Crucial words for abelian powers*

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Outline

1. Background
   - Repetitions & patterns in words
   - Crucial words & abelian powers
2. Minimal crucial words avoiding abelian cubes
   - Upper bound for length
   - Lower bound for length
3. Minimal crucial words avoiding abelian k-th powers
   - Upper bound for length
   - Lower bound for length
4. Further research

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gives (in the limit) the infinite word

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- Now called the *Thue-Morse word* as it was rediscovered by Morse in 1921 (in the context of symbolic dynamics).

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- Answer: **YES**. Existence was established for alphabets of size:
  - **25** and improved to **7** (A. Evdokimov, 1968 & 1971);
  - **5** (P.A.B. Pleasants, 1970);
  - **4** (Keränen, 1992), the **optimal result** (such a word does not exist over a 3-letter alphabet).

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- We are interested in a particular problem in relation to words avoiding abelian powers.

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Let $\mathcal{A}_n = \{1, 2, \ldots, n\}$ and let $k \geq 2$ be an integer.

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- Examples:
  - V contains the abelian square 43232 32324.
  - 123 312 213 is an abelian cube.

Crucial words with respect to abelian powers

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  In fact: W is a minimal crucial word over $\{1, 2\}$ with respect to abelian cubes.

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- The *k-generalised Zimin word* $Z_n^k = X_n$ is defined as

$$X_1 = 1^{k-1} = 11\ldots1, \; X_n = (X_{n-1}n)^{k-1}X_{n-1} = X_{n-1}nX_{n-1}n\ldots nX_{n-1}$$

where the number of 1's, as well as the number of n's, is $k-1$.

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$Z_2^3 = 11\underline{2}11\underline{2}11,$
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- Z_n^k gives the length of a minimal crucial word avoiding k-th powers.

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Much less is known in the case of abelian k-th powers ...

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- Now we extend the study of crucial abelian k-power-free words to the case of $k > 2$.
  - We provide a complete solution to the problem of determining the length of a minimal crucial abelian cube-free word (the case $k = 3$).
  - And we conjecture a solution in the general case.
- Let $\ell_k(n)$ denote the length of a minimal crucial word over $\mathcal{A}_n$ avoiding abelian k-th powers.

Outline

1. Background
   - Repetitions & patterns in words
   - Crucial words & abelian powers
2. Minimal crucial words avoiding abelian cubes
   - Upper bound for length
   - Lower bound for length
3. Minimal crucial words avoiding abelian k-th powers
   - Upper bound for length
   - Lower bound for length
4. Further research

Upper bound for $\ell_3(n)$

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For $n \geq 1$, we have $\ell_3(n) \leq 3 \cdot 2^{n-1} - 1$.

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Sketch Proof: "Greedy" construction of a crucial abelian cube-free word $X = X_n$ over $\mathcal{A}_n$, defined recursively as follows:

$$X_1 = 11 \quad \text{and} \quad X_n = \phi_1(\sigma(X_{n-1}))1 \quad \text{for } n \geq 2,$$

where $\sigma : x \mapsto x + 1$ and $\phi_1 : x \mapsto x1$ for all letters x.

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1. Increase all letters of X_{n-1} by 1 to obtain X'_{n-1}.
2. Insert the letter 1 to the right of each letter of X'_{n-1} and adjoin one extra 1 to the right of the resulting word to obtain X_n.

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- Hence $\ell_3(n) \leq 3 \cdot 2^{n-1} - 1$.

An optimal construction

- Let X be a crucial word over $\mathcal{A}_n$ with respect to abelian k-th powers.

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- If X is minimal, we may assume w.l.o.g. that Xn is an abelian k-th power, and we write:

$$X = \Omega_{n,1}\Omega_{n,2}\cdots\Omega'_{n,k}$$

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A construction of crucial abelian cube-free words over $\mathcal{A}_n$ for $n \geq 4$:

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A construction of crucial abelian cube-free words over $\mathcal{A}_n$ for $n \geq 4$:

- Basis: Minimal crucial abelian square-free words $W_n = W_{n,2}$ given by Evdokimov & Kitaev (2004). For $n = 4, 5, 6, 7$:

$W_{4,2} = 34231\ 3231,$

$W_{5,2} = 4534231\ 432341,$

$W_{6,2} = 564534231\ 54323451,$

$W_{7,2} = 67564534231\ 6543234561,$ where spaces separate the blocks.

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General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

- 1st block $\Omega_{n,1}$: adjoin the factors $i(i+1)$ for $i = n-1, n-2, \ldots, 2$, followed by the letter 1.

