BENDING, VIBRATION AND BUCKLING RESPONSE OF CONVENTIONAL AND MODIFIED EULER–BERNOULLI AND TIMOSHENKO BEAM THEORIES ACCOUNTING FOR THE VON KÁRMÁN GEOMETRIC NONLINEARITY

A Thesis

by

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ABSTRACT

Beams are among the most commonly used structural members that are encountered in virtually all systems of structural design at various scales. Mathematical models used to determine the response of beams under external loads are deduced from the three-dimensional elasticity theory through a series of assumptions concerning the kinematics of deformation and constitutive behavior. The kinematic assumptions exploit the fact that such structures do not experience significant transverse normal and shear strains and stresses. For example, the solution of the three-dimensional elasticity problem associated with a straight beam is reformulated as a one-dimensional problem in terms of displacements whose form is presumed on the basis of an educated guess concerning the nature of the deformation.

In many cases beam structures are subjected to compressive in-plane loads that may cause out-of-plane buckling of the beam. Typically, before buckling and during compression, the beam develops internal axial force that makes the beam stiffer. In the linear buckling analysis of beams, this internal force is not considered. As a result the buckling loads predicted by the linear analysis are not accurate. The present study is motivated by lack of suitable theory and analysis that considers the nonlinear effects on the buckling response of beams.

This thesis contains three new developments: (1) The conventional beam theories are generalized by accounting for nonlinear terms arising from ε_{zz} and ε_{xz} that are of the same magnitude as the von Kármán nonlinear strains appearing in ε_{xx}. The equations of motion associated with the generalized Euler–Bernoulli and Timoshenko beam theories with the von Kármán type geometric nonlinear strains are derived using Hamilton's principle. These equations form the basis of investigations to determine certain microstructural length scales on the bending, vibration and buckling response of beams used in micro- and nano-devices. (2) Analytical solutions of the conventional Timoshenko beam theory with the von Kármán nonlinearity are developed for the case where the inplane inertia is negligible when compared to other terms in the equations of motion. Numerical results are presented to bring out the effect of transverse shear deformation on the buckling response. (3) The development of a nonlinear finite element model for post-buckling behavior of beams.

To my loving family and friends

for their unending support and encouragement

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NOMENCLATURE

σ_{ij}	Stress components in Cartesian coordinates $(i, j = 1, 2)$
ε_{ij}	Normal strain components in Cartesian coordinates $(i, j = 1, 2)$
γ_{xz}	Shear strain component in the $x{-}z$ plane
θ_x	Slope of the beam in the deformed configuration
ϕ_x	Rotation of a transverse normal line about the $y{-}$axis
$\frac{\partial}{\partial x}$	Partial derivative with respect to displacement, x
$\frac{\partial}{\partial t}$	Partial derivative with respect to time, t
δ	Variational parameter used in the principle of virtual displacements
δK	Virtual kinetic energy
δU	Virtual strain energy
δV	Virtual external work done
K_s	Shear correction factor
λ_n	Eigenvalue for buckling application
$K_{ij}^{\alpha\beta}$	Stiffness matrix components in Cartesian coordinates $(i, j = 1, 2)$ for generalized displacements $(\alpha, \beta = 1, 2, 3)$
ψ_i	Lagrange interpolation function for virtual axial displacement, δu
ψ_j	Lagrange interpolation function for true axial displacement, u
ϕ_i	Hermite interpolation function for virtual transverse displacement, δw
ϕ_j	Hermite interpolation function for true transverse displacement, w
$\psi_i^{(\alpha)}$	Lagrange interpolation function $(\alpha = 1)$, Hermite cubic interpolation function $(\alpha = 2, 3)$ for virtual generalized displacements
$\psi_j^{(\alpha)}$	Lagrange interpolation function $(\alpha = 1)$, Hermite cubic interpolation function $(\alpha = 2, 3)$ for true generalized displacements

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1. INTRODUCTION

1.1 Microstructural Length Scale Studies

In the last two decades, a number of papers appeared on bringing microstructural length scales into the continuum description of beams and plates. Eringen's [1] nonlocal elasticity is based on the hypothesis that the stress field at a point in an elastic continuum not only depends on the strain field at the point but also on strains at all other points of the continuum (i.e., the stress–strain relation is an integral equation). His observation was based on the atomic theory of lattice dynamics and experimental observations on phonon dispersion. Eringen converted his nonlocal integral model to a differential model that contained a single length scale, which was used by Wang et al. [2], Lu et al. [3], and Reddy and his colleagues [4–10] to bring out the effect of a single internal characteristic parameter on the bending, buckling, and vibration characteristics of beams and plates.

Yang et al. [11] developed a modified couple stress theory where the authors considered an additional energy term due to couple stress tensor in the strain energy density function. They assumed that the couple stress tensor is related to the curvature tensor through a single length scale. Such a assumption makes the analysis simple but it is not a physically realistic assumption because different components of couple stress tensor are related, in general, to different components of the curvature tensor. Microstructure-dependent theories were developed by Park and Gao [12], [13] for the Bernoulli–Euler beam theory; Ma, Gao, and Reddy [14–16] for the Timoshenko and Reddy–Levinson beams and Mindlin plates; and Reddy and his colleagues [17–22], Simisek and Reddy [23], Roque et al. [24], [25], Xia, Wang, and Yin [26], Ke and Wang [27], and Gao, Huang, and Reddy [28] for the first-order and higher-order beam and plate theories using the modified couple stress theory proposed by Yang et al. [11].

The nonlocal model of Eringen [1] and the modified couple stress theory of Yang et al [11] are based on simply postulating the equations directly for small deformation elasticity, and no attempt was made at a systematic derivation from finite deformation elasticity. For example, as the modified couple stress theory is concerned, there is no corresponding finite deformation couple stress theory with constrained rotations. Recently, Srinivasa and Reddy [29] developed such a finite deformation gradient elasticity theory for a fully constrained finitely deforming hyperelastic cosserat continuum where the directors are constrained to rotate with the body rotation. This is a generalization of small deformation couple stress theories, such as the one considered by Yang et al. [11], and it contains several length scales, the number depending on the type of theory used. The Srinivasa–Reddy finite deformation couple stress theory is useful, for example, in modeling an elastic material with embedded stiff short fibers or inclusions, that is, materials with carbon nanotubes or nematic elastomers, cellular materials with oriented hard phases, open cell foams, and so on.

The commonality between Eringen's and Yang et al. models is that both bring a microstructural length scale into the governing equations of a continuum, although no relationship between the two length scales has been established. Microstructural length scale can also be brought into the discrete form of structural equations by the *discrete peridynamics* approach suggested by Reddy et al. [30]. The approach is yet to be explored completely.

Another approach through which microstructural length scales can be brought into the structural theory is to account for the additional terms in the strain–displacement relations. The strain components associated with the simplified Green-Lagrange strain tensor (see Reddy [31]) $\mathbf{E} \approx \boldsymbol{\varepsilon}$ includes small strains but moderately large rotations, and it is commonly called the von Kármán strain tensor, and the associated theories are termed von Kármán beam theories. Conventional von Kármán nonlinear beam theories only account for $(1/2)(\partial w/\partial x)^2$ in the membrane strain $\varepsilon_{xx}^{(0)}$. In this study we develop generalized Euler–Bernoulli and Timoshenko beam theories that account for all terms of the type $(1/2)(\partial w/\partial x)^2$ in ε_{xx}, ε_{zz}, and ε_{xz}.

1.2 Present Study

In a series of papers, Nayfeh and his colleagues [32–34] claim that they have obtained the post-buckling configurations of Euler–Bernoulli beams with clamped-clamped, clamped-hinged and hinged-hinged boundary conditions. They take into account the von Kármán nonlinear strain arising from midplane stretching. They eliminate the axial displacement from the governing equations under the assumption that the inplane inertia is negligible and that the ends of the beam are immovable in the horizontal direction (i.e., the assumption limits application of the resulting equations only to beams with hinged or fixed ends). Thus, the results they obtained are not valid for post-buckling response because during post-buckling the ends have to move due to the applied axial force. Therefore results are applicable only for the onset of buckling.

In this study, a generalization of the Euler–Bernoulli and Timoshenko beam theories using the simplified Green–Lagrange strain tensor is presented. These theories bring a couple of microstructural length scales into the beam theories. The theories are then specialized to conventional theories by omitting the length scale effects. A systematic approach to eliminate the axial displacement for both the Euler–Bernoulli and Timoshenko beam theories is presented. Analytical solutions for buckling of beams using the conventional Euler–Bernoulli beams and conventional Timoshenko beams are presented.

Finally, nonlinear finite element models are developed for the generalized theories for the buckling application. The primary buckling load is obtained for a variety of mesh sizes for the conventional and generalized theories of both Euler–Bernoulli and Timoshenko beams. An initial geometric perfection, of varying magnitude, is applied to the transverse deflection to initiate buckling instead of just axial displacement while under loading.

2. CONVENTIONAL BEAM THEORIES*

The work introduced in this section is reprinted with permission by American Scientific Publishers* and covers analytical formulations from Reddy and Mahaffey [35]. Several beam theories are applied for research purposes and taught in the classroom, however, two specific theories are more commonly employed, the Euler–Bernoulli beam theory (EBT) and the Timoshenko beam theory (TBT). The Euler–Bernoulli theory is considered the classical beam theory based on the assumptions/hypothesis of straight lines normal to the central axis [36], discussed next. However, the Timoshenko theory, discussed later, relaxes one of the EBT assumptions and takes into account a shear correction factor that accounts for the shear energy present in the beam while undergoing bending [36]. The conventional form of both theories considers the assumption of inextensible lines normal to the central axis by removing the Poisson effect, thus neglecting the ε_{zz} strain component, which allows for simpler derivations and the use of one-dimensional constitutive relations for the principle of virtual generalized displacements and equations of motion. Independent of which theory is used to characterize the beam under investigation for an application, the generalized displacement field, in Cartesian coordinates, is defined as

$$\mathbf{u}(x,z,t) = u_1(x,z,t)\hat{\mathbf{e}}_1 + u_3(x,z,t)\hat{\mathbf{e}}_3 \equiv u_x(x,z,t)\hat{\mathbf{e}}_x + u_z(x,z,t)\hat{\mathbf{e}}_z \tag{2.1}$$

*Reprinted with permission from J. N. Reddy and P. Mahaffey, "Generalized Beam Theories Accounting for von Kármán Nonlinear Strains with Application to Buckling," *Journal of Coupled Systems and Multiscale Dynamics*, Vol. 1, Copyright [2013] by American Scientific Publishers.

which is used to describe displacements in specified directions under kinematic deformation. The strain components associated with the simplified Green-Lagrange strain tensor ($\mathbf{E}$) $\mathbf{E} \approx \boldsymbol{\varepsilon}$ includes small strains but moderately large rotations, such that

$$E_{ij} \approx \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right) \tag{2.2}$$

and it is commonly referred to as the von Kármán strain tensor. Conventional von Kármán nonlinear beam theories only account for $(1/2)(\partial w/\partial x)^{(2)}$ in the membrane strain $\varepsilon_{xx}^{(0)}$.

We will first discuss the conventional EBT and take a look at the governing equations of motion which will lead to an analytical solution for the axial buckling application. Next we will go through the same process and derive the equations of motion for the conventional Timoshenko case in the same fashion, and present the differences that lead to a slightly different analytical solution, primarily in the differential equation coupling from the transverse displacement and additional rotation function. Plots and numerical tables will be provided to explicitly show the differences in buckling modes for both theories, and allow for the deflection–load behavior for the various boundary conditions of each theory to be observed.

2.1 Conventional Euler–Bernoulli Theory

The conventional EBT, or classical theory, is based on several assumptions concerning straight lines normal to the centerline axis after deformation:

i) rotate as rigid lines to remain normal

ii) no extension in the z-direction (inextensible)

iii) remain straight after deformation.

These three assumptions can be seen in Figure 2.1.

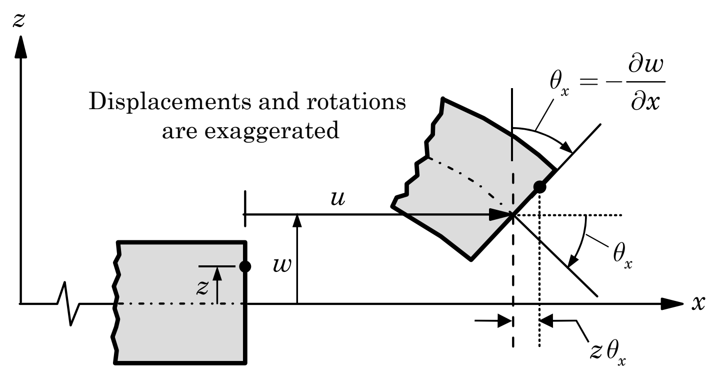

Figure 2.1: Kinematics of deformation in the Euler–Bernoulli beam theory.

The displacement field is given as

$$u_x(x,z,t) = u(x,t) + z\theta_x, \quad u_z(x,z,t) = w(x,t), \quad \theta_x \equiv -\frac{\partial w}{\partial x} \tag{2.3}$$

with the only strain component present

$$\boldsymbol{\varepsilon} = \varepsilon_{xx}\ \hat{\mathbf{e}}_x\hat{\mathbf{e}}_x \tag{2.4}$$

where

$$\varepsilon_{xx} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} \tag{2.5}$$

$$\varepsilon_{xx}^{(0)} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \varepsilon_{xx}^{(1)} = \frac{\partial \theta_x}{\partial x} \tag{2.6}$$

For the assumptions stated, we neglect the Poisson effect and transverse shear strain, such that neither ε_{zz} nor $\gamma_{xz} = 2\varepsilon_{xz}$ are present for the strain field.

2.1.1 Equations of Motion

For the governing equations of bending, natural vibrations and buckling due to an axially applied load, we make use of the principle of virtual displacements. This is carried out by applying Hamilton's principle [37] which uses the virtual kinetic and strain energy, as well as the virtual work done by externally applied forces. Hamilton's principle can be expressed as

$$0 = \int_{t_1}^{t_2} (-\delta K + \delta U + \delta V)\, dt \tag{2.7}$$

where δK is the virtual kinetic energy, δU is the virtual strain energy, and δV is the virtual work done by external forces. The kinetic energy expression is

$$\begin{aligned} \delta K &= \int_0^l \int_A \rho \dot{u}_i \, \delta \dot{u}_i \, dA\, dx = \int_0^l \int_A \rho \left[\left(\dot{u} + z\dot{\theta}_x \right) \left(\delta\dot{u} + z\delta\dot{\theta}_x \right) + \dot{w}\, \delta\dot{w} \right] dA\, dx \\ &= \int_0^l \left[\left(m_0 \dot{u} + m_1 \dot{\theta}_x \right) \delta\dot{u} + \left(m_1 \dot{u} + m_2 \dot{\theta}_x \right) \delta\dot{\theta}_x + m_0\, \dot{w}\, \delta\dot{w} \right] dx \end{aligned} \tag{2.8}$$

where

$$(m_0, m_1, m_2) = \int_A \rho(1, z, z^2) dA \tag{2.9}$$

and ρ is the mass density.

The expression for the virtual strain energy when the beam is subjected to an axial compressive force P is

$$\begin{aligned} \delta U &= \int_0^l \int_A \sigma_{xx}\, \delta\varepsilon_{xx} dA dx - P \int_0^l \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x}\, dx + \int_0^l F_v\, \delta w\, dx \\ &= \int_0^l \left[M_{xx}^{(0)} \left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right) + M_{xx}^{(1)} \frac{\partial \delta\theta_x}{\partial x} - P \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + F_v\, \delta w \right] dx \end{aligned} \tag{2.10}$$

where F_v is the viscous force. The viscous force is assumed to be proportional to the velocity $\dot{w}$, $F_v = \hat{\mu}\dot{w}$, where $\hat{\mu}$ is the viscous damping coefficient. Various stress resultants used in equation (2.10) are defined as

$$M_{xx}^{(0)} = \int_A \sigma_{xx}\, dA, \quad M_{xx}^{(1)} = \int_A z\, \sigma_{xx}\, dA \tag{2.11}$$

where $P(\partial w/\partial x)$ denotes the component of force along the deformed centerline of the beam, which is oriented at an angle of $\partial w/\partial x$. The virtual work done by the external forces is

$$\delta V = -\int_0^l \big(f_x\, \delta u + q\, \delta w\big) dx \tag{2.12}$$

where

$$f_x = \int_A \bar{f}_x\, dA, \quad q = (q^t + q^b) \tag{2.13}$$

and q^t and q^b denote the distributed load for top and bottom surfaces, respectively, and $\bar{f}_x$ is the force per unit volume. Substituting δU, δV, and δK into the Hamilton's principle (2.7), performing integration-by-parts with respect t as well as x to relieve the generalized displacements δu and δw of any differentiations, and using the fundamental lemma of calculus of variations, we obtain the following equations of motion:

$$-\frac{\partial M_{xx}^{(0)}}{\partial x} + m_0 \frac{\partial^2 u}{\partial t^2} + m_1 \frac{\partial^2 \theta_x}{\partial t^2} = f_x \tag{2.14}$$

$$-\frac{\partial}{\partial x}\left(M_{xx}^{(0)} \frac{\partial w}{\partial x}\right) + P\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 M_{xx}^{(1)}}{\partial x^2} + m_0 \frac{\partial^2 w}{\partial t^2} + m_1 \frac{\partial^3 u}{\partial x \partial t^2} + m_2 \frac{\partial^3 \theta_x}{\partial t^2 \partial x} + \hat{\mu}\frac{\partial w}{\partial t} = q \tag{2.15}$$

with the natural (or force) boundary conditions

$$\begin{aligned} \delta u &: \quad M_{xx}^{(0)} \\ \delta w &: \quad M_{xx}^{(0)} \frac{\partial w}{\partial x} - P\frac{\partial w}{\partial x} + \frac{\partial M_{xx}^{(1)}}{\partial x} + m_2 \frac{\partial^2 \theta_x}{\partial t^2} \\ \delta \theta_x &: \quad M_{xx}^{(1)} \end{aligned} \tag{2.16}$$

2.2 Conventional Timoshenko Beam Theory

The displacement field of the Timoshenko theory relaxes the normality constraint of the Euler–Bernoulli theory such that a line normal to the centerline in the undeformed configuration undergoes additional rotation in the deformed state such that the angle between the two lines is smaller than 90 degrees, as can be seen in Figure 2.2. This introduces a new variable $\phi_x(x,t)$ that accounts for the additional rotation due to shear. This new variable allows us to apply a shear correction factor [38, 39] when determining the buckling modes, as will be seen in the section concerning the analytical solution of the buckling application.

This additional rotation combines with θ_x to develop the shear strain γ_{xz}. The

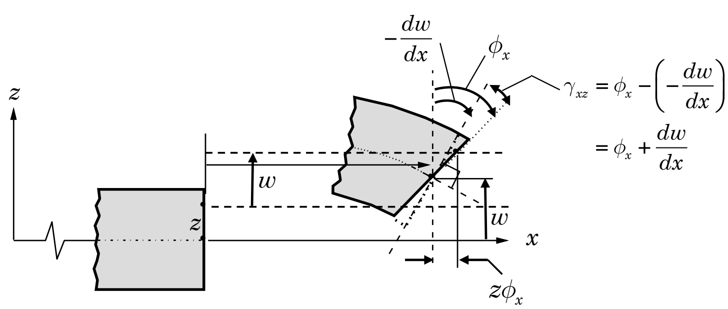

Figure 2.2: Kinematics of deformation in the Timoshenko beam theory.

displacement field can be stated as

$$u_x(x,z,t) = u(x,t) + z\phi_x(x,t), \quad u_z(x,z,t) = w(x,t) \tag{2.17}$$

with the strain field

$$\varepsilon_{xx} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)}, \quad \gamma_{xz} = \gamma_{xz}^{(0)} \tag{2.18}$$

$$\varepsilon_{xx}^{(0)} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_{xx}^{(1)} = \frac{\partial \phi_x}{\partial x}, \quad \gamma_{xz}^{(0)} = \phi_x + \frac{\partial w}{\partial x} \tag{2.19}$$

2.2.1 Equations of Motion

Following the same procedure as both forms of the EBT, the equations of motion for the conventional TBT become

$$\begin{aligned}\delta K &= \int_0^l \int_A \rho \dot{u}_i \delta \dot{u}_i \, dA \, dx = \int_0^l \int_A \rho \big[(\dot{u} + z\dot{\phi}_x)(\delta \dot{u} + z \delta \dot{\phi}_x) + \dot{w} \delta \dot{w} \big] dA \, dx \\ &= \int_0^l \big[(m_0 \dot{u} + m_1 \dot{\phi}_x) \delta \dot{u} + (m_1 \dot{u} + m_2 \dot{\phi}_x) \delta \dot{\phi}_x + m_0 \dot{w} \delta \dot{w} \big] dx\end{aligned} \tag{2.19}$$

where m_0, m_1 and m_2 are defined in Eq. (2.9).

The virtual strain energy δU is computed as

$$\begin{aligned}\delta U &= \int_0^l \int_A (\sigma_{xx} \, \delta\varepsilon_{xx} + \sigma_{xz} \, \delta\gamma_{xz}) \, dA \, dx - P \int_0^l \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} dx + \int_0^l F_v \, \delta w \, dx \\ &= \int_0^l \bigg[M_{xx}^{(0)} \left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right) + M_{xz}^{(0)} \left(\delta \phi_x + \frac{\partial \delta w}{\partial x} \right) + M_{xx}^{(1)} \frac{\partial \delta \phi_x}{\partial x} \\ &\qquad - P \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + F_v \delta w \bigg] dx\end{aligned} \tag{2.20}$$

where the stress resultants ($M_{xx}^{(0)}$ and $M_{xx}^{(1)}$) are defined in Eq. (2.11), and we add a new stress resultant

$$M_{xz}^{(0)} = \int_A \sigma_{xz} \, dA \tag{2.21}$$

The expression for the virtual work done by external forces for the TBT case is the same as in Eq. (2.12), with the exception of the displacement field being applied. We have

$$\delta V = - \int_0^l \big(f_x \, \delta u + q \, \delta w \big) dx \tag{2.22}$$

where (f_x, q) are the distributed axial and transverse loads, respectively.

Again, using Hamilton's principle, we obtain the following equations of motion of the conventional TBT:

$$-\frac{\partial}{\partial x}\Big(M_{xx}^{(0)} + M_{xz}^{(0)}\phi_x\Big) + m_0\frac{\partial^2 u}{\partial t^2} + m_1\frac{\partial^2 \phi_x}{\partial t^2} = f_x \tag{2.23}$$

$$-\frac{\partial}{\partial x}\left(M_{xz}^{(0)} + M_{xx}^{(0)}\frac{\partial w}{\partial x} - P\frac{\partial w}{\partial x}\right) + \hat{\mu}\frac{\partial w}{\partial t} + m_0\frac{\partial^2 w}{\partial t^2} = q \tag{2.24}$$

$$-\frac{\partial M_{xx}^{(1)}}{\partial x} + M_{xz}^{(0)} + m_1\frac{\partial^2 u}{\partial t^2} + m_2\frac{\partial^2 \phi_x}{\partial t^2} = 0 \tag{2.25}$$

The natural boundary conditions are

$$\begin{aligned} \delta u:&\quad M_{xx}^{(0)} \\ \delta w:&\quad M_{xz}^{(0)} + (M_{xx}^{(0)} - P)\,\frac{\partial w}{\partial x} \\ \delta \phi_x:&\quad M_{xx}^{(1)} \end{aligned} \tag{2.26}$$

2.3 Generalized Force-Displacement Relations

As previously mentioned, for conventional theories we have $\varepsilon_{zz} = 0$ and we use the one-dimensional stress strain relations (i.e., neglecting the Poisson effect, $\nu = 0$)

$$\sigma_{xx} = E\varepsilon_{xx}, \quad \sigma_{xz} = G\gamma_{xz} \tag{2.27}$$

Hence the stress resultants are related to the displacements by

$$\begin{Bmatrix} M_{xx}^{(0)} \\ M_{xx}^{(1)} \\ M_{xz}^{(0)} \end{Bmatrix} = \int_A \begin{Bmatrix} \sigma_{xx} \\ z\sigma_{xx} \\ \sigma_{xz} \end{Bmatrix} dA = \begin{Bmatrix} A_{11}\,\varepsilon_{xx}^{(0)} + B_{11}\,\varepsilon_{xx}^{(1)} \\ B_{11}\,\varepsilon_{xx}^{(0)} + D_{11}\,\varepsilon_{xx}^{(1)} \\ A_{xz}\gamma_{xz}^{(0)} \end{Bmatrix} - \begin{Bmatrix} X_T^{(0)} \\ X_T^{(1)} \\ 0 \end{Bmatrix} \tag{2.28}$$

where

$$
\begin{gathered}
A_{11} = \int_A E\, dA, \quad B_{11} = \int_A Ez\, dA, \quad D_{11} = \int_A Ez^2\, dA, \\
A_{xz} = K_s \int_A G\, dA, \quad X_T^{(0)} = \int_A E\, \alpha\, \Delta T\, dA, \quad X_T^{(1)} = \int_A E\, z\, \alpha\, \Delta T\, dA
\end{gathered}
\tag{2.29}
$$

Here K_s denotes the shear correction coefficient that appears only in the TBT. Note also that $\gamma_{xz}^{(0)} = 0$ for the conventional EBT.

2.4 Static Bending

2.4.1 Conventional EBT Model

For the case of static bending, we omit all terms that contain time derivatives. The equations for the conventional EBT are

$$
-\frac{dM_{xx}^{(0)}}{dx} = f_x \tag{2.30}
$$

$$
-\frac{d^2 M_{xx}^{(1)}}{dx^2} - \frac{d}{dx}\left(M_{xx}^{(0)} \frac{dw}{dx} - P\frac{dw}{dx}\right) = q \tag{2.31}
$$

where

$$
M_{xx}^{(0)} = A_{11}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - B_{11}\,\frac{d^2w}{dx^2} - X_T^{(0)} \tag{2.32}
$$

$$
M_{xx}^{(1)} = B_{11}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - D_{11}\,\frac{d^2w}{dx^2} - X_T^{(1)} \tag{2.33}
$$

2.4.2 Conventional TBT Model

Like the original equations of motion for the TBT case, there are three equations governing static bending since we must account for the extra rotation variable

$\phi_x(x,t)$. This yields the following equations for static bending:

$$-\frac{dM_{xx}^{(0)}}{dx} = f_x \tag{2.34}$$

$$-\frac{d}{dx}\left(M_{xz}^{(0)} + M_{xx}^{(0)}\frac{dw}{dx} - P\frac{dw}{dx}\right) = q \tag{2.35}$$

$$-\frac{dM_{xx}^{(1)}}{dx} + M_{xz}^{(0)} = 0 \tag{2.36}$$

where

$$M_{xx}^{(0)} = A_{11}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + B_{11}\left(\frac{d\phi_x}{dx}\right) - X_T^{(0)} \tag{2.37}$$

$$M_{xx}^{(1)} = B_{11}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + D_{11}\left(\frac{d\phi_x}{dx}\right) - X_T^{(1)} \tag{2.38}$$

$$M_{xz}^{(0)} = A_{xz}\left(\phi_x + \frac{dw}{dx}\right) \tag{2.39}$$

2.4.3 Natural Vibration

When dealing with the case of natural vibration for both theories, we set all externally applied forces except for the axial compressive load P to zero and seek a solution which is periodic in nature:

$$u(x,t) = U(x)\, e^{i\omega t}, \quad w(x,t) = W(x)\, e^{i\omega t}, \quad \phi_x(x,t) = \Phi(x)\, e^{i\omega t} \tag{2.40}$$

In addition, we assume that there is no damping (i.e., $\hat{\mu} = 0$) and we omit thermal effects. The resulting equations for the various theories are summarized next.

