IRREDUCIBLE INFEASIBLE SUBSYSTEM DECOMPOSITION FOR PROBABILISTICALLY CONSTRAINED STOCHASTIC INTEGER PROGRAMS

A Dissertation

by

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ABSTRACT

This dissertation explores methods for finding irreducible infeasible subsystems (IISs) of systems of inequalities with binary decision variables and for solving probabilistically constrained stochastic integer programs (SIP-C). Finding IISs for binary systems is useful in decomposition methods for SIP-C. SIP-C has many important applications including modeling of strategic decision-making problems in wildfire initial response planning.

New theoretical results and two new algorithms to find IISs for systems of inequalities with binary variables are developed. The first algorithm uses the new theory and the method of the alternative polyhedron within a branch-and-bound (BAB) approach. The second algorithm applies the new theory and the method of the alternative polyhedron to a system in which zero/one box constraints are appended. Decomposition schemes using IISs for binary systems can be used to solve SIP-C.

SIP-C is challenging to solve due to the generally non-convex feasible region. In addition, very weak lower (upper) bounds on the objective function are obtained from the linear programming (LP) relaxation of the deterministic equivalent problem (DEP) to SIP-C. This work develops a branch-and-cut (BAC) method based on IIS inequalities to solve SIP-C with random technology matrix and random righthand-side vector. Computational results show that the LP relaxation of the DEP to SIP-C can be strengthened by the IIS inequalities.

SIP-C modeling can be applied to wildfire initial response planning. A new methodology for wildfire initial response that includes a fire behavior simulation model, a wildfire risk model, and SIP-C is developed and tested. The new methodology assumes a known standard response needed to contain a fire of given size. Likewise, this methodology is used to evaluate deployment decisions in terms of the

number of firefighting resources positioned at each base, the expected number of escaped and contained fires, as well as the wildfire risk associated with fires not receiving a standard response. A study based on the Texas district 12 (TX12) that is one of the Texas A&M Forest Service (TFS) fire planning units in east Texas demonstrates the effectiveness of the new methodology towards making strategic deployment decisions for wildfire initial response planning.

DEDICATION

I dedicate this dissertation to my family, Jose Leonidas Gallego Romero, Maria del Socorro Arrubla de Gallego, Jose Mauricio Gallego Arrubla, and Doralba Bedoya.

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NOMENCLATURE

BAB	Branch-and-bound
BAC	Branch-and-cut
CPU	Central processing unit
DEP	Deterministic equivalent problem
FD	Fire day
FS	Fire scenario
ID	Identification
IIS	Irreducible infeasible subsystem
IP	Integer programming
JD	John Deere
LP	Linear programming
MIP	Mixed integer programming
NWE	Normalized wildfire exposure
NVC	Net value change
PPRI	Pine plantation response index
RFL	Representative fire location
SIP	Stochastic integer programming
TFS	Texas A&M forest service
TWRA	Texas wildfire risk assessment
VRI	Value response index
WE	Wildfire exposure
WT	Wildfire threat
WUI	Wildland urban interface
WUIRI	Wildland urban interface response index

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1. INTRODUCTION

1.1 Motivation and Problem Statement

1.1.1 Irreducible Infeasible Subsystems

An irreducible infeasible subsystem (IIS) is a subsystem of linear inequalities that is infeasible, but it could be made feasible by dropping any inequality from it. IISs have been well-studied for systems of inequalities with unrestricted decision variables, and several methods for identifying these subsystems using linear programming (LP) and duality theory have been developed. For instance, the method of the alternative polyhedron [23] has the property that every extreme point of the polyhedron corresponds with an IIS of the original system. This method, however, cannot be directly applied to systems of inequalities with binary variables. Finding IISs for systems with binary variables is useful in decomposition methods for probabilistically constrained stochastic integer programs (SIP).

This dissertation develops new theoretical results and two new algorithms to find IISs for systems of inequalities with binary variables. The first algorithm, called the *IIS-BAB algorithm*, uses the new theory and the method of the alternative polyhedron [23] within a branch-and-bound (BAB) approach. The second algorithm, termed the *IIS-Heuristic algorithm*, applies the new theory and the method of the alternative polyhedron [23] to a system in which zero/one box constraints are appended.

1.1.2 Probabilistically Constrained Stochastic Programming

Probabilistically constrained stochastic programming involves optimizing a function subject to certain constraints where at least one of them is satisfied with a prescribed probability. This class of program is best suited for optimization problems in which satisfying a set of constraints is desirable, but it may not be done almost surely. Consider the wildfire initial response planning problem for example. This problem involves making effective strategic resource deployment plans so that the total deployment, relocation, fire damage, and dispatch cost is minimized. Dispatch of resources to all fires during a fire season may be too expensive, or it may just not be possible, depending on the number of available resources to provide initial response. Thus, strategic deployment decisions should be made while minimizing the risk associated with those fires not receiving an initial response. The aim of probabilistically constrained stochastic programming is to find optimal solutions while allowing a subset of the constraints to be violated (or not included in the problem) an acceptable amount of time based on the decision maker's attitude towards risk. Thus, a probabilistically constrained stochastic program can be formulated as follows:

$$\begin{aligned} \textbf{SIP-C1: } \min \ & c^\top x && (1.1a)\\ \text{s.t. } \ & Ax \geq b && (1.1b)\\ & \mathrm{P}\{T(\tilde{\omega})x \geq r(\tilde{\omega})\} \geq 1-\beta && (1.1c)\\ & x \in \mathcal{X}. && (1.1d) \end{aligned}$$

Note that x is the decision variable vector, and the set $\mathcal{X}$ imposes either continuous, integer, or mixed-integer restrictions on x. The random vector $\tilde{\omega}$ with outcomes (scenarios) $\omega \in \Omega$ gives rise to $T(\omega) \in \mathbb{R}^{m\times n}$ and $r(\omega) \in \mathbb{R}^m$. $T(\tilde{\omega})$ is the random technology matrix and $r(\tilde{\omega})$ is the random righthand-side vector. $\beta \in (0, 1)$ is the decision maker's attitude towards risk. If the decision maker is either *risk-seeking*, *risk-averse*, or *risk-neutral*, he or she would choose a risk attitude level close to one, zero or 0.5, respectively. In addition, optimizing SIP-C1 is challenging due to the generally non-convex feasible region. Moreover, very weak lower bounds on the objective function are obtained from the LP relaxation of the deterministic equivalent problem (DEP) to SIP-C1.

SIP-C1 can be classified into the following categories:

1. $\tilde{\omega}$ has continuous probability distribution with $|\Omega| < \infty$.

2. $\tilde{\omega}$ has discrete probability distribution with $|\Omega| < \infty$.

R. $r(\tilde{\omega})$ is random, and $T(\tilde{\omega}) \equiv T$ is deterministic.

B. $T(\tilde{\omega})$ and $r(\tilde{\omega})$ are random.

C. $\mathcal{X} \subseteq \mathbb{R}^{n_1}$.

I. $\mathcal{X} \subseteq \mathbb{Z}^{n_2}$.

N. $\mathcal{X} \subseteq \mathbb{B}^{n_2}$.

M. $\mathcal{X} \subseteq \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}$ or $\mathcal{X} \subseteq \mathbb{R}^{n_1} \times \mathbb{B}^{n_2}$.

Most of the approaches found in the literature attempt to solve SIP-C1 for the 1RC case. This work develops an IIS decomposition method for solving SIP-C1 for the 2BN case. This method is based on a branch-and-cut (BAC) approach with IIS inequalities. The BAC approach can be used to solve problems in wildfire initial response planning that are modeled as SIP-C1.

1.1.3 Wildfire Initial Response Planning

Wildfires have been an integral part of the environment for centuries. Indeed, wildfires play an important role in maintaining balance in ecosystems and ensuring the survival of certain plants and animals. Unfortunately, more than 90 percent of the wildfires in Texas occur within two miles of a community [61], and more than

32,000 wildfires were reported in this state between 2011 and 2013. These wildfires burned over 4 million acres and destroyed around 5,800 homes and structures [61]. Due to a limited budget for strategic planning, Texas A&M Forest Service (TFS), an agency that directs forestry matters in Texas, does not have enough resource capacity to provide wildfire initial response to all reported fires.

This dissertation also introduces a new methodology for wildfire initial response planning that includes a fire behavior simulation, a wildfire risk, and a probabilistically constrained SIP model. The new methodology is used to evaluate deployment decisions in terms of the number of firefighting resources positioned at each base, the expected number of escaped and contained fires, as well as the wildfire risk associated with fires not receiving an initial response. Likewise, this work presents a study based on the Texas district 12 (TX12) that is one of the TFS fire planning units in east Texas. Computational results from this study provide several insights into the deployment decisions made by the new methodology. For instance, they show that the original distribution of resources at the time of this study is not consistent with the actual wildfire risk profile of TX12.

1.2 Research Contributions

The research contributions of this dissertation can be summarized as follows:

- A method for finding IISs of systems of linear inequalities with binary variables using BAB and heuristic approaches.

- A BAC method using IIS inequalities to solve probabilistically constrained SIP with random technology matrix and random righthand-side vector along with preliminary computational results obtained from an implementation using Microsoft Visual C++ and CPLEX 12×64 Callable Library [32].

- A methodology for wildfire initial response planning that integrates a fire behavior, a wildfire risk, and a probabilistically constrained SIP model in association with computational results obtained from an implementation using Microsoft Visual C++, CPLEX 12×64 Callable Library [32], and BehavePlus [5].

1.3 Dissertation Organization

The rest of this dissertation is organized as follows: Chapter 2 introduces literature review of methods to find IISs for systems of linear inequalities, to solve probabilistically constrained stochastic programs, and to make decisions for wildfire initial response planning. Chapter 3 describes a new method for finding IISs of systems of linear inequalities with binary variables. Specifically, Chapter 3 discusses methods for finding IISs for systems of inequalities with unrestricted decision variables, new theoretical results and algorithms to identify IISs of systems with binary variables, and it presents two example illustrations. Chapter 4 presents a new method to solve probabilistically constrained SIP. Particularly, Chapter 4 discusses preliminaries, a BAC algorithm, an example illustration, as well as computational results and a discussion. Chapter 5 details a new methodology for wildfire initial response planning. In particular, Chapter 5 presents wildfire risk, standard response, a probabilistically constrained SIP model, computational results, and a discussion. Finally, Chapter 6 considers the conclusion to this dissertation including a summary and suggestions for future research.

2. LITERATURE REVIEW

2.1 Irreducible Infeasible Subsystems

An IIS is a minimal set of infeasible inequalities. IISs have been well studied for systems of inequalities with unrestricted decision variables, and several methods for identifying these subsystems using LP, duality or heuristic methods have been developed. Irreducible infeasible subsystems were first discussed in [47]. In this thesis, the author proves that a system defined by a matrix A is infeasible if and only if the rank of the matrix A is equal to the number of rows minus one. The author also presents a method to check if a subsystem of linear inequalities with unrestricted decision variables is in fact an IIS. This method is based on the values of the slack variables after solving a standard phase I LP. If bounded variables are considered, it is required to explicitly define constraints associated with these bounds in the original formulation.

The authors of [9] present a variation of the phase I approach for finding IISs for systems of linear inequalities with unrestricted decision variables that is referred to as *elastic programming*. In this approach, each inequality is given both a slack and a surplus variable, called elastic variables, with initial guesses for the price in the objective. Just like in an ordinary phase I LP method, there is no infeasibility after including these elastic variables. Therefore, the problem of finding an IIS is transformed into one that seeks to diagnose an *anomalous* solution that is characterized by positive values associated with the elastic variables.

The problem of identifying IISs for systems of inequalities with unrestricted decision variables is reduced to the problem of finding vertices of an alternative polyhedron in [23]. The method of the alternative polyhedron has the property that every extreme point of the polyhedron corresponds with an IIS of the original system. The vertices of the alternative polyhedron can be identified using LP or efficient algorithms to enumerate all extreme vertices of a polyhedron such as the one described in [18]. This method provides similar results to those obtained in [40], and it considers extreme rays of an alternative cone instead of extreme points in the alternative polyhedron.

The authors of [12, 11] present a series of filtering algorithms that are called the deletion, the additive, the elastic, and the sensitive filtering algorithm. These algorithms guarantee the identification of at least one IIS for systems of inequalities with unrestricted decision variables. These routines combine heuristic approaches with LP methods, and they represent an alternative to methods based on phase I LP and duality as the ones described above.

Methods to detect IISs using techniques from *lexicographic goal programming* are developed in [59]. The authors present a new algorithm for detection of IISs using multiple conflicting objective functions or constraints that guarantees the isolation of at least one IIS for any system of linear inequalities with unrestricted decision variables.

The authors of [27] extend the ideas in [12] to systems of inequalities with integer and mixed-integer decision variables. In systems of linear inequalities, the authors include not only linear constraints but also bounds and integer restrictions on decision variables. Each of these elements is considered an independent source of infeasibility in a system and is independently included in an IIS. Since the authors assume that standard methods for identifying IISs for systems of inequalities with unrestricted variables can be directly applied to systems of inequalities with integer or mixed-integer variables, they only present filtering routines for identifying IISs of integer or mixed-integer systems when the corresponding continuous and unrestricted relaxation is feasible.

More recently, the work in [1] generalizes conflict graphs analysis of satisfiability problems (SAT) to BAB based mixed-integer programs (MIP). At a particular node, when a relaxation to a nodal problem is infeasible, conflict variables and bounds can be identified by constructing a conflict graph, choosing a cut in this graph, and by producing a conflict constraint that consists of the variables in the conflict set. These constraints can be used as cutting planes to strengthen the relaxations of other subproblems in the tree.

Approaches for finding IISs for systems of inequalities with unrestricted, integer, or mixed-integer decision variables have been presented. However, methods for obtaining IISs for systems of inequalities with binary decision variables in which the integer restrictions are implicitly included in a system have not been addressed yet in the literature. A summary of the literature review of methods for finding IISs for systems of linear inequalities is shown in Table 2.1. In addition, IISs for binary systems can be used to derive decomposition methods for probabilistically constrained SIP. A literature review on probabilistically constrained SIP is introduced in the next section.

Table 2.1: Summary of relevant literature on IIS

Paper	Methodology	Decision variables
[47]	Method based on the phase I LP	Unrestricted
[9]	Elastic programming	Unrestricted
[40]	Method of the alternative cone	Unrestricted
[18]	Method to enumerate all extreme vertices in a polyhedron	Unrestricted
[23]	Method of the alternative polyhedron	Unrestricted
[12]	Method based on filtering routines	Unrestricted
[59]	Goal lexicographic programming	Unrestricted
[27]	Method based on filtering routines	Mixed integer
[1]	Method based on conflict graphs analysis	Mixed integer

2.2 Probabilistically Constrained Stochastic Programming

Probabilistically constrained SIP can be classified into different cases as described in Chapter 1. This classification is summarized in Table 2.2. Approaches to solve probabilistically constrained SIP for the 1RC, 1R1, 1BC, 2RC, 2BC, or the 2RM case have been developed. However, a method to solve probabilistically constrained SIP for the 2BN case has not been addressed yet in the literature.

Table 2.2: Classification of probabilistically constrained stochastic programs

Case	Description
1RC	Random variable with continuous probability distribution. Random righthand-side vector. Continuous decision variables.
1RI	Random variable with continuous probability distribution. Random righthand-side vector. Integer decision variables.
1BC	Random variable with continuous probability distribution. Random technology matrix and random righthand-side vector. Continuous decision variables.
2RC	Random variable with discrete probability distribution. Random righthand-side vector. Continuous decision variables.
2RM	Random variable with discrete probability distribution. Random righthand-side vector. Mixed-integer decision variables.
2BC	Random variable with discrete probability distribution. Random technology matrix and random righthand-side vector. Continuous decision variables.
2BI	Random variable with discrete probability distribution. Random technology matrix and random righthand-side vector. Integer decision variables.
2BM	Random variable with discrete probability distribution. Random technology matrix and random righthand-side vector. Mixed-integer decision variables.
2BN	Random variable with discrete probability distribution. Random technology matrix and random righthand-side vector. Binary decision variables.

Methods to solve SIP-C1 for the 1RC case have been reported in the literature. For example, the authors of [10] studied the scheduling heating oil production problem in which individual probabilistic constraints are imposed on each constraint involving random variables. The authors of [46] went one step further by studying the case in which joint probabilistic constraints are imposed on constraints with dependent random variables. Afterwards, the work in [10] and [46] is generalized in [55] by considering SIP-C1 with joint probabilistic constraints and independent random variables. In this work, the author shows conditions in which SIP-C1 reduces to a convex (or quasi-convex) programming problem.

In addition to the solution methods for SIP-C1 for the 1RC case, approaches to solve SIP-C1 for the 1RI or the 1BC case have also been developed. For instance, [6] and [15] formulated DEPs to SIP-C1 using p-efficient points for the 1RI case. In both approaches, p-efficient points are computed via enumeration algorithms. Likewise, [52] developed a sample average approximation (SAA) approach to SIP-C1 for the 1BC case. Their main contribution is a theoretical foundation to approximate the joint probabilistic constraints in SIP-C1 using the SAA method. The authors also suggest a choice of parameters that can be used in an actual implementation of the SAA method while obtaining good candidate solutions.

Methods to solve SIP-C1 for the 2RM or the 2RC case have also been presented in the literature. For example, [39] presents reformulations for SIP-C1 for the the 2RM case using p-efficient points. In this work, the authors derived a mathematical programming framework to generate exact p-efficient points such that a sequence of increasingly tighter outer approximation problems are constructed. This new method represents an alternative to enumeration algorithms to find p-efficient points. In addition, [42] developed a MIP approach for SIP-C1 for the 2RC case. The authors leveraged a natural ordering in the random righthand-side vector to overcome the weakness of the big-M formulation so that two strengthened formulations can be obtained. As a byproduct of this analysis, the authors present new results for the previously studied mixing set subject to an additional knapsack inequality.

In addition to the methods described above, approaches for solving SIP-C1 for the 2BC case have also been devised. For instance, [56] developed new valid inequalities for DEP reformulations to SIP-C1 that can be applied in combination with dominance between different realizations of the random input. The author of [41] developed a way to reduce SIP-C1 for the 2BC case to the structure studied in [42] and ultimately apply the same types of valid inequalities for mixing sets with a knapsack constraint. Likewise, [60] derived a BAC algorithm using IIS inequalities. In fact, computational results for the optimal vaccine allocation problem provided empirical evidence that the IIS inequalities offer a significant increase in the strength of DEP reformulations to SIP-C1 for this type of problem.

IIS inequalities have also been used to generate combinatorial benders (CB) cuts for MIPs involving logical constraints modeled through big-M coefficients [14]. This approach closely resembles the benders decomposition method, but the cuts the method produces are purely combinatorial, and they do not depend on the big-M values used in the MIP formulation. Computational results show that the new method produces a reformulation that can be solved faster than the original MIP model in some orders of magnitude.

A summary of the literature review for probabilistically constrained stochastic programs is shown in Table 2.3. SIP with probabilistic constraints can be used to model strategic decision-making problems in wildfire initial response planning. A literature review on wildfire initial response planning is presented in the next section.

2.3 Wildfire Initial Response Planning

The first step towards wildfire containment is to effectively perform a wildfire initial response. Wildfire initial response is the action taken by the first resources to arrive at a wildfire to protect lives, homes, and property and to prevent further

Table 2.3: Summary of relevant literature on probabilistically constrained stochastic programs

Paper	Methodology	Case
[10]	SIP-C1 with single probabilistic constraints	1RC
[46]	SIP-C1 with joint probabilistic constraints and independent random variables	1RC
[55]	Convexity of SIP-C1 with joint probabilistic constraints and dependent random variables	1RC
[6]	DEP to SIP-C1 obtained using p-efficient points within a BAB approach	1RI
[15]	Methods to construct lower and upper bounds for SIP-C1 using p-efficient points	1RI
[56]	New valid inequalities for DEPs of SIP-C1 and lifting procedures for these inequalities	2BC
[36]	Reformulation for SIP-C1 as a minmax multidimensional knapsack problem (MKP)	2RM
[52]	SAA approach for SIP-C1	1BC
[42]	MIP reformulations for SIP-C1 with mixing sets as a substructure	2RC
[41]	MIP reformulations for SP-C with mixing sets as a substructure	2BC
[60]	BAC method using IIS inequalities to solve reformulation for SIP-C1	2BC
[39]	Reformulations for SIP-C1 using p-efficient points	2RM
[37]	Redefinition of p-points as p-patterns and reformulations for SIP-C1	2RC
[38]	Sequential method to derive the collection of p-patterns and reformulations for SIP-C1	2RC

extension of the fire. Wildfire initial response includes two sequential phases. The first phase (strategic phase) involves making decisions about deploying or relocating resources to operations bases. The second phase (operational phase) involves making decisions about dispatching resources to fire locations after fires are reported.

Deployment has been addressed using mathematical programming to assign resources to stations in order to minimize operational costs. For instance, [30] and [26] present models that assign firefighting resources to operations bases to minimize operation costs while meeting resource requirements. The authors in [45] present an integer program (IP) to deploy fire suppression resources to fires that have grown beyond the initial attack phase in order to maximize the expected total utility of all perimeter partitions. A model to assign resources to the dozer-line in order to minimize the total expected cost of a wildfire given uncertainty in both the flame length and the dozer-line width produced is described in [44]. As a last example, [43] developed a mathematical programming model that identifies a home-basing strategy that minimizes the average annual cost of daily airtanker deployment to bases.

Deployment has also been addressed using queueing theory, statistics, and heuristics. For example, [33] derived a queueing model to analyze airtanker performance associated with wildfire initial response range. The author of [25] developed a spreadsheet procedure for evaluating airtanker deployment prior to wildfire initial response. The spreadsheet allows for computing expectation and higher moments of the flight distance from an airtanker base to a random fire ignition point after specifying the spatial distribution of fire occurrence within the airtanker base's wildfire initial response zone. Likewise, [34] presents a wildfire initial response airtanker system (IAAS) model and a heuristic procedure to solve the daily airtanker deployment decision problem.

Simulation models have also been developed for deployment decisions in wildfire initial response planning. For example, [63] presents a stochastic simulation model in order to closely represent the dynamics of fire occurrence and suppression resource deployment of many federal and state protection programs in the western United States. In particular, this model can be used as a decision tool to help managers make decisions in both locally controlled initial attack and shared aerial resource programs.

Dispatching has been modeled to determine the number and type of containment resources to dispatch in order to minimize damage and suppression cost. For example, [54] presents a deterministic model based on analytical decision rules. The goal is to minimize the total cost of suppression plus fire damages by determining manpower requirements for dispatching. [35] devised two dynamic programming (DP) algorithms that minimize the cost of dispatching water bombers, helicopters, and firefighting crews to newly reported fires in Ontario. Likewise, [62] presents a deterministic DP model in order to find the economically efficient set of resources to suppress a wildfire in an initial attack setting.

Dispatching has also been modeled to provide an estimate of fire damage when making tactical decisions about fire containment. For instance, [16] developed an IP model that determines the optimal mix of firefighting resources to dispatch in order to achieve the minimum value of cost plus net value change (C+NVC) [24, 17]. The authors assume that fire growth is known at any given time in the future, and they do not take the stochastic nature of the fire spread process into consideration.

Simulation models have also been considered for dispatching decisions in wildfire initial response planning. For example, stochastic simulation models have been used to evaluate changes in the number and location of resources and dispatching rules [33, 21, 22]. More recently, an integrated simulation and optimization framework for initial response planning was developed in [31]. The authors use the DEVS-FIRE [50] model for fire behavior simulation and stochastic programming for computing an optimal mix of firefighting resources to dispatch to fires. Simulation of the fire dispatch decisions is done using an agent-based simulation model in order to evaluate the effectiveness of the decisions based on several firefighting tactics.

Deployment and dispatching have been generally addressed independently in the literature. More recently, these two phases in wildfire initial response planning have been jointly considered. [49] derived a simulation and a SIP methodology. This methodology considers firefighting resources constructing a perimeter around a fire at discrete time intervals over the wildfire initial response planning horizon. Moreover, the authors developed a model that minimizes the total fixed cost of deploying available fire resources to operations bases and the total travel cost for relocating resources between different operations bases. The model also seeks to minimize the expected operation cost of resources dispatched to fires and the total NVC.

Standard response models have also been developed for joint deployment and dispatching decisions in wildfire initial response planning. Urban planners define *standard response* for fire protection service in terms of expected fire size, maximum response distance and several types of suppression resources. [29] presents a standard response model that optimizes both daily deployment and dispatching of a single type of resource for a finite number of fire scenarios. This is an extension of the maximal covering location problem that handles standard response requirements. [48] devised a two-stage SIP standard response model for wildfire initial response in order to minimize total expected cost. This model considers multiple types of resources while assuming a standard response to contain a fire of known size.

Probabilistically constrained models are a type of optimization model in which a subset of the constraints regarding uncertainty are satisfied an acceptable amount of time depending on the decision maker's risk attitude level. Probabilistically constrained models have not been used to address wildfire initial response planning. Nevertheless, there are some approaches in the literature in which probabilistic constraints are considered for solving other wildfire planning problems. For instance, [8] presents a probabilistically constrained model in order to minimize total cost across possible treatment areas while satisfying the probability that the wildfire loss exceeds the total cost be less than a certain parameter. Likewise, [7] developed a probabilistically constrained SIP to allocate fire organization budgets to fire planning units. The aim of this model is to minimize an upper bound on the total fire cost such that the probability that the total random fire cost exceeds the upper bound is less than a certain level.

The methodologies discussed above contribute to efficient decision making regarding deployment and dispatching of resources for wildfire initial response planning (Table 2.4). However, these approaches do not take wildfire risk into account. One of the goals of this dissertation is to present a new methodology for making decisions to wildfire initial response planning while considering wildfire risk. Unlike previous approaches, the new methodology integrates a fire behavior simulation, a wildfire risk, and a standard response probabilistically constrained SIP model to make strategic deployment decisions regarding initial response planning. This work uses a probabilistically constrained SIP model since this approach gives more flexibility when making decisions about whether or not to provide standard response to a fire.

Table 2.4: Summary of relevant literature on wildfire initial response planning

Paper	Problem	Methodology
[54]	Dispatching	Analytical decision rules
[24]	Fire damage	Model for wildfire economics
[30]	Deployment	Mathematical programming model
[26]	Deployment	Mathematical programming model
[35]	Dispatching	DP model
[45]	Deployment	Mathematical programming model
[44]	Dispatching	Model to predict flame length and dozer-line width
[43]	Deployment	Models to identify home-basing strategie
[33]	Deployment	Queueing model
[33]	Dispatching	Stochastic simulation model
[21]	Dispatching	Stochastic simulation model
[62]	Dispatching	DP model
[25]	Deployment	Spreadsheet for modeling airtanker performance
[17]	Fire damage	Model for wildfire economics
[16]	Dispatching	IP model using NVC
[22]	Dispatching	Stochastic simulation model
[29]	Deployment and dispatching	Scenario based standard response model
[8]	Mitigation	Chance-constrained model
[7]	Budget planning	Chance-constrained model
[63]	Deployment	Stochastic simulation model
[50]	Fire simulation	DEVS-FIRE model
[31]	Dispatching	Optimization and simulation model
[34]	Deployment	Wildfire initial response airtanker system (IAAS)
[48]	Deployment and dispatching	2-stage standard response SIP model
[49]	Deployment and dispatching	2-stage explicit fire growth SIP model

3. IRREDUCIBLE INFEASIBLE SUBSYSTEMS FOR SYSTEMS OF INEQUALITIES WITH BINARY VARIABLES

3.1 Preliminaries

This chapter presents a new method for finding IISs for systems of inequalities with binary variables. Let us introduce first the notation that will be used throughout this chapter. Consider decision vectors x and y that are defined as $x^\top = (x_1, ..., x_n)$ and $y^\top = (y_1, ..., y_m)$. The support of a decision vector is the set of indices of its nonzero components. Let $\mathcal{V}$ be an index set of the components of x, $x_p,\ 1 \leq p \leq n$. Let $G \in \mathbb{R}^{m\times n}$ and $v \in \mathbb{R}^m$. Then, G, v, x and y define the systems of inequalities (constraints) $Gx \leq v$ and $y^\top G = 0, y^\top v \leq 0$. Let $\mathcal{A}$ be an index set of the constraints in $Gx \leq v$ such that $g_s x \leq v_s,\ 1 \leq s \leq m$, is an individual inequality in $Gx \leq v$. Let Ψ be an index such that $\Psi \subseteq \mathcal{A}$. Then, $g_s x \leq v_s, \forall s \in \Psi$ is a subsystem of $Gx \leq v$. Any subsystem $g_s x \leq v_s, \forall s \in \Psi$ such that $\Psi \subset \mathcal{A}$ is a relaxation of $Gx \leq v$. Furthermore, let $x_p \leq 1,\ -x_p \leq 0,\ \forall p \in \mathcal{V}$ be a system of inequalities that can be appended to the system $Gx \leq x$. This system of inequalities (or any subsystem related to it) will be referred to as box constraints. Let $\mathcal{U}$ be an index set of the inequalities $x_p \leq 1,\ \forall p \in \mathcal{V}$ appended to $Gx \leq x$ indexed $u = m + p,\ p = 1, ..., n$. Moreover, let $\mathcal{L}$ be an index set of the inequalities $-x_p \leq 0,\ \forall p \in \mathcal{V}$ appended to $Gx \leq x,\ x_p \leq 1,\ \forall p \in \mathcal{V}$ indexed $l = p + m + n,\ p = 1, ..., n$. Finally, $\mathcal{H} := \mathcal{U} \cup \mathcal{L}$. Now, let us formally introduce the systems of inequalities and concepts that will be used as part of the theory and methods in this chapter.

DEFINITION 3.1.1. *B is a system of inequalities that is defined as $B := \{Gx \leq v : x \in \{0,1\}^n\}$. A subsystem to B is denoted by B_Ψ, and it is defined as $B_\Psi := \{g_s x \leq v_s,\ \forall s \in \Psi :\ x \in \{0,1\}^n\}$.*

DEFINITION 3.1.2. *N is a system of inequalities that is defined as $N := \{Gx \leq v : x \in \mathbb{R}^n_+\}$. A subsystem to N is denoted by N_Ψ, and it is defined as $N_\Psi := \{g_s x \leq v_s,\ \forall s \in \Psi :\ x \in \mathbb{R}^n_+\}$.*

DEFINITION 3.1.3. *R is a system of inequalities that is defined as $R := \{Gx \leq v,\ x_p \leq 1,\ -x_p \leq 0, \forall p \in \mathcal{V} :\ x \in \mathbb{R}^n\}$. A subsystem to R is denoted by $\bar{R}$, and it is defined as $\bar{R} := \{x_p \leq 1,\ -x_p \leq 0,\ \forall p \in \mathcal{V} :\ x \in \mathbb{R}^n\}$.*

Let $\Phi \subseteq \mathcal{V}$ be an index set that is defined as $\Phi := \{p : ax_p \leq b,\ a \in \{1, -1\},\ b \in \{0, 1\}\}$. Observe that $ax_p \leq b,\ a \in \{1, -1\},\ b \in \{0, 1\}$ is a system of inequalities that can be appended to $Gx \leq x$, and it is a subsystem of the system defined by $\mathcal{U} \cup \mathcal{L}$ (box constraints). Φ is used to define the index sets Λ_1 and Λ_2 as $\Lambda_1 := \{\lambda_1 : \lambda_1 = p + m,\ \forall p \in \Phi \text{ s.t. } x_p \leq 1\}$ and $\Lambda_2 := \{\lambda_2 : \lambda_2 = p + m + n,\ \forall p \in \Phi \text{ s.t.} - x_p \leq 0\}$. Likewise, $\Lambda_1 \subseteq \mathcal{U}$ and $\Lambda_1 \subseteq \mathcal{L}$. Let Π be an index set that is defined as $\Pi := \Psi \cup \Lambda_1 \cup \Lambda_2$. Now, let us consider some additional systems of inequalities as follows.

DEFINITION 3.1.4. *C_Π is a system of inequalities that is defined as $C_\Pi := \{g_s x \leq v_s,\ \forall s \in \Psi,\ ax_p \leq b,\ a \in \{1, -1\},\ b \in \{0, 1\},\ \forall p \in \Phi :\ x \in \mathbb{R}^n\}$. A subsystem to C_Π is denoted by $\bar{C}_\Pi$, and it is defined as $\bar{C}_\Pi = \{ax_p \leq b,\ a \in \{1, -1\},\ b \in \{0, 1\},\ \forall p \in \Phi :\ x \in \mathbb{R}^n\}$.*

DEFINITION 3.1.5. *U is a system of inequalities that is defined as $U := \{Gx \leq v :\ x \in \mathbb{R}^n\}$. A subsystem to U is denoted by U_Ψ, and it is defined as $U_\Psi := \{g_s x \leq v_s,\ \forall s \in \Psi : x \in \mathbb{R}^n\}$.*

A system of inequalities $Gx \leq v$ is called infeasible if there is no x satisfying it.

DEFINITION 3.1.6. *An irreducible infeasible subsystem (IIS) S of B is a subsystem B_Ψ that is infeasible, but it could be made feasible by dropping any inequality from it. Note that $S = \Psi$. In addition, $\mathcal{S}$ is the set of IISs for $Gx \leq v$ that is defined as $\mathcal{S} := \{S_1, ..., S_i\}$, indexed j.*

The Definition 3.1.6 applies to other systems of inequalities such as N, R, C_{Π}, and U including their subsystems.

3.2 IISs for Systems of Inequalities with Unrestricted Variables

Consider the following variant of the Farkas Lemma of the alternative:

LEMMA 3.2.1. *(Farkas Lemma.) One of the following systems of inequalities has a feasible solution [19, 20]:*

System 1: *There exists $x \in \mathbb{R}^n$ such that $Gx \geq v$.*

System 2: *There exists $y \in \mathbb{R}^m_+$ with $y^\top G = 0$, $y^\top v < 0$.*

This version of the Farkas Lemma is used to derive Theorem 3.2.2 (see below). This theorem is the main result in [23], and it allows us to compute an IIS for a system of linear inequalities with unrestricted variables using LP or efficient algorithms to enumerate extreme vertices in a polyhedron.

THEOREM 3.2.2. *The indices of the IISs for an infeasible system of inequalities U are exactly the supports of the vertices of the alternative polyhedron:*

$$\mathcal{P} = \{y \in \mathbb{R}^m : \; y^\top G = 0, y^\top v \leq -1, y \geq 0\}.$$

The following theorem states that the maximum cardinality on each IIS of U is $n + 1$ [13].

THEOREM 3.2.3. *Assume U to be infeasible, and let Ψ be an IIS of U. Then, $|\Psi| \leq n + 1$.*

The following is an example of how to obtain an IIS for systems of inequalities with unrestricted variables.

$$
\begin{aligned}
U = \{&x_1 - x_2 \leq 0 \ (1),\\
&x_1 + x_2 \leq 1 \ (2),\\
&-x_1 - x_2 \leq -2 \ (3),\\
&-x_2 \leq -1 \ (4),\\
&-2x_1 - x_2 \leq -3 \ (5) : x \in \mathbb{R}^2\}.
\end{aligned}
$$

U is a system of *five* inequalities subject to 2-dimensional vectors with unrestricted real components. This system of inequalities is infeasible. By applying Theorem 3.2.2 to U, an alternative polyhedron $\mathcal{P}$ is obtained as

$$
\begin{aligned}
\mathcal{P} = \{y \in \mathbb{R}^5_+ : &1y_1 + 0y_2 - 1y_3 + 0y_4 - 2y_5 = 0\\
&-1y_1 + 2y_2 - 1y_3 - 1y_4 - 1y_5 = 0\\
&0y_1 + 1y_2 - 2y_3 - 1y_4 - 3y_5 \leq -1\}.
\end{aligned}
$$

The set of IISs of U are $\Psi_1 = \{1, 2, 3\}$, $\Psi_2 = \{1, 2, 5\}$, and $\Psi_3 = \{2, 4\}$ as shown in Figure 3.1. Likewise, the maximum cardinality of any IIS for U is *three* in agreement with Theorem 3.2.3. Notice also how U can become feasible by dropping constraint 2 from this system.

