PRESENTATIONS AND STRUCTURAL PROPERTIES OF SELF-SIMILAR GROUPS AND GROUPS WITHOUT FREE SUB-SEMIGROUPS

A Dissertation

by

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ABSTRACT

This dissertation is devoted to the study of self-similar groups and related topics. It consists of three parts.

The first part is devoted to the study of examples of finitely generated amenable groups for which every finitely presented cover contains non-abelian free subgroups. The study of these examples was motivated by natural questions about finiteness properties of finitely generated groups. We show that many examples of amenable self-similar groups studied in the literature cannot be covered by finitely presented amenable groups. We investigate the class of contracting self-similar groups from this perspective and formulate a general result which is used to detect this property. As an application we show that almost all known examples of groups of intermediate growth cannot be covered by finitely presented amenable groups. The latter is related to the problem of the existence of finitely presented groups of intermediate growth.

The second part focuses on the study of one important example of a self-similar group called *the first Grigorchuk group* $\mathcal{G}$, from the viewpoint of profinite groups. We investigate finite quotients of this group related to presentations and group (co)homology. As an outcome of this investigation we prove that the the profinite completion $\widehat{\mathcal{G}}$ of this group is not finitely presented as a profinite group.

The last part focuses on a class of recursive group presentations known as L-presentations, which appear in the study of self-similar groups. We investigate the relation of such presentations with the normal subgroup structure of finitely presented groups and show that normal subgroups with infinite cyclic quotient of finitely presented groups have such presentations. We apply this result to finitely presented indicable groups without free sub-semigroups.

DEDICATION

To my parents

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1. INTRODUCTION

The notion of self-similarity has various manifestations in nature, art, physics and mathematics. Although self-similarity established its place as a central notion in fractal geometry and dynamical systems long ago, its appearance in algebra is rather new and starts with the notion of a self-similar group. This class, which has origins in the theory of automata, caught attention at the end of 1970's when mathematicians started discovering examples with unusual properties belonging to this class. Such examples were either unknown or were not easy to construct. Roughly speaking a self-similar group is a group of automorphisms of a regular rooted tree which in some sense inherits the self-similar geometric structure of the tree. Such groups have been shown to play a central role in various problems in mathematics.

Among these is the celebrated Burnside Problem posed by the British mathematician William Burnside in 1902. He asked whether a group, generated by finitely many elements each having finite order, needed to be finite. This question, one of the most influential problems in the history of groups, was eventually answered to be negative by E.S. Golod [Gol64] using the Golod-Shafarevich construction. Although many researchers hinted at the importance of self-similar groups before, the first concrete constructions came in the beginning of 80's. One simple and ingenious construction due to R. Grigorchuk [Gri80] sparked the interest in the class of self-similar groups. This group, known today as the *first Grigorchuk Group*, serves as a prototype for the theory of self-similar groups in the years since. In [Gri80] it was observed that it is a very simple counter example to the Burnside problem. Its striking property of having intermediate growth was shown in [Gri84], answering a question posed by John Milnor in [Mil68a] and initiating the study of *groups of* *intermediate growth*.

Throughout the years, many connections with other areas of mathematics have been established and problems in diverse fields have been answered with the help of ideas emanating from the theory of self-similar groups.

It was observed that self-similar groups play an important role in dynamical systems and ergodic theory. One striking discovery was that such groups appear naturally in complex dynamics and provide tools for attacking unanswered problems in this area.

A notion, which has its origins in self-similar groups, is the notion of *branching*. This property, which in a sense is *dual* to self-similarity, led to the introduction of the class of *branch groups* which plays an important role in the theory of infinite groups.

Many problems regarding amenability, a key notion in group theory, have been answered via examples of self-similar groups. Among these is the problem posed by M. Day regarding elementary amenable groups [Day57].

Connections to other parts of mathematics such as spectral, algorithmic and combinatorial problems, random walks and coding theory have been established throughout the years. We refer to [GŠ07] and [Gri11a] for nice expositions about self-similar groups.

In this dissertation, we will touch upon various points of the theory of self-similar groups. A strong theme is problems regarding presentations of such groups from different viewpoints. The results are discussed in three main chapters, each having its own short introduction into the history of the problem under consideration.

The dissertation is organized as follows: Chapter 2 contains preliminaries and background on various topics related to the material that will be discussed in the forthcoming chapters. Chapter 3 is devoted to the discussion of the results published in [BGDLH13] written in collaboration with Rostislav Grigorchuk and Pierre De La Harpe. Although most proofs follow the lines of [BGDLH13], various points have been altered to unify it with the whole of the dissertation. More space has been devoted to various parts especially to automatically presented groups. Also the comprehensive appendix of [BGDLH13] is not reproduced here for considerations of integration and length, but the author encourages interested readers to utilize this well written appendix. Chapter 4 is related to the results of [Ben12b]. It follows the structure of [Ben12b] but some proofs have been shortened and some have been explained in more detail. Also small typos that were undetected in [Ben12b] have been corrected with clear indications. Chapter 5, is about the results published in [Ben12a]. It is completely rewritten with more insight into the topic and more examples. Yet, the proof of main theorem follows the lines of [Ben12a].

2. PRELIMINARIES

This chapter contains basics and preliminaries which will be used in the main parts of the dissertation. The author tried to be as self-contained as possible.

2.1 Rooted trees and their automorphisms

Let $X = \{0, \dots, d-1\}$ be a finite set. X will serve as an *alphabet* for our purposes and the numerical values of its elements are immaterial. We will denote by X^* the set consisting of finite sequences (or words) over X, including the empty sequence. In other words, X^* is the *free monoid* generated by X with the binary operation of *concatenation* of finite sequences. X^* has the geometric structure of a *regular rooted tree of degree d*: The empty sequence is at the root and sequences of length one are on the first level, each connected by an edge to the root vertex. In general, each sequence $w \in X^*$ is connected by an edge to its *children* $\{xw \mid x \in X\}$. This gives a bijection between X^* and the vertices of the graph $\mathcal{T}_d$ we just described. We will not distinguish between the set of sequences X^* and the rooted tree $\mathcal{T}_d$ and use them interchangeably throughout this thesis. When $X = \{0, 1\}$, the resulting tree is called the *binary rooted tree* (see Figure 2.1).

Many notions have different wordings whether one thinks X^* as a set of sequences or as a graph. For example, each vertex is on a unique *level* depending on its (combinatorial) distance from the root vertex. In terms of sequences, the n-th level consists of sequences of length n which we denote by X^n.

An important associated object is the *boundary* of the tree ∂X^*. It is the set of all infinite rays starting from the root vertex and can be identified with the set of infinite sequences $X^{\mathbb{N}}$. Therefore, the boundary has the structure of a Cantor set.

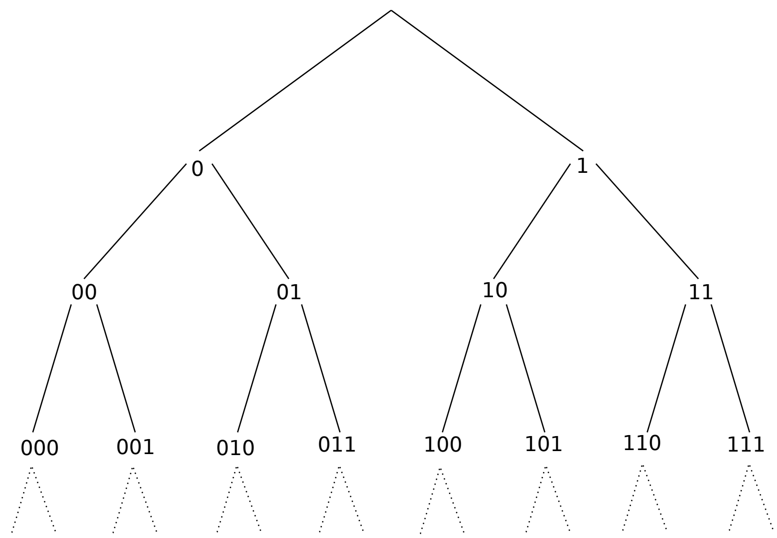

Figure 2.1: The binary rooted tree

Let us denote by $Aut(X^*)$ (or $Aut(\mathcal{T}_d)$ if $|X| = d$) the group of (graph) automorphism of X^*. In other words, a bijection f on X^* belongs to $Aut(X^*)$ if it preserves the incidence relation of vertices. In terms of sequences, this translates into f preserving prefixes of sequences. From this it follows easily that all automorphisms have to fix the root vertex and hence preserve levels of vertices. Therefore an automorphism permutes the vertices of the same level.

The d-ary rooted tree is a natural *self-similar* geometric object: Given a vertex $v \in X^*$ let us denote by vX^* the subtree hanging down at vertex v (i.e., sequences starting with v). X^* and vX^* are isomorphic via the map $\phi_v : w \mapsto vw$. The self-similarity of the tree also reflects upon its automorphism group as we shall see below. First we have a simple but very important definition:

Definition 2.1. Let $f \in Aut(X^*)$ and $v \in X^*$. The *section* of f at the vertex v is the automorphism defined by

$$f_v(w) = (\phi_{f(v)}^{-1} \circ f \circ \phi_v)(w) \text{ for all } w \in X^*$$

in other words, f_v is uniquely defined by the equality $f(vw) = f(v)f_v(w)$ for all $w \in X^*$.

It is easy to see that the section of an automorphism at a vertex v is indeed an automorphism of the tree.

Given $f \in Aut(X^*)$ where $X = \{0, \ldots, d-1\}$, let $\tau_f \in S_d$ be the permutation determined by f by its action on vertices of the first level. It is apparent that f is uniquely determined by the data $(\tau_f; f_0 \ldots, f_{d-1})$, where τ_f determines how f acts on the first level and $f_0, \ldots, f_{d-1}$ determine how it acts on the subtrees hanging down the first level vertices. The following equations are easy to observe from the definitions:

$$\begin{gathered}
(f_u)_v = f_{uv} \text{ for all } u, v \in X^*, \\
(fg)_u = f_{\tau_g(u)} g_u \text{ for all } f, g \in Aut(X^*),\ u \in X^*, \\
\tau_{fg} = \tau_f \tau_g \text{ for all } f, g \in Aut(X^*).
\end{gathered} \tag{2.1}$$

In the next section we will see algebraic consequences of self-similarity.

2.2 Self-similar groups and groups generated by automata

We refer to the book [Nek05] for a comprehensive source regarding self-similar groups.

The classical definition of a self-similar group is given only for groups of tree automorphisms as subgroups which are closed under taking sections of its elements. We will give here a more general definition which is based on iterated permutational wreath products. This generality will be used in Chapter 3. Our exposition borrows from [Bar13, Nek05]. We begin with basics about permutational wreath products.

Let G and H be two groups and let X be a set on which H acts from the left. Let G^X denote the group of functions $f : X \to G$ with finite support. We will write such a function f as a tuple indexed by X as $(f_x)_{x\in X}$ thinking it as an element of the restricted direct product $\prod_{x\in X} G$. There is a natural right action of H on G^X (by automorphisms) given by $(f_x)_{x\in X}.h = (f_{h.x})_{x\in X}$. The *permutational wreath product* corresponding to this data is the semi-direct product:

$$H \wr_X G := H \ltimes G^X$$

The product of two elements in $G \wr_X H$ is given by

$$(h; (g_x)_{x\in X})(h'; (g'_x)_{x\in X}),) = (hh'; (g_{h'x} g'_x)_{x\in X}))$$

If G has a left action on some set Z then $H \wr_X G$ has a natural left action on $X \times Z$ given by

$$(h; (g_x)_{x\in X}).(y, z) = (h.y, g_y.z)$$

If $\alpha : G \to G'$ is a homomorphism between two groups then there is a natural homomorphism

$$1_H \wr \alpha : H \wr_X G \longrightarrow H \wr_X G' \tag{2.2}$$

given by $(h; (g_x)_{x\in X}) \mapsto (h; (\alpha(g_x)_{x\in X}))$ where 1_H denotes the identity homomorphism of H.

The particular situation we will be interested is when $X = \{0, \ldots, d-1\}$ and $H = S_d$ where S_d is the symmetric group with its natural left action on X. In this situation let us write $S_d \wr G$ for $S_d \wr_X G$. Denote an arbitrary element by

$(\tau; g_0, \ldots, g_{d-1})$ where $\tau \in S_d$ and $g_0, \ldots, g_{d-1} \in G$.

The *iterated permutational wreath products* are defined inductively as

$$S_d \wr^n G = \begin{cases} G & \text{for } n = 0 \\ S_d \wr (S_d \wr^{n-1} G) & \text{for } n \geq 1. \end{cases}$$

We have the following *associativity of permutational wreath products*: for a H-set Y and a K-set X, the canonical mapping

$$\begin{array}{ccc} K \wr_X (H \wr_Y G) & \longrightarrow & (K \wr_X H) \wr_{X \times Y} G \\ \Big(k; \big(h_x; (g_{x,y})_{y \in Y}\big)_{x \in X}\Big) & \longmapsto & \Big(\big(k; (h_x)_{x \in X}\big); \big(g_{x,y}\big)_{(x,y) \in X \times Y}\Big) \end{array}$$

is an isomorphism of groups (this is standard, see e.g. [Mel95, Chapter 1, Theorem 3.2]). In particular, we have

$$S_d \wr^n G = S_d \wr (S_d \wr^{n-1} G) = (S_d \wr^{n-1} S_d) \wr G = S_d^{(n)} \ltimes G^{X^d}$$

where $S_d^{(n)}$ is a subgroup of S_{d^n} defined inductively by

$$S_d^{(n)} = \begin{cases} S_d & \text{for } n = 0 \\ S_d \wr S_d^{(n-1)} & \text{for } n \geq 1. \end{cases}$$

and acting onto X^n from the left. We write

$$(\tau; (g_v)_{v \in X^n}) \text{ with } g_v \in G \text{ for all } v \in X^n \text{ and } \tau \in S_d^{(n)}$$

for a typical element of $S^d \wr^n G$.

Definition 2.2. Let G be a group and $d \geq 2$ an integer. A *self-similar structure* of degree d on G is a homomorphism

$$\Phi : G \longrightarrow S_d \wr G$$

A *self-similar group* is a pair (G, Φ). When Φ is clear from the context, we will simply call G a self-similar group.

If (G, Φ) is a self-similar group, the construction (2.2) gives rise to a sequence of homomorphisms

$$\Phi_n \,:\, G \xrightarrow{\Phi_{n-1}} S_d \wr^{n-1} G \xrightarrow{1_{d^{n-1}} \wr \Phi} S_d \wr^{n-1} (S_d \wr G) = S_d \wr^n G \tag{2.3}$$

for $n \geq 2$; we write $\Phi_0 = \mathrm{id}_G$ and $\Phi_1 = \Phi$. Note that, if Φ is injective, so is Φ_n for all $n \geq 0$. It is routine to check that Φ_{m+n} is the composition

$$\Phi_{m+n} \,:\, G \xrightarrow{\Phi_n} S_d \wr^n G \xrightarrow{1_{d^n} \wr \Phi_m} S_d \wr^n (S_d \wr^m G) = S_d \wr^{m+n} G \tag{2.4}$$

for all $m, n \geq 0$. The composition of Φ_n and the quotient map $S_d \wr^n G \longrightarrow S_d^{(n)}$ is a homomorphism

$$G \longrightarrow S_d^{(n)}, \;\; g \longmapsto \tau_g^{(n)}. \tag{2.5}$$

Thus, introducing the v-coordinates of $\Phi_n(\cdot)$, we have

$$\Phi_n(g) \;=\; \left(\tau_g^{(n)}; (g_v)_{v \in X^n}\right) \;\in\; S_d^{(n)} \wr_{X^n} G = S_d \wr^n G$$

for all $g \in G$. Note that

$$\tau_g^{(n)} \;=\; \left(\tau_g^{(1)}; \left(\tau_{g_x}^{(n-1)}\right)_{x \in X}\right) \;\in\; S_d \wr S_d^{(n-1)} = S_d^{(n)} \tag{2.6}$$

for all $g \in G$ and $n \geq 1$.

A self-similar group (G, Φ) of degree d defines a left action of G onto the regular d-ary tree $X^* = \{0, \dots, d-1\}^*$ via the homomorphisms (2.5) i.e., for $v \in X^*$ with $|v| = n$ and $g \in G$ we have

$$g(v) = \tau_g^{(n)}(v)$$

For $n \geq 1$ and $v \in X^n$, we have the vertex and level stabilizers

$$\mathrm{St}_G(v) \;=\; \left\{g \in G \mid \tau_g^{(n)}(v) = v\right\},\; St_G(n) = \bigcap_{|v|=n} St_G(v) \tag{2.7}$$

and we have the homomorphisms

$$\Phi_v \;:\; \mathrm{St}_G(v) \longrightarrow G,\;\; g \longmapsto g_v \tag{2.8}$$

where $g_v = \Phi_v(g)$ is the v-coordinate of $\Phi_n(g)$.

Lemma 2.1. *With the notation above,*

$$g_{uv} = (g_u)_v, \;\; (gh)_u = g_{\tau_h(u)} h_u \;; \textit{and}\;\; (g^{-1})_v = \left(g_{\tau_{g^{-1}}(u)}\right)^{-1}$$

Proof. Straightforward from the definitions. □

Recall from Definition 2.1 that the section of an automorphism $f \in Aut(X^*)$ at the vertex $x \in X$ is an automorphism f_x uniquely determined by the equation $f(xv) = f(x)f_x(v)$ for all $v \in X^*$. Hence, given $g \in G$ and $x \in X, v \in X^*$, $|v| = n$, by virtue of equation (2.6) and the action of $S_d \wr S_d^{n-1}$ on X^n, we have

$$g(xv) = \tau_g^{(n+1)}(xv) = \tau_g^{(1)}(x)\tau_{g_x}^{(n)}(v) = g(x)g_x(v)$$

Therefore, it follows that the section of the automorphism defined by g at the vertex $x \in X$ is given by the action of g_x. This justifies the notational similarities of a section and of the x coordinate of $g \in G$ of its image under Φ.

Observe that the section of an arbitrary element $g \in G$ at an arbitrary vertex $x \in X$ belongs to G. This brings us to the classical definition of a self-similar group:

Definition 2.3. Let $X = \{0, \ldots, d-1\}$. A subgroup $G \leq Aut(X^*)$ is called self-similar if for every $g \in G$ and every $x \in X$, the section g_x (as in Definition 2.1) is again an element of G. i.e., G is closed under taking sections.

Note that equations (2.1) show that we have group isomorphism $\Phi : Aut(X^*) \longrightarrow S_d \wr Aut(X^*)$ given by

$$\Phi(f) = (\tau_f; f_0, \ldots, f_{d-1})$$

Hence, Definition 2.3 translates into the following: A subgroup $G \leq Aut(X^*)$ is self-similar if the restriction of $\Phi : Aut(X^*) \longrightarrow S_d \wr Aut(X^*)$ induces a monomorphism $G \longrightarrow S_d \wr G$. Note that, in general this will not be an isomorphism.

We see that every self-similar subgroup $G \leq Aut(X^*)$ (in the sense of Definition 2.3) gives a self-similar pair (G, Φ) in the sense of 2.2 via the canonical map Φ described above. We will always consider this canonical self-similarity structure when the group in consideration is perceived as as a subgroup of the automorphism group of a rooted tree.

The difference between the two definitions of self-similarity is that the first definition gives an action onto the tree which may or may not be faithful.

Definition 2.4. A self-similar group (G, Φ) of degree d is called faithful if the action induced onto the tree X^* is faithful, where $X = \{0, \ldots, d-1\}$.

Note that if (G, Φ) is faithful then necessarily Φ is injective but the converse is not true.

Finitely generated (faithful) self-similar groups can be constructed using the following general idea:

Start with a list of symbols $\{a_1, \ldots, a_n\}$ which will serve as generators. Also let $\{\sigma_1, \ldots, \sigma_n\} \subset S_d$. Consider the system:

$$\begin{array}{ccc} a_1 & = & (\sigma_1; w_{10}, \ldots, w_{1(d-1)}) \\ a_2 & = & (\sigma_2; w_{20}, \ldots, w_{2(d-1)}) \\ \cdots & \cdots & \cdots \\ a_n & = & (\sigma_n; w_{n0}, \ldots, w_{n(d-1)}) \end{array}$$

where w_{ij} are words over the alphabet $\{a_1, \ldots, a_n\}^{\pm}$. Such a system is called a *wreath recursion* and defines uniquely n elements in $Aut(X^*)$ where $X = \{0, \ldots, d-1\}$, which again will be denoted by $\{a_1, \ldots, a_n\}$. Then we can look at the subgroup $G = \langle a_1, \ldots, a_n \rangle \leq Aut(X^*)$ which by construction is clearly a self-similar group. A special case is when all w_{ij} are of length 1. In this case the group obtained belongs to a special family of self-similar groups, namely the *groups generated by finite automata*. The language of automata in the theory of self-similar groups is a fundamental one. We refer to [GNS00] for a comprehensive source related to groups generated by automata.

Definition 2.5. A *Mealy automaton* is a tuple $\mathcal{A} = (Q, X, \tau, \lambda)$ where Q is a set called the *set of states*, X is the alphabet, $\tau : Q \times X \to Q$ is a function, called the *transition function*, and $\lambda : Q \times X \to X$ is a function, called the *output function*.

A Mealy automaton can be thought as a machine operating on sequences over the alphabet X. Given a state $q \in Q$, it acts on a sequence $w = x_1 x_2 x_3 \ldots x_n \in X^*$ as follows: Initially the machine is at state $q_1 = q$ and x_1 is changed to $\lambda(q_1, x_1)$ while the state is changed to $q_2 = \tau(q_1, x_1)$. Then the machine proceeds to the next letter x_2 changing it to $\lambda(q_2, x_2)$ while also changing the current state to $q_3 = \tau(q_2, x_2)$ and continuing in this fashion until all letters of w have been processed. This gives for each $q \in Q$ a transformation $q : X^* \to X^*$ which is easily seen to preserve prefixes. If for all $q \in Q$ the induced transformation on X^* invertible then the automaton is said to be *invertible*. Hence for an invertible automaton, the set of states define automorphisms of the rooted tree X^*.

Definition 2.6. A group of automorphisms generated by the states of an invertible automaton $\mathcal{A}$ is called an *automaton group*.

A priori, the class of (faithful) self-similar groups and groups generated by invertible automata coincide: Given self-similar $G \leq Aut(X^*)$, one can build an automaton with the set of states as G over the alphabet X where the transition function is $\tau : G \times X \to X$ is given by $\tau(g, x) = g_x$ and the output function $\lambda : G \times X \to X$ is given by $\lambda(g, x) = g(x)$.

A more interesting class is the class of groups generated by *finite* invertible automata. In literature the term “automaton group” usually refers to this class. Classifying automaton groups and determining which groups can be realized as automaton groups is a very active area and there are various publications devoted for these problems some of which are [BGK$^+$07, BGK$^+$09, BGK$^+$08, SVV11, VV07, VV10].

2.2.1 Contracting self-similar groups

Definition 2.7. A self-similar group (G, Φ) is *contracting* if there is a finite subset $\mathcal{M} \subset G$ such that for all $g \in G$, there exists an integer $k \geq 0$, such that $g_v \in \mathcal{M}$ for all $v \in X^*$ with $|v| \geq k$.

The smallest such $\mathcal{M}$, namely

$$\mathcal{N} := \bigcup_{g \in G} \bigcap_{k \geq 0} \{g \in G \mid \exists\, h \in G,\ \ell \geq k,\ v \in X^{\ell} \text{ with } h_v = g\}$$

is called the *nucleus* of (G, Φ).

Contracting self-similar groups constitute a very important subclass of self-similar groups. Almost all interesting properties of self-similar groups are shown via the contracting property. In contrast, non-contracting groups are usually very hard to deal with. It has been conjectured that all contracting groups are amenable.

If G is finitely generated, the contracting property translates into length reduction under taking sections, hence justifying the name *contracting*:

Proposition 2.1. *[Nek05, Proposition 2.11.11] If G is a finitely generated contracting self-similar group then there exists $M, n > 0$ such that for every $g \in G$ and every $v \in X^n$ we have*

$$|g_v| \leq \frac{|g|}{2} + M$$

This allows proofs by induction on the length of elements. Various properties, such as periodicity or intermediate growth, are proven by inductive arguments.

All examples we are going to discuss belong to the class of contracting groups. Moreover, our examples satisfy the following property:

Definition 2.8. A self-similar group (G, Φ) is *self-replicating*, if for all $g \in G$ and $x \in X$ there is an element $h \in St_G(x)$ such that $h_x = g$. In other words, for all $x \in X$ the homomorphism Φ_x defined in (2.8) is onto.

Note that in literature the terminologies *fractal* or *recurrent* are also used for the self-replicating property. This property plays an important role and one use is the following proposition which will be used in Chapter 3:

Proposition 2.2. *Let (G, Φ) be a contracting self-similar group with nucleus $\mathcal{N}$, as above.*

(i) For $g \in \mathcal{N}$ and $x \in X$, we have $g_x \in \mathcal{N}$.

(ii) If (G, Φ) is self-replicating and G is finitely generated, then $\mathcal{N}$ generates G.

Proof. For $g \in \mathcal{N}$, there exist $h \in G$, $k \geq 0$, and $v \in X^k$ such that $h_v = g$ and $h_w \in \mathcal{N}$ for all $w \in X^*$ with $|w| \geq k$ (otherwise, $\mathcal{N}$ would not be minimal). Hence $g_x = (h_v)_x = h_{vx} \in \mathcal{N}$ for all $x \in X$. This proves (i).

For (ii), we paraphrase [Nek05, Lemma 2.11.3]. Denote by $\langle \mathcal{N} \rangle$ the subgroup of G generated by $\mathcal{N}$. Let S be a symmetric finite generating set of G. For all $s \in S$, there exists $k_s \geq 0$ such that $s_v \in \mathcal{N}$ for all $v \in X^*$ with $|v| \geq k_s$. Set $k = \max\{k_s \mid s \in S\}$.