2.4.3.1 Conventional EBT

The equations governing natural vibration according to the conventional EBT are obtained by setting

$$-\omega^2\left(m_0 U - m_1\frac{dW}{dx}\right) - \frac{d\tilde{M}_{xx}^{(0)}}{dx} = 0 \quad (2.41)$$

$$-\omega^2\left(m_0 W + m_1\frac{dU}{dx} - m_2\frac{d^2W}{dx^2}\right) - \frac{d^2\tilde{M}_{xx}^{(1)}}{dx^2} - \frac{d}{dx}\left[\left(\tilde{M}_{xx}^{(0)} - P\right)\frac{dW}{dx}\right] = 0 \quad (2.42)$$

where ($\tilde{M}_{xx}^{(0)}$ and $\tilde{M}_{xx}^{(1)}$) are defined by Eqs. (2.32) and (2.33) in which (u, w, θ_x) is replaced by $(U, W, -\frac{dW}{dx})$.

2.4.3.2 Conventional TBT

For the TBT case, we get:

$$-\omega^2\left(m_0 U + m_1\Phi_x\right) - \frac{d\tilde{M}_{xx}^{(0)}}{dx} = 0 \quad (2.43)$$

$$-\omega^2 m_0 W - \frac{d}{dx}\left(\tilde{M}_{xx}^{(0)}\frac{dW}{dx} + \tilde{M}_{xz}^{(0)} - P\frac{dW}{dx}\right) = 0 \quad (2.44)$$

$$-\omega^2\left(m_1 U + m_2\Phi_x\right) - \frac{d\tilde{M}_{xx}^{(1)}}{dx} + \tilde{M}_{xz}^{(0)} = 0 \quad (2.45)$$

2.4.4 Buckling

In the case of buckling under an axial compressive load P, we set all time derivative terms and externally applied mechanical and thermal forces to zero to obtain the governing equations. These are outlined for various theories in the following sections (one can obtain these equations directly from the governing equations of natural vibration by omitting the frequency terms). In this section (U, W, Φ_x) denote the solutions of the onset of buckling.

2.4.4.1 Conventional EBT

For the equations concerning the buckling application, they take the same form as the equations of bending, with the exception of the applied loading. We are now only concerned with the axially applied loading P with no transverse loading q and no $\bar{f}_x$ component. Our buckling equations become

$$-\frac{d\tilde{M}_{xx}^{(0)}}{dx} = 0 \tag{2.46}$$

$$-\frac{d^2\tilde{M}_{xx}^{(1)}}{dx^2} - \frac{d}{dx}\left[\left(\tilde{M}_{xx}^{(0)} - P\right)\frac{dW}{dx}\right] = 0 \tag{2.47}$$

2.4.4.2 Conventional TBT

The equations governing buckling of beams according to the conventional TBT become:

$$-\frac{d\tilde{M}_{xx}^{(0)}}{dx} = 0 \tag{2.48}$$

$$-\frac{d}{dx}\left(\tilde{M}_{xx}^{(0)}\frac{dW}{dx} + \tilde{M}_{xz}^{(0)} - P\frac{dW}{dx}\right) = 0 \tag{2.49}$$

$$-\frac{d\tilde{M}_{xx}^{(1)}}{dx} + \tilde{M}_{xz}^{(0)} = 0 \tag{2.50}$$

In the case of beams with material distribution symmetric about the x-axis (i.e., $B_{ij} = 0$ and $m_1 = 0$), the conventional theories can be shown to admit analytical solutions under certain conditions, as discussed in the next section.

2.5 Elimination of the Axial Displacement

2.5.1 Preliminary Comments

In this section we discuss a strategy to eliminate the axial displacement $u(x,t)$ from the governing equations of motion and absorb the von Kármán nonlinear terms (in terms of the transverse deflection) into a constant. Here we assume that the beams considered are such that $B_{11} = 0$ and $m_1 = 0$. We make the following assumptions:

(1) There are no thermal effects.

(2) Terms involving the time derivatives such as

$$m_0 \frac{\partial^2 u}{\partial t^2} \tag{2.51}$$

are very small compared to the rest of the terms in the equation of motion associated with the x-direction and, therefore, can be neglected.

(3) The beam is supported at $x = 0$ and $x = l$ such that $u(0) = u(l) = 0$.

2.5.2 Conventional EBT

For this theory Eqs. (2.14) and (2.15) simplify to

$$-\frac{\partial M_{xx}^{(0)}}{\partial x} = f_x \tag{2.52}$$

$$-\frac{\partial^2 M_{xx}^{(1)}}{\partial x^2} - m_2 \frac{\partial^4 w}{\partial t^2 \partial x^2} + m_0 \frac{\partial^2 w}{\partial t^2} + \hat{\mu} \frac{\partial w}{\partial t} - \frac{\partial}{\partial x}\left(M_{xx}^{(0)} \frac{\partial w}{\partial x} - P \frac{\partial w}{\partial x} \right) = q \tag{2.53}$$

Integrating Eq. (2.52) with respect to x, we obtain

$$M_{xx}^{(0)} + \int f_x \, dx + C(t) = 0 \tag{2.54}$$

or

$$\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{A_{11}}\left[\int f_x \, dx + C(t)\right] = 0 \tag{2.55}$$

Integrating the above expression from 0 to l and noting that $u(0) = u(l) = 0$, we obtain

$$\frac{1}{2}\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx + \int_0^l \left[\frac{1}{A_{11}} \int f_x \, dx\right] dx + SC(t) = 0$$

or

$$C(t) = -\frac{1}{S}\left\{\frac{1}{2}\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx + \int_0^l \left[\frac{1}{A_{11}} \int f_x \, dx\right] dx\right\} \tag{2.56}$$

where

$$S = \int_0^l \frac{1}{A_{11}} dx = \frac{l}{A_{11}} \tag{2.57}$$

Note the identity

$$\begin{aligned} -\frac{\partial}{\partial x}\left(M_{xx}^{(0)} \frac{\partial w}{\partial x}\right) &= -\frac{\partial M_{xx}^{(0)}}{\partial x}\frac{\partial w}{\partial x} - M_{xx}^{(0)} \frac{\partial^2 w}{\partial x^2} \\ &= f_x \frac{\partial w}{\partial x} + \left(\int f_x \, dx\right)\frac{\partial^2 w}{\partial x^2} + C(t)\frac{\partial^2 w}{\partial x^2} \end{aligned} \tag{2.58}$$

where Eqs. (2.52) and (2.54) are utilized in arriving at the second line of Eq. (2.58). Using Eqs. (2.56) and (2.58) in Eq. (2.53), we arrive at the following equation of motion governing the transverse displacement w:

$$m_0 \frac{\partial^2 w}{\partial t^2} - m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^2}{\partial x^2}\left(D_{11}\frac{\partial^2 w}{\partial x^2}\right) + P\frac{\partial^2 w}{\partial x^2} + \hat{\mu}\frac{\partial w}{\partial t}$$

$$-\frac{1}{S}\frac{\partial^2 w}{\partial x^2}\left\{\frac{1}{2}\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx + \int_0^l \frac{1}{A_{11}}\left(\int f_x\, dx\right) dx\right\}$$
$$+f_x\frac{\partial w}{\partial x} + \left(\int f_x\, dx\right)\frac{\partial^2 w}{\partial x^2} - q = 0 \qquad (2.59)$$

or

$$\frac{\partial^2}{\partial t^2}\left(m_0 w - m_2\frac{\partial^2 w}{\partial x^2}\right) + \frac{\partial^2}{\partial x^2}\left(D_{11}\frac{\partial^2 w}{\partial x^2}\right) + P\frac{\partial^2 w}{\partial x^2} - q$$
$$+\hat{\mu}\frac{\partial w}{\partial t} - \frac{1}{2S}\frac{\partial^2 w}{\partial x^2}\left\{\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx + \int_0^l \frac{1}{A_{11}}\left(\int f_x\, dx\right) dx\right\}$$
$$+\frac{\partial}{\partial x}\left[\left(\int f_x\, dx\right)\frac{\partial w}{\partial x}\right] = 0 \qquad (2.60)$$

Equation (2.60) is linear because of the fact

$$\frac{1}{2S}\left[\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx\right]$$

is a constant, which is not known because w is not known.

For the case of free vibrations, Eq. (2.60) reduces to

$$-\omega^2\left(m_0 W - m_2\frac{d^2 W}{dx^2}\right) + \frac{d^2}{dx^2}\left(D_{11}\frac{d^2 W}{dx^2}\right) + P\frac{d^2 W}{dx^2} + \hat{\mu} i\omega W$$
$$-\frac{1}{2S}\frac{d^2 W}{dx^2}\left[\int_0^l \left(\frac{dW}{dx}\right)^2 dx\right] = 0 \qquad (2.61)$$

For buckling under axial load P, Eq. (2.60) becomes (omitting the damping term)

$$\frac{d^2}{dx^2}\left(D_{11}\frac{d^2 W}{dx^2}\right) + P\frac{d^2 W}{dx^2} - \frac{1}{2S}\frac{d^2 W}{dx^2}\left[\int_0^l \left(\frac{dW}{dx}\right)^2 dx\right] = 0 \qquad (2.62)$$

2.5.3 Conventional TBT

For the TBT case, Eqs. (2.23)–(2.25) simplify to

$$\frac{\partial M_{xx}^{(0)}}{\partial x} = -f_x \tag{2.63}$$

$$-\frac{\partial}{\partial x}\left(M_{xz}^{(0)} + M_{xx}^{(0)}\frac{\partial w}{\partial x}\right) + P\frac{\partial^2 w}{\partial x^2} + \hat{\mu}\frac{\partial w}{\partial t} + m_0\frac{\partial^2 w}{\partial t^2} = q \tag{2.64}$$

$$-\frac{\partial M_{xx}^{(1)}}{\partial x} + M_{xz}^{(0)} + m_2\frac{\partial^2 \phi_x}{\partial t^2} = 0 \tag{2.65}$$

Eqs. (2.54)–(2.58) are also valid for the TBT. Hence, Eqs. (2.64) and (2.65) take the form

$$\begin{aligned}-\frac{\partial}{\partial x}\left[A_{xz}\left(\frac{\partial w}{\partial x} + \phi_x\right)\right] - \frac{1}{S}\frac{\partial^2 w}{\partial x^2}\left\{\frac{1}{2}\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx\right.\\ \left. + \int_0^l \frac{1}{A_{11}}\left(\int f_x\, dx\right) dx\right\} + P\frac{\partial^2 w}{\partial x^2} + \hat{\mu}\frac{\partial w}{\partial t} - q + m_0\frac{\partial^2 w}{\partial t^2}\\ + \frac{\partial}{\partial x}\left[\left(\int f_x\, dx\right)\frac{\partial w}{\partial x}\right] = 0\end{aligned} \tag{2.66}$$

$$-\frac{\partial}{\partial x}\left(D_{11}\frac{\partial \phi_x}{\partial x}\right) + \left[A_{xz}\left(\phi_x + \frac{\partial w}{\partial x}\right)\right] + m_2\frac{\partial^2 \phi_x}{\partial t^2} = 0 \tag{2.67}$$

For the free vibration case, Eqs. (2.66) and (2.67) reduce to

$$\begin{aligned}-\omega^2 m_0 W + \hat{\mu} i\omega W - \frac{1}{S}\frac{d^2 W}{dx^2}\left[\frac{1}{2}\int_0^l \left(\frac{dW}{dx}\right)^2 dx\right] + P\frac{d^2 W}{dx^2}\\ - \frac{d}{dx}\left[A_{xz}\left(\frac{dW}{dx} + \Phi_x\right)\right] = 0\end{aligned} \tag{2.68}$$

$$-\omega^2 m_2 \Phi_x - \frac{d}{dx}\left(D_{11}\frac{d\Phi_x}{dx}\right) + \left[A_{xz}\left(\frac{dW}{dx} + \Phi_x\right)\right] = 0 \tag{2.69}$$

and by omitting the damping term for buckling under axial loading P, we get

$$P\frac{d^2W}{dx^2} - \frac{1}{S}\frac{d^2W}{dx^2}\left[\frac{1}{2}\int_0^l \left(\frac{dW}{dx}\right)^2 dx\right] - \frac{d}{dx}\left[A_{xz}\left(\frac{dW}{dx} + \Phi_x\right)\right] = 0 \quad (2.70)$$

$$-\frac{d}{dx}\left(D_{11}\frac{d\Phi_x}{dx}\right) + \left[A_{xz}\left(\frac{dW}{dx} + \Phi_x\right)\right] = 0 \quad (2.71)$$

for both displacements w and ϕ_x, respectively.

2.6 Nondimensionalized Governing Equations

2.6.1 Nondimensional Variables and Parameters

We now consider a beam of uniform cross-sectional area A, moment of inertia I, length l, constant elastic modulus E, and mass density ρ, and is subjected to a transverse load ($f_x = 0$)

$$F_z^{(0)} = q = F(x)\,\cos\omega t \quad (2.72)$$

This gives us

$$m_0 = \rho A, \quad m_2 = \rho I, \quad A_{11} = EA, \quad D_{11} = EI, \quad A_{xz} = GAK_s, \quad S = \frac{l}{EA}$$

Let us introduce the following nondimensional quantities ([32] and [33]):

$$\xi = \frac{x}{l}\,, \quad v = \frac{w}{r}\,, \quad \psi = \frac{l}{r}\,\phi_x$$

$$\tau = t\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}\,, \quad \Omega = \omega\, l^2\sqrt{\frac{\rho A}{EI}}\,, \quad r = \sqrt{\frac{I}{A}}, \quad s^2 = \frac{EI}{K_s GAl^2} \quad (2.73)$$

$$\Lambda = \frac{Pl^2}{EI}\,, \quad q = \frac{Fl^4}{rEI}\,, \quad \mu = \frac{\hat{\mu}l^2}{\sqrt{\rho EAI}}$$

This aides in providing a better general solution that can be applied over a variety of studies, with the understanding that these variables are used for beams that are considered isotropic.

2.6.2 Conventional EBT

For this case, Eq. (2.60) takes the form

$$\rho A \frac{\partial^2 w}{\partial t^2} - \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} + EI \frac{\partial^4 w}{\partial x^4} + P \frac{\partial^2 w}{\partial x^2} + \hat{\mu} \frac{\partial w}{\partial t} - F_0(x) \cos \omega t$$
$$- \frac{EA}{2l} \frac{\partial^2 w}{\partial x^2} \left[\int_0^l \left(\frac{\partial w}{\partial x} \right)^2 dx \right] = 0 \qquad (2.74)$$

which, in terms of the nondimensionalized variables, takes the form (note that $\omega\, t = \Omega\, \tau$)

$$\frac{\partial^2 v}{\partial \tau^2} - \frac{r^2}{l^2} \frac{\partial^4 v}{\partial \xi^2 \partial \tau^2} + \frac{\partial^4 v}{\partial \xi^4} + \Lambda \frac{\partial^2 v}{\partial \xi^2} + \mu \frac{\partial v}{\partial \tau} - \frac{1}{2} \frac{\partial^2 v}{\partial \xi^2} \left[\int_0^1 \left(\frac{\partial v}{\partial \eta} \right)^2 d\eta \right] = q(\xi) \cos \Omega \tau \quad (2.75)$$

for $0 < \xi < 1$ and $\tau > 0$.

2.6.3 Conventional TBT

For the conventional Timoshenko case, Eqs. (2.66) and (2.67) become

$$\frac{\partial^2 v}{\partial \tau^2} - \frac{1}{s^2} \frac{\partial}{\partial \xi} \left(\frac{\partial v}{\partial \xi} + \psi \right) + \mu \frac{\partial v}{\partial \tau} + \Lambda \frac{\partial^2 v}{\partial \xi^2}$$
$$- \frac{1}{2} \frac{\partial^2 v}{\partial \xi^2} \left[\int_0^1 \left(\frac{\partial v}{\partial \eta} \right)^2 d\eta \right] = q(\xi) \cos \Omega \tau \qquad (2.76)$$
$$\frac{r^2}{l^2} \frac{\partial^2 \psi}{\partial \tau^2} - \frac{\partial^2 \psi}{\partial \xi^2} + \frac{1}{s^2} \left(\frac{\partial v}{\partial \xi} + \psi \right) = 0 \qquad (2.77)$$

Thus, one must solve for both v and ψ in the case of the Timoshenko beam theory.

2.7 Analytical Solutions for Buckling

2.7.1 Conventional EBT

2.7.1.1 Governing Equations

For the EBT case of buckling without damping, Eq. (2.75) reduces to

$$\frac{d^4 v}{d\xi^4} + \lambda^2 \frac{d^2 v}{d\xi^2} = 0 \tag{2.78}$$

where

$$\lambda^2 = \Lambda - \Gamma, \quad \Gamma = \frac{1}{2}\int_0^1 \left(\frac{dv}{d\eta}\right)^2 d\eta \tag{2.79}$$

The general solution to Eq. (2.78) for the transverse displacement is [40]

$$v(\xi) = c_1 \sin\lambda\xi + c_2 \cos\lambda\xi + c_3\,\xi + c_4 \tag{2.80}$$

where c_1, c_2, c_3, and c_4 are constants to be determined using the boundary conditions.

2.7.1.2 Types of Boundary Conditions

The following types of classical boundary conditions are considered first. Note that for the Euler–Bernoulli beam theory, two boundary conditions at each edge are required (see Figure 2.3).

Free:

$$\frac{d^2 v}{d\xi^2} = 0, \quad \frac{d^3 v}{d\xi^3} + \lambda^2 \frac{dv}{d\xi} = 0 \tag{2.81}$$

Hinged:

$$v = 0, \quad \frac{d^2 v}{d\xi^2} = 0 \tag{2.82}$$

Clamped:

$$v = 0, \quad \frac{dv}{d\xi} = 0 \tag{2.83}$$

Three specific examples, namely, hinged–hinged, clamped–clamped and clamped–hinged beams are considered next. Note that beams with a free edge cannot be analyzed because assumption (3) of Section 2.5 is violated.

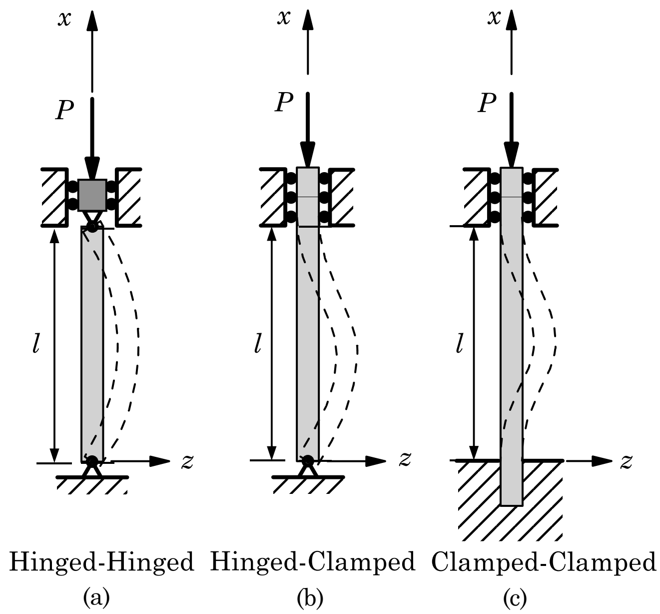

Figure 2.3: Buckling of beams with various boundary conditions.

2.7.1.3 Hinged–Hinged Beams

For a simply supported beam (at $\xi = 0, 1$), we have

$$v = 0, \quad \frac{d^2v}{d\xi^2} = 0 \tag{2.84}$$

Use of the boundary conditions on v gives

$$\begin{aligned}
v(0) = 0: &\quad c_2 + c_4 = 0 \ \text{ or } \quad c_4 = -c_2 \\
\left.\frac{d^2v}{d\xi^2}\right|_{\xi=0} = 0: &\quad -\lambda^2 c_2 = 0 \text{ or } \quad c_2 = 0 \\
v(1) = 0: &\quad c_1 \sin\lambda + c_3 = 0 \\
\left.\frac{d^2v}{d\xi^2}\right|_{x=1} = 0: &\quad -\lambda^2 c_1 \sin\lambda = 0
\end{aligned}$$

From the above equations, it follows that

$$c_1 \sin\lambda = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 0$$

The solution for the first constant implies that either $c_1 = 0$ or (and) $\sin\lambda = 0$. For a nontrivial solution v, we must have

$$\sin\lambda = 0 \quad \text{which implies} \quad \lambda_n = \sqrt{\Lambda - \Gamma} = n\pi \quad \text{or} \quad n^2\pi^2 = \Lambda - \Gamma \tag{2.85}$$

The mode shapes are ($c_{1n} = c_n$)

$$v_n(\xi) = c_n \sin\lambda_n\xi, \quad n = 1, 2, 3, \cdots \tag{2.86}$$

where c_n is a constant to be determined using the condition

$$\lambda_n^2 = \Lambda - \frac{1}{2}\int_0^1 \left(\frac{dv_n}{d\eta}\right)^2 d\eta \tag{2.87}$$

which, for the present case, takes the form

$$\lambda_n^2 = \Lambda - \frac{c_n^2\lambda_n^2}{2}\int_0^1 \cos^2\lambda_n\eta \, d\eta \quad \Rightarrow \quad c_n = \pm 2\sqrt{\frac{\Lambda}{\lambda_n^2} - 1}, \quad n = 1, 2, 3, \cdots \tag{2.88}$$

where we have used the identity

$$\int_0^1 \cos^2 \lambda_n \eta \, d\eta = \frac{1}{2} \int_0^1 (1 + \cos 2\lambda_n \eta) \, d\eta = \frac{1}{2} \left[\eta + \frac{\sin 2\lambda_n \eta}{\lambda_n} \right]_0^1 = \frac{1}{2} \left(1 + \frac{\sin 2\lambda_n}{\lambda_n} \right)$$

and $\sin \lambda_n = 0$. Note that $v_n(\xi)$ in Eq. (2.86) contains both symmetric ($n = 2, 4, \cdots$) and unsymmetric ($n = 1, 3, \cdots$) mode shapes.

2.7.1.4 Clamped–Clamped Beams

For a beam clamped at both ends (at $\xi = 0, 1$), we have

$$v = 0, \quad \frac{dv}{d\xi} = 0 \tag{2.89}$$

Use of this set of boundary conditions on v yields

$$\begin{aligned} v(0) = 0: &\quad c_2 + c_4 = 0 \quad \text{or} \quad c_4 = -c_2 \\ \left.\frac{dv}{d\xi}\right|_{\xi=0} = 0: &\quad \lambda c_1 + c_3 = 0 \quad \text{or} \quad c_3 = -\lambda c_1 \\ v(1) = 0: &\quad c_1 \sin\lambda + c_2 \cos\lambda + c_3 + c_4 = 0 \\ \left.\frac{dv}{d\xi}\right|_{\xi=1} = 0: &\quad \lambda \left(c_1 \cos\lambda - c_2 \sin\lambda \right) + c_3 = 0 \end{aligned} \tag{2.90}$$

Using the first two equations, c_3 and c_4 can be eliminated from the last two equations. We get

$$c_1 (\sin\lambda - \lambda) + c_2 (\cos\lambda - 1) = 0 \;\; \text{and} \;\; c_1 (\cos\lambda - 1) - c_2 \sin\lambda = 0 \tag{2.91}$$

For nonzero transverse deflection (i.e., for nonzero values of c_1 and c_2), we require that the determinant of the above pair of equations be zero:

$$\begin{vmatrix} \sin\lambda - \lambda & \cos\lambda - 1 \\ \cos\lambda - 1 & -\sin\lambda \end{vmatrix} = 0$$

or

$$\lambda \sin\lambda + 2 \cos\lambda - 2 = 0 \tag{2.92}$$

This nonlinear (transcendental) equation can be solved by an iterative (Newton's) method for various roots of the equation. The first five roots of this equation are $\lambda_1 = 2\pi$, $\lambda_2 = 8.9868$, $\lambda_3 = 4\pi$, $\lambda_4 = 15.4505$ and $\lambda_5 = 6\pi$.

Note that the expression in Eq. (2.92) can be expressed as a product of two expressions (so that the determination of the roots is made easier) using the identities

$$\cos 2\theta = 1 - 2\sin^2\theta, \quad \sin 2\theta = 2 \sin\theta \cos\theta \tag{2.93}$$

We obtain

$$\begin{aligned} 0 &= \lambda \sin\lambda + 2 \cos\lambda - 2 \\ &= 2\lambda \sin\frac{\lambda}{2} \cos\frac{\lambda}{2} + 2 - 4\sin^2\frac{\lambda}{2} - 2 \\ &= 4\sin\frac{\lambda}{2}\left(\frac{\lambda}{2}\cos\frac{\lambda}{2} - \sin\frac{\lambda}{2}\right) \end{aligned} \tag{2.94}$$

which shows that there are two sets of roots, one corresponding to symmetric modes

$$\sin\frac{\lambda}{2} = 0 \;\Rightarrow\; \lambda_n = 2n\pi, \quad n = 1, 2, 3, \cdots \tag{2.95}$$

and the other corresponds to unsymmetric modes

$$\tan\frac{\lambda}{2} = \frac{\lambda}{2} \tag{2.96}$$

The mode shapes are determined by using Eqs. (2.80), (2.95), and (2.96). In particular, we have [c_1 is expressed in terms of c_2 using the first expression in Eq. (120)]

$$c_4 = -c_2, \quad c_3 = -\lambda\, c_1, \quad c_1 = \left(\frac{1-\cos\lambda}{\sin\lambda - \lambda}\right) c_2 \tag{2.97}$$

so that we can write

$$\begin{aligned} v_n(\xi) &= c_{1n}\,\sin\lambda_n\xi + c_{2n}\,\cos\lambda_n\xi + c_{3n}\,\xi + c_{4n} \\ &= c_{1n}\,(\sin\lambda_n\xi - \lambda_n\xi) + c_{2n}\,(\cos\lambda_n\xi - 1) \\ &= c_n\left[1 - \cos\lambda_n\xi - (\sin\lambda_n\xi - \lambda_n\xi)\left(\frac{1-\cos\lambda_n}{\sin\lambda_n - \lambda_n}\right)\right] \end{aligned} \tag{2.98}$$

where $c_n = -c_{2n}$ is a constant to be determined by following the same procedure as in the case of a hinged–hinged beam. An alternative expression for $v_n(\xi)$ is [c_2 is expressed in terms of c_1 using the second equation in Eq. (2.91)]

$$v_n(\xi) = \hat{c}_n\left[1 + \cos\lambda_n(1-\xi) - \lambda_n\sin\lambda_n\xi - \cos\lambda_n\xi - \cos\lambda_n\right] \tag{2.99}$$

where $\hat{c}_n = c_{1n}/\sin\lambda_n$ is to be determined.

