corresponds to a fragment of agent i's gradient of utility function related to his own consumption, $i \in N$;
$\beta_m \in \mathbb{R}^{N_m \backslash \{i_0\}}, \quad m = 1, \ldots, k$ correspond to the budget restrictions;
$\sigma \in \mathbb{R}^l$ reflects the market clearing condition;
$(q_1, \ldots, q_k) \in \Theta$ corresponds to a hierarchic price representing non-standard equilibrium prices p;
$\varphi_i \in \mathbb{R}^{T_i}$ relates to the condition $x^i \in F^i$.

Equations (20) are necessary conditions for the utility maximization problem under the restrictions imposed by the face F and the budget.

We present now a summary of results from Abraham and Robbin (1967) that will be used in the proof of the theorem.

Definition 5.5 (Transversality)
Let X and Y be C^1 manifolds, $f : X \to Y$ a C^1 map and $W \subseteq Y$ a submanifold. We say that f is *transversal to W at a point* $x \in X$ (denoted by $f \pitchfork_x W$), if, where $y = f(x)$, either $y \notin W$ or $y \in W$ and

(1) the image $T_x f(T_x X)$ contains a closed complement to $T_y W$ in $T_y Y$, and

(2) the inverse image $(T_x f)^{-1}(T_y W)$ splits (has a closed complement to $T_x X$).

We say that f is *transversal to W* (in symbols $f \pitchfork W$) if $f \pitchfork_x W$ for every $x \in X$.

Let $\mathcal{V}, X, Y$ be C^r manifolds, $\mathcal{C}^r(X,Y)$ the set of C^r maps from X to Y and $\rho : \mathcal{V} \to \mathcal{C}^r(X,Y)$ a map. For $v \in \mathcal{V}$ we write ρ_v instead of $\rho(v)$; i.e., $\rho_v : X \to Y$ is a C^r map. We say ρ is a C^r *representation* if the *evaluation map* $\omega_\rho : \mathcal{V} \times X \to Y$ given as $\omega_\rho(v,x) = \rho_v(x)$ for $v \in \mathcal{V}$ and $x \in X$ is a C^r map from $\mathcal{V} \times X$ to Y.

Theorem 5.6 (Transversal Density Theorem)
Let $\mathcal{V}$, X, Y be C^r manifolds, $\rho : \mathcal{V} \to C^r(X,Y)$ a C^r representation, $W \subseteq Y$ a submanifold (not necessarily closed), and $\omega_\rho : \mathcal{V} \times X \to Y$ the evaluation map. Assume that

1. *X has finite dimension n and W has finite codimension m;*
2. *$\mathcal{V}$ and X are second countable[2];*
3. *$r > \max\{n-m, 0\}$;*
4. *$\omega_\rho \pitchfork W$.*

Then a set $\mathcal{V}_W = \{v \in \mathcal{V} \mid \rho_v \pitchfork W\}$ is residual (and hence dence) in $\mathcal{V}$.

Theorem 5.7 (Openness of Transversal Intersection)
Let $\mathcal{V}, X, Y$ be C^r manifolds, $\rho : \mathcal{V} \to C^1(X,Y)$ a C^1 representation, $W \subseteq Y$ a C^1 submanifold, $K \subseteq X$ a compact subset, and

1. *X is of finite dimension;*
2. *W is closed.*

Then $\mathcal{V}_{KW} = \{v \in \mathcal{V} \quad | \quad \rho_v \pitchfork_x W, \quad x \in K\}$ is open in $\mathcal{V}$.

Denote by $\mathcal{L}_i$ the linear hull of vectors $\{c_t^i\}$, $t \in T_i$ that constitute the rows of the matrix C_i.

Lemma 5.8
Consider an element $(u_0, z_0) = (u_0, x_0, q_1^0, \dots, q_k^0) \in U \times Z$ such that

$$q_m^0 \notin \mathcal{L}_i, \quad i \in N_m \quad , \quad m = 1, \dots, k. \tag{22}$$

Then $\Psi : U \times Z \to \mathbb{R}^T \times \Theta$ is transversal at (u_0, z_0) to any submanifold of $\mathbb{R}^T \times \Theta$.

[2]second countable means having a topology with a countable basis of open sets.