Consider an IIS Ψ of U obtained by applying Theorem 3.2.2. Consider also a subsystem U_Ψ to U. U_Ψ is infeasible since the corresponding y vector satisfies the system two in Lemma 3.2.1. Likewise, U_Ψ is irreducible since $y \geq 0,\ y^\top v \leq -1$ in Theorem 3.2.2. This means that at least one component in vector y has to be positive while indexing at least one constraint in U_Ψ. If the constraints corresponding to zero

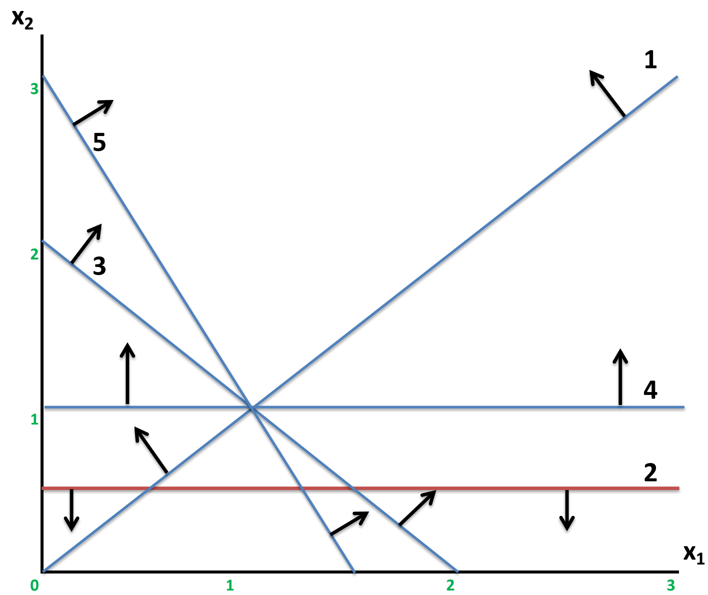

Figure 3.1: System of inequalities U

elements in $y \in \mathcal{P}$ are included in Ψ, then U_Ψ will be infeasible but not irreducible. Theorem 3.2.2 does not hold for a system of inequalities N since the subsystem N_Ψ for the IIS Ψ will be infeasible but not irreducible. Besides, Theorem 3.2.2 does not work for a system of inequalities B since the Farkas Lemma does not hold for this type of system.

3.3 Identifying IISs for Systems of Inequalities with Binary Variables

3.3.1 Theoretical Results

Even though the methods for finding an IIS for a system U cannot be directly applied to find an IIS for a system B, these methods can be used as a basis of approaches for obtaining an IIS of B. Consider the method of the alternative polyhedron described in [23] for finding an IIS of U. This method cannot be directly applied to B since Theorem 3.2.2 is proved by Lemma 3.2.1. However, this method can be used to find an IIS of B within a branch-and-bound (BAB) approach. Let us develop some theoretical results before the method for finding IISs of B is presented. First, Lemma 3.3.1 states the existence of at least one IIS for an infeasible system B whose minimal cardinality is *one*.

LEMMA 3.3.1. *Consider an infeasible system B. Then, there exists at least one IIS S of B such that $|S| \geq 1$.*

Proof. *Assume that there does not exist at least one IIS S of B such that $|S| \geq 1$. This would imply that any infeasible subsystem B_Ψ is not irreducible. Consider subsystem $B_\Psi, \Psi = \{s\}, 1 \leq s \leq m$ that is infeasible but not irreducible. By dropping the inequality associated with index s from B_Ψ, a feasible system B is obtained. This contradicts the initial assumption of B to be infeasible. Moreover, since there is a subsystem B_Ψ that is infeasible and irreducible, then $S = \Psi$ where $|S| \geq 1$.*

■

Proposition 3.3.2 states that given an infeasible system B with a known IIS S there exists an IIS Π to the corresponding system C_Π that contains the IIS S for B.

PROPOSITION 3.3.2. *Consider an infeasible system B with a known IIS S. Then, there exist an index set Π and an infeasible system C_Π such that Π is an IIS of C_Π and $\Pi \supseteq S$.*

Proof. *Consider the systems U and R associated with B as defined in Section 3.1. Observe that R is infeasible. Thus, by Theorems 3.2.2 and 3.2.3, there exists an IIS Π of R such that $|\Pi| \leq n+1$. Given the IIS S of B, let $U_S = \{g_s x \leq v_s,\ \forall s \in S:\ x \in \mathbb{R}^n\}$ be a subsystem to U. Then, consider the following* two *cases:*

1. *Assume U_S is feasible. There exists a nonempty index set Φ associated with components $x_p,\ p \in \Phi$ and a system of linear inequalities $g_s x \leq v_s,\ \forall s \in$*

$S,\ ax_p \leq b,\ a \in \{1, -1\},\ b \in \{0, 1\},\ \forall p \in \Phi$ *with unrestricted decision variables that is infeasible (by construction) and irreducible (because* S *is an IIS of* B*). Let* $\Lambda_1 \cup \Lambda_2$ *be an index set of the inequalities* $ax_p \leq b,\ a \in \{1, -1\},\ b \in \{0, 1\},\ p \in \Phi$ *in this system such that* $\Pi = S \cup \Lambda_1 \cup \Lambda_2$ *and* $|\Lambda_1 \cup \Lambda_2| > 0$*. Therefore, there exist an index set* Π *and an infeasible system* C_Π *such that* Π *is an IIS of* C_Π *and* $\Pi \supseteq S$*.*

2. *Assume* U_S *is infeasible. There exists an empty index set* Φ *associated with components* $x_p,\ p \in \Phi$ *and a system of linear inequalities* $g_s x \leq v_s,\ \forall s \in S$ *with unrestricted decision variables that is infeasible (by construction) and irreducible (because* S *is an IIS of* B*). Let* $\Lambda_1 \cup \Lambda_2$ *be an empty index set of the inequalities* $ax_p \leq b,\ a \in \{1, -1\},\ b \in \{0, 1\},\ p \in \Phi$ *in* C_Π *such that* $\Pi = S \cup \Lambda_1 \cup \Lambda_2$ *and* $|\Lambda_1 \cup \Lambda_2| = 0$*. Therefore, there exist an index set* Π *and an infeasible system* C_Π *such that* Π *is an IIS of* C_Π *and* $\Pi \supseteq S$*.*

■

Proof of Proposition 3.3.2 provides a definition of Π as $\Pi := S \cup \Lambda_1 \cup \Lambda_2$ such that $|\Pi| \leq n + 1$. Now, consider the reverse of Proposition 3.3.2 as follows. Let B be an infeasible binary system. There exists an infeasible system C_Π with an IIS Π such that the index subset $S = \Pi \setminus \{\Lambda_1 \cup \Lambda_2\}$ is an IIS of B. This result is formalized in Theorem 3.3.3.

THEOREM 3.3.3. *Consider an infeasible system* B*. There exist both an index set* Φ *(associated with components* $x_p,\ p \in \Phi$ *and the set* $\Lambda_1 \cup \Lambda_2$ *that indexes the inequalities* $ax_p \leq b,\ a \in \{1, -1\},\ b \in \{0, 1\},\ p \in \Phi$*) and an infeasible system* C_Π *with an IIS* Π *such that the index subset* $S = \Pi \setminus \{\Lambda_1 \cup \Lambda_2\}$ *is an IIS of* B*.*

Proof. *By Lemma 3.3.1, there exists an IIS* S *of* B*. By Proposition 3.3.2, there exist both an index set* Π *and an infeasible system* C_Π *such that* Π *is an IIS for* C_Π

and $\Pi = \Lambda_1 \cup \Lambda_2 \cup S$. *Therefore,* $S = \Pi \setminus \{\Lambda_1 \cup \Lambda_2\}$ *where* S *is an IIS of* B.

■

The following is an example of how to obtain an IIS S of a system of inequalities with binary variables via Theorem 3.3.3.

$$B = \{-2x_1 - 2x_2 + x_3 \leq -1, \ (1)$$
$$x_1 + x_2 \leq 0 \ (2) : \ x \in \mathbb{B}^3\}.$$

B is an infeasible system of *two* inequalities subject to 3-dimensional vectors with binary components. Thus, there exist both an index set $\Phi = \{3\}$ associated with (i) component x_3 and (ii) the sets $\Lambda_1 = \emptyset, \ \Lambda_2 = \{8\}$ indexing inequality $-x_3 \leq 0, \ 3 \in \Phi$ and an infeasible system C_Π (see below) with an IIS $\Pi = \{1, 2, 8\}$ such that the index subset $S = \{1, 2, 8\} \setminus \{8\} = \{1, 2\} \subseteq \Pi$ is an IIS of B as shown in Figure 3.2.

$$C_\Pi = \{-2x_1 - 2x_2 + x_3 \leq -1, \ (1)$$
$$x_1 + x_2 \leq 0, \ (2)$$
$$-x_3 \leq 0 \ (8) : \ x \in \mathbb{R}^3\}.$$

Corollary 3.3.4 states that the maximum number of indices associated with the inequalities $ax_p \leq b, \ a \in \{1, -1\}, \ b \in \{0, 1\}, \ p \in \Phi$ in a system C_Π is n.

COROLLARY 3.3.4. *Consider a system* B *and an associated system* C_Π. *Assume* C_Π *to be infeasible. Let* Π *be an IIS for* C_Π. *If there are* n *components in* x, *then* $|\Pi \cap \mathcal{H}| \leq n$.

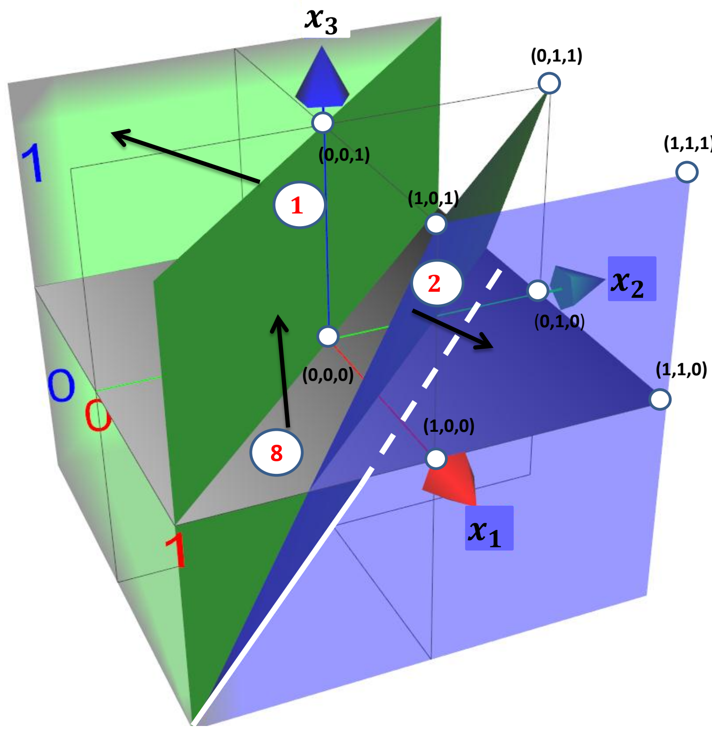

Figure 3.2: Systems of inequalities B and C_Π

Proof. *By Proposition 3.3.2,* $\Pi = S \cup \Lambda_1 \cup \Lambda_2$ *and* $|\Pi| \leq n + 1$. *By Lemma 3.3.1,* $|S| \geq 1$. *Since* $\Pi \cap \mathcal{H} = \Lambda_1 \cup \Lambda_2$ *and* $|\Lambda_1 \cup \Lambda_2| \leq n$, *then* $|\Pi \cap \mathcal{H}| \leq n$.

■

An IIS S of B can be obtained via an IIS Π of a system C_Π by Lemma 3.3.1, Proposition 3.3.2, Theorem 3.3.3, and Corollary 3.3.4. That is, there exists an associated system C_Π to B such that an IIS Π for C_Π will characterize an IIS S of B after removing from Π the indices associated with the inequalities $ax_p \leq b,\ a \in \{1, -1\},\ b \in \{0, 1\},\ p \in \Phi$. Two methods to find IISs of a system B are described in the next sections. First, the *IIS-BAB algorithm* considers the case in which the related system U to B is infeasible. Second, the *IIS-Heuristic algorithm* considers the case in which the associated system U to B is feasible. Let us introduce additional notation before the methods to find IISs of B are formally introduced.

3.3.2 Additional Notation

The solution methods for finding an IIS of B in this chapter are based on tree data structures. A tree data structure can be defined recursively as a collection of unique nodes starting at a root node r where each node k is a data structure consisting of a value and a list of children. Specifically, the tree data structures considered in this chapter are the BAB binary trees.

DEFINITION 3.3.5. *A BAB binary tree is a tree data structure in which each node has at most two children nodes usually distinguished as left and right. Nodes with children are parent nodes, and the node who is an ancestor to all nodes is the root node. A root node that has not been developed into a binary tree is called an open root node.*

Let $\mathcal{Q}$ be the set of open root nodes, indexed $r = 1, ..., n$. Likewise, $\mathcal{T}$ is the set of BAB binary trees associated with $x_p,\ \forall p \in \mathcal{V}$ indexed $t = 1, ..., |\mathcal{Q}|$. A leaf node in a BAB binary tree is a node that has no children.

DEFINITION 3.3.6. *A path is a sequence of nodes from a (leaf) node k to the root node r with no node repetitions. A path is represented by $\tau(r, k) = \{k, k-1, ..., j, ..., r\}$. The depth of a (leaf) node k in a BAB binary tree is $\alpha = |\tau(k, r)|$. Likewise, let Δ be the set of paths that is defined as $\Delta = \{\tau_1, ..., \tau_{|\Delta|}\}$.*

A node that is not part of a BAB binary tree is said to be *open*. Then, define the set of open nodes in a BAB tree $t \in \mathcal{T}$ as follows.

DEFINITION 3.3.7. *$\mathcal{N}(t)$ is the set of open nodes indexed $k_t = 0, ..., \sum_{i=1}^{n} 2^i$ where $k_t = 0$ indexes the root node in a BAB tree $t \in \mathcal{T}$. Besides, $p(k_t)$ is a function* *that returns the parent node of a node k_t. Likewise, μ_t is the maximum index node in a BAB tree $t \in \mathcal{T}$.*

At a node $k_t,\ t \in \mathcal{T}$ let $\mathcal{A}(k_t),\ \mathcal{U}(k_t),\ \mathcal{L}(k_t)$ be index sets such that $\mathcal{A}(k_t) \subseteq \mathcal{A}$, $\mathcal{U}(k_t) \subseteq \mathcal{U}$, and $\mathcal{L}(k_t) \subseteq \mathcal{L}$. Let us define the following additional systems of inequalities associated with B.

DEFINITION 3.3.8. *At node $k_t,\ t \in \mathcal{T}$, B^{k_t} is a system of inequalities that is defined as B^{k_t}:=$\{g_s x \leq v_s,\ \forall s \in \mathcal{A}(k_t),\ x_p = 1,\ p = \bar{u} - m,\ \forall \bar{u} \in \mathcal{U}(k_t),\ x_p = 0,\ p = \bar{l} - (m+n),\ \forall \bar{l} \in \mathcal{L}(k_t):\ x \in \{0,1\}^n\}$. B^{k_t} is denoted as the* nodal *system associated with B at node $k_t,\ t \in \mathcal{T}$.*

DEFINITION 3.3.9. *At node $k_t,\ t \in \mathcal{T}$, C^{k_t} is a system of inequalities that is defined as C^{k_t}:= $\{g_s x \leq v_s,\ \forall s \in \mathcal{A}(k_t),\ x_p \leq 1,\ p = \bar{u} - m,\ \forall \bar{u} \in \mathcal{U}(k_t),\ x_p \leq 0,\ p = \bar{l} - (m+n),\ \forall \bar{l} \in \mathcal{L}(k_t):\ x \in \mathbb{R}^n\}$. C^{k_t} is denoted as the* nodal *system with unrestricted variables associated with B^{k_t} at node $k_t,\ t \in \mathcal{T}$.*

Let us define some notation corresponding to an IIS of C^{k_t} as follows.

DEFINITION 3.3.10. *At node $k_t,\ t \in \mathcal{T}$, let Π^{k_t} be an IIS for C^{k_t}. Then, $\Gamma(\Pi^{k_t})$ is an index set of the inequalities $a x_p \leq b,\ a \in \{1,-1\},\ b \in \{0,1\},\ p \in \Phi$ in IIS Π^{k_t} such that $\Gamma(\Pi^{k_t}) \subseteq \mathcal{U}(k_t) \cup \mathcal{L}(k_t)$. Furthermore, $\Theta(\Pi^{k_t})$ is an index set of the original constraints in Π^{k_t} such that $\Theta(\Pi^{k_t}) \subseteq \mathcal{A}(k_t)$.*

Sets of index sets associated with Π^{k_t} are defined as follows.

DEFINITION 3.3.11. *At node $k_t,\ t \in \mathcal{T}$, $\mathcal{M}(k_t)$ is a set of index sets where $\mathcal{M}(k_t) := \{\Pi_1^{k_t}, ..., \Pi_j^{k_t}\}$ indexed j. In addition, $\mathcal{Y}(k_t)$ is a set of index sets where $\mathcal{Y}(k_t) := \{\Gamma_1(\Pi_1^{k_t}), ..., \Gamma_z(\Pi_j^{k_t})\}$ indexed z. Likewise, $\mathcal{O}(k_t)$ is a set of index sets where $\mathcal{O}(k_t) := \{\Theta_1(\Pi_1^{k_t}), ..., \Theta_q(\Pi_j^{k_t})\}$ indexed q.*

3.3.3 The IIS-BAB Algorithm

Consider a system B with related systems R and U that have the same system of inequalities to B. Provided that B and U are infeasible, the aim of the *IIS-BAB algorithm* is to find at least one IIS of B by applying Theorem 3.3.3 and Corollary 3.3.4 within a BAB framework. The algorithm terminates if either the desired number of IISs is reached or if the sets of open nodes and open root nodes are empty.

The *IIS-BAB algorithm* goes along the following lines. First, check the feasibility of the systems B and R. If B and R are infeasible, then check if the system U is infeasible. If U is feasible, then stop the *IIS-BAB algorithm*. Else, if U is infeasible, then select either a root node r to start a BAB binary tree or a node k to keep branching on the same tree. Then, find an IIS of the related system C^k to B^k, at the selected node k, using Theorem 3.2.2. Node k can be fathomed based on feasibility if the system C^k is feasible. Besides, node k can be fathomed if no improvement in the search of an IIS is seen. If $B^k \subset B^{p(k)}$ for the current node k and parent node $p(k)$, then improvement is seen in the search of an IIS. Likewise, node k can be fathomed if an IIS of B is found. In any of these cases, one should return to the termination step, and a new iteration might begin. Otherwise, if an IIS of C^k, which is not an IIS of B, represents an improvement with respect to the IIS found in the parent node $p(k)$ to k, then proceed to the branching step in which two nodes are created by setting one variable that has not been chosen yet to be equal to 0 and 1. Then, one should return to the termination step, and a new iteration might start. The *IIS-BAB algorithm* is now formalized.

Input: System B

IIS-Binary Algorithm

Step 0. Initialization. Consider a system B with the related systems R and U that have the same system of inequalities to B. Check the feasibility of the systems B and R.

Step 1. Categorization. Classify the systems B and R into one of the following cases:

1. If B and R are feasible, then *stop* algorithm.
2. If B is infeasible, but R is feasible, then *stop* algorithm.
3. If B and R are infeasible, then go to *step 2*.

Step 2. System U Check. Set $\mathcal{A} = \{1, ..., m\}$. Then, consider the following cases:

- If the system U is feasible, then *stop* algorithm.
- Else, if the system U is infeasible, then go to *step 3*.

Step 3. Formulation.

1. Set $\mathcal{H} = \{m+1, ..., m+2n\}$, $\mathcal{U} = \{m+1, ..., m+n\}$, $\mathcal{L} = \{m+n+1, ..., m+2n\}$, and $\mathcal{Q} = \{1, ..., n\}$.
2. Set $\mathcal{S} = \emptyset$, $\mathcal{T} = \emptyset$, $\delta \in \mathbb{Z}_+$, and $t = 0$. Likewise, choose a *node selection* and *node division* rule. Then, go to *step 4*.

Step 4. Termination. If $|\mathcal{S}| = \delta$ or $\mathcal{Q} = \emptyset$ and $\mathcal{N}(t) = \emptyset, \ \forall t \in \mathcal{T}$, then *stop* algorithm. Else, go to *step 5*.

Step 5. Node and Root Node Selection.

If $\mathcal{N}(t) \neq \emptyset, \forall t \in \mathcal{T}$:

1. Select and delete node $k_t \in \mathcal{N}(t)$. Let $\mu_t = \max k_t$.
2. Set the *nodal* systems B^{k_t} and C^{k_t}.
3. Find an IIS $\Pi_j^{k_t}$ of the *nodal* system C^{k_t} using Theorem 3.2.2. Then, add $\Pi_j^{k_t}$ to the set $\mathcal{M}(k_t)$. Go to *step 6*.

Else, If $\mathcal{N}(t) = \emptyset, \forall t \in \mathcal{T}$ and $\neq \emptyset$:

1. Select and delete root node r from $\mathcal{Q}$.
2. Set $t = |\mathcal{T}| + 1$ and add t to $\mathcal{T}$.
3. Add 1_t and 2_t to $\mathcal{N}(t)$. Set $\mu_t = 2_t$.
4. Set $p(1_t) = p(2_t) = 0_t$ and $\mathcal{O}(p(1_t)) = \mathcal{O}(p(2_t)) = \mathcal{A} \cup \mathcal{H}$.
5. Set $\mathcal{A}(1_t) = \mathcal{A}(2_t) = \mathcal{A}$, $\mathcal{U}(1_t) = \{m + r\}$, $\mathcal{L}(1_t) = \emptyset$, $\mathcal{U}(2_t) = \emptyset$, $\mathcal{L}(2_t) = \{m + n + r\}$.
6. Select and delete node $k_t \in \mathcal{N}(t)$. Set the *nodal* systems B^{k_t} and C^{k_t}.
7. Find an IIS $\Pi_j^{k_t}$ of the *nodal* system C^{k_t} using Theorem 3.2.2. Then, add $\Pi_j^{k_t}$ to the set $\mathcal{M}(k_t)$. Go to *step 6*.

Step 6. Fathoming. Set the index sets $\Gamma_z(\Pi_j^{k_t})$ and $\Theta_q(\Pi_j^{k_t})$ for every IIS $\Pi_j^{k_t} \neq \emptyset$ in the set $\mathcal{M}(k_t)$. Fathom node if either **a**, **b**, or **c**. Else, $\Theta_q(\Pi_j^{k_t}) \cup \mathcal{O}(k_t)$ and $\Gamma_z(\Pi_j^{k_t}) \cup \mathcal{Y}(k_t)$. Then, go to *step 7*.

(a). Feasibility: The *nodal* system C^{k_t} is feasible. Return to *step 4*.

(b). Non-improved IIS:

- $\Theta_q(\Pi_j^{k_t})$ is not an IIS of the system B and,
- $\Gamma_z(\Pi_j^{k_t}) \neq \mathcal{U}(k,t) \cup \mathcal{L}(k,t)$ and,
- $\Theta_q(\Pi_j^{k_t})$ is not a subset of some $\Theta_q(\Pi_j^{p(k_t)}) \in \mathcal{O}(p(k_t))$.

Then, $\Theta_q(\Pi_j^{k_t}) \cup \mathcal{O}(k_t)$ and $\Gamma_z(\Pi_j^{k_t}) \cup \mathcal{Y}(k_t)$ and return to *step 4*.

(c). IIS: $\Theta_q(\Pi_j^{k_t})$ is an IIS of the system B. Add $\Theta_q(\Pi_j^{k_t})$ to $\mathcal{S}$. Then, return to *step 4*.

Step 7. Node Division. Choose an index $p \in \mathcal{V} \setminus \{r\}$ such that $m + p \notin \mathcal{U}(k_t)$ and $m + n + p \notin \mathcal{L}(k_t)$, then:

1. Step (i):
   - Create two new nodes μ_t+1 and μ_t+2. Add these two nodes to the set of open nodes $\mathcal{N}(t)$. Set parent nodes $p(\mu_t+1) = p(\mu_t+2) = k_t$.
2. Step (ii):
   - If the index set $\mathcal{U}(k_t) \neq \emptyset$, then add $\mathcal{U}(k_t)$ and $m + i$ to $\mathcal{U}(k_t + 1)$. In addition, add the index set $\mathcal{L}(k_t)$ to $\mathcal{L}(k_t + 1)$.
   - Else, if the index set $\mathcal{U}(k_t) = \emptyset$, then add $m + i$ to $\mathcal{U}(k_t + 1)$. In addition, add the index set $\mathcal{L}(k_t)$ to $\mathcal{L}(k_t + 1)$.
3. Step (iii):
   - If the index set $\mathcal{L}(k_t) \neq \emptyset$, then add $\mathcal{L}(k_t)$ and $m+n+i$ to $\mathcal{L}(k_t+2)$. In addition, add the index set $\mathcal{U}(k_t)$ to $\mathcal{U}(k_t + 2)$.
   - Else, if the index set $\mathcal{L}(k_t) = \emptyset$, then add $m + n + i$ to $\mathcal{L}(k_t + 2)$. In addition, add the index set $\mathcal{U}(k_t)$ to $\mathcal{U}(k_t + 2)$.
4. Step (iv):
   - Set the index set $\mathcal{A}(k_t + 1) = \cup_{q=1,\ldots,|\mathcal{O}(k_t)|}\Theta_q(\Pi_j^{k_t})$. Return to *step 4*.

Output: $\mathcal{S}$ such that $1 \leq |\mathcal{S}| \leq \delta$.

At a particular node k, the size of the system C^k remains tractable by considering only original constraints associated with indices in the IISs that are present in the corresponding *parental* node. This greatly reduces the number of basic feasible solutions to explore in the alternative polyhedron $\mathcal{P}$ from Theorem 3.2.2. This result is formalized in Lemma 3.3.12

LEMMA 3.3.12. *The maximum number of basic feasible solutions to explore in the polyhedron $\mathcal{P}$ (Theorem 3.2.2) at each node k_t in BAB tree t in the IIS-BAB algorithm is*

$$\psi = C(n, |\mathcal{A}(k_t)| + |\mathcal{U}(k_t)| + |\mathcal{L}(k_t)|)$$

with $\psi < C(n, m + 2n)$.

Proof. *At node k_t, the maximum number of rows in C^{k_t} is $|\mathcal{A}(k_t)| + |\mathcal{U}(k_t)| + |\mathcal{L}(k_t)|$. Therefore, the maximum number of basic feasible solutions to explore in polyhedron $\mathcal{P}$ at node k_t is $\psi = C(n, |\mathcal{A}(k_t)| + |\mathcal{U}(k_t)| + |\mathcal{L}(k_t)|)$. Since $|\mathcal{A}(k_t)| + |\mathcal{U}(k_t)| + |\mathcal{L}(k_t)| \leq n + 1 < m + 2n$ by Theorem 3.2.3, then $\psi < C(n, m + 2n)$.*

■

Corollary 3.3.13 states that the maximum cardinality on each IIS S for B found by the *IIS-BAB algorithm* is $n + 1$.

COROLLARY 3.3.13. *Consider an infeasible system B. Let S be an IIS of B obtained by the IIS-BAB algorithm. Then, $|S| \leq n + 1$.*

Proof. *At node $k_t \in \mathcal{N}, t \in \mathcal{T}$, consider index sets $\mathcal{U}(k_t)$ and $\mathcal{L}(k_t)$. By Theorem 3.3.2, $|\mathcal{U}(k_t) \cup \mathcal{L}(k_t)| \geq 0$, and $|\mathcal{U}(k_t) \cup \mathcal{L}(k_t) \cup S| \leq n + 1$ for any node k in the* IIS-BAB algorithm. *Thus, $|S| \leq n + 1$.*

■

COROLLARY 3.3.14. *The maximum number of nodes to explore in the IIS-BAB algorithm is $\sum_{i=1}^{n} 2^i$.*

Proof. *There are n variables in B. The maximum number of feasible solutions in B are $\sum_{i=1}^{n} 2^i$. Thus, there are $\sum_{i=1}^{n} 2^i$ possibilities for assigning upper or lower bounds to variables in a system C_Π that implies the maximum number of nodes to explore in the IIS-BAB algorithm.*

■

Lemma 3.3.15 guarantees the isolation of an IIS of an infeasible system B by the *IIS-BAB algorithm*.

LEMMA 3.3.15. *Given an infeasible system B, then the IIS-BAB algorithm will find at least one IIS of B.*

Proof. *The existence of an IIS S to a system B is guaranteed by Lemma 3.3.1. Therefore, the IIS-BAB algorithm will find at least one IIS S of B by Proposition 3.3.2 and Theorem 3.3.3.*

■

The finite convergence of the *IIS-BAB algorithm* is guaranteed by Corollary 3.3.14 and Lemma 3.3.15.

THEOREM 3.3.16. *The IIS-BAB algorithm converges in a finite number of iterations.*

Proof. *The number of nodes to search in the IIS-BAB algorithm is finite by Corollary 3.3.14. The isolation of an IIS of B is guaranteed by Lemma 3.3.15. Therefore, the IIS-BAB algorithm will end in a finite number of iterations.*

■

3.3.4 Improving the Node Division Step

The performance of the *IIS-BAB algorithm* can be improved by sharing information between subsequent binary trees. Consider a system B with an associated system C_Π whose corresponding IIS Π leads to the IIS S of B. Let us assume that C_Π was found using the *IIS-BAB algorithm*. Therefore, C_Π can be associated with a leaf node $\bar{k}$ in a BAB binary tree $\bar{t}$ with root node $\bar{r}$. Moreover, there is a path $\tau(\bar{k}, \bar{r})$ from the leaf node $\bar{k}$ to the root node $\bar{r}$ that is associated with an index set $\bar{\Phi}$. Let us refer to this path as a *successful path*, and let $\bar{\alpha}$ be the the depth of a node $\bar{k}$ in the BAB tree $\bar{t}$. The elements in the index set $\bar{\Phi}$ can be used to represent $\bar{\alpha}$!

successful paths whose corresponding leaf and root nodes would have led us exactly to the same C_Π.

Given also a leaf node $\hat{k}$ and root node $\hat{r}$ in a BAB tree $\hat{t}$, an unsuccessful path $\tau(\hat{k}, \hat{t})$ is the one whose corresponding index set $\hat{\Phi}$ and system C_Π does not lead to an IIS S of B. The elements in the index set $\hat{\Phi}$ can be used to represent $\hat{\alpha}!$ *unsuccessful paths* whose corresponding leaf and root nodes would have led us exactly to the same C_Π. Therefore, once a *successful path* or an *unsuccessful path* is identified, then there is no need to traverse this same path in any order in subsequent BAB trees.

In the *IIS-BAB algorithm*, after all nodes have been fathomed in a particular BAB binary tree t, information related to *successful and unsuccessful paths* should be carried to subsequent trees. By keeping track of *successful and unsuccessful paths* in a BAB tree, the number of nodes to search in the *IIS-BAB algorithm* can be reduced. Let us now introduce a new tree data structure that will be refereed to as the *compass search tree.* The compass search tree will be used as vademecum by the *IIS-BAB algorithm* in order to share information on *successful and unsuccessful paths* between different BAB binary trees. Therefore, the compass search tree can be used as a strategy for node division (step 7) in the *IIS-BAB algorithm* as explained in the following example.

Consider an arbitrary system B with m inequalities subject to 4-dimensional decision vectors with binary components. The corresponding compass search tree for this system B is shown in Figure 3.3. By Corollary 3.3.4, the maximum depth α of any node in the compass search tree depicted in Figure 3.3 is 4. The goal is to find an IIS of B. Then, let us assume that after traversing the path $1, 5, 11$ in the compass search tree depicted in Figure 3.3 no IIS for B is found while applying the *IIS-BAB algorithm.* By fathoming the leaf node 11 in this compass search tree, then no other paths will visit this node again. For example, other possible paths down to the third level in the compass search tree would be $1, 6, 13$ or $2, 8, 14$ but never $1, 6, 11$ nor $2, 8, 11$. Furthermore, Lemma 3.3.1, Proposition 3.3.2, Theorem 3.3.3, and Corollary 3.3.4 still hold for the *IIS-BAB algorithm* using the compass search tree approach.

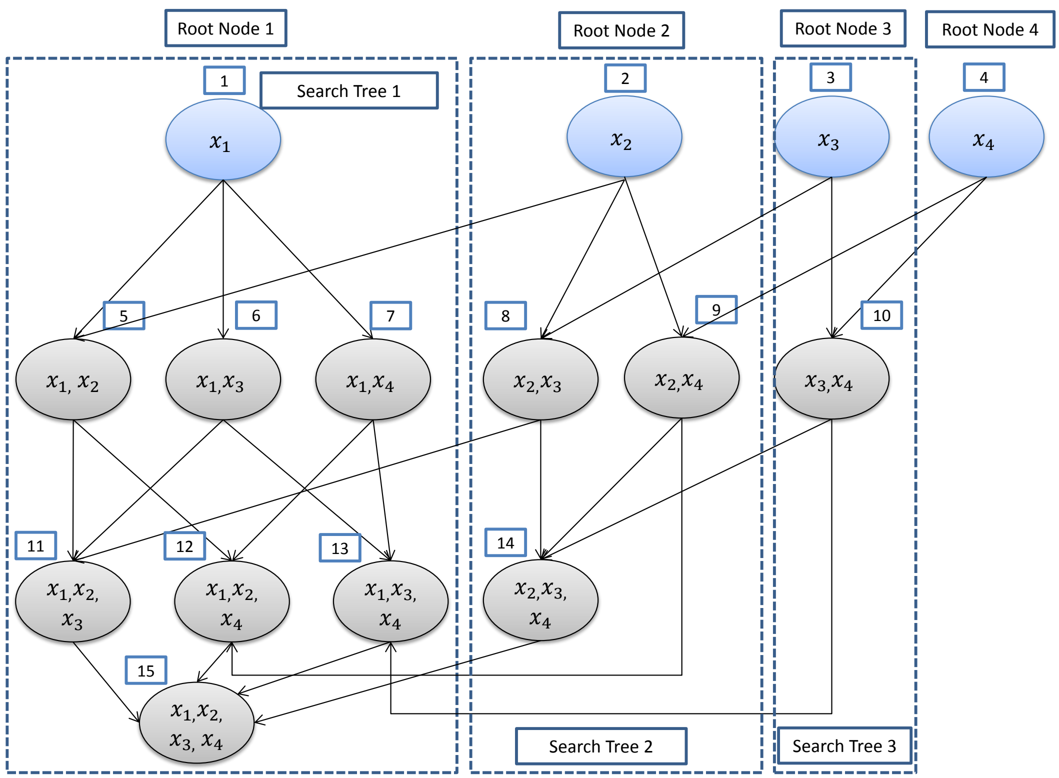

Figure 3.3: Compass search tree of system B

3.3.5 The IIS-Heuristic Algorithm

An important characteristic of the *IIS-BAB algorithm* is that it can only be applied to B if the associated system U is infeasible. In fact, if U is feasible, then Theorem 3.2.2 can not be applied to the corresponding system C^{k_t} at any node k_t

in a BAB tree t. Thus, the *IIS-Heuristic algorithm* represents an alternative to the *IIS-BAB algorithm* when the system U is feasible. The idea behind the proposed heuristics is that a set of inequalities $ax_p \leq b,\ a \in \{1, -1\},\ b \in \{0, 1\},\ p \in \Phi$ can be appended to the system U at the same time until the system U is infeasible. This can be done in an iterative manner until an IIS S of B is found. By following this heuristic approach, the original system B can be rapidly reduced to smaller relaxation subsystems where finding an IIS will be more likely to occur.

Consider a system B with related systems R and U that have the same system of inequalities to B. Given an infeasible system B with the feasible system U, the goal of the *IIS-Heuristic algorithm* is to find at least one IIS of B by applying Theorem 3.3.3 and Corollary 3.3.4 to an infeasible system that is related to B. Since the *IIS-Heuristic algorithm* deals with iterations instead of BAB binary trees and nodes, notation associated with node index $k_t,\ t \in \mathcal{T}$ will be replaced by the iteration index i. For example, $\mathcal{A}(i) \subseteq \mathcal{A}$, $\mathcal{U}(i) \subseteq \mathcal{U}$, and $\mathcal{L}(i) \subseteq \mathcal{L}$ will now refer to index sets that can define the inequalities in the system $C^i := \{g_s x \leq v_s,\ \forall s \in \mathcal{A}(i),\ x_p \leq 1,\ p = \bar{u} - m,\ \forall \bar{u} \in \mathcal{U}(i),\ x_p \leq 0,\ p = \bar{l} - (m+n),\ \forall \bar{l} \in \mathcal{L}(i) :\ x \in \mathbb{R}^n\}$ at iteration i. Similarly, this change will apply to all the notation defined in Section 3.3.2 by replacing k_t by i.