Let $g \in G$ and $v \in X^*$ with $|v| \geq k$. There exist $s_1 \ldots, s_m \in S$ with $g = s_1 \cdots s_m$, so that

$$\begin{aligned} g_v &= (s_1)_v (s_2 \cdots s_m)_{vs_1} = \cdots \\ &= (s_1)_v (s_2)_{vs_1} (s_3)_{vs_1 s_2} \cdots (s_m)_{vs_1 \cdots s_{m-1}} \in \langle N \rangle, \end{aligned}$$

where the last inclusion follows from $|v| = |vs_1| = \cdots |vs_1 \cdots s_{m-1}| \geq k$. In particular, the image of Φ_v, as defined in (2.8), is contained in $\langle \mathcal{N} \rangle$.

If (G, Φ) is self replicating, then Φ_v is onto for all $v \in X^*$ with $|v| \geq 1$. The conclusion follows. □

2.2.2 Examples of self-similar groups

Firstly, the full automorphism group $Aut(X^*)$ (with its canonical structure) is clearly a self-similar group. In fact, in this case the homomorphism Φ is actually an isomorphism $Aut(X^*) \cong S_d \wr Aut(X^*)$ and hence $Aut(X^*)$ has the form of an infinitely iterated wreath product of symmetric groups. Another simple example is the realization of the group of integers as a self-similar group via the wreath recursion $a = (\sigma : 1, \ldots, 1, a)$ where $\sigma \in S_d$ is the cyclic permutation $(01 \ldots d-1)$. Thus $\mathbb{Z}$

can be given the structure of a self-similar group of degree d for any $d \geq 2$.

A much more interesting example is the so called *first Grigorchuk Group* $\mathcal{G}$. This group, introduced in [Gri80], lies at the heart of the study of self-similar groups. $\mathcal{G}$ is the subgroup the automorphism group of the binary rooted tree generated by four elements denoted traditionally by $\{a, b, c, d\}$ which are given by the following wreath recursion:

$$\begin{aligned} a &= (\sigma : 1, 1) \\ b &= (e; a, c) \\ c &= (e; a, d) \\ b &= (e; 1, b) \end{aligned}$$

where $S_2 = \{e, \sigma\}$. As mentioned before, this particular group has various interesting properties. We will touch upon some of these properties and refer the reader to [Gri05, dlH00] for nice expositions about this group.

To begin with, $\mathcal{G}$ is a finitely generated infinite 2-group [Gri80], hence a counter example to the celebrated Burnside Problem which asks whether a finitely generated periodic group is necessarily finite. Although the first counter examples to the Burnside Problem were constructed by Golod and Shafarevich [GŠ64], $\mathcal{G}$ is one of the first "tangible" examples of such groups. More importantly, $\mathcal{G}$ is the first example of a group which has *intermediate growth*. That is, the size of the balls in its Cayley graph grows faster than any polynomial but slower than the exponential function (see Section 2.6). This was an answer to a question posed by J.Milnor [Mil68b] and has various consequences (see again Section 2.6). $\mathcal{G}$ is also important related to branching, amenability and recursive presentations, properties which will be discussed in Sections 2.5, 2.6 and 2.8 respectively. Also its profinite completion $\widehat{\mathcal{G}}$ is an interesting object with important properties which are discussed in Chapter 4. Let us also mention that $\mathcal{G}$ has uncountably many relatives (hence justifying the name first

Grigorchuk group) which will be introduced in Section 2.4. $\mathcal{G}$ (and its relatives) have also interesting properties related to random walks, spectral graph theory, dynamical systems, algorithmic problems and various other branches of mathematics.

Our next example is the so called *Basilica group* $\mathcal{B}$. It is a self-similar subgroup of the binary rooted tree given by the following wreath recursion:

$$\begin{aligned} a &= (\sigma; b, 1) \\ b &= (e; a, 1) \end{aligned}$$

This group was introduced in [GŻ02a, GŻ02b]. Its name comes from its relation to the Julia set of the quadratic polynomial $p(z) = z^2 - 1$ which resembles the *Basilica Cattedrale Pariarcale di San Marco* . The relation comes via the identification of $\mathcal{B}$ as the *iterated monodromy group* of the polynomial $p(z) = z^2 - 1$ (see [Nek05, BGN03]). Unlike $\mathcal{G}$, $\mathcal{B}$ is torsion free and has exponential growth. The most important properties of $\mathcal{B}$ are related to amenability and random walks which will be mentioned in Section 2.6 in detail.

The next example we will see is known as the *Gupta-Sidki Group* and will be denoted by $\mathcal{GS}$ throughout this dissertation. It is the 2-generated subgroup of automorphisms of the ternary rooted tree given by the wreath recursion:

$$\begin{aligned} a &= (\tau; 1, 1, 1) \\ b &= (e; a, a^{-1}, b) \end{aligned}$$

where τ is the cyclic permutation (012). This infinite 3-group, introduced in [GS83], shares various properties with $\mathcal{G}$, a major difference being that the growth type of $\mathcal{GS}$ remains unknown.

The next example we will introduce will be denoted by $\mathcal{I}$ and is the iterated monodromy group of the polynomial $z^2 + i$. It is a 3-generated group of automorphisms of the binary rooted tree given by the wreath recursion

$$\begin{aligned} a &= (\tau; 1, 1) \\ b &= (e; a, c) \\ c &= (e; b, 1) \end{aligned}$$

This group was studied in detail in [GSŠ07] and was shown to be of intermediate growth in [BP06].

The *Fabrykowski-Gupta group* $\mathcal{FG}$ is the 2-generated group of ternary tree automorphisms defined by the wreath recursion

$$\begin{aligned} a &= (\tau; 1, 1, 1) \\ b &= (e; a, 1, b) \end{aligned}$$

It was introduced and studied in [FG85, FG91, BP09] and shown to be of intermediate growth. Observed to be the iterated monodromy group of a cubic polynomial in [Nek11].

Our last example is the ternary *Hanoi Towers* group $\mathcal{H}$. It is the 3-generated group of automorphisms of the ternary rooted tree given by

$$\begin{aligned} a &= (\tau_{1,2}; a, 1, 1) \\ b &= (\tau_{0,2}; 1, b, 1) \\ c &= (\tau_{0,1}; 1, 1, c) \end{aligned}$$

where $\tau_{i,j}$ is the transposition of S_3 exchanging i and j. This group was introduced in [GŠ06] as a model for the well-known Hanoi Towers problem. $\mathcal{H}$ is known to have exponential growth.

2.3 Topology in the space of marked groups

A *marked k-generated group* is a pair (G, S) where G is a group with an ordered set $S = \{s_1, \dots, s_k\}$ of (not necessarily distinct) generators of G. The *canonical map* between two marked k-generated groups (G, S) and (H, T) is the map that sends s_i to t_i for $i = 1, 2, \dots, k$. Let $\mathcal{M}_k$ denote the *space of marked k-generated groups* consisting of marked k-generated groups where two marked groups are identified whenever the canonical map between them extends to an isomorphism of the groups.

There is a natural metric on $\mathcal{M}_k$: Two marked groups $(G, S), (H, T)$ are of distance 2^{-m} where m is the largest natural number such that the canonical map between (G, S) and (H, T) extends to an isomorphism (of labeled graphs) from the ball of radius m (around the identity) in the Cayley graph of (G, S) onto the ball of radius m in the Cayley graph of (H, T). This makes M_k into a compact, totally disconnected topological space see [Gri84]. The resulting topology is called by many names in literature usually associated with the names Cayley or Grigorchuk.

Another description of the topology on $\mathcal{M}_k$ is by identifying it with the set of normal subgroups of the free group F_k and using the idea of furnishing the space of subgroups of a group with a topology which goes back to Chabauty [Cha50]. Let F_k be a free group of rank k and $X = \{x_1, \dots, x_k\}$ a fixed ordered basis. To every point (G, S) in $\mathcal{M}_k$ there corresponds a unique normal subgroup $\ker \pi_G$ of F_k where $\pi_G : F_k \to G$ is the canonical surjection sending x_i to s_i, $i = 1, \dots, k$. It is easily seen that this gives a bijection from $\mathcal{M}_k$ to the set of normal subgroups of F_k. The latter has a natural topology inherited from the space of all subset of F_k which has a natural Tychonoff topology. A set of basis elements for this topology are given by sets of the form

$$\mathcal{O}_{A,B} = \{N \lhd F_k \mid A \subset N \text{ and } B \cap N = \varnothing\}$$

where A, B range over finite subsets of F_k. It is also completely metrizable with the following metric: $d(N, K) = 2^{-m}$ where m is the largest integer such that $N \cap B_X^{F_k}(m) = K \cap B_X^{F_k}(m)$ and $B_X^{F_k}(m)$ denotes the ball of radius m around the identity in the Cayley graph of F_k. This topology is usually called the Chabauty topology in literature. See [Gri84], [Cha00], [CG05], [CSC10] for more on this topology.

Amongst various occasions, one instance where this topology was used in crucial way was in [Gri84] by defining and proving structural results about a family of 2-groups (see Section 2.4). An important role is played by the following basic observation which is well known:

Proposition 2.3. *Let $\big((G_n, S_n)\big)_{n\geq 1}$ be a converging sequence in $\mathcal{M}_k$; set $(G, S) = \lim_{n\to\infty}(G_n, S_n)$. Let Γ be a finitely presented group; assume there exists a cover $\pi : \Gamma \twoheadrightarrow G$. Then Γ is a cover of G_n for n large enough.*

Proof. Denote as above by $(s_1, \dots, s_k)$ an ordered free basis of F_k. Let $p_n : F_k \twoheadrightarrow G_n$ and $p : F_k \twoheadrightarrow G$ be the free covers corresponding to (G_n, S_n) and (G, S) respectively. Set $N_n = \ker(p_n)$ and $N = \ker(p)$. Let $(t_1, \dots, t_\ell)$ an ordered generating set of Γ. Consider the free group F_ℓ on an ordered basis $U = (u_1, \dots, u_\ell)$ and the free cover $q : F_\ell \twoheadrightarrow \Gamma$ defined by $q(u_j) = t_j$ for $j = 1, \dots, \ell$.

Since Γ is finitely presented, there exists a finite subset $R \subset F_\ell$ of words $v_1, \dots, v_m$ in the letters of $U \cup U^{-1}$ such that $\ker(q)$ is the normal subgroup of F_ℓ generated by R, namely such that $\langle U \mid R\rangle$ is a presentation of Γ. For $j \in \{1, \dots, \ell\}$, choose a word w_j in the letters $p(s_1), \dots, p(s_k)$ and their inverses such that $\pi(t_j) = w_j$. Let $\tilde{w}_j$ be the word in $\{s_1, s_1^{-1}, \dots, s_k, s_k^{-1}\}$ obtained by substitution of $s_i^{\pm 1}$ in place of

$p(s_i)^{\pm 1}$; observe that $p(\tilde{w}_j) = w_j = \pi(t_j)$. Consider the homomorphism

$$h \,:\, F_\ell \longrightarrow F_k \quad \text{defined by} \quad h(u_j) \,=\, \tilde{w}_j \quad (1 \le j \le \ell).$$

Then $ph(u_j) = p(\tilde{w}_j) = w_j = \pi(t_j) = \pi q(u_j)$ for all j, so that $ph = \pi q$, and therefore $h(R) \subset N$.

The last inclusion means that the open subset

$$\mathcal{O}' \,:=\, \{M \lhd F_k \;\;:\;\; h(R) \subset M\} \,=\, \bigcap_{i=1}^{m} \mathcal{O}_{\emptyset,\{h(r_i)\}}$$

is a neighborhood of N in $\mathcal{M}_k$. Hence, for n large enough, we have $N_n \in \mathcal{O}'$ and therefore $h(R) \subset N_n$.

Denote by $\langle\langle T \rangle\rangle$ the *normal* subgroup of a group H generated by a subset $T \subset H$. Let

$$h_1 \,:\, \Gamma = F_\ell/\langle\langle R \rangle\rangle \longrightarrow F_k/\langle\langle h(R) \rangle\rangle$$

be the cover induced by h, and

$$h_2 \,:\, F_k/\langle\langle h(R) \rangle\rangle \longrightarrow F_k/N_n \,=\, G_n$$

that defined by the inclusion $\langle\langle h(R) \rangle\rangle \subset N_n$ (for $n \gg 1$). The composition $h_2 h_1$ is a cover $\Gamma \twoheadrightarrow G_n$, and this concludes the proof. $\square$

2.4 Grigorchuk 2-groups

In this section we will define a family of groups which are in a sense analogue to $\mathcal{G}$ and share many common properties with it. The original definition of these groups given in [Gri84] is in terms of (Lebesgue) measure preserving transformations of the unit interval. Here we will take the alternative approach and define them in terms of automorphism of the binary rooted tree. Although these groups are not necessarily self-similar, as we will see, are closely related to self-similar groups. The main importance of this family is that it not only provides examples of groups of intermediate growth, but also provides examples for which the growth function has a wide range of different behavior. We refer to [BGV13] for a detailed discussion of growth behavior of groups belonging to this family.

Let Ω denote the Cantor space $\{0,1,2\}^{\mathbf{N}}$ of all infinite sequences of 0's, 1's and 2's, with the product topology. Denote by Ω_- the countable subspace of eventually constant sequences, by Ω_+ its complement, and by Ω_0 the subspace of sequences with infinitely many occurrences of each of $0,1,2$; thus

$$\Omega_0 \subset \Omega_+ \subset \Omega = \Omega_+ \sqcup \Omega_-.$$

We denote by σ the shift on Ω, defined by $(\sigma(\omega))_n = \omega_{n+1}$ for all $n \geq 1$.

For each $\omega \in \Omega$ we will define a group $G_\omega \leq Aut(X^*)$ where $X = \{0,1\}$. Each group G_ω will be generated by four automorphisms denoted by $S_\omega = \{a, b_\omega, c_\omega, d_\omega\}$. The action of these generators are as follows: For $v \in \{0,1\}^*$

$$a(0v) = 1v \text{ and } a(1v) = 0v$$

$$\begin{array}{llllll} b_\omega(0v) = & 0\beta(\omega_1)(v) & c_\omega(0v) = & 0\zeta(\omega_1)(v) & d_\omega(0v) = & 0\delta(\omega_1)(v) \\ b_\omega(1v) = & 1b_{\sigma\omega}(v) & c_\omega(1v) = & 1c_{\sigma\omega}(v) & d_\omega(1v) = & 1d_{\sigma\omega}(v) \end{array}$$

where

$$\begin{array}{lll} \beta(0) = a & \beta(1) = a & \beta(2) = 1 \\ \zeta(0) = a & \zeta(1) = 1 & \zeta(2) = a \\ \delta(0) = 1 & \delta(1) = a & \delta(2) = a \end{array}$$

This action defines an embedding into the permutational wreath product

$$\begin{array}{rcccccccc} \Phi_\omega : & G_\omega & \to & S_2 & \ltimes & (G_{\sigma(\omega)} & \times & G_{\sigma(\omega)}) \\ & a & \mapsto & (\tau & ; & 1 & , & 1) \\ & b_\omega & \mapsto & (e & ; & \beta(\omega_1) & , & b_{\sigma(\omega)}) \\ & c_\omega & \mapsto & (e & ; & \zeta(\omega_1) & , & c_{\sigma(\omega)}) \\ & d_\omega & \mapsto & (e & ; & \delta(\omega_1) & , & d_{\sigma(\omega)}) \end{array}$$

where $S_2 = \{e, \tau\}$. It is clear from the definitions that we have the following equalities:

$$\begin{aligned} \Phi_\omega(ac_\omega a) &= \big(b_{\sigma(\omega)}, \beta(\omega_1)\big) \\ \Phi_\omega(ad_\omega a) &= \big(c_{\sigma(\omega)}, \zeta(\omega_1)\big) \\ \Phi_\omega(ab_\omega a) &= \big(d_{\sigma(\omega)}, \delta(\omega_1)\big) \\ a^2 = b_\omega^2 &= c_\omega^2 = d_\omega^2 = b_\omega c_\omega d_\omega = 1. \end{aligned} \tag{2.9}$$

It follows from the last line of (2.9) that any element of G_ω can be written as

$$(*)a * a * \cdots a(*) \tag{2.10}$$

with $* \in \{b_\omega, c_\omega, d_\omega\}$, $(*) \in \{1, b_\omega, c_\omega, d_\omega\}$, and $n \geq 0$ occurrences of a.

Although this does not give a self-similar structure in general, we always have an embedding $\Phi_\omega : G_\omega \longrightarrow S_2 \wr G_{\sigma(\omega)}$ which enables us to adapt many situation suited for self-similar groups for this family. Note that when $\omega = 012012012\ldots$ then the corresponding group G_ω is isomorphic to the first Grigorchuk group $\mathcal{G}$.

A desired property would be that the subset $\{(G_\omega, S_\omega) \mid \omega \in \Omega\} \subset \mathcal{M}_4$ is homeomorphic to Ω via $(G_\omega, S_\omega) \mapsto \omega$. It can be observed that this is not true and one needs to replace the countably many groups $\{G_\omega \mid \omega \in \Omega_-\}$, which consists of virtually free abelian groups, with appropriate limits. In [Gri84] this family was slightly modified into a family $\{\widetilde{G}_\omega \mid \omega \in \Omega\}$ where $\widetilde{G}_\omega = G_\omega$ if $\omega \in \Omega_+$. Regarding this new family we have the following:

Theorem 2.1.

(i) For $\omega \in \Omega$, $\widetilde{G}_\omega$ is an infinite 3-generated group.

(ii) For $\omega \in \Omega_+$, $\widetilde{G}_\omega$ is a 2-group.

(iii) For $\omega \in \Omega_-$, $\widetilde{G}_\omega$ is virtually metabelian and of exponential growth.

(iv) For $\omega \in \Omega \setminus \Omega_-$ the group $\widetilde{G}_\omega$ has intermediate growth.

(v) The mapping $\Omega \longrightarrow \mathcal{M}_4$, $\omega \longmapsto (\widetilde{G}_\omega, \widetilde{S}_\omega)$ is a homeomorphism onto its image.

(vi) For $\omega, \omega' \in \Omega$, the groups G_ω and $G_{\omega'}$ are isomorphic if and only if $\omega' = \eta(\omega)$ for some permutation η of $\{0, 1, 2\}$.

Proof. $i) - v)$ are proven in [Gri84]. For part $vi)$ see [Nek05, Theorem 2.10.13]. □

2.5 Branch groups

Another important class of groups of tree automorphisms is the class of *branch groups*. We refer to [Gri00, BGŠ03] for comprehensive sources regarding this class.

Definition 2.9. For an automorphism $g \in Aut(X^*)$ let $\mathrm{supp}(g)$ denote the set of vertices on which g acts non-trivially, i.e., its *support*. Given a subgroup G of $Aut(X^*)$ and a vertex $v \in X^*$, let $Rist_G(v) = \{g \in G \mid \mathrm{supp}(g) \subset vX^*\}$ be the subgroup consisting of elements of G which act trivially outside the subtree at v. This subgroup is called the *rigid stabilizer* of G at v.

For $n > 0$ let

$$Rist_G(n) = \left\langle \bigcup_{v \in X^n} Rist_G(v) \right\rangle$$

be the *rigid level stabilizer* of level n. Clearly rigid stabilizers corresponding to distinct vertices of the same level commute. Hence we have

$$Rist_G(n) = \prod_{v \in X^n} Rist_G(v)$$

Definition 2.10. A subgroup G of $Aut(X^*)$ is called a *branch group* (resp. *weakly branch group*) if it acts transitively on the levels of the tree X^* and the the rigid level stabilizers $Rist_G(n), n \geq 1$ have finite index in G (resp. are non-trivial).

We remark that such groups are sometimes called *geometric branch groups*. There is a more general notion of an *algebraic branch group* (see [Gri00]).

Definition 2.10 has two immediate algebraic consequences, firstly, every branch group has trivial center and every non-trivial homomorphic image of a branch group is virtually abelian [Gri00]. More interestingly, branch groups constitute one of the tree classes in the classification of just-infinite groups, i.e., infinite groups whose nontrivial homomorphic images are finite [Gri00].

Definition 2.11. A level transitive self-similar subgroup $G \leq Aut(X^*)$ is called a *regular branch group* over a finite index normal subgroup K if

$$K \times \ldots \times K \leq \Phi(St_G(1) \cap K)$$

where $\Phi : G \to S_d \wr G$ is as before. If K is only nontrivial then G is called a *weakly regular branch group*.

It is easy to see that a regular (weakly) branch group is a (weakly) branch group in the sense of Definition 2.10. Regarding our examples in Section 2.2 we have:

Theorem 2.2.

1. *$\mathcal{G}$ is regular branch group over its subgroup $K = \langle\langle (ab)^2 \rangle\rangle_{\mathcal{G}}$.*

2. *$\mathcal{B}$ is weakly regular branch over its commutator subgroup $\mathcal{B}'$ but is not a branch group.*

3. *$\mathcal{I}$ is regular branch group over the subgroup $N = \langle\langle [a,b],[b,c] \rangle\rangle_{\mathcal{I}}$*

4. *$\mathcal{GS}, \mathcal{FG}$ and $\mathcal{H}$ are regular branch groups over their commutator subgroups.*

Proof.

1. See [dlH00]

2. See [GŻ02a]

3. See [GSŠ07]

4. See [BGŠ03] and [GŠ06]

□

2.6 Amenable groups

The notion of amenability is perhaps one of the unique notions in mathematics which has deep connections and applications to a wide and seemingly unrelated areas. Thus, there is a long list of equivalent definitions of amenability, each being interesting from a certain point of view. The following is the classical definition for groups which is due to J. Von Neumann [vN29], it has various versions for other structures such as graphs, algebras, Banach spaces etc.

Definition 2.12. A group G is called *amenable* if there is a finitely additive probability measure μ defined on all subset of G which is left-invariant i.e.,

$$\mu(E) = \mu(gE)$$

for any subset E of G and any $g \in G$.

Since its introduction, the search for an "algebraic" description of the class of amenable groups was a driving force behind many developments in modern group theory. The following basic facts were already observed by Von Neumann:

Theorem 2.3. *[vN29]*

i) The class of amenable groups AG is closed under the taking subgroups, taking homomorphic images, taking extensions and taking directed unions.

ii) A free group of rank 2 is not amenable.

Let us denote by NF the class of groups not containing non-abelian free subgroups. Also let EG denote the class of *elementary amenable* groups which is the smallest class of groups containing finite and abelian groups and is closed under the operations of part i) of Theorem 2.3. A corollary of Theorem 2.3 are the inclusions

$$EG \subset AG \subset NF$$

Day asked in [Day57] whether any of these inclusions are actually equalities. The second inclusion was shown to be strict by Olshanskii [Ol′80] by constructing non-amenable groups for which every proper subgroup is cyclic. The first inclusion was also shown to be strict by Grigorchuk [Gri84]. The idea relies on group growth:

Definition 2.13. If G is a group with a finite generating set S, one has a length function given by $|g|_S = \min\{n \mid g = s_1 \ldots, s_n \; s_i \in S^{\pm}\}$. The *growth function* of G with respect to S is given by

$$\gamma_G^S(n) = |\{g \in G \mid |g|_S \leq n\}|$$

For two monotone functions $f_1, f_2 : \mathbb{N} \to \mathbb{N}$, let us write $f_1 \preceq f_2$ if there is a constant $C > 0$ such that $f_1(n) \leq C f_2(Cn)$ for all n. This induces an equivalence relation $\sim$ on such functions, for which $f_1 \sim f_2$ if $f_1 \preceq f_2$ and $f_2 \preceq f_1$. It is easy to observe that all growth functions of a group (corresponding to different finite generating sets) are $\sim$ equivalent and this allows one to speak about the growth function γ_G of a group, meaning the $\sim$ equivalence class.

There are basically three types of growth for groups: *polynomial growth* if $\gamma_G \preceq n^d$ for some $d > 0$, *exponential growth* if $\gamma_G \sim e^n$ and *intermediate growth* when neither of the previous happens.

Theorem 2.4.

i) Groups of subexponential growth are amenable.

ii) The class EG does not contain groups of intermediate growth.

Proof. i) is well known (see [CSC10]). ii) is proven in [Cho80]. $\square$

Therefore the groups G_ω, $\omega \in \Omega_+$ of Section 2.4 provide examples in the class $AG \setminus EG$ and are the first examples of such groups. See Section 2.8 for finitely presented examples in this class.

Motivated by this, Grigorchuk introduced the class SG of *subexponentially amenable groups* as the class containing all groups of subexponential growth and is closed under the operations as in part i) of Theorem 2.3 and asked whether the inclusion $SG \subset AG$ is strict or not.

Theorem 2.5. *The Basilica group $\mathcal{B}$ belongs to $AG \setminus BG$.*

Proof. The fact that $\mathcal{B} \notin SG$ was proven in [GŻ02a] and amenability of $\mathcal{B}$ was shown in [BV05]. $\square$

Let us mention that our other examples $\mathcal{GS}, \mathcal{I}, \mathcal{FG}, \mathcal{H}$ are amenable by a general result proven in [BKN10] which shows that all groups generated by a *bounded automaton* are amenable.

2.7 Profinite groups

We refer the reader to [Wil98, RZ10] for comprehensive sources about profinite groups.

Definition 2.14. Let $\mathcal{C}$ be a class of finite groups. A group G is called a *pro-$\mathcal{C}$* group if it is an inverse limit of $\mathcal{C}$ groups.

Throughout our discussion we will be mainly concerned with the case when $\mathcal{C}$ is the class of all finite groups or of finite p-groups where p is a prime number. In these cases we will talk about the class of *profinite* groups or the class of *pro-p* groups.

Let G be a group and let $\mathcal{N}$ be a family of finite index normal subgroups of G which is *filtered from below* i.e., for every $N_1, N_2 \in \mathcal{N}$, there is $N_3 \in \mathcal{N}$ such that $N_3 \subset N_1 \cap N_2$. This property implies that the set $\{G/N \mid N \in \mathcal{N}\}$ forms an inverse system with respect to the order $N_1 \preceq N_2$ if $N_2 \subset N_1$ and the canonical maps $\varphi_{N_2,N_2} : G/N_2 \to G/N_1$.

Definition 2.15. The completion of G with respect to $\mathcal{N}$ is the inverse limit

$$\mathcal{K}_{\mathcal{N}}(G) = \varprojlim_{N \in \mathcal{N}} G/N$$

If $\mathcal{C}$ is a class of finite groups and $\mathcal{N} = \{N \lhd G \mid G/N \in \mathcal{C}\}$, then the corresponding completion is called the *pro-$\mathcal{C}$* completion. In particular, one has the *profinite* completion, denoted by $\widehat{G}$ and the *pro-p* completion, denoted by $\widehat{G}_p$.