Proof. It is sufficient to show that the tangent correspondence $T_{(u_0,z_0)}\Psi$ is surjective. Let $\upsilon = (\upsilon_1, .., \upsilon_n, \upsilon'_1, .., \upsilon'_k, \upsilon'', \upsilon''', \upsilon^1, .., \upsilon^n) \in \mathbb{R}^T \times \Theta$. We need to find a differentiable path $(u(\tau), z(\tau))_{\tau\in[0,1]}$ such that

$$u(0) = u_0, \quad z(0) = z_0 = (x_0, q_1^0, .., q_k^0), \quad \frac{\partial}{\partial\tau}\Psi(u(\tau), z(\tau))|_{\tau=0} = \upsilon.$$

Components $x(\tau), q_m(\tau), m = 1, \ldots, k$ of a path $z(\tau)$ can be found in the following form:

$$x(\tau) = x_0 + \bar{x}\tau, \quad q_m(\tau) = q_m^0 + \bar{q}_m\tau, \quad m = 1, \ldots, k, \tag{23}$$

where $(\bar{x}, \bar{q}_1, \ldots, \bar{q}_k)$ are determined from the system of linear equations

$$\begin{cases} (\sum_{i\in N} x^i(\tau))'|_{\tau=0} = \upsilon'', \\ (x^i(\tau), q_m(\tau))'|_{\tau=0} = \upsilon'_{mi}, \quad i \in N_m\backslash\{i_0\}, \quad m = 1, \ldots, k, \\ (C_i x^i(\tau))'|_{\tau=0} = \upsilon^i, \quad i \in N, \\ (q_m(\tau))'|_{\tau=0} = \upsilon'''_m, \quad m = 1, \ldots, k. \end{cases}$$

Substituting representation (23) into this system and taking the first derivatives at $\tau = 0$ gives

$$\begin{cases} \sum_{i\in N} \bar{x}^i = \upsilon'', \\ (\bar{x}^i, q_m^0) = \upsilon'_{mi} - (x_0^i, \bar{q}_m), \quad i \in N_m\backslash\{i_0\}, \quad m = 1, \ldots, k, \\ C_i\bar{x}^i = \upsilon^i, \quad i \in N, \\ \bar{q}_m = \upsilon'''_m, \quad m = 1, \ldots, k. \end{cases}$$

Elements $\bar{q}_1, \ldots, \bar{q}_k$ can be found from the last kl equations. Take as $\bar{x}$ any solution of a system

$$\begin{cases} \sum_{i\in N} \bar{x}^i = \upsilon'', \\ (\bar{x}^i, q_m^0) = \tilde{\upsilon}_{mi}, \quad i \in N_m\backslash\{i_0\}, \quad m = 1, \ldots, k, \\ C_i\bar{x}^i = \upsilon^i, \quad i \in N, \end{cases} \tag{24}$$

where $\tilde{\upsilon}_{mi} = \upsilon'_{mi} - (x_0^i, \bar{q}_m)$. By the conditions of the lemma, all rows of the matrix D of this system (see Fig. 3) are linearly independent, hence a solution exists.

Let us determine a path $u(\tau)$ assuming $u_i(\tau, x) = u_0(x) + b_i x^i\tau$, where vectors $b_i \in \mathbb{R}^l$ are chosen to satisfy

$$\frac{\partial}{\partial\tau}\Big(\frac{\partial u_i}{\partial x_j^i}(x(\tau))\Big)\Big|_{\tau=0} = \upsilon_{ij}. \tag{25}$$

Since $x(\tau) = x_0 + \bar{x}\tau$, the equation (25) transforms to

$$b_{ij} = \upsilon_{ij} - \sum_{s\in N\times L}\frac{\partial^2(u_0)_i}{\partial x_j^i\partial x_s}(x_0)\bar{x}_s.$$

$$D = \begin{pmatrix}
E & E & \cdots & E & \cdots & E & \cdots & E \\
0 & q_1^0 & & & & & & \\
\vdots & & \ddots & & & & & \\
 & & & q_1^0 & & & & \\
 & & & & \ddots & & & \\
 & & & & & q_k^0 & & \\
 & & & & & & \ddots & \\
 & & & & & & & q_k^0 \\
 & C^i & & & & & & \\
 & & \ddots & & & & & \\
 & & & C^i & & & & \\
 & & & & \ddots & & & \\
 & & & & & C^i & & \\
\vdots & & & & & & \ddots & \\
0 & & & & & & & C^i
\end{pmatrix}$$

Figure 3: The matrix of the system (24). E is an $(l \times l)$ identity matrix. The first l columns of the matrix D correspond to the consumption of agent i_0.

Since the vector υ has been chosen arbitrarily, we have shown that the derivative mapping $T_{(u_0,z_0)}\Psi$ is surjective. To complete the proof, we need to establish that $(T_{(u_0,z_0)}\Psi)^{-1}(T_yW)$ splits. This follows from the surjectivity of $T_{(u_0,z_0)}\Psi$ and from the finite dimensionality of its range.