The *IIS-Heuristic algorithm* terminates when the desired number of IISs is reached. This algorithm starts by checking the feasibility of the systems B and R. If B and R are infeasible, then check if the system U is feasible. If U is infeasible, then stop the algorithm. Else, if U is feasible and bounded, then set the index of iterations i to *one* and solve the corresponding related system C^i to B. If C^i is feasible, then append inequalities of the form $ax_p \leq b,\ a \in \{1, -1\},\ b \in \{0, 1\},\ p \in \Phi$ to C^i until this system is infeasible. Otherwise, find an IIS for the system C^i using Theorem 3.2.2. If an IIS of B is found, then check the termination criteria. Else, if an IIS of C^i that is not an IIS of B represents an improvement with respect to the IIS found in the previous iteration, that is, if $B^i \subset B^{i-1}$, then update C^i, increase the iteration index i by one, return to the termination, and start a new iteration. The *IIS-Heuristic algorithm* is now formalized.

Input: System B

IIS-Heuristic Algorithm

Step 0. Initialization. Consider a system B with the related systems R and U that have the same system of inequalities to B. Check the feasibility of the systems B and R.

Step 1. Categorization. Classify the systems B and R into one of the following cases:

1. If B and R are feasible, then *stop* algorithm.
2. If B is infeasible, but R is feasible, then *stop* algorithm.
3. If B and R are infeasible, then go to *step 2*.

Step 2. System U Check. Set $\mathcal{A} = \{1, ..., m\}$. Then, consider the following cases:

- If the system U is infeasible, then *stop* algorithm.
- Else, if the system U is feasible and bounded, then set $i = 1$ and go to *step 3*.

Step 3. Formulation.

1. Set $\mathcal{H} = \{m+1, ..., m+2n\}$, $\mathcal{U} = \{m+1, ..., m+n\}$, and $\mathcal{L} = \{m+n+1, ..., m+2n\}$.
2. Set $\mathcal{A}(i) = \mathcal{A}$, $\mathcal{U}(i) = \emptyset$, and $\mathcal{L}(i) = \emptyset$.

3. Set $\mathcal{S} = \emptyset$ and $\delta \in \mathbb{Z}_+$. Then, go to *step 4.*

Step 4. Termination. If $|\mathcal{S}| = \delta$, then *stop* algorithm. Else, go to *step 5.*

Step 5. Solve Associated System. Solve the system C^i. If C^i is feasible, then go to *step 6.* Otherwise, if C^i is infeasible, then go to *step 7.*

Step 6. Variable Selection. If $x_p > 1$, then $u \cup \mathcal{U}(i)$ where $u = p+m, \forall p \in \mathcal{V}$. If $x_p < 0$, then $l \cup \mathcal{L}(i)$ where $l = p+m+n, \forall p \in \mathcal{V}$. Return to *step 5.*

Step 7. IIS Search. Find an IIS Π_j^i for the system C^i using Theorem 3.2.2. Then, add Π_j^i to the set $\mathcal{M}(i)$. Go to *step 8.*

Step 8. IIS Examination. If $\Theta_q(\Pi_j^i)$ is an IIS of the system B, then add $\Theta_q(\Pi_j^i)$ to $\mathcal{S}$ and go to *step 4.* Otherwise, if $\Theta_q(\Pi_j^i)$ is a subset of some $\Theta_q(\Pi_j^{i-1}) \in \mathcal{O}(i-1)$, then $\Theta_q(\Pi_j^i) \cup \mathcal{O}(i)$ and $\Gamma_z(\Pi_j^i) \cup \mathcal{Y}(i)$. Set $i = i+1$ and $\mathcal{A}(i) = \cup_{q=1,\ldots,|\mathcal{O}(i)|}\Theta_q(\Pi_j^i)$, $\mathcal{U}(i) = \emptyset$, $\mathcal{L}(i) = \emptyset$. Then, return to *step 4.*

Output: $\mathcal{S}$ such that $|\mathcal{S}| = \delta$.

LEMMA 3.3.17. *Given an infeasible system B, then the IIS-Heuristic algorithm will find at least one IIS of B.*

Proof. *Since B is infeasible and U is feasible, then there exists at least one vertex in U such that $x_p < 0$ or $x_p > 1$ for at least one $p \in \mathcal{V}$. Therefore, the system C^i at iteration i can become infeasible by appending inequalities of the form $ax_p \leq b,\ a \in \{1, -1\},\ b \in \{0, 1\},\ p \in \Phi$ to it. Given an infeasible system C^i, the existence of at least one IIS of this system is guaranteed by Theorem 3.2.2.*

■

THEOREM 3.3.18. *The IIS-Heuristic algorithm converges in a finite number of iterations.*

Proof. *Given an infeasible system B with an associated feasible and bounded system U, the finite convergence of the IIS-Heuristic algorithm is guaranteed by the finite number of extreme points in U.*

■

3.4 Example Illustration

3.4.1 The IIS-BAB Algorithm

Let us apply the *IIS-BAB algorithm* to the following system $\bar{B}$ in order to find one IIS.

$$
\begin{aligned}
\bar{B} = \{&-x_2 - 2x_3 + x_4 - 2x_5 \leq 0 \ (1),\\
&-x_1 - 2x_4 - x_5 \leq 0 \ (2),\\
&-2x_1 - 2x_2 + x_3 + x_4 - x_5 \leq -1 \ (3),\\
&-2x_1 - x_2 - x_3 - x_4 + x_5 \leq 0 \ (4)\\
&x_1 + x_2 + x_5 \leq 0 \ (5),\\
&-x_1 - 2x_2 - x_4 \leq -1 \ (6),\\
&-2x_1 - x_2 - 2x_3 \leq 0 \ (7),\\
&-x_2 - x_3 + x_4 - 2x_5 \leq 0 \ (8),\\
&x_2 + x_3 - 2x_4 - x_5 \leq 0 \ (9),\\
&x_1 - x_2 + x_3 + x_4 + x_5 \leq -1 \ (10) : \ x \in \{0,1\}^5\}.
\end{aligned}
$$

A system R related to $\bar{B}$ is shown below.

$$
\begin{aligned}
R = \{ & -x_2 - 2x_3 + x_4 - 2x_5 \leq 0 \ (1), \\
& -x_1 - 2x_4 - x_5 \leq 0 \ (2), \\
& -2x_1 - 2x_2 + x_3 + x_4 - x_5 \leq -1 \ (3), \\
& -2x_1 - x_2 - x_3 - x_4 + x_5 \leq 0 \ (4), \\
& \ x_1 + x_2 + x_5 \leq 0 \ (5), \\
& -x_1 - 2x_2 - x_4 \leq -1 \ (6), \\
& -2x_1 - x_2 - 2x_3 \leq 0 \ (7), \\
& -x_2 - x_3 + x_4 - 2x_5 \leq 0 \ (8), \\
& \ x_2 + x_3 - 2x_4 - x_5 \leq 0 \ (9) \\
& \ x_1 - x_2 + x_3 + x_4 + x_5 \leq -1 \ (10), \\
& \ x_1 \leq 1 \ (11), x_2 \leq 1 \ (12), x_3 \leq 1 \ (13), x_4 \leq 1 \ (14), x_5 \leq 1 \ (15) \\
& -x_1 \leq 0 \ (16), -x_2 \leq 0 \ (17), -x_3 \leq 0 \ (18), -x_4 \leq 0 \ (19), \\
& -x_5 \leq 0 \ (20) : \ x \in \mathbb{R}^5 \}.
\end{aligned}
$$

IIS-BAB Algorithm:

Step 0. Initialization. Check feasibility of the systems $\bar{B}$ and R.

Step 1. Categorization. Since the systems $\bar{B}$ and R are infeasible, then go to *step 2*.

Step 2. System U Check. Set $\mathcal{A} = \{1, ..., 10\}$. Since the system U is infeasible, then go to *step 3*.

Step 3. Formulation.

1. Set $\mathcal{H} = \{11, ..., 20\}$, $\mathcal{U} = \{11, ..., 15\}$, $\mathcal{L} = \{16, ..., 20\}$, and $\mathcal{Q} = \{1, ..., 5\}$.

2. Set $\mathcal{S} = \emptyset$, $\mathcal{T} = \emptyset$, $\delta = 1$, and $t = 0$. Choose *depth-first search plus backtracking* as the node selection rule while considering variables in decreasing order and right sons first.

<u>**Iteration 1**</u>

Step 4. Termination. Since $|\mathcal{S}| = 0$, then go to *step 5*.

Step 5. Node Selection. Since $\mathcal{N}(t) = \emptyset, \forall t \in \mathcal{T}$, and $\mathcal{Q} \neq \emptyset$, then:

1. Select root node $r = 5$ from $\mathcal{Q}$. In addition, $\mathcal{Q} \setminus \{5\}$.

2. Set $t = 1$ and $\mathcal{T} \cup \{1\}$.

3. Add 1_1 and 2_1 to $\mathcal{N}(1)$. Set $\mu_1 = 2_t$.

4. Set $p(1_1) = p(2_1) = 0_1$ and $\mathcal{O}(1_1) = \mathcal{O}(2_1) = \mathcal{A} \cup \mathcal{H}$.

5. Set $\mathcal{A}(1_1) = \mathcal{A}(2_1) = \mathcal{A}$, $\mathcal{U}(1_1) = \{15\}$, $\mathcal{L}(1_1) = \emptyset$, $\mathcal{U}(2_1) = \emptyset$, $\mathcal{L}(2_1) = \{20\}$.

6. Select node 2_1 from $\mathcal{N}(1)$ and $\mathcal{N}(1) \setminus \{2_1\}$.

7. Set the following *nodal* system:

$$
\begin{aligned}
B^{2_1} = \{&-x_2 - 2x_3 + x_4 - 2x_5 \leq 0 \ (1),\\
&-x_1 - 2x_4 - x_5 \leq 0 \ (2),\\
&-2x_1 - 2x_2 + x_3 + x_4 - x_5 \leq -1 \ (3),\\
&-2x_1 - x_2 - x_3 - x_4 + x_5 \leq 0 \ (4),\\
&x_1 + x_2 + x_5 \leq 0 \ (5),\\
&-x_1 - 2x_2 - x_4 \leq -1 \ (6),\\
&-2x_1 - x_2 - 2x_3 \leq 0 \ (7),\\
&-x_2 - x_3 + x_4 - 2x_5 \leq 0 \ (8),\\
&x_2 + x_3 - 2x_4 - x_5 \leq 0 \ (9),\\
&x_1 - x_2 + x_3 + x_4 + x_5 \leq -1 \ (10),\\
&x_5 = 0 (20) : \ x \in \mathbb{B}^5\}.
\end{aligned}
$$

8. Set the following *nodal* system:

$$
\begin{aligned}
C^{2_1} = \{ & -x_2 - 2x_3 + x_4 - 2x_5 <= 0 \ (1), \\
& -x_1 - 2x_4 - x_5 \leq 0 \ (2), \\
& -2x_1 - 2x_2 + x_3 + x_4 - x_5 \leq -1 \ (3), \\
& -2x_1 - x_2 - x_3 - x_4 + x_5 \leq 0 \ (4), \\
& \ \ x_1 + x_2 + x_5 \leq 0 \ (5), \\
& -x_1 - 2x_2 - x_4 \leq -1 \ (6), \\
& -2x_1 - x_2 - 2x_3 \leq 0 \ (7), \\
& -x_2 - x_3 + x_4 - 2x_5 \leq 0 \ (8), \\
& \ \ x_2 + x_3 - 2x_4 - x_5 \leq 0 \ (9), \\
& \ \ x_1 - x_2 + x_3 + x_4 + x_5 \leq -1 \ (10), \\
& -x_5 \leq 0 \ (20): \ x \in \mathbb{R}^5\}.
\end{aligned}
$$

9. Let $\Pi_1^{2_1} = \{2, 3, 5, 7, 10, 20\}$ be an IIS of C^{2_1}. Add $\Pi_1^{2_1}$ to $\mathcal{M}(2_1)$. Observe that $\Gamma_1(\Pi_1^{2_1}) = \Pi_1^{2_1} \cap \mathcal{H} = \{20\}$ and $|\Pi_1^{2_1}| = n + 1 = 6$. Furthermore, $|\Gamma_1(\Pi_1^{2_1})| = 1 \leq n = 5$.

Step 6. Fathoming Rules. $\Theta_1(\Pi_1^{2_1}) = \Pi_1^{2_1} \cap \mathcal{A} = \{2, 3, 5, 7, 10\}$. $\Theta_1(\Pi_1^{2_1})$ is not an IIS of $\bar{B}$, but $\Gamma_1(\Pi_1^{2_1}) = \mathcal{U}(2_1) \cup \mathcal{L}(2_1)$ and $\Theta_1(\Pi_1^{2_1})$ is a proper subset of $\Theta_1(p(2_1))$. $\Theta_1(\Pi_1^{2_1}) \cup \mathcal{O}(2_1)$ and $\Gamma_1(\Pi_1^{2_1}) \cup \mathcal{Y}(2_1)$. Go to *step 7*.

Step 7. Node Division. Choose $i = 4$ using the compass search tree for $\bar{B}$.

1. Step *(i)*: Create two new nodes $\mu_1 + 1 = 3_1$ and $\mu_1 + 2 = 4_1$. Add these two new nodes to $\mathcal{N}(1)$. Set $p(3_1) = 2_1,\ p(4_1) = 2_1$.

2. Step *(ii)*: Since $\mathcal{U}(2_1) = \emptyset$, then $\mathcal{U}(3_1) = \{14\}$ and $\mathcal{L}(3_1) = \{20\}$.

3. Step *(iii)*: Since $\mathcal{L}(2_1) \neq \emptyset$, then $\mathcal{L}(4_1) = \{19, 20\}$ and $\mathcal{U}(4_1) = \emptyset$.

4. Step *(iv)*: Let $\mathcal{A}(3_1) = \mathcal{A}(4_1) = O(\Pi_1^{2_1})$ and return to *step 4*.

<u>Iteration 2</u>

Step 4. Termination. Since $|\mathcal{S}| = 0$, then go to *step 5*.

Step 5. Node Selection. Since $\mathcal{N}(1) \neq \emptyset$, then:

1. Select and delete node 4_1 from $\mathcal{N}(1)$. Set $\mu_1 = 4_t$.

2. Set the following *nodal* system:

$$
\begin{aligned}
B^{4_1} = \{&-x_1 - 2x_4 - x_5 \leq 0 \text{ (2)},\\
&-2x_1 - 2x_2 + x_3 + x_4 - x_5 \leq -1 \text{ (3)},\\
&x_1 + x_2 + x_5 \leq 0 \text{ (5)},\\
&-2x_1 - x_2 - 2x_3 \leq 0 \text{ (7)},\\
&x_1 - x_2 + x_3 + x_4 + x_5 \leq -1 \text{ (10)},\\
&x_4 = 0 \text{ (19)}, x_5 = 0 \text{ (20)} : \ x \in \mathbb{B}^5\}.
\end{aligned}
$$

3. Set the following *nodal* system:

$$
\begin{aligned}
C^{4_1} = \{ & -x_1 - 2x_4 - x_5 \leq 0 \ (2), \\
& -2x_1 - 2x_2 + x_3 + x_4 - x_5 \leq -1 \ (3), \\
& x_1 + x_2 + x_5 \leq 0 \ (5), \\
& -2x_1 - x_2 - 2x_3 \leq 0 \ (7), \\
& x_1 - x_2 + x_3 + x_4 + x_5 \leq -1 \ (10), \\
& -x_4 \leq 0 \ (19), -x_5 \leq 0 \ (20) : \ x \in \mathbb{R}^5 \}.
\end{aligned}
$$

4. Let $\Pi_1^{4_1} = \{3, 5, 7, 10, 19, 20\}$ be an IIS of C^{4_1}. Add $\Pi_1^{4_1}$ to $\mathcal{M}(4_1)$. Observe that $\Gamma_1(\Pi_1^{4_1}) = \Pi_1^{4_1} \cap \mathcal{H} = \{19, 20\}$. Notice that $|\Pi_1^{4_1}| = n + 1 = 6$ and $|\Gamma_1(\Pi_1^{4_1})| = 2 \leq n = 5$.

Step 6. Fathoming Rules. $\Theta_1(\Pi_1^{4_1}) = \Pi_1^{4_1} \cap \mathcal{A} = \{3, 5, 7, 10\}$. $\Theta_1(\Pi_1^{4_1})$ is not an IIS of $\bar{B}$, but $\Gamma_1(\Pi_1^{4_1}) = \mathcal{U}(4_1) \cup \mathcal{L}(4_1)$ and $\Theta_1(\Pi_1^{4_1})$ is a proper subset of $\Theta_1(p(4_1))$. $\Theta_1(\Pi_1^{4_1}) \cup \mathcal{O}(4_1)$ and $\Gamma_1(\Pi_1^{4_1}) \cup \mathcal{Y}(4_1)$. Go to *step 7*.

Step 7. Node Division. Choose $i = 3$ using the compass search tree for $\bar{B}$.

1. Step *(i)*: Create two new nodes $\mu_1 + 1 = 5_1$ and $\mu_1 + 2 = 6_1$. Add these two new nodes to $\mathcal{N}(1)$. Set $p(5_1) = 4_1, \ p(6_1) = 4_1$.

2. Step *(ii)*: Since $\mathcal{U}(4_1) = \emptyset$, then $\mathcal{U}(5_1) = \{13\}$ and $\mathcal{L}(5_1) = \{19, 20\}$.

3. Step *(iii)*: Since $\mathcal{L}(4_1) \neq \emptyset$, then $\mathcal{L}(6_1) = \{18, 19, 20\}$ and $\mathcal{U}(6_1) = \emptyset$.

4. Step *(iv)*: Let $\mathcal{A}(5_1) = \mathcal{A}(6_1) = O(\Pi_1^{4_1})$ and return to *step 4*.

Iteration 3

Step 4. Termination. Since $\mathcal{S} = 0$, then go to *step 5*.

Step 5. Node Selection. Since $\mathcal{N}(1) \neq \emptyset$, then:

1. Select and delete node 6_1 from $\mathcal{N}(1)$. Set $\mu_1 = 6_t$.

2. Set the following *nodal* system:

$$
\begin{aligned}
B^{6_1} = \{ & -2x_1 - 2x_2 + x_3 + x_4 - x_5 \leq -1 \ (3), \\
& x_1 + x_2 + x_5 \leq 0 \ (5), \\
& -2x_1 - x_2 - 2x_3 \leq 0 \ (7), \\
& x_1 - x_2 + x_3 + x_4 + x_5 \leq -1 \ (10), \\
& x_3 = 0 \ (18), x_4 = 0 \ (19), x_5 = 0 \ (20) : \ x \in \mathbb{B}^5 \}.
\end{aligned}
$$

3. Set the following *nodal* system:

$$
\begin{aligned}
C^{6_1} = \{ & -2x_1 - 2x_2 + x_3 + x_4 - x_5 \leq -1 \ (3), \\
& x_1 + x_2 + x_5 \leq 0 \ (5), \\
& -2x_1 - x_2 - 2x_3 \leq 0 \ (7), \\
& x_1 - x_2 + x_3 + x_4 + x_5 \leq -1 \ (10), \\
& -x_3 \leq 0 \ (18), -x_4 \leq 0 \ (19), -x_5 \leq 0 \ (20) : \ x \in \mathbb{R}^5 \}.
\end{aligned}
$$

4. Let $\Pi_1^{6_1} = \{3, 5, 18, 19, 20\}$ be an IIS of C^{6_1}. Add $\Pi_1^{6_1}$ to $\mathcal{M}(6_1)$. Observe that $\Gamma_1(\Pi_1^{6_1}) = S_1^{6_1} \cap \mathcal{H} = \{18, 19, 20\}$. Notice that $|\Pi_1^{6_1}| = 5 \leq n + 1 = 6$ and $|\Gamma_1(\Pi_1^{6_1})| = 3 \leq n = 5$.

Step 6. Fathoming Rules. $\Theta_1(\Pi_1^{6_1}) = \Pi_1^{6_1} \cap \mathcal{A} = \{3, 5\}$ is an IIS of $\bar{B}$. Add $\Theta_1(\Pi_1^{6_1})$ to $\mathcal{S}$. Fathom corresponding node in the compass search tree. Go to *step 4*.

<u>Iteration 4</u>

Step 4. Termination. Since $|\mathcal{S}| = 1$, then stop algorithm.

A summary of the implementation of the *IIS-BAB algorithm* to B is depicted in Figures 3.4 and 3.5.

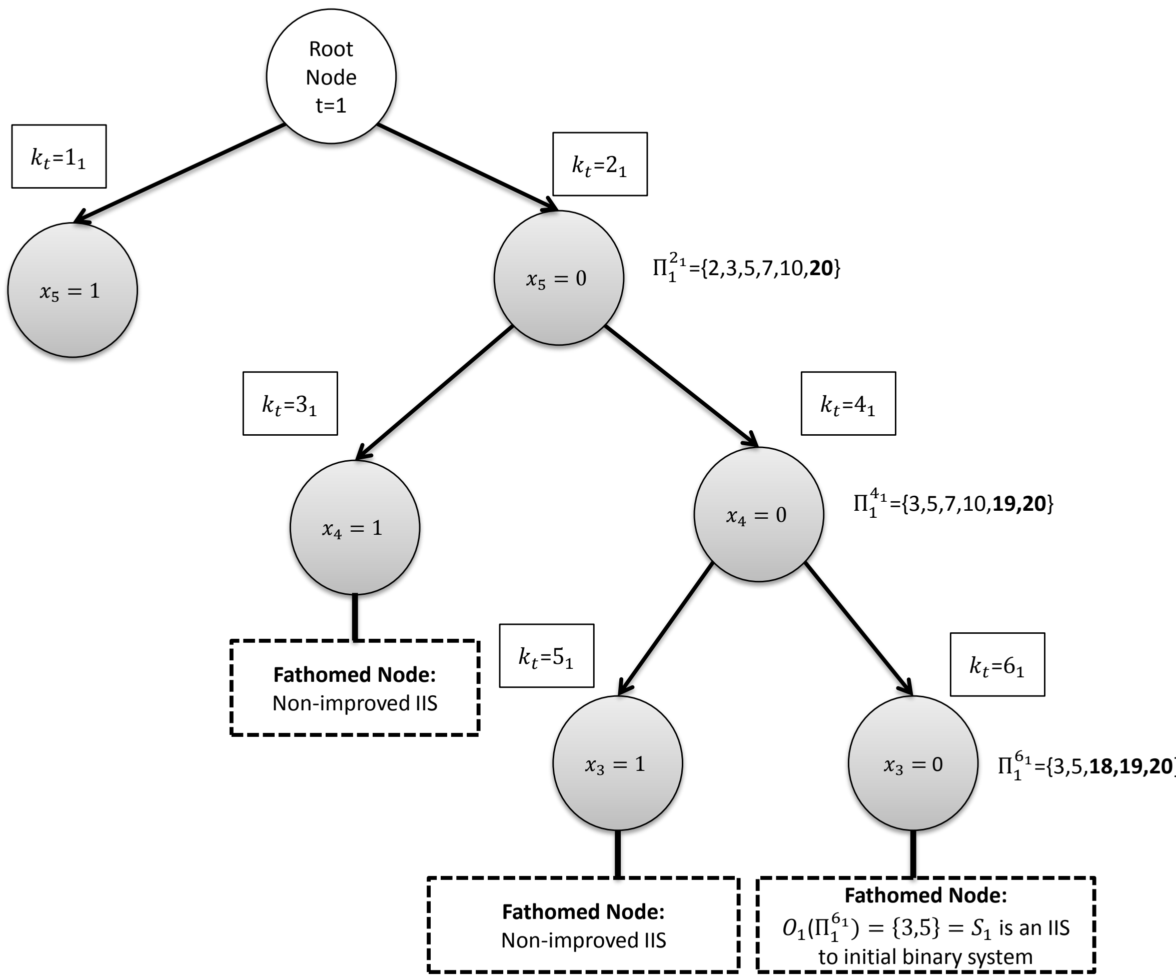

Figure 3.4: BAB tree $t = 1$ with root node $r = 5$ and $x_5 = 0$

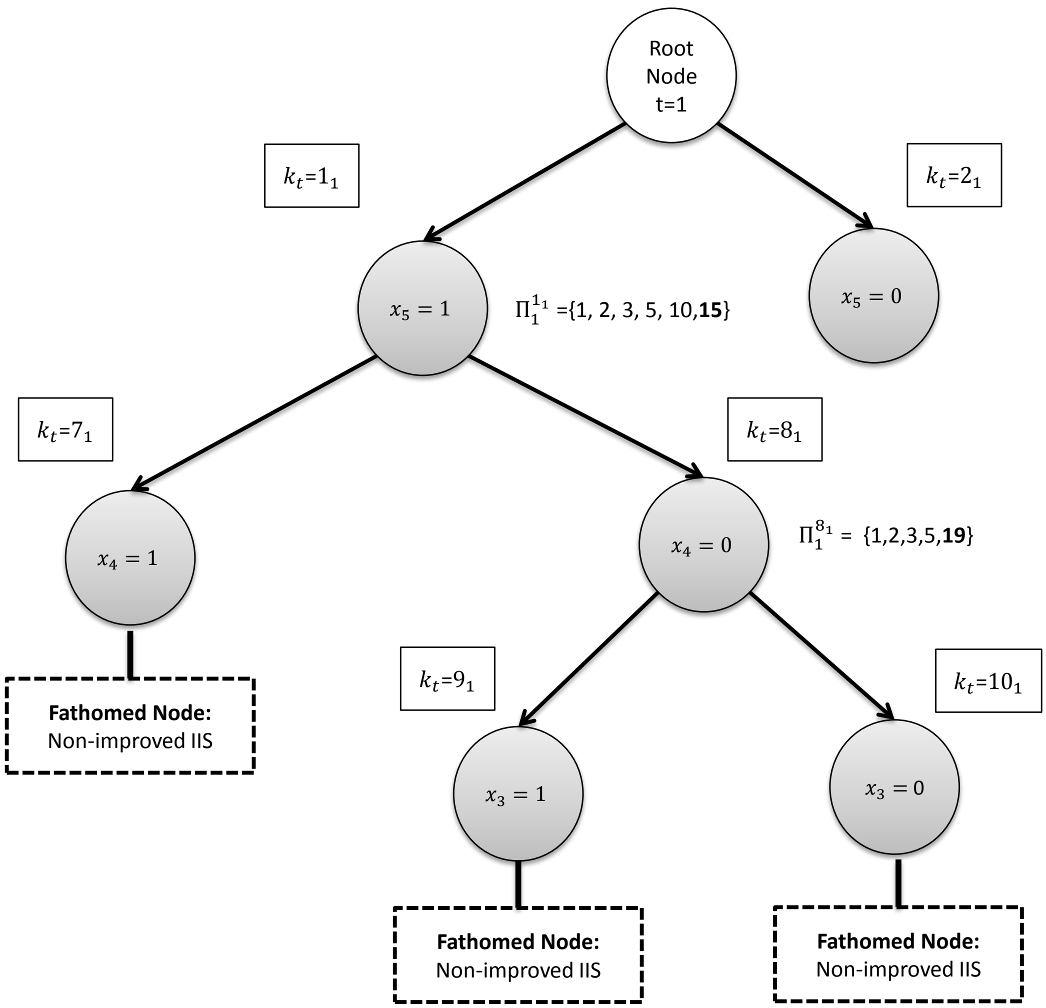

Figure 3.5: BAB tree $t = 1$ with root node $r = 5$ and $x_5 = 1$

3.4.2 The IIS-Heuristic Algorithm

Let us apply the *IIS-Heuristic algorithm* to the following system $\hat{B}$ in order to find one IIS.

$$
\begin{aligned}
\hat{B} = \{&3x_1 - 2x_2 + x_3 - 4x_4 - 2x_5 \le 0 \ (1),\\
&3x_1 + 2x_2 + x_3 - 3x_4 + 4x_5 \le -1 \ (2),\\
&-x_1 - 5x_2 + 3x_3 - 4x_5 \le -1 \ (3),\\
&x_1 + 4x_2 - 4x_3 + 3x_5 \le 0 \ (4),\\
&3x_2 - 4x_3 - 2x_4 + 4x_5 \le 0 \ (5),\\
&-5x_1 + 2x_2 + x_3 - 4x_4 + 3x_5 \le -2 \ (6),\\
&3x_2 - x_3 - x_4 + 3x_5 \le -2 \ (7),\\
&3x_1 - 5x_2 + 2x_3 - 4x_4 + 2x_5 \le 0 \ (8),\\
&-3x_1 - 2x_2 - 2x_3 + x_4 - 3x_5 \le -1 \ (9),\\
&-2x_1 - 3x_2 - 4x_3 - 5x_4 - x_5 \le -1 \ (10) : \ x \in \{0,1\}^5\}.
\end{aligned}
$$

A related system to $\hat{B}$ is shown below.

$$
\begin{aligned}
R = \{&3x_1 - 2x_2 + x_3 - 4x_4 - 2x_5 \leq 0 \ (1),\\
&3x_1 + 2x_2 + x_3 - 3x_4 + 4x_5 \leq -1 \ (2),\\
&-x_1 - 5x_2 + 3x_3 - 4x_5 \leq -1 \ (3),\\
&x_1 + 4x_2 - 4x_3 + 3x_5 \leq 0 \ (4),\\
&3x_2 - 4x_3 - 2x_4 + 4x_5 \leq 0 \ (5),\\
&-5x_1 + 2x_2 + x_3 - 4x_4 + 3x_5 \leq -2 \ (6),\\
&3x_2 - x_3 - x_4 + 3x_5 \leq -2 \ (7),\\
&3x_1 - 5x_2 + 2x_3 - 4x_4 + 2x_5 \leq 0 \ (8),\\
&-3x_1 - 2x_2 - 2x_3 + x_4 - 3x_5 \leq -1 \ (9),\\
&-2x_1 - 3x_2 - 4x_3 - 5x_4 - x_5 \leq -1 \ (10): \ x \in \mathbb{R}^5\}.
\end{aligned}
$$

IIS-Heuristic Algorithm:

Step 0. Initialization. Check feasibility of the systems $\hat{B}$ and R.

Step 1. Categorization. Since the systems $\hat{B}$ and R are infeasible, then go to *step 2*.

Step 2. System U Check. Set $\mathcal{A} = \{1, ..., 10\}$. Since the system U is feasible, then set $i = 1$ and go to *step 3*.

Step 3. Formulation.

1. Set $\mathcal{H} = \{11, ..., 20\}$, $\mathcal{U} = \{11, ..., 15\}$, and $\mathcal{L} = \{16, ..., 20\}$.
2. Set $\mathcal{A}(1) = \mathcal{A}$, $\mathcal{U}(1) = \emptyset$, and $\mathcal{L}(1) = \emptyset$.

3. Set $\mathcal{S} = \emptyset$ and $\delta = 1$. Then, go to *step 4*.

Iteration 1

Step 4. Termination. Since $|\mathcal{S}| = 0$, then go to *step 5*.

Step 5. Solve Associated System. Solve the following system:

$$
\begin{aligned}
C^1 = \{ & 3x_1 - 2x_2 + x_3 - 4x_4 - 2x_5 \leq 0 \ (1), \\
& 3x_1 + 2x_2 + x_3 - 3x_4 + 4x_5 \leq -1 \ (2), \\
& -x_1 - 5x_2 + 3x_3 - 4x_5 \leq -1 \ (3), \\
& x_1 + 4x_2 - 4x_3 + 3x_5 \leq 0 \ (4), \\
& 3x_2 - 4x_3 - 2x_4 + 4x_5 \leq 0 \ (5), \\
& -5x_1 + 2x_2 + x_3 - 4x_4 + 3x_5 \leq -2 \ (6), \\
& 3x_2 - x_3 - x_4 + 3x_5 \leq -2 \ (7), \\
& 3x_1 - 5x_2 + 2x_3 - 4x_4 + 2x_5 \leq 0 \ (8), \\
& -3x_1 - 2x_2 - 2x_3 + x_4 - 3x_5 \leq -1 \ (9), \\
& -2x_1 - 3x_2 - 4x_3 - 5x_4 - x_5 \leq -1 \ (10) : \ x \in \mathbb{R}^5 \}.
\end{aligned}
$$

Since C^1 is feasible, then go to *step 6*.

Step 6. Variable Selection. The solution to C^1 is shown in Table 3.1. Set $\mathcal{U}(1) = \{11, 12, 14\}$ and $\mathcal{L}(1) = \{20\}$. Go to *step 5*.

Step 5. Solve Associated System. Solve the following system:

Table 3.1: Solution to the system C^1 in Example 3.4.2

Variable Name	Solution Value
x_1	1.6909
x_2	1.5455
x_3	0.8909
x_4	1.4364
x_5	-1.4364

$$\begin{aligned} C^1 = \{ & 3x_1 - 2x_2 + x_3 - 4x_4 - 2x_5 \le 0 \ (1), \\ & 3x_1 + 2x_2 + x_3 - 3x_4 + 4x_5 \le -1 \ (2), \\ & -x_1 - 5x_2 + 3x_3 - 4x_5 \le -1 \ (3), \\ & x_1 + 4x_2 - 4x_3 + 3x_5 \le 0 \ (4), \\ & 3x_2 - 4x_3 - 2x_4 + 4x_5 \le 0 \ (5), \\ & -5x_1 + 2x_2 + x_3 - 4x_4 + 3x_5 \le -2 \ (6), \\ & 3x_2 - x_3 - x_4 + 3x_5 \le -2 \ (7), \\ & 3x_1 - 5x_2 + 2x_3 - 4x_4 + 2x_5 \le 0 \ (8), \\ & -3x_1 - 2x_2 - 2x_3 + x_4 - 3x_5 \le -1 \ (9), \\ & -2x_1 - 3x_2 - 4x_3 - 5x_4 - x_5 \le -1 \ (10), \\ & x_1 \le 1 (11), \\ & x_2 \le 1 (12), \\ & x_4 \le 1 (14), \\ & -x_5 \le 0 (20) : \ x \in \mathbb{R}^5 \}. \end{aligned}$$

Since C^1 is infeasible, then go to *step 7.*

Step 7. IIS Search. Let $\Pi_1^1 = \{3, 4, 7, 11, 12, 14\}$ be an IIS of C^1. Add Π_1^1 to

$\mathcal{M}(1)$. Observe that $\Gamma_1(\Pi_1^1) = \Pi_1^1 \cap \mathcal{H} = \{11, 12, 14\}$ and $|\Pi_1^1| = n+1 = 6$. Likewise, $|\Gamma_1(\Pi_1^1)| = 3 \leq n = 5$. Go to *step 8*.

Step 8. IIS Examination. $\Theta_1(\Pi_1^1) = \Pi_1^1 \cap \mathcal{A} = \{3, 4, 7\}$. $\Theta_1(\Pi_1^1)$ is not an IIS of $\hat{B}$, but $\Theta_1(\Pi_1^1)$ is a proper subset of $\Theta_1(p(1))$. Set $\Theta_1(\Pi_1^1) \cup \mathcal{O}(1)$ and $\Gamma_1(\Pi_1^1) \cup \mathcal{Y}(1)$. Likewise, set $i = 2$, $\mathcal{A}(2) = \{3, 4, 7\}$, $\mathcal{U}(2) = \emptyset$, and $\mathcal{L}(2) = \emptyset$. Return to *step 4*.

Step 4. Termination. Since $|\mathcal{S}| = 0$, then go to *step 5*.

Iteration 2

Step 5. Solve Associated System. Solve the following system:

$$\begin{aligned} C^2 = \{ & -x_1 - 5x_2 + 3x_3 - 4x_5 \leq -1 \ (3), \\ & x_1 + 4x_2 - 4x_3 + 3x_5 \leq 0 \ (4), \\ & 3x_2 - x_3 - x_4 + 3x_5 \leq -2 \ (7) : x \in \mathbb{R}^5 \}. \end{aligned}$$

Since C^2 is feasible, then go to *step 6*.