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General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

- 1st block $\Omega_{n,1}$: adjoin the factors $i(i+1)$ for $i = n-1, n-2, \ldots, 2$, followed by the letter 1.

  Example: For $n = 4$, we have $\Omega_{4,1} = 34231$.

- 2nd block $\Omega'_{n,2}$: adjoin the factors $(n-1)(n-2)\ldots 432$, then $34\ldots(n-2)(n-1)$, and finally the letter 1.

  Example: For $n = 4$, we have $\Omega_{4,2} = 3231$.

For $n = 4, 5, 6, 7$, we have:

$W_{4,2} = 34231\ 3231$,

$W_{5,2} = 4534231\ 43234$

An optimal construction . . .

General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

- 1st block $\Omega_{n,1}$: adjoin the factors $i(i+1)$ for $i = n-1, n-2, \ldots, 2$, followed by the letter 1.

  Example: For $n = 4$, we have $\Omega_{4,1} = 34231$.

- 2nd block $\Omega'_{n,2}$: adjoin the factors $(n-1)(n-2)\ldots 432$, then $34\ldots(n-2)(n-1)$, and finally the letter 1.

  Example: For $n = 4$, we have $\Omega_{4,2} = 3231$.

For $n = 4, 5, 6, 7$, we have:

$W_{4,2} = 34231\ 3231$,

$W_{5,2} = 4534231\ 432341$,

An optimal construction . . .

General construction of $W_{n,2} = \Omega_{n,1}\Omega'_{n,2}$ for $n \geq 4$:

- 1st block $\Omega_{n,1}$: adjoin the factors $i(i+1)$ for $i = n-1, n-2, \ldots, 2$, followed by the letter 1.

  Example: For $n = 4$, we have $\Omega_{4,1} = 34231$.

- 2nd block $\Omega'_{n,2}$: adjoin the factors $(n-1)(n-2)\ldots 432$, then $34\ldots(n-2)(n-1)$, and finally the letter 1.

  Example: For $n = 4$, we have $\Omega_{4,2} = 3231$.

For $n = 4, 5, 6, 7$, we have:

$W_{4,2} = 34231\ 3231$,

$W_{5,2} = 4534231\ 432341$,

$W_{6,2} = 564534231\ 54323451$,

$W_{7,2} = 67564534231\ 6543234561$.

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34231 34231 3231

Duplicate 1st block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow 34231\ 34231\underline{134}\ 3231$

Append 134 to 2nd block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow 34\underline{4}231\ 34231134\ 3231$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow\ 344233\underline{}1\ 34231134\ 3231$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow\ 3442331\underline{1}\ 34231134\ 3231$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow\ 34423311\ 34231134\ 32331$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 323311

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow\ 34423311\ 34231134\ 3233\underline{4}11$

Insert 4 before leftmost 1 in 3rd block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 4534231 4534231 432341

Duplicate 1st block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 4534231 4534231<u>1345</u> 432341

Append 1345 to 2nd block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534231 45342311345 432341

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 455344231 45342311345 432341

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 4553442331 45342311345 432341

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow\ 34423311\ 34231134\ 3233411 = W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow 4553442331\underline{1}\ 45342311345\ 432341$

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323341

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 43233441

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 43233441<u>1</u>

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323344<u>5</u>11

Insert 5 before leftmost 1 in 3rd block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323344511 $= W_{5,3}$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323344511 $= W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323344511 $= W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow$ <u>564534231</u> <u>564534231</u> 54323451

Duplicate 1st block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323344511 $= W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow$ 564534231 564534231<u>13456</u> 54323451

Append 13456 to 2nd block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323344511 $= W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow$ 566455344423311 56453423113456 5432334455 11

Duplicate rightmost x for each $x \neq 2$ in 1st & 3rd blocks

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323344511 $= W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow$ 56645534423311 56453423113456 5432334455611

Insert 6 before leftmost 1 in 3rd block

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323344511 $= W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow$ 56645534423311 56453423113456 5432334455611 $= W_{6,3}$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323344511 $= W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow$ 56645534423311 56453423113456 5432334455611 $= W_{6,3}$

$W_{7,2} = 67564534231\ 6543234561$

$\longrightarrow$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow\ 34423311\ 34231134\ 3233411 = W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3}$

$W_{7,2} = 67564534231\ 6543234561$

$\longrightarrow \underline{67564534231}\ \underline{67564534231}\ 6543234561$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323344511 $= W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow$ 56645534423311 56453423113456 5432334455611 $= W_{6,3}$