□

Linear dependence of the rows of a matrix $\begin{pmatrix} q_m \\ C_i \end{pmatrix}$ or, which is the same, the condition

$$q_m \in \mathcal{L}_i, \quad i \in N_m \tag{26}$$

implies that the budget restriction $q_m x^i = 0$ follows from the condition that the consumption plan x^i belongs to the face F_i. We are going now to remove the budget restrictions for agents from some set $H \subseteq N$, delete corresponding components $(\Psi^m)_i$ of the mapping Ψ and accordingly restrict the domain of Ψ in the part concerning prices. The modified mapping Ψ^H will be transversal to any submanifold $W \subseteq \mathbb{R}^l \times \Theta$ at any point $(u, x, q_1, \ldots, q_k) \in U \times \tilde{X} \times \Theta^H$ — the domain of Ψ^H.

Namely, let Θ^H be a set of all elements $(q_1, \ldots, q_k)$ in Θ such that $q_m \in \mathcal{L}_i$ *only* for $i \in H \cap N_m$. Denote $H \cap N_m$ by H_m and consider sets

$$W_m^H = \begin{cases} (\bigcap_{i\in H_m} \mathcal{L}_i)\backslash(\bigcup_{i\in N_m\backslash H_m} \mathcal{L}_i), & \text{if } H_m \neq \emptyset \ , \\ \mathbb{R}^l\backslash(\bigcup_{i\in N_m} \mathcal{L}_i) & \text{otherwise.} \end{cases} \tag{27}$$

Take sets

$$\Theta_m^H = \Theta^m \cap W_m^H, \quad m = 1, \dots, k$$

as components of the manifold Θ^H :

$$\Theta^H = \Theta_1^H \times \cdots \times \Theta_k^H.$$

Notice that sets Θ^H, $H \subseteq N$ are relatively open and form a partition of the manifold Θ.

Let $\bar{T} = T\backslash H$ and consider the mapping

$$\Psi_u^H = \prod_{i\in N} \Psi_i^u \times \Psi^{mc} \times \prod_{i\in N} \Psi_i^F \times \prod_{m=1}^{k} \Psi_H^m \times \Psi_H^q : \tilde{X} \times \Theta^H \to \mathbb{R}^{\bar{T}} \times cl\,\Theta^H,$$

where the mappings $\Psi_H^m : \tilde{X} \times \Theta_m^H \to \mathbb{R}^{N_m\backslash(H_m\cup\{i_0\})}$ are defined by

$$(\Psi_H^m(x, q_m))_i = q_m x^i, \quad i \in N_m\backslash(H_m \cup \{i_0\}), \quad m = 1, \dots, k,$$

and $\Psi_H^q(q_1, \dots, q_k) = (q_1, \dots, q_k)$ is an identity embedding from Θ^H to $cl\,\Theta^H$. All other mappings are defined as before. Put $\Psi_H(u, z) = \Psi_u^H(z)$. The manifold Δ_{HF} is the analogue of the manifold Δ_F :

$$\Delta_{HF} \stackrel{def}{=} \Big\{(\nu_1, \dots, \nu_n, \beta_1, \dots, \beta_k, \sigma, q, \varphi_1, \dots, \varphi_n) \in \mathbb{R}^{\bar{T}} \times cl\,\Theta^H|$$

$$\nu_i = \begin{cases} \lambda_i^m q_m + yC_i, \quad y \in \mathbb{R}^{T_i} & \text{if } i \in N_m'\backslash H_m, \quad m = 1, \dots, k, \\ yC_i, \quad y \in \mathbb{R}^{T_i} & \text{otherwise,} \end{cases} \tag{28}$$

$$(\beta_m)_i = \begin{cases} 0 & \text{if } i \in N_m'\backslash(H_m \cup \{i_0\}), \quad m = 1, \dots, k-1, \\ \delta_i\mu & \text{if } i \in N_k'\backslash(H_k \cup \{i_0\}), \quad m = k, \end{cases} \tag{29}$$

$$\varphi_i = 0, \quad i \in N, \quad \sigma = 0, \quad \|q_m\| = 1, \quad m = 1, \dots, k\Big\}.$$

Lemma 5.9

Suppose that $u \in U$ is such that for every $H \subseteq N$, Θ, every partition $(N_1, \dots, N_k)$ of the set N, and for all possible choices of subsets of satiated agents $N_1'', \dots, N_k''$, the mapping Ψ_u^H is transversal to the manifold Δ_{HF}. Then the number of proper δ-equilibria that belong to the face F is finite.

