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Twisted Weyl Group Multiple Dirichlet Series Over the Rational Function Field

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TWISTED WEYL GROUP MULTIPLE DIRICHLET SERIES OVER THE RATIONAL FUNCTION FIELD

A Dissertation Presented

by

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DEDICATION

To my sister Sydney, for sharing her mathematics lessons with me before I was old enough to realize that the game we were playing was called "arithmetic."

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ABSTRACT

TWISTED WEYL GROUP MULTIPLE DIRICHLET SERIES

OVER THE RATIONAL FUNCTION FIELD

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Let K be a global field. For each prime p of K, the p-part of a multiple Dirichlet series defined over K is a generating function in several variables for the p-power coefficients. Let Φ be an irreducible, reduced root system, and let n be an integer greater than 1. Fix a prime power $q \in \mathbb{Z}$ congruent to 1 modulo $2n$, and let $\mathbb{F}_q(T)$ be the field of rational functions in T over the finite field $\mathbb{F}_q$ of order q. In this thesis, we examine the relationship between Weyl group multiple Dirichlet series over $K = \mathbb{F}_q(T)$ and their p-parts, which we define using the Chinta–Gunnells method [10]. Our main result shows that Weyl group multiple Dirichlet series of type Φ over $\mathbb{F}_q(T)$ may be written as the finite sum of their p-parts (after a certain variable change), with "multiplicities" that are character sums. This result gives

an analogy between twisted Weyl group multiple Dirichlet series over the rational function field and characters of representations of semi-simple complex Lie algebras associated to Φ.

Because the p-parts and global series are closely related, the result above follows from a series of local results concerning the p-parts. In particular, we give an explicit recurrence relation on the coefficients of the p-parts, which allows us to extend the results of Chinta, Friedberg, and Gunnells [9] to all Φ and n. Additionally, we show that the p-parts of Chinta and Gunnells [10] agree with those constructed using the crystal graph technique of Brubaker, Bump, and Friedberg [4, 5] (in the cases when both constructions apply).

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CHAPTER 1

INTRODUCTION

In this chapter, we introduce Weyl group multiple Dirichlet series and summarize our results. Given a positive integer n, a global field K containing all $2n$th roots of unity, and an irreducible, reduced root system Φ of rank r, Weyl group multiple Dirichlet series are series in r complex variables, with analytic continuation to $\mathbb{C}^r$ and a group of functional equations isomorphic to the Weyl group of Φ. Section 1.1 provides the background information necessary to understand the heuristic construction of these objects and Section 1.2 describes our contributions. Section 1.3 concludes this chapter with a guide to the remainder of the text.

1.1 Background

A Dirichlet series is an infinite sum of the form

$$D(s) = \sum_{n \geq 1} \frac{a(n)}{n^s}, \qquad a(n) \in \mathbb{C}, \tag{1.1}$$

where s is a complex variable. The coefficients $a(n)$ often come from interesting arithmetic situations. For instance, for p prime, the $a(p)$ may count solutions to a fixed system of polynomial equations modulo p. Typically (1.1) only converges for $\Re(s) \gg 0$, but often can be shown to have nice analytic properties, like analytic continuation and a functional equation relating values at s to values at $1-s$. In such cases, studying the analytic properties of $D(s)$ frequently yields insight into the $a(n)$. In this section, we review examples of classical Dirichlet series and discuss the heuristic construction of Weyl group multiple Dirichlet series. We end with a non technical definition of Weyl group multiple Dirichlet series before describing our results in Section 1.2

1.1.1 Dirichlet Series

The famous Riemann zeta function $\zeta(s) = \sum_{n\geq 1} 1/n^s$ is the prototypical example of (1.1). Initially only defined for $\Re(s) > 1$, the series $\zeta(s)$ has analytic continuation to all of $\mathbb{C}$ and a functional equation that takes s to $1-s$. As a consequence of the fundamental theorem of arithmetic, we can write $\zeta(s) = \prod_p 1/(1-p^{-s})$ as an infinite product over all primes p; this expression is called an Euler product. We call the factor $1/(1-p^{-s})$ the "p-part" of $\zeta(s)$. In part due to its nice analytic properties, the Riemann zeta function is an invaluable tool for studying the distribution of prime numbers.