$W_{7,2} = 67564534231\ 6543234561$

$\longrightarrow$ 67564534231 67564534231134567 6543234561

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323344511 $= W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow$ 56645534423311 56453423113456 5432334455611 $= W_{6,3}$

$W_{7,2} = 67564534231\ 6543234561$

$\longrightarrow$ 67756645534423311 67564534231134567 6543233445566 11

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323344511 $= W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow$ 56645534423311 56453423113456 5432334455611 $= W_{6,3}$

$W_{7,2} = 67564534231\ 6543234561$

$\longrightarrow$ 67756645534423311 67564534231134567 6543233445566<u>7</u>11

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow\ 34423311\ 34231134\ 3233411 = W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow 45534423311\ 45342311345\ 4323344511 = W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow 56645534423311\ 56453423113456\ 5432334455611 = W_{6,3}$

$W_{7,2} = 67564534231\ 6543234561$

$\longrightarrow 67756645534423311\ 67564534231134567\ 654323344556711$

An optimal construction ...

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ ...

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 3233411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 4323344511 $= W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow$ 56645534423311 56453423113456 5432334455611 $= W_{6,3}$

$W_{7,2} = 67564534231\ 6543234561$

$\longrightarrow$ 67756645534423311 67564534231134567 654323344566711

Note:

$$W_{n,3} = \underbrace{(n-1)nn\Omega_{n-1,1}}_{\Omega_{n,1}}\underbrace{(n-1)n\Omega_{n-1,2}n}_{\Omega_{n,2}}\underbrace{(n-1)\Omega'_{n-1,3}[11]^{-1}(n-1)n11}_{\Omega'_{n,3}}.$$

An optimal construction . . .

For $n \geq 4$, we obtain crucial abelian cube-free words $W_{n,3}$ from $W_{n,2}$ . . .

Construction of $W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$:

$W_{4,2} = 34231\ 3231 \longrightarrow$ 34423311 34231134 32333411 $= W_{4,3}$

$W_{5,2} = 4534231\ 432341$

$\longrightarrow$ 45534423311 45342311345 43233344511 $= W_{5,3}$

$W_{6,2} = 564534231\ 54323451$

$\longrightarrow$ 56645534423311 56453423113456 54323344455611 $= W_{6,3}$

$W_{7,2} = 67564534231\ 6543234561$

$\longrightarrow$ 67756645534423311 67564534231134567 65432334455666711

Note:

$$W_{n,3} = \underbrace{(n-1)nn\Omega_{n-1,1}}_{\Omega_{n,1}}\underbrace{(n-1)n\Omega_{n-1,2}n}_{\Omega_{n,2}}\underbrace{(n-1)\Omega'_{n-1,3}[11]^{-1}(n-1)n11}_{\Omega'_{n,3}}.$$

An optimal construction . . .

- By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

where $\Omega_{n,3} = \Omega'_{n,3}n$ and each $\Omega_{n,i}$ contains two 1's, one 2, two n's, and three x's for $x = 3, \ldots, n-1$.

An optimal construction . . .

- By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

where $\Omega_{n,3} = \Omega'_{n,3}n$ and each $\Omega_{n,i}$ contains two 1's, one 2, two n's, and three x's for $x = 3, \ldots, n-1$.

- Hence, $|W_{n,3}| = 3(3(n-3) + 2 \cdot 2 + 1) - 1 = 9n - 13$.

An optimal construction . . .

- By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

  where $\Omega_{n,3} = \Omega'_{n,3}n$ and each $\Omega_{n,i}$ contains two 1's, one 2, two n's, and three x's for $x = 3, \ldots, n-1$.

- Hence, $|W_{n,3}| = 3(3(n-3) + 2 \cdot 2 + 1) - 1 = 9n - 13$.
- Moreover, $W_{n,3}$ is crucial with respect to abelian cubes.

An optimal construction . . .

- By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

  where $\Omega_{n,3} = \Omega'_{n,3}n$ and each $\Omega_{n,i}$ contains two 1's, one 2, two n's, and three x's for $x = 3, \ldots, n-1$.

- Hence, $|W_{n,3}| = 3(3(n-3) + 2 \cdot 2 + 1) - 1 = 9n - 13$.
- Moreover, $W_{n,3}$ is crucial with respect to abelian cubes.