Proof. The finiteness of proper δ-equilibria follows from the finiteness (with respect to a choice of parameters $\Theta, N_1, \dots, N_k, N_1'', \dots, N_k''$, and $H \subseteq N$) of the possibly different arrangements of the correspondence Ψ_u^H and the manifold

Δ_{HF}, as well as from the discreteness of the inverse images $(\Psi_u^H)^{-1}(\Delta_{HF})$, whose union covers all proper δ-equilibria from F. Therefore, it will suffice to establish finiteness of the sets $(\Psi_u^H)^{-1}(\Delta_{HF})$ to prove the lemma. By a well-known property of transversal correspondences,

$$\text{codim } \Delta_{HF} = \text{codim } (\Psi_u^H)^{-1}(\Delta_{HF}). \tag{30}$$

We show that

$$\text{codim } \Delta_{HF} \geq \dim \tilde{X} \times \Theta^H = nl + \dim \Theta^H. \tag{31}$$

Consider first only such equilibria that $N_1 = N_1'$ (i. e., there is no satiation in the usual sense). By the construction and the assumptions made, codim Δ_{HF} is equal to the difference between a number of restrictions and a number of free variables. Each restriction of type (29) or $\varphi_i = 0, \ \ i \in N$ corresponds to the free variable λ or y, respectively. The budget restriction $q_1 x^{i_0} = 0$ is omitted (i_0 necessarily belongs to the set N_1). Taking into account relations (28), $\sigma = 0$, $\|q_m\| = 1$, $m = 1, \ldots, k$, and a free variable μ, one gets

$$\text{codim } \Delta_{HF} = nl + l + k - 2.$$

Since $\dim \Theta^H \leq l$

$$\text{codim } \Delta_{HF} \geq nl + \dim \Theta^H \tag{32}$$

if $k > 1$. Note, that codim Δ_{HF} does not depend on the choice of H. If $k = 1$ then $N = N_1$, so we can put the free variable μ equal to zero (there are no satiated agents). This increases the codimension of the manifold by 1, and (32) is established again. In the case $N_1'' \neq \emptyset$ we proceed in the same way with the only difference that μ is not equated to zero but expressed through the values of the budget correspondences of the satiated agents.

By transversality of Ψ_u^H and relative openness of Θ_m^H in $\cap_{i \in H_m} \mathcal{L}_i$, we conclude that

$$\dim (\Psi_u^H)^{-1}(\Delta_{HF}) < 0, \text{ if } \dim \Theta^H \neq l;$$

$$\dim (\Psi_u^H)^{-1}(\Delta_{HF}) = 0, \text{ if } \dim \Theta^H = l.$$

Thus, $(\Psi_u^H)^{-1}(\Delta_{HF})$ is discrete whenever $\dim \ \Theta^H = l, \ k \leq 2$ or is empty otherwise. One can easily see that there is only one subset $\bar{H} \subseteq N$ such that $\dim \Theta^{\bar{H}} = l$.

Since Δ_F is closed, and X is, without the loss of generality, compact in $\mathbb{R}^{ln}$, the intersection $(\Psi_u)^{-1}(\Delta_F) \cap (X \times \Theta)$ is compact in $\mathbb{R}^{ln} \times \Theta$. We have the following relation

$$(\Psi_u)^{-1}(\Delta_F) \subseteq \bigcup_H (\Psi_u^H)^{-1}(\Delta_{HF}) = (\Psi_u^{\bar{H}})^{-1}(\Delta_{\bar{H}F}).$$

But $(\Psi_u^{\bar{H}})^{-1}(\Delta_{\bar{H}F})$ is discrete in $\tilde{X} \times \Theta^H$. Therefore, $(\Psi_u)^{-1}(\Delta_F) \cap (X \times S)$ is a discrete compact, which implies the finiteness of proper δ-equilibria from F.

□

Proof of Theorem 5.4. For each $i \in N$ choose a compact $K_i \in \mathbb{R}^{L \times \{i\}}$ such that $int\ K_i \supset X_i, K = \prod_{i \in N} K_i \subset \tilde{X}$, and let $\{S_t^H\}_{t=1}^{\infty}$ be a sequence of compact sets approximating Θ^H from within:

1. $S_t^H \subset \Theta^H, \quad t = 1, 2 \ldots,$
2. $\bigcup_{t=1}^{\infty} S_t^H = \Theta^H.$

Consider a sequence of compact sets $K_H^t = K \times S_t^H \subset \tilde{X} \times \Theta^H$, $t = 1, 2...,$ and apply the theorems of density and openness of transversal sections to the case $\mathcal{V} = U$, $X = \tilde{X} \times \Theta^H$, $Y = \mathbb{R}^{\bar{T}} \times cl\ \Theta^H$, $\rho_v = \Psi_u^H$, $\omega_\rho = \Psi_H$, $W = \Delta_{HF}$. By construction, Ψ_H is transversal to Δ_{HF}, and all other conditions of Thom's theorems are satisfied as well. Therefore, the set $\mathcal{V}_{KW}^t = \{u \in U \mid \Psi_u^H \pitchfork_z \Delta_{HF},\ z \in K_H^t\}$ is open and dense in U. Let

$$G = \bigcap_{t=1}^{\infty} \mathcal{V}_{KW}^t,$$

where the intersection is taken over $t = 1, 2...$ and over all admissible F, Θ^H, H, $N_1'..., N_k'$, i_0. Since G is a countable intersection of open dense sets, it is residual. Direct application of Lemma 5.9 completes the proof.