Step 6. Variable Selection. The solution to C^2 is shown in Table 3.2.
Set $\mathcal{U}(1) = \{14\}$ and $\mathcal{L}(1) = \{16\}$. Go to *step 5*.

Step 5. Solve Associated System. Solve the following system:

Table 3.2: Solution to the system C^2 in Example 3.4.2

Variable Name	**Solution Value**
x_1	-4.0000
x_2	1.0000
x_3	0.0000
x_4	5.0000
x_5	0.0000

$$
\begin{aligned}
C^2 = \{&-x_1 - 5x_2 + 3x_3 - 4x_5 \leq -1 \ (3),\\
&x_1 + 4x_2 - 4x_3 + 3x_5 \leq 0 \ (4),\\
&3x_2 - x_3 - x_4 + 3x_5 \leq -2 \ (7),\\
&x_4 \leq 1(14),\\
&-x_1 \leq 0(16): \ x \in \mathbb{R}^5\}.
\end{aligned}
$$

Since C^2 is feasible, then go to *step 6.*

Step 6. Variable Selection. The solution to C^2 is shown in Table 3.3.

Table 3.3: Revised solution to the system C^2 in Example 3.4.2

Variable Name	**Solution Value**
x_1	4.0000
x_2	-5.55×10^{-17}
x_3	1.0000
x_4	1.0000
x_5	0.0000

Set $\mathcal{U}(1) = \{11, 14\}$ and $\mathcal{L}(1) = \{16, 17\}$. Go to *step 5*.

Step 5. Solve Associated System. Solve the following system:

$$\begin{aligned} C^2 = \{&-x_1 - 5x_2 + 3x_3 - 4x_5 \leq -1 \ (3), \\ &x_1 + 4x_2 - 4x_3 + 3x_5 \leq 0 \ (4) \qquad\qquad , \\ &3x_2 - x_3 - x_4 + 3x_5 \leq -2 \ (7), \\ &x_1 \leq 1(11), \\ &x_4 \leq 1(14), \\ &-x_1 \leq 0(16), \\ &-x_2 \leq 0(17) : \ x \in \mathbb{R}^5\}. \end{aligned}$$

Since C^2 is feasible, then go to *step 6*.

Step 6. Variable Selection. The solution to C^2 is shown in Table 3.4.

Table 3.4: Improved solution to the system C^2 in Example 3.4.2

Variable Name	**Solution Value**
x_1	1.0000
x_2	3.0000
x_3	1.0000
x_4	1.0000
x_5	-3.0000

Set $\mathcal{U}(1) = \{11, 12, 14\}$ and $\mathcal{L}(1) = \{16, 17, 20\}$. Go to *step 5*.

Step 5. Solve Associated System. Solve the following system:

$$
\begin{aligned}
C^2 = \{&-x_1 - 5x_2 + 3x_3 - 4x_5 \leq -1 \ (3),\\
&x_1 + 4x_2 - 4x_3 + 3x_5 \leq 0 \ (4),\\
&3x_2 - x_3 - x_4 + 3x_5 \leq -2 \ (7),\\
&x_1 \leq 1(11),\\
&x_2 \leq 1(12),\\
&x_4 \leq 1(14),\\
&-x_1 \leq 0(16),\\
&-x_2 \leq 0(17),\\
&-x_5 \leq 0(20) : \ x \in \mathbb{R}^5\}.
\end{aligned}
$$

Since C^2 is infeasible, then go to *step 7*.

Step 7. IIS Search. Let $\Pi_1^2 = \{3, 7, 11, 14, 17, 20\}$ be an IIS of C^2. Add Π_1^2 to $\mathcal{M}(2)$. Observe that $\Gamma_1(\Pi_1^2) = \Pi_1^2 \cap \mathcal{H} = \{11, 14, 17, 20\}$ and $|\Pi_1^2| = n + 1 = 6$. Besides, $|\Gamma_1(\Pi_1^2)| = 4 \leq n = 5$. Go to *step 8*.

Step 8. IIS Examination. $\Theta_1(\Pi_1^2) = \Pi_1^2 \cap \mathcal{A} = \{3, 7\}$ is an IIS of $\hat{B}$. Add $\Theta_1(\Pi_1^2)$ to $\mathcal{S}$. Go to *step 4*.

<u>Iteration 3</u>

Step 4. Termination. Since $|\mathcal{S}| = 1$, then *stop* algorithm.

3.5 Summary

A method for finding IISs for systems of inequalities with binary variables was presented in this chapter. New theoretical results show that an IIS for a binary system can be obtained as subset of an IIS for an associated system of inequalities with both unrestricted decision variables and a subset of box constraints appended to it. Specifically, two new algorithms to find IISs for systems of inequalities with binary variables were developed, that is, the *IIS-BAB algorithm* and the *IIS-Heuristic algorithm*. The first algorithm uses the new theory and the method of the alternative polyhedron [23] within a branch-and-bound (BAB) approach. The second algorithm applies the new theory and the method of the alternative polyhedron [23] to a system in which zero/one box constraints are appended. Decomposition schemes using IISs for binary systems can be used to solve probabilistically constrained stochastic integer programs (SIP). A BAC method to solve probabilistically constrained SIP using IIS inequalities is introduced in the next chapter.

4. BRANCH-AND-CUT METHOD FOR STOCHASTIC INTEGER PROGRAMS WITH PROBABILISTIC CONSTRAINTS

4.1 Preliminaries

This chapter presents a method for solving probabilistically constrained SIP with random technology matrix and random righthand-side vector. The SIP model can be given as follows:

$$\textbf{SIP-C2:} \ \min \ c^\top x \tag{4.1a}$$

$$\text{s.t.} \ Ax \geq b \tag{4.1b}$$

$$\mathrm{P}\{T(\tilde{\omega})x \geq r(\tilde{\omega})\} \geq 1 - \beta \tag{4.1c}$$

$$x \in \mathbb{B}^n. \tag{4.1d}$$

If the random variable $\tilde{\omega}$ has a discrete probability distribution with $|\Omega| < \infty$, then a DEP to SIP-C2 can be formulated as follows:

$$\textbf{SIP-C3:} \ \min \ c^\top x \tag{4.2a}$$

$$\text{s.t.} \ Ax \geq b \tag{4.2b}$$

$$T(\omega)x + M^\omega e z_\omega \geq r(\omega), \forall \omega \in \Omega \tag{4.2c}$$

$$\sum_{\omega \in \Omega} p_\omega z_\omega \leq \beta \tag{4.2d}$$

$$x \in \mathbb{B}^n, z_\omega \in \mathbb{B}, \forall \omega \in \Omega. \tag{4.2e}$$

Note that M^ω is an appropriate large number, p_ω is the probability of occurrence of a scenario $\omega \in \Omega$, and e is an appropriate sized vector of ones. It is assumed that $p_\omega \leq \beta$ for each $\omega \in \Omega$. If there is some ω such that $p_\omega > \beta$, $T(\omega)x \geq r(\omega)$ must hold for any feasible x, and these inequalities can be included in the definition of $Ax \geq b$. Likewise, z_ω is a binary decision variable for each scenario ω that takes the value of 0 if the constraints corresponding to ω are included, and 1 if the constraints corresponding to ω are excluded from the programming problem. There is also a knapsack constraint (4.2d) to guarantee that the probabilistic constraint (4.1c) is satisfied.

Optimizing SIP-C3 is difficult since very weak lower bounds are obtained by optimizing its LP relaxation. The aim is to strengthen the LP relaxation of SIP-C3 by finding valid inequalities for the following set:

$$
\begin{aligned}
Q^1 = \{(x,z) \in \mathbb{B}^{n_1} \times \mathbb{B}^{|\Omega|} : & \; Ax \geq b,\ T(\omega)x + Mz_\omega \geq r(\omega), \\
& \sum_{\omega \in \Omega} p_\omega z_\omega \leq \beta,\ \forall \omega \in \Omega\}.
\end{aligned}
$$

Remove the knapsack constraint from Q^1 and drop the z_ω variable from it. Let Q^2 be a system of linear inequalities as follows:

$$
Q^2 = \{Ax \geq b,\ T(\omega)x \geq r(\omega),\ \forall \omega \in \Omega :\ x \in \mathbb{B}^{n_1}\}.
$$

Q^2 is generally infeasible due to the nature of the applications involving reliability requirements. Consider the wildfire initial response planning problem described in Chapter 1. If the decision's maker attitude towards risk in this problem is zero, that is, the decision maker is completely risk averse, then there might not be enough resources available to provide initial response to all fires during a fire season. We are interested in finding feasible subsystems of Q^2. A maximum feasible subsystem (MFS) is a subsystem of linear inequalities with maximum cardinality. The problem of finding a MFS is known as *the maximum feasible subsystem problem* (MAX FS) [2]. The MAX FS can be formulated for exact solution via integer programming (IP). Specifically, the MAX FS for the system Q^2 can be formulated as follows:

$$\textbf{IP1:}\ \min \sum_{\omega \in \Omega} z_\omega \tag{4.5a}$$

$$\text{s.t. } Ax \geq b \tag{4.5b}$$

$$T(\omega)x + M^\omega z_\omega \geq r(\omega),\ \forall \omega \in \emptyset \tag{4.5c}$$

$$x \in \mathbb{B}^n, z_\omega \in \mathbb{B}, \forall \omega \in \Omega. \tag{4.5d}$$

Note that M is an appropriate large number, and z_ω is a binary decision variable for each constraint in $T(\omega)$ that takes the value of 0 if the corresponding constraint is included in IP1 and 1 if the corresponding constraint is excluded from IP1. After solving IP1, the MAX FS to Q^2 is indicated by the constraints whose corresponding z_ω variables are all zero.

PROPOSITION 4.1.1. *Solving SIP-C3 is equivalent to optimizing $c^\top x$ over the MFS of Q^2 that satisfies the knapsack inequality.*

Proof. *Optimizing $c^\top x$ over the MFS of Q^2 that satisfies the knapsack inequality is equivalent to optimizing:*

$$\textbf{\textit{IP2}}: \ \min \ c^\top x + \sum_{\omega \in \Omega} z_\omega \tag{4.6a}$$

$$s.t. \ Ax \geq b \tag{4.6b}$$

$$T(\omega)x + M^\omega z_\omega \geq r(\omega), \ \forall \omega \in \Omega \tag{4.6c}$$

$$\sum_{\omega \in \Omega} p_\omega z_\omega \leq \beta \tag{4.6d}$$

$$x \in \mathbb{B}^n, z_\omega \in \mathbb{B}, \forall \omega \in \Omega. \tag{4.6e}$$

$\sum_{\omega \in \Omega} p_\omega z_\omega < \sum_{\omega \in \Omega} z_\omega$ since $p_\omega \leq \beta$, $\forall \omega \in \Omega$. Therefore, IP2 can be rewritten as follows:

$$\textbf{\textit{IP3}}: \ \min \ c^\top x \tag{4.7a}$$

$$s.t. \ Ax \geq b \tag{4.7b}$$

$$T(\omega)x + M^\omega z_\omega \geq r(\omega), \ \forall \omega \in \emptyset \tag{4.7c}$$

$$\sum_{\omega \in \Omega} p_\omega z_\omega \leq \beta \tag{4.7d}$$

$$x \in \mathbb{B}^n, z_\omega \in \mathbb{B}, \forall \omega \in \Omega. \tag{4.7e}$$

Note that IP3 is exactly SIP-C3.

■

An IIS S is a minimal set of infeasible constraints. Let Ψ be an index set defining some of the constraints in Q^2. Thus, an IIS of Q^2 is a subsystem Q^2_Ψ that is infeasible, but it could be made feasible by dropping any inequality from it. $\Psi = S$ when Ψ defines a subsystem than is an IIS for Q^2. Moreover, $\mathcal{S}$ is the set of IISs to Q^2 that is defined as $\mathcal{S} := \{S_1, ..., S_j\}$.

The MAX FS can also be viewed as finding the minimum number of linear constraints to remove such that the remaining constraints constitute a feasible system. Indeed, MAX FS is an integer covering problem. Consider an infeasible system Q^2 with corresponding set $\mathcal{S}$ of IISs such that $|\mathcal{S}| = r$. This system can be made feasible by deleting at least one constraint associated with an element of every $S_j \in \mathcal{S}$. Finding the smallest cardinality system of constraints to cover all IISs is known as the *minimum cardinality IIS set-covering problem* (MIN IIS COVER). Let $\mathcal{Z}_j$ be a subset of scenarios ω that is defined as $\mathcal{Z}_j := \{\omega \in \Omega : Q^2 \cap S_j \neq \emptyset\},\ j = 1, ..., r$. The MIN IIS COVER to Q^2 can be formulated as follows:

$$\textbf{IP4:}\ \min \sum_{\omega \in \Omega} z_\omega \tag{4.8a}$$

$$\text{s.t.} \sum_{\omega \in \mathcal{Z}_j} z_\omega \geq 1,\ j = 1, ..., r \tag{4.8b}$$

$$z_\omega \in \mathbb{B},\ \forall \omega \in \Omega. \tag{4.8c}$$

Constraints (4.8b) are called the IIS inequalities. These are facet-defining inequalities for the convex hull of feasible points of IP4 [53]. Likewise, z_ω is a binary decision variable whose value is 1 if the corresponding constraints associated with ω are chosen to be deleted, and 0 otherwise.

REMARK 4.1.2. *Consider an infeasible system of inequalities Q^2. If the set $\mathcal{S}$ of IISs of Q^2 is known, then solving the MIN IIS COVER for Q^2 is equivalent to solving the MAX FS for Q^2.*

The number of IISs can potentially be exponential in the size of Q^2. The authors of [53] developed a method to solve IP4 by generating IISs one at a time so that a new IIS that is not covered by the current MIN IIS COVER solution is found at each iteration. An outline of this algorithm is shown as follows:

Input: Infeasible system Q^2.

MIN IIS COVER Algorithm

Step 0. Initialization. Set $\mathcal{Z} = \{\omega : \forall \omega \in \Omega\}$, $\mathcal{S} = \emptyset$.

Step 1. Termination. Check feasibility of Q^2. If Q^2 is feasible, then *stop* algorithm. Else, go to *step 2*.

Step 2. IIS identification. Identify one IIS S_j to Q^2 such that $S_j \notin \mathcal{S}$.

Step 3. Solve IP2. Solve the MIN IIS COVER problem (IP4).

Step 4. Set Processing. Set $\mathcal{Z} \setminus \omega$ for every ω such that $z_\omega = 1$. Return to *step 1*.

Output: MIN IIS COVER.

An IIS decomposition approach for SIP-C3 is presented next. In particular, a BAC approach to solve SIP-C3 is developed using the IIS inequalities, Proposition (4.1.1), Remark (4.1.2), and a variation of the MIN IIS COVER algorithm. The BAC method branches on the z_ω variables and solves an LP relaxation of SIP-C3 at each BAB tree node. Moreover, IIS inequalities are added one at a time to the LP relaxation of SIP-C3 at each node of the BAB tree in order to tighten its feasible region while excluding the constraints associated with at least one scenario $\omega \in \Omega$.

4.2 IIS Decomposition Methodology

Consider SIP-C3 and a BAB tree in which the branching decision variable is z_ω. Let $\mathcal{N}$ be the set of open nodes in the BAB tree, indexed by k. At an arbitrary node of the BAB tree let

- a path be a sequence of nodes from a current node k to the root node r with no node repetitions. A path is represented by $\tau(r,k) = \{k, k-1, ..., j, ..., r\}$.
- $\mathcal{E}^k \subseteq \Omega$ be the set of all scenarios associated with the nodes in $\tau(r,k)$ such that $z_\omega = 1$ and $P(\mathcal{E}^k) \leq \beta$. This implies that the subsystem of constraints (4.11c) associated with ω are *excluded* from SIP-C6^k (see formulation below).
- $\mathcal{I}^k \subseteq \Omega$ be the set of all scenarios associated with the nodes in $\tau(r,k)$ such that $z_\omega = 0$. This implies that the subsystem of constraints (4.11c) associated with ω are *included* in SIP-C6^k (see formulation below).
- μ be the largest node index in a BAB tree.

Let us define a relaxation of SIP-C3 for node k as follows:

$$\textbf{SIP-C4}^k\textbf{:} \min \; c^\top x \tag{4.9a}$$

$$\text{s.t.} \; Ax \geq b \tag{4.9b}$$

$$T(\omega)x \geq r(\omega), \;\; \forall \omega \in \Omega \setminus \mathcal{E}^k \tag{4.9c}$$

$$x \in \mathbb{B}^n. \tag{4.9d}$$

Let S_j be an IIS of SIP-C4^k and $\mathcal{D} := \{\omega \in \Omega : T(\omega)x \geq r(\omega) \cap S_j \neq \emptyset\}$. $\mathcal{D}$ is a subset of scenarios $\omega \in \Omega$ such that the index of at least one constraint (4.9c) in SIP-C4^k associated with $\bar{\omega}$ is an element of S_j. Thus, Proposition 4.2.1 defines the IIS inequality as follows:

PROPOSITION 4.2.1. *The IIS inequality $\sum_{\omega \in \mathcal{D}} z_\omega \geq 1$ is valid for the set Q^1.*

Proof. *Let SIP-C4^k be infeasible with an IIS S_j such that the set $\mathcal{D}$ can be defined. Let also $p_\omega \leq \beta$, $\forall \omega \in \Omega$. Thus, min$\{\sum_{\omega \in \mathcal{D}} z_\omega : z \in Q^1\} \geq 1$, that is, $\sum_{\omega \in \mathcal{D}} z_\omega \geq 1$ is valid for Q^1.*

■

Observe that by adding an IIS inequality at a particular node k in the BAB tree, at least one scenario will be excluded from SIP-C3 so that the total number of nodes to search in the BAB tree is reduced. Let us define an additional relaxation of SIP-C3 for node k as follows:

$$\textbf{SIP-C5}^k\textbf{:} \min \; c^\top x \tag{4.10a}$$

$$\text{s.t.} \quad Ax \geq b \tag{4.10b}$$

$$T(\omega)x + M^\omega e z_\omega \geq r(\omega), \forall \omega \in \Omega \tag{4.10c}$$

$$\sum_{\omega \in \Omega} p_\omega z_\omega \leq \beta \tag{4.10d}$$

$$\sum_{\omega \in \mathcal{D}_i} z_\omega \geq 1, \forall i \in \tau(r,k) \tag{4.10e}$$

$$z_\omega = 0, \forall \omega \in \mathcal{I}^k \tag{4.10f}$$

$$z_\omega = 1, \forall \omega \in \mathcal{E}^k \tag{4.10g}$$

$$x \in \mathbb{B}^n, 0 \leq z_\omega \leq 1, \forall \omega \in \Omega. \tag{4.10h}$$

SIP-C5^k can be rewritten as follows:

$$
\begin{aligned}
\textbf{SIP-C6}^k\textbf{:}\ \min\ & c^\top x && \text{(4.11a)}\\
\text{s.t.}\ \ & Ax \geq b && \text{(4.11b)}\\
& T(\omega)x + M^\omega e z_\omega \geq r(\omega), \forall \omega \in \Omega \setminus \{\mathcal{E}^k \cup \mathcal{I}^k\} && \text{(4.11c)}\\
& T(\omega)x \geq r(\omega), \forall \omega \in \mathcal{I}^k && \text{(4.11d)}\\
& \sum_{\omega \in \Omega \setminus \{\mathcal{E}^k \cup \mathcal{I}^k\}} p_\omega z_\omega \leq \beta - \sum_{\omega \in \mathcal{E}^k} p_\omega z_\omega && \text{(4.11e)}\\
& \sum_{\omega \in \mathcal{D}_i} z_\omega \geq 1, \forall i \in \tau(r,k) && \text{(4.11f)}\\
& x \in \mathbb{B}^n, 0 \leq z_\omega \leq 1, \forall \omega \in \Omega \setminus \{\mathcal{E}^k \cup \mathcal{I}^k\}. && \text{(4.11g)}
\end{aligned}
$$

4.2.1 The IIS-BAC Algorithm

The aim of the *IIS-BAC algorithm* is to find optimal solutions to SIP-C2 by branching on the binary variable z_ω in SIP-C3 and generating IIS inequalities at a particular node k in a BAB tree. First, initialize the set of scenarios $\mathcal{I}$ and $\mathcal{E}$ to be empty. Initialize also the current node 1 with the sets $\mathcal{I}$ and $\mathcal{E}$. Then, initialize the set of open nodes $\mathcal{N}$ with the node 1 and the upper bound to infinity. This algorithm terminates if the set of open nodes is empty meaning that the incumbent solution is optimal. Then, select a node from the set of open nodes $\mathcal{N}$, solve SIP-C6^k, and check the fathoming rules.

The fathoming rules at a particular node are based, first, on *infeasibility* if constraints (4.11e) and (4.11f) are not satisfied, second, on *bound* if the objective function at the current node is greater than or equal to the incumbent solution, and, third, on *optimality* in which case the incumbent solution is updated. If any of these fathoming rules is satisfied, then return to the node selection step. Otherwise, go to the IIS inequality generation step. If SIP-C4^k is infeasible, then generate an

IIS inequality using the *IIS-BAB algorithm* or the *IIS-Heuristic algorithm*. Add the generated IIS inequality to SIP-C6^k. Solve SIP-C6^k and then branch on a non-integer z_ω variable. For non-integer z_ω variable, create two nodes, one in which z_ω takes the value of 0 and other in which z_ω takes the value of 1. Update the corresponding sets $\mathcal{I}$ and $\mathcal{E}$ and the set of open nodes $\mathcal{N}$, increase the number of total nodes in the BAB tree and then return to termination step. Let us now formalize a BAC algorithm for solving SIP-C2 for the 2BN case as follows:

IIS-BAC Algorithm

Step 0. Initialization . Set $k = 1$, $\mu = -\infty$, $\mathcal{I}^k = \emptyset$, $\mathcal{E}^k = \emptyset$, $\mathcal{N} = \{(\mathcal{I}^k, \mathcal{E}^k)\}$, and $\theta = \infty$.

Step 1. Termination. If $\mathcal{N} = \emptyset$, then the solution $\{x^*, \{z^*_\omega\}_{\forall\omega\in\Omega}\}$ that yielded the incumbent objective value $\theta = c^\top x^*$ is optimal. Then, stop algorithm. Otherwise, go to *step 2*.

Step 2. Node Selection. Set $\mu = \max\{k, \mu\}$. Furthermore, pick and delete node $k \in \mathcal{N}$ according to a node selection rule.

Step 3. Solve Node Problem. Solve SIP-C6^k.

Step 4. Fathoming Rules. Fathom node if

(a) **Infeasibility:** SIP-C6^k is *infeasible*. Return to *step 1*.

(b) **Optimality:** SIP-C6^k is feasible, z_ω and x^k are binary, and $c^\top x^k < \theta$. Then, set $\theta = c^\top x^k$ and return to *step 1*.

(c) **Bound:** $c^\top x^k \geq \theta$. Return to *step 1*.

Else, if

(d) **Feasibility:** SIP-C6^k is feasible, but not all of the z_ω variables have integer values, then continue to *step 5* if $\mathcal{D}_k = \emptyset$ or to *step 6* if $\mathcal{D}_k \neq \emptyset$.

Step 5. Inequality Generation.

- If SIP-C4^k is infeasible, then find an IIS S_k to SIP-C4^k using the *IIS-BAB algorithm* or the *IIS-Heuristic algorithm*. Update $\mathcal{D}_k$. Generate and add the IIS inequality S_k to SIP-C6^k. Return to *step 3*.
- Else, If Problem SIP-C4^k is feasible, then fathom the current node. In addition, if $c^\top x^k < \theta$, then set $\theta = c^\top x^k$. Return to *step 1*.

Step 6. Node Division. Pick non-integer $\bar{z}_\omega$ with the largest value. Create two new nodes $(\mathcal{I}^{\mu+1} \cup \{\omega : \bar{z}_\omega = 0\}, \mathcal{E}^{\mu+1})$ and $(\mathcal{I}^{\mu+2}, \mathcal{E}^{\mu+2} \cup \{\omega : \bar{z}_\omega = 1\})$. Add these *two* nodes to $\mathcal{N}$. Then, return to *step 1*.

4.2.2 Example Illustration

Consider an instance of SIP-C2 as

$$\textbf{SIP-C2:} \min \; c^\top x \tag{4.12a}$$

$$\text{s.t. } \mathrm{P}\{T(\tilde{\omega})x \geq r(\tilde{\omega})\} \geq 1 - \beta \tag{4.12b}$$

$$x \in \mathbb{B}^5 \tag{4.12c}$$

where

- $c^\top = [1 \;\; -2 \;\; -2 \;\; -2 \;\; 1]$,
- $\beta = 0.27$,
- $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ with $\omega_i = (T_i, r_i)$ for $i = 1, ..., 5$,

- $P(\omega_1) = 0.013$, $P(\omega_2) = 0.161$, $P(\omega_3) = 0.251$, $P(\omega_4) = 0.233$, and $P(\omega_5) = 0.343$,

- $r_i \in \mathbb{R}$ for $i = 1, ..., 5$ with $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, $r_3 = 0$, $r_4 = 0$, and $r_5 = 0$, and

- T_i is a 1×5 matrix with real components for $i = 1, ..., 5$ such that $T_1 = [0, 2, -1, 0, 0]$; $T_2 = [1, -1, -1, 1, 1]$; $T_3 = [0, -1, -1, 0, 0]$; $T_4 = [2, -1, 2, -1, -1]$; and $T_5 = [-1, -1, 2, -1, -1]$.

The corresponding SIP-C3 model can be stated as follows:

$$
\begin{aligned}
\textbf{SIP-C3:}\ \min\ & x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\
\text{s.t.}\ & 2x_2 - x_3 + 10z_1 \geq 0 \\
& x_1 - x_2 - x_3 + x_4 + x_5 + 10z_2 \geq 1 \\
& -x_2 - x_3 + 10z_3 \geq 0 \\
& 2x_1 - x_2 - x_3 - x_5 + 10z_4 \geq 0 \\
& -x_1 - x_2 + 2x_3 - x_4 - x_5 + 10z_5 \geq 0 \\
& 0.013z_1 + 0.161z_2 + 0.251z_3 + 0.233z_4 + 0.343z_5 \leq 0.27 \\
& z_1, z_2, z_3, z_4, z_5 \in \{0, 1\} \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.
\end{aligned}
$$

Let us apply the *IIS-BAC algorithm* to SIP-C3 in order to find an optimal solution for SIP-C2. The node selection rule is the *depth-first search*, and the node division rule is to select the binary variable z_ω with the maximum fractional value.

IIS-BAC Algorithm:

Step 0. Initialization. Set $\mathcal{I}^1 = \emptyset$, $\mathcal{E}^1 = \emptyset$, $1 = (\mathcal{I}^1, \mathcal{E}^1)$, $\mathcal{N} = \{1\}$, and $\theta = \infty$.

Iteration 1

Step 1. Termination. Since $\mathcal{N} = \{1\}$, then go to *step 2*.

Step 2. Node Selection. Set $\mu = 1$. Pick and delete node 1 from $\mathcal{N}$ according to the depth first selection rule.

Step 3. Solve Node Problem. Solve SIP-C6^1.

$$
\begin{aligned}
\textbf{SIP-C6:}^1 \ \min \ & x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\
\text{s.t. } & 2x_2 - x_3 + 10z_1 \geq 0 \\
& x_1 - x_2 - x_3 + x_4 + x_5 + 10z_2 \geq 1 \\
& -x_2 - x_3 + 10z_3 \geq 0 \\
& 2x_1 - x_2 - x_3 - x_5 + 10z_4 \geq 0 \\
& -x_1 - x_2 + 2x_3 - x_4 - x_5 + 10z_5 \geq 0 \\
& 0.013z_1 + 0.161z_2 + 0.251z_3 + 0.233z_4 + 0.343z_5 \leq 0.27 \\
& z_1, z_2, z_3, z_4, z_5 \geq 0 \\
& z_1, z_2, z_3, z_4, z_5 \leq 1 \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.
\end{aligned}
$$

The optimal solution to SIP-C6^1 is shown in Table 4.1.

Table 4.1: Optimal solution to SIP-C6[1] in Example 4.2.2

Variable Name	**Solution Value**
x_1	0.0000
x_2	1.0000
x_3	1.0000
x_4	1.0000
x_5	0.0000
z_1	0.0000
z_2	1.0000
z_3	0.2000
z_4	0.2539
z_5	0.0000

Step 4. Fathoming Rules

Since SIP-C6[1] is feasible, but not all the z_ω variables have integer values, and since $\mathcal{D}_1 = \emptyset$, then go to *step 5.*

Step 5. Inequality Generation. Consider SIP-C4[1] as follows:

$$\begin{aligned}
\textbf{SIP-C4:}^1\ \min\ & x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\
\text{s.t.}\ & 2x_2 - x_3 \geq 0 \\
& x_1 - x_2 - x_3 + x_4 + x_5 \geq 1 \\
& -x_2 - x_3 \geq 0 \\
& 2x_1 - x_2 - x_3 - x_5 \geq 0 \\
& -x_1 - x_2 + 2x_3 - x_4 - x_5 \geq 0 \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.
\end{aligned}$$

Notice that SIP-C4[1] is infeasible. Moreover, let $S_1 = \{2, 3, 5\}$ be an IIS of SIP-

C4^1. Let also $z_2 + z_3 + z_5 \geq 1$ be the corresponding IIS inequality. Let $\mathcal{D}_1 = \{2, 3, 5\}$. Add the IIS inequality to SIP-C6^1 and solve it.

$$
\begin{aligned}
\textbf{SIP-C6:}^1 \ \min\ & x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\
\text{s.t.}\ & 2x_2 - x_3 + 10z_1 \geq 0 \\
& x_1 - x_2 - x_3 + x_4 + x_5 + 10z_2 \geq 1 \\
& -x_2 - x_3 + 10z_3 \geq 0 \\
& 2x_1 - x_2 - x_3 - x_5 + 10z_4 \geq 0 \\
& -x_1 - x_2 + 2x_3 - x_4 - x_5 + 10z_5 \geq 0 \\
& 0.013z_1 + 0.161z_2 + 0.251z_3 + 0.233z_4 + 0.343z_5 \leq 0.27 \\
& z_2 + z_3 + z_5 \geq 1 \\
& z_1, z_2, z_3, z_4, z_5 \geq 0 \\
& z_1, z_2, z_3, z_4, z_5 \leq 1 \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.
\end{aligned}
$$

The updated optimal solution to SIP-C6^1 is shown in Table 4.2.

Step 4. Fathoming Rules. Since SIP-C6^1 is feasible, but not all the z_ω variables have integer values, and since $\mathcal{D}_1 \neq \emptyset$, then go to *step 6*.

Step 6. Node Division. Pick the non-integer $\bar{z}_\omega$ with the largest value, that is, $\bar{z}_3 = 0.6978$. Then, create two new nodes $2 = (\mathcal{I}^2 = \{\omega \in \Omega : z_3 = 0\},\ \mathcal{E}^2 = \emptyset)$ and $3 = (\mathcal{I}^3 = \emptyset,\ \mathcal{E}^3 = \{\omega \in \Omega : z_3 = 1\})$. Update $\mathcal{N} = \{2, 3\}$. Return to *step 1*.

Iteration 2

Step 1. Termination. Since $\mathcal{N} = \{2, 3\}$, then go to *step 2*.

Table 4.2: Updated optimal solution to SIP-C6^1 in Example 4.2.2

Variable Name	**Solution Value**
x_1	0.0000
x_2	1.0000
x_3	1.0000
x_4	1.0000
x_5	0.0000
z_1	0.0000
z_2	0.3022
z_3	0.6978
z_4	0.2000
z_5	0.0000

Step 2. Node Selection. Set $\mu = 3$. Pick and delete node 3 from $\mathcal{N}$ according to the depth first selection rule.

Step 3. Solve Node Problem. Solve SIP-C6^3.

$$
\begin{aligned}
\textbf{SIP-C6:}^3 \ \min\ & x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\
\text{s.t. }\ & 2x_2 - x_3 + 10z_1 \geq 0 \\
& x_1 - x_2 - x_3 + x_4 + x_5 + 10z_2 \geq 1 \\
& 2x_1 - x_2 - x_3 - x_5 + 10z_4 \geq 0 \\
& -x_1 - x_2 + 2x_3 - x_4 - x_5 + 10z_5 \geq 0 \\
& 0.013z_1 + 0.161z_2 + 0.233z_4 + 0.343z_5 \leq 0.019 \\
& z_2 + z_3 + z_5 \geq 1 \\
& z_1, z_2, z_3, z_5 \geq 0 \\
& z_1, z_2, z_3, z_5 \leq 1 \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.
\end{aligned}
$$

Step 4. Fathoming Rules. Since SIP-C6^3 is feasible, but not all the z_ω variables have integer values, and since $\mathcal{D}_3 = \emptyset$, then go to *step 5.*

Step 5. Inequality Generation. Consider SIP-C4^3 as follows:

$$
\begin{aligned}
\textbf{SIP-C4:}^3 \ \min\ & x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\
\text{s.t. }\ & 2x_2 - x_3 \geq 0 \\
& x_1 - x_2 - x_3 + x_4 + x_5 \geq 1 \\
& 2x_1 - x_2 - x_3 - x_5 \geq 0 \\
& -x_1 - x_2 + 2x_3 - x_4 - x_5 \geq 0 \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.
\end{aligned}
$$

Observe that SIP-C4^3 is infeasible. Moreover, let $S_3 = \{1, 2, 5\}$ be an IIS for SIP-C4^3. Let also $z_1 + z_2 + z_5 \geq 1$ be the corresponding IIS inequality. Let $\mathcal{D}_3 = \{1, 2, 5\}$. Add the IIS inequality to SIP-C6^3 and solve it.

$$
\begin{aligned}
\textbf{SIP-C6:}^3 \ \min \ & x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\
\text{s.t.} \ & 2x_2 - x_3 + 10z_1 \geq 0 \\
& x_1 - x_2 - x_3 + x_4 + x_5 + 10z_2 \geq 1 \\
& 2x_1 - x_2 - x_3 - x_5 + 10z_4 \geq 0 \\
& -x_1 - x_2 + 2x_3 - x_4 - x_5 + 10z_5 \geq 0 \\
& 0.013z_1 + 0.161z_2 + 0.233z_4 + 0.343z_5 \leq 0.019 \\
& z_2 + z_3 + z_5 \geq 1 \\
& z_1 + z_2 + z_5 \geq 1 \\
& z_1, z_2, z_3, z_5 \geq 0 \\
& z_1, z_2, z_3, z_5 \leq 1 \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.
\end{aligned}
$$

The updated optimal solution to SIP-C6^3 is shown in Table 4.3.

Step 4. Fathoming Rules. Since SIP-C6^3 is feasible, but not all the z_ω variables have integer values, and since $\mathcal{D}_3 \neq \emptyset$, then go to *step 6.*

Step 6. Node Division. Pick the non-integer $\bar{z}_\omega$ with the largest value, that is, $\bar{z}_4 = 0.0279$. Then, create two new nodes $4 = (\mathcal{I}^4 = \{\omega \in \Omega : z_4 = 0\},\ \mathcal{E}^4 = \{\omega \in \Omega : z_3 = 1\})$ and $5 = (\mathcal{I}^5 = \emptyset,\ \mathcal{E}^5 = \{\omega \in \Omega : z_3 = 1, z_4 = 1\})$. Update $\mathcal{N} = \{2, 4, 5\}$. Return to *step 1.*

Table 4.3: Updated optimal solution to SIP-C6^3 in Example 4.2.2

Variable Name	**Solution Value**
x_1	1.0000
x_2	0.0000
x_3	1.0000
x_4	1.0000
x_5	0.0000
z_1	1.0000
z_2	0.0000
z_3	1.0000
z_4	0.0279
z_5	0.0000

Iteration 3

Step 1. Termination. Since $\mathcal{N} = \{2, 4, 5\}$, then go to *step 2.*

Step 2. Node Selection. Set $\mu = 5$. Pick and delete node 5 from $\mathcal{N}$ according to the depth first selection rule.