In view of Eq. (2.98), the symmetric mode shapes are given by $(1-\cos 2n\pi = 0)$

$$v_n(\xi) = c_n(1-\cos\lambda_n\xi), \quad \lambda_n = 2n\pi, \quad n = 1, 2, 3, \cdots \tag{2.100}$$

The unsymmetric mode shapes can be determined using Eq. (2.96) and the following identity [where we use Eqs. (2.93) and (2.96)]:

$$\frac{1-\cos\lambda}{\sin\lambda-\lambda} = \frac{2\sin^2\frac{\lambda}{2}}{2\sin\frac{\lambda}{2}\cos\frac{\lambda}{2}-2\tan\frac{\lambda}{2}} = \frac{\sin\frac{\lambda}{2}\cos\frac{\lambda}{2}}{\cos^2\frac{\lambda}{2}-1} = -\cot\frac{\lambda}{2} = -\frac{2}{\lambda} \tag{2.101}$$

Therefore, the mode shapes for unsymmetric case are

$$v_n(\xi) = c_n\Big(1-2\xi-\cos\lambda_n\xi+\frac{2}{\lambda_n}\sin\lambda_n\xi\Big) \tag{2.102}$$

with λ_n determined from Eq. (2.96).

To determine the constant c_n appearing in Eq. (2.98) [the same as that appearing in Eqs. (2.100) and (2.102)], we use the condition in Eq. (2.87) and determine it with the help of $v_n(\xi)$ in Eq. (2.100) (because it is algebraically simpler)

$$\lambda_n^2 = \Lambda - \frac{c_n^2\lambda_n^2}{2}\int_0^1 \cos^2\lambda_n\eta\,d\eta \quad\Rightarrow\quad c_n = \pm 2\sqrt{\frac{\Lambda}{\lambda_n^2}-1}, \quad n=1,2,3,\cdots \tag{2.103}$$

2.7.1.5 Clamped–Hinged Beams

For a beam clamped at $\xi=0$ and hinged at $\xi=1$, the boundary conditions become

$$\text{At}\ \ \xi=0:\quad v=0,\quad \frac{dv}{d\xi}=0;\quad \text{At}\ \ \xi=1:\quad v=0,\quad \frac{d^2v}{d\xi^2}=0 \tag{2.104}$$

Use of this combination of boundary conditions on v gives

$$\begin{aligned} v(0) = 0: &\quad c_2 + c_4 = 0 \quad \text{or} \quad c_4 = -c_2 \\ \left.\frac{dv}{d\xi}\right|_{\xi=0} = 0: &\quad \lambda c_1 + c_3 = 0 \quad \text{or} \quad c_3 = -\lambda c_1 \\ v(1) = 0: &\quad c_1 \sin\lambda + c_2 \cos\lambda + c_3 + c_4 = 0 \\ \left.\frac{d^2v}{d\xi^2}\right|_{\xi=1} = 0: &\quad \lambda^2 \left(c_1 \sin\lambda + c_2 \cos\lambda\right) = 0 \end{aligned} \tag{2.105}$$

Thus we have $c_4 = -c_3 = -c_2 = \lambda\, c_1$, giving the characteristic equation

$$\tan\lambda - \lambda = 0 \tag{2.106}$$

The first five roots of this equation are $\lambda_1 = 4.4934$, $\lambda_2 = 7.7253$, $\lambda_3 = 10.9041$, $\lambda_4 = 14.0662$ and $\lambda_5 = 17.2208$.

The mode shapes for the clamped–hinged beam are

$$\begin{aligned} v_n(\xi) &= c_{1n}\, \sin\lambda_n\xi + c_{2n}\, \cos\lambda_n\xi + c_{3n}\,\xi + c_{4n} \\ &= c_n \left(\sin\lambda_n\xi - \lambda_n \cos\lambda_n\xi - \lambda_n\xi + \lambda_n\right) \end{aligned} \tag{2.107}$$

and the constant $c_n = c_{1n}$ is computed from Eq. (2.87) as

$$\begin{aligned} \lambda_n^2 = \Lambda - \frac{c_n^2\lambda_n^2}{2} \int_0^1 \big(&1 + \cos^2\lambda_n\eta + 2\lambda_n \cos\lambda_n\eta \sin\lambda_n\eta - 2\lambda_n \sin\lambda_n\eta \\ &- 2\cos\lambda_n\eta + \lambda_n^2 \sin^2\lambda_n\eta\big)\, d\eta \\ c_n = \pm\sqrt{\frac{2(\Lambda - \lambda_n^2)}{\lambda_n^2[2 + \frac{\lambda_n^2}{2} + \cos\lambda_n(2\lambda_n - \frac{1}{2}) - 2\lambda_n]}}&, \quad n = 1, 2, 3, \cdots \end{aligned} \tag{2.108}$$

where we have used the following identities

$$\int_0^1 \cos^2 \lambda_n\eta \, d\eta = \frac{1}{2}\int_0^1 (1 + \cos 2\lambda_n\eta)\, d\eta = \frac{1}{2}\left[\eta + \frac{\sin 2\lambda_n\eta}{\lambda_n}\right]_0^1 = \frac{1}{2}\left(1 + \frac{\sin 2\lambda_n}{\lambda_n}\right),$$

$$2\lambda_n \int_0^1 \cos\lambda_n\eta sin\lambda_n\eta \, d\eta = \lambda \int_0^1 \sin 2\lambda_n\eta \, d\eta = -\frac{1}{2}\Big[\cos\lambda_n\eta\Big]_0^1 = -\frac{1}{2}\cos\lambda_n + \frac{1}{2},$$

$$2\lambda_n \int_0^1 \sin\lambda_n\eta \, d\eta = -2\lambda_n \Big[\cos\lambda_n\eta\Big]_0^1 = -2\lambda_n\cos\lambda_n + 2\lambda_n,$$

$$2\int_0^1 \cos\lambda_n\eta \, d\eta = 2\Big[\sin\lambda_n\eta\Big]_0^1 = 0,$$

$$\lambda_n^2 \int_0^1 \sin^2\lambda_n\eta \, d\eta = \frac{\lambda_n^2}{2}\int_0^1 (1 - \cos 2\lambda_n\eta)\, d\eta = \frac{\lambda_n^2}{2}\left[\eta - \frac{\sin 2\lambda_n\eta}{2\lambda_n}\right]_0^1 = \frac{\lambda_n^2}{2}$$

The relations among the c_i for various classical boundary conditions are listed [taken from [40]] along with the equation governing λ_n in Table 1. Although clamped–free and free–free cases are listed in Table 1, they do not correspond to the case in which the von Kármán nonlinearity is included [because assumption (3) of Section 2.5 is violated].

Table 1: Values of the constants c_i $(i = 1, 2, 3, 4)$ and eigenvalues λ_n for buckling of beams with various boundary conditions.

End conditions at $\xi = 0$ and $\xi = 1$	Constants*	Characteristic equation and values of λ_n
Hinged–Hinged	$c_1 \neq 0$ $c_2 = c_3 = c_4 = 0$	$\sin \lambda_n = 0,\ \lambda_n = n\pi$
Clamped–Clamped	$c_1 = (1 - \cos \lambda_n)/(\sin \lambda_n - \lambda_n)c_2$ $c_3 = -\lambda_n c_1, \quad c_4 = -c_2 \neq 0$	$\lambda_n \sin \lambda_n + 2 \cos \lambda_n - 2 = 0$ $\lambda_n = 2\pi, 8.9868, 4\pi, 15.4505, 6\pi, \cdots$
Clamped–Hinged	$c_1 \neq 0$ $c_4 = -c_3 = -c_2 = \lambda\, c_1$	$\tan \lambda_n - \lambda_n = 0$ $\lambda_n = 4.4934, 7.7253, 10.9041, 14.0662, 17.2208 \cdots$
Clamped–Free	$c_1 = c_3 = 0$ $c_2 = -c_4 \neq 0$	$\cos \lambda_n = 0,\ \lambda_n = (2n-1)\pi/2$
Free–Free	$c_1 = c_3 = 0$ $c_2 \neq 0,\ c_4 \neq 0$	$\sin \lambda_n = 0,\ \lambda_n = n\pi$

* See Eq. (2.80): $v(\xi) = c_1 \sin \lambda\xi + c_2 \cos \lambda\xi + c_3\xi + c_4$.

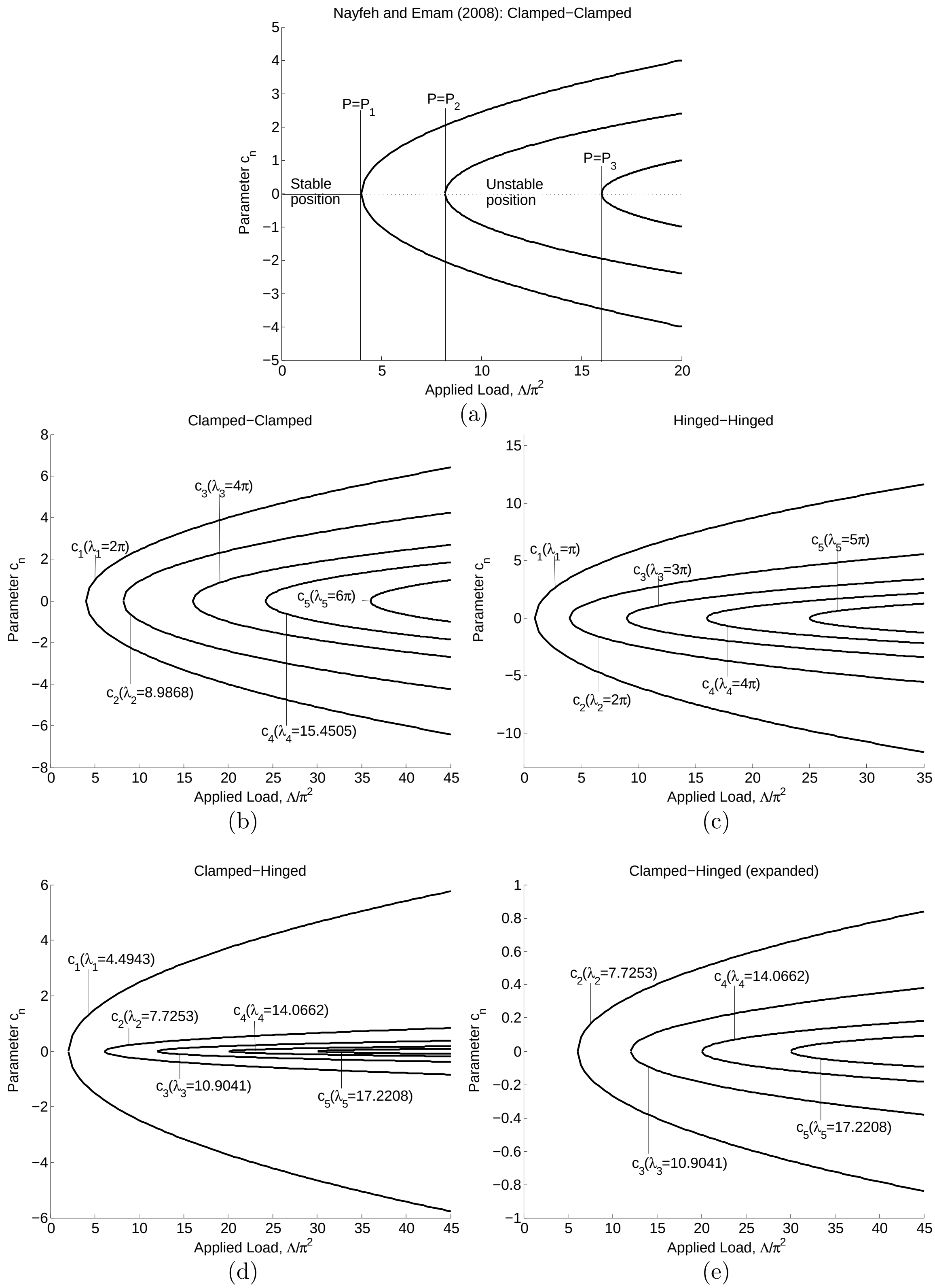

Figure 2.4: Constant parameter c_n vs. applied load for (a) clamped–clamped results from Nayfeh and Emam (2008), (b) clamped–clamped, (c) hinged–hinged, (d) clamped–hinged and (e) expanded view of clamped–hinged to see remaining bifurcation trends for the present study.

Figure 2.4 depicts the relation between the nondimensionalized applied load and the constant parameter c_n present for each general solution. The constant c_n can be seen as more of an amplitude constant for the transverse deflection $v_n(\xi)$. Due to the quadratic nature of c_n, there is zero amplitude for any applied load up to and equivalent to the analytical buckling load. Any applied load that is larger than the buckling load allows for a value of c_n to be obtained. From the plots, the bifurcation nature of c_n is evident, as seen from the derivations for the various boundary conditions. The clamped–clamped case in this study is in good agreement with Nayfeh and Emam (2008), and extends to include the fourth and fifth buckling modes. The deflection response provides good insight to the postbuckling behavior as a function of the end constraints.

To see the relationship between the applied load and the transverse deflection we can take a look at Figure 2.5. For the first five buckling modes of the clamped–hinged case we can see the transverse deflection of the conventional EBT beam at various locations along the beam.

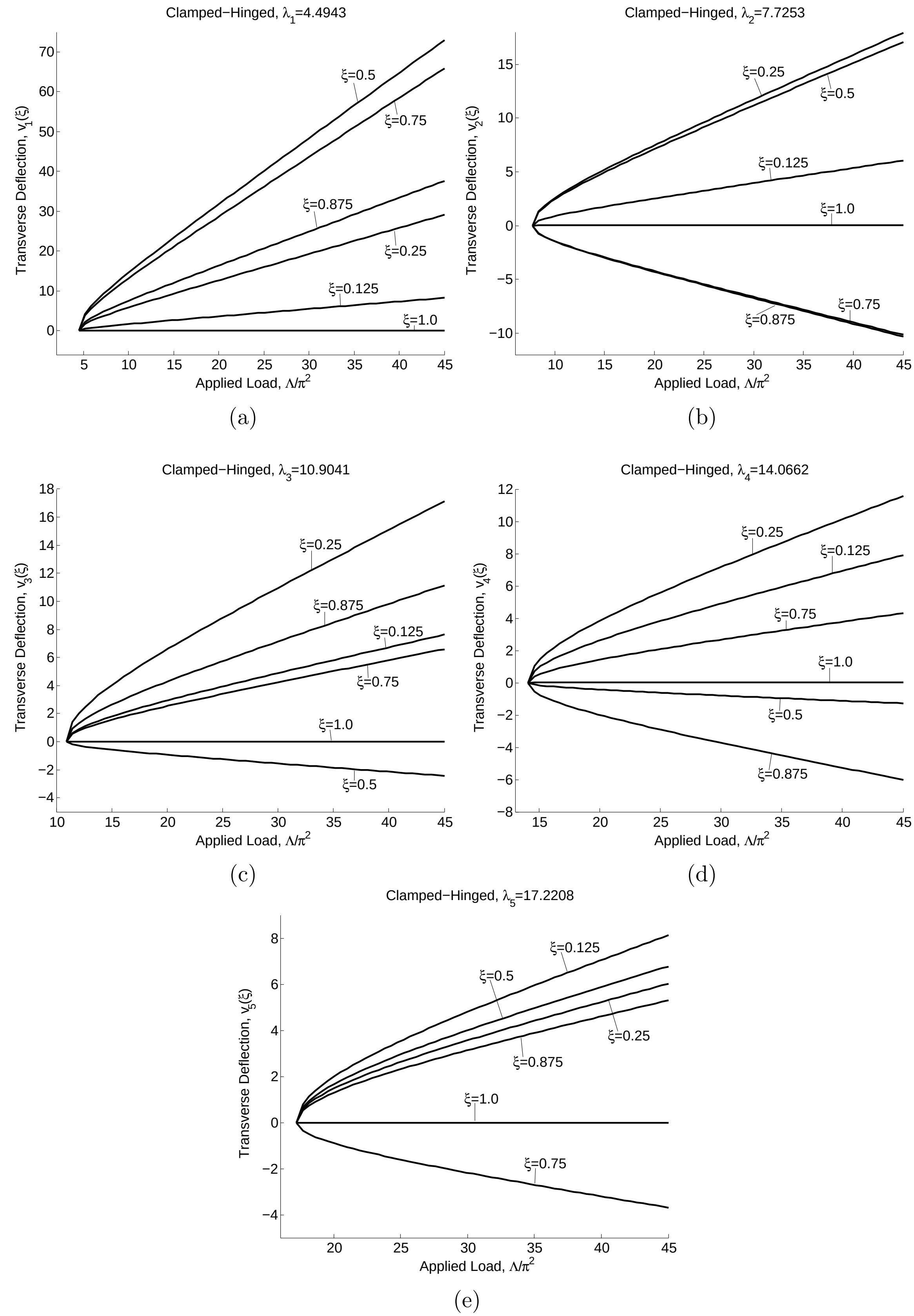

Figure 2.5: Load–deflection behavior for the first five buckling modes of the clamped–hinged case.

As well, some numerical results are presented in Tables 2–6 to explicitly describe the relation between the applied load, constant parameter c_n and transverse deflection $v_n(\xi)$ for the clamped–hinged case of the conventional EBT model:

Table 2: Transverse deflection as a function of loading Λ/π^2 and various values of ξ for the first buckling mode ($\lambda_1 = 4.4943$) of the clamped–hinged, conventional EBT beam.

		$v_1(\xi)$					
Λ/π^2	c_1	$\xi = 0.125$	$\xi = 0.25$	$\xi = 0.5$	$\xi = 0.75$	$\xi = 0.875$	$\xi = 1.0$
4.4943	0	0	0	0	0	0	0
5	0.6108	0.4042	1.4225	3.5674	3.2211	1.8393	0
7.5	1.6738	1.1075	3.8979	9.7754	8.8264	5.0401	0
10	2.4903	1.6477	5.7993	14.5439	13.1320	7.4986	0
12.5	3.2516	2.1515	7.5722	18.9901	17.1466	9.7910	0
15	3.9894	2.6397	9.2905	23.2992	21.0374	12.0127	0
17.5	4.7148	3.1197	10.9797	27.5358	24.8627	14.1970	0
20	5.4328	3.5947	12.6517	31.7288	28.6486	16.3589	0
22.5	6.1459	4.0665	14.3124	35.8935	32.4091	18.5062	0
25	6.8557	4.5362	15.9653	40.0388	36.1520	20.6435	0
27.5	7.5631	5.0042	17.6126	44.1702	39.8823	22.7735	0
30	8.2687	5.4711	19.2559	48.2911	43.6032	24.8982	0
32.5	8.9730	5.9371	20.8959	52.4042	47.3169	27.0188	0
35	9.6762	6.4024	22.5335	56.5110	51.0250	29.1363	0
37.5	10.3785	6.8671	24.1691	60.6129	54.7287	31.2511	0
40	11.0802	7.3314	25.8031	64.7107	58.4288	33.3639	0
42.5	11.7813	7.7953	27.4358	68.8053	62.1259	35.4751	0
45	12.4819	8.2589	29.0674	72.8972	65.8205	37.5848	0

Table 3: Transverse deflection as a function of loading Λ/π^2 and various values of ξ for the second buckling mode ($\lambda_2 = 7.7253$) of the clamped–hinged, conventional EBT beam.

		$v_2(\xi)$					
Λ/π^2	c_2	$\xi = 0.125$	$\xi = 0.25$	$\xi = 0.5$	$\xi = 0.75$	$\xi = 0.875$	$\xi = 1.0$
7.7253	0	0	0	0	0	0	0
10	0.2717	0.8659	2.5686	2.4464	-1.4555	-1.4780	0
12.5	0.4204	1.3401	3.9752	3.7860	-2.2526	-2.2874	0
15	0.5501	1.7534	5.2011	4.9536	-2.9473	-2.9929	0
17.5	0.6718	2.1413	6.3519	6.0497	-3.5994	-3.6551	0
20	0.7893	2.5157	7.4624	7.1073	-4.2287	-4.2941	0
22.5	0.9041	2.8817	8.5483	8.1415	-4.8440	-4.9189	0
25	1.0172	3.2423	9.6180	9.1603	-5.4501	-5.5344	0
27.5	1.1292	3.5991	10.6763	10.1682	-6.0498	-6.1434	0
30	1.2402	3.9530	11.7262	11.1682	-6.6448	-6.7476	0
32.5	1.3506	4.3049	12.7700	12.1623	-7.2363	-7.3482	0
35	1.4605	4.6551	13.8089	13.1518	-7.8250	-7.9460	0
37.5	1.5700	5.0041	14.8440	14.1376	-8.4115	-8.5416	0
40	1.6791	5.3520	15.8760	15.1205	-8.9964	-9.1355	0
42.5	1.7880	5.6990	16.9056	16.1011	-9.5798	-9.7279	0
45	1.8967	6.0454	17.9330	17.0796	-10.1620	-10.3191	0

Table 4: Transverse deflection as a function of loading Λ/π^2 and various values of ξ for the third buckling mode ($\lambda_3 = 10.9041$) of the clamped–hinged, conventional EBT beam.

		$v_3(\xi)$					
Λ/π^2	c_3	$\xi = 0.125$	$\xi = 0.25$	$\xi = 0.5$	$\xi = 0.75$	$\xi = 0.875$	$\xi = 1.0$
10.9041	0	0	0	0	0	0	0
12.5	0.1291	1.0677	2.3958	-0.3404	0.9226	1.5592	0
15	0.2176	1.7995	4.0380	-0.5737	1.5550	2.6279	0
17.5	0.2891	2.3912	5.3658	-0.7623	2.0663	3.4921	0
20	0.3542	2.9291	6.5726	-0.9337	2.5310	4.2775	0
22.5	0.4158	3.4383	7.7155	-1.0961	2.9711	5.0212	0
25	0.4752	3.9302	8.8192	-1.2529	3.3961	5.7395	0
27.5	0.5333	4.4105	9.8969	-1.4060	3.8112	6.4409	0
30	0.5904	4.8826	10.9563	-1.5565	4.2191	7.1303	0
32.5	0.6467	5.3487	12.0022	-1.7051	4.6219	7.8110	0
35	0.7026	5.8102	13.0379	-1.8522	5.0207	8.4850	0
37.5	0.7579	6.2682	14.0656	-1.9982	5.4165	9.1539	0
40	0.8130	6.7234	15.0870	-2.1433	5.8098	9.8186	0
42.5	0.8677	7.1763	16.1032	-2.2877	6.2011	10.4799	0
45	0.9223	7.6273	17.1152	-2.4315	6.5908	11.1386	0

Table 5: Transverse deflection as a function of loading Λ/π^2 and various values of ξ for the fourth buckling mode ($\lambda_4 = 14.0662$) of the clamped–hinged, conventional EBT beam.

		$v_4(\xi)$					
Λ/π^2	c_4	$\xi = 0.125$	$\xi = 0.25$	$\xi = 0.5$	$\xi = 0.75$	$\xi = 0.875$	$\xi = 1.0$
14.0662	0	0	0	0	0	0	0
15	0.0606	0.9640	1.4099	-0.1562	0.5259	-0.7328	0
17.5	0.1211	1.9264	2.8174	-0.3121	1.0510	-1.4644	0
20	0.1653	2.6307	3.8475	-0.4262	1.4352	-1.9999	0
22.5	0.2042	3.2493	4.7523	-0.5265	1.7727	-2.4701	0
25	0.2403	3.8241	5.5930	-0.6196	2.0863	-2.9071	0
27.5	0.2748	4.3723	6.3948	-0.7084	2.3854	-3.3238	0
30	0.3081	4.9029	7.1708	-0.7944	2.6748	-3.7272	0
32.5	0.3407	5.4211	7.9286	-0.8784	2.9575	-4.1211	0
35	0.3727	5.9300	8.6730	-0.9608	3.2352	-4.5080	0
37.5	0.4042	6.4320	9.4072	-1.0422	3.5090	-4.8896	0
40	0.4354	6.9285	10.1333	-1.1226	3.7799	-5.2670	0
42.5	0.4663	7.4206	10.8530	-1.2024	4.0484	-5.6411	0
45	0.4971	7.9091	11.5676	-1.2815	4.3149	-6.0125	0

Table 6: Transverse deflection as a function of loading Λ/π^2 and various values of ξ for the fifth buckling mode ($\lambda_5 = 17.2208$) of the clamped–hinged, conventional EBT beam.

		$v_5(\xi)$					
Λ/π^2	c_5	$\xi = 0.125$	$\xi = 0.25$	$\xi = 0.5$	$\xi = 0.75$	$\xi = 0.875$	$\xi = 1.0$
17.2208	0	0	0	0	0	0	0
17.5	0.0240	0.6078	0.4509	0.5069	-0.2764	0.3969	0
20	0.0783	1.9855	1.4729	1.6560	-0.9028	1.2966	0
22.5	0.1114	2.8270	2.0970	2.3577	-1.2854	1.8460	0
25	0.1395	3.5380	2.6245	2.9508	-1.6087	2.3103	0
27.5	0.1650	4.1856	3.1049	3.4909	-1.9031	2.7332	0
30	0.1890	4.7956	3.5574	3.9997	-2.1805	3.1316	0
32.5	0.2121	5.3808	3.9915	4.4877	-2.4465	3.5137	0
35	0.2345	5.9485	4.4126	4.9611	-2.7047	3.8844	0
37.5	0.2564	6.5032	4.8241	5.4238	-2.9569	4.2466	0
40	0.2778	7.0481	5.2283	5.8783	-3.2046	4.6024	0
42.5	0.2990	7.5853	5.6268	6.3263	-3.4489	4.9532	0
45	0.3200	8.1162	6.0206	6.7691	-3.6903	5.2999	0

2.7.1.6 Other Boundary Conditions

Some non-classical boundary conditions that have not been studied in the literature are listed next.

Elastically Hinged:

$$v + \alpha \left(\frac{d^3 v}{d\xi^3} + \lambda^2 \frac{dv}{d\xi} \right) = 0, \quad \frac{d^2 v}{d\xi^2} = 0 \tag{2.109}$$

where α is the inverse of a nondimensional elastic (spring) constant. When $\alpha = 0$

(i.e., the support is rigid), we recover the conventional simply supported boundary condition. When α is very large, the boundary condition approaches that of a free edge.