□

6 Hierarchic equilibria and constrained equilibria

Another interesting generic property of a δ-equilibrium is that unless it is a Walrasian equilibrium, it has to be a coupons equilibrium. Coupons equilibrium introduced in Drèze and Müller (1980) is a normative concept for a model with price rigidities and rationing. At such an equilibrium, each agent is given an element of his budget set such that the coupon value of his net trade is smaller than his coupon endowment. Formally, an allocation $x \in X$ is a *coupons equilibrium* if there exist $p \in Q$, $a \in \{v \in \mathbb{R}^l | v_1 = 0\}$, and $(c^i)_{i \in N} \in \mathbb{R}_+^N$ such that *i)* $\sum_{i \in N} x^i = \sum_{i \in N} w^i$; *ii)* $px^i = pw^i$ and $ax^i \leq c^i + aw^i$ for every $i \in N$; and *iii)* x^i maximizes ith utility over all consumption bundles that satisfy *ii)*. The first commodity is chosen as the numeraire, it is never rationed at a coupons equilibrium. Furthermore, we assume

A3. Commodity 1 is strictly desirable and $X_i + ce_1 \subseteq X_i$ for every $i \in N$, $c \in \mathbb{R}_+$, where $e_1 = (1, 0, \ldots, 0)$.

Theorem 6.1

Suppose that an economy satisfies assumptions A1–A3. Then, for any strictly positive vector δ, there exists a residual set $G' \subseteq U$ such that for each $u \in G'$ the following condition is true: every proper δ-equilibrium is either a Walrasian equilibrium, or a coupons equilibrium.

Proof. The proof is similar to that of Theorem 5.4. The result follows from the fact that $(\Psi_u^H)^{-1}(\Delta_{HF})$ is non-empty only if the number of components in Θ is less than or equal to two. Therefore, there exists a residual set G' such that for every $u \in G'$ each δ-equilibrium is sustained by a hierarchic price with $k \leq 2$.

Assumption $A3$ implies local non-satiation. It follows then by Theorem 2.6 that ${}^\circ\varepsilon = 0$ (we assume without loss of generality that λ_1 is near-standard). If $k = 1$ then it follows from Proposition 3.6 that each δ-equilibrium is a Walrasian equilibrium sustained by prices q_1. Suppose that x is a δ-equilibrium and (q_1, q_2) is a hierarchic representation of non-standard equilibrium prices. By local non-satiation, $q_1 x^i = q_1 w^i$ for every $i \in N$. Take q_1 as "money prices". Note that $(q_1)_1 > 0$. Denote by $\tilde{q_1}$ and $\tilde{q_2}$ the projections of hierarchic components on the last $l - 1$ coordinates. For any $y^i \in \{z^i \in X_i | q_1 z^i = q_1 w^i\}$ the condition $q_2 y^i \leq q_2 w^i + \mu_i$ is equivalent to

$$\left(\tilde{q_2} - \frac{(q_2)_1}{(q_1)_1}\tilde{q_1}\right) y^i \leq \left(\tilde{q_2} - \frac{(q_2)_1}{(q_1)_1}\tilde{q_1}\right) w^i + \mu_i.$$

Therefore, we can take $a = \tilde{q_2} - ((q_2)_1/(q_1)_1)\tilde{q_1}$ as a vector of coupon prices, and put $c^i = \mu_i = {}^\circ(\varepsilon\delta_i/\lambda_2)$ if the second budget constraint plays any role. Otherwise, we can take c^i big enough so that $ay^i \leq aw^i + c^i$ is satisfied for any $y^i \in \{z^i \in X_i | q_1 z^i = q_1 w^i\}$. Thus we have proved that x is a coupons equilibrium sustained by "money prices" q_1, coupon prices a and coupon endowments $(c^i)_{i \in N}$.

□

7 Optimality and core equivalence

Marakulin (1990) has shown that each non-standard dividend equilibrium is weakly Pareto optimal and, conversely, that each weakly Pareto optimal allocation can be decentralized with an appropriate non-standard price vector. For an economy with polyhedral consumption sets, Proposition 3.9 implies that such an allocation (sometimes called a Pareto equilibrium) is a non-standard equilibrium after some appropriate redistribution of initial endowments, so the second welfare theorem holds. Whether a Pareto equilibrium is always a non-standard equilibrium for an economy with consumption sets that are not necessarily polyhedral is still an open question.