Another class of Dirichlet series with arithmetic applications is that of Dirichlet L-functions. For example, let $m \in \mathbb{Z}$ be a square-free fundamental discriminant (a square-free integer congruent to 1 modulo 4), and consider the quadratic L-function

$$L(s, \chi_m) = \sum_{n\geq 1} \frac{\chi_m(n)}{n^s}. \tag{1.2}$$

Here $\chi_m(n) = \left(\frac{m}{n}\right)$ is the Kronecker symbol that, for $n = p$ an odd prime, records whether or not m is a square modulo p. More precisely,

$$\chi_m(p) = \left(\frac{m}{p}\right) := \begin{cases} 0 & \text{if} \quad (m,p) \neq 1, \\ 1 & \text{if} \quad m \equiv \square \pmod{p}, \\ -1 & \quad\text{otherwise.} \end{cases} \tag{1.3}$$

The symbol $\left(\frac{m}{2}\right)$ is defined similarly, but depends on m modulo 8 (we have $\left(\frac{m}{2}\right)$ is 0 if $(m,8) \neq 1$, is 1 if $m \equiv \pm 1 \pmod 8$, and is -1 otherwise). We extend $\left(\frac{m}{n}\right)$ to all $n \in \mathbb{Z}_{\geq 0}$ multiplicatively: if $n = p_1^{k_1} \cdots p_t^{k_t}$, then put $\left(\frac{m}{n}\right) = \prod_{i=1}^{t} \left(\frac{m}{p_i}\right)^{k_i}$. The quadratic Dirichlet series $L(s,\chi_m)$ has analytic continuation to $\mathbb{C}$ and a functional equation of the form $s \mapsto 1-s$. It follows from the multiplicativity of χ_m that we can also express (1.2) as an Euler product: $L(s,\chi_m) = \prod_p 1/(1-\chi_m(p)p^{-s})$. As before, we call the factor $1/(1-\chi_m(p)p^{-s})$ the p-part of $L(s,\chi_m)$. Dirichlet used the analytic properties of $L(s,\chi_m)$ to prove his famous theorem on primes in arithmetic progressions, which says that for $a, n \in \mathbb{Z}$ with $(a,m) = 1$, there are infinitely many primes p congruent to a modulo m. In other words, by studying the analytic properties of $L(s,\chi_m)$, Dirichlet obtained information about congruences modulo m.

We remark that $L(s,\chi_m)$ can be viewed as a twisted zeta function. In the classical sense, to twist a Dirichlet series by a character χ means replacing the coefficients $a(n)$ in (1.1) with $\chi(n)a(n)$. This suggests thinking of $L(s,\chi_m)$ as the zeta function $\zeta(s)$ twisted by χ_m. Note that for p prime, the value of $\chi_m(p)$ records information about the quadratic field $\mathbb{Q}(\sqrt{m})$: the situation $\chi_m(p) = 1$ is equivalent to p splits in $\mathbb{Q}(\sqrt{m})$, the situation $\chi_m(p) = -1$ is equivalent to p inert in $\mathbb{Q}(\sqrt{m})$, and if $\chi_m(p) = 0$, then p ramifies in $\mathbb{Q}(\sqrt{m})$. Thus, the twisted zeta function $L(s,\chi_m)$ provides information not just about the distribution of primes of $\mathbb{Q}$, but about the splitting behavior of primes in the extension $\mathbb{Q}(\sqrt{m})$.

As a third example of a classical Dirichlet series, we consider Kubota's Dirichlet series. The coefficients of Kubota's Dirichlet series are Gauss sums. Though perhaps not as familiar as Dirichlet characters, Gauss sums have many important applications in number theory. For example, one place they arise is in the functional equations of L-functions. Gauss sums are essentially finite Fourier transforms. They are the convolution of a multiplicative character with an additive character over a finite field. We describe Gauss sums and their properties in detail in Section 2.1. For now, it is enough to know that these are functions $g : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ of the form

$$g(n) = \sum_{a \mod n} \chi(n)e(n),$$

where χ is a multiplicative character and e is an additive character. For example, we may take $\chi = \chi_m$ to be the quadratic character defined by (1.3) and e to be the additive character $a \mapsto e^{2\pi i a}$. With this definition, Kubota's Dirichlet series takes the form

$$Z(s) = \sum_{n \geq 1} \frac{g(n)}{n^s}. \tag{1.4}$$

Unlike Dirichlet characters, Gauss sums are not in general multiplicative, but they are *twisted* multiplicative: if $(a, b) = 1$, then we have

$$g(ab) = g(a)g(b)\varphi(a, b), \tag{1.5}$$

where $\varphi(a, b)$ is a product of roots of unity depending on a and b. This failure of multiplicativity means that in general $Z(s)$ does not have an Euler product. However, (1.5) shows that we do have a twisted analogue of an Euler product. As with $\zeta(s)$ and $L(s, \chi_m)$, the p-parts of $Z(s)$ are generating functions

$$\sum_{k \geq 0} g(p^k)p^{-ks} = 1 + g(p)p^{-s},$$

in p^{-s} for the p-power coefficients.