For the case of $\alpha = 0$ (or as α approaches 0), the boundary conditions become that of the clamped–hinged case from the previous section, with the constants c_n being the same. However, as α increases the beam takes on the boundary conditions of a clamped–free system. With the constants $c_1 = c_3 = 0, c_2 = -c_4 \neq 0$, the mode shapes become

$$\begin{aligned} v_n(\xi) &= c_{1n}\, \sin \lambda_n \xi + c_{2n}\, \cos \lambda_n \xi + c_{3n}\, \xi + c_{4n} \\ &= c_n(\cos \lambda_n \xi - 1) \end{aligned} \tag{2.110}$$

and the constant c_n is computed as

$$\lambda_n^2 = \Lambda - \frac{c_n^2 \lambda_n^2}{2} \int_0^1 \sin^2 \lambda_n \eta \, d\eta \quad \Rightarrow \quad c_n = \pm 2 \sqrt{\frac{\Lambda}{\lambda_n^2} - 1}\,, \quad n = 1, 2, 3, \cdots \tag{2.111}$$

Elastically Clamped:

$$v = 0, \quad \frac{dv}{d\xi} + \beta \frac{d^2 v}{d\xi^2} = 0 \tag{2.112}$$

where β is the inverse of the torsional spring constant. When $\beta = 0$ (i.e., the restraint is rigid), we recover the conventional clamped boundary condition. This gives us a clamped–clamped system with the same mode shapes $v_n(\xi)$ and constants c_n given in Section 2.7.1.4.

On the other hand, if β is very large (i.e., the restraint is very flexible), the condition approaches that of a simply supported case which makes use of the boundary conditions for a clamped–hinged system. This form of an elastically clamped beam will yield the same mode shapes $v_n(\xi)$ and constants c_n as those in Section 2.7.1.5.

2.7.2 Conventional TBT

For the TBT case, Eq. (2.76) and (2.77) give us the pair of second-order coupled differential equations:

$$-\frac{1}{s^2}\left(\frac{d^2v}{d\xi^2}+\frac{d\psi}{d\xi}\right)+\Lambda\frac{d^2v}{d\xi^2}-\frac{1}{2}\frac{d^2v}{d\xi^2}\left[\int_0^1\left(\frac{dv}{d\eta}\right)^2 d\eta\right]=0$$
$$-\frac{d^2\psi}{d\xi^2}+\frac{1}{s^2}\left(\frac{dv}{d\xi}+\psi\right)=0$$

or

$$-\frac{1}{s^2}\frac{d\psi}{d\xi}-\lambda_s^2\frac{d^2v}{d\xi^2}=0 \tag{2.113}$$
$$-\frac{d^2\psi}{d\xi^2}+\frac{1}{s^2}\left(\frac{dv}{d\xi}+\psi\right)=0 \tag{2.114}$$

where

$$\lambda_s^2=\frac{1}{s^2}+\Gamma-\Lambda=\frac{1}{s^2}-\lambda^2,\quad \Gamma=\frac{1}{2}\left[\int_0^1\left(\frac{dv}{d\eta}\right)^2 d\eta\right],\quad \frac{1}{s^2}=\frac{GAK_s l^2}{EI} \tag{2.115}$$

For a rectangular cross-section beam with height h width b, and length l, we have

$$s^2=\frac{1+\nu}{5}\left(\frac{h}{l}\right)^2=0.26\left(\frac{h}{l}\right)^2 \quad \text{for } \nu=0.3 \tag{2.116}$$

Solving Eq. (2.113) for $d\psi/d\xi$,

$$\frac{d\psi}{d\xi}=-s^2\lambda_s^2\frac{d^2v}{d\xi^2}$$

and substituting the result into Eq. (2.114) (after differentiating it once), we obtain

$$s^2\lambda_s^2\frac{d^4v}{d\xi^4}+\lambda^2\frac{d^2v}{d\xi^2}=0 \quad \text{or} \quad \frac{d^4v}{d\xi^4}+\beta^2\frac{d^2v}{d\xi^2}=0 \tag{2.117}$$

where

$$\beta^2=\frac{\lambda^2}{s^2\lambda_s^2}=\frac{\lambda^2}{1-s^2\lambda^2} \quad \text{or} \quad \lambda^2=\frac{\beta^2}{1+s^2\beta^2} \tag{2.118}$$

The general solution to this equation is of the form

$$v(\xi)=c_1\sin\beta\xi+c_2\cos\beta\xi+c_3\xi+c_4 \tag{2.119}$$

The solutions for the TBT can be obtained in the same manner as the EBT solutions for the various boundary conditions using the general solution in Eq. (2.119). The solutions are discussed here for the same various boundary conditions.

2.7.2.1 Hinged–Hinged Beams

Using the boundary conditions we obtain

$$\begin{aligned}
v(0)=0: &\quad c_2+c_4=0 \quad \text{or} \quad c_2=-c_4\\
\left.\frac{d\psi}{d\xi}\right|_{\xi=0}=0 \Rightarrow \left.\frac{d^2v}{d\xi^2}\right|_{\xi=0}=0: &\quad c_2=c_4=0\\
v(1)=0: &\quad c_1\sin\beta+c_3=0\\
\left.\frac{d^2v}{d\xi^2}\right|_{x=1}=0: &\quad c_1\sin\beta=0\Rightarrow c_3=0
\end{aligned}$$

From the above equations, it follows from Eq. (2.118) that

$$\sin\beta=0 \quad \to \quad \beta_n=n\pi \quad \text{or} \quad \lambda_n^2=\frac{\beta_n^2}{1+s^2\beta_n^2} \tag{2.120}$$

Thus we have

$$\lambda_n^2 = \frac{n^2\pi^2}{1+n^2\pi^2\, s^2} = \Lambda - \Gamma \tag{2.121}$$

The mode shapes are

$$v_n(\xi) = c_n \, \sin\beta_n\xi, \quad n = 1, 2, 3, \cdots \tag{2.122}$$

where c_n is given by

$$c_n = \pm 2\sqrt{\frac{\Lambda}{\lambda_n^2} - 1}, \quad n = 1, 2, 3, \cdots \tag{2.123}$$

2.7.2.2 Clamped–Clamped Beams

For a beam clamped at both ends (at $\xi = 0, 1$), we have

$$v = 0, \quad \psi = 0 \;\; \text{at} \;\; \xi = 0, 1 \tag{2.124}$$

In order to use the boundary conditions on ψ, we note the following relationship [see [41], p. 196]

$$\psi(\xi) = -s^2\lambda_s^2\frac{dv}{d\xi} - s^2\lambda^2 c_3 \tag{2.125}$$

Use of the boundary conditions gives

$$\begin{aligned}
v(0) = 0: &\quad c_2 + c_4 = 0 \quad \text{or} \quad c_4 = -c_2 \\
\psi(0) = 0: &\quad s^2\lambda_s^2\beta c_1 + c_3 = 0 \quad \text{or} \quad c_3 = -s^2\lambda_s^2\beta c_1 \\
v(1) = 0: &\quad c_1\sin\beta + c_2\cos\beta + c_3 + c_4 = 0 \\
\psi(1) = 0: &\quad s^2\lambda_s^2\beta\,(c_1\cos\beta - c_2\sin\beta) + c_3 = 0
\end{aligned}$$

Using the first two equations, c_3 and c_4 can be eliminated from the last two equations. We have

$$c_1\left(\sin\beta - s^2\lambda_s^2\beta\right) + c_2\left(\cos\beta - 1\right) = 0 \quad \text{and} \quad c_1\left(\cos\beta - 1\right) - c_2\sin\beta = 0 \tag{2.126}$$

For nonzero transverse deflection (i.e., for nonzero values of c_1 and c_2), we require that the determinant of the above pair of equations be zero:

$$\begin{vmatrix} \sin\beta - s^2\lambda_s^2\beta & \cos\beta - 1 \\ \cos\beta - 1 & -\sin\beta \end{vmatrix} = 0$$

or

$$s^2\lambda_s^2\,\beta\,\sin\beta + 2\,\cos\beta - 2 = 0 \tag{2.127}$$

This transcendental equation takes the same form as Eq. (2.92) and must also be solved iteratively for the proper values of β_n which can then be used to determine the corresponding values of λ_n. Using the same method and identities from Eq. (2.93) and (2.94), we have

$$\begin{aligned} 0 &= s^2\lambda_s^2\beta\sin\beta + 2\,\cos\beta - 2 \\ &= 2s^2\lambda_s^2\beta\,\sin\frac{\beta}{2}\cos\frac{\beta}{2} + 2 - 4\sin^2\frac{\beta}{2} - 2 \\ &= 4\sin\frac{\beta}{2}\left(\frac{s^2\lambda_s^2\beta}{2}\cos\frac{\beta}{2} - \sin\frac{\beta}{2}\right) \end{aligned} \tag{2.128}$$

which shows that there are two sets of roots, one corresponding to symmetric modes

$$\sin\frac{\beta}{2} = 0 \;\Rightarrow\; \beta = 2n\pi, \quad n = 1, 2, 3, \cdots \tag{2.129}$$

and the other corresponds to unsymmetric modes

$$\tan\frac{\beta}{2} = \frac{s^2\lambda_s^2\beta}{2} \tag{2.130}$$

Applying the constants obtained from the boundary conditions and

$$c_1 = \left(\frac{1-\cos\beta}{\sin\beta - s^2\lambda_s^2\beta}\right)c_2 \tag{2.131}$$

we can obtain the general solution as

$$\begin{aligned} v_n(\xi) &= c_{1n}\,\sin\beta_n\xi + c_{2n}\,\cos\beta_n\xi + c_{3n}\,\xi + c_{4n} \\ &= c_{1n}\left(\sin\beta_n\xi - s^2\lambda_s^2\beta_n\xi\right) + c_{2n}\left(\cos\beta_n\xi - 1\right) \\ &= c_n\left[1-\cos\beta_n\xi - \left(\sin\beta_n\xi - s^2\lambda_s^2\beta_n\xi\right)\left(\frac{1-\cos\beta_n}{\sin\beta_n - s^2\lambda_s^2\beta_n}\right)\right] \end{aligned} \tag{2.132}$$

The alternative expression for $v_n(\xi)$ is [c_2 is expressed in terms of c_1 using the second equation in Eq. (2.126)]

$$v_n(\xi) = \hat{c}_n\left[1+\cos\beta_n(1-\xi) - s^2\lambda_s^2\beta_n\sin\beta_n\xi - \cos\beta_n\xi - \cos\beta_n\right] \tag{2.133}$$

where $\hat{c}_n = c_{1n}/\sin\beta_n$ is to be determined.

In view of Eq. (2.132), the symmetric mode shapes are given by $(1-\cos 2n\pi = 0)$

$$v_n(\xi) = c_n(1-\cos\beta_n\xi), \quad \beta_n = 2n\pi, \quad n = 1,2,3,\cdots \tag{2.134}$$

The unsymmetric mode shapes can be determined using Eq. (2.130) and the following identity [where we use Eqs. (2.93) and (2.130)]:

$$\frac{1-\cos\beta}{\sin\beta - s^2\lambda_s^2\beta} = \frac{2\sin^2\frac{\beta}{2}}{2\sin\frac{\beta}{2}\cos\frac{\beta}{2} - 2\tan\frac{\beta}{2}} = \frac{\sin\frac{\beta}{2}\cos\frac{\beta}{2}}{\cos^2\frac{\beta}{2} - 1} = -\cot\frac{\beta}{2} = -\frac{2}{\beta} \tag{2.135}$$

Therefore the mode shapes for the unsymmetric case are

$$v_n(\xi) = c_n\Big(1 - 2s^2\lambda_s^2\xi - \cos\beta_n\xi + \frac{2}{\beta_n}\sin\beta_n\xi\Big) \tag{2.136}$$

with λ_n determined from Eq. (2.133).

Since this follows so closely to the clamped–clamped case for the EBT, we can use the form applied to obtain c_n in Eq. (2.103) for the TBT case, so that

$$c_n = \pm 2\sqrt{\frac{\Lambda}{\lambda_n^2} - 1}, \quad n = 1, 2, 3, \cdots \tag{2.137}$$

In Table 7 the constants c_n as well as the characteristic equation and respective eigenvalues λ_n associated with the different boundary conditions are presented for the conventional Timoshenko case. As well, Tables 8 and 9 compare the Euler–Bernoulli and Timoshenko buckling modes for the hinged–hinged and clamped–clamped cases, respectively. The Timoshenko results are dependent upon the aspect ratio, as can be seen in the columns presented.

Table 7: Values of the constants c_i $(i = 1, 2, 3, 4)$ and eigenvalues λ_n for buckling of beams with various boundary conditions for the Timoshenko case.

End conditions at $\xi = 0$ and $\xi = 1$	Constants*	Characteristic equation and values of λ_n
Hinged–Hinged	$c_1 \neq 0$ $c_2 = c_3 = c_4 = 0$	$\sin\beta_n = 0 \quad \Rightarrow \quad \beta_n = n\pi$ $\lambda_n = \frac{\beta_n}{\sqrt{1+\beta_n^2 s^2}}$
Clamped–Clamped**	$c_1 = (1 - \cos\beta_n)/(\sin\beta_n - s^2\lambda_s^2\beta_n)c_2$ $c_3 = -s^2\lambda_s^2\beta_n\, c_1, \quad c_4 = -c_2 \neq 0$	$s^2\lambda_s^2\beta_n \sin\beta_n + 2\cos\beta_n - 2 = 0$ $\lambda_n = \frac{\beta_n}{\sqrt{1+\beta_n^2 s^2}}$

* See Eq. (2.119): $v(\xi) = c_1 \sin\beta\xi + c_2 \cos\beta\xi + c_3\xi + c_4$.

** β_n must be determined by solving the transcendental equation.

Table 8: Comparison of λ_n for the hinged–hinged case for both Euler–Bernoulli and Timoshenko beams. Various aspect ratios (h/l) are given for the Timoshenko theory.

		TBT			
Mode	EBT	$h/l = 0.001$	$h/l = 0.01$	$h/l = 0.05$	$h/l = 0.1$
1	3.14159	3.14159	3.14119	3.13156	3.10204
2	6.28319	6.28315	6.279963	6.204089	5.9836
3	9.42478	9.42467	9.413914	9.16394	8.49476
4	12.56637	12.56611	12.54065	11.96719	10.58063
5	15.70796	15.70746	15.65782	14.58208	12.26016
6	18.84956	18.84869	18.76309	16.98952	13.59007
7	21.99115	21.98977	21.85418	19.18196	14.63676
8	25.13274	25.13068	24.92887	21.16127	15.46139
9	28.27433	28.2714	27.98499	22.93627	16.11462
10	31.41593	31.4119	31.02044	24.52032	16.63617
11	34.55752	34.55216	34.0332	25.92932	17.05639
12	37.69911	37.69215	37.02131	27.18014	17.39823
13	40.8407	40.83185	39.98293	28.28955	17.67896
14	43.9823	43.97124	42.91629	29.27351	17.91164
15	47.12389	47.11029	45.8197	30.1468	18.10621

Table 9: Comparison of λ_n for the clamped–clamped case for both Euler–Bernoulli and Timoshenko beams. Various aspect ratios (h/l) are given for the Timoshenko theory.

		TBT			
Mode	EBT	$h/l = 0.001$	$h/l = 0.01$	$h/l = 0.05$	$h/l = 0.1$
1	6.2832	6.2832	6.2800	6.2041	5.9836
2	8.9868	8.9867	8.9765	8.7383	8.1009
3	12.5664	12.5661	12.5407	11.9672	10.5807
4	15.4505	15.4500	15.4012	14.3432	12.0600
5	18.8496	18.8487	18.7632	16.9896	13.5901
6	21.8082	21.8069	21.6725	19.0225	14.5157
7	25.1327	25.1306	24.9289	21.1612	15.4614
8	28.1324	28.1295	27.8446	22.8212	16.0344
9	31.4159	31.4119	31.0206	24.5203	16.6362
10	34.4415	34.4362	33.9191	25.8424	16.9998
11	37.6991	37.6921	37.0216	27.1801	17.3982
12	40.7426	40.7338	39.8872	28.2217	17.6372
13	43.9823	43.9712	42.9167	29.2735	17.9116
14	47.0389	47.0254	45.7375	30.0925	18.0742
15	50.2655	50.2490	48.6922	30.9228	18.2703

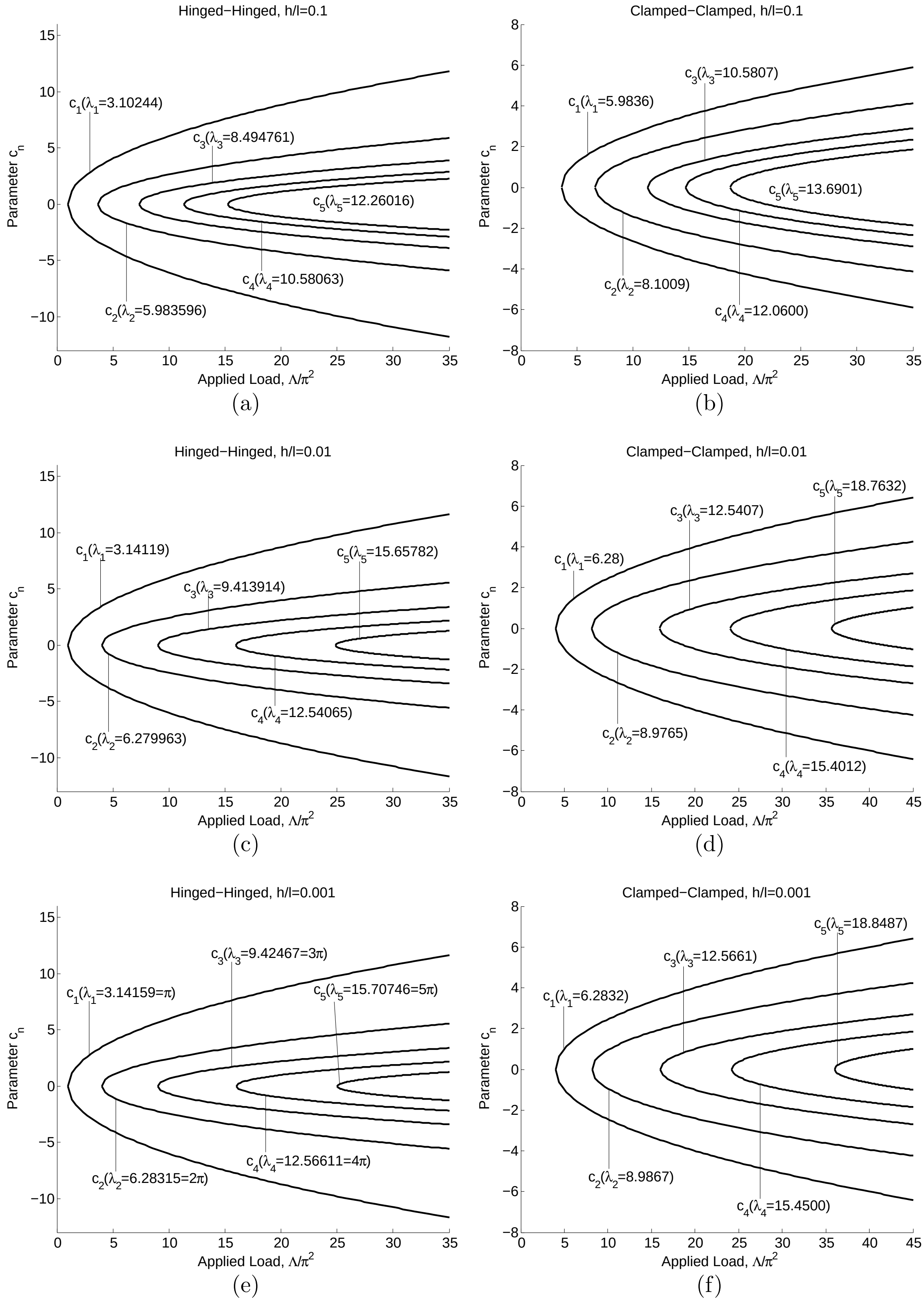

Figure 2.6: Constant parameter c_n vs. applied load for hinged–hinged and clamped–clamped boundary conditions for the different aspect ratios.

Figure 2.6 gives a comparison for the load–deflection behavior based on the aspect ratio for the TBT case. As the beam becomes thinner, it approaches the EBT, which is most evident in the hinged–hinged case for $h/l = 0.001$ where the λ_n values approach $n\pi$. This can also be seen in Tables 3 and 4.

3. MODIFIED BEAM THEORIES*

The work introduced in this section is reprinted with permission by American Scientific Publishers* and elaborates on theory and formulations from Reddy and Mahaffey [35]. The conventional theories presented in the last section do not make use of the complete von Kármán strain field. The complete set of von Kármán components allow for the nonlinear terms in the normal strain tensor $\mathbf{E}$ to be applied for a more complete account of the geometric behavior of beams. Including the additional nonlinear components in the strain tensor for the investigation of the behavior of systems and structures [42–49] allows for a more elaborate set of governing equations, such as the equations of motion, to be developed for more accurate solutions to large scale applications, as well as micro- and nano-level applications due to their scale.

For the modified beam theories discussed in this section, we expand the strain fields to include the ε_{zz} component for both theories, and add the shear strain component γ_{xz} to the Euler–Bernoulli theory. This will allow an adjustment to the kinematic deformations by keeping the small strains and involving moderate rotations brought on by both bending and buckling. The advantage to elaborating the conventional theories to include the Poisson effect are apparent when modelling functionally graded or laminated beams. As will be seen, we will make use of two-dimensional constitutive relations instead of the one-dimensional relations seen in the conventional theories.

*Reprinted with permission from J. N. Reddy and P. Mahaffey, "Generalized Beam Theories Accounting for von Kármán Nonlinear Strains with Application to Buckling," *Journal of Coupled Systems and Multiscale Dynamics*, Vol. 1, Copyright [2013] by American Scientific Publishers.

3.1 Modified Euler–Bernoulli Theory

To reiterate the displacement field being used for the Euler–Bernoulli theory, whether it is for the conventional or modified model, we have:

$$u_x(x,z,t) = u(x,t) + z\theta_x, \quad u_z(x,z,t) = w(x,t), \quad \theta_x \equiv -\frac{\partial w}{\partial x}$$

The strain tensor that includes the additional rotation terms of the $(1/2)(\partial w/\partial x)^2$ type in the ε_{zz} and γ_{xz} components becomes [i.e., $(\partial u_1/\partial x)^2 \approx 0$]

$$\boldsymbol{\varepsilon} = \varepsilon_{xx}\ \hat{\mathbf{e}}_x\hat{\mathbf{e}}_x + \varepsilon_{xz}\left(\hat{\mathbf{e}}_x\hat{\mathbf{e}}_z + \hat{\mathbf{e}}_z\hat{\mathbf{e}}_x\right) + \varepsilon_{zz}\ \hat{\mathbf{e}}_z\hat{\mathbf{e}}_z \tag{3.1}$$

where

$$\varepsilon_{xx} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)}, \quad \gamma_{xz} = 2\varepsilon_{xz} = \gamma_{xz}^{(0)} + z\gamma_{xz}^{(1)}, \quad \varepsilon_{zz} = \varepsilon_{zz}^{(0)} \tag{3.2}$$

with

$$\begin{aligned} \varepsilon_{xx}^{(0)} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_{xx}^{(1)} = \frac{\partial \theta_x}{\partial x} \\ \gamma_{xz}^{(0)} &= \frac{\partial u}{\partial x}\theta_x, \quad \gamma_{xz}^{(1)} = \theta_x\frac{\partial \theta_x}{\partial x}, \quad \varepsilon_{zz}^{(0)} = \frac{1}{2}\theta_x^2 \end{aligned} \tag{3.3}$$

3.1.1 Equations of Motion

We develop the equations of motion for the modified theories using the same process as we did for the conventional models. However, additional stress and strain terms will appear in the virtual strain energy function since we have now included the γ_{xz} and ε_{zz} components. Just as was done for the conventional theories in the previous section, we apply Hamilton's principle

$$0 = \int_{t_1}^{t_2} \left(-\delta K + \delta U + \delta V\right) dt$$

to capture the total energy of the system for deriving the equations of motion for the modified bending, buckling and vibration governing equations. We can derive the virtual kinetic and strain energy, and the virtual applied work done based on Eq. (3.3). The virtual kinetic energy expression remains the same as the conventional model, but for clarity we will include it as

$$\begin{aligned}\delta K &= \int_0^l \int_A \rho \dot{u}_i \, \delta \dot{u}_i \, dA \, dx = \int_0^l \int_A \rho \left[\left(\dot{u} + z\dot{\theta}_x \right) \left(\delta \dot{u} + z\delta\dot{\theta}_x \right) + \dot{w} \, \delta \dot{w} \right] dA \, dx \\ &= \int_0^l \left[\left(m_0 \dot{u} + m_1 \dot{\theta}_x \right) \delta \dot{u} + \left(m_1 \dot{u} + m_2 \dot{\theta}_x \right) \delta \dot{\theta}_x + m_0 \, \dot{w} \, \delta \dot{w} \right] dx\end{aligned}$$

Our virtual strain energy will now differ compared to that of the conventional EBT with the included additional stress and strain components. The expression for the virtual strain energy, with applied axial compressive force P, now becomes

$$\begin{aligned}\delta U &= \int_0^l \int_A \left(\sigma_{xx} \, \delta\varepsilon_{xx} + \sigma_{xz} \, \delta\gamma_{xz} + \sigma_{zz} \, \delta\varepsilon_{zz} \right) dA dx - P \int_0^l \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \, dx + \int_0^l F_v \, \delta w \, dx \\ &= \int_0^l \left[M_{xx}^{(0)} \left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right) + M_{xx}^{(1)} \frac{\partial \delta\theta_x}{\partial x} + M_{xz}^{(0)} \left(\frac{\partial u}{\partial x} \delta\theta_x + \frac{\partial \delta u}{\partial x} \theta_x \right) \right. \\ &\quad \left. + M_{xz}^{(1)} \left(\delta\theta_x \frac{\partial \theta_x}{\partial x} + \theta_x \frac{\partial \delta\theta_x}{\partial x} \right) + M_{zz}^{(0)} \, \theta_x \delta\theta_x - P \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + F_v \, \delta w \right] dx \qquad (3.4)\end{aligned}$$

We have the same stress resultants used in Eqs. (2.11) and (2.21), but with additional stress resultants to account for the stress in the z-direction as well as the shear stress in the $x-z$ direction. For completeness, all stress resultants used for the modified EBT virtual strain energy are

$$\begin{gathered}M_{xx}^{(0)} = \int_A \sigma_{xx} \, dA, \quad M_{xx}^{(1)} = \int_A z \, \sigma_{xx} \, dA \\ M_{xz}^{(0)} = \int_A \sigma_{xz} \, dA, \quad M_{xz}^{(1)} = \int_A z\sigma_{xz} \, dA, \quad M_{zz}^{(0)} = \int_A \sigma_{zz} \, dA\end{gathered} \qquad (3.5)$$

where $P(\partial w/\partial x)$ denotes the component of force along the deformed centerline of the beam, which is oriented at an angle of $\partial w/\partial x$. The virtual work done by the external forces will be the same as the conventional EBT as

$$\delta V = -\int_0^l \big(f_x\,\delta u + q\,\delta w\big)dx$$

Substituting the virtual energies $(\delta K, \delta U)$ and virtual external work done δV into Hamilton's principle and minimizing the functional we can obtain the proper equations of motion for the modified EBT, we get:

$$-\frac{\partial}{\partial x}\Big(M_{xx}^{(0)} + M_{xz}^{(0)}\theta_x\Big) + m_0\frac{\partial^2 u}{\partial t^2} + m_1\frac{\partial^2 \theta_x}{\partial t^2} = f_x \quad (3.6)$$

$$-\frac{\partial}{\partial x}\left(M_{xx}^{(0)}\frac{\partial w}{\partial x} - M_{zz}^{(0)}\theta_x - M_{xz}^{(0)}\frac{\partial u}{\partial x} - M_{xz}^{(1)}\frac{\partial \theta_x}{\partial x}\right) + P\frac{\partial^2 w}{\partial x^2}$$
$$-\frac{\partial^2}{\partial x^2}\Big(M_{xx}^{(1)} + M_{xz}^{(1)}\theta_x\Big) + m_0\frac{\partial^2 w}{\partial t^2} + m_1\frac{\partial^3 u}{\partial x\partial t^2} + m_2\frac{\partial^3 \theta_x}{\partial t^2\partial x} + \hat{\mu}\frac{\partial w}{\partial t} = q \quad (3.7)$$

with the resulting natural boundary conditions

$$\begin{aligned}
\delta u &: \; M_{xx}^{(0)} + M_{xz}^{(0)}\theta_x \\
\delta w &: \; M_{xx}^{(0)}\frac{\partial w}{\partial x} - M_{zz}^{(0)}\theta_x - M_{xz}^{(0)}\frac{\partial u}{\partial x} - M_{xz}^{(1)}\frac{\partial \theta_x}{\partial x} - P\frac{\partial w}{\partial x} \\
&\quad + \frac{\partial}{\partial x}\Big(M_{xx}^{(1)} + M_{xz}^{(1)}\theta_x\Big) + m_2\frac{\partial^2 \theta_x}{\partial t^2} \\
\delta\theta_x &: \; M_{xx}^{(1)} + M_{xz}^{(1)}\theta_x
\end{aligned} \quad (3.8)$$

Note that the transverse normal strain is *not* zero because of the geometric nonlinearity, requiring us to use the two-dimensional stress-strain relations. Most beam theories that include the von Kármán nonlinearity omit the nonlinear terms in the transverse normal strain so that one can use one-dimensional constitutive relations.