Florig (2001) has proved that the set of hierarchic equilibria coincides with the fuzzy rejective core of an economy. The latter concept is a refinement of the weak core introduced first in Konovalov (1998). It follows from Theorem 3.8 that the set of non-standard dividend equilibria coincides with the fuzzy rejective core as well. In this section we will show how this result can be proved directly by use of the non-standard separation argument.

We say that a coalition $S \subseteq N$ *rejects* a feasible allocation $\bar{x}$ if there exist a partition of S consisting of two subcoalitions $S_1, S_2 \subseteq S$ and consumption bundles $y^i \in X_i$, $i \in S$, such that

$$\sum_{i \in S} y^i = \sum_{i \in S_1} \bar{x}^i + \sum_{i \in S_2} w^i,$$

and $y^i \in \mathcal{P}_i(\bar{x}^i), \quad i \in S$. Rejecting can be viewed as performed in three steps. First, each member i of S receives an offer $\bar{x}^i$. Second, a subcoalition S_1 of agents who accept an offer is formed. At the third stage, trade within coalition S occurs. If it is possible to make all agents in S better off, then $\bar{x}$ is said to be rejected by the coalition S.

It is clear that rejection implies strong blocking in a usual sense. For this reason, the *rejective core* — the set of all feasible allocation that are not rejected by any coalition — is usually a proper subset of the weak core[3].

A fuzzy coalition $\xi \in [0,1]^N \backslash \{0\}$ rejects an allocation $\bar{x} \in X$, if there exist fuzzy coalitions t and s with $t + s = \xi$ and consumption plans $y^i \in X_i$, $y^i \in \mathcal{P}_i(\bar{x}^i)$ for all $i \in supp\, \xi$, such that

$$\sum_{i \in N} \xi_i y^i = \sum_{i \in N} t_i \bar{x}^i + \sum_{i \in N} s_i w^i.$$

The set of all feasible allocations that cannot be rejected by any fuzzy coalition is called *the fuzzy rejective core* of an economy $\mathcal{E}$ and is denoted by $C_{fr}(\mathcal{E})$.

[3]For economies with satiation, the concept of weak blocking lead to the core which is usually empty, see Aumann and Drèze (1986).

The interpretation of fuzzy coalitions and fuzzy rejection can be as follows. Each fuzzy coalition ξ can be viewed as a coalition in $\mathcal{E}_{[0,1]}$, a $[0,1]$-replica of the initial economy $\mathcal{E}$. In $\mathcal{E}_{[0,1]}$ every agent i is replaced by a continuum of agents identical to i, each set of agents of same type is of Lebesgue measure 1. The number ξ_i represents the measure of the set of agents of type i participating in the coalition ξ. Suppose that an allocation $\bar{x}$ is chosen as a solution by a social planner or by the society as a whole. Furthermore, suppose that it takes time to pass on from the initial allocation w to $\bar{x}$, and at some point only a fraction $\alpha \in (0,1)$ of the society has managed to get the prescribed consumption plans. Denote this coalition by S_1 (S_1 corresponds to a fuzzy coalition $(\alpha, \dots, \alpha)$). Recall that by feasibility of $\bar{x}$

$$\sum_{i \in N} \alpha \bar{x}^i = \sum_{i \in N} \alpha w^i.$$

Meanwhile, the complementary coalition S_2 represented by the fuzzy coalition $(1-\alpha, \dots, 1-\alpha)$ keeps the initial endowments. At this moment, some members of S_1 and S_2 can get involved in a mutually beneficial[4] trade. Specifically, take some $\beta \leq \min\{\alpha, 1-\alpha\}$. A fuzzy coalition $\beta\xi$ can provide the consumption bundle y^i for every its member of type i :

$$\sum_{i \in N} \beta \xi_i y^i = \sum_{i \in N} \beta t_i \bar{x}^i + \sum_{i \in N} \beta s_i w^i.$$

Here $\beta t_i \leq \alpha$, $\beta s_i \leq 1 - \alpha$ for every $i \in \mathit{supp}\, \xi$. In short, if reallocation of resources requires time, then $\bar{x}$ is unstable against possibility of recontracting. This is the meaning of fuzzy rejection.

Denote the set of non-standard dividend equilibria of an economy $\mathcal{E}$ by $W_{ns}(\mathcal{E})$.