Kubota [20] discovered (1.4) as the Fourier coefficient of an Eisenstein series on a metaplectic group (metaplectic groups are covers of reductive groups). The analytic continuation and functional equations of (1.4) then follow directly from the analytic properties of the Eisenstein series. The residues of Kubota's Dirichlet series are coefficients of generalized theta functions, which are higher order versions of the quadratic theta function $\theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z}$. Consequently, Kubota's Dirichlet series has deep connections with higher order reciprocity laws.

1.1.2 Multiple Dirichlet Series

Multiple Dirichlet series are the natural generalization of (1.1) to several variables:

$$D(s_1, \ldots, s_r) = \sum_{(n_1,\ldots,n_r)\in(\mathbb{Z}_{\geq 0})^r} \frac{a(n_1, \ldots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}}, \qquad a(n_1, \ldots, n_r) \in \mathbb{C}, \tag{1.6}$$

where the s_i are complex variables. Like classical Dirichlet series, we hope that (1.6) have nice analytic properties — not only should they converge in some complex half space, they should also have analytic continuation to $\mathbb{C}^r$ and a group of functional equations that intermixes all the variables $s_1, \ldots, s_r$. In this case, the series (1.6) are a tool for studying the coefficients $a(n_1, \ldots, n_r)$.

Like classical Dirichlet series, arithmetic questions often motivate the study of multiple Dirichlet series. For example, suppose we are interested in the average value of the class number in quadratic extensions $\mathbb{Q}(\sqrt{m})$ (the class number of a number field F is a statistic that measures how far the ring of integers $\mathcal{O}_F$ is from being a principal ideal domain — the class number is one if and only if every ideal in $\mathcal{O}_F$ is generated by a single element). The special value of $L(s, \chi_m)$ at $s = 1$ is closely related to the class number of the quadratic field $\mathbb{Q}(\sqrt{m})$. To study the values $L(1, \chi_m)$, it makes sense to replace $a(m)$ with $L(1, \chi_m)$ in (1.1) and instead study the analytic properties of this new series.

If we go further and replace the value 1 with a second complex variable s_2, we obtain a function of two complex variables

$$\mathcal{L}(s_1, s_2) = \sum_{m\geq 1} \frac{L(s_2, \chi_m)}{m^{s_1}}. \tag{1.7}$$

Unwinding (1.7) using (1.2), we obtain a double Dirichlet series of the form (1.6):

$$\mathcal{L}(s_1, s_2) = \sum_{n,m\geq 1} \frac{\left(\frac{m}{n}\right)}{m^{s_1} n^{s_2}}. \tag{1.8}$$

Thus, one natural way to construct multiple Dirichlet series is to consider Dirichlet series whose coefficients themselves come from Dirichlet series.

A double Dirichlet series similar to (1.8) and a method for its analytic continuation first appeared in a 1956 paper of Siegel [26]. In fact, Goldfeld and Hoffstein [15] used a Dirichlet series roughly of this form to prove that there exists a constant c such that $\sum_{|m|<X} L(1, \chi_m) \sim cX \log X$, where the sum is taken over all fundamental discriminants (here "roughly" means that the coefficients include correction factors to ensure the desired analytic properties). Put another way, Goldfeld and Hoffstein [15] used a double Dirichlet series to obtain information about the asymptotic behavoir of the class number in quadratic extensions of $\mathbb{Q}$.

We emphasize that the definition (1.8) does not meet the needs of first paragraph of this section because it does not have the right analytic properties; we have not correctly defined $a(n, m)$ when n and m are not relatively prime or square free. But, when the coefficients of (1.8) are assigned certain correction factors, e.g. [15], the resulting double Dirichlet series satisfies a group of functional equations isomorphic to the symmetric group $\mathcal{S}_3$ on three letters, which is the Weyl group of the root system A_2. These six functional equations are generated by the two relations:

$$\sigma_1 : \begin{cases} s_1 & \mapsto & 2 - s_1 \\ s_2 & \mapsto & s_1 + s_2 - 1 \end{cases} \quad \text{and } \sigma_2 : \begin{cases} s_1 & \mapsto & s_1 + s_2 - 1 \\ s_2 & \mapsto & 2 - s_2 \end{cases}. \tag{1.9}$$

Figure 1 shows the A_2 root system (with base given by α_1 and α_2) and its Weyl group (generated by the simple reflections σ_1 and σ_2 through the lines orthogonal to α_1 and α_2 respectively). One easily checks that the set of reflections in the complex plane generated by σ_1 and σ_2 form a group isomorphic to $\mathcal{S}_3$.