3.2 Modified Timoshenko Beam Theory

3.2.1 Displacements and Strains

For the Timoshenko theory, the displacement field from Eq. (2.17) is

$$u_x(x,z,t) = u(x,t) + z\phi_x(x,t), \quad u_z(x,z,t) = w(x,t)$$

The simplified Green–Lagrange strain tensor components are

$$\varepsilon_{xx} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)}, \quad \gamma_{xz} = \gamma_{xz}^{(0)} + z\gamma_{xz}^{(1)}, \quad \varepsilon_{zz} = \varepsilon_{zz}^{(0)} \tag{3.9}$$

with

$$\begin{aligned} \varepsilon_{xx}^{(0)} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_{xx}^{(1)} = \frac{\partial \phi_x}{\partial x} \\ \gamma_{xz}^{(0)} &= \phi_x + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\phi_x, \quad \gamma_{xz}^{(1)} = \phi_x \frac{\partial \phi_x}{\partial x}, \quad \varepsilon_{zz}^{(0)} = \frac{1}{2}\phi_x^2 \end{aligned} \tag{3.10}$$

3.2.2 Equations of Motion

The virtual kinetic energy δK for the modified TBT model is stated as

$$\begin{aligned} \delta K &= \int_0^l \int_A \rho \dot{u}_i \delta \dot{u}_i \, dA\, dx = \int_0^l \int_A \rho\big[(\dot{u} + z\dot{\phi}_x)(\delta\dot{u} + z\delta\dot{\phi}_x) + \dot{w}\delta\dot{w}\big]\, dA\, dx \\ &= \int_0^l \big[(m_0\dot{u} + m_1\dot{\phi}_x)\delta\dot{u} + (m_1\dot{u} + m_2\dot{\phi}_x)\delta\dot{\phi}_x + m_0\dot{w}\delta\dot{w}\big]\, dx \end{aligned}$$

where m_0, m_1 and m_2 are defined in Eq. (2.9).

The virtual strain energy δU is computed as

$$\delta U = \int_0^l \int_A (\sigma_{xx}\,\delta\varepsilon_{xx} + \sigma_{xz}\,\delta\gamma_{xz} + \sigma_{zz}\,\delta\varepsilon_{zz})\, dA\, dx - P\int_0^l \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} dx + \int_0^l F_v\, \delta w\, dx$$

$$
\begin{aligned}
= \int_0^l \Bigg[& M_{xx}^{(0)} \left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right) + M_{xx}^{(1)} \frac{\partial \delta \phi_x}{\partial x} + M_{zz}^{(0)} \, \phi_x \, \delta \phi_x \\
& + M_{xz}^{(0)} \left(\delta \phi_x + \frac{\partial \delta w}{\partial x} + \frac{\partial \delta u}{\partial x} \phi_x + \frac{\partial u}{\partial x} \delta \phi_x \right) \\
& + M_{xz}^{(1)} \left(\delta \phi_x \frac{\partial \phi_x}{\partial x} + \phi_x \frac{\partial \delta \phi_x}{\partial x} \right) - P \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} + F_v \delta w \Bigg] dx
\end{aligned} \tag{3.11}
$$

where the stress resultants ($M_{xx}^{(0)}$, $M_{xx}^{(1)}$, $M_{zz}^{(0)}$, $M_{xz}^{(0)}$, $M_{xz}^{(1)}$) are defined in Eq. (3.5).

The expression for the virtual work done by external forces for the TBT model is the same as the conventional model in Eq. (2.22).

Applying Hamilton's principle, the equations of motion for the modified TBT model become:

$$
-\frac{\partial}{\partial x} \left(M_{xx}^{(0)} + M_{xz}^{(0)} \phi_x \right) + m_0 \frac{\partial^2 u}{\partial t^2} + m_1 \frac{\partial^2 \phi_x}{\partial t^2} = f_x \tag{3.12}
$$

$$
-\frac{\partial}{\partial x} \left(M_{xz}^{(0)} + M_{xx}^{(0)} \frac{\partial w}{\partial x} - P \frac{\partial w}{\partial x} \right) + \hat{\mu} \frac{\partial w}{\partial t} + m_0 \frac{\partial^2 w}{\partial t^2} = q \tag{3.13}
$$

$$
\begin{aligned}
\left(M_{xz}^{(0)} + M_{xz}^{(0)} \frac{\partial u}{\partial x} + M_{zz}^{(0)} \phi_x + M_{xz}^{(1)} \frac{\partial \phi_x}{\partial x} \right) - \frac{\partial}{\partial x} \left(M_{xx}^{(1)} + M_{xz}^{(1)} \phi_x \right) & \\
+ m_1 \frac{\partial^2 u}{\partial t^2} + m_2 \frac{\partial^2 \phi_x}{\partial t^2} = 0 &
\end{aligned} \tag{3.14}
$$

The natural boundary conditions become

$$
\begin{aligned}
\delta u : & \quad M_{xx}^{(0)} + M_{xz}^{(0)} \phi_x \\
\delta w : & \quad M_{xz}^{(0)} + (M_{xx}^{(0)} - P) \frac{\partial w}{\partial x} \\
\delta \phi_x : & \quad M_{xx}^{(1)} + M_{xz}^{(1)} \phi_x
\end{aligned} \tag{3.15}
$$

3.3 Generalized Force-Displacement Relations

The strains defined in Eqs. (3.3) and (3.10) are clearly two dimensional, requiring us to use two-dimensional constitutive relations. We have [$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + (\lambda\,\varepsilon_{kk} - c_T)\delta_{ij}$]

$$\begin{aligned}
\sigma_{xx} &= (2\mu + \lambda)\,\varepsilon_{xx} + \lambda\varepsilon_{zz} - c_T\Delta T = c_{11}\varepsilon_{xx} + c_{13}\varepsilon_{zz} - c_T\Delta T \\
\sigma_{zz} &= (2\mu + \lambda)\,\varepsilon_{zz} + \lambda\varepsilon_{xx} - c_T\Delta T = c_{31}\varepsilon_{xx} + c_{33}\varepsilon_{zz} - c_T\Delta T \\
\sigma_{xz} &= \mu\gamma_{xz} = G\gamma_{xz}
\end{aligned} \tag{3.16}$$

where μ and λ are Lame's constants,

$$\mu = G = \frac{E}{2(1+\nu)} \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{3.17}$$

E, G and ν denote Young's modulus, shear modulus, and Poisson's ratio, respectively, and c_{ij} are defined as

$$c_{11} = c_{33} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}, \quad c_{13} = c_{31} = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad c_T = \frac{E\alpha}{(1-2\nu)} \tag{3.18}$$

The stress resultants defined in Eq. (3.5) can be related to the displacements (u, w) and their derivatives as

$$\left\{\begin{array}{c} M_{xx}^{(0)} \\ M_{xx}^{(1)} \\ M_{zz}^{(0)} \\ M_{xz}^{(0)} \\ M_{xz}^{(1)} \end{array}\right\} = \int_A \left\{\begin{array}{c} \sigma_{xx} \\ z\sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \\ z\sigma_{xz} \end{array}\right\} dA = \int_A \left\{\begin{array}{c} c_{11}\varepsilon_{xx} + c_{13}\varepsilon_{zz} - c_T\Delta T \\ z\,(c_{11}\varepsilon_{xx} + c_{13}\varepsilon_{zz} - c_T\Delta T) \\ c_{31}\varepsilon_{xx} + c_{33}\varepsilon_{zz} - c_T\Delta T \\ G\gamma_{xz} \\ zG\gamma_{xz} \end{array}\right\} dA$$

$$= \left\{ \begin{array}{c} A_{11}\,\varepsilon_{xx}^{(0)} + B_{11}\,\varepsilon_{xx}^{(1)} + A_{13}\,\varepsilon_{zz}^{(0)} \\ B_{11}\,\varepsilon_{xx}^{(0)} + D_{11}\,\varepsilon_{xx}^{(1)} + B_{13}\varepsilon_{zz}^{(0)} \\ A_{13}\,\varepsilon_{xx}^{(0)} + B_{13}\,\varepsilon_{xx}^{(1)} + D_{33}\,\varepsilon_{zz}^{(0)} \\ A_{xz}\,\gamma_{xz}^{(0)} + B_{xz}\,\gamma_{xz}^{(1)} \\ B_{xz}\,\gamma_{xz}^{(0)} + D_{xz}\,\gamma_{xz}^{(1)} \end{array} \right\} - \left\{ \begin{array}{c} X_T^{(0)} \\ X_T^{(1)} \\ Z_T^{(0)} \\ 0 \\ 0 \end{array} \right\} \tag{3.19}$$

where

$$\begin{aligned} (A_{ij}, B_{ij}, D_{ij}) &= \int_A c_{ij}(1, z, z^2)\, dA \quad (i, j = 1, 3) \\ (A_{xz}, B_{xz}, D_{xz}) &= K_s \int_A G(1, z, z^2)\, dA \\ X_T^{(0)} = Z_T^{(0)} &= \int_A c_T \Delta T\, dA, \quad X_T^{(1)} = \int_A c_T z \Delta T\, dA \end{aligned} \tag{3.20}$$

In writing the constitutive relations, we accounted for the possibility that the moduli vary through the beam thickness (for functionally graded beams).

3.4 Specialization of Equations for Bending, Vibration, and Buckling

3.4.1 Static Bending

3.4.1.1 Modified EBT model

The equations of motion presented in Eqs. (3.6) and (3.7) are valid for static bending and transient analysis. For the static case, we omit all time derivative terms. Then the equations simplify to

$$-\frac{d}{dx}\left(M_{xx}^{(0)} + M_{xz}^{(0)}\theta_x\right) = f(x) \tag{3.21}$$

$$-\frac{d}{dx}\left(M_{xx}^{(0)}\frac{dw}{dx} - M_{zz}^{(0)}\theta_x - P\frac{dw}{dx} - M_{xz}^{(0)}\frac{du}{dx} - M_{xz}^{(1)}\frac{d\theta_x}{dx}\right) - \frac{d^2}{dx^2}\left(M_{xx}^{(1)} + M_{xz}^{(1)}\theta_x\right) = q(x) \tag{3.22}$$

The stress resultants are known in terms of the displacements (u, w) as [see Eqs. (3.3) and (3.19)]

$$M_{xx}^{(0)} = A_{11}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - B_{11}\frac{d^2w}{dx^2} + A_{13}\left[\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - X_T^{(0)} \tag{3.23}$$

$$M_{xx}^{(1)} = B_{11}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - D_{11}\frac{d^2w}{dx^2} + B_{13}\left[\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - X_T^{(1)} \tag{3.24}$$

$$M_{zz}^{(0)} = A_{13}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - B_{13}\frac{d^2w}{dx^2} + D_{33}\left[\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - Z_T^{(0)} \tag{3.25}$$

$$M_{xz}^{(0)} = -A_{xz}\frac{du}{dx}\frac{dw}{dx} + B_{xz}\frac{dw}{dx}\frac{d^2w}{dx^2} \tag{3.26}$$

$$M_{xz}^{(1)} = -B_{xz}\frac{du}{dx}\frac{dw}{dx} + D_{xz}\frac{dw}{dx}\frac{d^2w}{dx^2} \tag{3.27}$$

3.4.1.2 Modified TBT

The equations of equilibrium become

$$-\frac{d}{dx}\left(M_{xx}^{(0)} + M_{xz}^{(0)}\phi_x\right) = f \tag{3.28}$$

$$-\frac{d}{dx}\left(M_{xz}^{(0)} + M_{xx}^{(0)}\frac{dw}{dx} - P\frac{dw}{dx}\right) = q \tag{3.29}$$

$$-\frac{d}{dx}\left(M_{xx}^{(1)} + M_{xz}^{(1)}\phi_x\right) + \left(M_{xz}^{(0)} + M_{xz}^{(0)}\frac{du}{dx} + M_{zz}^{(0)}\phi_x + M_{xz}^{(1)}\frac{d\phi_x}{dx}\right) = 0 \tag{3.30}$$

where the stress resultants are related to the displacements as

$$M_{xx}^{(0)} = A_{11}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + B_{11}\frac{d\phi_x}{dx} + A_{13}\left(\frac{1}{2}\phi_x^2\right) - X_T^{(0)} \tag{3.31}$$

$$M_{xx}^{(1)} = B_{11}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + D_{11}\frac{d\phi_x}{dx} + B_{13}\left(\frac{1}{2}\phi_x^2\right) - X_T^{(1)} \tag{3.32}$$

$$M_{zz}^{(0)} = A_{13}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + B_{13}\frac{d\phi_x}{dx} + D_{33}\left(\frac{1}{2}\phi_x^2\right) - Z_T^{(0)} \tag{3.33}$$

$$M_{xz}^{(0)} = A_{xz}\left(\phi_x + \frac{dw}{dx} + \frac{du}{dx}\phi_x\right) + B_{xz}\left(\phi_x \frac{d\phi_x}{dx}\right) \tag{3.34}$$

$$M_{xz}^{(1)} = B_{xz}\left(\phi_x + \frac{dw}{dx} + \frac{du}{dx}\phi_x\right) + D_{xz}\left(\phi_x \frac{d\phi_x}{dx}\right) \tag{3.35}$$

3.4.2 Natural Vibration

For the natural vibration application, we set all externally applied forces except for the axial compressive load P to zero and seek a solution in periodic form, just as in the conventional theories, as

$$u(x,t) = U(x)\, e^{i\omega t}, \quad w(x,t) = W(x)\, e^{i\omega t}, \quad \phi_x(x,t) = \Phi(x)\, e^{i\omega t}$$

In addition, we assume that there is no damping (i.e., $\hat{\mu} = 0$) and also omit thermal effects. The resulting equations for the various theories are summarized next.

3.4.2.1 Modified EBT

For investigation of the natural vibrations, the equations of motion in Eqs. (3.6) and (3.7) of the modified EBT reduce to the following set:

$$-\omega^2\Big(m_0 U - m_1 \frac{dW}{dx}\Big) - \frac{d\tilde{M}_{xx}^{(0)}}{dx} + \frac{d}{dx}\Big(\tilde{M}_{xz}^{(0)} \frac{dW}{dx}\Big) = 0 \tag{3.36}$$

$$-\omega^2\Big(m_0 W + m_1 \frac{dU}{dx} - m_2 \frac{d^2W}{dx^2}\Big) - \frac{d^2}{dx^2}\Big(\tilde{M}_{xx}^{(1)} - \tilde{M}_{xz}^{(1)} \frac{dW}{dx}\Big) - \frac{d}{dx}\left[\Big(\tilde{M}_{xx}^{(0)} + \tilde{M}_{zz}^{(0)} - P\Big)\frac{dW}{dx} - \tilde{M}_{xz}^{(0)} \frac{dU}{dx} + \tilde{M}_{xz}^{(1)} \frac{d^2W}{dx^2}\right] = 0 \tag{3.37}$$

where $(\tilde{M}_{xx}^{(0)}, \tilde{M}_{xx}^{(1)}, \tilde{M}_{zz}^{(0)}, \tilde{M}_{xz}^{(0)}, \tilde{M}_{xz}^{(1)})$ are defined by Eqs. (3.23)–(3.27); the tilde over the Ms is used to indicate that (u, w, θ_x) is to be substituted by $(U, W, -\frac{dW}{dx})$ in these definitions.

3.4.2.2 Modified TBT

The equations of motion presented in Eqs. (3.12)–(3.14) of the modified TBT model reduce, for natural vibration, to the following:

$$-\omega^2 \left(m_0 U + m_1 \Phi_x\right) - \frac{d}{dx}\left(\tilde{M}_{xx}^{(0)} + \tilde{M}_{xz}^{(0)} \Phi_x\right) = 0 \tag{3.38}$$

$$-\omega^2 m_0 W - \frac{d}{dx}\left(\tilde{M}_{xx}^{(0)} \frac{dW}{dx} + \tilde{M}_{xz}^{(0)} - P\frac{dW}{dx}\right) = 0 \tag{3.39}$$

$$\begin{aligned} -\omega^2 \left(m_1 U + m_2 \Phi_x\right) - \frac{d}{dx}\left(\tilde{M}_{xx}^{(1)} + \tilde{M}_{xz}^{(1)} \Phi_x\right) + \tilde{M}_{xz}^{(0)} + \tilde{M}_{xz}^{(0)} \frac{dU}{dx} \\ + \tilde{M}_{zz}^{(0)} \Phi_x + \tilde{M}_{xz}^{(1)} \frac{d\Phi_x}{dx} = 0 \end{aligned} \tag{3.40}$$

3.4.3 Buckling

In the case of buckling under axial compressive load P, we set all time derivative terms and externally applied mechanical and thermal forces to zero and obtain the governing equations. One can obtain these equations directly from the governing equations of natural vibration by omitting the frequency terms. For these modified theories, (U, W, Φ_x) denotes the solutions on the onset of buckling.

3.4.3.1 Modified EBT

The governing set of equations concerning buckling of beams according to the modified EBT are

$$-\frac{d\tilde{M}_{xx}^{(0)}}{dx} + \frac{d}{dx}\left(\tilde{M}_{xz}^{(0)} \frac{dW}{dx}\right) = 0 \tag{3.41}$$

$$\begin{aligned} -\frac{d}{dx}\left[\left(\tilde{M}_{xx}^{(0)} + \tilde{M}_{zz}^{(0)} - P\right)\frac{dW}{dx} - \tilde{M}_{xz}^{(0)} \frac{dU}{dx} + \tilde{M}_{xz}^{(1)} \frac{d^2W}{dx^2}\right] \\ - \frac{d^2}{dx^2}\left(\tilde{M}_{xx}^{(1)} - \tilde{M}_{xz}^{(1)} \frac{dW}{dx}\right) = 0 \end{aligned} \tag{3.42}$$

3.4.3.2 Modified TBT

For the modified TBT model, the derived governing equations become

$$-\frac{d}{dx}\left(\tilde{M}_{xx}^{(0)} + \tilde{M}_{xz}^{(0)}\Phi_x\right) = 0 \tag{3.43}$$

$$-\frac{d}{dx}\left(\tilde{M}_{xx}^{(0)}\frac{dW}{dx} + \tilde{M}_{xz}^{(0)} - P\frac{dW}{dx}\right) = 0 \tag{3.44}$$

$$-\frac{d}{dx}\left(\tilde{M}_{xx}^{(1)} + \tilde{M}_{xz}^{(1)}\Phi_x\right) + \tilde{M}_{xz}^{(0)} + \tilde{M}_{xz}^{(0)}\frac{dU}{dx} + \tilde{M}_{zz}^{(0)}\Phi_x + \tilde{M}_{xz}^{(1)}\frac{d\Phi_x}{dx} = 0 \tag{3.45}$$

The equations developed in this section are nonlinear and, in general, cannot be solved analytically. Numerical solutions are the best way to seek their solutions.

3.5 Concluding Comments

Although an analytical solution was obtained for the conventional theories in the previous section, it turns out that it is not possible to eliminate the axial displacement for the modified theories. Therefore, only numerical results for the modified theories can be obtained by means of computational efforts. The next section concerning nonlinear finite element analysis will elaborate on the use of this specific method for obtaining numerical results for nonlinear systems of equations, such as those seen in the two modified theories presented in this section.

4. NONLINEAR FINITE ELEMENT ANALYSIS

For the conventional theories in Section 2, an analytical solution was obtained due to the ability to remove the axial displacement from the governing equations of motion. However, the modified theories would not allow for an analytical solution due to the nature of the nonlinear terms present. The nonlinear nature of the theories requires computational efforts that utilize an iterative mathematical process so that the terms (du/dx), (dw/dx), ϕ_x and $(d\phi_x/dx)$ will yield an initial value, after the first iteration, that can be applied for the remaining iterations.

One such method that has been successfully applied for various applications is the finite element method (FEM). This method allows for the entire physical domain to be broken up into a finite number of smaller domains, or elements, where the appropriate constitutive and physical laws are applied to each element to allow for proper simulation of the modeled environment. The externally applied forces, initial boundary conditions associated with the domain and application at hand, and the specific material properties are applied to the model so that any unknown generalized displacements or resulting forces can be obtained after postprocessing of output data [36]. For the various types of theories being considered in this work, the FEM code provides general displacements for the axial and transverse displacement, as well as the rotation θ_x at each node.

The principle of virtual displacements, applied earlier but now omitting the virtual kinetic energy, will be applied to characterize the beam elements through virtual strains for the internal energy, and virtual displacements for the external work done. We will focus on the buckling application that was previously presented for the conventional theories, since the analytical solution was obtained specifically for buckling response.

Before developing the appropriate finite element models, the weak form of both the virtual strain energy and the virtual work done must be derived. This is carried out in the same manner as obtaining the equations of motion and natural boundary conditions of the beam based on the strain fields and resulting stress components. The principle of virtual displacements is of the form

$$\delta W^e \equiv \delta W_I^e - \delta W_E^e = 0 \tag{4.1}$$

where δW_I^e represents the virtual strain energy due to stresses moving through their virtual strains , and δW_E^e represents the work done by externally applied loads moving through their virtual displacements. The process and formulation of a nonlinear finite element model for both modified Euler–Bernoulli and Timoshenko beam theories will be discussed next. As well, for completeness sake the respective stiffness matrices for the conventional theories will be provided for comparison purposes.

4.1 Modified Euler–Bernoulli Theory

For completeness of the section and reference sake, the displacement and strain fields for the modified EBT, from Section 3, are

$$u_x(x,z,t) = u(x,t) + z\theta_x, \quad u_z(x,z,t) = w(x,t), \quad \theta_x \equiv -\frac{dw}{dx}$$

$$\varepsilon_{xx} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)}, \quad \gamma_{xz} = 2\varepsilon_{xz} = \gamma_{xz}^{(0)} + z\gamma_{xz}^{(1)}, \quad \varepsilon_{zz} = \varepsilon_{zz}^{(0)}$$

with

$$\varepsilon_{xx}^{(0)} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_{xx}^{(1)} = \frac{\partial \theta_x}{\partial x}$$

$$\gamma_{xz}^{(0)} = \frac{\partial u}{\partial x}\theta_x, \quad \gamma_{xz}^{(1)} = \theta_x \frac{\partial \theta_x}{\partial x}, \quad \varepsilon_{zz}^{(0)} = \frac{1}{2}\theta_x^2$$

The virtual internal strain energy and external work done are written, for this case, as

$$\begin{aligned} \delta W_I^e &= \int_{V^e} \sigma_{ij}\delta\varepsilon_{ij} dV \\ \delta W_E^e &= \int_{x_a}^{x_b} P \frac{dw}{dx}\frac{d\delta w}{dx} dx + \sum_{i=1}^{6} Q_i^e \delta\Delta_i^e \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \delta W_I^e &= \int_{x_a}^{x_b}\int_{A^e} \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xz}\delta\gamma_{xz} + \sigma_{zz}\delta\varepsilon_{zz}\right) dAdx \\ &= \int_{x_a}^{x_b}\int_{A^e} \left[\sigma_{xx}\left(\delta\varepsilon_{xx}^{(0)} + z\delta\varepsilon_{xx}^{(1)}\right) + \sigma_{xz}\left(\delta\gamma_{xz}^{(0)} + z\delta\gamma_{xz}^{(1)}\right) + \sigma_{zz}\delta\varepsilon_{zz}^{(0)}\right] dAdx \\ &= \int_{x_a}^{x_b}\int_{A^e} \left\{\sigma_{xx}\left[\left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}\right) + z\frac{d\delta\theta_x}{dx}\right] + \sigma_{zz}\theta_x\delta\theta_x \right. \\ &\qquad \left. + \sigma_{xz}\left[\frac{d\delta u}{dx}\theta_x + \frac{du}{dx}\delta\theta_x + z\left(\delta\theta_x\frac{d\theta_x}{dx} + \theta_x\frac{d\delta\theta_x}{dx}\right)\right]\right\} dAdx \\ &= \int_{x_a}^{x_b}\left[M_{xx}^{(0)}\left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}\right) + M_{xx}^{(1)}\frac{d\delta\theta_x}{dx} + M_{xz}^{(0)}\left(\frac{d\delta u}{dx}\theta_x + \frac{du}{dx}\delta\theta_x\right)\right. \\ &\qquad \left. + M_{xz}^{(1)}\left(\delta\theta_x\frac{d\theta_x}{dx} + \theta_x\frac{d\delta\theta_x}{dx}\right) + M_{zz}^{(0)}\,\theta_x\delta\theta_x\right] dx \end{aligned} \tag{4.3}$$

and the stress resultants used, that were previously defined, are

$$M_{xx}^{(0)} = \int_A \sigma_{xx}\, dA, \quad M_{xx}^{(1)} = \int_A z\,\sigma_{xx}\, dA$$
$$M_{xz}^{(0)} = \int_A \sigma_{xz}\, dA, \quad M_{xz}^{(1)} = \int_A z\sigma_{xz}\, dA, \quad M_{zz}^{(0)} = \int_A \sigma_{zz}\, dA$$

It should be noted that the time domain that is present within the displacement field is not included for any derivations, including the strain field, since we are only concerned with static loading for this model. As well, since F_v represents the viscous damping coefficient for a general solution that was discussed for the conventional and modified theories in Sections 2 and 3, it will not be included in the finite element model.