Given $\mathcal{N}$ as above, one can make G into a topological group by considering $\mathcal{N}$ as a neighborhood basis for the identity element in G. Then one has a natural continuous homomorphism $\iota : G \to \mathcal{K}_{\mathcal{N}}(G)$ which has dense image and is injective if and only if $\bigcap_{N \in \mathcal{N}} N = \{1\}$. In the case when $\mathcal{N} = \{N \lhd G \mid G/N \in \mathcal{C}\}$, the latter condition is equivalent to G being residually $\mathcal{C}$.

The following well known lemma will be used in the sequel.

Lemma 2.2. *Suppose $\mathcal{N}_1 \subset \mathcal{N}_2$ are two families of finite index normal subgroups of G which are filtered from below. Suppose $\mathcal{N}_1$ is cofinal in $\mathcal{N}_2$, that is, for every $N \in \mathcal{N}_1$, there is $K \in \mathcal{N}_2$ such that $K \subset N$. Then $\mathcal{K}_{\mathcal{N}_1}(G) \cong \mathcal{K}_{\mathcal{N}_2}(G)$.*

Proof. See [RZ10, Lemma 1.1.9] □

2.7.1 $Aut(\mathcal{T}_k)$ as a profinite group

The automorphism group of the k-ary rooted tree has a natural structure of a profinite group and this structure can be defined in various ways: Let $\mathcal{T}_k^{[n]}$ denote the finite k-ary tree consisting of levels up to n and $Aut(\mathcal{T}_k^{[n]})$ denote its automorphism group. For $n > m$ we have the map $\varphi_{nm} : Aut(\mathcal{T}_k^{[n]}) \to Aut(\mathcal{T}_k^{[m]})$ which is given by restriction to the smaller tree. It is easy to see that $Aut(\mathcal{T}_k)$ is isomorphic to the the inverse limit of the inverse system which is given by $\{Aut(\mathcal{T}_k^{[n]}), \varphi_{nm}\}$ and hence is a profinite group.

More concretely, one has a natural metric on $Aut(\mathcal{T}_k)$: Two automorphisms f and g are of distance $2^{-m(f,g)}$, where $m(f,g)$ is the largest natural number such that the actions of f and g on $\mathcal{T}_k$ agree up to level $m(f,g)$. This defines a metric (indeed an ultrametric) which makes $Aut(\mathcal{T}_k)$ into a compact totally disconnected topological group. It is not difficult to observe that the topology arising in this way coincides with the topology defined in the previous paragraph.

Given a subgroup $G \leq Aut(\mathcal{T}_k)$ one has its closure in $\bar{G} \leq Aut(\mathcal{T}_k)$ which is a profinite group containing G as a dense subgroup. Similarly, one can consider the profinite completion $\widehat{G}$ of G. In general these two groups are different. However the following property ensures that these two groups are isomorphic:

Definition 2.16. A group $G \leq Aut(\mathcal{T}_k)$ is said to have the *congruence subgroup property* if any finite index subgroup $N \leq G$ contains the subgroup $St_G(n)$ for some n.

Since all the subgroups $\{St_G(n) \mid n \geq 1\}$ are of finite index, the congruence subgroup property ensures that $\{St_G(n) \mid n \geq 1\}$ is cofinal in the set of all finite index normal subgroups of G and hence by Lemma 2.2 we have $\bar{G} \cong \widehat{G}$. Therefore groups with the congruence subgroup property are particularly interesting since their profinite completions can be concretely realized as automorphism groups of trees.

More generally, one can discuss other completions of groups $G \leq Aut(\mathcal{T}_k)$, such as completions with respect to the rigid stabilizers $\{Rist_G(N) \mid n \geq 1\}$ when the group is a branch group. A rigorous discussion about different completions of branch groups can be found in [BSZ12].

2.8 L-Presentations

Immediately after its discovery it was observed that the first Grigorchuk group $\mathcal{G}$ is not finitely presented. An infinite recursive presentation was found by I.Lysenok:

Theorem 2.6. *[Lys85]*

$$\mathcal{G} = \left\langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, \sigma^i((ad)^4), \sigma^i((adacac)^4), i \geq 0 \right\rangle$$

where σ is the substitution given by $a \mapsto aca, b \mapsto d, c \mapsto b, d \mapsto c$.

This discovery led to the following general definition due to L.Bartholdi:

Definition 2.17. An L-presentation (or an endomorphic presentation) is an expression

$$\langle X \mid Q \mid R \mid \Phi \rangle \tag{2.11}$$

where X is a set, Q, R are subsets of the free group $F(X)$ on the set X and $\Phi \subset End(F(X))$ is a set of endomorphisms of $F(X)$. The expression (2.11) defines a group

$$G = F(X)/N$$

where

$$N = \left\langle\left\langle Q \cup \bigcup_{\phi \in \Phi^*} \phi(R) \right\rangle\right\rangle$$

and Φ^* denotes sub-monoid generated by Φ in $End(F(X))$. Here X are the *generators*, Q the fixed *relators*, R the *iterated relators* and Φ the *endomorphisms* of the presentation. An L-presentation is called finite X, Q, R, Φ are all finite. It is called *ascending* if Q is empty and *invariant* if the endomorphisms in Φ induce endomorphisms of the group defined by the presentation. Note that an ascending L-presentation is invariant and an invariant L-presentation is equivalent to an ascending L-presentation where one replaces R by $Q \cup R$.

It is clear that all finitely presented groups are finitely L-presented. Also a counting argument shows that there are groups which are not finitely L-presented. Relations of such presentations with general recursive presentations are discussed in Chapter 5.

After Lysenok's presentation for $\mathcal{G}$, examples of L-presentations for various other groups were found (see [Bar03],[GŻ02b], [GSŠ07]). A general theorem due L.Bartholdi asserts that all self-similar, self-replicating regular branch groups have a finite L-presentation [Bar03].

General and algorithmic properties of L-presentations were investigated in the papers [Bar03], [Har12a], [Har12b], [Har11].

Knowing a finite L-presentation allows one to embed the group into a finitely presented group. This idea was firstly utilized in [Gri98] to embed the first Grigorchuk group $\mathcal{G}$ into a finitely presented group (see Section 5.2). We will explore such presentations in more detail in Chapter 5.

3. AMENABLE GROUPS WITHOUT FINITELY PRESENTED AMENABLE COVERS*

3.1 Introduction

The results presented in this chapter are discussed in the paper [BGDLH13] written in collaboration with Rostislav Grigorchuk and Pierre De La Harpe.

Let G and H be two groups. Let us call G *a cover* of H if there exists an epimorphism $G \longrightarrow H$. The main motivating question behind the results of this chapter is a variation of the following general question:

Question 1. *Given a property $\mathcal{P}$ of groups, does every finitely generated group with property $\mathcal{P}$ have a finitely presented cover with property $\mathcal{P}$?*

When free groups have the property $\mathcal{P}$ or groups with the property $\mathcal{P}$ are necessarily finitely presented, this question has a trivially positive answer. Also a non-trivial result of Shalom [Sha00] states that if $\mathcal{P}$ is the Kazhdan's property $(\mathcal{T})$ the answer is again positive.

The main aim of this chapter is to focus on negative answers to Question 1 with the property $\mathcal{P}$ being amenability. More precisely, we will focus on examples of groups for which every finitely presented cover contains non-abelian free subgroups. Therefore, the following definition is natural (recall that a group G is called *large* if it has a finite index subgroup H which surjects onto a non-abelian free group).

*Amenable groups without finitely presented amenable covers, by M. G. Benli, R. Grigorchuk and P. De La Harpe, Bulletin of Mathematical Sciences, Volume 3, Issue 01, pp. 73-131, 2012,

Definition 3.1. Let G be a finitely generated group. We will say that G has the property

($\star$) if every finitely presented cover of G contains non-abelian free subgroups.

($\star\star$) if every finitely presented cover of G is large.

A finitely presented group G has property ($\star$) (resp. ($\star\star$)) if and only if it contains a non-abelian free subgroup (resp. is large). Therefore Definition 3.1 is more interesting for finitely generated groups which are not finitely presented. Since large groups contain non-abelian free subgroups and there are finitely presented groups which contain a non-abelian free subgroup but are not large (see [EP84, Theorem 6]), property ($\star\star$) is strictly stronger than property ($\star$).

The following question is one of the main problems left unanswered regarding growth of finitely generated groups:

Question 2. *Does there exists a finitely presented group of intermediate growth?*

A possibly easier question is the following:

Question 3. *Does there exists a finitely generated group of intermediate growth which is a quotient of a finitely presented group without non-abelian free subgroups? or a quotient of a finitely presented amenable group?*

This question was investigated for the first Grigorchuk group $\mathcal{G}$ in [GdlH01]:

Theorem 3.1. *[GdlH01] $\mathcal{G}$ has property ($\star\star$).*

The results of this chapter show that almost all known examples of groups with intermediate growth have the same property and hence are not answering Question 3. Our main result is:

Theorem 3.2 (Section 3.5)**.** *Let G be an infinite finitely generated self-similar group. Assume that G is contracting, faithful and self-replicating. Let G_0 denote a standard contracting cover of G, as in Definition 3.5.*

If G_0 has non-abelian free subgroups, then G has property $(\star)$.

If G_0 is large, then G has property $(\star\star)$.

Hence for self-similar groups with appropriate properties, Theorem 3.2 reduces the problem of showing property $(\star)$ (or $(\star\star)$) to checking whether the *standard contracting cover* G_0 (which happens to be finitely presented) has non-abelian free subgroups (or is large). For all the examples mentioned in Section 2.2 we will see that this cover is in fact large and hence all of them have property $(\star\star)$. As mentioned in Section 2.6 some of these groups have intermediate growth and all of them are amenable. Therefore this gives answers to questions 1 and 3 for these groups.

We extend Theorem 3.1 for the whole family of Grigorchuk groups from Section 2.4:

Theorem 3.3 (Section 3.6)**.** *For $\omega \in \Omega_+$, the group G_ω has property $(\star\star)$.*

It is clear that properties $(\star)$ and $(\star\star)$ pass to covers, i.e., if G is a cover of H and H has property $(\star)$ (resp. $(\star\star)$) then G has property $(\star)$ (resp. $(\star\star)$). This together with Theorem 3.3 shows that the following list of groups of intermediate growth have property $(\star\star)$:

(i) The uncountably many groups of [Ers04], which are finitely generated, of intermediate growth and not residually finite, each one being a *central* cover of $\mathcal{G}$.

(ii) The groups of [BE12], which are finitely generated groups of intermediate growth, with exactly known growth functions, each one being a cover of $\mathcal{G}$.

(iii) Permutational wreath products of the form $A \wr_X G_\omega$, where $A \neq \{1\}$ is a finite group and G_ω is as in Theorem 3.3 [BE11].

Recall that finitely generated elementary amenable groups cannot have intermediate growth, therefore problems 2, 3 do not arise for elementary amenable groups. As mentioned in Section 2.6, the class $\mathcal{AG} \setminus \mathcal{EG}$ is not fully understood and there are only sporadic finitely presented examples of groups known to be in this class. One could try to find new examples by looking at finitely presented covers, but our results show that this does not look very promising.

Let us interpret the properties $(\star)$ and $(\star\star)$ in terms of converging sequences in the space of marked groups:

Lemma 3.1. *Let G be a finitely generated group and S a generating set of k elements. Then G has property $(\star)$ (resp. $(\star\star)$) iff there exists a sequence $\{(G_n, S_n)\}_{n\geq 0}$ in $\mathcal{M}_k$ such that*

i) $\lim_{n\to\infty}(G_n, S_n) = (G, S)$,

ii) each G_n contains non-abelian free subgroups (resp. is large).

Proof. Suppose G has property $(\star)$ (resp. $(\star\star)$). If G is finitely presented one can take the constant sequence $\{(G, S)\}_{n\geq 0}$. If G is not finitely presented, let $\langle s_1, \ldots, s_k \mid r_2, r_2, \ldots\rangle$ be an infinite presentation of G. Considering the sequence $G_n = \langle s_1, \ldots, s_k \mid r_1, \ldots r_n\rangle$, by assumption each G_n has non-abelian free subgroups (resp. is large) and clearly $\{(G_n, S_n)\}_{n\geq 0}$ converges to (G, S).

Conversely, if such sequence $\{(G_n, S_n)\}$ exists and H is a cover of G, by Lemma 2.3, H covers G_n for n large enough. Therefore H contains non-abelian free subgroups (resp. is large). □

Let us see an example where one can show property $(\star)$ relatively easily. Let $G = \mathbb{Z} \wr \mathbb{Z}$. We have a presentation

$$G = \langle s, t \mid [s^{t^i}, s^{t^j}] \ \forall i, j \in \mathbb{Z}\rangle;$$

indeed, any element in the right-hand side can be written as

$$\begin{aligned} & s^{m_1}t^{n_1}s^{m_2}t^{n_2}s^{m_3}t^{n_3}\cdots s^{m_\ell}t^{n_\ell} \\ & \quad = s^{m_1}\left(s^{t^{-n_1}}\right)^{m_2}\left(s^{t^{-n_1-n_2}}\right)^{m_3}\cdots\left(s^{-n_1-\cdots-n_{\ell-1}}\right)^{m_\ell}t^{n_1+\cdots n_\ell} \end{aligned}$$

for some $m_1, n_1, \ldots, m_\ell, n_\ell \in \mathbb{Z}$, and therefore as

$$\left(s^{t^{j_1}}\right)^{i_1}\left(s^{t^{j_2}}\right)^{i_2}\cdots\left(s^{t^{j_k}}\right)^{i_k}t^N$$

for appropriate $i_1, j_1, \ldots, i_k, j_k, N \in \mathbb{Z}$ with $t_1 < t_2 < \cdots < t_k$. It follows that the natural homomorphism

$$\langle s, t \mid [s^{t^i}, s^{t^j}] \ \forall i, j \in \mathbb{Z}\rangle \longrightarrow G$$

is an isomorphism.

Since $t^{-i}[s, s^{t^k}]t^i = [s^{t^i}, s^{t^{i+k}}]$, we have a second presentation

$$G = \langle s, t \mid [s, s^{t^i}] \ \forall i \in \mathbf{N}\rangle.$$

For a positive integer n, define

$$G_n = \langle s, t \mid [s, s^{t^i}], \ i = 0, \ldots, n\rangle.$$

Note that $\lim_{n\to\infty} G_n = G$ in $\mathcal{M}_2$. We have a third presentation

$$G_n = \left\langle s_0, \ldots, s_n, t \ \middle| \ \begin{array}{l} [s_i, s_j], \ 0 \le i, j \le n, \\ s_k^t = s_{k+1}, \ 0 \le k \le n-1 \end{array} \right\rangle.$$

Indeed, it can be checked that the assignments

$$\varphi_1 : s \longmapsto s_0, \;\; t \longmapsto t$$
$$\varphi_2 : s_i \longmapsto s^{t^i}, \;\; t \longmapsto t \qquad (0 \le i \le n)$$

define, between the groups of the two previous presentations, isomorphisms that are inverse to each other.

Let H_n be the free abelian subgroup of G_n generated by $s_0, \ldots, s_n$. Denote by K_n the subgroup of H_n generated by $s_0, \ldots, s_{n-1}$, and by L_n that generated by $s_1, \ldots, s_n$; observe that $K_n \simeq L_n \simeq \mathbb{Z}^n$. Let $\psi_n : K_n \longrightarrow L_n$ be the isomorphism defined by $\psi(s_{i-1}) = s_i$ for $i = 1, \ldots, n$. Then G_n is clearly the HNN-extension corresponding to the data $(H_n, \psi_n : K_n \overset{\simeq}{\rightarrow} L_n)$. By Britton's lemma, G_n contains non-abelian free subgroups. Now Lemma 3.1 shows that any finitely presented cover of G contains non-abelian free groups.

Similar finitely presented covering sequence $\mathcal{G}_n$ for the first Grigorchuk group $\mathcal{G}$, obtained by truncating the Lysenok's presentation of Section 2.8, were investigated in [GdlH01] to prove Theorem 3.1. More precisely, they have shown that for all n, $\mathcal{G}_n$ is virtually a direct product of finitely generated non-abelian free groups. This was sharpened in [BdC06] quantitatively and can be further improved:

Theorem 3.4 (Section 3.7)**.** *Let $\mathcal{G}$ and $\mathcal{G}_n$ be as above. For each $n \ge 0$, the group $\mathcal{G}_n$ has a normal subgroup H_n of index $2^{2^{n+1}+2}$ that is isomorphic to the direct product of 2^n free groups of rank 3.*

Let us survey some known methods for showing properties ($\star$) and ($\star\star$).

3.2 The Bieri-Neumann-Strebel invariant

Negative answers to Question 1 for various properties can be obtained via a theorem due to Bieri and Strebel [BS80, Theorem 5.5 and Corollary 5.6]. We state it here in a slightly weaker form for finitely presented covers instead of covers with the property FP_2:

Theorem 3.5. *(Bieri-Strebel) Let G be a finitely generated virtually metabelian group which is not finitely presented. Then G has property $(\star)$.*

This gives simultaneously a negative answer to Question 1 when the property $\mathcal{P}$ is being solvable, being amenable or not containing non-abelian free subgroups. The ideas behind Theorem 3.5 in [BS80] were further developed in [BNS87] and put into a framework which is known as the *Bieri-Neumann-Strebel invariant* (and generalizations known as Σ-invariants). These powerful tools are used to answer various questions related to metabelian groups. We refer to [BGDLH13, Appendix C] for a very short exposition about this invariant and to [Str12] for an extensive source. The BNS-invariant can also be used to show property $(\star)$ for other groups which are not necessarily metabelian. For example in [BGDLH13, Corollary C.6] it was observed that the uncountably many 2-generated groups constructed by B.H. Neumann [Neu37] have property $(\star)$. In particular since all the involved groups are elementary amenable, this shows that none of them is finitely presented. Let us remark that Theorem 3.5 is no longer true for groups of higher solvability degree. We also want to remark that for some groups for which Theorem 3.2 applies, the BNS invariants were computed by Zoran Šunić (unpublished, private communication) and show that this invariant cannot be used to show these groups have property $(\star)$.

3.3 Automatically presented groups

A different method was used by A. Erschler in [Ers07] to prove the following:

Theorem 3.6 (Erschler)**.** *The Basilica group $\mathcal{B}$ has property $(\star)$.*

The idea is to introduce the following notion:

Definition 3.2. Given an invertible automaton (A, τ) over a finite alphabet X, the *automatically presented group* $G^*(A, \tau)$ generated by the automaton is defined in the following way: Let $F = F(A)$ be the free group on the set of states of the automaton (A, τ). let N be the normal subgroup generated by all words w over $A \cup A^{-1}$ such that there exists a level n such that w represents an element in the stabilizer of level n in $G(A, \tau)$ and all sections $w_v, v \in X^n$ represent the identity in the free group $F(A)$. Define $G^*(A, \tau)$ by the quotient F/N.

Informally, instead of using the automaton to generate a group (as in Section 2.2), we use the automaton to define the relations of a presentation. It is clear from the definition that $G(A, \tau)$ is a homomorphic image of $G^*(A, \tau)$. Regarding such groups Erschler proves the following:

Theorem 3.7. *[Ers07, Theorem 1] Let G be an automatically presented group. If G is not virtually abelian, then any finitely presented cover of G contains non-abelian free subgroups.*

For an automaton (A, τ) the groups $G(A, \tau)$ and $G^*(A, \tau)$ are in general different but (as shown in [Ers07]) for the Basilica automaton they coincide and hence Theorem 3.6 is a corollary of Theorem 3.7. As also remarked in [Ers07], for the Grigorchuk automaton these groups are essentially different and hence a similar result cannot be deduced for the Grigorchuk group $\mathcal{G}$. Let us prove two small lemmas which can be used to detect when the groups $G(A, \tau)$ and $G^*(A, \tau)$ are different for an automaton. The first lemma is essentially contained in [Ers07].

Lemma 3.2. *Let (A, τ) be a finite invertible automaton over the alphabet X. Suppose there exists two non-identity states g, h and a vertex $v \in X^*$ such that $g(v) = h(v) = v$ and $g_v = g,\ h_v = h$. Then the subgroup of $G^*(A, \tau)$ generated by g and h is free.*

Proof. Let w be a word over $\{g,h\}^{\pm}$. It is clear from the assumptions on g and h that we have $w_v = w$. Hence if $w = 1$ in $\mathcal{G}(\mathcal{A})$, there is $n \in \mathbb{N}$ such that w is in the n-th level stabilizer and $w_u = 1$ freely for all u with $|u| = n$ (and hence for all u with $|u| \geq n$). Let k be large enough, so that $|v^k| \geq n$. Then $w = w_{v^k} = 1$ freely. $\square$

For the automaton generating the Grigorchuk group $\mathcal{G}$ one has $b_{111} = b$ and $c_{111} = c$ and also $b(111) = c(111) = 111$. Therefore, from Lemma 3.2 we see that the automatically presented group $G^*(A,\tau)$ contains non-abelian free groups. Therefore $G^*(A,\tau)$ is not isomorphic to the automaton group $\mathcal{G} = G(A,\tau)$.

Lemma 3.3. *Let (A,τ) be a finite invertible automaton over the alphabet X. Suppose there exists two non-identity states g,h and a vertex $v \in X^*$ such that $g(v) = h(v) = v$ and $g_v = g,\ h_v = 1$. Then the semi-group generated by g and hgh in $G^*(A,\tau)$ is free.*

Proof. Let w be a nontrivial word in the alphabet $\{g, hgh\}$ with positive powers. It is easy to see that w_v is obtained from w by deleting the occurrences of the letter h. If w represents the identity in $G^*(A,\tau)$, then there is $n \in \mathbb{N}$ such that w is in the n-th level stabilizer and $w_u = 1$ freely for all u with $|u| = n$ (and hence for all u with $|u| \geq n$). Let k be large enough, so that $|v^k| \geq n$. Then $w_{v^k} = 1$ freely. But $w_{v^k} = w_v$ is of the form g^ℓ, where ℓ is the number of occurrences of g in w. Therefore we have $g^\ell = 1$ freely, which implies that $\ell = 0$. This means that g does not occur in w, which is only possible if w is the empty word. $\square$

If (A,τ) is the automaton generating the group $\mathcal{I}$ of Section 2.2 then one has $b_{10} = b$ and $c_{10} = 1$. Also $b(10) = c(10) = 10$. Therefore by the previous lemma b and cbc generate a free semi-group in $G^*(A,\tau)$, in particular it is of exponential growth. Therefore we see that $G^*(A,\tau)$ cannot be isomorphic to $\mathcal{I}$ since the latter has intermediate growth. For our other examples similar ideas can be used to show that these groups are different.

In the next section we will see the underlying reason why these differences occur.

3.4 Self-similar covers of self-similar groups

Definition 3.3. Let (G, Φ) and (H, Ψ) be two self-similar groups of degree d. We say that (G, Φ) is a self-similar cover of (H, Ψ) if there exists an epimorphism $\pi : G \to H$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & S_d \wr G \\ \pi \downarrow & & \downarrow \widehat{\pi} \\ H & \xrightarrow{\Psi} & S_d \wr H \end{array}$$

commutes where $\widehat{\pi} = 1 \wr \pi$ is as defined in equation (2.2) of Section 2.2.

We will show that every contracting, self-replicating self-similar group has a *finitely presented* contracting self-similar cover using ideas from [Nek05]. Our exposition borrows from [Bar13].

Let (G, Φ) be a self-replicating contracting self-similar group, with nucleus $\mathcal{N} = \{n_1, \ldots, n_\ell\}$. Let $S = \{s_1, \ldots, s_\ell\}$ be a finite set given with a bijection $s_j \leftrightarrow n_j$ with $\mathcal{N}$. Let R be the set of relators in the letters of S of one the forms

$$\begin{aligned} s_i &= 1 && \text{if } n_i &&= 1 \in G, \\ s_i s_j &= 1 && \text{if } n_i n_j &&= 1 \in G, \\ s_i s_j s_k &= 1 && \text{if } n_i n_j n_k &&= 1 \in G. \end{aligned}$$

Note that these relators are of length at most 3; they are indexed by a subset of $\mathcal{N} \sqcup \mathcal{N}^2 \sqcup \mathcal{N}^3$.

Definition 3.4. The universal contracting cover of G is the finitely presented group G_0^{un} defined by the presentation with S as set of generators and R as set of relators.

The assignment $\pi^{\mathrm{un}}(s_i) = n_i$ extends to a group homomorphism

$$\pi^{\mathrm{un}} : G_0^{\mathrm{un}} = \langle S \mid R \rangle \longrightarrow G, \tag{3.1}$$

because $\pi^{\mathrm{un}}(r) = 1$ for any $r \in R$. Note that π^{un} is onto, by Proposition 2.2. We define finally

$$\widehat{\pi}^{\mathrm{un}} = 1_d \wr \pi^{\mathrm{un}} : S_d \wr G_0^{\mathrm{un}} \longrightarrow S_d \wr G \tag{3.2}$$

Remark 3.1. In particular examples, and for simplicity, it is often convenient to delete from S the generator corresponding to $1 \in \mathcal{N}$, to delete s_k if there exist $i, j \in \{1, \ldots, \ell\}$ with $n_k = n_i n_j$, and to delete one generator of every pair corresponding to $\{n, n^{-1}\} \subset \mathcal{N}$. For example, in Example 3.2, we have $\mathcal{N} = \{1, a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$ with 7 elements, and $c = a^{-1}b$, but $S = \{a, b\}$ with 2 elements. Note however that, in Example 3.1, we keep d in the generating set $\{a, b, c, d\}$ of G_0, even though $d = bc$.