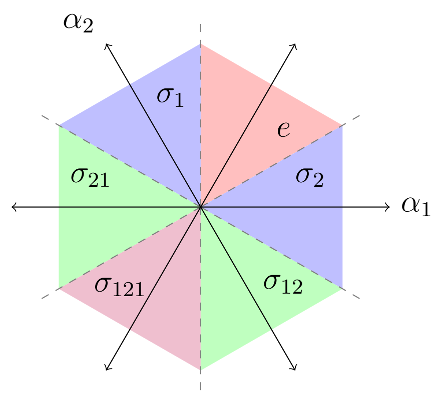

Figure 1: A_2 root system

We are now in a position to describe the construction of Weyl group multiple Dirichlet series. From now on, fix an integer $n \geq 1$, a global field K containing all $2n$th roots of unity, and an irreducible, reduced root system Φ of rank r. To this data we associate a multiple Dirichlet series of the form (1.6), with analytic continuation to $\mathbb{C}^r$ and a group of functional equations isomorphic to the Weyl group of Φ. As we will see, Kubota's Dirichlet series (1.4) is an example of a rank one Weyl group multiple Dirichlet series and the double Dirichlet series (1.8) corresponds, heuristically, to the case $n = 2$ and $\Phi = A_2$.

To build Weyl group multiple Dirichlet series for more general root systems, we start by interpreting (1.8) more directly in terms of data attached to the root system $\Phi = A_2$. For general n, this type of Weyl group multiple Dirichlet series takes the form

$$Z(s_1, s_2; A_2, n) \approx \sum_{c_i \in (\mathcal{O}_K \setminus \{0\})/\mathcal{O}_K^\times} \frac{g(c_1)g(c_2)\left(\frac{c_1}{c_2}\right)_n^{-1}}{|c_1|^{s_1}|c_2|^{s_2}}, \tag{1.10}$$

where $\mathcal{O}_K$ is the ring of integers of K with unit group $\mathcal{O}_K^\times$. Here $\left(\frac{\cdot}{\cdot}\right)_n$ is an nth order power residue symbol, which we also use as the multiplicative character for the Gauss sums $g(c_i)$, and $|c_i|$ denotes the norm of c_i (see Section 2.1 for details). Again, we caution the reader that (1.10) is only heuristic — the coefficients as written do not yield the desired analytic properties when c_1 and c_2 are not relatively prime or square free. Nonetheless, we can still use (1.10) to gain intuition about how the construction of Weyl group multiple Dirichlet series might be axiomatized. We do this by interpreting (1.10) as a multiple Dirichlet series arising from a labelled graph that depends on Φ and n. We describe this construction below.

Associate a multiple Dirichlet series $Z(s_1, \ldots, s_r)$ with an r-vertex graph in the following way: label the vertices 1 through r — one for each Gauss sum $g(c_i)$ that appears in the heuristic coefficient of (c_1, c_2). If $\left(\frac{c_i}{c_j}\right)^{-1}$ also appears in this coefficient, then join vertex i and j with an edge. In this way, one may think of the Gauss sum $g(c_i)$ as the contribution of vertex i and of $\left(\frac{c_i}{c_j}\right)^{-1}$ as the contribution of the ij-edge. Then, (1.10) corresponds to the graph shown in Figure 2, which (not coincidentally) is the Dynkin diagram for the root system A_2. Different Dynkin

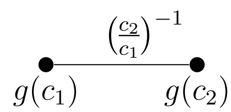

Figure 2: Graph heuristic for $Z(s_1, s_2; A_2)$

diagrams give rise to different Weyl group multiple Dirichlet series. As another example, consider the Dynkin diagram shown in Figure 3 for the root system D_4. If we follow the description above, we are led to the series

$$Z(s_1, s_2, s_3, s_4; D_4) \approx \sum_{c_i \in (\mathcal{O}_K \setminus \{0\})/\mathcal{O}_K^\times} \frac{g(c_1)g(c_2)g(c_3)g(c_4)\left(\frac{c_2}{c_1}\right)^{-1}\left(\frac{c_3}{c_2}\right)^{-1}\left(\frac{c_4}{c_2}\right)^{-1}}{|c_1|^{s_1}|c_2|^{s_2}|c_3|^{s_3}|c_4|^{s_4}}.$$

This definition will be correct when the c_i satisfy certain conditions. In Section 2.1, we give a precise construction of Weyl group multiple Dirichlet series such that the resulting series have both meromorphic continuation and the desired group of functional equations.