Theorem 7.1
Assume that all consumption sets X_i are polyhedral, and that the sets $\mathcal{P}_i(x^i)$ are open, convex and do not contain x^i for all $x^i \in X_i, i \in N$. Then the fuzzy rejective core of an economy $\mathcal{E}$ coincides with the set of non-standard dividend equilibria:

$$C_{fr}(\mathcal{E}) = W_{ns}(\mathcal{E}).$$

Proof. First, we show that

$$C_{fr}(\mathcal{E}) \subseteq W_{ns}(\mathcal{E}).$$

Let $\bar{x} = (\bar{x}^i)_{i \in N} \in C_{fr}(\mathcal{E})$ and consider the sets

$$\mathcal{G}^i(\bar{x}^i) = \{y^i - \bar{x}^i | y^i \in \mathcal{P}_i(\bar{x}^i)\}, \quad i \in N,$$

[4]Strictly speaking, we need $\bar{x}^i$ to be individually rational and preference order transitive for a trade to be beneficial for an agent i who kept his initial endowments.

and

$$\mathcal{G}^i(w^i) = \{y^i - w^i | y^i \in \mathcal{P}_i(\bar{x}^i)\}, \quad i \in N.$$

Denote the convex hull of the union of these sets by G

$$G = conv \bigcup_{i \in N} (\mathcal{G}^i(\bar{x}^i) \cup \mathcal{G}^i(w^i))$$

and show that G does not contain zero. Suppose it does. Then, by convexity of $\mathcal{G}^i(\bar{x}^i)$ and $\mathcal{G}^i(w^i)$, there exist $t = (t_i)_{i \in N}$, $s = (s_i)_{i \in N}$, $y, z \in X$ such that

$$\sum_{i \in N} t_i + \sum_{i \in N} s_i = 1, \quad t_i, s_i \geq 0, \quad i \in N,$$

$$\sum_{i \in N} t_i y^i + \sum_{i \in N} s_i z^i = \sum_{i \in N} t_i \bar{x}^i + \sum_{i \in N} s_i w^i, \tag{33}$$

where $y^i \in \mathcal{P}_i(\bar{x}^i)$ if $i \in supp\ t$, and $z^i \in \mathcal{P}_i(\bar{x}^i)$ if $i \in supp\ s$. This implies that $\bar{x}$ is rejected by a fuzzy coalition $t + s$, a contradiction. Therefore, a zero point does not belong to the convex set G. By the non-standard separating hyperplane theorem (see Konovalov (2001), Theorem 2.9.3), there exists $p \in {}^*\mathbb{R}^l$ such that for every $y^i \in \mathcal{P}_i(\bar{x}^i),\ i \in N$, the following conditions hold simultaneously

$$py^i > pw^i, \tag{34}$$

and

$$py^i > p\bar{x}^i. \tag{35}$$

Define the components of the vector $d \in {}^*\mathbb{R}^n_+$ by

$$d_i = \max\{0, p\bar{x}^i - pw^i\}, \quad i \in N. \tag{36}$$

Then $p\bar{x}^i \leq pw^i + d_i, \quad i \in N$, which implies attainability of $\bar{x}$. To prove the validity of the required inclusion we have to show that $\bar{x}$ satisfies the equilibrium property of individual rationality:

$$\bar{B}_i(p, d_i) \cap \mathcal{P}_i(\bar{x}^i) = \emptyset. \tag{37}$$

What we do have so far is

$$\{x \in {}^*X_i \ : \ px \leq pw^i + d_i\} \cap \mathcal{P}_i(\bar{x}^i) = \emptyset, \quad i \in N.$$

But then (37) is a consequence of Proposition 3.9 and openness of the sets $\mathcal{P}_i(\bar{x}^i)$, $i \in N$.

To prove the converse inclusion $W_{ns}(\mathcal{E}) \subseteq C_{fr}(\mathcal{E})$, let $\bar{x} \in W_{ns}(\mathcal{E})$ and assume that $p \in {}^*\mathbb{R}^l$ and $d \in {}^*\mathbb{R}^n_+$ are corresponding non-standard equilibrium prices and dividends. Suppose that there exists a fuzzy coalition $\xi = (\xi_i)_{i \in N}$ that blocks $\bar{x}$. Then, there exist $y^i \in \mathcal{P}_i(\bar{x}^i)$, $t_i, s_i \in \mathbb{R}_+$, $i \in supp\ \xi$ such that

$$\sum_{i \in N} t_i y^i + \sum_{i \in N} s_i y^i = \sum_{i \in N} t_i \bar{x}^i + \sum_{i \in N} s_i w^i, \tag{38}$$

where $t_i + s_i = \xi_i$. By individual rationality,

$$y^i \notin \bar{B}_i(p, d_i), \quad i \in supp\,\xi, \tag{39}$$

which implies that

$$py^i > pw^i, \quad i \in supp\,\xi.$$

Fix $i \in supp\,\xi$ and consider a hierarchic representation $(q_1, \ldots, q_k)$ of prices p. For every y^i find a number $h = h(i) \in \{1, \ldots, k\}$ such that