Proposition 3.1. *Let (G, Φ) be a self-replicating contracting self-similar group of degree d, with nucleus $\mathcal{N}$. Assume that G is finitely generated. Let $G_0^{\mathrm{un}} = \langle S \mid R \rangle$ and $\pi^{\mathrm{un}} : G_0^{\mathrm{un}} \twoheadrightarrow G$ be the universal contracting cover and the projection of Definition 3.4. Then there exists a homomorphism*

$$\varphi_1^{\mathrm{un}} : G_0^{\mathrm{un}} \longrightarrow S_d \wr G_0^{\mathrm{un}}$$

such that the self-similar group $(G_0^{\mathrm{un}}, \varphi_1^{\mathrm{un}})$ is contracting, with nucleus S. Moreover the diagram

$$\begin{array}{ccc} G_0^{\mathrm{un}} & \xrightarrow{\varphi_1^{\mathrm{un}}} & S_d \wr G_0^{\mathrm{un}} \\ \pi^{\mathrm{un}} \downarrow & & \downarrow \widehat{\pi}^{\mathrm{un}} \\ G & \xrightarrow{\Phi} & S_d \wr G \end{array} \tag{3.3}$$

commutes, i.e., $(G_0^{\mathrm{un}}, \varphi_1^{\mathrm{un}})$ is a self-similar cover of (G, Φ).

Proof. *Step 1*, definition of φ_1^{un} Denote by ℓ the order of $\mathcal{N}$, and write $\mathcal{N} = \{n_1, \ldots, n_\ell\}$, as above. Let $i \in \{1, \ldots, \ell\}$. By Proposition 2.2, there exist $i_0, \ldots, i_{d-1} \in \{1, \ldots, \ell\}$ and $\tau_i \in S_d$ such that

$$\Phi(n_i) = (\tau_i; n_{i_0}, \ldots, n_{i_{d-1}}).$$

We set

$$\varphi_1^{\mathrm{un}}(s_i) = (\tau_i; s_{i_0}, \ldots, s_{i_{d-1}}) \in S_d \wr G_0^{\mathrm{un}}$$

and we claim that this extends to a group homomorphism φ_1^{un} as in (3.3).

Consider a relator as in Definition 3.4, say $s_i s_j s_k = 1$ (shorter generators are dealt with similarly); hence $n_i n_j n_k = 1 \in G$. Choose $x \in X$; recall that X stands for $\{0, \ldots, d-1\}$. There exist $p, q, r \in \{1, \ldots, \ell\}$ and $\tau_p, \tau_q, \tau_r \in S_d$ such that the x-coordinate and the last coordinate of $\Phi(n_i n_j n_k)$ can be written as

$$\left(n_i n_j n_k\right)_x = n_p n_q n_r \text{ and } \tau_{n_i n_j n_k} = \tau_p \tau_q \tau_r.$$

Since $n_i n_j n_k = 1 \in G$, we have

$$n_p n_q n_r = 1 \in G \ \forall x \in X \text{ and } \tau_p \tau_q \tau_r = 1 \in S_d.$$

Hence $\varphi_1^{\mathrm{un}}(s_i)\varphi_1^{\mathrm{un}}(s_j)\varphi_1^{\mathrm{un}}(s_k) = 1 \in G_0^{\mathrm{un}}$. The claim is proven.

Step 2: $(G_0^{\mathrm{un}}, \varphi_1^{\mathrm{un}})$ is a contracting group with nucleus S. For any word w in the letters of S, we have to show that there exists a vertex $v \in X^*$ such that $(w)_v \in S$. By induction on the word length, and by Lemma 2.1, it is enough to show this for a word of length 2.

Let $s_i, s_j \in S$ and $v \in X^*$ be such that $(n_i n_j)_v \in \mathcal{N}$, say $(n_i n_j)_v = n_k$. We have

$$(n_i)_{\tau_{n_j}^{(|v|)}(v)}(n_j)_v = n_k \quad \text{in } G,$$

which is a relator of length at most 3. Hence the corresponding relator $(s_i s_j)_v = s_k$ holds in S.

It follows that S is the nucleus of the group $(G, \varphi_1^{\text{un}})$.

Step 3, commutativity of the diagram. This can be checked on the set S of generators of G_0^{un}. $\square$

The universal contracting cover $(G_0^{\text{un}}, \varphi_1^{\text{un}})$ of (G, Φ) is uniquely defined by (G, Φ), and is contracting. But we believe it need not be self-replicating (even though we do not know of any specific example). In all cases, $(G_0^{\text{un}}, \varphi_1^{\text{un}})$ has quotients by finite sets of relations that are self-replicating contracting covers of (G, Φ), as described in the Definition 3.5 and Proposition 3.2. Note however that these quotients are no more uniquely defined by (G, Φ), since choices are involved. In all our examples this latter modification is not needed since the universal contracting cover of our examples are already self-replicating.

Definition 3.5. Let (G, Φ) be a self-replicating contracting self-similar group, with nucleus $\mathcal{N} = \{n_1, \ldots, n_\ell\}$; assume that G is finitely generated. Let $S = \{s_1, \ldots, s_\ell\}$ be in bijection with $\mathcal{N}$, and $(G_0^{\text{un}}, \varphi_1^{\text{un}})$ the universal contracting cover of (G, Φ), as in Definition 3.4. Let $\pi^{\text{un}} : G_0^{\text{un}} \longrightarrow G$ be as in (3.1).

Let $x \in X$ and $n_i \in \mathcal{N}$. Since the pair (G, Φ) is self-replicating, there exists[1] $g(x, n_i) \in \mathrm{Stab}_G(x)$ such that $(g(x, n_i))_x = n_i$. Since π^{un} is onto, there exists $h(x, n_i) \in G_0^{\text{un}}$ such that $\pi^{\text{un}}(h(x, n_i)) = g(x, n_i)$; moreover, since $\widehat{\pi}^{\text{un}}$ is the iden-

[1] Note that a choice is involved here.

tity on the permutations of the wreath product, we have $h(x, n_i) \in \mathrm{Stab}_{G_0^{\mathrm{un}}}(x)$. By commutativity of Diagram (3.3), we have $\pi^{\mathrm{un}}((h(x, n_i))_x) = n_i$. Set $w(x, n_i) = (h(x, n_i))_x s_i^{-1}$; then $w(x, n_i)$ belongs to the kernel of π^{un}.

Again, by commutativity of (3.3), we have $(w(x, n_i))_v \in \ker(\pi^{\mathrm{un}})$ for all $v \in X^*$. Since $(G_0^{\mathrm{un}}, \varphi_1^{\mathrm{un}})$ is contracting, the subset

$$E(x, n_i) = \{g \in G_0^{\mathrm{un}} \mid g = (w(x, n_i))_v \text{ for some } v \in X^*\}$$

of G_0^{un} is finite. Define

$$E = \bigcup_{x \in X, n \in \mathcal{N}} E(x, n) \quad \text{and} \quad H = \langle\langle E \rangle\rangle \subset G_0^{\mathrm{un}},$$

where $\langle\langle E \rangle\rangle$ denote the *normal* subgroup of G_0^{un} generated by E.

A *standard contracting cover* of G is a quotient group of the form $G_0 = G_0^{\mathrm{un}}/H$, with H as above; the image of S in G is a generating set, that we denote again (abusively) by S. Note that E is a finite subset of G_0^{un}, and consequently that G_0 is a finitely presented group.

The epimorphism π^{un} factors through a homomorphism $\pi : G_0 \longrightarrow G$, because E is a subset of $\ker \pi^{\mathrm{un}}$. It follows from the definition that the restriction of π to the generating set S of G_0 is injective.

The following proposition is the analogue of proposition 3.1 for G_0.

Proposition 3.2. *Let (G, Φ) be a self-replicating contracting self-similar group of degree d, with nucleus $\mathcal{N}$. Assume that G is finitely generated. Let G_0 and $\pi : G_0 \twoheadrightarrow G$ be a standard contracting cover of (G, Φ) and its projection to G, as in Definition 3.5.*

Then there exists a homomorphism

$$\varphi_1 : G_0 \longrightarrow S_d \wr G_0$$

such that the self-similar group (G_0, φ_1) *is contracting and self-replicating with nucleus* S*. Moreover the diagram*

$$\begin{array}{ccc} G_0 & \xrightarrow{\varphi_1} & S_d \wr S_d \\ \pi \downarrow & & \downarrow \widehat{\pi} \\ G & \xrightarrow{\Phi} & S_d \wr G \end{array} \tag{3.4}$$

commutes that is (G_0, φ_1) *is a self-similar cover of* (G, Φ)*.*

Proof. By construction of the set E, for any element $g \in E$ and any $x \in X$, we have $g_x \in E$. Hence the homomorphism $\varphi_1^{\mathrm{un}} : G_0^{\mathrm{un}} \longrightarrow S_d \wr G_0^{\mathrm{un}}$ induces a homomorphism $\varphi_1 : G_0 \longrightarrow S_d \wr G_0$. Since $(G_0^{\mathrm{un}}, \varphi_1^{\mathrm{un}})$ is contracting with nucleus S, the self-similar group (G_0, φ_1) is contracting with nucleus S.

Let $x \in X$, and $n_i \in \mathcal{N}$. We continue with the notation of Definition 3.5. By construction of G_0, the relation $h(x, n_i) = s_i$ holds in G_0; moreover $h(x, n_i)$ is an element of $\mathrm{Stab}_{G_0}(x)$. This shows that the pair (G_0, φ_1) is self-replicating.

The commutativity of diagram (3.4) can be checked on the generators of $\tilde{G}_0$. □

3.5 Finitely presented covers of contracting self-similar groups

From here to Corollary 3.1, we keep the same notation as in Definition 3.5 and Proposition 3.2 for G_0, π, and φ_1, in relation with a given contracting self-replicating self-similar group (G, Φ), with G finitely generated.

Definition 3.6. For an integer $n \geq 0$, define

(i) the homomorphism $\varphi_n : G_0 \longrightarrow S_d \wr^n G_0$ as in (2.3),

(ii) its kernel $N_n = \ker(\varphi_n)$ and the quotient $G_n = G_0/N_n$,

(iii) the homomorphism

$$\widehat{\pi}_n = \pi \wr 1_{d^n} \; : \; S_d \wr^n G_0 \longrightarrow S_d \wr^n G \tag{3.5}$$

as in (2.2); note that $\widehat{\pi}_1$ is the $\widehat{\pi}$ of (3.2).

We have $\Phi_n \pi = \widehat{\pi}_n \varphi_n$, i.e. the diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{\varphi_n} & S_d \wr^n G_0 \\ \pi \downarrow & & \downarrow \widehat{\pi}_n \\ G & \xrightarrow{\Phi_n} & S_d \wr^n G \end{array} \tag{3.6}$$

commutes. Observe that $N_0 \subset \cdots \subset N_n \subset N_{n+1} \subset \cdots$ and define

$$N = \bigcup_{n=0}^{\infty} N_n.$$

Remark 3.2. As noted in Definition 3.5, the restriction of π to S is injective. More generally, in Definition 3.6, the restriction of $\widehat{\pi}_n$ to the subset $(S^{X^n}; 1)$ of $S_d \wr^n G_0 = S_d^{(n)} \ltimes G_0^{X^n}$ is injective.

Lemma 3.4. *Let (G, Φ) be a self-similar group; assume that G is finitely generated and that (G, Φ) is faithful contracting self-replicating. With the notation above, we have*

$$N \;=\; \ker \pi, \;\; \textit{namely } G_0/N = G,$$

so that

$$\lim_{n \to \infty} G_n \;=\; G$$

in the space of marked groups on $|S|$ generators (in the sense of Section 2.3).

Proof. Let $g \in N$. Let $n \geq 1$ be such that $g \in \ker(\varphi_n)$. Then $\Phi_n \pi(g) = \widehat{\pi}_n \varphi_n(g) = 1$, hence $g \in \ker(\pi)$ by the faithfulness assumption.

Conversely, let $k \in \ker(\pi)$. On the one hand, since (G_0, φ) is contracting, there exists $n \geq 0$ such that $k_v \in S$ for all $v \in X^n$. On the other hand, $\pi(k) = 1$ implies $\pi(k_v) = 1$ for all $v \in X^n$; moreover, the $S_d^{(n)}$-coordinate of $\varphi_n(k)$ is 1, by commutativity of Diagram (3.6). Hence, by Remark (3.2), we have $k_v = 1$ for all $v \in X^n$, and therefore $k \in N_n = \ker(\varphi_n)$, a fortiori $k \in N$. $\square$

Remark 3.3. Lemma 3.4 asserts that if one starts with a standard contracting cover of a faithful, finitely generated, self-similar group (G, Φ), then one obtains a sequence of groups which converge to G in the appropriate space of finitely generated marked groups.

In general one can follow the same procedure starting with an arbitrary self-similar cover of (G, Φ). In this case one will obtain a sequence of groups whose limit is a cover of G but may not be equal to G. In particular one can start with a free group of appropriate rank $(F_k, \bar{\Phi})$, where $\bar{\Phi}$ is induced by Φ. The limiting group is easily seen to be equal to the automatically presented group defined by an automaton whenever G is defined by an automaton (see Definition 3.2). This gives another viewpoint of automatically presented groups.

Lemma 3.5. *In the situation of the previous lemma, for all $n \geq 1$, we have*

$$N_n = \varphi_1^{-1}(N_{n-1}^d)$$

so that $\varphi_1 : G_0 \longrightarrow G_0 \wr S_d$ induces a homomorphism

$$\psi_n : \begin{cases} G_n \longrightarrow & S_d \wr G_{n-1} \\ gN_n \longmapsto & \left(\tau_g^{(1)}; \left(\varphi_1(g)_x N_{n-1}\right)_{x \in X}\right). \end{cases}$$

Moreover ψ_n is injective.

Proof. For $g \in G$, write

$$\varphi(g) = \left(\tau_g^{(1)}; (g_x)_{x\in X}\right) \quad \text{and} \quad \varphi_n(g) = \left(\tau_g^{(n)}; (g_v)_{v\in X^n}\right). \tag{3.7}$$

Assume first that $g \in N_n$. Thus $(g_x)_{v'} = 1$ and $\tau_{g_x}^{(n-1)} = 1$ for all $x \in X$ and $v' \in X^{n-1}$. This can be written

$$\varphi_{n-1}(g_x) = \left(\tau_{g_x}^{(n-1)}; ((g_x)_{v'})_{v'\in X^{n-1}}\right) = 1 \quad \forall x \in X,$$

namely $g_x \in N_{n-1}$ $\forall x \in X$. We have checked that $\varphi_1(N_n) \subset N_{n-1}^d$, and $N_n \subset \varphi_1^{-1}(N_{n-1}^d)$ follows.

Assume now that $g \in \varphi_1^{-1}(N_{n-1}^d)$, namely that $(g_x)_{v'} = 1$ and $\tau_{g_x}^{(n-1)} = 1$ for all $x \in X$ and $v' \in X^{n-1}$. This can be written $g_v = 1$ for all $v \in X^n$ and $\tau_g^{(n)} = 1$, namely $g \in N_n$. Hence $\varphi_1^{-1}(N_{n-1}^d) \subset N_n$. $\square$

The next theorem is a detailed version of Theorem 3.2.

Theorem 3.8. *Let (G, Φ) be a self-similar group; assume that G is finitely generated and that (G, Φ) is faithful contracting self-replicating. Let G_0 be a standard contracting cover, as in Definition 3.5.*

Assume that G_0 contains non-abelian free subgroups. Then, for each $n \geq 0$, the group G_n of Definition 3.6 contains non-abelian free subgroups. More generally, every finitely presented cover of G contains non-abelian free subgroups.

Assume moreover that G_0 is large. Then every finitely presented cover of G is large.

Proof. Let $S_d^{(0)}$ be the subgroup of S_d of permutations fixing the letter $x = 0$. For

$n \geq 1$, let H_n be the finite index subgroup of G_n defined by

$$H_n \ = \ \psi_n^{-1}(G_{n-1}^{\{0,1,\ldots,d-1\}} \rtimes S_d^{(0)}),$$

where ψ_n is as in Lemma 3.5. Projection onto the first coordinate (i.e. the coordinate $x = 0$)

$$p_n^{(0)} \ : G_n^{\{0,1,\ldots,d-1\}} \rtimes S_d^{(0)} \longrightarrow G_n$$

defined by

$$p_n^{(0)}((g_x N_n)_{x \in X}, \tau) \ = \ g_0 N_n$$

is a group homomorphism. It turns out that the composition

$$q_n^{(0)} \ : H_n \xrightarrow{\psi_n} \psi_n(H_n) \xrightarrow{p_{n-1}^{(0)}} G_{n-1}$$

defines a group homomorphism from H_n to G_{n-1}.

Given a generator sN_{n-1} of G_{n-1} (where s is a generator of G_0), using the self-replicating property of (G_0, φ_1) let $h \in St_{G_0}(0)$ such that $\varphi_1(h)_0 = s$. It turns out that $q^{(0)}(hN_n) = sN_{n-1}$ which shows that $q_n^{(0)}$ is onto G_{n-1}. The conclusion is that for each $n \geq 1$, G_n contains a finite index subgroup H_n which maps onto G_{n-1}.

Therefore, if G_0 contains non-abelian free subgroups (respectively is large), by induction on n, each G_n will contain non-abelian free subgroups (respectively will be large). Then by Lemma 3.4 and corollary 3.1 below, every finitely presented cover of G will contain non-abelian free subgroups (respectively will be large). □

Corollary 3.1. *Let G be as in Theorem 3.8. If G_0 contains non-abelian free subgroups, then G is infinitely presented.*

Proof. Since G_0 does contain non-abelian free subgroups, by assumption, and G

does not, by [Nek05, Theorem 4.2], G cannot be finitely presented, by the previous theorem. $\square$

Let us apply Theorem 3.8 to the groups mentioned in Section 2.2.

Example 3.1. The first Grigrochuk group $\mathcal{G}$ is contracting with nucleus $\mathcal{N} = \{1, a, b, c, d\}$. The universal contracting cover of Definition 3.4 has the presentation

$$G_0 = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd \rangle \simeq C_2 * V,$$

where C_2 is now the group $\{1, a\}$ and V the Klein Vierergruppe $\{1, b, c, d\}$, isomorphic to $C_2 \times C_2$. It is easily seen that the universal contracting cover is self-replicating and hence is a standard contracting cover in the sense of Definition 3.5.

In general, if A, B are two non-trivial finite groups, then then kernel of the map $A * B \to A \times B$ is free with basis $\{[a, b] \mid a \in A \setminus \{1\}, b \in B \setminus \{1\}\}$ (see [Ser03, Page 6]). Hence G_0 contains a free group of rank 3 of index 8 hence is large. Therefore Theorem 3.1 is a particular case of 3.8.

Example 3.2. The Basilica group $\mathcal{B}$ is also contracting with nucleus given by $\mathcal{N} = \{1, a^{\pm}, b^{\pm}, (a^{-1}b)^{\pm}\}$.

In general there is no known algorithm for deciding whether a given self-similar group is contracting or not and finding its nucleus. But there is a partial algorithm due to Y.Muntyan and D.Savchuk implemented into a beautiful GAP package which can be found at `http://www.gap-system.org/Packages/automgrp.html`. Let us demonstrate the functionality of this package for finding the nucleus of the Basilica group:

```
gap> B:=AutomatonGroup("a=(b,1)(1,2),b=(a,1)");
< a, b >
```

```
gap> IsContracting(B);
true
gap> FindNucleus(B);
Trying generating set with 5 elements
Elements added:[ a^-1*b, b^-1*a ]
Trying generating set with 7 elements
[ [ 1, a, b, a^-1, b^-1, a^-1*b, b^-1*a ],
[ 1, a, b, a^-1, b^-1, a^-1*b, b^-1*a ],
  <automaton> ]
```

Note that this package uses right hand notation for denoting wreath recursions. It can also be used to find relations of length at most 3 between the elements in the nucleus:

```
gap> FindGroupRelations(B,3);
[  ]
```

Hence we see that there are no relations of length at most 3 between the elements in the nucleus. Therefore the universal contracting cover has the presentation

$$G_0 \;=\; \langle a, b \mid \emptyset \rangle \;\cong\; F_2.$$

which again is self-replicating.

Recalling Remark 3.3, this is another proof of the fact that the automatically presented group defined by the Basilica automaton is isomorphic to the Basilica group.

Example 3.3. For the Gupta-Sidki group $\mathcal{GS}$ the nucleus is $\mathcal{N} = \{1, a, a^{-1}, b, b^{-1}\}$ and the universal contracting cover (which is self-replicating)

$$G_0 \;=\; \langle a, b \mid a^3, b^3 \rangle \;\simeq\; C_3 * C_3,$$

is again large. Therefore $\mathcal{GS}$ has property $(\star\star)$.

Example 3.4. For $\mathcal{I}$ its nucleus is

$$\mathcal{N} \;=\; \{1, a, b, c\}$$

and the only non-trivial relators of length ≤ 3 among elements of $\mathcal{N}$ are $a^2 = b^2 = c^2 = 1$. Hence the universal contracting cover

$$G_0 \;=\; \langle a, b, c \mid a^2, b^2, c^2 \rangle \;\simeq\; C_2 * C_2 * C_2$$

is large.

Example 3.5. For the Fabrykowski-Gupta group $\mathcal{FG}$ the nucleus is

$$\mathcal{N} \;=\; \{1, a, a^{-1}, b, b^{-1}\}.$$

The universal contracting cover is

$$G_0 \;=\; \langle a, b \mid a^3, b^3 \rangle \;\simeq\; C_3 * C_3.$$

It is self-replicating. Hence this group also has property $(\star\star)$.

Example 3.6. For the Hanoi Towers group $\mathcal{H}$ the nucleus is

$$\mathcal{N} = \{1, a, b, c\}$$

The universal contracting cover is

$$G_0 = \langle a, b, c \mid a^2, b^2, c^2 \rangle \simeq C_2 * C_2 * C_2$$

is self-replicating and large. Hence $\mathcal{H}$ has property ($\star\star$).

Let us end this section with two questions:

Question 4. *Let (G, Φ) be a faithful, contracting, self-replicating self-similar group. Assume that G is not virtually nilpotent. Is it true that a standard contracting cover of G has non-abelian free subgroups? or is large?*

Question 5. *Let (G, Φ) be a faithful, contracting, self-replicating self-similar group. Assume moreover that G is a (weakly) branch group. Is it true that a standard contracting cover of G has non-abelian free subgroups? or is large?*

A positive answer will show that all such groups will have property ($\star$) (or ($\star\star$)). All our examples are either branch or weakly branch (see Section 2.5). Note that such groups are never finitely presented by a theorem of Bartholdi [Bar03].

3.6 The analogue of Theorem 3.1 for the Grigorchuk 2-groups

In this section we will prove an analogue of Theorem 3.1 for the Grigorchuk 2-groups of Section 2.4.

We keep the notations of Section 2.4.

Definition 3.7. Recall we have an injective homomorphism

$$\Phi_\omega^{(1)} = \Phi_\omega \, : \, G_\omega \longrightarrow S_2 \wr G_{\sigma(\omega)}$$

which induces a sequence of homomorphisms $\left(\Phi_\omega^{(n)}\right)_{n\geq 1}$ defined inductively by:

$$\Phi_\omega^{(n)} \, : \, G_\omega \overset{\Phi_\omega^{(n-1)}}{\longrightarrow} S_2 \wr^{n-1} G_{\sigma^{n-1}(\omega)} \overset{\Phi_{\sigma^{n-1}(\omega)}^{(1)} \wr 1_{d^{n-1}}}{\longrightarrow} S_2 \wr^n G_{\sigma^n(\omega)}$$

Lemma 3.6 (contraction in G_ω)**.** *Let $\omega \in \Omega$. We keep the notation above.*

(i) For each $n \geq 1$, the homomorphism $\Phi_\omega^{(n)}$ is injective.

(ii) For all $g \in G_\omega$, there exists an integer $n \geq 1$ such that

$$\Phi_\omega^{(n)}(g) = \left(\tau_g^{(n)}; (g_v)_{v \in X^n}\right)$$

with $g_v \in \{1, a, b_{\sigma^n(\omega)}, c_{\sigma^n(\omega)}, d_{\sigma^n(\omega)}\} \; \forall \, v \in X^n$ and $\tau_g^{(n)} \in S_2$.

Proof. By induction on the length of g, in the sense of (2.10). □

The main theorem of this section is the following:

Theorem 3.9. *For $\omega \in \Omega_+$, any finitely presented cover of G_ω is large.*

Remark 3.4. (1) Let $\omega \in \Omega_-$. Any finitely presented cover of the infinitely presented group $\widetilde{G}_\omega$ contains non-abelian free groups, by Theorem 3.5. As recorded in Theorem 2.1, the group G_ω is virtually free abelian, and finitely presented. For example, if ω is the constant sequence $000\cdots$, then G_ω is the infinite dihedral group.

(2) If we replace "is large" by "contains non-abelian free subgroups" in Theorem 3.9, the resulting statement has a short proof. More precisely:

> *For any $\omega \in \Omega$, any finitely presented cover of $\widetilde{G}_\omega$ has non-abelian free subgroups.*

Indeed, let $(\omega_n)_{n\geq 1}$ be a sequence of eventually constant sequences converging to ω in Ω. Then $\widetilde{G}_{\omega_n}$ is virtually metabelian and infinitely presented for all $n \geq 1$ (Theorem 2.1), and $(\widetilde{G}_{\omega_n})_{n\geq 1}$ converges to $\widetilde{G}_{\omega}$. Let E be a finitely presented cover of $\widetilde{G}_{\omega}$. Then E is a cover of G_{ω_n} for n large enough (Proposition 2.3). Hence Bieri-Strebel Theorem 3.5 shows that E contains non-abelian free subgroups.

From now on, we assume that

$$\omega \in \Omega_+.$$

Our strategy for the proof of Theorem 3.9 is to adapt to the present context the steps of Section 3.5.

The following definition should be compared with Definition 3.6. Note however that G_0 has not quite the same meaning here and there.

Definition 3.8. Set again

$$G_0 = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd\rangle \simeq C_2 * V,$$

as in Example 3.1. Observe that any element of G_0 can be written as

$$(*)a * a * \cdots a(*) \tag{3.8}$$

with $* \in \{b, c, d\}$, $(*) \in \{1, b, c, d\}$, and $n \geq 0$ occurrences of a (compare with Equation (2.10).