Including the external work done with the virtual strain energy, the following total weak form becomes:

$$0 = \int_{x_a}^{x_b} \left[M_{xx}^{(0)} \left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} \right) + M_{xx}^{(1)} \frac{d\delta\theta_x}{dx} + M_{xz}^{(0)} \left(\frac{du}{dx}\delta\theta_x + \frac{d\delta u}{dx}\theta_x \right) + M_{xz}^{(1)} \left(\delta\theta_x \frac{d\theta_x}{dx} + \theta_x \frac{d\delta\theta_x}{dx} \right) + M_{zz}^{(0)}\, \theta_x \delta\theta_x \right] dx - \int_{x_a}^{x_b} P \frac{dw}{dx}\frac{d\delta w}{dx} dx - \sum_{i=1}^{6} Q_i^e \delta\Delta_i^e \tag{4.4}$$

In order to relieve the virtual displacements δu, δw and $\delta\theta_x$ of any differentiation, and to obtain the natural (or force) boundary conditions, we perform integration by parts on the weak form that was just obtained. This becomes

$$0 = \int_{x_a}^{x_b} \Bigg\{ -\frac{d}{dx}\left(M_{xx}^{(0)} + M_{xz}^{(0)}\theta_x \right)\delta u - \frac{d}{dx}\left(M_{xx}^{(0)}\frac{dw}{dx} - P\frac{dw}{dx} \right)\delta w + \left[-\frac{d}{dx}\left(M_{xx}^{(1)} + M_{xz}^{(1)}\theta_x \right) + M_{xz}^{(0)}\frac{du}{dx} + M_{xz}^{(1)}\frac{d\theta_x}{dx} + M_{zz}^{(0)}\theta_x \right]\delta\theta_x \Bigg\} dx + \left[\left(M_{xx}^{(0)} + M_{xz}^{(0)}\theta_x \right)\delta u + \left(M_{xx}^{(0)}\frac{dw}{dx} - P\frac{dw}{dx} \right)\delta w + \left(M_{xx}^{(1)} + M_{xz}^{(1)}\theta_x \right)\delta\theta_x \right]_{x_a}^{x_b} - \sum_{i=1}^{6} Q_i^e \delta\Delta_i^e \tag{4.5}$$

which gives the natural boundary conditions

$$
\begin{aligned}
&Q_1^e + \left[M_{xx}^{(0)} + M_{xz}^{(0)}\theta_x\right]_{x_a} = 0, \qquad Q_4^e - \left[M_{xx}^{(0)} + M_{xz}^{(0)}\theta_x\right]_{x_b} = 0 \\
&Q_2^e + \left[M_{xx}^{(0)}\frac{dw}{dx} - P\frac{dw}{dx}\right]_{x_a} = 0, \qquad Q_5^e - \left[M_{xx}^{(0)}\frac{dw}{dx} - P\frac{dw}{dx}\right]_{x_b} = 0 \\
&Q_3^e + \left[M_{xx}^{(1)} + M_{xz}^{(1)}\theta_x\right]_{x_a} = 0, \qquad Q_6^e - \left[M_{xx}^{(1)} + M_{xz}^{(1)}\theta_x\right]_{x_b} = 0
\end{aligned}
\tag{4.6}
$$

Before we proceed to formulate the general form of the model, we need to write the stress resultants in terms of the displacements (u, w, θ_x):

$$
\begin{aligned}
M_{xx}^{(0)} &= \int_{A^e} \sigma_{xx} dA = \int_{A^e} E^e \varepsilon_{xx} dA = \int_{A^e} E^e \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\theta_x}{dx}\right] dA \\
&= A_{xx}^e \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + B_{xx}^e \frac{d\theta_x}{dx} && (4.7a) \\
M_{xx}^{(1)} &= \int_{A^e} z\sigma_{xx} dA = \int_{A^e} zE^e \varepsilon_{xx} dA = \int_{A^e} zE^e \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\theta_x}{dx}\right] dA \\
&= B_{xx}^e \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + D_{xx}^e \frac{d\theta_x}{dx} && (4.7b) \\
M_{xz}^{(0)} &= \int_{A^e} \sigma_{xz} dA = \int_{A^e} G^e \gamma_{xz} dA = \int_{A^e} G^e \left[\frac{du}{dx}\theta_x + z\theta_x\frac{d\theta_x}{dx}\right] dA \\
&= A_{xz}^e \frac{du}{dx}\theta_x + B_{xz}^e \theta_x \frac{d\theta_x}{dx} && (4.7c) \\
M_{xz}^{(1)} &= \int_{A^e} z\sigma_{xz} dA = \int_{A^e} zG^e \gamma_{xz} dA = \int_{A^e} zG^e \left[\frac{du}{dx}\theta_x + z\theta_x\frac{d\theta_x}{dx}\right] dA \\
&= B_{xz}^e \frac{du}{dx}\theta_x + D_{xz}^e \theta_x \frac{d\theta_x}{dx} && (4.7d) \\
M_{zz}^{(0)} &= \int_{A^e} \sigma_{zz} dA = \int_{A^e} E^e \sigma_{zz} dA = \int_{A^e} E^e \left(\frac{1}{2}{\theta_x}^2\right) dA \\
&= A_{xx}^e \left(\frac{1}{2}{\theta_x}^2\right) && (4.7e)
\end{aligned}
$$

Although these resultants, in terms of the new constants (A_{ij}, B_{ij}, D_{ij}) $(i, j = x, z)$, are exactly identical to the previously obtained results, the derivations in this section are more for clarity since we are using a different approach for analysis. As stated before, the notation for the constants can be written as

$$
\begin{aligned}
(A^e_{xx}, B^e_{xx}, D^e_{xx}) &= \int_{A^e} E^e\big(1, z, z^2\big)dA \\
(A^e_{xz}, B^e_{xz}, D^e_{xz}) &= K_s \int_{A^e} G^e\big(1, z, z^2\big)dA
\end{aligned}
$$

Using these explicit stress resultants in terms of the stiffness coefficients, we can substitute Eq. (4.7) into Eq. (4.4) so that the virtual displacements are expressed in terms of the generalized displacements, as

$$
\begin{aligned}
0 = \int_{x_a}^{x_b} \left\{ A_{xx}\frac{d\delta u}{dx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + A_{xz}\theta_x^2\frac{du}{dx}\frac{d\delta u}{dx}\right\}dx - Q_1\delta u(x_a) \\
- Q_4\delta u(x_b)
\end{aligned} \tag{4.8a}
$$

$$
\begin{aligned}
0 = \int_{x_a}^{x_b} \left\{ A_{xx}\frac{dw}{dx}\frac{d\delta w}{dx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\}dx - \int_{x_a}^{x_b} P\frac{dw}{dx}\frac{d\delta w}{dx}dx \\
- Q_2\delta w(x_a) - Q_5\delta w(x_b)
\end{aligned} \tag{4.8b}
$$

$$
\begin{aligned}
0 = \int_{x_a}^{x_b} \left\{ D_{xx}\frac{d\theta_x}{dx}\frac{d\delta\theta_x}{dx} + A_{xx}\frac{1}{2}{\theta_x}^3\delta\theta_x + A_{xz}\theta_x\left(\frac{du}{dx}\right)^2\delta\theta_x \right. \\
\left. + D_{xz}\theta_x\frac{d\theta_x}{dx}\left(\delta\theta_x\frac{d\theta_x}{dx} + \theta_x\frac{d\delta\theta_x}{dx}\right)\right\}dx - Q_3\delta\theta_x(x_a) - Q_6\delta\theta_x(x_b)
\end{aligned} \tag{4.8c}
$$

Since we are treating the beam elements in these models as isotropic, B_{xx} will become zero since the x-coordinate will coincide with the geometric centroidal axis of the beam, such that $\int_{A^e} z dA = 0$. As well, since θ_x is defined as the derivative of the transverse displacement, we must combine Eqs. (4.8b) and (4.8c) to create a single equation dedicated to the unknown displacement w, and to reduce any confusion about the number of independent functions being used. The new equation in terms of the unknown displacement w becomes

$$
\begin{aligned}
0 = \int_{x_a}^{x_b} & \left\{ A_{xx} \frac{dw}{dx}\frac{d\delta w}{dx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + D_{xx}\frac{d^2 w}{dx^2}\frac{d^2 \delta w}{dx^2} + A_{xz}\frac{dw}{dx}\left(\frac{du}{dx}\right)^2\frac{d\delta w}{dx} \right. \\
& \left. + D_{xz}\frac{dw}{dx}\frac{d^2 w}{dx^2}\left(\frac{d\delta w}{dx}\frac{d^2 w}{dx^2} + \frac{dw}{dx}\frac{d^2 \delta w}{dx^2}\right) + A_{xx}\frac{1}{2}\left(\frac{dw}{dx}\right)^3\frac{d\delta w}{dx}\right\} dx \\
& - \int_{x_a}^{x_b} P \frac{dw}{dx}\frac{d\delta w}{dx} dx - Q_2 \delta w(x_a) - Q_5 \delta w(x_b) \\
& + Q_3 \left(\frac{d\delta w}{dx}\right)\Big|_{x_a} + Q_6 \left(\frac{d\delta w}{dx}\right)\Big|_{x_b} \qquad (4.9)
\end{aligned}
$$

We interpolate the axial and transverse displacement as

$$
u(x) = \sum_{j=1}^{2} u_j \psi_j(x), \qquad w(x) = \sum_{j=1}^{4} \bar{\Delta}_j \phi_j(x) \qquad (4.10)
$$

$$
\bar{\Delta}_1 \equiv w(x_a), \quad \bar{\Delta}_2 \equiv \theta_x(x_a), \quad \bar{\Delta}_3 \equiv w(x_b), \quad \bar{\Delta}_4 \equiv \theta_x(x_b) \qquad (4.11)
$$

where ψ_j are the linear Lagrange interpolation functions and ϕ_j are Hermite cubic interpolation functions.

If we substitute Eqs. (4.10) and (4.11) into Eqs. (4.8a) and (4.9), we can obtain the following two equations written in terms of the generalized displacements:

$$
0 = \sum_{j=1}^{2} K_{ij}^{11} u_j + \sum_{J=1}^{4} K_{iJ}^{12} \bar{\Delta}_J - F_i^1 \quad (i = 1, 2) \qquad (4.12a)
$$

$$
0 = \sum_{j=1}^{2} K_{Ij}^{21} u_j + \sum_{J=1}^{4} K_{IJ}^{22} \bar{\Delta}_J - F_I^2 \quad (I = 1, 2, 3, 4) \qquad (4.12b)
$$

where

$$
\begin{aligned}
K_{ij}^{11} &= \int_{x_a}^{x_b} \left[A_{xx} \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} + A_{xz} \left(\frac{dw}{dx} \right)^2 \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} \right] dx \\
K_{iJ}^{12} &= \frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\psi_i}{dx} \frac{d\phi_J}{dx} dx \\
K_{Ij}^{21} &= \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\phi_I}{dx} \frac{d\psi_j}{dx} dx \\
K_{IJ}^{22} &= \int_{x_a}^{x_b} \left[A_{xx} \left(\frac{dw}{dx} \right)^2 \frac{d\phi_I}{dx} \frac{d\phi_J}{dx} + D_{xz} \frac{dw}{dx} \frac{d^2 w}{dx^2} \left(\frac{d\phi_I}{dx} \frac{d^2 \phi_J}{dx^2} + \frac{d^2 \phi_I}{dx^2} \frac{d\phi_J}{dx} \right) \right. \\
&\qquad \left. + D_{xx} \frac{d^2 \phi_I}{dx^2} \frac{d^2 \phi_J}{dx^2} \right] dx
\end{aligned} \tag{4.13a}
$$

and

$$
\begin{aligned}
F_i^1 &= \tilde{Q}_i, \quad F_I^2 = F_1^2 + F_2^2 + F_3^2 + F_4^2 \\
F_1^2 &= Q_2, \quad F_2^2 = \int_{x_a}^{x_b} P \frac{dw}{dx} \frac{d\phi_2}{dx} dx - Q_3 \\
F_3^2 &= Q_5, \quad F_4^2 = \int_{x_a}^{x_b} P \frac{dw}{dx} \frac{d\phi_4}{dx} dx - Q_6
\end{aligned} \tag{4.13b}
$$

for $(i, j = 1, 2)$ and $(I, J = 1, 2, 3, 4)$, where $\tilde{Q}_1 = Q_1$ and $\tilde{Q}_2 = Q_4$.

We can compact Eqs. (4.12a,b) to be written as

$$
\sum_{p=1}^{2} K_{ip}^{\alpha 1} u_p + \sum_{P=1}^{4} K_{iP}^{\alpha 2} \bar{\Delta}_P = F_i^\alpha \tag{4.14}
$$

or in matrix and vector notation as

$$
\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{\Delta^1\} \\ \{\Delta^2\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix} \tag{4.15}
$$

where we have

$$
\Delta_i^1 = u_i, \quad i = 1, 2; \quad \Delta_i^2 = \bar{\Delta}_i, \quad i = 1, 2, 3, 4 \tag{4.16}
$$

Since our model is nonlinear, we cannot apply the direct stiffness matrix due to the lack of symmetry. To overcome this, we need to develop a tangent stiffness matrix that contains components that are the linearized set of equations of the direct stiffness matrix components. From [36] we can either apply a direct iteration procedure or the Newton–Raphson method, which differentiates the residual vector of each component with respect to the generalized displacements. The Newton–Raphson method is widely used [50], [51], [52], [53], [54], [55] in nonlinear FE models, and the derivations for the tangent matrix components of the modified Euler–Bernoulli theory are carried out next.

As mentioned, the tangent matrix components are functions of the residual vector of each direct stiffness component that is differentiated with respect to the generalized displacements. This definition is given by

$$T_{ij}^{\alpha\beta} = \left(\frac{\partial R_i^\alpha}{\partial \Delta_j^\beta} \right)^{(r-1)} \tag{4.17}$$

where

$$\begin{aligned} R_i^\alpha &= \sum_{\gamma=1}^{2} \sum_{p=1} K_{ip}^{\alpha\gamma} \Delta_p^\gamma - F_i^\alpha \\ &= \sum_{p=1}^{2} K_{ip}^{\alpha 1} \Delta_p^1 + \sum_{P=1}^{4} K_{iP}^{\alpha 2} \Delta_P^2 - F_i^\alpha \\ &= \sum_{p=1}^{2} K_{ip}^{\alpha 1} u_p + \sum_{P=1}^{4} K_{iP}^{\alpha 2} \bar{\Delta}_P - F_i^\alpha \end{aligned} \tag{4.18}$$

This gives us the full explicit form of the tangent components as

$$T_{ij}^{\alpha\beta} = \left(\frac{\partial R_i^\alpha}{\partial \Delta_j^\beta} \right) = \frac{\partial}{\partial \Delta_j^\beta} \left(\sum_{\gamma=1}^{2} \sum_{p=1} K_{ip}^{\alpha\gamma} \Delta_p^\gamma - F_i^\alpha \right)$$

$$= \sum_{\gamma=1}^{2} \sum_{p=1} \left(K_{ip}^{\alpha\gamma} \frac{\partial \Delta_p^{\gamma}}{\partial \Delta_j^{\beta}} + \frac{\partial K_{ip}^{\alpha\gamma}}{\partial \Delta_j^{\beta}} \Delta_p^{\gamma} \right)$$

$$= K_{ij}^{\alpha\beta} + \sum_{p=1}^{2} \frac{\partial}{\partial \Delta_j^{\beta}} \left(K_{ip}^{\alpha 1} \right) u_p + \sum_{P=1}^{4} \frac{\partial}{\partial \Delta_j^{\beta}} \left(K_{ip}^{\alpha 2} \right) \bar{\Delta}_P \tag{4.19}$$

For our tangent matrix, the components can be obtained from Eq. (4.19) as:

$$\begin{aligned} T_{ij}^{11} &= K_{ij}^{11} + \sum_{p=1}^{2} \left(\frac{\partial K_{ip}^{11}}{\partial u_j} \right) u_p + \sum_{P=1}^{4} \left(\frac{\partial K_{iP}^{12}}{\partial u_j} \right) \bar{\Delta}_p \\ &= K_{ij}^{11} + \sum_{p=1}^{2} 0 \cdot u_p + \sum_{P=1}^{4} 0 \cdot \bar{\Delta}_P \\ T_{ij}^{11} &= K_{ij}^{11} \end{aligned} \tag{4.20}$$

Since our derivatives are with respect to generalized displacements, and the superscript β governs which displacement we are concerned with, any component that requires $\beta = 1$ will result in differentiation with respect to u_j. Since none of our components contain du/dx terms, certain tangent components will be equal to their respective initial direct stiffness components, such that:

$$\left[T^{11}\right] = \left[K^{11}\right], \qquad \left[T^{21}\right] = \left[K^{21}\right] \tag{4.21}$$

For the remaining components, T_{iJ}^{12} and T_{IJ}^{22}, that are not equivalent to their initial direct stiffness matrix components, the derivations are condensed in this section but the full derivations can be found in Appendix A for reference.

$$
\begin{aligned}
T^{12}_{iJ} &= K^{12}_{iJ} + 2\int_{x_a}^{x_b} A_{xz}\frac{du}{dx}\frac{dw}{dx}\frac{d\psi_i}{dx}\frac{d\phi_J}{dx}dx + \frac{1}{2}\int_{x_a}^{x_b} A_{xx}\frac{dw}{dx}\frac{d\psi_i}{dx}\frac{d\phi_J}{dx}dx \\
&= 2K^{12}_{iJ} + 2\int_{x_a}^{x_b} A_{xz}\frac{du}{dx}\frac{dw}{dx}\frac{d\psi_i}{dx}\frac{d\phi_J}{dx}dx \\
T^{22}_{IJ} &= K^{22}_{IJ} + \int_{x_a}^{x_b} A_{xx}\frac{du}{dx}\frac{d\phi_I}{dx}\frac{d\phi_J}{dx}dx + \int_{x_a}^{x_b}\left\{2A_{xx}\left(\frac{dw}{dx}\right)^2\frac{d\phi_I}{dx}\frac{d\phi_J}{dx}\right. \\
&\left. + D_{xz}\left(\frac{d\phi_J}{dx}\frac{d^2w}{dx^2} + \frac{dw}{dx}\frac{d^2\phi_J}{dx^2}\right)\left(\frac{d\phi_I}{dx}\frac{d^2w}{dx^2} + \frac{d^2\phi_I}{dx^2}\frac{dw}{dx}\right)\right\}dx
\end{aligned}
\tag{4.22}
$$

If we compare the resulting tangent components with the direct stiffness components we can see that our stiffness matrix is not symmetric like the conventional theories have. This is attributed to the additional shear component

$$
\int_{x_a}^{x_b} A_{xz}\left(\frac{dw}{dx}\right)^2\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx
$$

that appears in K^{11}_{ij}. Since this term does not disappear from differentiation, we are left with an unsymmetric stiffness matrix. Although this results in more computation time, it can be handled with an unsymmetric banded equations solver. For the sake of completeness, and to give the reader a better understanding of the symmetric nature of the direct and tangent stiffness matrices, both matrices of the conventional

EBT are listed as [36]:

$$
\begin{aligned}
K_{ij}^{11} &= \int_{x_a}^{x_b} A_{xx} \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} dx \\
K_{iJ}^{12} &= \frac{1}{2} \int_{x_a}^{x_b} \left(A_{xx} \frac{dw}{dx} \right) \frac{d\psi_i}{dx} \frac{d\phi_J}{dx} dx \\
K_{Ij}^{21} &= 2K_{jI}^{12} \\
K_{IJ}^{22} &= \int_{x_a}^{x_b} \left\{ D_{xx} \frac{d^2\phi_I}{dx^2} \frac{d^2\phi_J}{dx^2} + \frac{1}{2} \left[A_{xx} \left(\frac{dw}{dx} \right)^2 \right] \frac{d\phi_I}{dx} \frac{d\phi_J}{dx} \right\} dx \\
T_{ij}^{11} &= K_{ij}^{11} \\
T_{iJ}^{12} &= K_{iJ}^{12} + \int_{x_a}^{x_b} \left(\frac{1}{2} A_{xx} \frac{dw}{dx} \right) \frac{d\psi_i}{dx} \frac{d\phi_J}{dx} dx = 2K_{iJ}^{12} = K_{Ji}^{21} \\
T_{IJ}^{22} &= K_{IJ}^{22} + \int_{x_a}^{x_b} A_{xx} \left(\frac{du}{dx} + \frac{dw}{dx} \frac{dw}{dx} \right) \frac{d\phi_I}{dx} \frac{d\phi_J}{dx} dx
\end{aligned}
\tag{4.23}
$$

For the stiffness components, we require a different combination of linear Lagrange and Hermite cubic interpolation functions. This results in a set of submatrices that each contain a different order: 2×2, 2×4, 4×2 and 4×4 for $[K^{11}]$, $[K^{12}]$, $[K^{21}]$ and $[K^{22}]$, respectively. This yields a stiffness matrix of the form

$$
\begin{bmatrix}
K_{11}^{11} & K_{12}^{11} & K_{11}^{12} & K_{12}^{12} & K_{13}^{12} & K_{14}^{12} \\
K_{21}^{11} & K_{22}^{11} & K_{21}^{12} & K_{22}^{12} & K_{23}^{12} & K_{24}^{12} \\
K_{11}^{21} & K_{12}^{21} & K_{11}^{22} & K_{12}^{22} & K_{13}^{22} & K_{14}^{22} \\
K_{21}^{21} & K_{22}^{21} & K_{21}^{22} & K_{22}^{22} & K_{23}^{22} & K_{24}^{22} \\
K_{31}^{21} & K_{32}^{21} & K_{31}^{22} & K_{32}^{22} & K_{33}^{22} & K_{34}^{22} \\
K_{41}^{21} & K_{42}^{21} & K_{41}^{22} & K_{42}^{22} & K_{43}^{22} & K_{44}^{22}
\end{bmatrix}
\begin{Bmatrix} u_1^e \\ u_2^e \\ \bar{\Delta}_1^e \\ \bar{\Delta}_2^e \\ \bar{\Delta}_3^e \\ \bar{\Delta}_4^e \end{Bmatrix}
=
\begin{Bmatrix} F_1^1 \\ F_2^1 \\ F_1^2 \\ F_2^2 \\ F_3^2 \\ F_4^2 \end{Bmatrix}
\tag{4.24}
$$

Reorganizing the matrix such that the displacement vector is in order according to nodal displacements, the resulting stiffness matrix becomes

$$\begin{bmatrix} K_{11}^{11} & K_{11}^{12} & K_{12}^{12} & K_{12}^{11} & K_{13}^{12} & K_{14}^{12} \\ K_{11}^{21} & K_{11}^{22} & K_{12}^{22} & K_{12}^{21} & K_{13}^{22} & K_{14}^{22} \\ K_{21}^{21} & K_{21}^{22} & K_{22}^{22} & K_{22}^{21} & K_{23}^{22} & K_{24}^{22} \\ K_{21}^{11} & K_{21}^{12} & K_{22}^{12} & K_{22}^{11} & K_{23}^{12} & K_{24}^{12} \\ K_{31}^{21} & K_{31}^{22} & K_{32}^{22} & K_{32}^{21} & K_{33}^{22} & K_{34}^{22} \\ K_{41}^{21} & K_{41}^{22} & K_{42}^{22} & K_{42}^{21} & K_{43}^{22} & K_{44}^{22} \end{bmatrix} \begin{Bmatrix} u_1^e \\ \bar{\Delta}_1^e \\ \bar{\Delta}_2^e \\ u_2^e \\ \bar{\Delta}_3^e \\ \bar{\Delta}_4^e \end{Bmatrix} = \begin{Bmatrix} F_1^1 \\ F_1^2 \\ F_2^2 \\ F_2^1 \\ F_3^2 \\ F_4^2 \end{Bmatrix} \tag{4.25}$$

4.2 Modified Timoshenko Theory

From Section 3, the displacement field and resulting modified strain field of the modified TBT are

$$u_x(x,z,t) = u(x,t) + z\phi_x(x,t), \quad u_z(x,z,t) = w(x,t)$$

and

$$\varepsilon_{xx} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)}, \quad \gamma_{xz} = \gamma_{xz}^{(0)} + z\gamma_{xz}^{(1)}, \quad \varepsilon_{zz} = \varepsilon_{zz}^{(0)}$$

with

$$\varepsilon_{xx}^{(0)} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_{xx}^{(1)} = \frac{\partial \phi_x}{\partial x}$$

$$\gamma_{xz}^{(0)} = \phi_x + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\phi_x, \quad \gamma_{xz}^{(1)} = \phi_x \frac{\partial \phi_x}{\partial x}, \quad \varepsilon_{zz}^{(0)} = \frac{1}{2}\phi_x^2$$

The external work done for the modified TBT case is the same as δW_E^e in Eq. (4.2), however the virtual internal strain energy differs slightly, giving us

$$\delta W_I^e = \int_{x_a}^{x_b} \int_{A^e} \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xz}\delta\gamma_{xz} + \sigma_{zz}\delta\varepsilon_{zz}\right) dA dx$$

$$= \int_{x_a}^{x_b} \int_{A^e} \left[\sigma_{xx} \left(\delta\varepsilon_{xx}^{(0)} + z\delta\varepsilon_{xx}^{(1)} \right) + \sigma_{xz} \left(\delta\gamma_{xz}^{(0)} + z\delta\gamma_{xz}^{(1)} \right) + \sigma_{zz}\delta\varepsilon_{zz}^{(0)} \right] dAdx$$

$$= \int_{x_a}^{x_b} \left[M_{xx}^{(0)} \left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} \right) + M_{xx}^{(1)} \frac{d\delta\phi_x}{dx} + M_{xz}^{(1)} \left(\delta\phi_x \frac{d\phi_x}{dx} + \phi_x \frac{d\delta\phi_x}{dx} \right) \right.$$
$$\left. + M_{zz}^{(0)} \phi_x\delta\phi_x + M_{xz}^{(0)} \left(\delta\phi_x + \frac{d\delta w}{dx} + \frac{d\delta u}{dx}\phi_x + \frac{du}{dx}\delta\phi_x \right) \right] dx \tag{4.26}$$

Combining our virtual internal strain energy and virtual external work done, we get the following form for the principle of virtual displacements, or Hamilton's principle as seen before, for the modified TBT case:

$$0 = \int_{x_a}^{x_b} \left[M_{xx}^{(0)} \left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} \right) + M_{xx}^{(1)} \frac{d\delta\phi_x}{dx} + M_{xz}^{(1)} \left(\delta\phi_x \frac{d\phi_x}{dx} + \phi_x \frac{d\delta\phi_x}{dx} \right) \right.$$
$$\left. + M_{xz}^{(0)} \left(\delta\phi_x + \frac{d\delta w}{dx} + \frac{d\delta u}{dx}\phi_x + \frac{du}{dx}\delta\phi_x \right) + M_{zz}^{(0)} \phi_x\delta\phi_x \right] dx$$
$$- \int_{x_a}^{x_b} P\frac{dw}{dx}\frac{d\delta w}{dx}dx - \sum_{i=1}^{6} Q_i^e \delta\Delta_i^e \tag{4.27}$$