We now introduce our series of interest, saving the details for later sections. Let

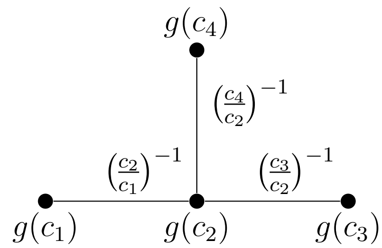

Figure 3: Graph heuristic for $Z(s_1, s_2, s_3, s_4; D_4)$

n and K be as above, and let $\mathbf{s} = (s_1, \ldots, s_r)$ be an r-tuple of complex variables. Fix an r-tuple $\mathbf{m} = (m_1, \ldots, m_r)$ of integers in $\mathcal{O}_K$. Define the degree n Weyl group multiple Dirichlet series of type Φ over K with twisting parameter $\mathbf{m}$ by

$$Z(\mathbf{s}; \mathbf{m}; K, \Phi, n) = Z(\mathbf{s}; \mathbf{m}, K) \approx \sum_{c_i \in (\mathcal{O}_K \setminus \{0\})/\mathcal{O}_K^\times} \frac{H(c_1, \ldots, c_r; \mathbf{m})}{|c_1|^{s_1} \cdots |c_r|^{s_r}}. \tag{1.11}$$

The coefficients $H(\mathbf{c}; \mathbf{m}) = H(c_1, \ldots, c_r; \mathbf{m})$ are complex numbers involving products of Gauss sums and nth roots of unity and are built using the combinatorics of Φ. Like the quadratic Dirichlet characters χ_m, they depend on both the c_i and the twisting parameters m_i; and like Gauss sums, they satisfy a twisted multiplicativity. This means that all $H(\mathbf{c}; \mathbf{m})$ are determined by the p-power coefficients, for each prime p dividing the c_i, using a twisted analogue of an Euler product (cf. Section 2.3). For each prime $p \in \mathcal{O}_K^\times$ and tuple $\ell = (l_1, \ldots, l_r) \in (\mathbb{Z}_{\geq 0})^r$ of nonnegative integers (the tuple ℓ depends on p and $\mathbf{m}$), the p-part of $Z(\mathbf{s}; \mathbf{m}, K)$ is a generating function in the variables $|p|^{-s_i}$ for the p-power coefficients $H(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \ldots, p^{l_r})$. These p-parts are constructed using an analogue of the Weyl character formula. We describe this construction in detail in Section 2.2.

We remark that (1.11) is still not precise. In general, some care must be taken when $\mathcal{O}_K$ has class number greater than one and to ensure that the coefficients $H(\mathbf{c}; \mathbf{m})$ are well defined up to multiplication of the c_i by units. For now, (1.11) is sufficient to describe our results. We make (1.11) precise in Chapter 2.

When $\mathbf{m} = (1, \ldots, 1)$, we say that $Z(\mathbf{s}; \mathbf{m}, K)$ is untwisted. Otherwise, we say it is twisted. This is meant to mirror the classical setting. Heuristically, the twisted coefficients $H(\mathbf{c}; \mathbf{m})$ are obtained by multiplying the untwisted coefficients $H(\mathbf{c}; 1, \ldots, 1)$ by characters depending on the m_i. As with the twisted zeta function $L(s, \chi_m)$, we expect that twisted Weyl group multiple Dirichlet series encode additional arithmetic information related to extensions of K by the twisting parameters m_i.

1.2 Overview of Results

The results of this thesis are of two flavors: local and global. The local results concern the p-parts of Weyl group multiple Dirichlet series — these are the analogues of the Euler factors of classical Dirichlet series. The global results are statements about the full multiple Dirichlet series (1.11).

Let $n \geq 1$ be an integer, and let $q \in \mathbb{Z}$ be a prime power such that $q \equiv 1 \pmod{2n}$. Our global results require that $K = \mathbb{F}_q(T)$ is the field of rational functions in T over the finite field $\mathbb{F}_q$ of order q. In this case — in fact, if $K/\mathbb{F}_q$ is any algebraic function field — Weyl group multiple Dirichlet series are rational functions in several variables: $q^{-s_1}, \ldots, q^{-s_r}$. Given Φ and n, the denominators of such functions are known, but the numerators are not fully understood. Fisher and Friedberg [13, 14] and Chinta [8] compute examples for $\Phi = A_2$ and $\Phi = A_3$, and Chinta and Mohler [11] treat the case $\Phi = A_r$ and $n \gg r$, but few other examples appear in the literature. An understanding of these rational functions would have several applications. For instance, like the Kubota series, in some cases the residues of such series are coefficients of generalized theta functions. Having an explicit rational function representation for these series would provide a testing ground for conjectured values of these mysterious coefficients [8, 16, 24].