  For instance: it is easy to check that

$$W_{4,3} = 34423311\ 34231134\ 3233411$$

  is an abelian cube-free crucial word on 4 letters.
  Use same arguments for $n > 4$.

An optimal construction . . .

- By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

  where $\Omega_{n,3} = \Omega'_{n,3}n$ and each $\Omega_{n,i}$ contains two 1's, one 2, two n's, and three x's for $x = 3, \ldots, n-1$.
- Hence, $|W_{n,3}| = 3(3(n-3) + 2 \cdot 2 + 1) - 1 = 9n - 13$.
- Moreover, $W_{n,3}$ is crucial with respect to abelian cubes.
- Thus, a minimal crucial word avoiding abelian cubes has length at most $9n - 13$ for $n \geq 4$.

An optimal construction . . .

- By construction:

$$W_{n,3} = \Omega_{n,1}\Omega_{n,2}\Omega'_{n,3}$$

  where $\Omega_{n,3} = \Omega'_{n,3}n$ and each $\Omega_{n,i}$ contains two 1's, one 2, two n's, and three x's for $x = 3, \ldots, n-1$.

- Hence, $|W_{n,3}| = 3(3(n-3) + 2 \cdot 2 + 1) - 1 = 9n - 13$.
- Moreover, $W_{n,3}$ is crucial with respect to abelian cubes.
- Thus, a minimal crucial word avoiding abelian cubes has length at most $9n - 13$ for $n \geq 4$.

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This upper bound is optimal . . .

Lower bound for $\ell_3(n)$

Let X be a crucial abelian cube-free word over $\mathcal{A}_n$ such that Xn is an abelian cube.

- Sort in non-decreasing order the # of occurrences of the letters $1, 2, \ldots, n-1$ in X to obtain the sequence $(a_1 \leq a_2 \leq \ldots \leq a_{n-1})$.

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We prove that for $n \geq 5$ this sequence **cannot** be "improved" by decreasing one or more of its terms, no matter how the crucial word is constructed.

Lower bound for $\ell_3(n)$ . . .

That is, for a crucial abelian cube-free word X over $\mathcal{A}_n$:

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For example: 11, 21211, 11231321211, 42131214231211321211.

Outline

Upper bound for $\ell_k(n)$

- Z_n^k crucial with respect to abelian k-th powers $\implies \ell_k(n) \leq k^n - 1$.

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For $n \geq 4$ and $k \geq 2$, we construct a crucial abelian k-power-free word $W_{n,k}$ of length $k^2(n-1) - k - 1$ using the same method as before.

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Note:

- $|W_{n,2}| = 4n - 7$ and $|W_{n,3}| = 9n - 13 \longrightarrow W_{n,2}$ and $W_{n,3}$ are minimal crucial words over $\mathcal{A}_n$ avoiding abelian squares and abelian cubes, respectively.

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- $|W_{n,2}| = 4n - 7$ and $|W_{n,3}| = 9n - 13 \longrightarrow W_{n,2}$ and $W_{n,3}$ are minimal crucial words over $\mathcal{A}_n$ avoiding abelian squares and abelian cubes, respectively.
- In the case of $k \geq 4$, we make the following conjecture.

Conjecture (G.-Halldórsson-Kitaev, 2008)

For $k \geq 4$ and sufficiently large n, the length of a minimal crucial word over $\mathcal{A}_n$ avoiding abelian k-th powers is given by $k^2(n-1) - k - 1$.

Lower bound for $\ell_k(n)$

- Trivial lower bound: $\ell_k(n) \geq nk - 1$ as all letters except n must occur at least k times, whereas n must occur at least $k - 1$ times.

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- A slight improvement using results in the case of abelian cubes ...

Theorem (G.-Halldórsson-Kitaev, 2008)

For $n \geq 5$ and $k \geq 4$, we have $\ell_k(n) \geq k(3n - 4) - 1$.

Outline

1. Background
   - Repetitions & patterns in words
   - Crucial words & abelian powers
2. Minimal crucial words avoiding abelian cubes
   - Upper bound for length
   - Lower bound for length
3. Minimal crucial words avoiding abelian k-th powers
   - Upper bound for length
   - Lower bound for length
4. Further research

Problem 1 – Prove or disprove the conjecture: $\ell_k(n) = k^2(n-1) - k - 1$.

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- Example: 323121 is a maximal abelian square-free word over $\{1, 2, 3\}$ of minimal length.
- The length of a minimal crucial word gives a lower bound for the length of a shortest maximal word.

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Question: Can our approach improve Bullock's result or can it provide an alternative solution?

Takk Fyrir!