$$q_h y^i > q_h w^i, \quad q_r y^i = q_r w^i, \quad r < h.$$

We claim that

$$q_h y^i > q_h \bar{x}^i, \quad i \in supp\,\xi. \tag{40}$$

To prove it, use the characterization of the budget set $\bar{B}_i(p, d_i)$ given by Proposition 3.6. Since $y^i \notin \bar{B}_i(p, d_i)$, there exists $m = m(i) \in \{1, \ldots, k\}$ such that at least one of the conditions

$$\begin{gathered} q_1 y^i = q_1 w^i, \\ \vdots \\ q_{m-1} y^i = q_{m-1} w^i, \\ (q_m y^i \leq q_m w^i + \mu_i) \vee (q_m y^i \leq q_m w^i), \end{gathered}$$

where $\mu_i = {}^\circ(d_i/\lambda_{j(p,d_i)})$, is violated, so it must be $h \leq m$. Taking into account $\bar{x}^i \in \bar{B}_i(p, d_i)$ one obtains $q_h y^i > q_h \bar{x}^i$. Moreover, it is easy to see that $q_r \bar{x}^i = q_r w^i = q_r y^i$ for all $r < h$. Define

$$\bar{h} = \min_{i \in supp\,\xi} \quad h(i).$$

Then for every $i \in supp\,\xi$

$$q_{\bar{h}} y^i \geq q_{\bar{h}} w^i, \tag{41}$$

$$q_{\bar{h}} y^i \geq q_{\bar{h}} \bar{x}^i, \tag{42}$$

and there exists $i \in supp\,\xi$ such that both inequalities are strict. Multiplying (41) and (42) by s_i and t_i respectively, and summing over all i gets

$$(q_{\bar{h}}, \sum_{i \in N} t_i y^i) + (q_{\bar{h}}, \sum_{i \in N} s_i y^i) > (q_{\bar{h}}, \sum_{i \in N} t_i \bar{x}^i) + (q_{\bar{h}}, \sum_{i \in N} s_i w^i),$$

which contradicts (38). Consequently, the inclusion $W_{ns}(\mathcal{E}) \subseteq C_{fr}(\mathcal{E})$ is true and this completes the proof of the theorem.

□

Konovalov (1998) has shown by standard mathematical methods that the set of dividend equilibria is equivalent to the fuzzy rejective core of an economy if the survival assumption is met for every agent. As we know, non-standard dividend equilibria become dividend equilibria in this case. Thus Theorem 7.1 provides a generalization of Konovalov's equivalence theorem to the case where the survival assumption may fail. It is easy to check that in Gale's (1976) example no dividend equilibria exist, while there is a whole continuum of fuzzy rejective core allocations. However, it is possible to restore the core equivalence once dividend equilibria with non-standard prices are brought into consideration.

8 Appendix

Proof of Proposition 2.4. It is sufficient to show that $B_i(\bar{p}) = \bar{B}_i(p)$ for every $i \in N$. For a fixed i, assume $x \in \bar{B}_i(p)$ and let $(q_1, \ldots, q_k)$ be a hierarchic representation of the price vector p. Note that $\bar{p} = {}^\circ(p/\|p\|) = q_1$. There exists $\tilde{x} \in {}^*X_i$ such that $p\tilde{x} \leq pw^i$ and $\tilde{x} \approx x$. Assume inequality $q_1 x > q_1 w^i$. Then

$$q_1\tilde{x} + \frac{\lambda_2}{\lambda_1}q_2\tilde{x} + \cdots + \frac{\lambda_k}{\lambda_1}q_k\tilde{x} > q_1 w^i + \frac{\lambda_2}{\lambda_1}q_2 w^i + \cdots + \frac{\lambda_k}{\lambda_1}q_k w^i,$$

which yields a contradiction with $p\tilde{x} \leq pw^i$. Therefore $q_1 x \leq q_1 w^i$ and $x \in B_i(\bar{p})$.

Let $x \in B_i(\bar{p})$ and find $y \in B_i(\bar{p})$ such that $\bar{p}y < \bar{p}w^i$. Consider an internal sequence $\alpha : {}^*\mathbb{N} \to {}^*X_i$ defined by $\alpha(n) = (1/n)y + (1 - 1/n)x$. Since $p\alpha(n) < pw^i$ for all $n \in \mathbb{N}$, it should be also true for some hyperfinite natural number $\tilde{n} \in {}^*\mathbb{N}\backslash\mathbb{N}$. Then $\alpha(\tilde{n}) \approx x$, which implies $x \in st\ {}^*B_i(p) = \bar{B}_i(p)$. We have shown that $B_i(\bar{p}) = \bar{B}_i(p)$ for each $i \in N$. Hence $\bar{x}$ is a Walrasian equilibrium with equilibrium prices $\bar{p}$.

□

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