Step 3. Solve Node Problem. Solve SIP-C6^5.

$$\begin{aligned}
\textbf{SIP-C6:}^{5} \quad \min\ & x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\
\text{s.t. }\ & 2x_2 - x_3 + 10z_1 \geq 0 \\
& x_1 - x_2 - x_3 + x_4 + x_5 + 10z_2 \geq 1 \\
& -x_1 - x_2 + 2x_3 - x_4 - x_5 + 10z_5 \geq 0 \\
& 0.013z_1 + 0.016z_2 + 0.343z_5 \leq -0.214 \\
& z_2 + z_3 + z_5 \geq 1 \\
& z_1 + z_2 + z_5 \geq 1 \\
& z_1, z_2, z_3, z_5 \geq 0 \\
& z_1, z_2, z_3, z_5 \leq 1 \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.
\end{aligned}$$

Step 4. Fathoming Rules. Since SIP-C6^5 is infeasible, then fathom the current node 5 and return to *step 1*.

Iteration 4

Step 1. Termination. Since $\mathcal{N} = \{2, 4\}$, then go to *step 2*.

Step 2. Node Selection. Pick and delete node 4 from $\mathcal{N}$ according to the depth first selection rule.

Step 3. Solve Node Problem. Solve SIP-C6^4.

$$
\begin{aligned}
\textbf{SIP-C6:}^4 \ \min\ & x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\
\text{s.t.}\ & 2x_2 - x_3 + 10z_1 \geq 0 \\
& x_1 - x_2 - x_3 + x_4 + x_5 + 10z_2 \geq 1 \\
& 2x_1 - x_2 - x_3 - x_5 \geq 0 \\
& -x_1 - x_2 + 2x_3 - x_4 - x_5 + 10z_5 \geq 0 \\
& 0.013z_1 + 0.016z_2 + 0.233z_4 + 0.343z_5 \leq 0.019 \\
& z_2 + z_3 + z_5 \geq 1 \\
& z_1 + z_2 + z_5 \geq 1 \\
& z_1, z_2, z_3, z_5 \geq 0 \\
& z_1, z_2, z_3, z_5 \leq 1 \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.
\end{aligned}
$$

Step 4. Fathoming Rules. Since SIP-C6^4 is feasible, but not all the z_ω variables have integer values, and since $\mathcal{D}_4 = \emptyset$, then go to *step 5.*

Step 5. Inequality Generation. Consider SIP-C4^4 as follows:

$$\textbf{SIP-C4:}^4 \quad \begin{aligned} \min\ & x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\ \text{s.t.}\ & 2x_2 - x_3 \geq 0 \\ & x_1 - x_2 - x_3 + x_4 + x_5 \geq 1 \\ & 2x_1 - x_2 - x_3 - x_5 \geq 0 \\ & -x_1 - x_2 + 2x_3 - x_4 - x_5 \geq 0 \\ & x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}. \end{aligned}$$

Note that SIP-C4^4 is infeasible. Moreover, let $S_4 = \{1, 2, 5\}$ be an IIS of SIP-C4^4. Let also $z_1 + z_2 + z_5 \geq 1$ be the corresponding IIS inequality. Let $\mathcal{D}_4 = \{1, 2, 5\}$. Add the IIS inequality to SIP-C6^4 and solve it.

$$\textbf{SIP-C6:}^4 \quad \begin{aligned} \min\ & x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\ \text{s.t.}\ & 2x_2 - x_3 + 10z_1 \geq 0 \\ & x_1 - x_2 - x_3 + x_4 + x_5 + 10z_2 \geq 1 \\ & 2x_1 - x_2 - x_3 - x_5 \geq 0 \\ & -x_1 - x_2 + 2x_3 - x_4 - x_5 + 10z_5 \geq 0 \\ & 0.013z_1 + 0.161z_2 + 0.343z_5 \leq 0.019 \\ & z_2 + z_3 + z_5 \geq 1 \\ & z_1 + z_2 + z_5 \geq 1 \\ & z_1, z_2, z_3, z_5 \geq 0 \\ & z_1, z_2, z_3, z_5 \leq 1 \\ & x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}. \end{aligned}$$

The updated optimal solution to SIP-C6^4 is shown in Table 4.4.

Table 4.4: Updated optimal solution to SIP-C6^4 in Example 4.2.2

Variable Name	**Solution Value**
x_1	1.0000
x_2	0.0000
x_3	1.0000
x_4	1.0000
x_5	0.0000
z_1	0.9803
z_2	0.0000
z_3	1.0000
z_4	0.0000
z_5	0.0197

Step 4. Fathoming Rules. Since SIP-C6^4 is feasible, but not all the z_ω variables have integer values, and since $\mathcal{D}_4 \neq \emptyset$, then go to *step 6*.

Step 6. Node Division. Pick the non-integer $\bar{z}_\omega$ with the largest value, that is, $\bar{z}_1 = 0.9803$. Create two new nodes $6 = (\mathcal{I}^6 = \{\omega \in \Omega : z_4 = 0, z_1 = 0\},\ \mathcal{E}^6 = \{\omega \in \Omega : z_3 = 1\})$ and $7 = (\mathcal{I}^7 = \{\omega \in \Omega : z_4 = 0\},\ \mathcal{E}^7 = \{\omega \in \Omega : z_3 = 1, z_1 = 1\})$. Update $\mathcal{N} = \{2, 6, 7\}$. Return to *step 1*.

<u>**Iteration 5**</u>

Step 1. Termination. Since $\mathcal{N} = \{2, 6, 7\}$, then go to *step 2*.

Step 2. Node Selection. Set $\mu = 7$. Pick and delete node 7 from $\mathcal{N}$ according to the depth first selection rule.

Step 3. Solve Node Problem. Solve SIP-C6^7.

$$
\begin{aligned}
\textbf{SIP-C6:}^7 \ \min \ & x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\
\text{s.t.} \ & x_1 - x_2 - x_3 + x_4 + x_5 + 10z_2 \geq 1 \\
& 2x_1 - x_2 - x_3 - x_5 \geq 0 \\
& -x_1 - x_2 + 2x_3 - x_4 - x_5 + 10z_5 \geq 0 \\
& 0.016z_2 + 0.233z_4 + 0.343z_5 \leq 0.007 \\
& z_2 + z_3 + z_5 \geq 1 \\
& z_1 + z_2 + z_5 \geq 1 \\
& z_1, z_2, z_3, z_5 \geq 0 \\
& z_1, z_2, z_3, z_5 \leq 1 \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.
\end{aligned}
$$

Step 4. Fathoming Rules. Since SIP-C6^7 is feasible, but not all the z_ω variables have integer values, and since $\mathcal{D}_7 = \emptyset$, then go to *step 5.*

Step 5. Inequality Generation. Consider SIP-C4^7 as follows:

$$\begin{aligned}
\textbf{SIP-C4:}^7 \ \min \ & x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\
\text{s.t.} \ & x_1 - x_2 - x_3 + x_4 + x_5 \geq 1 \\
& 2x_1 - x_2 - x_3 - x_5 \geq 0 \\
& -x_1 - x_2 + 2x_3 - x_4 - x_5 \geq 0 \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.
\end{aligned}$$

Since SIP-C4^7 is feasible, then fathom the current node 7. Likewise, since the objective value of SIP-C6^7 is less than the current incumbent solution, that is, $-3 < \infty$, then $\theta = -3$. The updated feasible solution to SIP-C6^7 is shown in Table 4.5. Return to *step 1*.

Table 4.5: Updated feasible solution to SIP-C6^7 in Example 4.2.2

Variable Name	**Solution Value**
x_1	1.0000
x_2	0.0000
x_3	1.0000
x_4	1.0000
x_5	0.0000
z_1	1.0000
z_2	0.0000
z_3	1.0000
z_4	0.0000
z_5	0.0000

Iteration 6

Step 1. Termination. Since $\mathcal{N} = \{2, 6\}$, then go to *step 2*.

Step 2. Node Selection. Pick and delete node 6 from $\mathcal{N}$ according to the depth first selection rule.

Step 3. Solve Node Problem. Solve SIP-C6^6.

$$
\begin{aligned}
\textbf{SIP-C6:}^6 \quad & \min \; x_1 - 2x_2 - 2x_3 - 2x_4 + x_5 \\
& \text{s.t.} \; 2x_2 - x_3 \geq 0 \\
& \quad x_1 - x_2 - x_3 + x_4 + x_5 + 10z_2 \geq 1 \\
& \quad 2x_1 - x_2 - x_3 - x_5 \geq 0 \\
& \quad -x_1 - x_2 + 2x_3 - x_4 - x_5 + 10z_5 \geq 0 \\
& \quad 0.016z_2 + 0.233z_4 + 0.343z_5 \leq 0.007 \\
& \quad z_2 + z_3 + z_5 \geq 1 \\
& \quad z_1 + z_2 + z_5 \geq 1 \\
& \quad z_1, z_2, z_3, z_5 \geq 0 \\
& \quad z_1, z_2, z_3, z_5 \leq 1 \\
& \quad x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.
\end{aligned}
$$

Step 4. Fathoming Rules. Since SIP-C6^6 is infeasible, then fathom the current node 6 and return to *step 1*.

Iteration 7

Step 1. Termination. Since $\mathcal{N} = \{2\}$, then go to *step 2.*

Step 2. Node Selection. Pick and delete node 2 from $\mathcal{N}$ according to the depth first selection rule. The objective value of SIP-C6^2 is -2. Since $-2 > \theta = -3$, then fathom the current node 2. Return to *step 1.*

Iteration 8

Step 1. Termination. Since $\mathcal{N} = \emptyset$, then stop the *IIS-BAC algorithm.* The solution shown in Table 4.5 is optimal with an optimal objective value of -3.

A summary of the *IIS-BAC algorithm* implementation to the current example is shown in Table 4.6. The total number of nodes to explore in the *IIS-BAC algorithm* is 7 when compared to 26 if a standard branch-and-bound method without the IIS inequalities is used. Note that SIP-C3 as stated in the beginning of the section with $z_\omega = 0$ for $\omega = 1, ..., 5$ has two IISs, that is, $S_1 = \{2, 3, 5\}$ and $S_3 = S_4 = \{1, 2, 5\}$.

Table 4.6: Summary of the IIS-BAC algorithm in Example 4.2.2

Iteration	Node Selection	Fathoming Rule	IIS Inequality	Branching Variable	New Node	Incumbent Solution
1	1	Cut generation	2,3,5	z_3	2,3	–
2	3	Cut generation	1,2,5	z_4	4,5	–
3	5	Infeasible SIP-C6	–	–	–	–
4	4	Cut generation	1,2,5	z_1	6,7	–
5	7	Feasible SIP-C4	–	–	–	-3
6	6	Infeasible SIP-C6	–	–	– .	–
7	2	Bounding function	–	–	–	–

4.3 Preliminary Computational Results

Computational results showing the effectiveness of the *IIS-BAC algorithm* in solving SIP-C3 are now presented. The *IIS-BAC algorithm* was implemented using Microsoft Visual C++ and the CPLEX 12×64 Callable Library. We developed instances of SIP-C3 with single probabilistic constraints for different numbers of scenarios and decision variables. In fact, we fixed the number of scenarios per instance to 90, 100, 150, 200, 250, 300, and 350 and the number of decision variables to 5, 10, 15, 20, 25, and 30. The constant M used in the inequalities (4.11c) was chosen for each instance to be the smallest value to ensure feasibility without causing computational instability.

All the test instances were solved on a Dell X5355 computer with 2 Intel(R) Xeon(R) X processors at 2.66 GHz each with 12.0 GB of RAM. For these computational results, all solution times are given in seconds, and a time limit of 7200 seconds (2 hours) is imposed. The average and standard deviation for computing an IIS were 8 seconds and 7 seconds, respectively. The maximum time for obtaining an IIS for an infeasible SIP-C4^k was 32 seconds.

Three different sets of experiments were studied. First, we solved all SIP-C3 instances using the CPLEX MIP solver directly in order to provide a benchmark solution. Second, we solved all SIP-C3 instances using the *IIS-BAC algorithm*, so the IIS inequalities were added at some nodes of the corresponding BAB tree. Finally, we solved all SIP-C3 instances using a standard *BAB algorithm*. This algorithm is similar to the *IIS-BAC algorithm* described in section 4.2.1 but without the inequality generation step, that is, without adding IIS inequalities at any node of a BAB tree.

Depth-first search plus backtracking and breadth-first search were considered as the node selection rules for each set of experiments. In the depth-first search, if the current node was not fathomed, the next node to consider was one of its two sons. Backtracking means that when a node was fathomed, we went back on the path from this node towards the root until we found the first open node. In the breadth-first search, all of the nodes at a given level in a BAB tree were considered before any other nodes at the next lower level. In addition, two different node rules were applied for each set of experiments. First, we considered the component z_ω with the maximum fractional value. Second, we selected the component z_ω with the minimum fractional.

Tables 4.8, 4.9, 4.10, and 4.11 show the results of the SIP-C3 instances for the three different sets of experiments as well as for the different node selection and the node creation rules. In all tables, the first column gives the name of the test instance while the second and third column show the number of rows and binary decision variables, respectively. The next four columns give the CPLEX results. Particularly, the fourth column gives the best solution found by CPLEX; the fifth column gives the optimality gap returned by CPLEX; the sixth column gives the number of nodes searched in the BAB tree; and the seventh column gives the time to prove optimality.

The next four columns, in Tables 4.8, 4.9, 4.10, and 4.11, give the results of the *IIS-BAC algorithm* on the test instances. Specifically, the eighth column gives the best objective value found; the ninth column gives the number of nodes searched; the tenth column gives the number of IIS inequalities added to the formulation; and the eleventh column gives the wall-clock time in seconds.

The final three columns give the results of the implementation of a standard *BAB algorithm*, that is, without adding the IIS inequalities. The first of these columns gives the best solution value found; the second of these columns gives the number of nodes searched in the BAB tree; and the third column gives the wall-clock time in seconds.

If Tables 4.8, 4.9, 4.10, or 4.11 show a dash, then it means that either CPLEX, the

IIS-BAC algorithm, or the standard *BAB algorithm* was not able to find a feasible solution within the 2-hour time limit. Likewise, if the tables show that the number of nodes searched is greater than some number, then it means that the algorithm was unable to prove optimality within the 2-hour time limit.

Table 4.7 shows a summary of the computational results presented in Tables 4.8, 4.9, 4.10, and 4.11. These computational results provide empirical evidence that shows that the IIS inequalities offer an increase in the strength of the LP relaxation of SIP-C3. When compared to the implementation of a standard *BAB algorithm*, the *IIS-BAC algorithm* finds a solution for a larger percentage of instances. When compared to CPLEX, the BAC method reduces the number of nodes to explore in a BAB tree in order to find a solution. The results presented in Table 4.7 are dissected in the next section.

Table 4.7: Summary of the computational results

DEPTH	**Nodes**		**% of Problems Solved**		**IIS Inequalities**		**Time[sec.]**	
MAX	**Avg.**	**Std. Dev.**	**Avg.**	**Std. Dev.**	**Avg.**	**Std. Dev.**	**Avg.**	**Std. Dev.**
CPLEX	37,820	112,551	77%	0%	–	–	25	61
IIS-BAC	116	177	60%	0%	19	32	2,761	3,398
BAB	363	976	43%	0%	–	–	567	1,788
DEPTH	**Nodes**		**% of Problems Solved**		**IIS Inequalities**		**Time[sec.]**	
MIN	**Avg.**	**Std. Dev.**	**Avg.**	**Std. Dev.**	**Avg.**	**Std. Dev.**	**Avg.**	**Std. Dev.**
CPLEX	37,820	112,551	77%	0%	–	–	26	64
IIS-BAC	32	83	37%	0%	3	6	159	406
BAB	191	581	37%	0%	–	–	223	736
BREADTH	**Nodes**		**% of Problems Solved**		**IIS Inequalities**		**Time[sec.]**	
MAX	**Avg.**	**Std. Dev.**	**Avg.**	**Std. Dev.**	**Avg.**	**Std. Dev.**	**Avg.**	**Std. Dev.**
CPLEX	37,820	112,551	77%	0%	–	–	29	73
IIS-BAC	34	83	40%	0%	6	7	551	1,759
BAB	109	339	37%	0%	–	–	62	204
BREADTH	**Nodes**		**% of Problems Solved**		**IIS Inequalities**		**Time[sec.]**	
MIN	**Avg.**	**Std. Dev.**	**Avg.**	**Std. Dev.**	**Avg.**	**Std. Dev.**	**Avg.**	**Std. Dev.**
CPLEX	37,820	112,551	77%	0%	–	–	25	62
IIS-BAC	32	94	37%	0%	5	6	578	1,807
BAB	127	424	40%	0%	–	–	730	1,942

Table 4.8: The IIS-BAC algorithm using the depth-first node selection rule while branching on the z_ω variable with maximum fractional value

INSTANCES			CPLEX				IIS				BAB		
Name	**Rows**	**Bin. var.**	**Objval**	**Gap (%)**	**Nodes**	**Time (sec.)**	**Objval**	**Nodes**	**IIS v.i.**	**Time (sec.)**	**Objval**	**Nodes**	**Time (sec.)**
SIP-C90c5v	91	95	-8	0.0	0	0.0	-8	0	1	0.2	-8	1	0.2
SIP-C90c10v	91	100	-45	0.0	0	0.0	-45	1	1	0.5	-45	1	3.5
SIP-C90c20v	91	110	-77	0.0	8,749	7.3	-16	>338	51	>7200	–	–	–
SIP-C90c25v	91	115	–	–	–	–	–	–	–	–	–	–	–
SIP-C90c30v	91	120	–	–	–	–	–	–	–	–	–	–	–
SIP-C100c5v	101	105	-6	0.0	0	0.0	-6	0	1	0.3	-6	1	0.7
SIP-C100c10v	101	110	-45	0.0	0	0.0	-45	22	3	50.8	-45	30	17.8
SIP-C100c20v	101	120	-60	0.0	4,756	8.9	–	–	–	–	–	–	–
SIP-C100c25v	101	125	-234	0.0	1,567	4.5	-91	>40	0	>7200	–	–	–
SIP-C100c30v	101	130	-178	0.0	28,608	108.9	–	–	–	–	–	–	–
SIP-C150c5v	151	155	-3	0.0	0	0.0	-3	1	1	0.7	-3	2	1.9
SIP-C150c10v	151	160	-31	0.0	0	0.0	-31	1	1	0.8	-31	1	5.1
SIP-C150c20v	151	170	-63	0.0	2,324	3.3	34	>56	0	>700	–	–	–
SIP-C150c25v	151	175	-107	0.0	802	0.6	–	–	–	–	–	–	–
SIP-C150c30v	151	180	-176	0.0	391,872	179.0	–	–	–	–	-99	>3,870	>7,200
SIP-C200c5v	201	205	8	0.0	0	1.0	8	328	24	1,404.8	8	1,016	883.1
SIP-C200c10v	201	210	-2	0.0	427	1.5	20	>501	20	>7200	–	–	–
SIP-C200c20v	201	220	-6	0.0	459,502	111.4	–	–	–	–	–	–	–
SIP-C200c25v	201	225	–	–	–	–	–	–	–	–	–	–	–
SIP-C200c30v	201	230	–	–	–	–	–	–	–	–	–	–	–
SIP-C250c5v	251	255	-10	0.0	0	0.0	-10	1	1	0.7	-10	2	0.7
SIP-C250c10v	251	260	-38	0.0	0	0.0	-38	1	1	0.9	-38	1	0.5
SIP-C250c20v	251	270	-131	0.0	0	0.0	-52	>81	72	>7200	–	–	–
SIP-C250c25v	251	275	–	–	–	–	–	–	–	–	–	–	–
SIP-C250c30v	251	280	–	–	–	–	–	–	–	–	–	–	–
SIP-C300c5v	301	305	-2	0.0	0	0.0	-2	1	1	0.9	-2	2	0.8
SIP-C300c10v	301	310	-43	0.0	0	0.0	-43	1	1	1.0	-43	1	0.6
SIP-C300c20v	301	320	-116	0.0	0	0.0	-69	>145	80	>7200	–	–	–
SIP-C300c25v	301	325	-144	0.0	122,532	246.1	–	–	–	–	–	–	–
SIP-C300c30v	301	330	–	–	–	–	–	–	–	–	–	–	–
SIP-C350c5v	351	355	-10	0.0	0	0.0	-10	1	1	0.9	-10	1	0.6
SIP-C350c10v	351	360	-37	0.0	0	0.0	-37	2	2	1.6	-37	2	0.9
SIP-C350c20v	351	370	-102	0.0	0	0.0	-102	535	75	5,991.5	-102	518	391.9
SIP-C350c25v	351	375	-177	0.0	0	0.0	-32	>388	103	>7200	–	–	–
SIP-C350c30v	351	380	–	–	–	–	–	–	–	–	–	–	–

Table 4.9: The IIS-BAC algorithm using the depth-first node selection rule while branching on the z_ω variable with minimum fractional value

INSTANCES	CPLEX						IIS				BAB		
Name	Rows	Bin. var.	Objval	Gap (%)	Nodes	Time (sec.)	Objval	Nodes	IIS v.i.	Time (sec.)	Objval	Nodes	Time (sec.)
SIP-C90c5v	91	95	-8	0.0	0	0.00	-8	0	1	0.2	-8	1	0.8
SIP-C90c10v	91	100	-45	0.0	0	0.00	-45	1	1	0.5	-45	1	3.4
SIP-C90c20v	91	110	-77	0.0	8749	7.82	–	–	–	–	–	–	–
SIP-C90c25v	91	115	–	–	–	–	–	–	–	–	–	–	–
SIP-C90c30v	91	120	–	–	–	–	–	–	–	–	–	–	–
SIP-C100c5v	101	105	-6	0.0	0	0.00	-6	0	1	0.3	-6	1	0.7
SIP-C100c10v	101	110	-45	0.0	0	0.02	-45	109	2	611.0	-45	281	113.7
SIP-C100c20v	101	120	-60	0.0	4,756	9.38	–	–	–	–	–	–	–
SIP-C100c25v	101	125	-234	0.0	1,567	4.38	–	–	–	–	–	–	–
SIP-C100c30v	101	130	-178	0.0	28,608	116.17	–	–	–	–	–	–	–
SIP-C150c5v	151	155	-3	0.0	0	0.02	-3	1	1	0.7	-3	2	1.7
SIP-C150c10v	151	160	-31	0.0	0	0.00	-31	1	1	0.7	-31	1	5.0
SIP-C150c20v	151	170	-63	0.0	2324	3.23	–	–	–	–	–	–	–
SIP-C150c25v	151	175	-107	0.0	802	0.61	–	–	–	–	–	–	–
SIP-C150c30v	151	180	-176	0.0	39,1872	176.69	–	–	–	–	–	–	–
SIP-C200c5v	201	205	8	0.00	0	0.9	8	303	22	1,446.8	8	2,188	2770.64
SIP-C200c10v	201	210	-2	0.00	427	1.3	–	–	–	–	–	–	–
SIP-C200c20v	201	220	-6	0.00	459,502	110.7	–	–	–	–	–	–	–
SIP-C200c25v	201	225	–	–	–	–	–	–	–	–	–	–	–
SIP-C200c30v	201	230	–	–	–	–	–	–	–	–	–	–	–
SIP-C250c5v	251	255	-10	0.00	0	0.0	-10	1	1	1.0	-10	2	1.061
SIP-C250c10v	251	260	-38	0.00	0	0.0	-38	1	1	1.1	-38	1	0.671
SIP-C250c20v	251	270	-131	0.00	0	0.0	–	–	–	–	–	–	–
SIP-C250c25v	251	275	–	–	–	–	–	–	–	–	–	–	–
SIP-C250c30v	251	280	–	–	–	–	–	–	–	–	–	–	–
SIP-C300c5v	301	305	-2	0.00	0	0.0	-2	1	1	0.8	-2	2	0.811
SIP-C300c10v	301	310	-43	0.00	0	0.0	-43	1	1	0.9	-43	1	0.546
SIP-C300c20v	301	320	-116	0.00	0	0.0	–	–	–	–	–	–	–
SIP-C300c25v	301	325	-144	0.00	122,532	265.4	–	–	–	–	–	–	–
SIP-C300c30v	301	330	–	–	–	–	–	–	–	–	–	–	–
SIP-C350c5v	351	355	-10	0.00	0	0.0	-10	1	1	1.1	-10	1	0.671
SIP-C350c10v	351	360	-37	0.00	0	0.0	-37	2	2	1.6	-37	2	0.936
SIP-C350c20v	351	370	-102	0.00	0	0.0	–	–	–	–	–	–	–
SIP-C350c25v	351	375	-177	0.00	0	0.0	–	–	–	–	–	–	–
SIP-C350c30v	351	380	–	–	–	–	–	–	–	–	–	–	–

Table 4.10: The IIS-BAC algorithm using the breadth-first node selection rule while branching on the z_ω variable with maximum fractional value

INSTANCES			**CPLEX**				**IIS**				**BAB**		
Name	**Rows**	**Bin. var.**	**Objval**	**Gap (%)**	**Nodes**	**Time (sec.)**	**Objval**	**Nodes**	**IIS v.i.**	**Time (sec.)**	**Objval**	**Nodes**	**Time (sec.)**
SIP-C90c5v	91	95	-8	0.0	0	0.0	-8	2	0	0.6	-8	2	0.7
SIP-C90c10v	91	100	-45	0.0	0	0.0	-45	2	12	0.6	-45	2	3.0
SIP-C90c20v	91	110	-77	0.0	8,749	7.0	–	–	–	–	–	–	–
SIP-C90c25v	91	115	–	–	–	–	–	–	–	–	–	–	–
SIP-C90c30v	91	120	–	–	–	–	–	–	–	–	–	–	–
SIP-C100c5v	101	105	-6	0.0	0	0.0	-6	2	0	0.7	-6	2	0.7
SIP-C100c10v	101	110	-45	0.0	0	0.0	-45	45	12	9.3	-45	105	17.6
SIP-C100c20v	101	120	-60	0.0	4,756	10.2	–	–	–	–	–	–	–
SIP-C100c25v	101	125	-234	0.0	1,567	3.9	–	–	–	–	–	–	–
SIP-C100c30v	101	130	-178	0.0	28,608	112.2	–	–	–	–	–	–	–
SIP-C150c5v	151	155	-3	0.0	0	0.0	-3	6	2	2.0	-3	6	1.6
SIP-C150c10v	151	160	-31	0.0	0	0.0	-31	2	15	1.1	-31	2	4.6
SIP-C150c20v	151	170	-63	0.0	2,324	2.9	–	–	–	–	–	–	–
SIP-C150c25v	151	175	-107	0.0	802	0.5	–	–	–	–	–	–	–
SIP-C150c30v	151	180	-176	0.0	391,872	173.3	–	–	–	–	–	–	–
SIP-C200c5v	201	205	8	0.0	0	0.9	8	324	25	1,589.6	8	1,280	766.9
SIP-C200c10v	201	210	-2	0.0	427	1.8	–	–	–	–	–	–	–
SIP-C200c20v	201	220	-6	0.0	459,502	245.4	–	–	–	–	–	–	–
SIP-C200c25v	201	225	–	–	–	–	–	–	–	–	–	–	–
SIP-C200c30v	201	230	–	–	–	–	–	–	–	–	–	–	–
SIP-C250c5v	251	255	-10	0.0	0	0.0	-10	2	1	1.3	-10	6	1.7
SIP-C250c10v	251	260	-38	0.0	0	0.0	-38	2	1	1.4	-38	2	0.7
SIP-C250c20v	251	270	-131	0.0	0	0.0	–	–	–	–	–	–	–
SIP-C250c25v	251	275	–	–	–	–	–	–	–	–	–	–	–
SIP-C250c30v	251	280	–	–	–	–	–	–	–	–	–	–	–
SIP-C300c5v	301	305	-2	0.0	0	0.0	-2	2	1	1.5	-2	6	1.7
SIP-C300c10v	301	310	-43	0.0	0	0.0	-43	2	1	1.6	-43	2	0.8
SIP-C300c20v	301	320	-116	0.0	0	0.0	–	–	–	–	–	–	–
SIP-C300c25v	301	325	-144	0.0	122,532	246.1	–	–	–	–	–	–	–
SIP-C300c30v	301	330	–	–	–	–	–	–	–	–	–	–	–
SIP-C350c5v	351	355	-10	0.0	0	0.0	-10	2	1	1.6	-10	2	0.9
SIP-C350c10v	351	360	-37	0.0	0	0.0	-37	6	2	4.0	-37	6	2.0
SIP-C350c20v	351	370	-102	0.0	0	0.0	-59	>71	8	>7200	–	–	–
SIP-C350c25v	351	375	-177	0.00	0	0.0	–	–	–	–	–	–	–
SIP-C350c30v	351	380	–	–	–	–	–	–	–	–	–	–	–

Table 4.11: The IIS-BAC algorithm using the breadth-first node selection rule while branching on the z_ω variable with minimum fractional value

INSTANCES			**CPLEX**				**IIS**				**BAB**		
Name	**Rows**	**Bin. var.**	**Objval**	**Gap (%)**	**Nodes**	**Time (sec.)**	**Objval**	**Nodes**	**IIS v.i.**	**Time (sec.)**	**Objval**	**Nodes**	**Time (sec.)**
SIP-C90c5v	91	95	-8	0.0	0	0.00	–	–	–	–	-8	2	0.8
SIP-C90c10v	91	100	-45	0.0	0	0.00	-45	2	0	0.8	-45	2	3.5
SIP-C90c20v	91	110	-77	0.0	8749	7.13	–	–	–	–	–	–	–
SIP-C90c25v	91	115	–	–	–	–	–	–	–	–	–	–	–
SIP-C90c30v	91	120	–	–	–	–	–	–	–	–	–	–	–
SIP-C100c5v	101	105	-6	0.0	0	0.00	-6	2	0	0.5	-6	2	0.5
SIP-C100c10v	101	110	-45	0.0	0	0.02	-45	11	0	2.3	-45	11	103.3
SIP-C100c20v	101	120	-60	0.0	4,756	8.58	–	–	–	–	–	–	–
SIP-C100c25v	101	125	-234	0.0	1,567	3.76	–	–	–	–	–	–	–
SIP-C100c30v	101	130	-178	0.0	28,608	108.23	–	–	–	–	–	–	–
SIP-C150c5v	151	155	-3	0.0	0	0.00	-3	6	0	1.9	-3	6	1.8
SIP-C150c10v	151	160	-31	0.0	0	0.02	-31	2	0	0.8	-31	2	5.2
SIP-C150c20v	151	170	-63	0.0	2,324	2.51	–	–	–	–	–	–	–
SIP-C150c25v	151	175	-107	0.0	802	0.53	–	–	–	–	–	–	–
SIP-C150c30v	151	180	-176	0.0	39,1872	175.22	–	–	–	–	–	–	–
SIP-C200c5v	201	205	8	0.0	0	1.20	8	342	21	1454.0	8	1596	2894.6
SIP-C200c10v	201	210	-2	0.0	427	0.95	–	–	–	–	–	–	–
SIP-C200c20v	201	220	-6	0.0	459502	114.91	–	–	–	–	–	–	–
SIP-C200c25v	201	225	–	–	–	–	–	–	–	–	–	–	–
SIP-C200c30v	201	230	–	–	–	–	–	–	–	–	–	–	–
SIP-C250c5v	251	255	-10	0.0	0	0.00	-10	2	1	1.5	-10	6	1.6
SIP-C250c10v	251	260	-38	0.0	0	0.00	-38	2	1	1.4	-38	2	0.9
SIP-C250c20v	251	270	-131	0.0	0	0.03	–	–	–	–	–	–	–
SIP-C250c25v	251	275	–	–	–	–	–	–	–	–	–	–	–
SIP-C250c30v	251	280	–	–	–	–	–	–	–	–	–	–	–
SIP-C300c5v	301	305	-2	0.0	0	0.00	-2	2	1	1.4	-2	6	1.6
SIP-C300c10v	301	310	-43	0.0	0	0.00	-43	2	1	1.5	-43	2	0.7
SIP-C300c20v	301	320	-116	0.0	0	0.03	–	–	–	–	–	–	–
SIP-C300c25v	301	325	-144	0.0	122,532	246.1	–	–	–	–	–	–	–
SIP-C300c30v	301	330	–	–	–	–	–	–	–	–	–	–	–
SIP-C350c5v	351	355	-10	0.0	0	0.00	-10	2	1	1.9	-10	2	1.0
SIP-C350c10v	351	360	-37	0.0	0	0.00	-37	6	2	3.7	-37	6	1.9
SIP-C350c20v	351	370	-102	0.0	0	0.02	–	–	–	–	–	–	–
SIP-C350c25v	351	375	-177	0.00	0	0.0	–	–	–	–	–	–	–
SIP-C350c30v	351	380	–	–	–	–	–	–	–	–	–	–	–

4.4 Discussion

Figure 4.1 shows the percentage of instances solved by CPLEX, the *IIS-BAC algorithm*, and the *BAB algorithm*. For the case that considered the depth-first node selection rule while branching on the z_ω variable with maximum fractional value, CPLEX was able to find a solution for 77% of the instances followed by the *IIS-BAC algorithm* with 60% and the *BAB algorithm* with 43% of the instances. This situation is observed also for other node selection and node division rules considered in the computational experiments as shown in Figure 4.1.

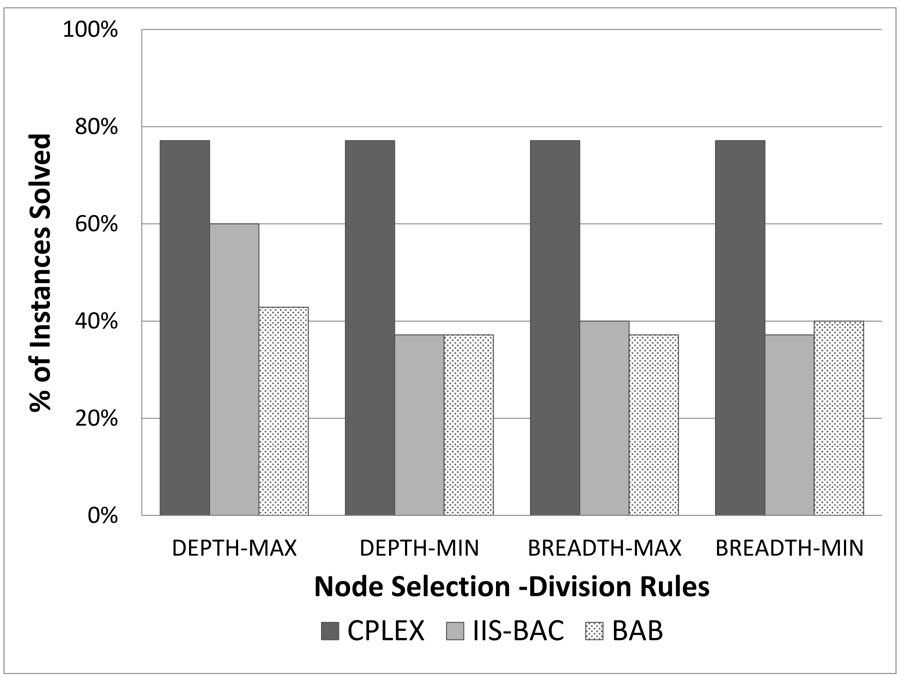

Figure 4.1: Percentage of instances solved by CPLEX, IIS-BAC and BAB

Figure 4.2 shows the average number of nodes explored by CPLEX, the *IIS-BAC algorithm*, and the *BAB algorithm* in order to find a solution. For the case that considered the depth-first node selection rule while branching on the z_ω variable with maximum fractional value, the *IIS-BAC algorithm* had to search the least number of nodes, that is, 116. For this same case, the average number of IIS inequalities added at a particular node where a solution can be found was 19. the *IIS-BAC algorithm* is followed by the *BAB algorithm* that explored an average of 363 nodes and by CPLEX that searched an average of 38,720 nodes. This same situation is observed for other node selection and node division rules considered in the computational experiments as shown in Figure 4.2.