Our main global contribution is Theorem 5.2.1, which shows that, after a variable change, Weyl group multiple Dirichlet series over $\mathbb{F}_q(T)$ can be written as finite weighted sums of their p-parts. The weights on these sums are as simple as possible; they are explicit character sums that come directly from the defining series. We suggest thinking of Theorem 5.2.1 as an analogy between Weyl group multiple Dirichlet series over $\mathbb{F}_q(T)$ and characters of representations of the complex semisimple Lie algebra $\mathfrak{g}$ associated to Φ. This analogy is discussed in more detail in Section 2.2.

Theorem 5.2.1 is actually a generalization of the well-documented [8, 11] similarity between untwisted Weyl group multiple Dirichlet series over the rational function field and their p-parts. Indeed, Chinta and Gunnells [10] use this similarity as a basis for their combinatorial construction of the p-parts. Proposition 5.1.1 shows that in the untwisted case, a simple change of variables transforms the local p-part into the global series. The situation is more complicated in the twisted case, but Theorem 5.2.1 shows that, after this same variable change, the global series is still in the $\mathbb{C}$-span of the p-parts. After multiplying by a normalizing product of zeta functions, the weights on the p-parts are actually certain coefficients of the original series $Z(\mathbf{s}; \mathbf{m}, \mathbb{F}_q(T))$. This fact is a consequence of Lemma 4.1.5, which describes the support of a rational function closely related to the p-parts.

Proposition 5.1.1 allows us to apply results about the local p-parts to the global series. Chinta, Friedberg, and Gunnells [9] previously detailed several properties of the p-parts in the case $n = 2$ and Φ simply laced. In order for Theorem 5.2.1 to hold in the most general context, we extend the results of [9] to general n and Φ. To this end, Theorem 4.1.1 gives an explicit recurrence relation on the coefficients of the p-parts. It is the main tool for proving Theorem 4.1.2, which extends [9, Theorem 3.2] and shows that the support of any p-part is contained in a shifted weight polytope. In addition, Theorem 4.3.3 addresses the extent to which these p-parts are uniquely determined by the relations given in Theorem 4.1.1. Using the relationship between Weyl group multiple Dirichlet series and their p-parts, we are also able to apply Theorem 4.3.3 to the global series.

As a final application of Theorem 4.1.1, we connect the two main approaches to define Weyl group multiple Dirichlet series. These approaches are that of Brubaker, Bump, and Friedberg [4–6] and Chinta and Gunnells [10]. Brubaker, Bump, and Friedberg's method builds off [7] and uses combinatorial techniques related to crystal graphs to define the p-parts. Their approach applies to $\Phi = A_r$ for all n [6] and to general Φ when $n \gg r$ [4, 5]. The method we adopt is that of Chinta and Gunnells [10], which defines the p-parts by means of an averaging technique analogous to the Weyl character formula; their construction produces a global object with the correct analytic properties for all n and all Φ.

The equivalence of [4–6] and [10] has been an open problem. For $\Phi = A_r$, the combined works of Chinta and Offen [12] and McNamara [21] resolve this question affirmatively. For $n = 2$ and Φ simply laced, Chinta, Friedberg, and Gunnells [9] provide further evidence by showing that the stable (see Section 4.2) coefficients of the p-parts of (1.11) agree. (Mohler [23] also compares the stable coefficients for $\Phi = A_r$ and $n \gg r$.) Our Theorem 4.1.1 allows us to compare the stable coefficients of the p-parts of [4, 5] and [10], for general n and Φ, and Theorem 4.2.1 shows they do indeed agree. In fact, for $n \gg r$, the stable coefficients are the only nonzero coefficients of the p-parts. Thus, Theorem 4.2.1 shows that, in the cases when both constructions apply, the methods to construct (1.11) of [4, 5] and [10] agree in general.

1.3 Guide to this Thesis

Chapter 2 explains how to build Weyl group multiple Dirichlet series using the Chinta and Gunnells [10] construction of the p-parts. To begin, Section 2.1 provides the preliminary definitions and facts needed for this construction. Because we are ultimately interested in global series defined over the rational function field, we will focus our definitions on this case. However, we note that the p-part construction given in Section 2.2 is purely local — the same method applies to build Weyl group multiple Dirichlet series over any global field K. We end this chapter with Section 2.3, which describes how to build $Z(\mathbf{s}; \mathbf{m}, \mathbb{F}_q(T))$ from its p-parts using a twisted analogue of an Euler product.