For $i \in \{0, 1, 2\}$, set

$$\varphi_i(a) = \tau(1, 1) \text{ for all } i \in \{0, 1, 2\}$$

and

$$\begin{aligned}
\varphi_0(b) &= (a,b) & \varphi_1(b) &= (a,b) & \varphi_2(b) &= (1,b)\\
\varphi_0(c) &= (a,c) & \varphi_1(c) &= (1,c) & \varphi_2(c) &= (a,c)\\
\varphi_0(d) &= (1,d) & \varphi_1(d) &= (a,d) & \varphi_2(d) &= (a,d).
\end{aligned}$$

It is easy to check that these formulas define homomorphisms

$$\varphi_i : G_0 \longrightarrow S_2 \wr G_0 \quad (i = 0, 1, 2).$$

Set $\varphi_\omega^{(1)} = \varphi_{\omega_1}$ and define, inductively for $n \geq 2$, homomorphisms

$$\varphi_\omega^{(n)} : G_0 \xrightarrow{\varphi_\omega^{(n-1)}} S_2 \wr^{n-1} G_0 \xrightarrow{\varphi_{\omega_n} \wr 1_{2^n}} S_2 \wr^n G_0$$

For $n \geq 1$, set

$$N_{\omega,n} = \ker(\varphi_\omega^{(n)}) \text{ and } G_{\omega,n} = G_0/N_{\omega,n}.$$

We have natural homomorphisms

$$\begin{aligned}
\pi_\omega &: G_0 \longrightarrow G_\omega\\
\widehat{\pi}_\omega = \widehat{\pi}_{\omega,1} &: S_2 \wr G_0 \longrightarrow S_2 \wr G_{\sigma(\omega)}\\
\widehat{\pi}_{\omega,n} &: S_2 \wr^n G_0 \longrightarrow S_2 \wr^n G_{\sigma^n(\omega)}
\end{aligned}$$

(Compare with (3.1), (3.2), and (3.5), but note that $\widehat{\pi}_{\omega,1} = \pi_\omega \wr 1_2$ *does not hold* here.)

The next lemma is about diagrams analogous to (3.3) and (3.6). Its proof uses an argument similar to one in the proof of Proposition 3.1, and will be omitted.

Lemma 3.7. *The diagram*

$$\begin{array}{ccc} G_0 & \stackrel{\varphi_\omega^{(n)}}{\longrightarrow} & S_2 \wr^n G_0 \\ \pi_\omega \downarrow & & \downarrow \widehat{\pi}_{\omega,n} \\ G & \stackrel{\Phi_\omega^{(n)}}{\longrightarrow} & S_2 \wr^n G_{\sigma^n(\omega)} \end{array} \tag{3.9}$$

commutes for each $n \geq 1$.

The next lemma is analogous to Step 2 in the proof of Proposition 3.1.

Lemma 3.8 (contraction in G_0)**.** *For all $k \in G_0$, there exists an integer $n \geq 1$ such that*

$$\varphi_\omega^{(n)}(k) = \left(\tau_k^{(n)}; (k_v)_{v \in X^n}\right)$$

with $k_v \in \{1, a, b, c, d\}\ \forall\, v \in X^n$ and $\tau_k^{(n)} \in S_2$.

Proof. by induction on the length of k, in the sense of (3.8). □

Define now

$$N_\omega = \bigcup_{n \geq 1} N_{\omega,n}$$

(compare with Definition 3.6). The two following lemmas are appropriate modifications of Lemmas 3.4 and 3.5; we repeat the proof for the first one, and not for the second one.

Lemma 3.9. *We have*

$$N_\omega = \ker\left(\pi_\omega : G_0 \longrightarrow G_\omega\right), \ \textit{namely}\ G_\omega \simeq G_0/N_\omega,$$

so that

$$\lim_{n\to\infty} G_{\omega,n} = G_\omega$$

in the space of marked groups on 4 *generators.*

Proof. Let $g \in N$. Let $n \geq 1$ be such that $g \in \ker(\varphi_\omega^{(n)})$. Since $\Phi_\omega^{(n)}\pi_\omega(g) = \widehat{\pi}_{\omega,n}\varphi_\omega^{(n)}(g)$, we have $\pi_\omega(g) = 1$ by Lemma 3.6.i.

Conversely, let $k \in G_0$. There exists $n \geq 0$ such that $\left(\varphi_\omega^{(n)}(k)\right)_v \in \{1, a, b, c, d\}$ for all $v \in X^n$, by Lemma 3.8. Assume that $k \in \ker(\pi_\omega)$. Then $\widehat{\pi}_{\omega,n}(\varphi_\omega^{(n)}(k)) = 1$. As $\widehat{\pi}_{\omega,n}$ is injective "on generators" (in a sense similar to that of Remark 3.2), we have $\varphi_\omega^{(n)}(k) = 1$, and therefore $k \in N_{\omega,n} \subset N_\omega$.

(Note that the hypothesis "$\omega \in \Omega_+$" is necessary for the previous argument. If ω_n were eventually constant, one of $b_{\sigma^n(\omega)}, c_{\sigma^n(\omega)}, d_{\sigma^n(\omega)}$ would be the identity of $G_{\sigma^n(\omega)}$ for n large enough). □

Lemma 3.10. *In the situation of the previous lemma, we have for all* $n \geq 1$

$$\varphi_\omega^{(1)}(N_{\omega,n}) \subset N_{\omega,n-1}^2 \subset G_0 \wr S_2 \ \text{ and } \ \left(\varphi_\omega^{(1)}\right)^{-1}(N_{\omega,n-1}^2) \subset N_{\omega,n}.$$

It follows that $\varphi_\omega^{(1)} : G_0 \longrightarrow G_0 \wr S_2$ *induces a homomorphism*

$$\psi_\omega^{(n)} : \begin{cases} G_{\omega,n} \longrightarrow & S_2 \wr G_{\omega,n-1} \\ gN_{\omega,n} \longmapsto & \left(\tau_g^{(1)}; \left((\varphi_{\omega_n}(g))_x N_{\omega,n-1}\right)_{x\in X}\right) \end{cases}$$

which is injective.

Proposition 3.3. *For each* $\omega \in \Omega_+$ *and* $n \geq 0$*, the group* $G_{\omega,n}$ *is large.*

Proof. The group $G_0 = C_2 * V$ has a free subgroup of finite index, indeed a subgroup isomorphic to F_3 of index 8. For $n \geq 1$, because of the previous lemma and as in the proof of Theorem 3.8, there exists a subgroup of index 2 in $G_{\omega,n}$ and a homomorphism from this subgroup onto $G_{\omega,n-1}$. It follows by induction on n that $G_{\omega,n}$ is large. □

End of proof of Theorem 3.9. Since $G_{\omega,n}$ is large for n large enough, it follows from Lemma 3.9 and Corollary 3.1 that any cover of G_ω is large. $\square$

Definition 3.9. For $\omega \in \Omega$, let M_ω denote the kernel of the defining cover $F_4 \twoheadrightarrow G_\omega$; in other terms, M_ω is the inverse image of N_ω by the epimorphism $F_4 \twoheadrightarrow G_0$ mapping the four generators of F_4 onto $a, b, c, d \in G_0$. For a subset Ψ of Ω, the **Ψ-universal group** is the group

$$\mathcal{U}_\Psi \;=\; F_4 / \bigcap_{\omega \in \Psi} M_\omega.$$

For example, $\mathcal{U}_\emptyset = \{1\}$, and $\mathcal{U}_{\{\omega\}} = G_\omega$ for all $\omega \in \Omega$.

The terminology is justified by cases that have appeared in the literature, with Ψ large. For example, let Λ denote the subset of Ω_0 of sequences that are concatenations of blocks $012, 120, 201$. Then $\mathcal{U}_\Lambda$ has uncountably many quotients (a consequence of Proposition 2.1.iv); it has intermediate growth, and therefore is amenable (established in [Gri11b, Theorem 9.7]).

Suppose that Ψ contains some $\omega \in \Omega_+$. Then any cover of $\mathcal{U}_\Psi$ is a cover of G_ω. Theorem 3.9 implies:

Corollary 3.2. *For any $\Psi \subset \Omega$ such that $\Psi \cap \Omega_+ \neq \emptyset$, the Ψ-universal group $\mathcal{U}_\Psi$ is infinitely presented, and any finitely presented cover of it is large.*

In particular, this corollary solves the first part of Problem 9.5 in [Gri05], by showing that $\mathcal{U}_\Omega$ is infinitely presented.

3.7 The first Grigorchuk group

In this section we will prove Theorem 3.4. We will describe a sequence of finitely presented groups $\mathcal{G}_n$ which converge to $\mathcal{G}$. Note however that the sequence we are going to describe is different from the sequence obtained via the general result obtained in Section 3.5.

Recall the Lysenok's presentation for $\mathcal{G}$ discussed in Section 2.8. Let us set

$$\mathcal{G}_{-1} = \left\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1 \right\rangle \simeq C_2 * V,$$

and denote by S the system of four involutions $\{a, b, c, d\}$ generating $\mathcal{G}_{-1}$.

Elements in $\mathcal{G}_{-1}$ are in natural bijection with "reduced words" of the form

$$t_0 a t_1 a \cdots a t_{k-1} a t_k$$

with $k \geq 0$, $t_1, \ldots, t_{k-1} \in \{b, c, d\}$, and $t_0, t_k \in \{\emptyset, b, c, d\}$. Throughout the remainder of this section, we use the same symbol to denote an element of $\mathcal{G}_{-1}$ and its image in any quotient of $\mathcal{G}_{-1}$, in particular in $\mathcal{G}$; thus, $S = \{a, b, c, d\}$ denotes a set of generators in $\mathcal{G}_{-1}$ *and* in any quotient of $\mathcal{G}_{-1}$.

The substitution σ defined by

$$\sigma(a) = aca, \;\; \sigma(b) = d, \;\; \sigma(c) = b, \;\; \sigma(d) = c$$

extends to reduced words, for example $\sigma(abac) = acadacab$, and the resulting map

$$\sigma \,:\, \mathcal{G}_{-1} \longrightarrow \mathcal{G}_{-1}$$

is a group endomorphism. Define

$$\begin{aligned} u_0 &= (ad)^4 & u_n &= \sigma^n(u_0) \quad \forall n \geq 0 \\ v_0 &= (adacac)^4 & v_n &= \sigma^n(v_0) \quad \forall n \geq 0 \end{aligned}$$

Definition 3.10. For $n \geq 0$, define a pair $(\mathfrak{G}_n, S) \in \mathcal{M}_4$ by

$$\mathcal{G}_n = \left\langle a, b, c, d \;\middle|\; \begin{array}{l} a^2 = b^2 = c^2 = d^2 = bcd = 1 \\ \\ u_0 = \cdots = u_n = v_0 = \cdots = v_{n-1} = 1 \end{array} \right\rangle$$

$$S = \{a, b, c, d\} \subset \mathcal{G}_n.$$

Observe that $\lim_{n\to\infty}(\mathcal{G}_n, S) = (\mathcal{G}, S)$ in $\mathcal{M}_4$, and that there are natural surjections $\mathcal{G}_{-1} \twoheadrightarrow \mathcal{G}_n \twoheadrightarrow \mathcal{G}$ for all $n \geq 0$.

Theorem 3.10. *For each $n \geq 0$, the group $\mathcal{G}_n$ has a normal subgroup H_n of index $2^{2^{n+1}+2}$ which is isomorphic to the direct product of 2^n free groups of rank 3.*

Remark 3.5. (i) A weaker result was first established in [GdlH01]: For each $n \geq 0$, $\mathcal{G}_n$ contains a subgroup of finite index isomorphic to the direct product of 2^n copies of finitely generated non-abelian free groups. This by itself implies that any finitely presented cover of $\mathcal{G}$ contains non-abelian free subgroups.

(ii) The result of [GdlH01] was improved in [BdC06]: For each $n \geq 0$, the group $\mathfrak{G}_n$ has a normal subgroup H_n of index 2^{α_n}, where $\alpha_n \leq (11 \cdot 4^n + 1)/3$, and H_n is a subgroup of index 2^{β_n} in a finite direct product of 2^n non-abelian free groups of rank 3, where $\beta_n \leq (11 \cdot 4^n - 8)/3 - 2^n$.

(iii) Our proof of Theorem 3.10 is split in several lemmas, until 3.16.

Define first

$$\begin{aligned} B_0 &= \langle\langle b \rangle\rangle_{\mathcal{G}_0}, \\ \Xi_0 &= \langle b, c, d, aba, aca, ada \rangle_{\mathcal{G}_0}, \\ D_0 &= \langle a, d \rangle_{\mathcal{G}_0}, \\ D_0^{\mathrm{diag}} &= \langle (a, d), (d, a) \rangle_{\mathcal{G}_0}. \end{aligned}$$

It is easy to check that $D_0^{\mathrm{diag}} \cap (B_0 \times B_0) = \{1\}$, and that D_0^{diag} normalizes $B_0 \times B_0$. The assignment

$$\begin{array}{llcll} b & \mapsto (a,c) & & aba & \mapsto \quad (c,a) \\ c & \mapsto (a,d) & & aba & \mapsto \quad (d,a) \\ d & \mapsto (1,b) & & aba & \mapsto \quad (b,1) \end{array}$$

extends to a group homomorphism $\psi_0 : \Xi_0 \longrightarrow \mathcal{G}_0 \times \mathcal{G}_0$ [GdlH01, Proposition 1].

For each $n \geq 0$, define now

$$\begin{aligned} N_n &= \langle\langle u_0, \ldots, u_n, v_0, \ldots, v_{n-1} \rangle\rangle_{\mathcal{G}_0}; \text{ observe that } N_n \subset \Xi_0; \\ \mathcal{G}_n &= \mathcal{G}_0 / N_n \text{ and } \pi_n : \mathcal{G}_0 \twoheadrightarrow \mathcal{G}_n \text{ the canonical projection;} \\ B_n &= \langle\langle b \rangle\rangle_{\mathcal{G}_n} = \pi_n(B_0); \\ \Xi_n &= \langle b, c, d, aba, aca, ada \rangle_{\mathcal{G}_n} = \pi_n(\Xi_0); \\ D_n^{\mathrm{diag}} &= \langle (a,d), (d,a) \rangle_{\mathcal{G}_n \times \mathcal{G}_n}; \\ \sigma_n &: \mathcal{G}_{n-1} \longrightarrow \mathcal{G}_n, \ gN_{n-1} \longmapsto \sigma(g) N_n \quad \text{(for } n \geq 1 \text{ only).} \end{aligned}$$

For the definition of the homomorphism σ_n, note that $\sigma(N_{n-1}) \subset N_n$.

Lemma 3.11 ([GdlH01], Lemma 3)**.** *Let B_0 denote the* normal *subgroup of $\mathcal{G}_0$ generated by b. Then:*

(i) B_0 is of index 8 in $\mathcal{G}_0$;

(ii) B_0 is generated by the four elements

$\xi_1 := b$, $\xi_2 := aba$, $\xi_3 := dabad$, $\xi_4 = adabada$;

(iii) B_0 has the presentation $\langle \xi_1, \xi_2, \xi_3, \xi_4 \mid \xi_1^2 = \xi_2^2 = \xi_3^2 = \xi_4^2 = 1 \rangle$;

(iv) B_0 contains N_n for all $n \geq 1$.

Lemma 3.12 ([GdlH01], mostly Proposition 10). *(i) The kernel and the image of the homomorphism ψ_0 are given by*

$$\begin{aligned}\ker(\psi_0) &= \langle\langle u_1, v_0\rangle\rangle_{\Xi_0}, \\ \mathrm{Im}(\psi_0) &= (B_0 \times B_0) \rtimes D_n^{\mathrm{diag}} \ \textit{of index } 8 \textit{ in } \mathcal{G}_0 \times \mathcal{G}_0.\end{aligned}$$

(ii) For $n \geq 1$, the homomorphism ψ_0 induces an isomorphism

$$\psi_n : \Xi_n \xrightarrow{\simeq} (B_{n-1} \times B_{n-1}) \rtimes D_{n-1}^{\mathrm{diag}} <_8 \mathcal{G}_{n-1} \times \mathcal{G}_{n-1}$$

where $<_8$ indicates that the left-hand side is a subgroup of index 8 in the right-hand side.

Set $K_0 = \langle\langle (ab)^2 \rangle\rangle_{\mathcal{G}_0}$; observe that $K_0 \subset B_0$.

Lemma 3.13. *(i) The subgroup K_0 is of index 2 in B_0. It is generated by*

$$t = (ab)^2 \quad v = (bada)^2 \quad w = (abad)^2$$

Moreover K_0 contains N_n for $n \geq 1$.

(ii) The group K_0 is a free group of rank 3.

Proof. (i) This follows from [dlH00, Page 230]. Since B_0 contains N_n and each u_n, v_n is a fourth power, necessarily N_n is contained in K_0.

For (ii), see [BdC06, Proposition 4], where the proof uses Kurosh's Theorem. Alternatively one can use the Reidemeister-Schreier method to find a presentation for K_0 and see that it is indeed free of rank 3. □

Lemma 3.14. *If g is an element of B_{n-1} then*

$$\psi_n(\sigma_n(g)) \;=\; (1,g) \quad \textit{and} \quad \psi_n(a\sigma_n(g)a) \;=\; (g,1).$$

Proof. For the generators of B_{n-1} that are images of those of Lemma 3.11 for B_0, we have

$$\begin{aligned}
\psi_n(\sigma_n(b)) &= \psi_n(d) = (1,b),\\
\psi_n(\sigma_n(aba)) &= \psi_n(acadaca) = (d^2,aba) = (1,aba),\\
\psi_n(\sigma_n(dabad)) &= \psi_n(cacadacac) = (ad^2a,dabad) = (1,dabad),\\
\psi_n(\sigma_n(adabada)) &= \psi_n(acacacadacacaca) = (1,adabada),
\end{aligned}$$

and this shows the first equality. The second follows because, if $\psi_n(h) = (h_0,h_1)$, then $\psi_n(aha) = (h_1,h_0)$. □

Let $K_n = K_0/N_n$. It is a normal subgroup of $\mathcal{G}_n$ contained in B_n.

Lemma 3.15. *Let $n \geq 1$.*

(i) We have $\sigma_n(K_{n-1}) \subset K_n \subset B_n$.

(ii) If H_{n-1} is a subgroup of K_{n-1}, then $\psi_n^{-1}(H_{n-1}\times H_{n-1}) \subset K_n$.

Proof. (i) Let t,v,w be now the canonical images in K_n of the elements of K_0 denoted by the same symbols in Lemma 3.13. On the one hand, we have $\psi_n(\sigma_n(t)) = (1,t)$ by Lemma 3.14. On the other hand, we have

$$\psi_n(w) = \psi_n(aba)\psi_n(d)\psi_n(aba)\psi_n(d) = (cc,abab) = (1,t)$$

by the definitions of ψ_n and w. Hence $\sigma_n(t) = w \in K_n$ by Lemma 3.12.ii.

Let $g_1 \in \mathcal{G}_{n-1}$. From the definition of ψ_n, we see that the composition $\Xi_n \longrightarrow \mathcal{G}_{n-1}$

of ψ_n with a projection onto one of the factors is onto. Hence there exists $g \in \Xi_n$ and $g_0 \in \mathcal{G}_{n-1}$ such that $\psi_n(g) = (g_0, g_1)$. We have as above[2] $\psi_n(\sigma_n(t^{g_1})) = (1, t^{g_1})$ and

$$\psi_n(w^g) = \psi_n(w)^{\psi_n(g)} = (1, t)^{\psi_n(g)} = (1, t^{g_1}),$$

and therefore $\sigma_n(t^{g_1}) = w^g$. Since K_n is a normal subgroup of $\mathcal{G}_n$ containing w, we have $\sigma_n(t^{g_1}) \in K_n$ for all $g_1 \in \mathcal{G}_{n-1}$. The inclusion $\sigma_n(K_{n-1}) \subset K_n$ follows, because K_{n-1} is generated by t as a normal subgroup of $\mathcal{G}_{n-1}$.

(ii) Let $(h_0, h_1) \in H_{n-1} \times H_{n-1}$. We have

$$\psi_n^{-1}(h_0, h_1) \;=\; a\sigma_n(h_0)a\,\sigma_n(h_1)$$

by Lemma 3.14, and the right-hand side is in K_n by (i). $\square$

Set $H_0 = K_0$. For $n \geq 1$, define inductively

$$H_n \;=\; \psi_n^{-1}(H_{n-1} \times H_{n-1}).$$

The definition makes sense by Lemma 3.15.ii. The following lemma finishes the proof of Theorem 3.10.

Lemma 3.16. *Let $n \geq 0$, and the notation be as above.*

(i) H_n is a normal subgroup of $\mathcal{G}_n$ contained in K_n.

(ii) The group H_n is a direct product of 2^n free groups of rank 3.

(iii) Its index is given by $[\mathcal{G}_n : H_n] = 2^{(2^{n+1}+2)}$.

Proof. For $n = 0$, the three claims follow from Lemmas 3.11 and 3.13. We suppose now that $n \geq 1$ and that the lemma holds for $n - 1$.

[2]Remember that $t^h = h^{-1}th$.

(i) The group H_n is clearly normal in Ξ_n, by Lemma 3.12.ii. To show that H_n is normal in $\mathcal{G}_n$, it suffices to check that $aH_na \subset H_n$, because $\mathcal{G}_n$ is generated by Ξ_n (of index 2 in $\mathcal{G}_n$) and a. Let $h \in H_n$. Let $h_0, h_1 \in H_{n-1}$ be defined by $\psi_n(h) = (h_0, h_1)$. Then $\psi_n(aha) = (h_1, h_0) \in H_{n-1} \times H_{n-1}$, and therefore $aha \in H_n$.

(ii) This is a straightforward consequence of the isomorphism $H_n \simeq H_{n-1} \times H_{n-1}$, see again Lemma 3.12.

(iii) By the induction hypothesis, we have

$$\begin{aligned}
&[(B_{n-1} \times B_{n-1}) \rtimes D^{\mathrm{diag}}_{n-1} : H_{n-1} \times H_{n-1}] \\
&\qquad = \frac{[\mathcal{G}_{n-1} \times \mathcal{G}_{n-1} : H_{n-1} \times H_{n-1}]}{[\mathcal{G}_{n-1} \times \mathcal{G}_{n-1} : (B_{n-1} \times B_{n-1}) \rtimes D^{\mathrm{diag}}_{n-1}]} \\
&\qquad = \frac{2^{2^n+2} \times 2^{2^n+1}}{2^3} = 2^{2^{n+1}+1}.
\end{aligned}$$

Thus the commutative diagram

$$\begin{tikzcd}
 & \mathcal{G}_n \arrow[d, dash, "2"] & & \mathcal{G}_{n-1} \times \mathcal{G}_{n-1} \arrow[d, dash, "2^3"] \\
\psi_n : & \Xi_n \arrow[rr, "\cong"] \arrow[d, dash, "2^{(2^{n+1}+1)}"] & & (B_{n-1} \times B_{n-1}) \rtimes D^{diag}_{n-1} \arrow[d, dash, "2^{(2^{n+1}+1)}"] \\
 & H_n \arrow[rr] & & H_{n-1} \times H_{n-1}
\end{tikzcd}$$

shows that H_n has index $2^{(2^{n+1}+2)}$ in $\mathcal{G}_n$. $\square$

The proof of Theorem 3.10 is now complete.

4. PROFINITE COMPLETION OF THE FIRST GRIGORCHUK GROUP*

4.1 Introduction

The results presented in this chapter are published in the paper [Ben12b]. As it was mentioned on various occasions, the first Grigorchuk group $\mathcal{G}$ possesses various properties which makes it an important object to study. Similarly, its profinite completion $\widehat{\mathcal{G}}$ is interesting from the perspective of profinite groups. Recall that the Grigorchuk group is a 2- group, therefore its pro-2 completion $\widehat{\mathcal{G}}_2$ coincides with its profinite completion $\widehat{\mathcal{G}}$. Also due to the congruence subgroup property these coincide with its closure in $Aut(\mathcal{T}_2)$:

Theorem 4.1. *[Gri85]* $\mathcal{G}$ *has the congruence subgroup property.*

As mentioned in Section 2.7, this gives a concrete description of $\widehat{\mathcal{G}}$ as a compact and topologically finitely generated subgroup of the automorphism group of the binary rooted tree. Moreover, we will see that the congruence subgroup property is used in a crucial way in the proof of Theorem 4.2 since it allows one to work with the congruence quotients $\mathcal{G}/St_{\mathcal{G}}(n),\ n \geq 1$ instead of arbitrary quotients.

Let us mention some properties of $\widehat{\mathcal{G}}$. First of all it is a just-infinite branch group [Gri00]. It has the following universal property: $\widehat{\mathcal{G}}$ contains any countably based pro-2 group as a subgroup. Here countably based means that the group is an inverse limit of countably many finite 2-groups. Another important property is that $\widehat{\mathcal{G}}$ has finite width, that is, its lower central factors have bounded width [BG00]. Moreover

*Profinite completion of Grigorchuk's group is not finitely presented, by Mustafa Gökhan Benli International Journal of Algebra and Computation, Volume 22, Issue 05, 2012, Copyright ©2012 World Scientific Publishing Company. Reprinted with the permission of World Scientific Publishing Company.

in [BG00] it was shown that $\widehat{\mathcal{G}}$ is a counter example to a conjecture about just-infinite pro-p groups of finite width.

The group $\widehat{\mathcal{G}}$ and its subgroups were studied on various occasions. In [Gri05, Šun11] it was given a combinatorial description in terms of so called *forbidden patterns* in the automorphism group of the binary rooted tree. In [AdlHKŠ07] it was investigated with relations to pro-soluble completions of groups and also a combinatorial condition was given for an automorphism $g \in Aut(\mathcal{T}_2)$ to belong to $\widehat{\mathcal{G}}$. A similar condition was used in [Leo10] to show that $\widehat{\mathcal{G}}$ contains the closure of an automaton group W for which the underlying semigroup has no torsion. Also in [BS10] an automaton group $\mathcal{G}_t$ called *the twisted twin* of Grigorchuk group was investigated and was shown to have the same closure as $\mathcal{G}$ in $Aut(\mathcal{T}_2)$ but is not isomorphic to $\mathcal{G}$.

Our aim in this chapter is to investigate the presentation problem of $\widehat{\mathcal{G}}$ in the category of profinite groups.

4.2 Profinite presentations

The classical notion of a presentation has a natural profinite (or more generally a pro-$\mathcal{C}$) analogue:

Let F be a discrete free group and $\widehat{F}$ be its profinite completion. It can be observed that $\widehat{F}$ is in fact a *free profinite* group, i.e., a free object in the category of profinite groups (see [Wil98, Chapter 5]).