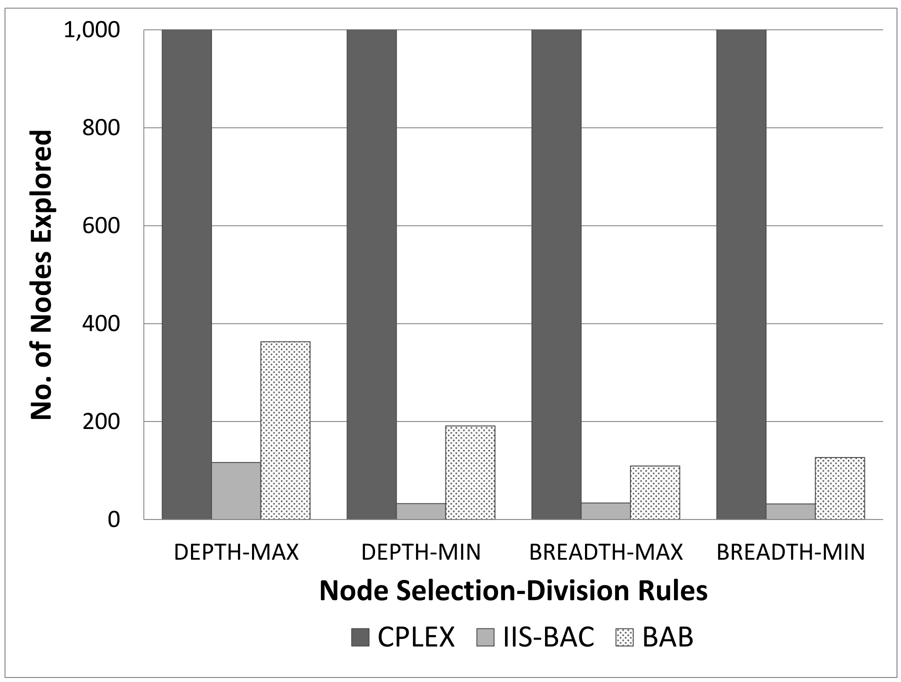

Figure 4.2: Average number of nodes explored by CPLEX, IIS-BAC and BAB

Figure 4.3 shows the average solution time by CPLEX, the *IIS-BAC algorithm*, and the *BAB algorithm*. For the case that considered the depth-first node selection rule while branching on the z_ω variable with maximum fractional value, CPLEX could find a solution in 25 seconds on average followed by the *BAB algorithm* with an average of 567 seconds and the *IIS-BA algorithm* with an average of 2,761 seconds. The *IIS-BAC algorithm* is the approach with the largest time since the time required to generate an IIS of SIP-C4^k was considered in the solution time. This situation is observed also for other node selection and node division rules considered in the computational experiments as shown in Figure 4.3.

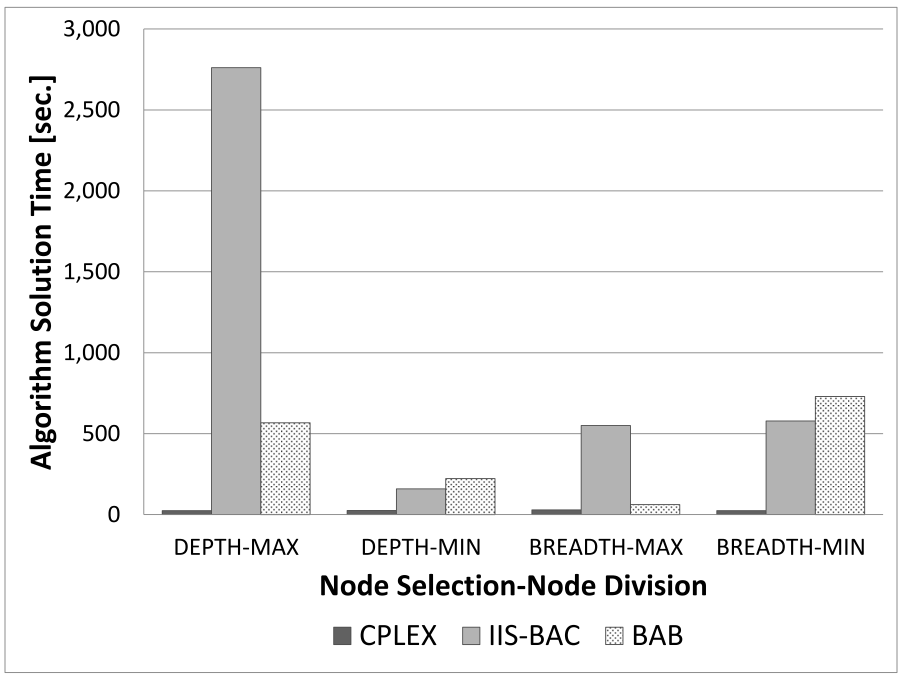

Figure 4.3: Average solution time by CPLEX, IIS-BAC and BAB

Decomposition methods based on IIS inequalities provide promising computational results for solving SIP-C2. The *IIS-BAC algorithm* outperformed the *BAB algorithm* in terms of the percentage of instances solved and both CPLEX and the

BAB algorithm in terms of the average number of nodes to explore in order to find a solution. However, CPLEX outperformed the *IIS-BAC algorithm* in terms of both the percentage of instances solved and the solution time.

Even though CPLEX solved the largest number of problems, there were instances for which CPLEX did not find a solution. Note that the instances considered in the computational experiments are smaller than instances of practical applications such as the wildfire initial response planning. In fact, instances with only 20 scenarios for this type of problem can have more than 500,000 constraints and decision variables.

The IIS decomposition methods presented in this chapter can be improved by strengthening the IIS inequality or by using additional inequalities that consider the decision variable x. The IIS inequality can be strengthened by including the decision variable x in it such that inequalities of the form $\sum_{i \in I} t_i x_i + \sum_{\omega \in \mathcal{D}} z_\omega \geq h$ can be obtained. This can be achieved by performing linear combinations of the IIS inequality with other valid inequalities such as the n-step MIR for the n-mixing set [57]. Likewise, disjunctive decomposition inequalities [58] can be appended to SIP-C6^k at a particular node k in the *IIS-BAC algorithm* along with the IIS inequalities discussed in this chapter.

Another path to consider when solving probabilistically constrained SIP would be to obtain new valid inequalities for the set Q^1. Consider a reformulation of SIP-C3 as follows:

$$\textbf{SIP-C7:} \min\ c^\top x \tag{4.27a}$$

$$\text{s.t. } y(\omega) = T(\omega)x, \forall \omega \in \Omega \tag{4.27b}$$

$$y(\omega) + a_\omega z_\omega \geq r(\omega), \forall \omega \in \Omega \tag{4.27c}$$

$$\sum_{\omega \in \Omega} p_\omega z_\omega \leq 1 - \beta \tag{4.27d}$$

$$x \in \mathbb{B}^n \cap \bar{X}, z_\omega \in \mathbb{B}, \forall \omega \in \Omega. \tag{4.27e}$$

Note that $y(\omega) \in \mathbb{R}^m, \forall \omega \in \Omega$. Likewise, $\bar{X} = \{x \in \mathbb{R}^n : Ax \leq b\}$ and $a_\omega = M^\omega e$. The aim is to strengthen SIP-C7 by finding strong formulations for the set $Q^3 = \{(y, z) \in \mathbb{R}^m \times \mathbb{B}^{|\Omega|} : \sum_{\omega \in \Omega} p_\omega z_\omega \leq 1 - \beta,\ y(\omega) + a_\omega z_\omega \geq r(\omega), \forall \omega \in \Omega\}$. If $T(\omega) \in \mathbb{R}^{1 \times n}, \forall \omega \in \Omega$, then Q^3 can be rewritten as $Q^4 = \{(y, z) \in \mathbb{R} \times \mathbb{B}^{|\Omega|} : \sum_{\omega \in \Omega} p_\omega z_\omega \leq 1 - \beta,\ y(\omega) + a_\omega z_\omega \geq r(\omega), \forall \omega \in \Omega\}$. Consider also $Q^5 = \{(y, z) \in \mathbb{R} \times \mathbb{B}^{|\Omega|} : y(\omega) + a_\omega z_\omega \geq r(\omega), \forall \omega \in \Omega\}$ that is a relaxation of Q^4.

Similar sets to Q^5 have been studied in the literature such as the knapsack ($Q^6 = \{z \in \times \mathbb{B}^{|\Omega|} : \sum_{\omega \in \Omega} a_\omega z_\omega \leq r(\omega), \forall \omega \in \Omega\}$), the mixing set ($Q^7 = \{(y, z) \in \mathbb{R} \times \mathbb{B}^{|\Omega|} : y + a_\omega z_\omega \geq r(\omega), \forall \omega \in \Omega\}$), and the n-mixing set ($Q^8 = \{(y, z) \in \mathbb{R} \times \mathbb{B}^{|\Omega|} : y + \sum_{i \in I} a^i_\omega z^i_\omega \geq r(\omega), \forall \omega \in \Omega\}$). In fact, facet-defining inequalities have been developed for these sets [28, 57, 4]. However, valid inequalities with facet-defining properties have not been presented in the literature neither for the set Q^5 nor for the set $Q^9 = \{(y, z) \in \mathbb{R}^m \times \mathbb{B}^{|\Omega|} :\ \sum_{\omega \in \Omega} p_\omega z_\omega \leq 1 - \beta,\ y(\omega) + \sum_{i \in I} a^i_\omega z^i_\omega \geq r(\omega),\ \forall \omega \in \Omega\}$. Therefore, the IIS decomposition method presented in this dissertation could be improved by obtaining valid inequalities for the set Q^1 via facet-defining inequalities for the sets Q^5 or Q^9.

5. WILDFIRE INITIAL RESPONSE PLANNING USING PROBABILISTICALLY CONSTRAINED STOCHASTIC INTEGER PROGRAMMING

5.1 Introduction

This chapter presents a new methodology for wildfire initial response planning. The new methodology integrates a fire behavior simulation, a wildfire risk, and a probabilistically constrained SIP model as shown in Figure 5.1. Both the fire behavior simulation and the wildfire risk model provide the required input data to the SIP model. The fire behavior model simulates fire growth and computes burned area and fire perimeter during the planning horizon. The wildfire risk model identifies the areas that would have the most negative impact on people in case of a wildfire. The SIP model that assumes a known standard response needed to contain a fire yields different outputs including deployment plans, dispatch plans, and the risk associated with fires not receiving a standard response. Details of the fire behavior, the wildfire risk, and the SIP model are provided in the next sections. Likewise, this chapter presents the application of the new methodology to a real fire planning unit in east Texas.

5.2 Wildfire Risk Model

The wildfire risk model is an important component of the new methodology for wildfire initial response planning. The aim of this model is to identify which areas within a region of interest would have the most negative impact on people and valuable resources in case of a wildfire. This work uses the Texas Wildfire Risk Assessment (TWRA) system to define the wildfire risk model since our study area is located in Texas. However, the new methodology can be extended to allow for other

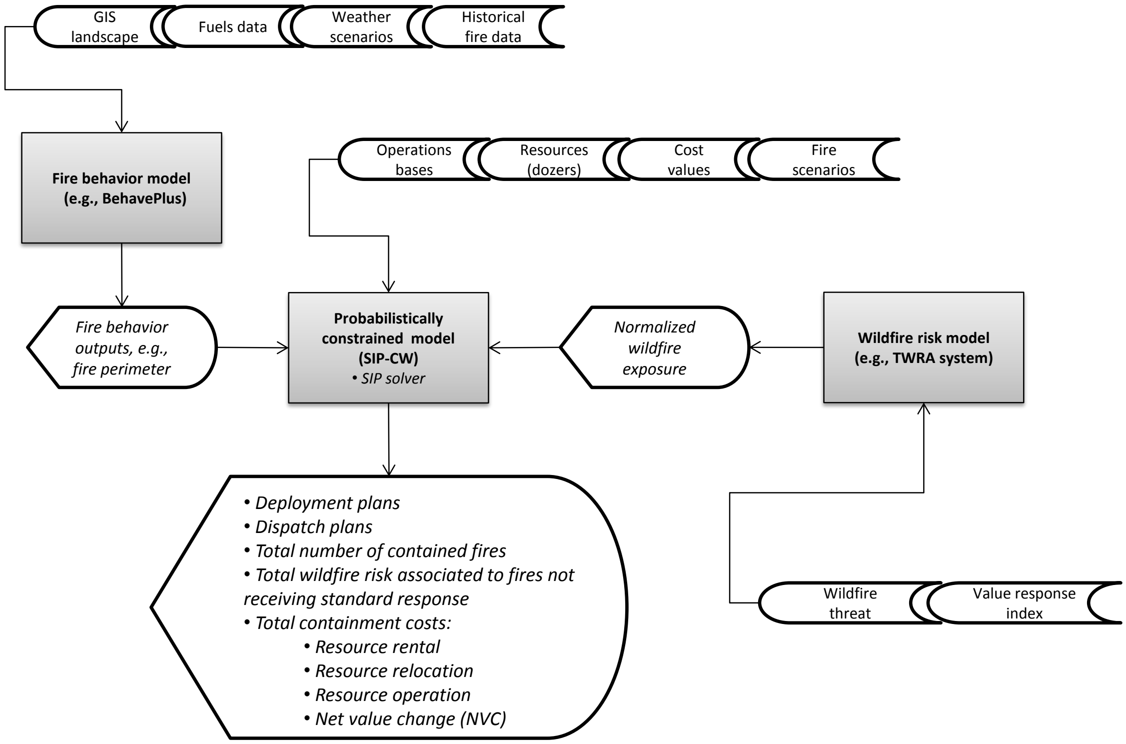

Figure 5.1: Schematic diagram of the simulation and probabilistically constrained optimization methodology

wildfire risk criteria.

Texas A&M Forest Service (TFS) established the TWRA system in response to increasing demand for more accurate and up-to-date wildfire risk information across the state of Texas. The goal of the TWRA system is to provide a consistent and comparable set of scientific results to be used as a foundation for wildfire emergency response planning activities in Texas. These activities are mitigation, initial and extended response, and evacuation planning.

The main components of the TWRA system are: wildfire threat (WT), value response index (VRI), and wildfire exposure (WE). There is a relation between these components and the definition of wildfire risk. Wildfire risk is the likelihood of suffering loss of lives, homes and property due to wildfires not receiving wildfire initial response. Thus, WT is related to the likelihood of a fire happening, VRI is associated with the potential negative impact that a fire would have on people and resources, and WE is the overall measure of risk associated with a wildfire not receiving an initial response.

A summary of the TWRA system is depicted in Figure 5.2. The grey boxes indicate input data such as percentile weather, topography, landscape, and historic fire report data and locations. Input data is used to compute output data, represented by the white boxes, such as WT, VRI, and WE, which are discussed in detail next.

5.2.1 Wildfire Threat

WT is the likelihood of a fire occurring by igniting in an area. It is calculated for different percentile weather categories and then summed to obtain an overall WT for a particular area. Percentile weather is described in terms of percentage of fire occurrence and environmental values for four categories: low, moderate, high, and extreme. The percentage of fire occurrence for each category are: low (0-15%), moderate (16-90%), high (91-97%), and extreme (98-100%).

WT is categorized into seven classes from 0 to 7 with 0 corresponding to non-burnable areas and 7 corresponding to areas with very high WT. The actual probability range of values for each of the seven WT categories are shown in Table 5.1.

5.2.2 Value Response Index

VRI is an overall rating of the potential impact of a wildfire. VRI combines the impact ratings for wildland urban interface response index (WUIRI) and pine plantation response index (PPRI). The calculation of WUIRI depends on the values

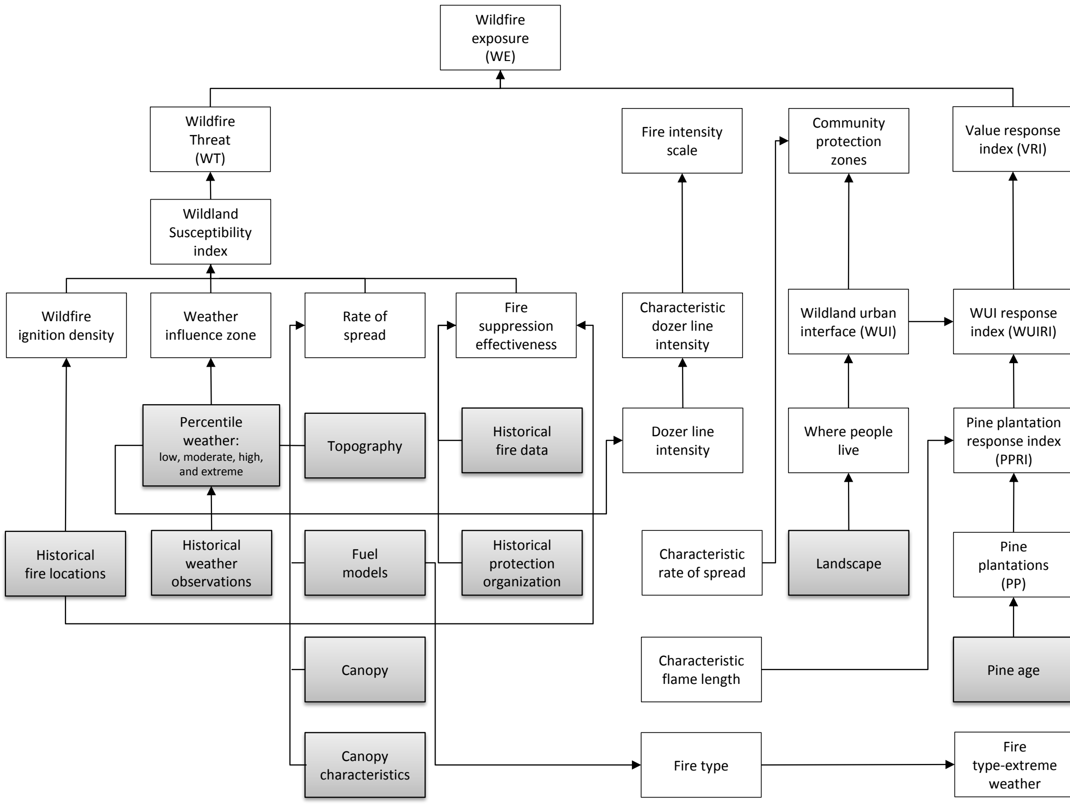

Figure 5.2: The TWRA system for Texas

Table 5.1: Wildfire threat categories and probabilities

WT Classes	Interval
0	Non-burnable
1	0.000 - 0.002
2	0.002 - 0.004
3	0.004 - 0.006
4	0.006 - 0.008
5	0.008 - 0.010
6	0.010 - 0.020
7	0.020 - 0.061

for wildland urban interface (WUI), and the estimation of PPRI depends on the values for pine plantations (PPs) (Figure 5.2).

WUI is the area where structures and other human improvements meet with undeveloped wildland, and it is measured in houses per acre. Population growth within WUI substantially increases the risk from wildfires. Flame length is used as the measure of wildfire risk at WUI since the higher the flame length the worse the impact on people and their homes. Thus, WUIRI is a rating of the potential impact of a wildfire on WUI, and it assigns values to areas where WUI and flame length overlap. WUIRI ranges from -1 to -9 with -1 representing the least negative impact and -9 representing the most negative impact.

PPs are pine stands that are planted and actively managed for financial gain. PPs are recognized as a key economic resource in Texas. Thus, PPRI is a rating of the potential impact of a wildfire on PPs. In other words, PPRI measures potential economic impact due to wildfires. PPRI ranges from 3 to -9 with 3 representing the most positive impact and -9 representing the most negative impact.

WUI and PPRI are used to calculate VRI. VRI ranges from 1 to -9 for the state of Texas. A value of 1 represents the most positive impact while a value of -9 represents the worst possible impact. This worst possible impact may happen at regions where both the flame length can potentially be greater than 12 feet due to PPs and the house and population density are very high.

5.2.3 Wildfire Exposure

WE is an overall metric that defines the possibility of loss or harm due to a wildfire not receiving an initial response. WE is calculated by multiplying WT and VRI. Thus, WE ranges between -0.549 and 0.061 with -0.549 representing the maximum negative impact and 0.061 corresponding to the least negative impact. Let a_f be the WE associated with the area around an arbitrary fire f. Thus, WE can be normalized as follows:

$$g(a_f) = \frac{0.061 - a_f}{0.61}$$

where $g(a_f)$ is referred to as the normalized wildfire exposure (NWE) around the area of fire f such that $0 \leq g(a_f) \leq 1$.

5.3 Standard Response

The capacity to provide wildfire initial response to a fire is characterized in terms of both wildfire risk and the number of firefighting resources (dozers) required to contain a fire. Fires representing a higher wildfire risk are given a higher priority in relation to the number of dozers dispatched. When a dozer arrives at an incident, it can construct a dozer-line or barrier scraped to mineral soil around the fire to contain it. If the perimeter of a dozer-line is greater than or equal to the fire perimeter, then it is said that wildfire initial response is achieved. An escaped fire occurs when it exceeds the initial response capabilities, for instance when a fire jumps a dozer-line.

In this work, standard response will be defined as the combination of dozers located within a maximum response time that are required to fully contain a fire that is receiving wildfire initial response. A wildfire receives standard response if the total dozer-line rate is greater than or equal to its fire perimeter. On the contrary, a wildfire *cannot receive* standard response if it has very long flames under strong fire-induced winds, or if there is a lack in response capacity due to resource availability and response time restrictions. The standard response values used in the probabilistically constrained SIP model are calculated following the same guidelines described in [48].

A *let it burn* policy is not assumed in the new methodology, that is, all fires can receive standard response even if these fires occur in remote areas. However, fires occurring in less densely populated areas are given lower priority when dozers are needed to respond to other fires that represent a higher wildfire risk. The new probabilistically constrained SIP model, which is presented next, makes deployment and dispatching decisions to wildfires according to our definition of standard response and the wildfire risk they represent.

5.4 Probabilistically Constrained Model

Wildfire initial response planning requires making strategic deployment and tactical dispatching plans of firefighting resources while prioritizing areas where fire occurrences would have a higher impact on homes and other valuable resources. Thus, we developed a probabilistically constrained standard response model that determines the resources to deploy to each operations base and the optimal mix of resources to dispatch to each fire from operations bases every day during a fire season. The goal of the probabilistically constrained model is to provide standard response to as many fires as possible while minimizing both total cost and wildfire risk of fires not receiving a standard response during a fire season. Throughout this chapter, we make the following assumptions on the probabilistically constrained model:

A1. Firefighting resource characteristics (e.g. dozer-line rate, arrival time to the fire, operation cost) are known (but stochastic) or can be estimated.

A2. Firefighting resources are dispatched to wildfires after the fires have been reported. Each resource can only be dispatched to one fire on a given day.

A3. The distribution of final fire sizes (perimeter and area burned) for the fire planning unit is known or can be estimated.

A4. The distribution of wildfire risk for the fire planning unit is known or can be estimated.

A5. Standard dozer-line rates required to contain a given fire size within standard response time are known or can be estimated.

A6. The multivariate random vector that characterizes the uncertainties in the probabilistically constrained model has a discrete probability distribution and finite support.

Firefighting resource characteristics can be obtained from the literature to satisfy assumption (A1). Assumption (A2) allows a resource to be committed to fight a single fire in a given day. Historical wildfire data and a wildfire behavior simulation can be used to generate possible final fire sizes to meet assumption (A3). Wildfire risk values can be obtained from databases created by forest service agencies, or it can be estimated by experts based on their experience. This assumption enables the probabilistically constrained model to make deployment plans based on the potential impact of wildfires on people and resources. Regarding assumption (A5), standard dozer-line rates required to contain a given fire size within a specified amount of response time can be estimated either through simulation or expert experience. This assumption enables the probabilistically constrained SIP model to compute a mix of different types of resources (with different dozer-line rates) to dispatch to a given fire. Assumption (A6) will allow for devising a deterministic equivalent problem (DEP) to the probabilistically constrained model.

5.4.1 Mathematical Formulation

The notation required for the probabilistically constrained model is described in Tables 5.2, 5.3, and 5.4.

Table 5.2: Sets

Notation	Definition
$\mathcal{R}$:	Set of all resources, indexed i.
$\mathcal{B}$:	Set of all bases, indexed j and k.
$\mathcal{F}$:	Set of all fires, indexed f.
$\mathcal{D}$:	Set of all days during a fire season, indexed d.
$\mathcal{T}$:	$\mathcal{T} = \{1, \cdots, T\}$ is the set of discrete time periods (in hours or hr) during a fire season day, indexed t.
$\hat{\mathcal{R}}(j)$:	Set of resources (initially) located at base j such that $\hat{\mathcal{R}}(j) \subseteq \mathcal{R}$.
$\bar{\mathcal{R}}(j,k)$:	Set of resources located at base j that can be relocated to base k such that $\bar{\mathcal{R}}(j,k) \subseteq \hat{\mathcal{R}}(j)$.
$\hat{\mathcal{B}}(i)$:	Set of bases to which resource i can be relocated such that $\hat{\mathcal{B}}(i) \subseteq \mathcal{B}$.
$\hat{\mathcal{F}}(d)$:	Set of fires during fire season day d such that $\hat{\mathcal{F}}(d) \subseteq \mathcal{F}$.
$\bar{\mathcal{F}}(k,d)$:	Set of fires that can be reached by resources dispatched from base $k \in \mathcal{B}$ within maximum response time $t_{\max}$ during fire season day d such that $\bar{\mathcal{F}}(k,d) \subseteq \hat{\mathcal{F}}(d)$.
Ω:	Set of fire scenarios, indexed ω. A fire scenario ω is a sequence of fire days d during a fire season. Each day in ω has a unique pattern of fire occurrences f.
$\mathcal{B}'(\omega,d)$:	Set of bases from which resources can reach a fire f in scenario ω during fire season day d within the maximum response time $t_{\max}$ such that $\mathcal{B}'(\omega,d) \subseteq \mathcal{B}$.
$\mathcal{F}'(\omega,d)$:	Set of fires f during fire season day d under scenario ω such that $\mathcal{F}'(\omega,d) \subseteq \hat{\mathcal{F}}(d)$.

Table 5.3: Parameters

Notation	Definition
b:	Fixed deployment, relocation cost, operations cost and net value change (NVC) budget for all scenarios $\omega \in \Omega$.
β:	Decision maker's risk attitude level associated with scenarios $\omega \in \Omega$ with $\beta \in (0, 1)$.
ρ_1:	Fraction of the overall NWE for scenario ω with $\rho_1 \in [0, 1]$.
ρ_2:	Fraction of fires receiving a standard response for scenario ω with $\rho_2 \in [0, 1]$.
M:	Large constant.
e:	Appropriately sized vector of ones.
c_{ik}:	Fixed rental cost (\$) of resource i assigned to based k.
c_{ijk}:	Relocation cost (\$) of resource i from its initial base j to base k, $c_{ijk} = 0$ if $j = k$.
h^d_{if}:	Standard operation time (hr) of resource i at fire f on fire season day d that takes the value of $\lvert\mathcal{T}\rvert$.
q_{ik}:	Hourly cost (\$/hr) of operation resource i from base k.
$\alpha^{d,\omega}_f$:	Minimum standard dozer-line rate required at fire f on fire season day d in scenario ω in (km/hr).
α_i:	Dozer line rate of resource i in (km/hr).
p_ω:	Probability of occurrence of scenario ω.
μ^d_f:	NVC for fire f on fire season day d if it receives standard response.
a^d_f:	WE value associated with the area around fire f in fire season day d
t_{max}:	Maximum response time of any resource to reach fire f under any scenario $\omega \in \Omega$ and any fire season day $d \in \mathcal{D}$.
n_k:	Maximum number of resources that can be deployed to base k.
$g(a^d_f)$:=	NWE for fire f during fire season day d.
$e(\omega)$:=	Overall NWE for scenario ω such that $e(\omega) = \sum_{d \in \mathcal{D}} \sum_{f \in \mathcal{F}'(\omega,d)} g(a^d_f)$.
$p(\omega)$:=	Probability of occurrence for scenario ω.
$h(\omega)$:=	Weighted NWE for scenario ω such that $h(\omega) = p(\omega)e(\omega)$.
$r(\omega)$:=	Normalized risk value associated with scenario ω such that $r(\omega) = \frac{h(\omega)}{\sum_{\hat{\omega} \in \Omega} h(\hat{\omega})}$.

Table 5.4: Decision variables

Notation	Definition
x_{ijk}:	$x_{ijk} = 1$ if resource i initially at base j is deployed to base k, $x_{ijk} = 0$ otherwise.
z^d_{ijkf}:	$z^d_{ijkf} = 1$ if resource i from base k (initially at base j) is dispatched to fire f during fire season day d, $z^d_{ijkf} = 0$ otherwise.
y^d_f:	$y^d_f = 1$ if fire f receives standard response during fire season day d, $y^d_f = 0$ otherwise.
υ_ω:	$\upsilon_\omega = 1$ if the constraints associated with scenario ω are included in the formulation, $\upsilon_\omega = 0$ otherwise.

The probabilistically constrained model for wildfire initial response is now presented.

$$\textbf{SIP-CW:} \min \sum_{j\in\mathcal{B}}\sum_{i\in\hat{\mathcal{R}}(j)}\sum_{k\in\hat{\mathcal{B}}(i)} c_{ik}x_{ijk} + \sum_{j\in\mathcal{B}}\sum_{i\in\hat{\mathcal{R}}(j)}\sum_{k\in\hat{\mathcal{B}}(i)} c_{ijk}x_{ijk}$$

$$+\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{B}}\sum_{k\in\mathcal{B}}\sum_{i\in\bar{\mathcal{R}}(j,k)}\sum_{f\in\bar{\mathcal{F}}(k,d)} h_{if}q_{ik}z^d_{ijkf} + \sum_{d\in\mathcal{D}}\sum_{f\in\hat{\mathcal{F}}(d)} \mu_f y^d_f \tag{5.2a}$$

$$\text{s.t.} \sum_{k\in\hat{\mathcal{B}}(i)} x_{ijk} = 1,\ \forall j\in\mathcal{B}, \forall i\in\hat{\mathcal{R}}(j) \tag{5.2b}$$

$$\sum_{f\in\bar{\mathcal{F}}(k,d)} z^d_{ijkf} \leq x_{ijk},\ \forall d\in\mathcal{D},\ \forall j\in\mathcal{B},\ \forall k\in\mathcal{B},\ \forall i\in\bar{\mathcal{R}}(j,k) \tag{5.2c}$$

$$\sum_{j\in\mathcal{B}}\sum_{i\in\hat{\mathcal{R}}(j)}\sum_{k\in\hat{\mathcal{B}}(i)} c_{ik}x_{ijk} + \sum_{j\in\mathcal{B}}\sum_{i\in\hat{\mathcal{R}}(j)}\sum_{k\in\hat{\mathcal{B}}(i)} c_{ijk}x_{ijk}$$

$$+\sum_{d\in\mathcal{D}}\sum_{j\in\mathcal{B}}\sum_{k\in\mathcal{B}}\sum_{i\in\bar{\mathcal{R}}(j,k)}\sum_{f\in\bar{\mathcal{F}}(k,d)} h_{if}q_{ik}z^d_{ijkf} + \sum_{d\in\mathcal{D}}\sum_{f\in\hat{\mathcal{F}}(d)} \mu_f y^d_f \leq b \tag{5.2d}$$

$$\mathbf{P}\begin{pmatrix} \sum_{j\in\mathcal{B}}\sum_{k\in\mathcal{B}'(\tilde{\omega},d)}\sum_{i\in\bar{\mathcal{R}}(j,k)} \alpha_i z^d_{ijkf} \geq \alpha^{d,\tilde{\omega}}_f y^d_f,\ \forall d\in\mathcal{D},\ \forall f\in\mathcal{F}'(\tilde{\omega},d) \\ \sum_{d\in\mathcal{D}}\sum_{f\in\mathcal{F}'(\tilde{\omega},d)} g(a^d_f)y^d_f \geq \rho_1 e(\omega) \\ \sum_{d\in\mathcal{D}}\sum_{f\in\mathcal{F}'(\tilde{\omega},d)} y^d_f \geq \rho_2 \sum_{d\in\mathcal{D}} \mid \mathcal{F}'(\tilde{\omega},d) \mid \\ y^d_f \geq z^d_{ijkf},\ \forall d\in\mathcal{D},\ \forall f\in\mathcal{F}'(\tilde{\omega},d),\ \forall j\in\mathcal{B},\ \forall k\in\mathcal{B},\ \forall i\in\bar{\mathcal{R}}(j,k) \end{pmatrix} \geq 1-\beta \tag{5.2e}$$

$$x_{ijk}\in\{0,1\},\ \forall j\in\mathcal{B},\ \forall i\in\hat{\mathcal{R}}(j),\ \forall k\in\hat{\mathcal{B}}(i) \tag{5.2f}$$

$$z^d_{ijkf}\in\{0,1\},\ y^d_f\in\{0,1\},\ \forall d\in\mathcal{D},\ \forall j\in\mathcal{B},\ \forall k\in\mathcal{B},\ \forall i\in\bar{\mathcal{R}}(j,k),\ \forall f\in\mathcal{F} \tag{5.2g}$$

SIP-CW aims to minimize the total fixed cost of deploying available resources to operations bases, the total travel cost for relocating resources between bases, the total resource operation cost and the total cost of fire damages, as shown in the objective function (5.2a). Constraint (5.2b) ensures that each resource i can only be assigned to exactly one operations base. It also implies that either a resource i remains at its initial base j, when $j = k$, or it is deployed to operations base k, when $j \neq k$. Constraint (5.2c) guarantees that a resource i can only respond to one fire in a day, and it is returned to its home base k by the end of the day. Likewise, (5.2d) imposes a budgetary constraint on the total deployment, relocation, operation, and fire damages cost. The joint probabilistic constraint is represented in (5.2e). This ensures that a set of four different constraints are satisfied with probability $1 - \beta$. These four constraints are: the minimum standard dozer-line rate required to contain a fire; the maximum NWE associated with fires not receiving a standard response during a fire season; the minimum number of fires receiving a standard response; and a logic constraint used to tighten the formulation. Likewise, constraints (5.2f) and (5.2g) ensure binary restrictions on the decision variables.

The random vector $\widetilde{\omega}$ is discretely distributed with $|\Omega| < \infty$ by assumption (A6). Thus, a DEP to SIP-CW can be written as DEP-W (Formulation 5.3) where v_ω is a binary decision variable for each scenario ω that takes the value of 0 if the constraints corresponding to ω are included, and 1 if the constraints corresponding to ω are excluded from the formulation. There is also a knapsack constraint (5.3i) to guarantee that the probabilistic constraint (5.2e) is satisfied. DEP-W can be directly solved using an off-the-shelf solver such as CPLEX. We selected CPLEX since this MIP solver allows for solving instances of DEP-W up to 320,000 constraints and 300,000 variables within a reasonable amount of time (about an hour).

$$
\begin{aligned}
\textbf{DEP-W:} \min & \sum_{j\in\mathcal{B}} \sum_{i\in\hat{\mathcal{R}}(j)} \sum_{k\in\hat{\mathcal{B}}(i)} c_{ik}x_{ijk} + \sum_{j\in\mathcal{B}} \sum_{i\in\hat{\mathcal{R}}(j)} \sum_{k\in\hat{\mathcal{B}}(i)} c_{ijk}x_{ijk} \\
& + \sum_{d\in\mathcal{D}} \sum_{j\in\mathcal{B}} \sum_{k\in\mathcal{B}} \sum_{i\in\bar{\mathcal{R}}(j,k)} \sum_{f\in\bar{\mathcal{F}}(k,d)} h_{if}q_{ik}z^d_{ijkf} + \sum_{d\in\mathcal{D}} \sum_{f\in\hat{\mathcal{F}}(d)} \mu_f y^d_f && \text{(5.3a)}
\end{aligned}
$$

$$
\text{s.t.} \sum_{k\in\hat{\mathcal{B}}(i)} x_{ijk} = 1,\ \forall j\in\mathcal{B}, \forall i\in\hat{\mathcal{R}}(j) \tag{5.3b}
$$

$$
\sum_{f\in\bar{\mathcal{F}}(k,d)} z^d_{ijkf} \le x_{ijk},\ \forall d\in\mathcal{D},\ \forall j\in\mathcal{B},\ \forall k\in\mathcal{B},\ \forall i\in\bar{\mathcal{R}}(j,k) \tag{5.3c}
$$

$$
\begin{aligned}
& \sum_{j\in\mathcal{B}} \sum_{i\in\hat{\mathcal{R}}(j)} \sum_{k\in\hat{\mathcal{B}}(i)} c_{ik}x_{ijk} + \sum_{j\in\mathcal{B}} \sum_{i\in\hat{\mathcal{R}}(j)} \sum_{k\in\hat{\mathcal{B}}(i)} c_{ijk}x_{ijk} \\
& + \sum_{d\in\mathcal{D}} \sum_{j\in\mathcal{B}} \sum_{k\in\mathcal{B}} \sum_{i\in\bar{\mathcal{R}}(j,k)} \sum_{f\in\bar{\mathcal{F}}(k,d)} h_{if}q_{ik}z^d_{ijkf} + \sum_{d\in\mathcal{D}} \sum_{f\in\hat{\mathcal{F}}(d)} \mu_f y^d_f \le b && \text{(5.3d)}
\end{aligned}
$$

$$
\sum_{j\in\mathcal{B}} \sum_{k\in\mathcal{B}'(\omega,d)} \sum_{i\in\bar{\mathcal{R}}(j,k)} \alpha_i z^d_{ijkf} + Mev_\omega \ge \alpha^{d,\omega}_f y^d_f,\ \forall d\in\mathcal{D},\ \forall f\in\mathcal{F}'(\omega,d),\ \forall\omega\in\Omega \tag{5.3e}
$$

$$
\sum_{d\in\mathcal{D}} \sum_{f\in\mathcal{F}'(\omega,d)} g(a^d_f)y^d_f + Mev_\omega \ge \rho_1 e(\omega),\ \forall\omega\in\Omega \tag{5.3f}
$$

$$
\sum_{d\in\mathcal{D}} \sum_{f\in\mathcal{F}'(\omega,d)} y^d_f + Mev_\omega \ge \rho_2 \sum_{d\in\mathcal{D}} \mid \mathcal{F}'(\omega,d) \mid,\ \forall\omega\in\Omega \tag{5.3g}
$$

$$
y^d_f + Mev_\omega \ge z^d_{ijkf},\ \forall d\in\mathcal{D},\ \forall f\in\mathcal{F}'(\omega,d),\ \forall j\in\mathcal{B},\ \forall k\in\mathcal{B},\ \forall i\in\bar{\mathcal{R}}(j,k),\ \forall\omega\in\Omega \tag{5.3h}
$$

$$
\sum_{\omega\in\Omega} p_\omega v_\omega \le \beta \tag{5.3i}
$$

$$
x_{ijk}\in\{0,1\},\ \forall j\in\mathcal{B},\ \forall i\in\hat{\mathcal{R}}(j),\ \forall k\in\hat{\mathcal{B}}(i) \tag{5.3j}
$$

$$
z^d_{ijkf}\in\{0,1\},\ y^d_f\in\{0,1\},\ \forall d\in\mathcal{D},\ \forall j\in\mathcal{B},\ \forall k\in\mathcal{B},\ \forall i\in\bar{\mathcal{R}}(j,k),\ \forall f\in\mathcal{F} \tag{5.3k}
$$

5.5 Application

5.5.1 Overview

An overview of the new methodology is shown in Figure 5.1. This methodology integrates three models: a wildfire risk model (e.g., TWRA system), SIP-CW, and a fire behavior simulation model (e.g., BehavePlus). The fire behavior model simulates fire growth and computes burned area and fire perimeter for a six-hour wildfire initial response horizon that becomes an input to SIP-CW. Fire growth is simulated in a fire planning unit under different weather scenarios (low, medium, high and extreme). The input data for the fire behavior simulation includes GIS landscape, fuels data, weather scenarios, and historical fire data. Likewise, NWE is computed using the TWRA system that is also an input to SIP-CW. The input data for the TWRA system includes WT and VRI.