Chapter 3 derives the functional equations for Weyl group multiple Dirichlet series $Z(\mathbf{s}; \mathbf{m}, \mathbb{F}_q(T))$ over the rational function field. Arguments to obtain these functional equations have previously appeared in the literature, but with limitations on Φ. For example, [8] treats the case $\Phi = A_2$ and n arbitrary. To extend the arguments of [8] to general Φ, we reference [10], which treats all Φ and n, but for K any number field. It is clear that the arguments of [10] apply to function fields as well, and the goal of this chapter is to make explicit how the arguments of [10] work in the much simpler case when $K = \mathbb{F}_q(T)$.

Chapter 4 describes our results that pertain to the p-parts of Weyl group multiple Dirichlet series. The goal of this chapter is to extend to general Φ and n the results of [9], which address the local factors in the case that $n = 2$ and Φ is simply laced. To this end, Theorem 4.1.1 gives a recurrence relation on the coefficients of the p-parts that allows us to generalize all of the results of [9]. Theorem 4.2.1 shows that the support of the p-parts are contained in shifted weight polytopes, and Theorem 4.3.3 shows that, up to finitely many coefficients, the p-parts are completely determined by the recurrence relations on their coefficients. Theorem 4.2.1 compares p-parts constructed via techniques of [4, 5] and [10], and shows that, in the cases when both constructions apply, the stable coefficients agree. Finally, Lemma 4.1.5 is a vanishing result that describes the support of a rational function closely related to the p-parts. This lemma is used to prove Theorem 5.2.1 in the next chapter.

Chapter 5 gives our global results. Proposition 5.1.1 and Theorem 5.2.1 relate Weyl group multiple Dirichlet series over the rational function field and their p-parts. Theorem 5.2.1 draws an analogy between the rational functions associated to twisted Weyl group multiple Dirichlet series over the rational function field and characters of representations of the semisimple complex Lie algebra associated to Φ. In particular, we show that Weyl group multiple Dirichlet series can be written as a sum of local p-parts after a certain variable change, with weights given by certain character sums. We end this last chapter with an example.

CHAPTER 2

CONSTRUCTION OF WEYL GROUP MULTIPLE DIRICHLET SERIES

This chapter explains how to construct Weyl group multiple Dirichlet series over the rational function field. After reviewing preliminary definitions and facts in Section 2.1, we describe how to construct the p-parts using the Chinta–Gunnells method in Section 2.2. We end with Section 2.3, which explains how to build Weyl group multiple Dirichlet series over the rational function field from their p-parts using a twisted analogue of an Euler product.

2.1 Preliminaries

2.1.1 Notation

As in the introduction, let Φ be an irreducible, reduced root system of rank r, and let n be an integer greater than one. Fix a prime power $q \in \mathbb{Z}$ such that $q \equiv 1 \pmod{2n}$. Let $K = \mathbb{F}_q(T)$ be the field of rational functions in T over $\mathbb{F}_q$. Let $\mathcal{O} = \mathbb{F}_q[T]$ be the polynomial ring over $\mathbb{F}_q$, and denote by $\mathcal{O}_{mon} \subset \mathcal{O}$ the set of monic polynomials. For $c \in \mathcal{O}$, define the norm to be $|c| := q^{\deg c}$.

Let $K_\infty = \mathbb{F}_q((T^{-1}))$ denote the field of Laurent series in $\pi_\infty := T^{-1}$; this is the completion of K at the place corresponding to π_∞. We have $\mathcal{O} \subset K \subset K_\infty$. For $f = \sum_{i=-k}^{\infty} a_i \pi_\infty^i \in K_\infty$, define $\deg f$ to be the smallest i such that $a_i \neq 0$. Note that K_∞ has ring of integers $\mathbb{F}_q[[\pi_\infty]]$ and maximal ideal $(\pi_\infty) := \pi_\infty \mathbb{F}_q[[\pi_\infty]]$.

2.1.2 Gauss Sums

The coefficients of Weyl group multiple Dirichlet series involve nth order Gauss sums. To define these coefficients, we first must assign a Gauss sum $g(c)$ to each $c \in \mathcal{O}_{mon}$. After an appropriate identification of the residue field of c with the finite field $\mathbb{F}_{q^{\deg c}}$, one sees that the $g(c)$ can be identified with finite field Gauss sums; we first define these.