Definition 4.1. A (profinite) presentation of a profinite group G, is an exact sequence

$$1 \longrightarrow N \longrightarrow \widehat{F} \longrightarrow G \longrightarrow 1$$

where F is a (discrete) free group. G is said to be finitely presented (as a profinite group) if it has a presentation where F has finite rank and N is the closed normal subgroup of $\widehat{F}$ generated by a finite subset R.

The following is well known:

Lemma 4.1. *If $1 \to N \to F \to G \to 1$ is a presentation of a discrete group G, then $1 \to \bar{N} \to \widehat{F} \to G \to 1$ gives a presentation for the completion of $\widehat{G}$, where $\bar{N}$ denotes the closure of N in $\widehat{G}$ under the identification of G with a subgroup of $\widehat{G}$.*

Proof. Let $\iota : G \to \widehat{G}$, $j : F \to \widehat{F}$ be the canonical maps and let $\pi : F \to G$ be the quotient map defined by the presentation. By the universal property of profinite completions, there is a continuous map $\phi : \widehat{F} \to \widehat{G}$ such that $\phi \circ j = \iota \circ \pi$. ϕ is surjective since its image is compact and contains the dense subset $\iota(G)$ of $\widehat{G}$. It is easy to see that $\ker(\phi) = \varprojlim_{H \lhd_f F} NH/H$ which is equal to the closure $\bar{N}$ by [RZ10, Corollary 1.1.8]. □

It follows that the profinite completion of a finitely presented group is a finitely presented profinite group. However the converse of this fact is not true (see [Lub05]). Two finitely generated residually finite groups, one finitely presented the other not, can have isomorphic profinite completions. In general we have the following definition:

Definition 4.2. A group theoretic property $\mathcal{P}$ is called a profinite property whenever for two groups G_1 and G_2 we have $\widehat{G}_1 \cong \widehat{G}_2$ and G_1 has the property $\mathcal{P}$, then G_2 has property $\mathcal{P}$.

Examples of profinite properties include being virtually nilpotent or having a certain subgroup growth. These can be easily seen since the set of finite index normal subgroups of a group G are in bijection with the set of open normal subgroups of its completion $\widehat{\mathcal{G}}$ (see [RZ10, Proposition 3.2.2]). More interestingly, a result of Lackenby [Lac10] shows that being large and finitely presented is a profinite property. Examples of properties which are not profinite include being residually finite, which is easily seen since $\widehat{G} \cong \widehat{G \times H}$ where H is an infinite simple group. Also a result of Aka [Aka12] shows that the Kazhdan's property (T) is not a profinite property.

4.3 Presentation problem for $\widehat{\mathcal{G}}$

Our main theorem of this chapter is the following:

Theorem 4.2. *$\widehat{\mathcal{G}}$ is not finitely presented as a profinite group.*

This theorem follows from the following well known cohomological criterion for finite presentability of pro-p groups:

Theorem 4.3. *[Wil98, Page 242] A finitely generated pro-p group G is finitely presented (as a pro-p group) if and only if $H^2(G, \mathbb{F}_p)$ is finite.*

and the following result due to Lubotzky:

Theorem 4.4. *[Lub01] A finitely generated pro-p group is finitely presented as a pro-p group if and only if it is finitely presented as a profinite group.*

Hence Theorem 4.2 is a consequence of the following theorem whose proof is presented in Section 4.4.3.

Theorem 4.5. *$H^2(\widehat{\mathcal{G}}, \mathbb{F}_2)$ is infinite.*

The proof of Theorem 4.5 follows from various intermediate results which bear significance themselves for other reasons. A step-by-step scheme can be summarized as follows:

1. Finding presentations for the finite quotients $\mathcal{G}_n = \mathcal{G}/St_{\mathcal{G}}(n)$ (Theorem 4.6).
2. Using these presentations for computing the multipliers $M(\mathcal{G}_n) = (C_2)^{2n-2}$ and the cohomology groups $H^2(\mathcal{G}_n, \mathbb{F}_2)^{2n+1}$ (Theorem 4.8 and Lemma 4.24).

3. Using Lemma 4.24 and the fact that $\mathcal{G}$ is a regular branch group to show that $H^2(\widehat{\mathcal{G}}, \mathbb{F}_2)$ is infinite (Section 4.4.3).

Also as byproduct we show in Theorem 4.10 that the presentations found in Theorem 4.6 are independent and moreover exhibit minimal presentations for each $\mathcal{G}_n$ in Theorem 4.11.

We will discuss these steps and also intermediate results in subsections:

Presentations, Schur multipliers and independence of relators

At the first step we have the following theorem whose proof is given in Section 4.4.1:

Theorem 4.6. *For $n \geq 3$ we have*

$$\mathcal{G}_n = \left\langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, u_0, \ldots, u_{n-3}, v_0, \ldots, v_{n-4}, w_n, t_n \right\rangle$$

where

$$u_i = \sigma^i((ad)^4),\ v_i = \sigma^i((adacac)^4),\ w_n = \sigma^{n-3}((ac)^4),\ t_n = \sigma^{n-3}((abac)^4)$$

and σ is the substitution given by

$$\sigma = \begin{cases} a & \mapsto & aca \\ b & \mapsto & d \\ c & \mapsto & b \\ d & \mapsto & c \end{cases}$$

Recall that given a group G, the *Schur multiplier* of G (denoted by $M(G)$) is the second integral homology group $H_2(G,\mathbb{Z})$. If G is given by a presentation $F/R \cong G$ where F is a free group, the Hopf's formula (obtained first by Schur for finite groups and generalized to infinite groups by Hopf) gives

$$M(G) \cong R \cap F'/[R,F]$$

Hence the abelian group $R \cap F'/[R,F]$ is independent of the presentation of the group. If the given presentation is finite (i.e., F has finite rank and R is the normal closure of finitely many elements $\{r_1,\ldots,r_m\}$ in F), then it is easy to see that the abelian group $R/[R,F]$ is generated by the images of $\{r_1,\ldots,r_m\}$ and hence its subgroup $R \cap F'/[R,F]$ is a finitely generated abelian group. Therefore the Schur multiplier of a finitely presented group is necessarily finitely generated. The converse of this is not true. Baumslag in [Bau71] gave an example of an infinitely presented group with trivial multiplicator. For generalities about Schur multipliers of groups see [Kar87].

The computation of the Schur multiplier of $\mathcal{G}$ using the Lysenok's presentation was done in [Gri99]:

Theorem 4.7. *[Gri99]* $M(\mathcal{G}) \cong (C_2)^\infty$.

Using similar ideas we prove an analogue for the finite groups $\mathcal{G}_n$ via the presentations found in Theorem 4.6:

Theorem 4.8. *For all* $n \geq 1$ *we have* $M(\mathcal{G}_n) \cong C_2^{2n-2}$.

The proof of Theorem 4.8 is presented in Section 4.4.2.

If one wants to talk about independence of relators of a presentation $\langle X \mid R\rangle$ one could talk about two notions[1]:

[1] I am indebted to P. De La Harpe for pointing out this important distinction.

1. For any $r \in R$ the canonical map $\langle X \mid R - \{r\}\rangle \to \langle X \mid R\rangle$ is not an isomorphism.

2. For any $r \in R$, $\langle X \mid R - \{r\}\rangle$ and $\to \langle X \mid R\rangle$ are not isomorphic.

Clearly 2 implies 1 but in general the converse is not true as witnessed by a non-hopfian group. But for hopfian groups these notions coincide. Therefore for hopfian groups one can talk about a presentation having independent relators which means verbally that removal of any relator changes the isomorphism type of the presentation. Another outcome of [Gri99] was the independence of the relators in the Lysenok's presentation:

Theorem 4.9. *[Gri99] The relator's in the Lysenok's presentation for $\mathcal{G}$ are independent.*

Similarly we obtain the analogue for the presentations found in Theorem 4.6:

Theorem 4.10. *The relators in the presentations of $\mathcal{G}_n$ for $n \geq 3$ found in Theorem 4.6 are independent.*

The proof of Theorem 4.10 will be presented in Section 4.4.2.

4.3.1 Minimality of presentations and deficiency

Definition 4.3. If G is a group, let $d(G)$ denote the minimal number of generators required to generate G. The minimal number m such that G has a presentation with m relators will be denoted by $r(G)$. The deficiency of G (denoted by $def(G)$) is defined to be the minimal difference $m - t$ such that G has a presentation with t generators and m relators. A presentation $G = \langle x_1, \ldots, x_t \mid r_1, \ldots, r_m\rangle$ is called minimal if $t = d(G)$ and $m = r(G)$.

Regarding minimal presentations, the following question is open (see [Gru76]):

Question 6. *Does every finite group have a minimal presentation?*

A related question is the following, which again is open:

Question 7. *Does every finite group have a presentation realizing its deficiency with $d(G)$ number of generators?*

It is clear that an affirmative answer to question 7 gives an affirmative answer to question 6. Rapaport [Rap73] proves that the second question has a positive answer for nilpotent groups. Also Evans [Eva93] gives an affirmative answer for finite solvable groups. Lubotzky [Lub01] gives an affirmative answer for question 7 in the category of profinite groups (see Section 4.2).

Therefore the groups $\mathcal{G}_n$ being finite 2-groups have minimal presentations. Theorems 4.6 and 4.8 allow us produce a minimal presentation for these groups. It relies on the following well known inequality:

Lemma 4.2. *For a finite group G we have $d(M(G)) \leq def(G)$.*

Proof. Given a presentation $G = \langle x_1, \ldots, x_t \mid r_1, \ldots, r_m \rangle = F/R$ of a finite group G, we have the quotient map

$$R/[R, F] \to R/(R \cap F')$$

whose kernel is the Schur multiplier $M(G)$. But $R/(R \cap F') \cong RF'/F'$ which is free abelian of rank t because R has finite index in F. Hence

$$d(M(G)) = d(R/[F, R]) - t \leq m - t$$

and since $M(G)$ does not depend on the presentation we have $d(M(G)) \leq def(G)$.

□

Theorem 4.11. *For $n \geq 3$ we have*

$$\mathcal{G}_n = \langle a, b, c \mid a^2, b^2, c^2, (bc)^2, u_0, \ldots, u_{n-3}, v_0, \ldots, v_{n-4}, w_n, t_n \rangle$$

where

$$u_i = \sigma^i((abc)^4),\ v_i = \sigma^i((abcacac)^4),\ w_n = \sigma^{n-3}((ac)^4),\ t_n = \sigma^{n-3}((abac)^4)$$

and σ is the substitution given by

$$\sigma = \begin{cases} a & \mapsto & aca \\ b & \mapsto & bc \\ c & \mapsto & b \end{cases}$$

and this presentation is minimal and realizes the deficiency $def(\mathcal{G}_n) = 2n - 2$.

Proof. The presentations found in Theorem 4.6 contain the relator $d = bc$. Hence applying Tietze transformations we get the asserted presentations. By Theorem 4.8 we have $d(M(\mathcal{G}_n)) = 2n - 2$. By Lemma 4.2 and counting generators and relators in the above presentation we get $def(\mathcal{G}_n) = 2n - 2$.

We have $\mathcal{G}_3^{ab} \cong (C_2)^3$ and $\mathcal{G}_n$ maps onto $\mathcal{G}_3$. Also $\mathcal{G}^{ab} \cong (C_2)^3$ and $\mathcal{G}$ maps onto $\mathcal{G}_n$. These show that $\mathcal{G}_n^{ab} \cong (C_2)^3$ and $d(\mathcal{G}_n) = 3$. Hence the above presentation realizes the deficiency with minimal number of generators. Therefore it is necessarily minimal. □

4.4 Proofs of the theorems

4.4.1 Presentations for $\mathcal{G}_n$

The aim of this section is to prove Theorem 4.6.

Let $\Gamma = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, (ad)^4 \rangle$ and let us denote by $\pi : \Gamma \longrightarrow \mathcal{G}$ the canonical surjection. Consider the subgroup $\Xi = \langle b, c, d, b^a, c^a, d^a \rangle_\Gamma$ which is the lift of the first level stabilizer $St_{\mathcal{G}}(1)$ to Γ.

We have a homomorphism

$$\begin{array}{rccc} \bar{\varphi}: & \Xi & \longrightarrow & \Gamma \times \Gamma \\ & b & \mapsto & (a, c) \\ & c & \mapsto & (a, d) \\ & d & \mapsto & (1, b) \\ & b^a & \mapsto & (c, a) \\ & c^a & \mapsto & (d, a) \\ & d^a & \mapsto & (b, 1) \end{array}$$

which is analogous to $\varphi : St_{\mathcal{G}}(1) \longrightarrow \mathcal{G} \times \mathcal{G}$. (The fact that $\bar{\varphi}$ is well defined can be checked by first finding a presentation for Ξ using Reidemeister-Schreier process and checking that it maps relators to relators.)

Given $w \in \Xi$ let us write $\bar{\varphi}(w) = (w_0, w_1)$ which is consistent with the section notation of tree automorphisms. Recall the substitution σ given by

$$\sigma = \begin{cases} a & \mapsto & aca \\ b & \mapsto & d \\ c & \mapsto & b \\ d & \mapsto & c \end{cases}$$

It is easy to check that given $w \in \Xi$ one has

$$\bar{\varphi}(\sigma(w)) = (v, w)$$

and $v \in \langle a, d\rangle_\Gamma \cong D_8$ where the latter denotes the dihedral group of order 8. Since all u_i, v_i, w_i, t_i (for appropriate index i) are 4-th powers (as elements of Ξ) and D_8 has exponent 4, we have the following equalities:

$$\begin{aligned}
\bar{\varphi}(u_i) &= (1, u_{i-1}) \\
\bar{\varphi}(v_i) &= (1, v_{i-1}) \\
\bar{\varphi}(w_i) &= (1, w_{i-1}) \\
\bar{\varphi}(t_i) &= (1, t_{i-1})
\end{aligned} \tag{4.1}$$

Let $\Omega = Ker(\pi)$ so that $\mathcal{G} = \Gamma/\Omega$. It is known (for example see [dlH00]) that Ω is a strictly increasing union $\Omega = \bigcup_n \Omega_n$, $\Omega_n \subset \Omega_{n+1}$. The subgroups Ω_n can be defined recursively as follows:

$$\Omega_1 = Ker(\bar{\varphi}) \quad \text{and} \quad \Omega_n = \{w \in \Xi \mid w_0, w_1 \in \Omega_{n-1}\} \text{ for } n \geq 2.$$

It is known that $\Omega_n = \langle\langle u_1, \ldots, u_n, v_0, \ldots . v_{n-1}\rangle\rangle_\Gamma$ (see [Gri98]). The subgroups Ω_n are related to the "branch algorithm" which solves the word problem in $\mathcal{G}$ (See [Gri05]). Roughly speaking Ω_n consists of elements for which the algorithm stops after n steps.

Similarly, we have subgroups Υ_n of Γ such that $\mathcal{G}_n = \mathcal{G}/St_{\mathcal{G}}(n) \cong \Gamma/\Upsilon_n$ where $\Upsilon_{n+1} \subset \Upsilon_n$ and $\bigcap_n \Upsilon_n = \Omega$. Hence $St_{\mathcal{G}}(n) = \Upsilon_n/\Omega$. A recursive definition for Υ_n is as follows:

$$\Upsilon_1 = \Xi \quad \text{and} \quad \Upsilon_n = \{w \in \Xi \mid w_0, w_1 \in \Upsilon_{n-1}\} \text{ for } n \geq 2.$$

We will prove Theorem 4.6 by showing that for $n \geq 3$ we have

$$\Upsilon_n = \langle\langle u_1, \ldots, u_{n-3}, v_0, \ldots, v_{n-4}, w_n, t_n \rangle\rangle_\Gamma .$$

This will be done by induction on n and the case $n = 3$ follows from the following 3 lemmas:

Lemma 4.3. *We have $\mathcal{G}_3 \cong (C_2 \wr C_2) \wr C_2$ which has the presentation*

$$\left\langle x, y, z \mid x^2, y^2, z^2, [x, x^y], [y, y^z], [x, x^z], [x, y^z], [y, x^z] \right\rangle \tag{4.2}$$

where $x = ada, y = c, z = a$.

Proof. Direct inspection of the action of a, c, ada on the tree consisting of the first 3 levels (See [dlH00] page 226). □

Lemma 4.4. *Presentation (4.2) is equivalent to*

$$\mathcal{G}_3 = \left\langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, (ad)^4, (ac)^4, (adac)^4 \right\rangle \tag{4.3}$$

Proof. Follows from the following equations and applying Tietze transformations to (4.2).

$$[x, x^y] = [ada, cadac] = (adac)^4$$
$$[y, y^z] = [c, aca] = (ca)^4$$
$$[x, x^z] = [ada, d] = (ad)^4$$
$$[x, y^z] = [ada, aca]$$
$$[y, x^z] = [c, d]$$

□

Lemma 4.5. *Presentation (4.3) is equivalent to*

$$\mathcal{G}_3 = \left\langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, (ad)^4, (ac)^4, (abac)^4 \right\rangle$$

and hence $\Upsilon_3 = \langle w_3, t_3 \rangle_{\Gamma}^{\#}$.

Proof. Follows from the following equalities:

$$(adac)^4 = (adacadac)^2 = (adcacacdac)^2 = (abacabac)^2 = (abac)^4$$

where in step 3 we use the equality $aca = cacac$. □

Recall that $\mathcal{G}$ is regular branch over the subgroup $K = \langle\langle (ab)^2 \rangle\rangle_{\mathcal{G}}$.

Lemma 4.6. $St_{\mathcal{G}}(3) \leq K$ *and hence for* $n \geq 4$ *we have*

$$\varphi(St_{\mathcal{G}}(n)) = St_{\mathcal{G}}(n-1) \times St_{\mathcal{G}}(n-1)$$

and therefore for $n \geq 4$

$$\bar{\varphi}(\Upsilon_n) = \Upsilon_{n-1} \times \Upsilon_{n-1}.$$

Proof. A proof of the fact $St_{\mathcal{G}}(3) \leq K$ can be found in [dlH00, Page 230]. Since $\mathcal{G}$ is a regular branch group over K (i.e $K \times K \leq \psi(K)$) we get the remaining equalities. □

Proof of Theorem 4.6: We have to show that for $n \geq 3$ we have

$$\Upsilon_n = \langle\langle u_1, \dots, u_{n-3}, v_0, \dots, v_{n-4}, w_n, t_n \rangle\rangle_{\Gamma}$$

Now induction on n, equations (4.1) and the fact $\bar{\varphi}(\Upsilon_n) = \Upsilon_{n-1} \times \Upsilon_{n-1}$ show that

$$\Upsilon_n = Ker(\bar{\varphi}) \langle\langle u_2, \ldots, u_{n-3}, v_1, \ldots, v_{n-4}, w_n, t_n \rangle\rangle_\Gamma$$

But $Ker(\bar{\varphi}) = \Omega_1 = \langle\langle u_1, v_0 \rangle\rangle_\Gamma$, as mentioned earlier, from this we obtain

$$\Upsilon_n = \langle\langle u_1, \ldots, u_{n-3}, v_0, \ldots, v_{n-4}, w_n, t_n \rangle\rangle_\Gamma .$$

4.4.2 Computation of the Schur multipliers of $\mathcal{G}_n$

This section is devoted to the proofs of theorems 4.8 and 4.10. The ideas are analogous to [Gri99] with slight modifications where needed.

Let F be the free group on $\{a, b, c, d\}$ and for $n \geq 3$ let

$$K_n = \left\langle\left\langle a^2, b^2, c^2, d^2, bcd, u_0, \ldots, u_{n-3}, v_0, \ldots, v_{n-4}, w_n, t_n \right\rangle\right\rangle$$

where u_i, v_j, w_n are as in Theorem 4.6. By Theorem 4.6 we have $F/K_n \cong \mathcal{G}_n$. Note that K_n is defined for $n \geq 3$, hence we will always assume that the index $n \geq 3$ throughout the reminder.

As mentioned before, the Schur multiplier can be found using Hopf's formula by:

$$M(\mathcal{G}_n) \cong K_n \cap F'/[K_n, F]$$

Recall the substitution σ mentioned on various occasions. Interpreting it as an endomorphism of F, we have the following basic fact which will be used in the remainder:

Lemma 4.7.

$$K_{n+1} \subset K_n$$

$$\sigma(K_n) \subset K_n$$

$$\sigma([K_n, F]) \subset [K_n, F]$$

Proof. The first inclusion follows from the fact that $\mathcal{G}_{n+1}$ maps onto $\mathcal{G}_n$. The images of generators of K_n under σ clearly lie in K_{n+1} and hence the second inclusion follows from the first one. Finally the third inclusion follows directly from the second. □

For computational reasons we need to change the relators in presentation of theorem 1 slightly to the ones given in the next lemma. The rationale behind this will be apparent when we will do computations modulo the subgroup $[K_n, F]$.

Lemma 4.8. [2] $K_n = \langle\langle B_1, B_2, B_3, B_4, L, U_0, \ldots, U_{n-3}, V_0, \ldots, V_{n-4}, W_n, T_n \rangle\rangle$ *where*

$$B_1 = a^2, B_2 = b^2, B_3 = c^2, B_4 = bcd$$

$$L = b^2c^2d^2(bcd)^{-2}$$

$$U_i = \sigma^i((ad)^4a^{-4}d^{-4}), V_i = \sigma^i((adacac)^4a^{-12}c^{-8}d^{-4})$$

$$W_n = \sigma^{n-3}((ac)^4a^{-4}c^{-4}), T_n = \sigma^{n-3}((abac)^4a^{-8}b^{-4}c^{-4})$$

Proof. Let K_n' be the subgroup on the right hand side. Clearly $B_i, L \in K_n$. Also since $U_0, V_0 \in K_n$ and $\sigma(K_n) \subset K_n$ we see that U_i, V_i are elements of K_n. We have

$$W_n = w_n\sigma^{n-3}(a^{-4}c^{-4}) \in K_n$$

[2] U_i was written incorrectly in [Ben12b, Page 13]

similarly $T_n \in K_n$. The reverse inclusion can be shown similarly using $\sigma(K'_n) \subset K'_n$. $\square$

Let $\approx$ denote equivalence modulo $[K_n, F]$.

Lemma 4.9. *In the group $K_n/[K_n, F]$ we have the equalities*

$$L^2 \approx U_i^2 \approx V_i^2 \approx W_n^2 \approx T_n^2 \approx 1$$

Proof. Observe that $x^2, [x, y] \in K_n$ where $x, y \in \{b, c, d\}^{\pm}$ and therefore we have

$$1 \approx [x^2, y] = [x, y]^x [x, y] \approx [x, y]^2$$

for any $x, y \in \{b, c, d\}^{\pm}$. Hence moving elements of the form x^2 and $[x, y]$ we get

$$L = b^2c^2d^2d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}b^{-1} \approx dcbd^{-1}c^{-1}b^{-1} = dcb[d, c]c^{-1}d^{-1}b^{-1}$$

$$\approx [d, c]dcbc^{-1}[d, b]b^{-1}d^{-1} \approx [d, c][d, b]dcb[c, b]b^{-1}c^{-1}d^{-1} \approx [d, c][d, b][c, b]$$

therefore, since $K_n/[K_n, F]$ is abelian we have $L^2 \approx 1$.

$$\begin{aligned} U_0^2 &= (ad)^4a^{-4}d^{-4}(ad)^4a^{-4}d^{-4} \\ &\approx a^{-1}d^{-1}a^{-1}d^{-1}a^{-1}d^{-1}a^{-1}d^{-1}a^{-1}aadadadad \\ &= [a, (ad)^4] = [a, u_0] \approx 1 \end{aligned}$$

By virtue of Lemma 4.7 it is enough to show $U_0^2 \approx V_0^2 \approx 1$.

Note that $(cacada)^4 = ((adacac)^4)^{(cac)^{-1}} = v_0^{(cac)^{-1}} \in K_n$.

$$\begin{aligned} V_0^2 &= (adacac)^4 a^{-12} c^{-8} d^{-4} (adacac)^4 a^{-12} c^{-8} d^{-4} \\ &\approx (cacada)^{-4} (adacac)^4 \\ &= (cacada)^{-4} ada (cacada)^4 (ada)^{-1} \\ &= [(cacada)^4, (ada)^{-1}] \approx 1 \end{aligned}$$

Since $\sigma([K_n, F]) \subset [K_n.F]$ we see that $U_i^2 = V_i^2 \approx 1$ for every i.

For W_n, a similar calculation (like the one for U_0) gives the following:

$$W_n^2 \approx [\sigma^{n-3}(a), \sigma^{n-3}((ac)^4)] = [\sigma^{n-3}(a), w_n] \approx 1$$

Also similar computation (like the one for V_0) shows that T_n^2 is conjugate to

$$[\sigma^{n-3}(d), \sigma^{n-3}((baca)^4)] \approx 1$$

□

Recall that C denotes the infinite cyclic group.

Lemma 4.10. *In $K_n/[K_n, F]$ we have $\langle B_1, B_2, B_3, B_4 \rangle \cong C^4$.*

Proof. We have the quotient map

$$K_n/[K_n, F] \longrightarrow K_n/(K_n \cap F')$$

and the right hand side is a free abelian group since

$$K_n/(K_n \cap F') \cong K_n F'/F' \leq F/F' \cong C^4$$

Now B_1, B_2, B_3, B_4 are mapped onto the vectors

$$(2,0,0,0),(0,2,0,0),(0,0,2,0),(0,1,1,1)$$

respectively. Linear independence of these vectors proves the assertion. $\square$

Lemma 4.11. *We have the following isomorphism:*

$$K_n/[K_n, F] \cong C^4 \times M_n$$

where C^4 is freely generated by B_1, B_2, B_3, B_4 and is isomorphic to $K_n/(K_n \cap F')$ and M_n is the torsion part generated by $\{L, U_i, V_i, W_n, T_n\}$, which is an elementary abelian 2-group isomorphic to $(K_n \cap F')/[K_n, F]$.