The stress resultants for the modified TBT case are now defined as

$$M_{xx}^{(0)} = \int_{A^e} \sigma_{xx} dA = \int_{A^e} E^e \varepsilon_{xx} dA = \int_{A^e} E^e \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\phi_x}{dx} \right] dA$$
$$= A_{xx}^e \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \right] + B_{xx}^e \frac{d\phi_x}{dx} \tag{4.28a}$$

$$M_{xx}^{(1)} = \int_{A^e} z\sigma_{xx} dA = \int_{A^e} zE^e \varepsilon_{xx} dA = \int_{A^e} zE^e \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + z\frac{d\phi_x}{dx} \right] dA$$
$$= B_{xx}^e \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \right] + D_{xx}^e \frac{d\phi_x}{dx} \tag{4.28b}$$

$$Q_x^{(0)} = \int_{A^e} \sigma_{xz} dA = K_s \int_{A^e} G^e \gamma_{xz} dA = K_s \int_{A^e} G^e \left[\phi_x + \frac{dw}{dx} + \frac{du}{dx}\phi_x + z\phi_x\frac{d\phi_x}{dx} \right] dA$$
$$= S_{xz}^e \left(\phi_x + \frac{dw}{dx} + \frac{du}{dx}\phi_x \right) + \tilde{S}_{xz}^e \phi_x \frac{d\phi_x}{dx} \tag{4.28c}$$

$$Q_x^{(1)} = \int_{A^e} z\sigma_{xz} dA = K_s \int_{A^e} zG^e \gamma_{xz} dA = K_s \int_{A^e} zG^e \left[\phi_x + \frac{dw}{dx} + \frac{du}{dx}\phi_x + z\phi_x\frac{d\phi_x}{dx} \right] dA$$

$$= \tilde{S}^e_{xz}\left(\phi_x + \frac{dw}{dx} + \frac{du}{dx}\phi_x\right) + \hat{S}^e_{xz}\phi_x\frac{d\phi_x}{dx} \tag{4.28d}$$

$$M^{(0)}_{zz} = \int_{A^e}\sigma_{zz}dA = \int_{A^e}E^e\sigma_{zz}dA = \int_{A^e}E^e\left(\frac{1}{2}\phi_x^2\right)dA$$

$$= A^e_{xx}\left(\frac{1}{2}{\phi_x}^2\right) \tag{4.28e}$$

where we must now account for the shear correction coefficient K_s that appears in $M^{(0)}_{xz}$ and $M^{(1)}_{xz}$. For these two stress resultants we changed the notation from $(M^{(0)}_{xz}, M^{(1)}_{xz})$ to $(Q^{(0)}_x, Q^{(1)}_x)$, and defined new coefficients as

$$(S^e_{xz}, \tilde{S}^e_{xz}, \hat{S}^e_{xz}) = K_s\int_{A^e}G^e\big(1, z, z^2\big)dA \tag{4.29}$$

For the general finite element model we will add a third equation to account for the generalized displacement ϕ_x. This will increase the order of the stiffness matrix from 2×2 to 3×3. The three equations in terms of the generalized displacements (u, w, ϕ_x) become:

$$0 = \int_{x_a}^{x_b}\left\{A_{xx}\frac{d\delta u}{dx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + S_{xz}\frac{d\delta u}{dx}\phi_x\left[\phi_x + \frac{dw}{dx} + \frac{du}{dx}\phi_x\right]\right\}dx$$
$$- Q^e_1\delta u(x_a) - Q^e_4\delta u(x_b) \tag{4.30a}$$

$$0 = \int_{x_a}^{x_b}\left\{A_{xx}\frac{d\delta w}{dx}\frac{dw}{dx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + S_{xz}\frac{d\delta w}{dx}\left[\phi_x + \frac{dw}{dx} + \frac{du}{dx}\phi_x\right]\right\}dx$$
$$- \int_{x_a}^{x_b}P\frac{dw}{dx}\frac{d\delta w}{dx}dx - Q^e_2\delta w(x_a) - Q^e_5\delta w(x_b) \tag{4.30b}$$

$$0 = \int_{x_a}^{x_b}\left\{A_{xx}\left(\frac{1}{2}{\phi_x}^2\right)\phi_x\delta\phi_x + D_{xx}\frac{d\phi_x}{dx}\frac{d\delta\phi_x}{dx} + \hat{S}_{xz}\phi_x\frac{d\phi_x}{dx}\left(\delta\phi_x\frac{d\phi_x}{dx} + \phi_x\frac{d\delta\phi_x}{dx}\right)\right.$$
$$\left. + \left[S_{xz}\left(\phi_x + \frac{dw}{dx} + \frac{du}{dx}\phi_x\right)\right]\left(\delta\phi_x + \frac{du}{dx}\delta\phi_x\right)\right\}dx$$
$$- Q^e_3\delta\phi_x(x_a) - Q^e_6\delta\phi_x(x_b) \tag{4.30c}$$

Combining these three equations and integrating by parts we are able to weaken the differentiability of the generalized displacements, allowing us to obtain the natural boundary conditions. Although the full derivation can be found in Appendix A, for the sake of clarity and ease of reference, the natural boundary conditions become

$$\begin{aligned}
&Q_1^e + \left(A_{xx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + S_{xz}\phi_x\left[\phi_x + \frac{dw}{dx} + \frac{du}{dx}\phi_x\right]\right)_{x_a} = 0\\
&Q_4^e - \left(A_{xx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + S_{xz}\phi_x\left[\phi_x + \frac{dw}{dx} + \frac{du}{dx}\phi_x\right]\right)_{x_b} = 0\\
&Q_2^e + \left(A_{xx}\frac{dw}{dx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + S_{xz}\left[\phi_x + \frac{dw}{dx} + \frac{du}{dx}\phi_x\right]\right)_{x_a} = 0\\
&Q_5^e - \left(A_{xx}\frac{dw}{dx}\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + S_{xz}\left[\phi_x + \frac{dw}{dx} + \frac{du}{dx}\phi_x\right]\right)_{x_b} = 0\\
&Q_3^e + \left(D_{xx}\frac{d\phi_x}{dx} + \hat{S}_{xz}\left(\phi_x\right)^2\frac{d\phi_x}{dx}\right)_{x_a} = 0\\
&Q_6^e - \left(D_{xx}\frac{d\phi_x}{dx} + \hat{S}_{xz}\left(\phi_x\right)^2\frac{d\phi_x}{dx}\right)_{x_b} = 0
\end{aligned} \tag{4.31}$$

Since we have one more generalized displacement in the Timoshenko theory, compared to the Euler–Bernoulli theory, we must use three independent interpolation functions. The three generalized displacements u, w and ϕ_x can be approximated as

$$u(x) = \sum_{j=1}^{m} u_j^e \psi_j^{(1)}, \quad w(x) = \sum_{j=1}^{n} w_j^e \psi_j^{(2)}, \quad \phi_x(x) = \sum_{j=1}^{p} s_j^e \psi_j^{(3)} \tag{4.32}$$

where $\psi_j^{(\alpha)}(x)$ $(\alpha = 1, 2, 3)$ are the Lagrange interpolation functions of degree $(m - 1)$, $(n - 1)$ and $(p - 1)$, respectively. Now, if the substitution $(\delta u, \delta w, \delta\phi_x) = (\psi_i^{(1)}, \psi_i^{(2)}, \psi_i^{(3)})$ is made into Eqs. (4.30a,b,c), we get the following nonlinear finite element model:

$$0 = \sum_{j=1}^{m} K_{ij}^{11} u_j^e + \sum_{j=1}^{n} K_{ij}^{12} w_j^e + \sum_{j=1}^{p} K_{ij}^{13} s_j^e - F_i^1 \tag{4.33}$$

$$0 = \sum_{j=1}^{m} K_{ij}^{21} u_j^e + \sum_{j=1}^{n} K_{ij}^{22} w_j^e + \sum_{j=1}^{p} K_{ij}^{23} s_j^e - F_i^2 \tag{4.34}$$

$$0 = \sum_{j=1}^{m} K_{ij}^{31} u_j^e + \sum_{j=1}^{n} K_{ij}^{32} w_j^e + \sum_{j=1}^{p} K_{ij}^{33} s_j^e - F_i^3 \tag{4.35}$$

where

$$\begin{aligned}
K_{ij}^{11} &= \int_{x_a}^{x_b} \left[A_{xx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(1)}}{dx} + S_{xz} {\phi_x}^2 \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(1)}}{dx} \right] dx \\
K_{ij}^{12} &= \int_{x_a}^{x_b} \left[\frac{1}{2} A_{xx} \frac{dw}{dx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(2)}}{dx} + S_{xz} \phi_x \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(2)}}{dx} \right] dx \\
K_{ij}^{13} &= \int_{x_a}^{x_b} S_{xz} \frac{du}{dx} \phi_x \frac{d\psi_i^{(1)}}{dx} \psi_j^{(3)} dx \\
K_{ij}^{21} &= \int_{x_a}^{x_b} \left[A_{xx} \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(1)}}{dx} + S_{xz} \phi_x \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(1)}}{dx} \right] dx \\
K_{ij}^{22} &= \int_{x_a}^{x_b} \left[\frac{1}{2} A_{xx} \left(\frac{dw}{dx} \right)^2 \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} + S_{xz} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} \right] dx \\
K_{ij}^{23} &= \int_{x_a}^{x_b} S_{xz} \frac{d\psi_i^{(2)}}{dx} \psi_j^{(3)} dx \\
K_{ij}^{31} &= \int_{x_a}^{x_b} S_{xz} \left(\phi_x + \frac{dw}{dx} + \frac{du}{dx} \phi_x \right) \psi_i^{(3)} \frac{d\psi_j^{(1)}}{dx} dx \\
K_{ij}^{32} &= \int_{x_a}^{x_b} S_{xz} \psi_i^{(3)} \frac{d\psi_j^{(2)}}{dx} dx \\
K_{ij}^{33} &= \int_{x_a}^{x_b} \left[\frac{1}{2} A_{xx} {\phi_x}^2 \psi_i^{(3)} \psi_j^{(3)} + D_{xx} \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(3)}}{dx} + S_{xz} \frac{du}{dx} \psi_i^{(3)} \psi_j^{(3)} \right. \\
&\qquad \left. + S_{xz} \psi_i^{(3)} \psi_j^{(3)} + S_{xz} \phi_x \frac{d\phi_x}{dx} \frac{d\psi_i^{(3)}}{dx} \psi_j^{(3)} + \hat{S}_{xz} \phi_x \frac{d\phi_x}{dx} \psi_i^{(3)} \frac{d\psi_j^{(3)}}{dx} \right] dx
\end{aligned} \tag{4.36a}$$

and

$$
\begin{aligned}
F_i^1 &= Q_1^e \psi_i^{(1)}(x_a) + Q_4^e \psi_i^{(1)}(x_b) \\
F_i^2 &= \int_{x_a}^{x_b} P \frac{dw}{dx}\frac{d\psi_i^{(2)}}{dx} dx + Q_2^e \psi_i^{(2)}(x_a) + Q_5^e \psi_i^{(2)}(x_b) \\
F_i^3 &= Q_3^e \psi_i^{(3)}(x_a) + Q_6^e \psi_i^{(3)}(x_b)
\end{aligned}
\tag{4.36b}
$$

Similary like Eq. (4.15), we can combine Eqs. (4.33), (4.34) and (4.35) to form the matrix and vector notation as

$$
\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] \\ [K^{21}] & [K^{22}] & [K^{23}] \\ [K^{31}] & [K^{32}] & [K^{33}] \end{bmatrix}
\begin{Bmatrix} \{u\} \\ \{w\} \\ \{s\} \end{Bmatrix}
=
\begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \end{Bmatrix}
\tag{4.37}
$$

The components $[K^{ij}]\,(i,j=1,2,3)$ in this stiffness matrix are condensed into this form to save space since the size of the component matrices are dependent upon the order of the interpolation functions [36].

Similar to determining the Euler–Bernoulli tangent matrix components by means of the Newton–Rhapson method, the tangent stiffness components for the Timoshenko case are determined from the following equation:

$$
\begin{aligned}
T_{ij}^{\alpha\beta} &= K_{ij}^{\alpha\beta} + \sum_{\gamma=1}^{3}\sum_{k=1}^{n} \frac{\partial}{\partial \Delta_j^{\beta}} \left(K_{ik}^{\alpha\gamma}\right) \Delta_k^{\gamma} \\
T_{ij}^{\alpha\beta} &= K_{ij}^{\alpha\beta} + \sum_{r=1}^{m} \frac{\partial}{\partial \Delta_j^{\beta}} \left(K_{ir}^{\alpha 1}\right) u_r + \sum_{t=1}^{n} \frac{\partial}{\partial \Delta_j^{\beta}} \left(K_{it}^{\alpha 2}\right) w_t \\
&\quad + \sum_{v=1}^{p} \frac{\partial}{\partial \Delta_j^{\beta}} \left(K_{iv}^{\alpha 3}\right) s_v
\end{aligned}
\tag{4.38}
$$

Using the definition that

$$\begin{Bmatrix} \Delta_j^1 \\ \Delta_j^2 \\ \Delta_j^3 \end{Bmatrix} = \begin{Bmatrix} u_j \\ w_j \\ s_j \end{Bmatrix} \tag{4.39}$$

we get the following components:

$$\begin{aligned}
T_{ij}^{11} &= K_{ij}^{11} + \int_{x_a}^{x_b} S_{xz}{\phi_x}^2 \frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(1)}}{dx}dx \\
T_{ij}^{12} &= K_{ij}^{12} + \frac{1}{2}\int_{x_a}^{x_b} A_{xx}\frac{dw}{dx}\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(2)}}{dx}dx \\
T_{ij}^{13} &= K_{ij}^{13} + 3\int_{x_a}^{x_b} S_{xz}\frac{du}{dx}\phi_x\frac{d\psi_i^{(1)}}{dx}\psi_j^{(3)}dx + \int_{x_a}^{x_b} S_{xz}\frac{dw}{dx}\frac{d\psi_i^{(1)}}{dx}\psi_j^{(3)}dx \\
T_{ij}^{21} &= K_{ij}^{21} \\
T_{ij}^{22} &= K_{ij}^{22} + \int_{x_a}^{x_b} A_{xx}\frac{du}{dx}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx}dx + \int_{x_a}^{x_b} A_{xx}\left(\frac{dw}{dx}\right)^2\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx}dx \\
T_{ij}^{23} &= K_{ij}^{23} + \int_{x_a}^{x_b} S_{xz}\frac{du}{dx}\frac{d\psi_i^{(2)}}{dx}\psi_j^{(3)}dx \\
T_{ij}^{31} &= K_{ij}^{31} + \int_{x_a}^{x_b} S_{xz}\frac{du}{dx}\phi_x\psi_i^{(3)}\frac{d\psi_j^{(1)}}{dx}dx + \int_{x_a}^{x_b} S_{xz}\phi_x\psi_i^{(3)}\frac{d\psi_j^{(1)}}{dx}dx \\
T_{ij}^{32} &= K_{ij}^{32} + \int_{x_a}^{x_b} S_{xz}\frac{du}{dx}\psi_i^{(3)}\frac{d\psi_j^{(2)}}{dx}dx \\
T_{ij}^{33} &= K_{ij}^{33} + \int_{x_a}^{x_b} S_{xz}\frac{du}{dx}\left(\psi_i^{(3)}\psi_j^{(3)} + \frac{du}{dx}\psi_i^{(3)}\psi_j^{(3)}\right)dx \\
&\quad + \int_{x_a}^{x_b}\Bigg\{A_{xx}{\phi_x}^2\psi_i^{(3)}\psi_j^{(3)} + S_{xz}\phi_x\left(\frac{d\phi_x}{dx}\frac{d\psi_i^{(3)}}{dx}\psi_j^{(3)} + \phi_x\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(3)}}{dx}\right) \\
&\qquad + \hat{S}_{xz}\frac{d\phi_x}{dx}\left(\frac{d\phi_x}{dx}\psi_i^{(3)}\psi_j^{(3)} + \phi_x\psi_i^{(3)}\frac{d\psi_j^{(3)}}{dx}\right)\Bigg\}dx
\end{aligned} \tag{4.40}$$

with the conventional TBT components being:

$$
\begin{aligned}
K_{ij}^{11} &= \int_{x_a}^{x_b} A_{xx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(1)}}{dx} dx \\
K_{ij}^{12} &= \frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx \\
K_{ij}^{13} &= 0 \\
K_{ij}^{21} &= \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(1)}}{dx} dx \\
K_{ij}^{22} &= \int_{x_a}^{x_b} S_{xx} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx + \frac{1}{2} \int_{x_a}^{x_b} A_{xx} \left(\frac{dw}{dx}\right)^2 \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx \\
K_{ij}^{23} &= \int_{x_a}^{x_b} S_{xx} \frac{d\psi_i^{(2)}}{dx} \psi_j^{(3)} dx = K_{ji}^{32} \\
K_{ij}^{31} &= 0 \\
K_{ij}^{33} &= \int_{x_a}^{x_b} \left(D_{xx} \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(3)}}{dx} + S_{xx} \psi_i^{(3)} \psi_j^{(3)} \right) dx \\
T_{ij}^{11} &= K_{ij}^{11} \\
T_{ij}^{12} &= K_{ij}^{12} + \frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx = 2K_{ij}^{12} \\
T_{ij}^{13} &= K_{ij}^{13} = 0 \\
T_{ij}^{21} &= K_{ij}^{21} \\
T_{ij}^{22} &= K_{ij}^{22} + \int_{x_a}^{x_b} A_{xx} \left[\frac{du}{dx} + \left(\frac{dw}{dx}\right)^2 \right] \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx \\
T_{ij}^{23} &= K_{ij}^{23} \\
T_{ij}^{31} &= K_{ij}^{31} \\
T_{ij}^{32} &= K_{ij}^{32} \\
T_{ij}^{33} &= K_{ij}^{33}
\end{aligned}
\tag{4.41}
$$

4.3 Results

To allow for a comparison of the analytical solution results that were obtained in the second section, a computer program that contained a finite element model for the conventional EBT was used. Results were obtained to reflect the transverse deflection experienced by axially applied loading for the buckling application. Due to the nature of the program, loading requirements, and input for specific applications, the results were obtained as more of a parametric study on the incremental loading and required initial imperfection. The conventional EBT model in the program contained the stiffness and tangent matrices, as well as the natural boundary conditions, that were stated in the previous section.

For the case of axially applied loading and the uniform and isotropic nature of the beam, there are no initial geometric conditions that allows the code to recognize that a physical transverse deflection occurs along the length of the beam. This is due to the fact that the loading is prescribed at a node such that it is considered to be uniformly distributed about the cross sectional area of that node. Unfortunately this alters the physical nature of the problem to that of a bar under axial loading, resulting strictly in axial displacement and no transverse deflection. Therefore an initial imperfection in the form of a very small initial transverse displacement around the center of the beam is applied. This initial condition is applied to the first iteration during loading and is then removed so that the resulting deflection can be applied for the second, and consecutive, iterations. Since the deflection is so small, and goes through an iterative process, it does not appear in the output results. The axially applied loading is given an incremental value, while a load step size is specified to properly capture the deflections of the associated loading value at each step, before and after buckling has occurred. At the onset of buckling small transverse deflections are observed, and the load value at that incremental step is characterized as the resulting buckling load.

Table 10 gives the various boundary conditions used for the conventional EBT from Section 2, and lists the number of elements used in the mesh as well as the initial imperfection value, resulting buckling load, and the buckling load obtained from the analytical solution. The difference in incremental loading size can be seen with respect to the different boundary conditions. The goal was to obtain a range of small incremental sizes and observe what imperfection value was needed to obtain the buckling load that resulted from the analytical solution of the respective boundary conditions. For the clamped–hinged case, the load increment size had to be increased in order to obtain a critical value close to that of the analytical solution, whereas for the clamped–clamped case, there was only one value that allowed for both a convergent solution and the proper buckling load.

Although the initial goal was to obtain the buckling load independent of any variables that can be modified by the user, the use of an initial transverse displacement as an imperfection dictates the fact that a convergent solution of the critical load will eventually be achieved based on the magnitude of the initial displacement. This, in turn, calls for a study that reflects the use of an imperfection and the effects that it, as well as the incremental loading, has on computational results of a buckling application based on the program being employed. Due to the complexity of not only the conventional Timoshenko model but both modified theories presented in Section 3, further work must be carried out to investigate the numerical results of the buckling application by using the finite element method. Although likely to be in somewhat of a good agreement with the analytical solution, the difference in response due to the aspect ratio should be carefully examined.

Table 10: FEM results for various boundary conditions of conventional EBT model; Newton–Raphson iterative process.

Boundary Conditions	Elements in Mesh	Load Increment	Trans. Defl. Imperfection	Buckling Load	Analytical
H–H	24	0.0125	7.25×10^{-3}	3.15	3.14159
		0.025	3.5×10^{-8}	3.15	
	32	0.0125	6.05×10^{-3}	3.15	
		0.025	4.5×10^{-8}	3.15	
	48	0.0125	4.68×10^{-3}	3.15	
		0.025	6.8×10^{-8}	3.15	
	50	0.0125	4.4×10^{-3}	3.15	
		0.025	6.8×10^{-8}	3.15	
C–C	24	0.1	9.99×10^{-2}	7.5	6.2832
	32	0.1	9.9×10^{-2}	6.8	
	48	0.1	9×10^{-2}	6.4	
	50	0.1	9×10^{-2}	6.4	
C–H	24	0.09	9.999×10^{-2}	6.12	4.4943
		0.1	3.25×10^{-2}	4.5	
		0.3	4.35×10^{-5}	4.5	
	32	0.09	9.85×10^{-2}	5.76	
		0.1	7.3×10^{-2}	4.5	
	48	0.09	5×10^{-2}	4.68	
		0.1	5.5×10^{-2}	4.5	
	50	0.09	8.5×10^{-2}	5.76	
		0.1	5.3×10^{-2}	4.5	

5. CONCLUSIONS

This work has two major contributions: First, the conventional Euler–Bernoulli and Timoshenko beam theories are generalized to include nonlinear terms arising from ε_{zz} and ε_{xz}, which are of the same magnitude as the von Kármán nonlinear terms appearing in ε_{xx}. The additional terms can be interpreted as microstructural length scale effects. The associated equations of motion, derived using Hamilton's principle, make use of two-dimensional constitutive relations. These equations can be used to determine bending, vibration and dynamic stability of beams. Second, analytical solutions for the onset of buckling of both classical Euler–Bernoulli and traditional Timoshenko beams for various boundary conditions are presented. The analytical solutions are developed by eliminating the axial displacement from the equations and reducing the nonlinearity to a constant. Numerical results for buckling loads are presented to show the effect of transverse shear deformation as a function of the beam height-to-length ratio for beams with rectangular cross sections. The buckling loads predicted by the Timoshenko beam theory are lower than those predicted by the conventional Euler–Bernoulli beam theory as the height-to-length ratio increases, indicating that the effect of shear deformation is significant in short beams.

Nonlinear finite element models for both Euler–Bernoulli and Timoshenko beams are developed, and results for the conventional Euler–Bernoulli theory to compare the buckling loads against those obtained in the analytical solutions are presented. Various initial imperfection values, applied as initial transverse deflections at the midspan of the beam, are presented to show the need for an initial geometry that differs from that of a perfectly straight beam in computational buckling applications.

Additional studies to investigate length scale effects of bending, vibration, and post-buckling behavior using numerical methods including generalized beam theories, as well as finite element results of the modified theories, are awaiting attention.

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APPENDIX A

FULL DERIVATIONS REMOVED FROM THE TEXT

Variation of principle of virtual displacements for modified Timoshenko theory to obtain natural boundary conditions. (Section 4: Nonlinear Finite Element Analysis.)

$$
\begin{aligned}
0 = \int_{x_a}^{x_b} \Bigg\{ & -\frac{d}{dx}\left(A_{xx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right)\delta u - \frac{d}{dx}\left(S_{xz}\phi_x\left[\phi_x+\frac{dw}{dx}+\frac{du}{dx}\phi_x\right]\right)\delta u \\
& - \frac{d}{dx}\left(A_{xx}\frac{dw}{dx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right)\delta w - \frac{d}{dx}\left(S_{xz}\left[\phi_x+\frac{dw}{dx}+\frac{du}{dx}\phi_x\right]\right)\delta w \\
& + A_{xx}\left(\frac{1}{2}{\phi_x}^3\right)\delta\phi_x - \frac{d}{dx}\left(D_{xx}\frac{d\phi_x}{dx}\right)\delta\phi_x + \hat{S}_{xz}\phi_x\left(\frac{d\phi_x}{dx}\right)^2\delta\phi_x \\
& - \frac{d}{dx}\left(\hat{S}_{xz}\left(\phi_x\right)^2\frac{d\phi_x}{dx}\right)\delta\phi_x + S_{xz}\left(\phi_x+\frac{dw}{dx}+\frac{du}{dx}\phi_x\right)\left(1+\frac{du}{dx}\right)\delta\phi_x\Bigg\}dx \\
& + \Bigg[\left(A_{xx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + S_{xz}\phi_x\left[\phi_x+\frac{dw}{dx}+\frac{du}{dx}\phi_x\right]\right)\delta u \\
& + \left(A_{xx}\frac{dw}{dx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + S_{xz}\left[\phi_x+\frac{dw}{dx}+\frac{du}{dx}\phi_x\right]\right)\delta w \\
& + \left(D_{xx}\frac{d\phi_x}{dx}+\hat{S}_{xz}\left(\phi_x\right)^2\frac{d\phi_x}{dx}\right)\delta\phi_x\Bigg]_{x_a}^{x_b} - \int_{x_a}^{x_b} P\frac{dw}{dx}\frac{d\delta w}{dx}dx - \sum_{i=1}^{6} Q_i^e\delta\Delta_i^e
\end{aligned}
$$

Full derivations for the tangent stiffness matrix components T_{ij} that are not equivalent to their initial direct stiffness matrix components. (Section 4: Nonlinear Finite Element Analysis.)