SIP-CW computes different outputs that can be used to make effective decisions regarding wildfire initial response at the beginning of a fire season before fires occur. Outputs of SIP-CW include resource deployment and dispatch plans, total number of fires contained, total wildfire risk associated with fires not receiving a standard response, and total containment cost.

This chapter also describes the application of the new methodology to a real fire planning unit in east Texas. The goals of the computational experiments can be summarized as follows:

1. Demonstrate the effectiveness of the new methodology for making strategic deployment decisions for wildfire initial response.

2. Compare the deployment decisions provided by the new methodology with the current deployment plan of firefighting resources in TX12.

3. Estimate future budgetary needs to effectively provide standard response.

4. Identify the regions in TX12 with the highest wildfire risk and the highest density of historical fires.

5.5.2 *Study Area*

We applied SIP-CW to a real fire planning unit that is managed by TFS. This fire planning unit that is called TX12 is located on the northeastern part of Texas as shown in Figure 5.3. TX12 has seven operations bases from which firefighting resources are dispatched to reported fires. TX12 is one of the districts in Texas with the highest historical density of fires per square mile over the past 20 years and the largest historical fires with fire sizes as large as 4000 acres. Likewise, TX12 is one of the districts in Texas with the highest wildfire risk with VRI, PPRI, and WT values as high as -8, -7 and 3, respectively. All input data used in the study including historical fire data was provided by TFS.

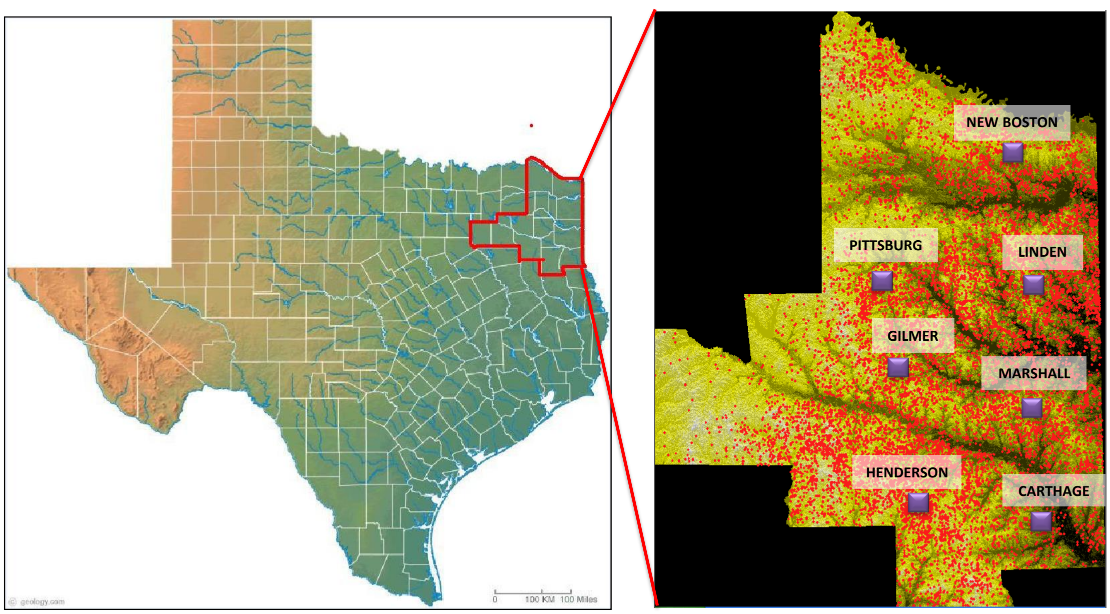

Figure 5.3: Location of TX12 in east Texas

We consider historical fire data between 1985 and 2006. There was a total of 13,163 fire occurrences during this time period in TX12. We implemented a clustering algorithm and generated representative fire locations (RFLs) in order to bring the problem size down to something tractable (Figure 5.4). A RFL is a single location that represents a fire or group of fires that actually occurred within the same geographic area and shared not only similar fuel models but also similar wildfire risk. We created 46 RFLs for the study in TX12 as suggested in the literature [29, 48, 49]. This number is a good and an uniform representation of fuels, weather, and wildfire risk over TX12 since more RFLs are located on areas with high wildfire risk. In addition, this number allows RFLs to be well-spread over the region of interest. We used a clustering algorithm that constructs a covering perimeter with constant incremental value around arbitrary historical fire location until the number of desired RFLs is achieved.

Standard response models for wildfire initial response require specifying the final fire size (burned area and perimeter) for each RFL for different weather scenarios. TFS defines four weather scenarios: low, moderate, high and extreme. Final fire size also depends on fuel models and landscape. TFS provided weighted fuel model and landscape samples around each RFL within a circular area of 200 acres. This selection was made through experimentation to ensure that the fuel models at each RFL were well represented and that the final fire sizes obtained by BehavePlus were within the range of historical values for the RFL. Using the weighted fuel models, the landscape data, and the four different weather scenarios, we performed fire simulations using BehavePlus to obtain final fire size at each RFL for a 6-hour wildfire initial response planning horizon.

SIP-CW for wildfire initial response requires specifying the NWE values around each RFL on any day during a fire season. TFS provided the WE sample values around each RFL within a circular area of 200, 2000, and 8000 acres. Since WE is

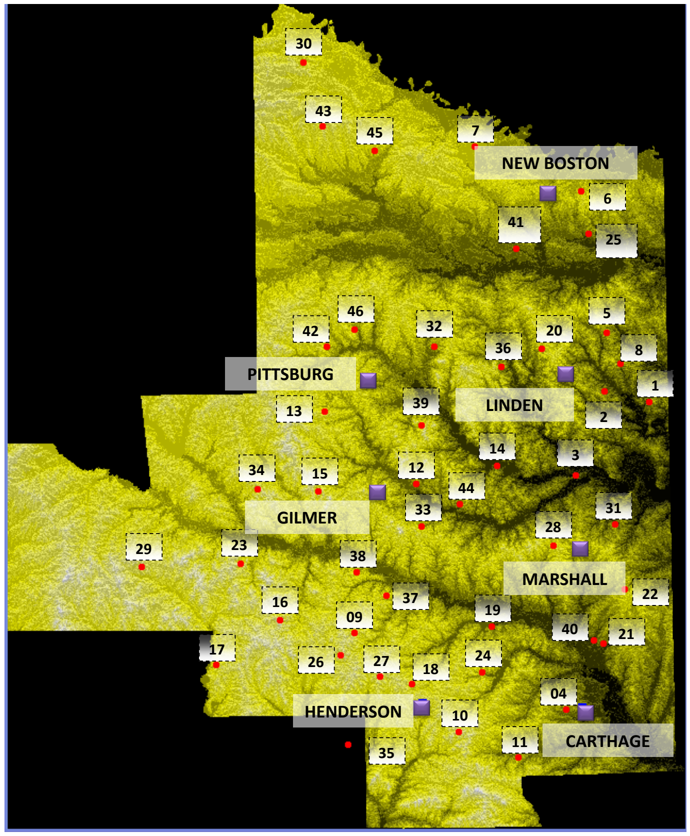

Figure 5.4: Representative fire locations and operations bases in TX12

calculated for each 30 m by 30 m cell on burnable surface within the circular areas, the WE values can be either summed or averaged to obtain an overall estimate of the WE value at each RFL. These estimates are then used to calculate the associated NWE at each RFL.

Fire seasons are the periods of the year during which wildfires are likely to occur and spread while affecting resource values such that organized fire management activities are needed. Fire seasons are composed of fire days (FDs) $d \in \mathcal{D}$. Each FD during a fire season has a *unique pattern* of fire occurrences $f \in \mathcal{F}$ at RFLs. Note that there could be FDs with no fire occurrences. Likewise, a fire scenario (FS) ω is a sequence of FDs during a fire season with an associated probability of occurrence $p(\omega)$ and an overall NWE $e(\omega)$. We focused our study on the summer fire seasons in TX12 for the 22-year period from 1986 and 2006 because of the high frequency of occurrence of multiple fires (2 or more) on a single day (Figure 5.5). The summer fire season for TX12 consists of 92 FD from July 1 to September 30.

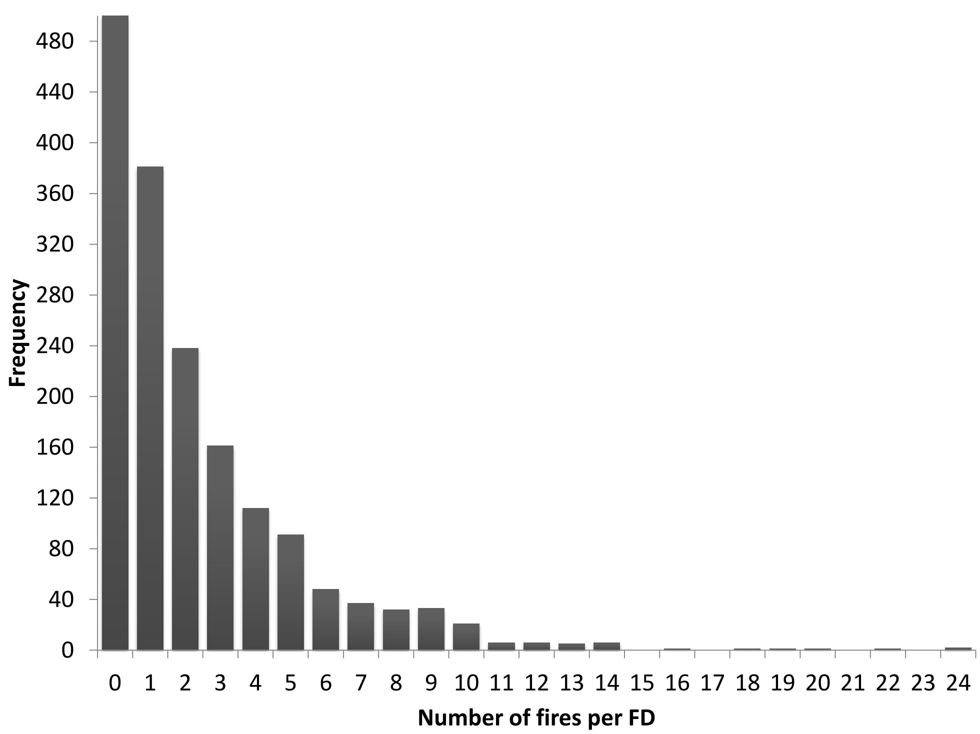

Figure 5.5: Empirical distribution of multiple fires on a single day in TX12

FSs can be created using different methods such as forecasting models or historic data. In this work, we created 22 FS using the 46 RFLs and the historical fire data for the period between 1985 and 2006 such that each year represents a FS. FSs can be sampled using a uniform distribution since there are not two or more years with exactly the same sequence of FDs at the RFLs, that is, each FSs is unique. In addition, FSs are weighted using a risk value $r(\omega)$ representing an overall measure of wildfire risk for a particular FS ω. This value is computed by multiplying the overall NWE times its probability of occurrence ($p(\omega) \times e(\omega)$) and then normalized to be between zero and one. The probability distribution of the normalized risk values associated with all FSs $\omega \in \Omega$ is the one imposed to the constraint (5.2e) in SIP-CW. Figure 5.6 summarizes how this probability distribution is computed.

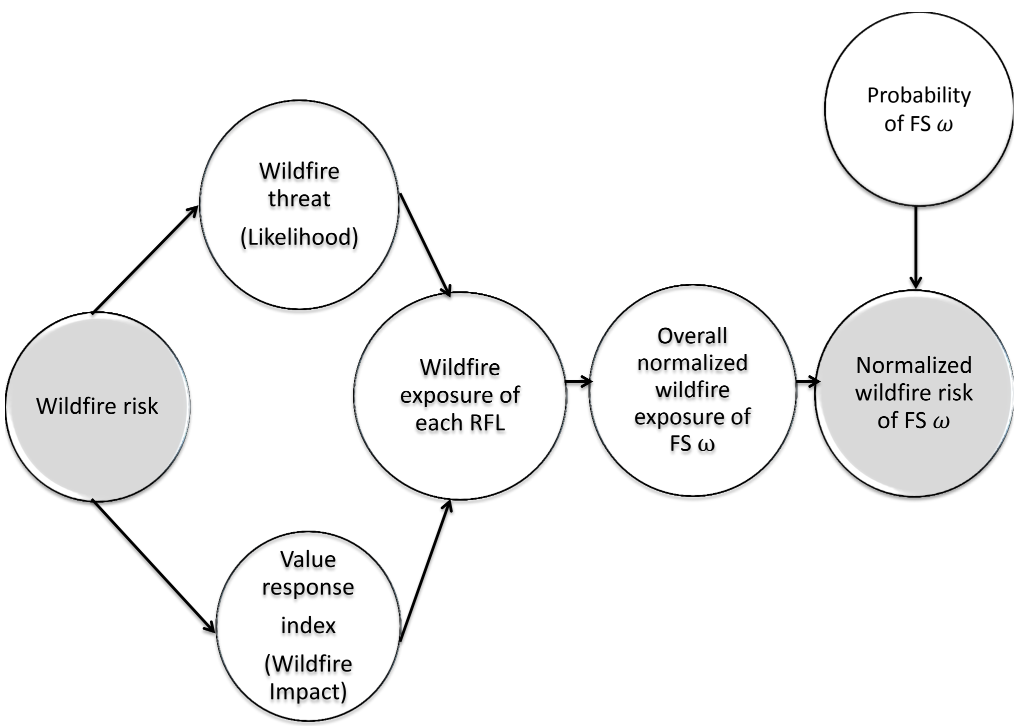

Figure 5.6: Normalized wildfire risk measure

SIP-CW for wildfire initial response requires the selection of a risk attitude level β. This establishes the decision maker's tendency towards risk when making decisions for wildfire initial response planning. For example, a *risk-seeking* decision maker would choose a risk attitude level close to one. Consequently, wildfire initial response planning decisions would be made using the fewest number of scenarios. Likewise, a *risk-averse* decision maker would choose a risk attitude level close to zero. As a result, wildfire initial response planning decisions would be made using as many scenarios as possible. The size of SIP-CW in terms of the number of decision variables and constraints is proportional to the number of scenarios considered for decision making. In other words, the higher the number of scenarios the larger the size of SIP-CW, and the harder it is to solve. Therefore, it is up to the decision maker to decide the appropriate risk attitude level depending on the number of available resources and the number of fires during a fire season receiving a standard response or not.

5.5.3 Experimental Design and Software Implementation

We designed four different sets of experiments to test the research goals stated above. First, we considered DEP-W while excluding the budgetary constraint (5.3d) and the constraints related to the minimum number of fires receiving a standard response (5.3g). Second, we studied DEP-W in which the budgetary constraint (5.3d) is included and the constraints (5.3g) are excluded. Third, we considered DEP-W while only excluding the budgetary constraint (5.2d). We studied an additional set of experiments in which all constraints in DEP-W are included.

We based the computational study on the seven operations bases and resources available in TX12. These bases are located at Pittsburg, Marshall, Carthage, Linden, Gilmer, Henderson, and New Boston. We used an initial set of 18 dozers (resources) in TX12 with dozer-line rates ranging from 1.03 to 3.18 kilometers per hour (km/hr) (Table 5.5). These are standard values from the National Wildfire Coordinating

Group *Fire Handbook* [51]. We allowed each resource to be redeployed from any of the seven operations bases to any other base within TX12. We consider an average relocation cost of $100 per resource since, within its limits, any location in TX12 can be reached from any other location in less than 60 minutes.

In our model we also allow for 20 additional resources located at a dummy base (Table 5.6). Since this dummy base does not exist, the additional resources assigned to it must be deployed to any of the other seven operations bases before they can be actually dispatched to any RFL. The relocation of resources from the dummy base to any other operations base in TX that can be done at zero cost allows us to avoid any bias when comparing the deployment plans provided by the new methodology with the current deployment plan for firefighting resources in TX12.

Table 5.5: Set of initial resources in TX12 at the operations bases

Base	**Resource** ID	**Description**	**Dozer** Type	**Prod.** (km/hr)	**Fixed Cost** ($)	**Supp. Cost** ($/hr)
Pittsburg	1	JD 400G 1989	III	1.03	350	250
	2	JD 450D 1984	III	1.77	350	250
Marshall	3	JD 400G 1990	III	1.03	350	250
	4	JD 450G 1996	III	1.77	350	250
Carthage	5	JD 400G 1991	III	1.03	350	250
	6	JD 450E 1984	III	1.77	350	250
Linden	7	JD 400G 1992	III	1.03	350	250
	8	JD 450G 1991	III	1.77	350	250
Gilmer	9	JD 400G 1992	III	1.03	350	250
	10	JD 450H 2000	III	1.77	350	250
Henderson	11	JD 400G 1991	III	1.03	350	250
	12	JD 450E 1991	III	1.77	350	250
	13	JD 450G 1996	III	1.77	350	250
New Boston	14	JD 400G 1992	III	1.03	350	250
	15	JD 400G 1989	III	1.03	350	250
	16	JD 450G 1994	III	1.77	350	250
	17	JD 550 2002	III	2.15	350	250
	18	JD 750 2002	I	3.18	350	250

All four sets of experiments assume that both the initial 18 dozers and the additional 20 dozers are available for deployment and dispatching in TX12. The experiments also assume a 60-minute response time restriction such that all RFLs in TX12 can be reached by at least one dozer since the maximum of the minimum traveling times from all bases to any RFL is 54 minutes (we approximated this to 60 minutes for consistency). Since fire growth data is required by SIP-CW, we simulate fire behavior at each RFL using the four weather scenarios: low, moderate, high and extreme. Afterwards, the simulated fire behavior is sampled for each RFL using a uniform distribution.

Table 5.6: Set of additional resources in TX12 at the dummy base

Base	**Resource** ID	**Description**	**Dozer** Type	**Prod.** (km/hr)	**Fixed Cost** ($)	**Supp. Cost** ($/hr)
Dummy base	19	JD 400G 1989	III	1.03	350	250
	20	JD 400G 1989	III	1.03	350	250
	21	JD 450G 1994	III	1.77	350	250
	22	JD 400G 1989	III	1.03	350	250
	23	JD 400G 1989	III	1.03	350	250
	24	JD 450G 1994	III	1.77	350	250
	25	JD 400G 1989	III	1.03	350	250
	26	JD 400G 1992	III	1.03	350	250
	27	JD 400G 1992	III	1.03	350	250
	28	JD 450G 1994	III	1.77	350	250
	29	JD 400G 1989	III	1.03	350	250
	30	JD 400G 1989	III	1.03	350	250
	31	JD 450G 1994	III	1.77	350	250
	32	JD 400G 1989	III	1.03	350	250
	33	JD 400G 1989	III	1.03	350	250
	34	JD 450G 1994	III	1.77	350	250
	35	JD 400G 1989	III	1.03	350	250
	36	JD 400G 1992	III	1.03	350	250
	37	JD 400G 1992	III	1.03	350	250
	38	JD 450G 1994	III	1.77	350	250

NVC is the net wildfire damage to a given area in monetary terms. We adopt the cost plus net value change (C+NVC) model for wildfire economics [17] to estimate the cost of fire damages. Even though we use the C+NVC concept, SIP-CW can be extended to allow for other fire damage valuation criteria. Following recommendations made by TFS, we assumed an average NVC of $500 per acre. The C+NVC model for wildfire economics will allow us to estimate the future budgetary needs to effectively provide standard response to fires during a summer fire season in TX12.

TFS provided wildfire risk values in terms of aggregated WE corresponding to circular areas of 8000 acres around each RFL. We chose to use aggregated WE values for areas of 8000 acres since wildfire risk around each RFL can be underestimated if either smaller areas or other statistics such as the average are considered instead. Thus, these aggregated WE values can be used to calculate the corresponding NWE at each RFL. Figure 5.7 shows a comparison of the NWE values at each RFL using the sum of WE values within circular areas of 200, 2000, and 8000 acres. The NWE values along with the number of relocated resources at operations bases will help us to clearly identify those regions within TX12 with the highest wildfire risk.

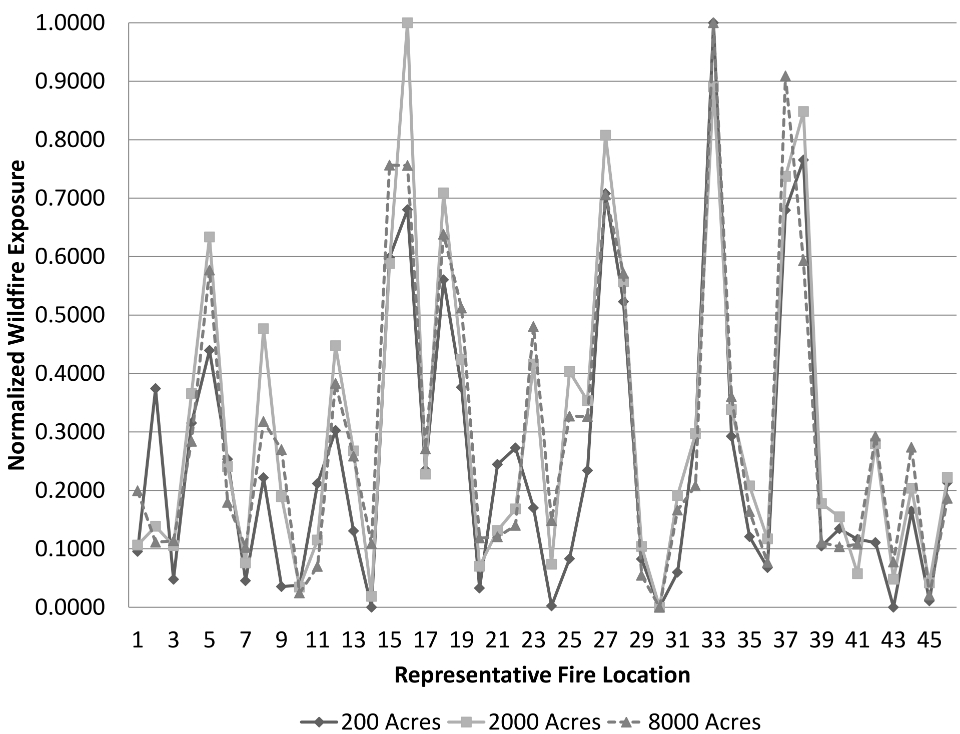

Figure 5.7: NWE within circular areas of 200, 2000, and 8000 acres

We implemented DEP-W in Microsoft Visual C++ 2010 and the CPLEX 12×64 Callable Library [32]. To create problem instances for each of the four sets of experiments, we established different value for FSs, ρ_1, ρ_2, and β. Since the total number of available FSs is 22, we fixed the number of scenarios per instance to 5, 10, 15, 20, and 22. Since $\rho_1 \in [0, 1]$ is the fraction of the overall NWE for a FS ω, and $\rho_2 \in [0, 1]$ is the fraction of the total number of fires receiving a standard response for FS ω, it is desirable to choose ρ_1 and ρ_2 close to one. Thus, we set the values of both ρ_1 and ρ_2 to 0.97, 0.98, 0.99, and 1.00. Since wildfires in Texas are very likely to occur within 2 miles of a community, we assume a *risk averse* tendency with risk attitude levels close to zero. Therefore, β values were set to 0.100, 0.050, 0.025, 0.001 and 0.000 in the experiments.

Each instance of DEP-W was solved using CPLEX 12×64 on a Dell Intel (R) Core (TM) at 2.66 Ghz each with 4.00 GB of RAM. We ran CPLEX with the following settings: node file indicator was set to 3; node selection strategy was set to *best-bound search*; variable selection was set to *strong branching*; clique cut was set to aggressive; and cover cut was set to aggressive. Each instance ran to optimality or was stopped when a CPU time limit of 3,600 seconds (1 hour) was reached.

5.5.4 Computational Results

Computational results for the four different sets of experiments and related instances are shown next. We evaluate deployment and dispatching decisions provided by SIP-CW in terms of the expected number of contained and escaped fires, NWE associated with contained and escaped fires, and total expected cost in terms of fixed rental, relocation and operation cost for different risk attitude levels. We also report on the number of initial and positioned firefighting resources or dozers at each base. The resources initially located at a base before applying SIP-CW are referred to as *initial* resources, and the resources assigned to a base after applying SIP-CW are referred to as *positioned* resources.

We report on instances with 15 FSs and ρ_1, ρ_2 values of 0.97, 0.98, and 0.99 since these computational results are typical for all the different sets of experiments. Instances with 20 and 22 FSs and risk attitude levels greater than or equal to 0.5, and they could not not be solved by CPLEX within the time limit of one hour. In fact, these instances have more than 500,000 constraints and decision variables.

Table 5.7: DEP-W output values for ρ_1=0.97 and 15 fire scenarios

DEP-W Output	β**=0.100**	β**=0.050**	β**=0.025**	β**=0.001**	β**=0.000**
Fires contained	1,288	1,380	1,431	1,464	1,464
Escaped fires	263	282	292	297	297
% of contained fires	83.04	83.03	83.05	83.13	83.13
Normalized wildfire exposure of fires contained	456.38	483.11	494.14	506.50	506.23
Normalized wildfire exposure of escaped fires	13.61	14.42	14.82	15.07	15.34
Fixed rental cost c_1 ($)	13,300	13,300	13,300	13,300	13,300
Relocation cost c_2 ($)	200	200	200	200	200
Operation cost	1,936,500	2,074,500	2,155,500	2,202,000	2,202,000
NVC cost c_{nvc} ($)	772,800	828,000	858,600	878,400	878,400
Total cost ($)	2,722,800	2,916,000	3,027,600	3,093,900	3,093,900
CPLEX CPU time (sec.)	3,602	3,601	625	429	667

We first consider DEP-W while excluding the budgetary constraint (5.2d) and the constraints related to the minimum number of fires receiving a standard response (5.3g). Table 5.7 shows the computational results for this case with ρ_1=0.97 and 15 FSs. The total NWE of fires receiving a standard response increases when the risk attitude level decreases with values ranging from 456.38 to 506.23. This is a consequence of an increasing number of fires contained with values ranging from 1,288 to 1,464. Consequently, the total expected cost increases up to $3,093,900 mainly because of an increasing operation cost since more dozers are utilized to provide standard response.

The fixed rental cost remains constant at \$13,300 (\$350 $\times$ 38) for the different risk attitude levels in Table 5.7. This happens because constraint (5.3b) in DEP-CW ensures that all resources including the additional ones are deployed to at least one operations base. Likewise, the relocation cost is \$200 for all instances due to the relocation of resources 17 and 18 from New Boston to other operations bases since these are the two available resources with the highest dozer-line rates, that is, 2.15 km/hr and 3.18 km/hr, respectively.

The number of escaped fires also increases when the risk attitude level decreases as shown in Table 5.7. Every time the risk attitude level is reduced then more information, in terms of a larger number of FSs, is considered in DEP-W for deployment decisions. Therefore, the percentage of fires contained remains stable around 83% even though more fires are contained. All instances for this set of experiments were solved to optimality within the time limit of 3,600 seconds (1 hour).

Figure 5.8 shows the initial and positioned number of resources at each operations base for the first set of experiments with ρ_1=0.97 and 15 FSs for different risk attitude levels. Pittsburg, Henderson, and Gilmer are in the top three of operations bases with the largest number of positioned resources. For instance, the number of positioned resources at Pittsburg ranges from 7 to 13 with only 2 initial resources while the number of positioned resources at Henderson ranges from 4 to 10 with only 3 initial resources. New Boston is another operations base with a high number of positioned resources. However, unlike Pittsburg, Henderson and Gilmer, this is due to the high number of initial resources at this base, that is, 5. Most of the relocated resources come from the dummy base since the number of positioned resources at this base is zero for all the risk attitude levels.

Table 5.8 shows the computational results for the first set of experiments with ρ_1=0.98 and 15 FSs. The total NWE associated with fires receiving a standard

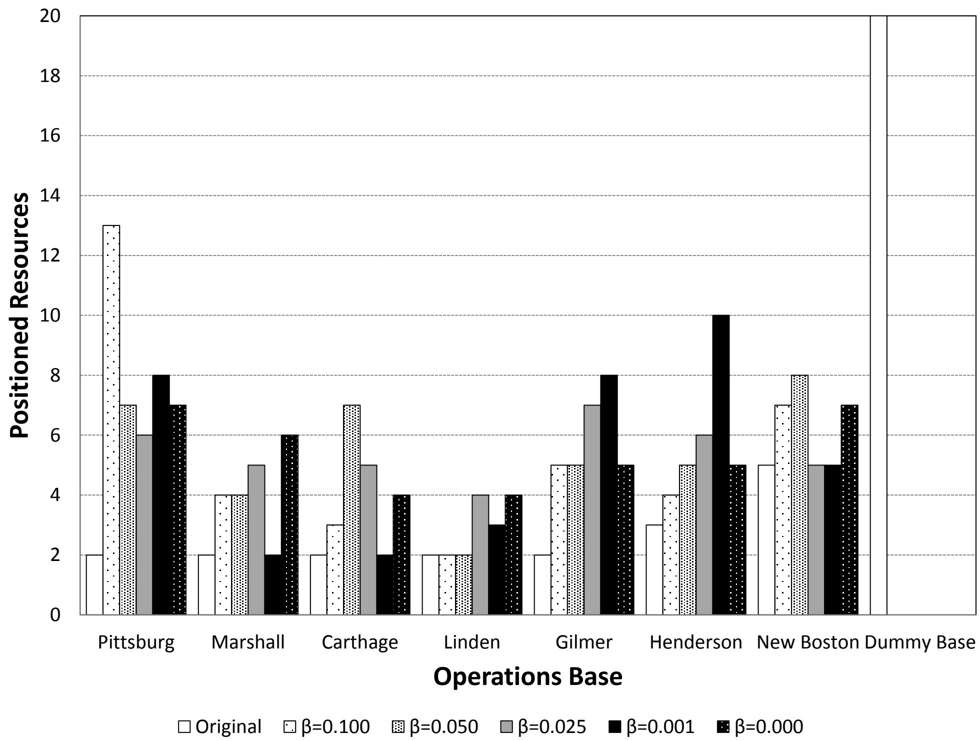

Figure 5.8: Initial versus positioned resources for ρ_1=0.97 and 15 fire scenarios

Table 5.8: DEP-W output values for ρ_1=0.98 and 15 fire scenarios

DEP-W Output	β**=0.100**	β**=0.050**	β**=0.025**	β**=0.001**	β**=0.000**
Fires contained	1,343	1,440	1,493	1,527	1,527
Escaped fires	208	222	230	234	234
% of contained fires	86.59	86.64	86.65	86.71	86.71
Normalized wildfire exposure of fires contained	460.91	488.01	499.02	511.54	511.53
Normalized wildfire exposure of escaped fires	9.09	9.52	9.94	10.03	10.04
Fixed rental cost c_1 ($)	13,300	13,300	13,300	13,300	13,300
Relocation cost c_2 ($)	200	200	200	200	200
Operation cost	2,025,000	2,172,000	2,247,000	2,295,000	2,304,000
NVC cost c_{nvc} ($)	805,800	864,000	895,800	916,200	916,200
Total cost ($)	2,844,300	3,049,500	3,156,300	3,224,700	3,233,700
CPLEX CPU time (sec.)	3,601	3,632	2,087	836	1,456

response increases when the risk attitude level decreases with values ranging from 460.91 to 511.53. Therefore, the number of fires contained also increases with values ranging from 1,343 to 1,464. The latter represents a 4% increase not only in the number of fires contained but also in the total expected cost compared to the case with ρ_1=0.97.

Figure 5.9 shows the initial and positioned number of resources at each operations base for the first set of experiments with ρ_1=0.98 and 15 FSs for different risk attitude levels. Gilmer is now the operations base with the largest number of positioned resources with values ranging from 7 to 9 followed by New Boston, Pittsburg, and Henderson or Carthage in that order. Most of the relocated resources come from the dummy base since the number of positioned resources at this base is zero for all the risk attitude levels.

Table 5.9: DEP-W output values for ρ_1=0.99 and 15 fire scenarios

DEP-W Output	β**=0.100**	β**=0.050**	β**=0.025**	β**=0.001**	β**=0.000**
Fires contained	1,403	1,502	1,557	1,593	1,593
Escaped fires	148	160	166	168	168
% of contained fires	90.46	90.37	90.37	90.46	90.46
Normalized wildfire exposure of fires contained	465.63	492.83	504.21	516.71	516.53
Normalized wildfire exposure of Escaped fires	4.36	4.70	4.75	4.86	5.04
Fixed rental cost c_1 ($)	13,300	13,300	13,300	13,300	13,300
Relocation cost c_2 ($)	200	200	200	200	200
Operation cost	2,116,500	2,263,500	2,356,500	2,398,500	2,401,500
NVC cost c_{nvc} ($)	841,800	901,200	934,200	955,800	955,800
Total cost ($)	2,971,800	3,178,200	3,304,200	3,367,800	3,370,800
CPLEX CPU time (sec.)	3,600	3,155	578	668	810

Table 5.9 shows the computational results for the first set of experiments with ρ_1=0.99 and 15 FSs. The total NWE of fires receiving a standard response increases when the risk attitude level decreases with values ranging from 465.63 to 516.53.