Proof. We have the split exact sequence of abelian groups

$$1 \to (K_n \cap F')/[K_n, F] \to K_n/[K_n, F] \to K_n/(K_n \cap F') \to 1$$

From the previous lemma we have $K_n/(K_n \cap F') \cong C^4$. Hence

$$M_n \cong (K_n \cap F')/[K_n, F]$$

and is elementary abelian 2-group by Lemma 7. $\square$

Lemma 4.12. *The elements L, U_0, W_3, T_3 are independent in M_3.*

Proof. Clearly M_3 maps to the abelian group

$$Q_3 = F'/([K_3, F]\gamma_5(F)F^{(2)})$$

The result follows from the next lemma. $\square$

Lemma 4.13. *Q_3 has the following presentation:*

Generators:

- $[a,b],[a,c],[a,d],[b,c]$
- $[a,x,y],\quad x\neq y,\quad x,y\in\{b,c,d\}$
- $[a,x,y,z],\quad x\neq y,\quad y\neq z$ *and* (x,y,z) *is not a permutation of* (b,c,d) *where* $x,y\in\{b,c,d\}$ *and* $z\in\{a,b,c,d\}$

Relations:

- *commutativity relations*
- $[a,b]^8=[a,c]^4=[a,d]^4=[b,c]^2=1$
- $[a,b,c]^4=[a,b,d]^4=[a,c,b]^2=[a,c,d]^2=[a,d,b]^2=[a,d,c]^2=1$
- $[a,x,y,z]^2=1$

Moreover, the the images of L,U_0,W_3,T_3 *in* Q_3 *are* $[b,c],\ [a,d]^2,\ [a,c]^2$ *and* $[a,b,c]^{-2}$ *respectively.*

Proof. ($\approx$ denotes equivalence modulo $[K_3,F]\gamma_5(F)F^{(2)}$ throughout this proof)

Since $a^2,b^2,c^2,d^2\in K_3$, using standard commutator calculus and the fact that $\gamma_5(F)$ appears in the denominator of Q_3, it is easy to see that Q_3 is generated by elements of the form

- $[x,y],\quad x\neq y.\quad x,y\in\{a,b,c,d\}$
- $[x,y,z],\quad x\neq y,x,y,z\in\{a,b,c,d\}$
- $[x,y,z,w].\quad x\neq y,\quad x,y,z,w\in\{a,b,c,d\}$

Before beginning calculations, we wish to write two equalities which will be frequently used in the remainder [3]:

$$[x, yz] = [x, z][x, y][x, y, z] \tag{4.4}$$

$$[xy, z] = [x, z][x, z, y][y, z] \tag{4.5}$$

Clearly, in Q_3 we have the following relations:

$$[x, B_i] = [x, L] = [x, U_0] = [x, W_3] = [x, T_3] = 1 \; x \in \{a, b, c, d\}$$

Using these we will further reduce the system of generators. Firstly, from equation (4.4) we have:

$$[x, a]^2[x, a, a] = [x, a^2] \approx 1, \quad x \in \{b, c, d\}$$

Hence

$$[x, a, a] \approx [a, x]^2 \tag{4.6}$$

and we can omit the generators $[x, a, a]$ where $x \in \{b, c, d\}$. Since

$$[x, a, y] = [a, x][y, [a, x]][x, a] \approx [y, [a, x]] = [a, x, y]^{-1} \tag{4.7}$$

we also can omit generators $[a, x, a]$ and $[x, a, y]$ where $x, y \in \{b, c, d\}$. Next, again using equation (4.4) we have

$$[x, y]^2[x, y, y] = [x, y^2] \approx 1 \quad x \in \{a, b, c, d\}, y \in \{b, c, d\}$$

[3] Equation (4.5) was written incorrectly in [Ben12b, Page 16, eqn (4.6)]

Hence

$$[x, y, y] \approx [y, x]^2 \tag{4.8}$$

and therefore the generators $[a, y, y]$ where $y \in \{b, c, d\}$ can be omitted.

Since $[x, y] \in K_3$ for $x, y \in \{b, c, d\}$, we also omit generators of the form $[x, y, a]$.

Using equations (4.7), (4.6) and (4.5) we have

$$[a, x, a, y] \approx [[x, a, a]^{-1}, y] \approx [[a, x]^{-2}, y] \approx [[a, x]^{-1}, y]^2 = [x, a, y]^2$$

which enables us to omit generators of the form $[a, x, a, y]$ where $x \in \{b, c, d\}$ and $y \in \{a, b, c, d\}$.

Similarly, using equations (4.8) and (4.5) we have

$$[a, x, x, z] \approx [[x, a]^2, z] \approx [[x, a], z]^2 = [x, a, z]^2$$

which enables us to omit generators of the form $[a, x, x, z]$ where $x \in \{b, c, d\}$ and where $z \in \{a, b, c, d\}$.

The following equation holds:

$$\begin{aligned} [x, yzt] &= [x, zt][x, y][x, y, zt] \\ &= [x, t][x, z][x, z, t][x, y][x, y, t][x, y, z][x, y, z, t] \end{aligned} \tag{4.9}$$

Substituting $x = a$ yields the omission of the generators of the form $[a, y, z, t]$ where (y, z, t) is a permutation of (b, c, d). Similarly, substituting different letters into equation (4.9) we get the following identities:

$$1 \approx [b, bcd] \approx [b, c][b, d]$$

$$1 \approx [c, bcd] \approx [c, d][c, b]$$

$$1 \approx [d, bcd] \approx [d, b][d, c]$$

Which yield the identities $[c, d] \approx [b, c] \approx [d, b]$. Thus $[c, d]$ and $[b, d]$ can be omitted from the system of generators. Hence Q_3 has the asserted set of generators. We proceed to showing it has the asserted relators.

Using equation (4.8) we have $[b, c]^2 \approx 1$. Using similar calculations as in Lemma 4.9 we get (note that $(ab)^8 \in K_3$) :

$$1 \approx [a, (ad)^4] \approx [a, d]^4$$

$$1 \approx [a, (ac)^4] \approx [a, c]^4$$

$$1 \approx [a, (ab)^8] \approx [a, b]^8$$

We have

$$1 \approx [x, (ad)^4] \approx [x, [a, d]^2] = [x, [a, d]]^2[[x, [a, d]], [a, d]] \approx [x, [a, d]]^2 \approx [a, d, x]^{-2}$$

hence $[a, d, b]^2 = [a, d, c]^2 \approx 1$. Similarly

$$1 \approx [x, (ac)^4] \approx [a, c, x]^{-2}$$

$$1 \approx [x, (ab)^8] \approx [a, b, x]^{-4}$$

yield $[a, c, b]^2 \approx [a, c, d]^2 \approx 1$ and $[a, b, c]^4 \approx [a, b, d]^4 \approx 1$. Finally

$$1 \approx [a, x, y, z^2] \approx [a, x, y, z]^2[a, x, y, z, z] \approx [a, x, y, z]^2$$

where $x, y \in \{b, c, d\}$ and $z \in \{a, b, c, d\}$.

Let us show that the images of L, U_0, W_3, T_3 in Q_3 are $[b, c], [a, d]^2, [a, c]^2$ and $[a, b, c]^{-2}$ respectively:

By Lemma 4.9, $L \approx [d, c][d, b][c, b] \approx [b, c]$. Also similar to earlier computations we have;

$$U_0 \approx [a, d]^2$$

$$W_3 \approx [a, c]^2$$

$$\begin{aligned} T_3 &= (abac)^4a^{-8}b^{-4}c^{-4} = abacabacabacabacabaca^{-8}b^{-4}c^{-4} \\ &\approx [a, b]b^{-1}c^{-1}a^{-1}bac[a, b]b^{-1}c^{-1}a^{-1}bac \\ &= ([a, b][b, ac])^2 \\ &= ([a, b][b, c][b, a][b, a, c])^2 \\ &\approx [b, a, c]^2 \approx [a, b, c]^{-2}(\text{by equation 4.7}) \end{aligned}$$

Therefore

$$1 \approx [a, T_3] \approx [a, [a, b, c]^{-2}] \approx [a, [a, b, c]]^{-2} \approx [a, b, c, a]^2$$

Similarly

$$1 \approx [x, T_3] \approx [x, b, c, a]^2$$

Now the following argument finishes the proof of the lemma:

It is clear that all relations of Q_3 can be derived from the relations

$$[x, B_i] = [x, L] = [x, U_0] = [x, W_3] = [x, T_3] = 1, \ x \in \{a, b, c, d\}$$

together with relations of the form $y = 1$ where $y \in \gamma_5(F)$ or $y \in F^{(2)}$.

The relations $y \in \gamma_5(F)$ can be disregarded by omitting the generators of commutator length 5 or more from the generating system. The relations $y \in F^{(2)}$ translate to commutativity relations among generators. Finally above computations show that one can further reduce the generating set to the one asserted in the lemma. Also the calculations imply that the relations are equivalent to the system of relators given in the lemma. Hence Q_3 has the given presentation, and clearly L, U_0, W_3, T_3 are independent in Q_3.

$\square$

Lemma 4.14. *The elements L, U_0, U_1, V_0 are independent in M_n where $n \geq 4$.*

Proof. M_n maps to the abelian group $Q_n = F'/([K_n, F]\gamma_5(F)F^{(2)})$. The result is a corollary of the next lemma. $\square$

Lemma 4.15. *Q_n has the following presentation:*

Generators:

- $[a, b], [a, c], [a, d], [b, c]$
- $[a, x, y], \quad x \neq y, \quad x, y \in \{b, c, d\}$
- $[a, x, y, z], \quad x \neq y, \quad y \neq z$ *and* (x, y, z) *is not a permutation of* (b, c, d) *where* $x, y \in \{b, c, d\}$ *and* $z \in \{a, b, c, d\}$

Relations:

- *commutativity relations*
- $[a, b]^{16} = [a, c]^8 = [a, d]^4 = [b, c]^2 = 1$
- $[a, b, c]^8 = [a, b, d]^8 = [a, c, b]^4 = [a, c, d]^4 = [a, d, b]^2 = [a, d, c]^2 = 1$

- $[a, x, y, z]^2 = 1$

Moreover the the images of L, U_0, U_1, V_0 in Q_n are $[b, c]$, $[a, d]^2$, $[a, c]^4$ and $[a, d]^2[a, c, d]^2$ respectively.

Proof. Most of the proof is similar to the proof of Lemma (4.13). Additionally we only need to show that the relations:

$$[x, U_1] = \ldots = [x, U_{n-3}] = [x, V_0] = \cdots = [x, V_{n-4}] = [x, W_n] = [x, T_n] = 1$$

are consequences of the given system of relators.

Let $\cong$ mean equality in F' modulo the subgroup $[K_n, F]F^{(2)}$. Using equation (4.8) we have:

$$[a, c]^2 \cong [a, c, c]^{-1}$$

Also using equation (4.4)

$$1 \cong [a, c, c^2] \cong [a, c, c]^2[a, c, c, c]$$

which implies

$$[a, c, c]^{-2} \cong [a, c, c, c]$$

and hence

$$U_1 \cong [a, c]^4 \cong [a, c, c]^{-2} \cong [a, c, c, c]$$

which yields $U_1 \in \gamma_4(F) \bmod [K_n, F]F^{(2)}$. Therefore $[x, U_1] \in \gamma_5(F) \bmod [K_n, F]F^{(2)}$. It follows that relations of the form $[x, U_1]$ are consequences of previous relations. Since $\sigma(\gamma_5(F)) \leq \gamma_5(F)$ and $\sigma(F^{(2)}) \leq F^{(2)}$, we also see that relations of the form $[x, U_i]$ where $i = 2, \ldots, n-3$ are consequences of previous relations.

For V_0 we have:

$$
\begin{aligned}
V_0 &= (adacac)^4 a^{-12} c^{-8} d^{-4} \\
&= (ad)^2 (ac)^2 [(ac)^2, ad](ac)^2 (ad)^2 (ac)^2 [(ac)^2, ad](ac)^2 a^{-12} c^{-8} d^{-4} \\
&\approx [a,d]^2 [a,c]^4 [[a,c], ad]^2 \\
&\approx [a,d]^2 [a,c]^4 [a,c,d]^2 [a,c,a]^2 [a,c,a,d]^2 \\
&\approx [a,d]^2 [a,c]^4 [a,c,d]^2 [c,a,a]^{-2} [a,c,a,d]^2 \\
&\approx [a,d]^2 [a,c]^4 [a,c,d]^2 [a,c]^4 [a,c,a,d]^2 \\
&\approx [a,d]^2 [a,c,d]^2
\end{aligned}
$$

hence

$$[x, V_0] \approx [x, [a,d]^2 [a,c,d]^2] \approx [x, [a,d]]^2 [a, [a,c,d]]^2 = [a,d,x]^2 [a,c,d,x]^2$$

is a consequence of previous relations.

For V_1 we get:

$$
\begin{aligned}
V_1 &= (acacacabacab)^4 (aca)^{-12} b^{-8} c^{-4} \\
&\cong ((ac)^4 ab[ab, ac][[ab, ac], ab])^4 (aca)^{-12} b^{-8} c^{-4} \\
&\cong (ac)^{16} (ab)^8 [[ab, ac], ab]^4 (aca)^{-12} b^{-8} c^{-4} \\
&\cong [c,a,a,a,a][b,a,a,a]
\end{aligned}
$$

therefore $V_1 \in \gamma_4(F) \bmod [K_n, F]F^{(2)}$ hence relations of the form $[x, V_i]$ where

$i = 1, \ldots, n-4$ are consequences of previous relations.

$$\begin{aligned} W_4 &= (acab)^4(aca)^{-4}b^{-4} \\ &\cong ([a,c][c,ab])^2 \\ &\cong [a,c]^2([c,b][c,a][c,a,b])^2 \\ &\cong [c,a,b]^2 \end{aligned}$$

But by equation (4.4),

$$1 \cong [c,a,b^2] \cong [c,a,b]^2[c,a,b,b]$$

therefore

$$W_4 \cong [c,a,b,b]^{-1}$$

hence $W_n = \sigma^{n-4}(W_4) \in \gamma_4(F)$ modulo $[K_n, F]F^{(2)}$ and relations $[x, W_n]$ follow from the previous relations.

Finally, for T_3 from previous computations we get:

$$T_3 \cong [a,b,c]^{-2}$$

using

$$1 \cong [[a,b],c^2] \cong [a,b,c]^2[a,b,c,c]$$

we get

$$T_3 \cong [a,b,c]^{-2} \cong [a,b,c,c]$$

hence $T_n = \sigma^{n-3}(T_3) \in \gamma_4(F)$ modulo $[K_n, F]F^{(2)}$ and relations $[x, T_n]$ follow from the previous relations. $\square$

Let $P = \langle a^2, b, c, d, b^a, c^a, d^a \rangle_F$ be the lift[4] of $St_{\mathcal{G}}(1)$ to F and let

$$\psi : P \to \Gamma \times \Gamma$$

be the homomorphism similar to $\bar{\varphi}$.

Lemma 4.16. *We have* $Ker(\psi) = \langle\langle B_1, B_2, B_3, B_4, L, U_0, U_1, V_0 \rangle\rangle$

Proof. Restatement of [Gri99] lemma 11. □

Lemma 4.17. *For* $n \geq 4$ *the following isomorphism holds:*

$$K_n/([K_n, F]) \cong C^4 \times C_2^4 \times \psi(K_n)/\psi([K_n, F])$$

where the factor C^4 *is generated by* $B_1,\ B_2,\ B_3,\ B_4$ *and* C_2^4 *is generated by* L, U_0, U_1, V_0.

Proof. Let

$$\psi_* : K_n/[K_n, F] \to \psi(K_n)/\psi([K_n, F])$$

be the homomorphism induced by ψ. Then by Lemma 4.16 we have:

$$\begin{aligned} Ker(\psi_*) &= (Ker(\psi))[K_n, F]/[K_n, F] \\ &= \langle B_1, B_2, B_3, B_4, L, U_0, U_1, V_0 \rangle_{K_n/[K_n,F]} \\ &\cong C^4 \times C_2^4 \end{aligned}$$

Hence from Lemma 4.11 the result follows. □

Lemma 4.18. *We have:*

$$\psi(U_i) = (1, U_{i-1})$$

[4]Note that the generator a^2 is missing here in [Ben12b, Page 20]

$$\psi(V_i) = (1, V_{i-1})$$

$$\psi(W_n) = (1, W_{n-1})$$

$$\psi(T_n) = (1, T_{n-1})$$

for $i \geq 1$ *and* $n \geq 4$.

Proof. We have

$$\psi\sigma \begin{cases} a \mapsto (d, a) \\ b \mapsto (1, b) \\ c \mapsto (a, c) \\ d \mapsto (a, d) \end{cases}$$

Hence the image of $\pi_1\psi\sigma$ lies in the subgroup $\langle a, d\rangle_\Gamma$ and $\pi_2\psi\sigma$ is the identity map. Since $\langle a, d\rangle_\Gamma$ has exponent 4 we get the asserted equalities. □

Let

$$\Theta_n = \langle\langle U_1, \ldots, U_{n-3}, V_0, \ldots, V_{n-4}, W_n, T_n\rangle\rangle_\Gamma$$

so that $\Gamma/\Theta_n \cong \mathcal{G}_n$.

Lemma 4.19. *The following relations hold:*

$$\psi(K_n) = \Theta_{n-1} \times \Theta_{n-1}$$

$$\psi([K_n, F]) = ([\Theta_{n-1}, \Gamma] \times [\Theta_{n-1}, \Gamma])\Psi$$

where $\Psi \leq \Gamma \times \Gamma$ *is the subgroup consisting of elements of the form* (w^{-1}, w) $w \in \Theta_{n-1}$.

Proof. Similar to [Gri99] lemma 14. □

Lemma 4.20. *We have the isomorphism:*

$$\psi(K_n)/\psi([K_n, F]) \cong \Theta_{n-1}/([\Theta_{n-1}, \Gamma])$$

and the generators $\psi(U_i)$, $\psi(V_i)$, $\psi(W_n)$, $\psi(T_n)$ *are mapped to the generators* $U_{i-1}, V_{i-1}, W_{n-1}, T_{n-1}$ *respectively.*

Proof. By Lemma 4.19, $\psi(K_n)/\psi([K_n, F])$ is isomorphic to

$$(\Theta_{n-1} \times \Theta_{n-1})/(([\Theta_{n-1}, \Gamma] \times [\Theta_{n-1}, \Gamma])\Psi) \tag{4.10}$$

Since $(1, x)^{-1}(x, 1) = (x, x^{-1}) \in \Xi$, (4.10) is generated by elements of the form $(1, x)$ where $x \in \{U_1, \ldots, U_{n-4}, V_0, \ldots, V_{n-5}, W_{n-1}, T_{n-1}\}$. It is easy to check that the map

$$(1, x) \mapsto x$$

gives an isomorphism between (4.10) and $\Theta_{n-1}/([\Theta_{n-1}, \Gamma])$. $\square$

Lemma 4.21. W_3, T_3 *are independent in* $\Theta_3/([\Theta_3, \Gamma])$

Proof. The proof is analogous to the proof of lemma (4.13) and is omitted. $\square$

Lemma 4.22. *For* $n \geq 4$, U_1, V_0 *are independent in* $\Theta_n/([\Theta_n, \Gamma])$

Proof. The proof is analogous to the proof of lemma (4.14) and is omitted. $\square$

Lemma 4.23. *For* $n \geq 4$ *we have*

$$\Theta_n/([\Theta_n, \Gamma]) \cong C_2^2 \times \Theta_{n-1}/([\Theta_{n-1}, \Gamma])$$

where U_1, V_0 are generators of the factor C_2^2 and the images of elements

$$U_2, \dots, U_{n-3}, V_1, \dots, V_{n-4}, W_n, T_n$$

are generators of the second factor.

Proof. Similar to the proof of lemma (4.17) using lemma (4.22). □

Proof of Theorem 4.8. We need to show that for $n \geq 3$ we have

$$H_2(\mathcal{G}_n, \mathbb{Z}) \cong (C_2)^{2n-2}$$

We claim that for $n \geq 3$

$$\Theta_n/([\Theta_n, \Gamma]) \cong C_2^{2n-4}$$

The case $n = 3$ follows from lemma (4.21). Assume it holds for $n > 3$. Then by lemma (4.23)

$$\Theta_{n+1}/([\Theta_{n+1}, \Gamma]) \cong C_2^2 \times \Theta_n/([\Theta_n, \Gamma])$$

and the claim follows from the induction hypothesis. Hence

$$K_n/([K_n, F]) \cong C^4 \times C_2^{2n-2}$$

and the result follows from lemma (4.11) and Hopf's formula.

Proof of Theorem 4.10. Let $\bar{K}_n = K_n/[K_n, F]$ and let $\acute{K}_n$ denote its quotient

$\bar{K}_n / \langle B_1^2, B_2^2, B_3^2, B_4^2 \rangle$. We have the following homomorphism:

$$\begin{array}{ccc}
\bar{K}_n & \longrightarrow & \acute{K}_n \\
a^2 & \mapsto & B_1 \\
b^2 & \mapsto & B_2 \\
c^2 & \mapsto & B_3 \\
d^2 & \mapsto & B_3 B_2 L \\
u_i & \mapsto & U_i \\
v_i & \mapsto & V_i \\
w_n & \mapsto & W_n \\
t_n & \mapsto & T_n
\end{array}$$

Hence any dependence among the initial relators will produce dependence among generators of $\acute{K}_n$.

4.4.3 Computation of $H^2(\widehat{\mathcal{G}}, \mathbb{F}_2)$

This subsection is devoted to the proof of Theorem 4.5. It is well known (see for example [Kar87]) that for a finite abelian group A one has

$$H^2(G, A) \cong (G/G' \otimes A) \times (M(G) \otimes A)$$

Using this, we have :

Lemma 4.24. *For $n \geq 3$ $H^2(\mathcal{G}_n, \mathbb{F}_2) \cong C_2^{2n+1}$.*

Proof. As mentioned before we have $\mathcal{G}_n/\mathcal{G}_n' \cong (C_2)^3$. Since by Theorem 4.8 $M(\mathcal{G}_n) \cong C_2^{2n-2}$ and $C_2 \otimes C_2 \cong C_2$ it follows that $H^2(\mathcal{G}_n, \mathbb{F}_2) \cong C_2^{2n+1}$. $\square$

Lemma 4.25. *For natural numbers n, k with $n \geq 3$, let*

$$q_{n,k} : \mathcal{G}_{n+k} \to \mathcal{G}_n$$

be the canonical quotient map. Then, there is $N \in \mathbb{N}$ such that for all n, k the dimension of the kernel of the induced map

$$q_{n,k}* : H^2(\mathcal{G}_n, \mathbb{F}_2) \to H^2(\mathcal{G}_{n+k}, \mathbb{F}_2)$$

is bounded above by N.

Proof. We have an exact sequence

$$1 \to Ker(q_{n,k}) \to \mathcal{G}_{n+k} \to \mathcal{G}_n \to 1 \tag{4.11}$$

and clearly $Ker(q_{n,k}) \cong St_{\mathcal{G}}(n)/St_{\mathcal{G}}(n+k)$. The sequence (4.11) induces the five term exact sequence (See [Wei94])

$$\begin{array}{cccccc} 0 \to & Hom(\mathcal{G}_n, \mathbb{F}_2) & \xrightarrow{\alpha} & Hom(\mathcal{G}_{n+k}, \mathbb{F}_2) & \xrightarrow{\beta} & Hom(Ker(q_{n,k}), \mathbb{F}_2)^{\mathcal{G}_n} \\ & & \xrightarrow{\partial} & H^2(\mathcal{G}_n, \mathbb{F}_2) & \xrightarrow{q_{n,k}*} & H^2(\mathcal{G}_{n+k}, \mathbb{F}_2) \end{array}$$

where $Hom(Ker(q_{n,k}), \mathbb{F}_2)^{\mathcal{G}_n}$ is the set of all homomorphisms invariant under the action of $\mathcal{G}_n$ on $Ker(q_{n,k})$ by conjugation. Since, $\mathcal{G}_n$ and $\mathcal{G}_{n+k}$ have the same abelianization α is an isomorphism. Therefore, β is the zero map and hence ∂ is an injection.

As noted earlier, for $n \geq 4$ we have $St_{\mathcal{G}}(n) \cong St_{\mathcal{G}}(n-1) \times St_{\mathcal{G}}(n-1)$. Therefore,

$$Ker(q_{n,k}) \cong \frac{St_{\mathcal{G}}(n)}{St_{\mathcal{G}}(n+k)} \cong \frac{St_{\mathcal{G}}(3)}{St_{\mathcal{G}}(k+3)} \times \ldots \times \frac{St_{\mathcal{G}}(3)}{St_{\mathcal{G}}(k+3)}$$

The action of $\mathcal{G}$ (and hence of $\mathcal{G}_n$) by conjugation on this decomposition agrees with the action to the corresponding level of the tree. Since the action of $\mathcal{G}$ is level transitive, it follows that the conjugation action of $\mathcal{G}_n$ on the factors of the above decomposition is transitive. Hence any homomorphism belonging to the group $Hom(Ker(q_{n,k}), \mathbb{F}_2)^{\mathcal{G}_n}$ is uniquely determined by its values in the first factor of this decomposition. But $St_{\mathcal{G}}(3)$ is finitely generated, hence the dimension of $Hom(Ker(q_{n,k}), \mathbb{F}_2)^{\mathcal{G}_n}$ is no more than a fixed number and in particular independent of k and n. □

Lemma 4.26. *Suppose $\{G_i, \varphi_{ij} \mid i, j \in I\}$ is a direct system consisting of finitely generated elementary abelian p-groups. Suppose also that the sequence $dim(G_i)$ is monotone increasing and there is a uniform bound N such that* $\dim(Ker(\varphi_{ij})) \leq N$. *Then the direct limit $\varinjlim G_i$ is infinite and hence isomorphic to $(C_p)^\infty$ where C_p is the cyclic group of order p.*

Proof. Recall that the direct limit can be defined as the disjoint union $\bigsqcup G_i$ factored by the equivalence relation:

$$g_i \sim g_j \iff \exists k \geq i, j \text{ such that } \varphi_{ik}(g_i) = \varphi_{jk}(g_j)$$

Suppose that $\varinjlim G_i$ has M elements. Select i large enough so that $dim(G_i) > NM$. So for large j we have

$$|G_i : Ker(\varphi_{ij})| \geq \frac{dim(G_i)}{N} > \frac{NM}{N} = M$$

which shows that the direct limit has more than M elements.