Modified Euler–Bernoulli theory:

$$
\begin{aligned}
T_{iJ}^{12} &= K_{iJ}^{12} + \sum_{p=1}^{2} \left(\frac{\partial K_{ip}^{11}}{\partial \bar{\Delta}_J} \right) u_p + \sum_{P=1}^{4} \left(\frac{\partial K_{iP}^{12}}{\partial \bar{\Delta}_J} \right) \bar{\Delta}_p \\
&= K_{iJ}^{12} + \sum_{p=1}^{2} \left[\int_{x_a}^{x_b} A_{xz} \frac{\partial}{\partial \bar{\Delta}_J} \left(\frac{dw}{dx} \right)^2 \frac{d\psi_i}{dx} \frac{d\psi_p}{dx} dx \right] u_p \\
&\quad + \sum_{P=1}^{4} \left[\frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{\partial}{\partial \bar{\Delta}_J} \left(\frac{dw}{dx} \right) \frac{d\psi_i}{dx} \frac{d\phi_P}{dx} dx \right] \bar{\Delta}_P \\
&= K_{iJ}^{12} + \sum_{p=1}^{2} \left[\int_{x_a}^{x_b} A_{xz} \frac{\partial}{\partial \bar{\Delta}_J} \left(\sum_{K}^{4} \bar{\Delta}_K \frac{\partial \phi_K}{\partial x} \right)^2 \frac{d\psi_i}{dx} \frac{d\psi_p}{dx} dx \right] u_p \\
&\quad + \sum_{P=1}^{4} \left[\frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{\partial}{\partial \bar{\Delta}_J} \left(\sum_{K}^{4} \bar{\Delta}_K \frac{\partial \phi_K}{\partial x} \right) \frac{d\psi_i}{dx} \frac{d\phi_P}{dx} dx \right] \bar{\Delta}_P \\
&= K_{iJ}^{12} + 2 \int_{x_a}^{x_b} A_{xz} \frac{dw}{dx} \frac{d\psi_i}{dx} \frac{d\phi_J}{dx} \left(\sum_{p=1}^{2} \frac{d\psi_p}{dx} u_p \right) dx \\
&\quad + \frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{d\psi_i}{dx} \frac{d\phi_J}{dx} \left(\sum_{P=1}^{4} \frac{d\phi_P}{dx} \bar{\Delta}_P \right) dx \\
&= K_{iJ}^{12} + 2 \int_{x_a}^{x_b} A_{xz} \frac{du}{dx} \frac{dw}{dx} \frac{d\psi_i}{dx} \frac{d\phi_J}{dx} dx + \frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\psi_i}{dx} \frac{d\phi_J}{dx} dx \\
&= 2K_{iJ}^{12} + 2 \int_{x_a}^{x_b} A_{xz} \frac{du}{dx} \frac{dw}{dx} \frac{d\psi_i}{dx} \frac{d\phi_J}{dx} dx
\end{aligned}
$$

$$
\begin{aligned}
T_{IJ}^{22} =& K_{IJ}^{22} + \sum_{p=1}^{2} \left(\frac{\partial K_{Ip}^{21}}{\partial \bar{\Delta}_J} \right) u_p + \sum_{P=1}^{4} \left(\frac{\partial K_{IP}^{22}}{\partial \bar{\Delta}_J} \right) \bar{\Delta}_p \\
=& K_{IJ}^{22} + \sum_{p=1}^{2} \left[\int_{x_a}^{x_b} A_{xx} \frac{\partial}{\partial \bar{\Delta}_J} \left(\frac{dw}{dx} \right) \frac{d\phi_I}{dx} \frac{d\psi_p}{dx} dx \right] u_p \\
& + \sum_{P=1}^{4} \left[\int_{x_a}^{x_b} \left\{ A_{xx} \frac{\partial}{\partial \bar{\Delta}_J} \left(\frac{dw}{dx} \right)^2 \frac{d\phi_I}{dx} \frac{d\phi_P}{dx} \right. \right. \\
& \left. \left. + D_{xz} \frac{\partial}{\partial \bar{\Delta}_J} \left(\frac{dw}{dx} \frac{d^2w}{dx^2} \right) \left(\frac{d\phi_I}{dx} \frac{d^2\phi_P}{dx^2} + \frac{d^2\phi_I}{dx^2} \frac{d\phi_P}{dx} \right) \right\} dx \right] \bar{\Delta}_P \\
=& K_{IJ}^{22} + \sum_{p=1}^{2} \left[\int_{x_a}^{x_b} A_{xx} \frac{\partial}{\partial \bar{\Delta}_J} \left(\sum_{K}^{4} \bar{\Delta}_K \frac{\partial \phi_K}{\partial x} \right) \frac{d\phi_I}{dx} \frac{d\psi_p}{dx} dx \right] u_p \\
& + \sum_{P=1}^{4} \left[\int_{x_a}^{x_b} \left\{ A_{xx} \frac{\partial}{\partial \bar{\Delta}_J} \left(\sum_{K}^{4} \bar{\Delta}_K \frac{\partial \phi_K}{\partial x} \right)^2 \frac{d\phi_I}{dx} \frac{d\phi_P}{dx} \right. \right. \\
& + D_{xz} \left[\frac{\partial}{\partial \bar{\Delta}_J} \left(\sum_{K}^{4} \bar{\Delta}_K \frac{\partial \phi_K}{\partial x} \right) \frac{d^2 w}{dx^2} + \frac{dw}{dx} \frac{\partial}{\partial \bar{\Delta}_J} \left(\sum_{K}^{4} \bar{\Delta}_K \frac{\partial^2 \phi_K}{\partial x^2} \right) \right] \\
& \left. \left. \times \left(\frac{d\phi_I}{dx} \frac{d^2\phi_P}{dx^2} + \frac{d^2\phi_I}{dx^2} \frac{d\phi_P}{dx} \right) \right\} dx \right] \bar{\Delta}_P \\
=& K_{IJ}^{22} + \int_{x_a}^{x_b} A_{xx} \frac{d\phi_J}{dx} \frac{d\phi_I}{dx} \left(\sum_{p=1}^{2} \frac{d\psi_p}{dx} u_p \right) dx + \int_{x_a}^{x_b} \left\{ 2A_{xx} \frac{dw}{dx} \frac{d\phi_J}{dx} \frac{d\phi_I}{dx} \right. \\
& \times \left(\sum_{P=1}^{4} \frac{d\phi_P}{dx} \right) \bar{\Delta}_P + D_{xz} \left(\frac{d\phi_J}{dx} \frac{d^2 w}{dx^2} + \frac{dw}{dx} \frac{d^2\phi_J}{dx^2} \right) \left(\frac{d\phi_I}{dx} \left(\sum_{P=1}^{4} \frac{d^2\phi_P}{dx^2} \right) \bar{\Delta}_P \right. \\
& \left. \left. + \frac{d^2\phi_I}{dx^2} \left(\sum_{P=1}^{4} \frac{d\phi_P}{dx} \right) \bar{\Delta}_P \right) \right\} dx \\
=& K_{IJ}^{22} + \int_{x_a}^{x_b} A_{xx} \frac{du}{dx} \frac{d\phi_I}{dx} \frac{d\phi_J}{dx} dx + \int_{x_a}^{x_b} \left\{ 2A_{xx} \left(\frac{dw}{dx} \right)^2 \frac{d\phi_I}{dx} \frac{d\phi_J}{dx} \right. \\
& \left. + D_{xz} \left(\frac{d\phi_J}{dx} \frac{d^2 w}{dx^2} + \frac{dw}{dx} \frac{d^2\phi_J}{dx^2} \right) \left(\frac{d\phi_I}{dx} \frac{d^2 w}{dx^2} + \frac{d^2\phi_I}{dx^2} \frac{dw}{dx} \right) \right\} dx
\end{aligned}
$$

Modified Timoshenko theory:

$$
\begin{aligned}
T_{ij}^{11} =& K_{ij}^{11} + \sum_{r=1}^{m} \left(\frac{\partial K_{ir}^{11}}{\partial u_j} \right) u_r + \sum_{t=1}^{n} \left(\frac{\partial K_{it}^{12}}{\partial u_j} \right) w_t + \sum_{v=1}^{p} \left(\frac{\partial K_{iv}^{13}}{\partial u_j} \right) s_v \\
=& K_{ij}^{11} + \sum_{r=1}^{m} 0 \cdot u_r + \sum_{t=1}^{n} 0 \cdot w_t + \sum_{v=1}^{p} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial u_j} \left(\frac{du}{dx} \right) \phi_x \frac{d\psi_i^{(1)}}{dx} \psi_v^{(3)} dx \right] s_v \\
=& K_{ij}^{11} + \sum_{v=1}^{p} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial u_j} \left(\sum_{K}^{m} u_K \frac{\partial \psi_K^{(1)}}{\partial x} \right) \phi_x \frac{d\psi_i^{(1)}}{dx} \psi_v^{(3)} dx \right] s_v \\
=& K_{ij}^{11} + \int_{x_a}^{x_b} S_{xz} \frac{d\psi_j^{(1)}}{dx} \phi_x \frac{d\psi_i^{(1)}}{dx} \left(\sum_{v=1}^{p} \psi_v^{(3)} s_v \right) dx \\
=& K_{ij}^{11} + \int_{x_a}^{x_b} S_{xz} {\phi_x}^2 \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(1)}}{dx} dx
\end{aligned}
$$

$$
\begin{aligned}
T_{ij}^{12} =& K_{ij}^{12} + \sum_{r=1}^{m} \left(\frac{\partial K_{ir}^{11}}{\partial w_j} \right) u_r + \sum_{t=1}^{n} \left(\frac{\partial K_{it}^{12}}{\partial w_j} \right) w_t + \sum_{v=1}^{p} \left(\frac{\partial K_{iv}^{13}}{\partial w_j} \right) s_v \\
=& K_{ij}^{12} + \sum_{r=1}^{m} 0 \cdot u_r + \sum_{t=1}^{n} \left[\frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{\partial}{\partial w_j} \left(\frac{dw}{dx} \right) \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_t^{(2)}}{dx} dx \right] w_t + \sum_{v=1}^{p} 0 \cdot s_v \\
=& K_{ij}^{12} + \sum_{t=1}^{n} \left[\frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{\partial}{\partial w_j} \left(\sum_{K}^{n} w_K \frac{\partial \psi_K^{(2)}}{\partial x} \right) \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_t^{(2)}}{dx} dx \right] w_t \\
=& K_{ij}^{12} + \frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{d\psi_j^{(2)}}{dx} \frac{d\psi_i^{(1)}}{dx} \left(\sum_{t=1}^{n} \frac{d\psi_t^{(2)}}{dx} w_t \right) dx \\
=& K_{ij}^{12} + \frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx
\end{aligned}
$$

$$
\begin{aligned}
T_{ij}^{13} =& K_{ij}^{13} + \sum_{r=1}^{m} \left(\frac{\partial K_{ir}^{11}}{\partial s_j} \right) u_r + \sum_{t=1}^{n} \left(\frac{\partial K_{it}^{12}}{\partial s_j} \right) w_t + \sum_{v=1}^{p} \left(\frac{\partial K_{iv}^{13}}{\partial s_j} \right) s_v \\
=& K_{ij}^{13} + \sum_{r=1}^{m} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial s_j} \left(\phi_x\right)^2 \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_r^{(1)}}{dx} dx \right] u_r \\
& + \sum_{t=1}^{n} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial s_j} \left(\phi_x\right) \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_t^{(2)}}{dx} dx \right] w_t \\
& + \sum_{v=1}^{p} \left[\int_{x_a}^{x_b} S_{xz} \frac{du}{dx} \frac{\partial}{\partial s_j} \left(\phi_x\right) \frac{d\psi_i^{(1)}}{dx} \psi_v^{(3)} dx \right] s_v \\
=& K_{ij}^{13} + \sum_{r=1}^{m} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial s_j} \left(\sum_{K}^{p} s_K \psi_K^{(3)} \right)^2 \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_r^{(1)}}{dx} dx \right] u_r \\
& + \sum_{t=1}^{n} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial s_j} \left(\sum_{K}^{p} s_K \psi_K^{(3)} \right) \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_t^{(2)}}{dx} dx \right] w_t \\
& + \sum_{v=1}^{p} \left[\int_{x_a}^{x_b} S_{xz} \frac{du}{dx} \frac{\partial}{\partial s_j} \left(\sum_{K}^{p} s_K \psi_K^{(3)} \right) \frac{d\psi_i^{(1)}}{dx} \psi_v^{(3)} dx \right] s_v \\
=& K_{ij}^{13} + 2 \int_{x_a}^{x_b} S_{xz} \phi_x \psi_j^{(3)} \frac{d\psi_i^{(1)}}{dx} \left(\sum_{r=1}^{m} \frac{d\psi_r^{(1)}}{dx} u_r \right) dx \\
& + \int_{x_a}^{x_b} S_{xz} \psi_j^{(3)} \frac{d\psi_i^{(1)}}{dx} \left(\sum_{t=1}^{n} \frac{d\psi_t^{(2)}}{dx} w_t \right) dx \\
& + \int_{x_a}^{x_b} S_{xz} \frac{du}{dx} \psi_j^{(3)} \frac{d\psi_i^{(1)}}{dx} \left(\sum_{v=1}^{p} \psi_v^{(3)} s_v \right) dx \\
=& K_{ij}^{13} + 3 \int_{x_a}^{x_b} S_{xz} \frac{du}{dx} \phi_x \frac{d\psi_i^{(1)}}{dx} \psi_j^{(3)} dx + \int_{x_a}^{x_b} S_{xz} \frac{dw}{dx} \frac{d\psi_i^{(1)}}{dx} \psi_j^{(3)} dx
\end{aligned}
$$

$$
\begin{aligned}
T_{ij}^{21} &= K_{ij}^{21} + \sum_{r=1}^{m} \left(\frac{\partial K_{ir}^{21}}{\partial u_j} \right) u_r + \sum_{t=1}^{n} \left(\frac{\partial K_{it}^{22}}{\partial u_j} \right) w_t + \sum_{v=1}^{p} \left(\frac{\partial K_{iv}^{23}}{\partial u_j} \right) s_v \\
&= K_{ij}^{21} + \sum_{r=1}^{m} 0 \cdot u_r + \sum_{t=1}^{n} 0 \cdot w_t + \sum_{v=1}^{p} 0 \cdot s_v \\
&= K_{ij}^{21}
\end{aligned}
$$

$$
\begin{aligned}
T_{ij}^{22} =& K_{ij}^{22} + \sum_{r=1}^{m} \left(\frac{\partial K_{ir}^{21}}{\partial w_j} \right) u_r + \sum_{t=1}^{n} \left(\frac{\partial K_{it}^{22}}{\partial w_j} \right) w_t + \sum_{v=1}^{p} \left(\frac{\partial K_{iv}^{23}}{\partial w_j} \right) s_v \\
=& K_{ij}^{22} + \sum_{r=1}^{m} \left[\int_{x_a}^{x_b} A_{xx} \frac{\partial}{\partial w_j} \left(\frac{dw}{dx} \right) \frac{d\psi_j^{(2)}}{dx} \frac{d\psi_r^{(1)}}{dx} dx \right] u_r \\
&+ \sum_{t=1}^{n} \left[\frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{\partial}{\partial w_j} \left(\frac{dw}{dx} \right)^2 \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_t^{(2)}}{dx} dx \right] w_t + \sum_{v=1}^{p} 0 \cdot s_v \\
=& K_{ij}^{22} + \sum_{r=1}^{m} \left[\int_{x_a}^{x_b} A_{xx} \frac{\partial}{\partial w_j} \left(\sum_{K}^{n} w_K \frac{\partial \psi_K^{(2)}}{\partial x} \right) \frac{d\psi_j^{(2)}}{dx} \frac{d\psi_r^{(1)}}{dx} dx \right] u_r \\
&+ \sum_{t=1}^{n} \left[\frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{\partial}{\partial w_j} \left(\sum_{K}^{n} w_K \frac{\partial \psi_K^{(2)}}{\partial x} \right)^2 \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_t^{(2)}}{dx} dx \right] w_t \\
=& K_{ij}^{22} + \int_{x_a}^{x_b} A_{xx} \frac{d\psi_j^{(2)}}{dx} \frac{d\psi_i^{(2)}}{dx} \left(\sum_{r=1}^{m} \frac{d\psi_r^{(1)}}{dx} u_r \right) dx \\
&+ \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\psi_j^{(2)}}{dx} \frac{d\psi_i^{(2)}}{dx} \left(\sum_{t=1}^{n} \frac{d\psi_t^{(2)}}{dx} w_t \right) dx \\
=& K_{ij}^{22} + \int_{x_a}^{x_b} A_{xx} \frac{du}{dx} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx + \int_{x_a}^{x_b} A_{xx} \left(\frac{dw}{dx} \right)^2 \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx
\end{aligned}
$$

$$
\begin{aligned}
T_{ij}^{23} =& K_{ij}^{23} + \sum_{r=1}^{m} \left(\frac{\partial K_{ir}^{21}}{\partial s_j} \right) u_r + \sum_{t=1}^{n} \left(\frac{\partial K_{it}^{22}}{\partial s_j} \right) w_t + \sum_{v=1}^{p} \left(\frac{\partial K_{iv}^{23}}{\partial s_j} \right) s_v \\
=& K_{ij}^{23} + \sum_{r=1}^{m} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial s_j} (\phi_x) \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_r^{(1)}}{dx} dx \right] u_r + \sum_{t=1}^{n} 0 \cdot w_t + \sum_{v=1}^{p} 0 \cdot s_v \\
=& K_{ij}^{23} + \sum_{r=1}^{m} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial s_j} \left(\sum_{K}^{p} s_K \psi_K^{(3)} \right) \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_r^{(1)}}{dx} dx \right] u_r \\
=& K_{ij}^{23} + \int_{x_a}^{x_b} S_{xz} \psi_j^{(3)} \frac{d\psi_i^{(2)}}{dx} \left(\sum_{r=1}^{m} \frac{d\psi_r^{(1)}}{dx} u_r \right) dx \\
=& K_{ij}^{23} + \int_{x_a}^{x_b} S_{xz} \frac{du}{dx} \frac{d\psi_i^{(2)}}{dx} \psi_j^{(3)} dx
\end{aligned}
$$

$$
\begin{aligned}
T_{ij}^{31} =& K_{ij}^{31} + \sum_{r=1}^{m} \left(\frac{\partial K_{ir}^{31}}{\partial u_j} \right) u_r + \sum_{t=1}^{n} \left(\frac{\partial K_{it}^{32}}{\partial u_j} \right) w_t + \sum_{v=1}^{p} \left(\frac{\partial K_{iv}^{33}}{\partial u_j} \right) s_v \\
=& K_{ij}^{31} + \sum_{r=1}^{m} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial u_j} \left(\frac{du}{dx} \right) \phi_x \psi_i^{(3)} \frac{d\psi_r^{(1)}}{dx} dx \right] u_r + \sum_{t=1}^{n} 0 \cdot w_t \\
& + \sum_{v=1}^{p} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial u_j} \left(\frac{du}{dx} \right) \psi_i^{(3)} \psi_v^{(3)} dx \right] s_v \\
=& K_{ij}^{31} + \sum_{r=1}^{m} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial u_j} \left(\sum_{K}^{m} u_K \frac{\partial \psi_K^{(1)}}{dx} \right) \phi_x \psi_i^{(3)} \frac{d\psi_r^{(1)}}{dx} dx \right] u_r \\
& + \sum_{v=1}^{p} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial u_j} \left(\sum_{K}^{m} u_K \frac{\partial \psi_K^{(1)}}{dx} \right) \psi_i^{(3)} \psi_v^{(3)} dx \right] s_v \\
=& K_{ij}^{31} + \int_{x_a}^{x_b} S_{xz} \frac{d\psi_j^{(1)}}{dx} \phi_x \psi_i^{(3)} \left(\sum_{r=1}^{m} \frac{d\psi_r^{(1)}}{dx} u_r \right) dx \\
& + \int_{x_a}^{x_b} S_{xz} \frac{d\psi_j^{(1)}}{dx} \psi_i^{(3)} \left(\sum_{v=1}^{p} \psi_v^{(3)} s_v \right) dx \\
=& K_{ij}^{31} + \int_{x_a}^{x_b} S_{xz} \frac{du}{dx} \phi_x \psi_i^{(3)} \frac{d\psi_j^{(1)}}{dx} dx + \int_{x_a}^{x_b} S_{xz} \phi_x \psi_i^{(3)} \frac{d\psi_j^{(1)}}{dx} dx
\end{aligned}
$$

$$
\begin{aligned}
T_{ij}^{32} =& K_{ij}^{32} + \sum_{r=1}^{m} \left(\frac{\partial K_{ir}^{31}}{\partial w_j} \right) u_r + \sum_{t=1}^{n} \left(\frac{\partial K_{it}^{32}}{\partial w_j} \right) w_t + \sum_{v=1}^{p} \left(\frac{\partial K_{iv}^{33}}{\partial w_j} \right) s_v \\
=& K_{ij}^{32} + \sum_{r=1}^{m} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial w_j} \left(\frac{dw}{dx} \right) \psi_i^{(3)} \frac{d\psi_r^{(1)}}{dx} dx \right] u_r + \sum_{t=1}^{n} 0 \cdot w_t + \sum_{v=1}^{p} 0 \cdot s_v \\
=& K_{ij}^{32} + \sum_{r=1}^{m} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial w_j} \left(\sum_{K}^{n} w_K \frac{\partial \psi_K^{(2)}}{\partial x} \right) \psi_i^{(3)} \frac{d\psi_r^{(1)}}{dx} dx \right] u_r \\
=& K_{ij}^{32} + \int_{x_a}^{x_b} S_{xz} \frac{d\psi_j^{(2)}}{dx} \psi_i^{(3)} \left(\sum_{r=1}^{m} \frac{d\psi_r^{(1)}}{dx} u_r \right) dx \\
=& K_{ij}^{32} + \int_{x_a}^{x_b} S_{xz} \frac{du}{dx} \psi_i^{(3)} \frac{d\psi_j^{(2)}}{dx} dx
\end{aligned}
$$

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\begin{aligned}
T_{ij}^{33} =& K_{ij}^{33} + \sum_{r=1}^{m} \left(\frac{\partial K_{ir}^{31}}{\partial s_j} \right) u_r + \sum_{t=1}^{n} \left(\frac{\partial K_{it}^{32}}{\partial s_j} \right) w_t + \sum_{v=1}^{p} \left(\frac{\partial K_{iv}^{33}}{\partial s_j} \right) s_v \\
=& K_{ij}^{33} + \sum_{r=1}^{m} \left[\int_{x_a}^{x_b} S_{xz} \frac{\partial}{\partial s_j} \left(\phi_x + \frac{du}{dx} \phi_x \right) \psi_i^{(3)} \frac{d\psi_r^{(3)}}{dx} dx \right] u_r + \sum_{t=1}^{n} 0 \cdot w_t \\
& + \sum_{v=1}^{p} \left[\int_{x_a}^{x_b} \left\{ \frac{1}{2} A_{xx} \frac{\partial}{\partial s_j} (\phi_x)^2 \psi_i^{(3)} \psi_v^{(3)} + S_{xz} \frac{\partial}{\partial s_j} \left(\phi_x \frac{d\phi_x}{dx} \right) \frac{d\psi_i^{(3)}}{dx} \psi_v^{(3)} \right.\right. \\
& \left.\left. \qquad + \hat{S}_{xz} \frac{\partial}{\partial s_j} \left(\phi_x \frac{d\phi_x}{dx} \right) \psi_i^{(3)} \frac{d\psi_v^{(3)}}{dx} \right\} dx \right] s_v \\
=& K_{ij}^{33} + \sum_{r=1}^{m} \left[\int_{x_a}^{x_b} S_{xz} \left[\frac{\partial}{\partial s_j} \left(\sum_{K}^{p} s_K \psi_K^{(3)} \right) + \frac{du}{dx} \frac{\partial}{\partial s_j} \left(\sum_{K}^{p} s_K \psi_K^{(3)} \right) \right] \right. \\
& \left. \qquad \times \psi_i^{(3)} \frac{d\psi_r^{(1)}}{dx} dx \right] u_r \\
& + \sum_{v=1}^{p} \left[\int_{x_a}^{x_b} \left\{ \frac{1}{2} A_{xx} \frac{\partial}{\partial s_j} \left(\sum_{K}^{p} s_K \psi_K^{(3)} \right)^2 \psi_i^{(3)} \psi_v^{(3)} \right.\right. \\
& + S_{xz} \left[\frac{\partial}{\partial s_j} \left(\sum_{K}^{p} s_K \psi_K^{(3)} \right) \frac{d\phi_x}{dx} + \phi_x \frac{\partial}{\partial s_j} \left(\sum_{K}^{p} s_K \frac{\partial \psi_K^{(3)}}{\partial x} \right) \right] \frac{d\psi_i^{(3)}}{dx} \psi_v^{(3)} \\
& \left.\left. + \hat{S}_{xz} \left[\frac{\partial}{\partial s_j} \left(\sum_{K}^{p} s_K \psi_K^{(3)} \right) \frac{d\phi_x}{dx} + \phi_x \frac{\partial}{\partial s_j} \left(\sum_{K}^{p} s_K \frac{\partial \psi_K^{(3)}}{\partial x} \right) \right] \psi_i^{(3)} \frac{d\psi_v^{(3)}}{dx} \right\} dx \right] s_v \\
=& K_{ij}^{33} + \int_{x_a}^{x_b} S_{xz} \left(\psi_j^{(3)} + \frac{du}{dx} \psi_j^{(3)} \right) \psi_i^{(3)} \left(\sum_{r=1}^{m} \frac{d\psi_r^{(1)}}{dx} u_r \right) dx \\
& + \int_{x_a}^{x_b} \left\{ A_{xx} \phi_x \psi_j^{(3)} \psi_i^{(3)} \left(\sum_{v=1}^{p} \psi_v^{(3)} s_v \right) \right. \\
& + S_{xz} \left(\psi_j^{(3)} \frac{d\phi_x}{dx} + \phi_x \frac{d\psi_j^{(3)}}{dx} \right) \frac{d\psi_i^{(3)}}{dx} \left(\sum_{v=1}^{p} \psi_v^{(3)} s_v \right) \\
& \left. + \hat{S}_{xz} \left(\psi_j^{(3)} \frac{d\phi_x}{dx} + \phi_x \frac{d\psi_j^{(3)}}{dx} \right) \psi_i^{(3)} \left(\sum_{v=1}^{p} \frac{d\psi_v^{(3)}}{dx} s_v \right) \right\} dx
\end{aligned}
$$

$$
\begin{aligned}
=&K_{ij}^{33} + \int_{x_a}^{x_b} S_{xz}\frac{du}{dx}\left(\psi_i^{(3)}\psi_j^{(3)} + \frac{du}{dx}\psi_i^{(3)}\psi_j^{(3)}\right)dx \\
&+ \int_{x_a}^{x_b}\left\{A_{xx}{\phi_x}^2\psi_i^{(3)}\psi_j^{(3)} + S_{xz}\phi_x\left(\frac{d\phi_x}{dx}\frac{d\psi_i^{(3)}}{dx}\psi_j^{(3)} + \phi_x\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(3)}}{dx}\right)\right. \\
&\left. + \hat{S}_{xz}\frac{d\phi_x}{dx}\left(\frac{d\phi_x}{dx}\psi_i^{(3)}\psi_j^{(3)} + \phi_x\psi_i^{(3)}\frac{d\psi_j^{(3)}}{dx}\right)\right\}dx
\end{aligned}
$$