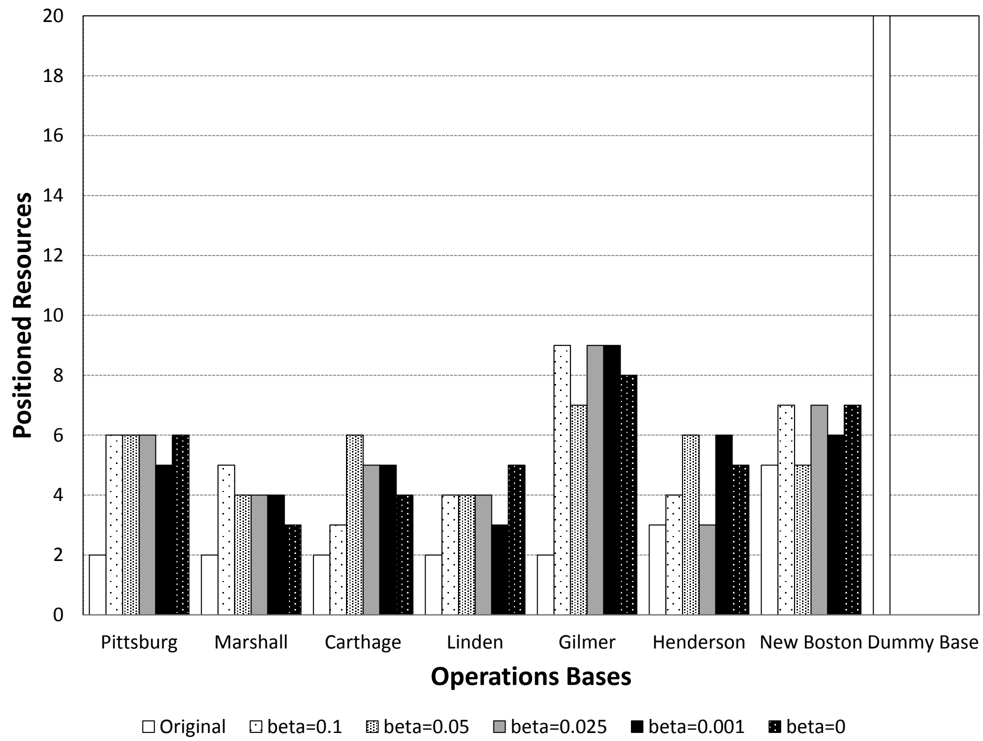

Figure 5.9: Initial versus positioned resources for ρ_1=0.98 and 15 fire scenarios

Therefore, the number of fires contained also increases with values ranging from 1,403 to 1,593. The latter represents a 4% increase not only in the number of fires contained but also in the total expected cost compared to the case with ρ_1=0.98 and almost a 8% increase in both categories for the case with ρ_1=0.97.

Figure 5.10 shows the initial and positioned number of resources at each operations base for the case with ρ_1=0.99 and 15 FSs for different risk attitude levels. Gilmer remains as the operations base with the largest number of positioned resources with values ranging from 4 to 13 followed now by Pittsburg, New Boston, and Henderson in that order. Since the difference in the number of positioned resources is so large between the top two bases and the rest, this result suggests that the region with the highest wildfire risk in TX12 is the one exactly located around and between

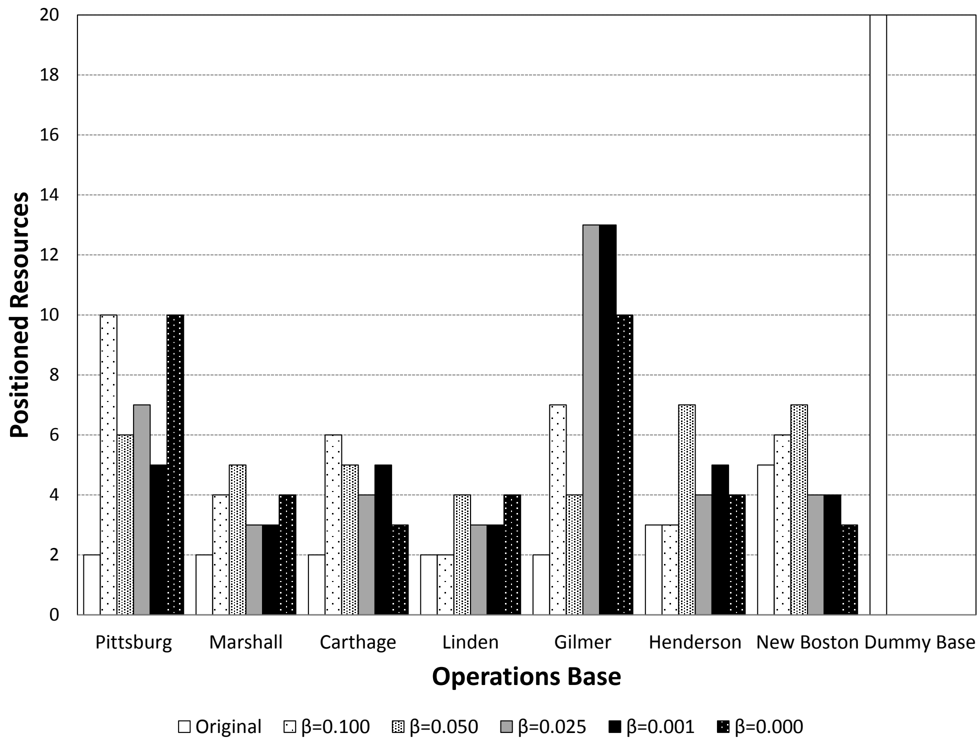

Figure 5.10: Initial versus positioned resources for ρ_1=0.99 and 15 fire scenarios

the operations bases of Gilmer and Pittsburg. Most of the relocated resources come from the dummy base since the number of positioned resources at this base is zero for all the risk attitude levels. The latter suggests that the current deployment plan of resources at TX12 is not sufficient to provide wildfire standard response to all fires during the summer fire season. In fact, the current deployment plan at TX12 has more resources assigned to bases located in regions with a lower wildfire risk such as New Boston.

We now report results on the second set of experiments. Here we consider DEP-W while excluding only the constraint associated with the minimum number of fires receiving a standard response (5.3g). Table 5.10 shows the computational results for this case with ρ_1=0.99, 15 FSs, and a budget of \$3,321,000. Values for the total

Table 5.10: DEP-W output values for ρ_1=0.99, 15 fire scenarios, and a budget of \$3,321,000

DEP-W Output	β**=0.100**	β**=0.050**	β**=0.025**	β**=0.001**	β**=0.000**
Fires contained	1,403	1,502	1,557	0	0
Escaped fires	148	160	166	0	0
% of contained fires	0.90	0.90	0.90	0	0
Normalized wildfire exposure				0	0
of fires contained	465.67	492.93	504.20	0	0
Normalized wildfire exposure					
of escaped fires	4.32	4.60	4.76	0	0
Fixed rental cost c_1 (\$)	13,300	13,300	13,300	0	0
Relocation cost c_2 (\$)	200	200	200	0	0
Operation cost	2,115,000	2,268,000	2,346,000	0	0
NVC cost c_{nvc} (\$)	841,800	901,200	934,200	0	0
Total cost (\$)	2,970,300	3,182,700	3,293,700	0	0
CPLEX CPU time (sec.)	3,715	1,513	1,878	0	0

NWE, number of fires contained, and total expected cost exhibit a similar pattern as the one seen for the case in which the budgetary constraint (5.2d) is excluded. Observe how all these values increase when the risk attitude level decreases, and how no solutions are obtained for instances with a risk attitude level less than or equal to 0.001 since the total cost for for these cases exceed the available budget.

Figure 5.11 shows the initial and positioned number of resources at each operations base for the case with ρ_1=0.99, 15 FSs, and a budget of \$3,321,000 for different risk attitude levels. Similarly to the case with unrestricted budget, Gilmer is the operations base with the largest number of positioned resources with values ranging from 10 to 12 followed by Pittsburg, Marshall, and New Boston in that order. This shows that when a budget is imposed, then the model prioritizes resource deployment to the base located in the region with the highest wildfire risk, in this case Gilmer, while evenly distributing resources among those bases located at regions with a medium and low-level of wildfire risk. As before, most of the relocated resources come from the dummy base since the number of positioned resources at this base is zero for all the risk attitude levels.

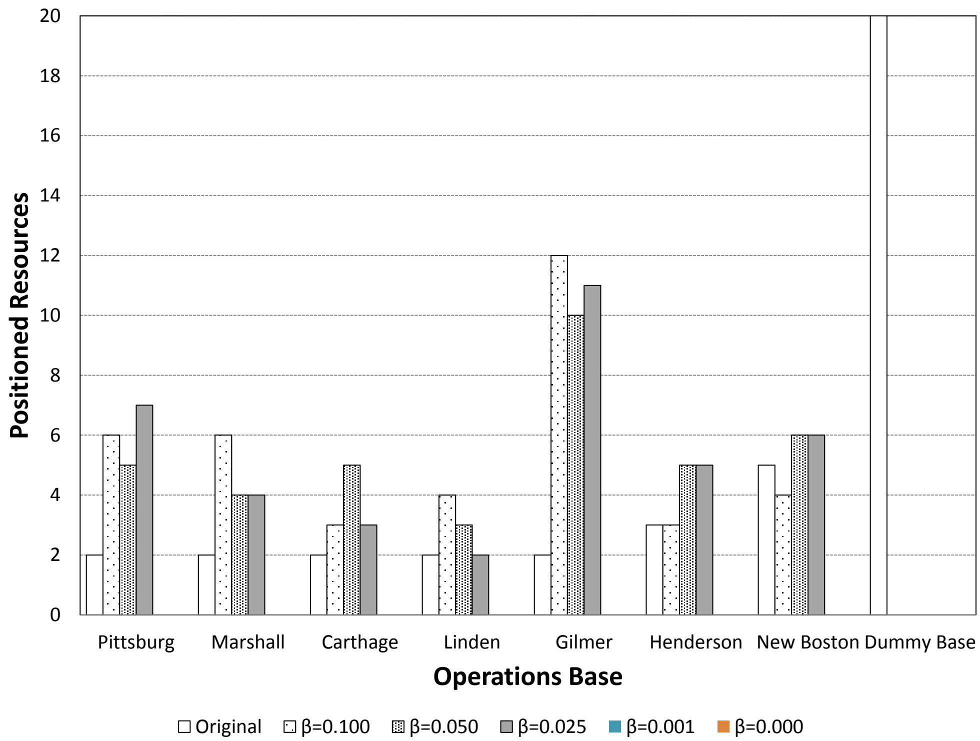

Figure 5.11: Initial versus positioned resources for ρ_1=0.99, 15 fire scenarios, and a budget of \$3,321,000

Table 5.11: DEP-W output values for ρ_1=0.97, ρ_2=0.97, and 15 fire scenarios

DEP-W Output	β**=0.100**	β**=0.050**	β**=0.025**	β**=0.001**	β**=0.000**
Fires contained	1,510	1,618	1,678	1,715	1,715
Escaped fires	41	44	45	46	46
% of contained fires	97.36	97.35	97.39	97.39	97.39
Normalized wildfire exposure of fires contained	459.22	485.91	496.59	509.17	509.41
Normalized wildfire exposure of escaped fires	10.77	11.62	12.37	12.40	12.16
Fixed rental cost c_1 (\$)	13,300	13,300	13,300	13,300	13,300
Relocation cost c_2 (\$)	100	100	100	100	200
Operation cost	2,265,000	2,427,000	2,517,000	2,572,500	2,572,500
NVC cost c_{nvc} (\$)	906,000	970,800	1,006,800	1,029,000	1,029,000
Total cost (\$)	3,184,400	3,411,200	3,537,200	3,614,900	3,615,000
CPLEX CPU time (sec.)	4,162	908	2,085	1,468	1,591

We now present results for the third set of experiments. Here only the budgetary constraint (5.2d) in DEP-W is excluded. Observe that constraints (5.3e), (5.3f), (5.3g), and (5.3h) are now included in the model. Table 5.11 shows the computational results for this case with ρ_1=0.97, ρ_2=0.97, and 15 FSs. The total NWE of fires receiving a standard response increases when the risk attitude level decreases with values ranging from 459.22 to 509.23. This is a consequence of an increasing number of fires contained with values ranging from 1,510 to 1,715. This constitutes a 17% increase in the number of fires contained with respect to the case in which only ρ_1=0.97 is considered in the first set of experiments. In addition, the total expected cost increases up to $3,615,000 for a *zero* risk attitude level. This value is 17% larger than the total expected cost obtained for the case in which only ρ_1=0.97 is considered. As expected, the percentage of fires contained is around 97% for all instances due to $\rho_2 = 0.97$. All instances were solved to optimality within the time limit of 3,600 seconds (1 hour) with the exception of the instance corresponding to β =0.1000 whose solution was obtained after 4,162 seconds.

Figure 5.12 shows the initial and positioned number of resources at each operations base for the third set of experiments with ρ_1=0.97, ρ_2=0.97, and 15 FSs for different risk attitude levels. Gilmer and Henderson are the bases with the largest number of relocated resources followed closely by Pittsburg. For instance, the number of positioned resources at Gilmer ranges from 6 to 9 with only 2 initial resources while the number of positioned resources at Henderson ranges from 5 to 7 with only 3 initial resources. This result suggests that the area between and around Gilmer and Henderson is the region with the highest density of historical fires in TX12 and one of the areas with the highest wildfire risk within the same Texas district. Most of the relocated resources come from the dummy base since the number of positioned resources at this base is zero for all the risk attitude levels.

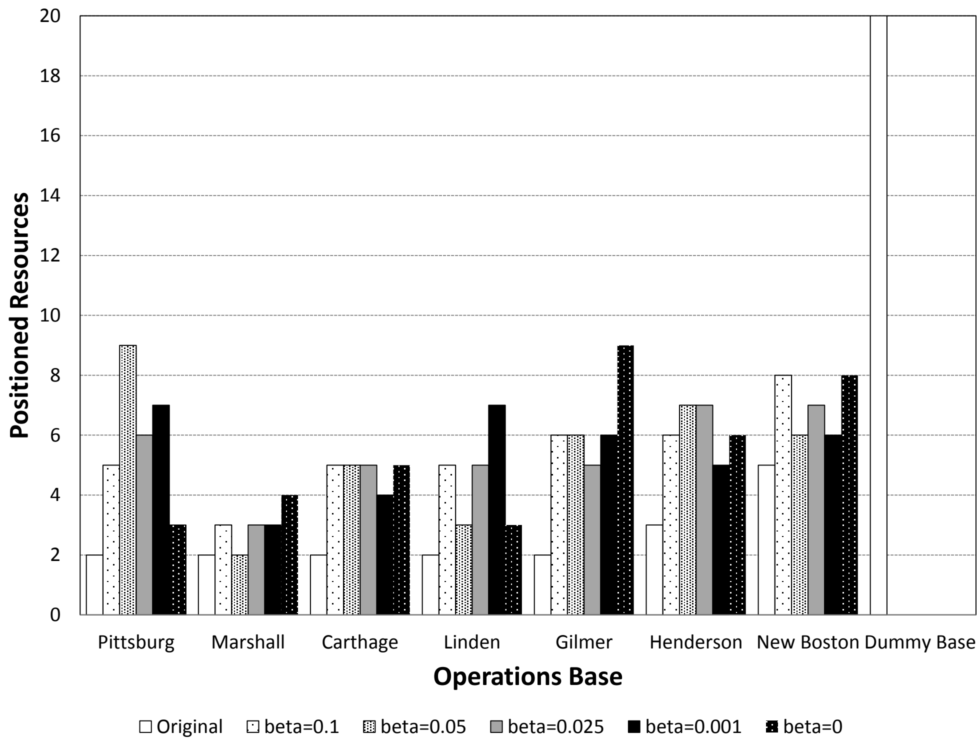

Figure 5.12: Initial versus positioned resources for ρ_1=0.97, ρ_2=0.97, and 15 fire scenarios

Table 5.12: DEP-W output values for ρ_1=0.99, ρ_2=0.99, and 15 fire scenarios

DEP-W Output	β**=0.100**	β**=0.050**	β**=0.025**	β**=0.001**	β**=0.000**
Fires contained	1,542	1,652	1,713	1,751	1,751
Escaped fires	9	10	10	10	10
% of contained fires	99.42	99.40	99.42	99.43	99.43
Normalized wildfire exposure of fires contained	468.28	495.91	507.06	518.67	519.08
Normalized wildfire exposure of escaped fires	1.71	1.62	1.90	2.89	2.49
Fixed rental cost c_1 ($)	13,300	13,300	13,300	13,300	13,300
Relocation cost c_2 ($)	200	200	200	200	200
Operation cost	2,313,000	2,484,000	2,586,000	2,637,000	2,647,500
NVC cost c_{nvc} ($)	925,200	991,200	1,027,800	1,050,600	1,050,600
Total cost ($)	3,251,700	3,488,700	3,627,300	3,701,100	3,711,600
CPLEX CPU time (sec.)	1,310	988	261	311	311

Table 5.12 shows the computational results for the third set of experiments with ρ_1=0.99, ρ_2=0.99, and 15 FSs. The total NWE of fires receiving a standard response increases when the risk attitude level decreases with values ranging from 468.28 to 519.08. Therefore, the number of fires contained also increases with values ranging from 1,542 to 1,751. The latter represents almost a 20% increase in the number of fires contained compared to the case where only ρ_1=0.97 is considered in the first set of experiments. Besides, the total expected cost increases up to $3,711,600 for a *zero* risk attitude level. As expected, the percentage of fires contained is now more than 99% for all instances due to $\rho_2 = 0.99$.

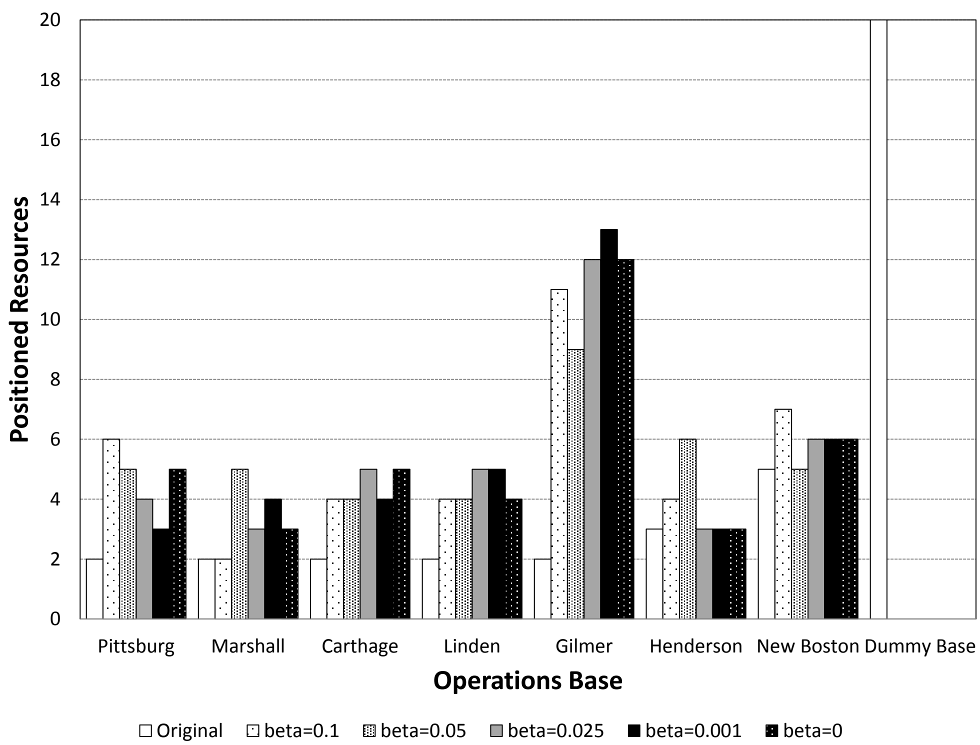

Figure 5.13: Initial versus positioned resources for ρ_1=0.99, ρ_2=0.99, and 15 fire scenarios

Figure 5.13 shows the initial and positioned number of resources at each operations base for the third set of experiments with ρ_1=0.99, ρ_2=0.99, and 15 FSs for different risk attitude levels. It is clear that Gilmer is the operations base with the largest number of positioned resources while the remaining resources are evenly distributed among the other operations bases. This result suggests that the area around Gilmer is the region in TX12 not only with the highest wildfire risk but also with the highest density of historical fires. Most of the relocated resources come from the dummy base since the number of positioned resources at this base is zero for all the risk attitude levels. The latter suggests a current deployment plan of resources for TX12 that is not consistent with the actual wildfire risk profile of TX12.

Table 5.13: DEP-W output values for ρ_1=0.99, ρ_2=0.99, 15 fire scenarios, and a budget of $3,600,000

DEP-W Output	β=**0.100**	β=**0.050**	β=**0.025**	β=**0.001**	β=**0.000**
Fires contained	1,542	1,652	0	0	0
Escaped fires	9	10	0	0	0
% of contained fires	0.99	0.99	0	0	0
Normalized wildfire exposure of fires contained	467.54	495.88	0	0	0
Normalized wildfire exposure of escaped fires	2.45	1.65	0	0	0
Fixed rental cost c_1 ($)	13,300	13,300	0	0	0
Relocation cost c_2 ($)	200	200	0	0	0
Operation cost	2,316,000	2,485,500	0	0	0
NVC cost c_{nvc} ($)	925,200	991,200	0	0	0
Total cost ($)	3,254,700	3,490,200	0	0	0
CPLEX CPU time (sec.)	1,313	787	0	0	0

We now report results on the fourth set of experiments. Here we consider DEP-W without excluding any of the constraint. Table 5.13 shows the computational results for this case with ρ_1=0.99, ρ_2=0.99, 15 FSs, and a budget of $3,600,000. Values for the total NWE, number of fires contained, and total expected cost increase when the risk attitude level decreases. No solutions are obtained for instances with a risk attitude level less than or equal to 0.025 since the total cost for for these cases exceed the available budget. All feasible instances were solved to optimality within the time limit of 3,600 seconds (1 hour).

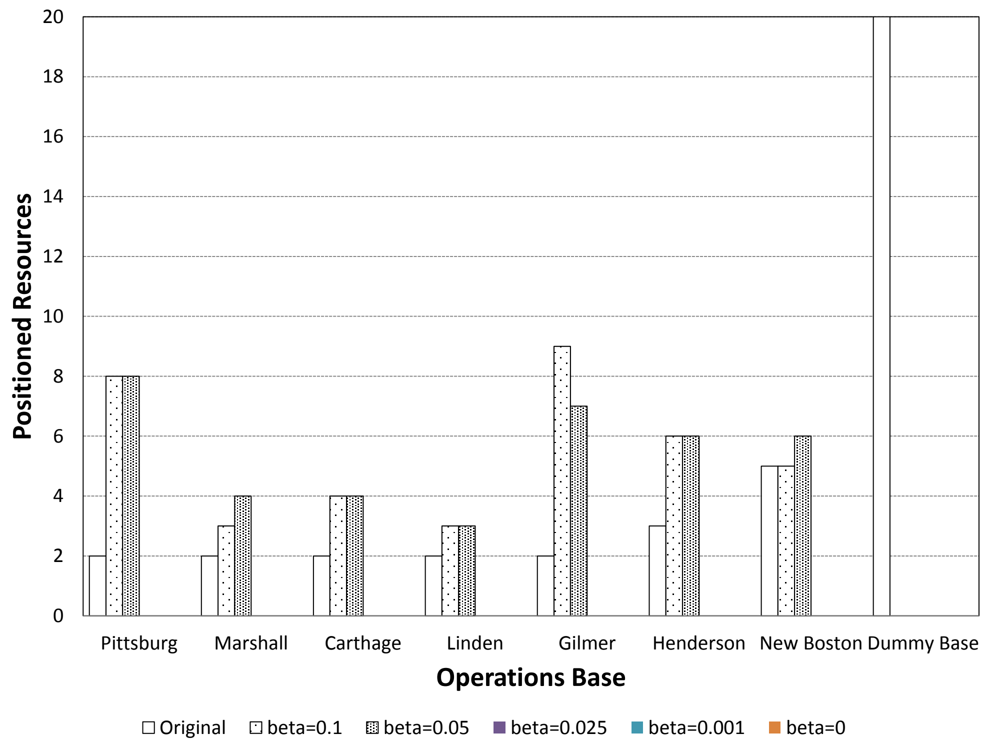

Figure 5.14: Initial versus positioned resources for ρ_1=0.99, ρ_2=0.99, 15 fire scenarios, and a budget of \$ 3,600,000

Figure 5.14 shows the initial and positioned number of resources at each operations base for the fourth set of experiments with ρ_1=0.99, ρ_2=0.99, 15 FSs, and a budget of \$3,600,000 for different risk attitude levels. Results show that Gilmer is the operations base with the largest number of positioned resources with values ranging from 7 to 9 followed by Pittsburg, Henderson, and New Boston in that order. This shows that when a budget is imposed, then the model prioritizes resource deployment to bases located in the region with the highest wildfire risk, in this case between Gilmer and Pittsburg, while evenly distributing resources among those bases with a medium and low-level of wildfire risk. As before, most of the relocated resources come from the dummy base since the number of positioned resources at this base is zero for all the risk attitude levels.

5.6 Discussion

The methodology presented in this work yields quantitative plans regarding the deployment of firefighting resources or dozers to operations bases in a fire planning unit and the dispatching of the deployed resources from these bases to fires before they occur. We used the probabilistically constrained SIP model (SIP-CW) to allow for making decisions that hedge against uncertain future fire occurrences and behavior. Specifically, SIP-CW makes decision plans regarding deployment and dispatching while considering wildfire risk, that is, by prioritizing assignment of resources to operations bases located in areas where potential fire occurrences would have a higher impact on homes and other valuable resources.

SIP-CW also calculates the fixed rental costs, relocation costs, operations cost and NVC associated with the deployment and dispatch decisions. In addition, SIP-CW computes the expected number of contained fires, expected number of escaped fires due to lack of response capacity, and wildfire risk associated with fires receiving and not receiving a standard response. The information produced by the new methodology is useful for planning future resource needs for a fire planning unit based on weather forecasts, and for projecting future budgetary needs.

A DEP-W formulation to SIP-CW can be written since the multivariate random vector that characterizes the uncertainties in SIP-CW has discrete probability distribution and finite support. Four different set of experiments were designed using DEP-W. First, we considered DEP-W while relaxing the budgetary constraint (5.3d) and the constraints related to the minimum number of fires receiving a standard response within the probabilistic constraint (5.3g). Second, we studied the same formulation considered in the first set of experiments but now including the budgetary constraint (5.3d). Then, we considered DEP-W while only relaxing the budgetary constraint (5.3d). Finally, we considered an additional set of experiments in which we studied DEP-W including all the constraints. Afterwards, DEP-W to several instances for each set of experiments were solved using CPLEX 12×64. These instances considered different number of scenarios, various risk attitude levels, and different fractional values of the total NWE and number of fires associated with a FS.

We experimented with varying the risk attitude level for different instances. In all cases, we found that both the total NWE and the total number of fires contained increase when the risk attitude level decreases with percentages of fires contained up to 99.39%. As expected, every time the risk attitude level is reduced, due to a tendency towards *risk averseness*, then more fires receive standard response especially those occurring at regions where the wildfire risk is higher. Therefore, the total expected cost increased in all instances when the risk attitude level was decreased with values up to $ 3,700,000. In fact, if more fires receive standard response, then more firefighting resources are used which makes the operation cost to increase as well. Almost all instances were solved to optimality within the time limit of 3,600 seconds (1 hour).

Computational results showed that the initial distribution of resources at the time of this study is not consistent with the wildfire risk profile of TX12. This also shows that the current deployment plan of resources is not sufficient to provide wildfire standard response to all fires during the summer fire season in TX12. Indeed, the number of positioned resources at either Gilmer, Pittsburg, or Henderson goes up to 13 with only 2 or 3 initial resources while the number of positioned resources at other operations bases such as New Boston goes up to 8 with 5 initial resources. In addition, results show that when a budget is imposed, then the model prioritizes resource deployment to bases located in the region with the highest wildfire risk, in this case between Gilmer and Pittsburg, while evenly distributing resources among those bases with a medium and low-level of wildfire risk

We identified three important regions within TX12 regarding wildfire risk and density of historical fires based on the distribution of positioned resources described above. First, the area between and around Gilmer and Pittsburg is identified as the region with the highest wildfire risk in TX12. Observe that this area is not necessarily the region with the highest density of historical fires. Second, the area between and around Gilmer and Henderson is recognized as the region with the highest density of historical fires in TX12. This area is not necessarily the region with the highest wildfire risk. Therefore, the area around Gilmer is identified not only as the region with the highest density of historical fires but also with the highest wildfire risk. This is exactly the region where the two areas described above intersect as shown in Figure 5.15.

There are several possible directions for future work. For example, this methodology can be extended to include other stages in wildfire emergency response planning such as mitigation, extended response and evacuation planning. In addition, the probability of escaped fires or response failure probability [3] can be incorporated in the definition of standard response. This methodology can also be easily extended to allow for the location of new operations bases due to storage capacity or wildfire initial response time restrictions.

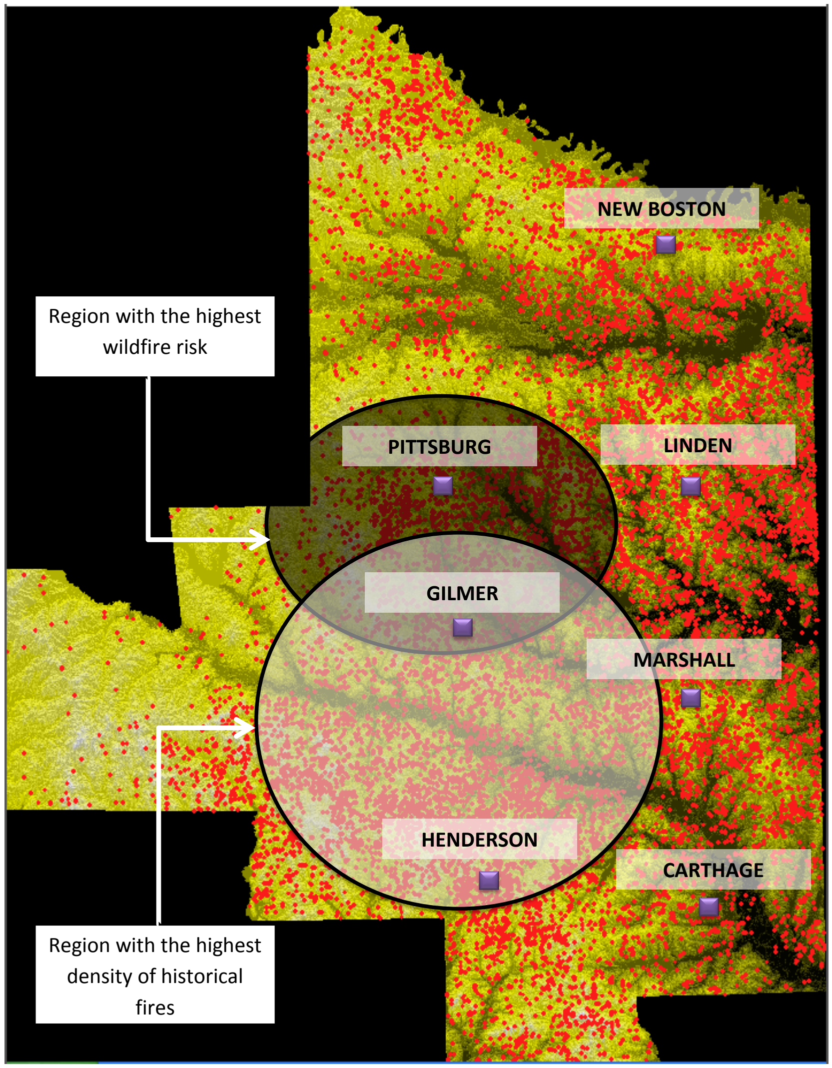

Figure 5.15: Wildfire risk profile of TX12

6. CONCLUSION

6.1 Summary

This dissertation presents methods for finding irreducible infeasible subsystems (IISs) of systems of inequalities with binary decision variables and for solving probabilistically constrained stochastic integer programs (SIP-C). Finding IISs for binary systems is useful in decomposition methods for SIP-C. SIP-C has many important applications including modeling of strategic decision-making problems in wildfire initial response planning. The research contributions of this dissertation can be summarized as follows:

- A method for finding IISs of systems of linear inequalities with binary variables using branch-and-bound (BAB) and heuristic approaches.
- A branch-and-cut (BAC) method using IIS inequalities to solve SIP-C with random technology matrix and random righthand-side vector along with preliminary computational results obtained from an implementation using Microsoft Visual C++ and CPLEX 12×64 Callable Library [32].
- A methodology for wildfire initial response planning that integrates a fire behavior simulation model, a wildfire risk model, and SIP-C in association with computational results obtained from an implementation using Microsoft Visual C++, CPLEX 12×64 Callable Library [32], and BehavePlus [5].

New theoretical results and two new algorithms to find IISs for systems of inequalities with binary variables were developed. The theoretical results show that an IIS for a binary system can be obtained as subset of an IIS to an associated system of inequalities with unrestricted decision variables. The first algorithm, called the *IIS-BAB algorithm*, uses the new theory and the method of the alternative polyhedron [23] within a branch-and-bound (BAB) approach. The second algorithm, termed the *IIS-Heuristic algorithm*, applies the new theory and the method of the alternative polyhedron [23] to a system in which zero/one box constraints are appended.

IISs for binary systems can be used to derive decomposition schemes to solve SIP-C. A BAC method based on IIS inequalities for SIP-C with random technology and random righthand-side vector was devised in this work. The IIS inequalities were first used to model the *minimum cardinality IIS set-covering problem* (MIN IIS COVER). Computational results provided empirical evidence that the IIS inequalities strengthen the LP relaxation of the DEP to SIP-C.

SIP-C can be used to model strategic decision-making problems in wildfire initial response planning. A new methodology for wildfire initial response was presented in this dissertation. This methodology includes a fire behavior simulation model, a wildfire risk model, and SIP-C, and it assumes a known standard response needed to contain a fire of given size. The new methodology can be used to evaluate deployment decisions in terms of the number of firefighting resources positioned at each base, the expected number of escaped and contained fires, as well as the wildfire risk associated with fires not receiving a standard response.

A study based on Texas district 12 (TX12), which is one of the Texas A&M Forest Service (TFS) fire planning units in east Texas, provided several insights into the deployment decisions made by the new methodology. For instance, computational results showed that the original distribution of resources at the time of this study is not consistent with the actual wildfire risk profile of TX12. Indeed, more resources are deployed to operations bases located in regions with low wildfire risk while less resources are allocated to operations bases located in regions with high wildfire risk.

6.2 Future Research

6.2.1 IIS Decomposition Method

Although the BAC method using IIS inequalities showed promising computational results for solving probabilistically constrained SIP, there is still room for improvement since neither this method nor CPLEX was able to find feasible or optimal solutions for all the instances considered in the computational experiments. Thus, the IIS decomposition method presented in Chapter 4 can be improved by strengthening the IIS inequality or by using additional inequalities that include the decision variable x. The IIS decomposition method can be also enhanced by obtaining new inequalities with facet-defining properties for the set of feasible points of the DEP to SIP-C.

6.2.2 Wildfire Initial Response Planning

There are several possible directions for future work regarding wildfire initial response. For example, the methodology described in this work can be extended to allow actions such as staging and/or the location of new operations bases due to storage capacity or initial response time restrictions. Likewise, the probability of escaped fires or response failure probability [3] can be also incorporated in the definition of standard response. Another direction would be to extend the methodology for wildfire initial response planning to a multi-stage decision-making setting. In this case, deployment decisions can be made in the first stage and relocation and dispatching decisions can be made in subsequent stages based on wildfire growth and risk.

Wildfire emergency response planning involves several phases: mitigation, initial response planning, which is the one considered in this work, extended response planning and evacuation planning. The goal of the mitigation phase is to prevent potentially harmful fires. Decisions related to fire containment are made during the initial response planning phase. This phase lasts during the first eight hours after a fire is reported. The extended response phase deals with all the decisions made after the initial response phase, and its goal is to fully suppress a fire or group of fires. Finally, decisions associated with evacuation planning represent an integral aspect of protecting public safety in locations where intense, fast-spreading forest fires co-occur with human populations

Methods for wildfire emergency response planning that involves all its phases have not been presented in the literature yet. Note that strategic decision planning regarding deployment of resources at any stage will affect the effectiveness of the decisions made at any other phase within the wildfire emergency planning process. For example, the deployment plans made for initial response will determine the immediate number of resources available for extended response planning. Therefore, there is a need for new models to assist fire managers in making deployment and dispatching plans for emergency response planning.

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