□

Proof of Theorem 4.5: The following is well known: (See [Wil98, Page 178])

If G is the inverse limit of the inverse system $\{G_i, \phi_{ij}\}$ then

$$H^n(G, \mathbb{F}_p) \cong \varinjlim H^n(G_i, \mathbb{F}_p)$$

i.e., $H^n(G, \mathbb{F}_p)$ is the direct limit of the direct system $\{H^n(G_i, \mathbb{F}_p), \phi_{ij}^*\}$ where ϕ_{ij}^* is the inflation map induced by ϕ_{ij}. Hence

$$H^2(\widehat{\mathcal{G}}, \mathbb{F}_2) \cong \varinjlim H^2(\mathcal{G}_n, \mathbb{F}_2)$$

Now lemmas 4.24 and 4.25 show that the hypotheses of Lemma 4.26 are satisfied and therefore $H^2(\widehat{\mathcal{G}}, \mathbb{F}_2) \cong (C_2)^\infty$.

5. INDICABLE GROUPS AND ENDOMORPHIC PRESENTATIONS*

This chapter is concerned with results published in the paper [Ben12a]. As already mentioned in the introduction and Chapter 4, L-presentations play an important role in the study of self-similar groups. The result that will be presented in this chapter suggests that such presentations could play also a role outside of the realm of self-similar groups.

5.1 Introduction

It is a well known fact that finite index subgroups of finitely presented groups are also finitely presented. More generally, finite index subgroups of a group usually share many properties of the group itself. But once the subgroup is of infinite index, various possibilities occur. For example, a finitely generated group may have a subgroup which is not finitely generated. Classical examples are non-abelian free groups with their commutator groups or the wreath products of the form $A \wr B$ where both A, B are nontrivial finitely generated groups with B infinite. It may also happen that a finitely presented group contains a finitely generated subgroup which is not finitely presented, even if the subgroup has cyclic quotient. A classical example (see [BR84]) is when $G = F_2 \times F_2$ and $H \leq G$ is the kernel of the homomorphism $F_2 \times F_2 \to \mathbb{Z}$ which sends each generator to 1.

In this chapter we will investigate exactly this situation. We will look at normal subgroups of finitely presented groups which have infinite cyclic quotient. The main result is the following:

*Indicable Groups and Endomorphic Presentations, by Mustafa Gökhan Benli, Glasgow Mathematical Journal, Volume 45, Issue 02 (May 2012), pp. 335-344,

Theorem 5.1. *Let G be a finitely presented group containing a finitely generated normal subgroup H such that the quotient G/H is cyclic. Then H has an ascending finite L-presentation with at most two free group endomorphisms, and the endomorphisms induce automorphisms of H.*

This theorem has the following significance: It is not difficult to observe that a finitely generated subgroup of a finitely presented group must be recursively presented. The celebrated *Higman embedding theorem* [Hig61] states that the converse is also true. Namely, any recursively presented group can be embedded into a finitely presented group. This gives a complete picture (in terms of presentations) of the structure of finitely generated subgroups of finitely presented groups.

Related to the Higman embedding theorem one can ask the following question: Does a recursively presented group embed *as a normal subgroup* into a finitely presented group? Interestingly, it was pointed out to the author by Mark Sapir that the first Grigorchuk group could not be embedded in this way. Here is M.Sapir's argument:

Proposition 5.1. *Let G be a normal subgroup of a finitely presented group H. Assume that G has trivial center and locally finite outer automorphism group $Out(G)$. Then G is finitely presented.*

Proof. Given $h \in H$ let f_h denote the automorphism $x \mapsto x^h$ of G. Let $\phi : H \to Out(G)$ denote the composition of the canonical map $H \to Aut(G)$, sending h to f_h, with the projection $Aut(G) \to Out(G)$. Let $Z_H(G)$ denote the centralizer of G in H and note that it is a normal subgroup of H. We claim that $\ker \phi = G \cdot Z_H(G)$. One inclusion is clear, for the other let $h \in \ker(\phi)$. This means that $f_h \in Inn(G)$ and there exists $g \in G$ such that $f_h = f_g$. This shows that $g^{-1}h \in Z_H(G)$ i.e., $h \in G \cdot Z_H(G)$. Since $Out(G)$ is locally finite, we see that $H/\ker(\phi)$ is finite and hence $\ker(\phi)$ is finitely presented. Since G has trivial center we have $G \cap Z_H(G) = \{1\}$ and therefore $\ker(\phi) \cong G \times Z_H(G)$. $Z_H(G)$ (being a homomorphic image of $\ker(\phi)$) is finitely generated. Therefore, since $\ker(\phi)$ is finitely presented, we obtain that G must be finitely presented. □

Corollary 5.1. *The first Grigorchuk group $\mathcal{G}$ cannot be embedded into a finitely presented group as a normal subgroup.*

Proof. As mentioned in the introduction, $\mathcal{G}$ is not finitely presented and has trivial center. Also, a result of Sidki and Grigorchuk [GS04] states that $Out(\mathcal{G})$ is elementary abelian 2-group of infinite rank and hence locally finite. Therefore by Proposition 5.1, $\mathcal{G}$ cannot be embedded into a finitely presented group as a normal subgroup. □

This shows that even a finitely L-presented group may fail to be a normal subgroup of a finitely presented group. Therefore, finitely generated normal subgroups of finitely presented groups must have very special kind of recursive presentations. To this extend we have the following definition:

Definition 5.1. An invariant L-presentation $\langle X \mid R \mid \Phi \rangle$ is called an AL- presentation if the endomorphisms $\phi \in \Phi$ induce automorphisms of the group defined by the presentation.

The reason for having this definition is the following:

Proposition 5.2. *If a group has a finite AL-presentation, then it can be embedded into a finitely presented group as a normal subgroup.*

Proof. Let $G = \langle X \mid R \mid \Phi \rangle$ be a finite AL-presentation and let $\Phi = \{\phi_1, \ldots \phi_n\}$. Let $Q = \bigcup_{\phi \in \Phi^*} \phi(R)$. By assumption, ϕ_i induces an automorphism φ_i of G where $i = 1, \ldots, n$. Form the HNN extension with the data $(G, \varphi_1 : G \to G)$ to obtain a group $G_1 = \langle X, t_1 \mid Q, x^{t_1} = \phi_1(x), x \in X \rangle$. Next, form the HNN extension with the data $(G_1, \varphi_2 : G \to G)$ to obtain $G_2 = \langle X, t_1, t_2 \mid Q, x^{t_1} = \phi_1(x), x^{t_2} = \phi_2(x), x \in X \rangle$. Continuing this process we obtain the group

$$G_n = \left\langle X, t_1, t_2, \ldots, t_n \mid Q, x^{t_i} = \phi_i(x), x \in X, i = 1, \ldots, n \right\rangle$$

Let $r = x_1 \ldots x_m \in R$ where $x_i \in X^{\pm}$ and $\phi = \phi_{j_1} \ldots \phi_{j_s} \in \Phi^*$ where $j_k \leq n$. We have $\phi(r) = \phi(x_1) \ldots \phi(x_m)$ which is equal to $(x_1 \ldots x_m)^{t_{j_1} \cdots t_{j_s}} = r^{t_{j_1} \cdots t_{j_s}} = 1$ in G_n using the relators of the form $x^{t_j} = \phi_j(x)$ in G_n. Therefore, G_n has the presentation

$$G_n = \left\langle X, t_1, t_2, \ldots, t_n \mid R,\ x^{t_i} = \phi_i(x),\ x \in X, i = 1, \ldots, n \right\rangle$$

and hence is finitely presented. Now G is a normal subgroup of G_n. To see this we only need to verify that $x^{t_j^{-1}} \in G$ whenever $x \in X$ and $j \leq n$. Let $y \in G$ be such that $\varphi_j(y) = x$. This shows that $\phi_j(y) = x$ and hence $x^{t_j^{-1}} = y$.

□

This proposition was indicated in [Ben12a] without a proof. Our proof above follows the lines of [Har12b, Proposition 5.2].

Theorem 5.1 now can be thought as a partial converse to Proposition 5.2. It is natural to ask to what extend Theorem 5.1 can be generalized:

Question 8. *Is it true that a finitely generated normal subgroup of a finitely presented group has a finite AL-presentation?*

A positive answer to Question 8 would give a characterization of finitely generated normal subgroups of finitely generated groups as groups which have finite AL-presentation.

At the time of writing this dissertation R. Hartung's pre-print [Har12b] contains the following generalizations of Theorem 5.1:

Theorem 5.2. *[Har12b] If H is a finitely generated normal subgroup of a finitely presented group G such that H splits over G, or G/H is abelian of torsion-free rank at most 2, then H has finite AL-presentation.*

5.2 Application to indicable groups

Recall that a group is called *indicable* if it can be mapped onto the infinite cyclic group. Indicable groups play an important role in the study of orderable groups, amenability and bounded cohomology. Recall the Lysenok presentation of the first Grigorchuk group introduced in the introduction:

$$\mathcal{G} = \left\langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, \sigma^i((ad)^4), \sigma^i((adacac)^4), i \geq 0 \right\rangle$$

where σ is the endomorphism induced by $a \mapsto aca, b \mapsto d, c \mapsto b, d \mapsto c$.

It can be observed from the above presentation that σ induces a homomorphism $\tilde{\sigma} : \mathcal{G} \to \mathcal{G}$. It is also true that $\tilde{\sigma}$ is injective (see [Gri98]). Therefore we can form the HNN extension $\tilde{\mathcal{G}}$ corresponding to the data $(\mathcal{G}, \tilde{\sigma} : \mathcal{G} \to \sigma(\tilde{\mathcal{G}}))$ and one can observe that $\tilde{\mathcal{G}}$ is finitely presented with the following presentation:

$$\tilde{\mathcal{G}} = \left\langle a, b, c, d, t \mid a^2, b^2, c^2, d^2, bcd, (ad)^4, (adacac)^4, a^t = aca, b^t = d, c^t = b, d^t = c \right\rangle$$

$\tilde{\mathcal{G}}$ was the first example of a finitely presented amenable group which is not elementary amenable.

$\tilde{\mathcal{G}}$ is a finitely presented group which is indicable. Since it is amenable it does not contain any non-abelian free subgroups. Also, by construction it is an ascending HNN extension of a finitely generated group. All these properties are in alignment with the following theorem due to Bieri and Strebel [BS78]:

Theorem 5.3. *A finitely presented indicable group which does not contain non-abelian free subgroups is an ascending HNN extension with a finitely generated base group.*

A natural question was raised by Grigorchuk in [Gri05] regarding this alignment:

Question 9. *Is it true that a finitely presented indicable group which does not contain a non-abelian free subgroup is an ascending HNN extension with a base group having finite L-presentation?*

Question 2, which is the motivating question behind [Ben12a], remains open. But with the weaker assumption when the group does not contain free sub-semigroups of rank at least 2, we can answer it positively. The crucial point is that when the group does not contain free sub-semigroups, the kernel of any epimorphism onto the infinite cyclic group is itself finitely generated, allowing us to apply Theorem 5.1. To see this we will need some auxiliary lemmas:

Lemma 5.1. *[LMR95] If a group G does not contain a free sub-semigroup of rank 2, then for all $a, b \in G$ the subgroup $\left\langle a^{b^n} \mid n \in \mathbb{Z} \right\rangle$ is finitely generated.*

Lemma 5.2. *[Ros76] Let G be a finitely generated group such that for all $a, b \in G$, the subgroup $\left\langle a^{b^n} \mid n \in \mathbb{Z} \right\rangle$ is finitely generated. If H is a normal subgroup of G such that G/H is solvable, then H is finitely generated.*

The following is a corollary of the previous lemmas and Theorem 5.1:

Theorem 5.4. *Let G be a finitely presented indicable group not containing any free sub-semigroups of rank 2. Then G has the form of a semi-direct product $H \rtimes \mathbb{Z}$ where H has a finite AL-presentation.*

5.3 Schur multipliers of finitely L-presented groups

Recall, from previous chapters, that the Schur multiplier $M(G)$ of a group G is the homology group $H_2(G,\mathbb{Z})$ and can be calculated using the Schur-Hopf formula $M(G) = \dfrac{R \cap F'}{[F,R]}$ where F/R is any presentation of G. As mentioned earlier, if G is a finitely presented group then its Schur multiplier is a finitely generated abelian group. A natural question is to ask what special structure (if any) do Schur multipliers of finitely L-presented groups have.

A theorem proven by L.Bartholdi in [Bar03] asserts that the Schur multiplier of a finitely L-presented group is necessarily a direct sum of finitely generated abelian groups. Unfortunately it turns out that this theorem is false. To see this we will give an example of a group G which is finitely L-presented and for which $M(G) = \mathbb{Z}[\frac{1}{2}]$. I am grateful to Ian Agol for providing this example with all the details.

Example 5.1. Let G be the Baumslag-Solitar group $BS(1,2) = \left\langle a, b \mid a^b = a^2 \right\rangle$. Let K be the kernel of the homomorphism onto $\mathbb{Z}$ induced by $a \mapsto 0, b \mapsto 1$. Clearly $K = \left\langle a^{b^k} \mid k \in \mathbb{Z} \right\rangle \cong \mathbb{Z}[\frac{1}{2}]$. Let H be the amalgamated free product $G \star_K G$. We have the following Mayer-Vietoris sequence (see [Bro82, Corollary 7.7])

$$H_2(G,\mathbb{Z}) \oplus H_2(G,\mathbb{Z}) \xrightarrow{\alpha} H_2(H,\mathbb{Z}) \xrightarrow{\beta} H_1(K,\mathbb{Z}) \xrightarrow{\delta} H_1(G,\mathbb{Z}) \oplus H_1(G,\mathbb{Z})$$

It is well known that $H_2(G,\mathbb{Z}) = \{0\}$ (see for example [Rob10]). Also $\mathbb{Z}[\frac{1}{2}] \xrightarrow{\delta} \mathbb{Z}^2$ is the the zero map because its domain is 2 divisible and its range is not. Therefore β is an isomorphism, and hence $H_2(H,\mathbb{Z}) \cong \mathbb{Z}[\frac{1}{2}]$. Clearly H has the presentation

$$H = \left\langle a, b, c \mid a^b = a^2, a^c = a^2, a^{b^k} = a^{c^k}, k \in \mathbb{Z} \right\rangle$$

from which an AL-presentation directly follows:

$$H = \left\langle a, b, c \mid a^b = a^2, a^c = a^2 \mid \phi \right\rangle$$

where $\phi : a \mapsto a^2, b \mapsto b, c \mapsto c$. Clearly ϕ induces an automorphism and H embeds into the finitely presented group $H \rtimes \mathbb{Z}$ which has the presentation

$$\left\langle a, b, c, t \mid a^b = a^2, a^c = a^2, b^t = b, c^t = c, a^t = a^2 \right\rangle$$

It turns out that the Schur multiplier cannot be used to distinguish finite L-presentations from general recursively presented groups. Consequently one may consider the following question:

Question 10. *Are there recursively presented groups which are not finitely L- presented?*

5.4 Proof of Theorem 5.1

This section is devoted to the proof of Theorem 5.1.

Let G be a finitely presented group and H a finitely generated normal subgroup such that G/H is infinite cyclic. Suppose that for $t \in G$ we have $G/H = \langle tH \rangle$, then G has the form of a semi direct product $G = H \rtimes \langle t \rangle$.

From Neumann's Theorem [Bau93, Page 52] it follows that G has a presentation of the form

$$G = \langle t, a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle$$

where

$$H = \langle \langle a_1, \ldots, a_m \rangle \rangle$$

and $exp_t(r_k) = 0$ for $k = 1, \ldots, n$. Consequently, the set $T = \{t^i \mid i \in \mathbb{Z}\}$ is a right

Schreier transversal for H in G. Following the Reidemeister-Schreier process for H, we can take the elements

$$a_{j,i} = t^{-i} a_j t^i \qquad j = 1, \ldots, m \;\; i \in \mathbb{Z}$$

as generators for H and the words

$$r_{k,i} = \rho(t^{-i} r_k t^i) \qquad k = 1, \ldots, n \;\; i \in \mathbb{Z}$$

as relators, where ρ is the rewriting of $t^{-i} r_k t^i$ as a word in the $a_{j,i}$' s. So, H has the presentation

$$H = \langle a_{j,i} \quad (j = 1, \ldots, m \;\; i \in \mathbb{Z}) \mid r_{k,i} \quad (k = 1, \ldots, n \;\; i \in \mathbb{Z}) \rangle \tag{5.1}$$

Each r_k is a word of the form

$$r_k = \prod_{s=1}^{n_k} t^{-l_s} a_{z_s} t^{l_s}$$

where $a_{z_s} \in \{a_j, j = 1, \ldots, m\}^{\pm}$ and $n_k \in \mathbb{N}, \;\; l_s \in \mathbb{Z}$. Therefore we have

$$r_{k,0} = \rho(r_k) = \rho(\prod_{s=1}^{n_k} t^{-l_s} a_{z_s} t^{l_s}) = \prod_{s=1}^{n_k} a_{z_s, l_s}$$

and

$$r_{k,i} = \rho(t^{-i} r_k t^i) = \prod_{s=1}^{n_k} a_{z_s, l_s + i} \quad i \in \mathbb{Z} \tag{5.2}$$

Let F be the free group over the set $\{a_{j,i} \mid j = 1, \ldots, m, i \in \mathbb{Z}\}$. Let s denote automorphism of F sending $a_{j,i}$ to $a_{j,i+1}$. Clearly s induces on H the automorphism $h \mapsto t^{-1} h t$. Since by assumption H is finitely generated, we can select a big enough natural number N with the following properties:

- $H = \langle a_{j,i} \mid j = 1, \ldots, m, |i| \leq N \rangle$
- Each word $r_{k,0}$ is a word in $\{a_{j,i} \mid j = 1, \ldots, m, |i| \leq N\}^{\pm}$

So, each $a_{j,i}$ can be represented by a word in the finite generating set $\{a_{j,i} \mid j = 1, \ldots, m, |i| \leq N\}^{\pm}$.

For each $a_{j,i}$ we will recursively construct a word $\gamma(a_{j,i})$ in this new finite generating set such that $\gamma(a_{j,i})$ represents $a_{j,i}$ in H. For $a_{j,i}$ with $|i| \leq N$ we simply define $\gamma(a_{j,i})$ to be $a_{j,i}$. Pick $\gamma(a_{j,N+1})$ and $\gamma(a_{j,-(N+1)})$ two words in $\{a_{j,i} \mid j = 1, \ldots, m \quad |i| \leq N\}^{\pm}$ representing $a_{j,N+1}$ and $a_{j,-(N+1)}$ in H, respectively.

For $i \geq N+1$ we define $\gamma(a_{j,i+1})$ recursively as follows: Assume that $\gamma(a_{j,i})$ has already been defined, let

$$\gamma(a_{j,i+1}) = \gamma(s(\gamma(a_{j,i})))$$

(for a word w, we define $\gamma(w)$ as the word obtained by applying γ to each letter of w). Note that $s(\gamma(a_{j,i}))$ is a word in $\{a_{j,i} \mid j = 1, \ldots, m, |i| \leq N+1\}^{\pm}$ and for these letters $\gamma(a_{j,i})$ is already defined, therefore we can apply γ to $s(\gamma(a_{j,i}))$.

Similarly, for $i \leq -(N+1)$ we define $\gamma(a_{j,i-1})$ as

$$\gamma(a_{j,i-1}) = \gamma(s^{-1}(\gamma(a_{j,i})))$$

Defining γ as above gives the following equalities in the free group F:

$$\gamma(a_{j,i+1}) = \gamma(s(\gamma(a_{j,i}))) \quad \text{for} \quad i \geq -N \tag{5.3}$$

$$\gamma(a_{j,i-1}) = \gamma(s^{-1}(\gamma(a_{j,i}))) \quad \text{for} \quad i \leq N \tag{5.4}$$

Lemma 5.3. *H has the following presentation with finitely many generators*

$$\langle a_{j,i}, j = 1, \ldots, m, |i| \leq N \mid \gamma(r_{k,i}), k = 1, \ldots, n, i \in \mathbb{Z} \rangle .$$

Proof. This follows by Tietze transformations, but we will explicitly construct an isomorphism between these presentations. In order to avoid confusion we denote elements in the asserted presentation with bars and set

$$\overline{H} = \left\langle \overline{a_{j,i}}, j = 1, \ldots, m, |i| \leq N \mid \overline{\gamma(r_{k,i})}, k = 1, \ldots, n, i \in \mathbb{Z} \right\rangle$$

We will show that $\overline{H} \cong H$ using the presentation (5.1) of H. For this, define

$$\begin{array}{rccc} \varphi : & H & \longrightarrow & \overline{H} \\ & a_{j,i} & \mapsto & \overline{\gamma(a_{j,i})} \end{array}$$

We have $\varphi(r_{k,i}) = \overline{\gamma(r_{k,i})} = 1$ in $\overline{H}$. So φ maps relators of H to relators in $\overline{H}$ and hence is a well defined group homomorphism. Conversely, define

$$\begin{array}{rccc} \psi : & \overline{H} & \longrightarrow & H \\ & \overline{a_{j,i}} & \mapsto & a_{j,i} \end{array}$$

Since $\gamma(a_{j,i}) = a_{j,i}$ in H we have

$$\psi(\overline{\gamma(r_{k,i})}) = \gamma(r_{k,i}) = r_{k,i} = 1 \quad \text{in} \quad H$$

which shows that ψ is a well defined group homomorphism. Finally the following equalities show that φ and ψ are mutual inverses:

$$(\varphi \circ \psi)(\overline{a_{j,i}}) = \varphi(a_{j,i}) = \overline{\gamma(a_{j,i})} = \overline{a_{j,i}}$$

(where the last equality is true since $|i| \leq N$ in this case.)

$$(\psi \circ \varphi)(a_{j,i}) = \psi(\overline{\gamma(a_{j,i})}) = \gamma(a_{j,i}) = a_{j,i} \quad \text{in} \quad H.$$

Hence $\overline{H}$ is isomorphic to H.

□

Let F_r be the free group over the set $\{a_{j,i} \mid j = 1, \ldots, m \quad |i| \leq N\}$. Define two endomorphisms η and τ of F_r as follows:

$$\eta(a_{j,i}) = \gamma(s(a_{j,i})) = \gamma(a_{j,i+1})$$

and

$$\tau(a_{j,i}) = \gamma(s^{-1}(a_{j,i})) = \gamma(a_{j,i-1})$$

where γ is as above. Note that η and τ induce the automorphisms s and s^{-1} of H respectively.

Lemma 5.4. *In F_r we have the equality*

$$\gamma(r_{k,i}) = \begin{cases} \eta^i(r_{k,0}) & \text{if} \quad i \geq 0 \\ \tau^{-i}(r_{k,0}) & \text{if} \quad i < 0 \end{cases}$$

Proof. Suppose $i \geq 0$. We use induction on i.

If $i = 0$, $\gamma(r_{k,0}) = r_{k,0}$ by choice of γ and the natural number N. Suppose the equality holds for i. Then

$$\begin{aligned}
\eta^{i+1}(r_{k,0}) &= \eta(\eta^i(r_{k,0})) \\
&= \eta(\gamma(r_{k,i})) && \text{(by induction hypothesis)} \\
&= \eta(\gamma(\prod a_{z_s,l_s+i})) && \text{(using equation (5.2))} \\
&= \prod \eta(\gamma(a_{z_s,l_s+i})) \\
&= \prod \gamma s \gamma(a_{z_s,l_s+i}) \\
&= \prod \gamma(a_{z_s,l_s+i+1}) && \text{(using equation (5.3), since } |l_s| \leq N) \\
&= \gamma(\prod a_{z_s,l_s+i+1}) \\
&= \gamma(r_{k,i+1})
\end{aligned}$$

A similar argument with induction on $-i$ (and using equation (5.4)) shows the required identity for $i < 0$. □

Lemma 5.5. *H has the following ascending finite AL-presentation:*

$$\langle a_{j,i} \quad (j = 1, \ldots, m \quad |i| \leq N) \mid r_{k,0} \quad (k = 1, \ldots, n) \mid \{\eta, \tau\}\rangle$$

Proof. Again not to cause confusion we denote the asserted presentation with bars and set

$$\overline{H} = \langle \overline{a_{j,i}} \quad (j = 1, \ldots, m \quad |i| \leq N) \mid \overline{r_{k,0}} \quad (k = 1, \ldots, n) \mid \{\overline{\eta}, \overline{\tau}\}\rangle$$

where $\overline{\eta}, \overline{\tau}$ are endomorphisms of the free group $\overline{F_r}$ analogous to η and τ. More precisely:

$$\overline{\eta}(\overline{a_{j,i}}) = \overline{\eta(a_{j,i})}$$

$$\overline{\tau}(\overline{a_{j,i}}) = \overline{\tau(a_{j,i})}$$

We will show that $\overline{H} \cong H$ and we will use the presentation of H

$$\langle a_{j,i}(j = 1, \ldots, m \quad |i| \leq N) \mid \gamma(r_{k,i})(k = 1, \ldots, n \quad i \in \mathbb{Z})\rangle$$

which was found in Lemma 5.3. To this end define

$$\begin{array}{cccc} \phi: & H & \longrightarrow & \overline{H} \\ & a_{j,i} & \mapsto & \overline{a_{j,i}} \end{array}$$

We have

$$\phi(\gamma(r_{k,i})) = \overline{\gamma(r_{k,i})} = \begin{cases} \overline{\eta}^i(\overline{r_{k,0}}) & \text{if} \quad i \geq 0 \\ \overline{\tau}^{-i}(\overline{r_{k,0}}) & \text{if} \quad i < 0 \end{cases}$$

by Lemma 5.4. Hence ϕ is a well defined group homomorphism. Conversely define:

$$\begin{array}{cccc} \chi: & \overline{H} & \longrightarrow & H \\ & \overline{a_{j,i}} & \mapsto & a_{j,i} \end{array}$$

To show that χ is well defined, we need to prove that for all $\overline{f} \in \{\overline{\eta}, \overline{\tau}\}^*$ and for all $k = 1, \ldots, n$ we have

$$\chi(\overline{f}(\overline{r_{k,0}})) = 1 \quad \text{in} \quad H$$

This is true since η and τ (and hence f) induce isomorphisms on H. This shows that χ is a well defined group homomorphism. Clearly ϕ and χ are mutual inverses. $\square$

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