Assume that $q = p^k$, where p is necessarily an odd prime. Let $\mu_n = \{a \in \mathbb{F}_q : a^n = 1\}$ be the nth roots of unity in $\mathbb{F}_q^\times$, and fix an embedding $\epsilon : \mu_n \to \mathbb{C}$. Define the multiplicative character $\chi : \mathbb{F}_q^\times \to \mu_n$ by $a \mapsto a^{(q-1)/n}$ and the additive character $e_0 : \mathbb{F}_p^\times \to \mathbb{C}$ by $a \mapsto \exp 2\pi i a/p$. To extend e_0 to $\mathbb{F}_q$, put $e_* = e_0 \circ Tr_{\mathbb{F}_q/\mathbb{F}_p}$. By definition, we have $Tr_{\mathbb{F}_q/\mathbb{F}_p} : a \mapsto a + a^p + a^{p^2} + \cdots + a^{p^{k-1}}$. For $t \in \mathbb{Z}$, define the finite field Gauss sum

$$\tau(\epsilon^t) = \sum_{a \in \mathbb{F}_q^\times} \epsilon(\chi(a))^t e_*(a). \tag{2.1}$$

A detailed listing of the properties of these sums see can be found in [19, Section 8.2]. In particular, one can show that $\tau(\epsilon^t)\tau(\epsilon^{-t}) = q$.

For the remainder of the text, we assume that $P \in \mathcal{O}_{mon}$ is irreducible of norm $|P|$. As we will no longer need to refer explicitly to the characteristic of $\mathbb{F}_q$, from now on we let $p := |P|$. We abuse notation and refer to the P-parts of our multiple Dirichlet series as p-parts. Recall that with this notation, the p-parts are generating functions in p^{-s_i} for the P-power coefficients of $Z(\mathbf{s}; \mathbf{m}, \mathbb{F}_q(T))$.

Let $c \in \mathcal{O}_{mon}$. The multiplicative character we use to define the Gauss sums $g(c)$ is the nth order power residue symbol cf. [25, Chapter 3]. This symbol is defined in the following way: let $P \in \mathcal{O}_{mon}$ be irreducible and let $a \in \mathcal{O}$. If P does not divide a, define $\left(\frac{a}{P}\right)_n$ to be the unique element in $\mathbb{F}_q^\times$ such that

$$\left(\frac{a}{P}\right)_n \equiv a^{\frac{|P|-1}{n}} \mod P.$$

If P divides a, define $\left(\frac{a}{P}\right)_n = 0$. Using this definition, it is a simple exercise to show that if $\alpha \in \mathbb{F}_q^\times$, we have

$$\left(\frac{\alpha}{P}\right) = \alpha^{\frac{q-1}{n}\deg P}.$$

We extend this symbol multiplicatively to all $b \in \mathcal{O}$: if $b = \alpha P_1^{f_1} \cdots P_t^{f_t}$, put $\left(\frac{a}{b}\right)_n = \prod_{i=1}^t \left(\frac{a}{P_i}\right)_n^{f_i}$. In particular, if $\alpha \in \mathbb{F}_q^\times$, then $\left(\frac{a}{\alpha b}\right)_n = \left(\frac{a}{b}\right)_n$.

The nth order reciprocity law cf. [25, Theorem 3.5] relates $\left(\frac{a}{b}\right)_n$ to $\left(\frac{b}{a}\right)_n$. For $c \in \mathcal{O}$, let $\mathrm{sgn}(c)$ be the leading coefficient of c raised to the power $\frac{q-1}{n}$. If $a, b \in \mathcal{O}$ are relatively prime and non-zero, then

$$\left(\frac{a}{b}\right)_n = (b, a)_\infty \left(\frac{b}{a}\right)_n,$$

where

$$(b, a)_\infty := (-1)^{\frac{q-1}{n}\deg(a)\deg(b)} \mathrm{sgn}_n(a)^{\deg(b)} \mathrm{sgn}_n(b)^{-\deg(a)}, \tag{2.2}$$

is the Hilbert symbol at ∞. Note that for $a, b \in \mathcal{O}_{mon}$, our assumption $q \equiv 1 \pmod{2n}$ gives

$$\left(\frac{a}{b}\right)_n = \left(\frac{b}{a}\right)_n.$$

The additive character used to define $g(c)$ acts on F_∞. For $a \in F_\infty$, write $a = \sum_{i=-N}^{\infty} a_i \pi^i$, and let $\psi(a) = a_{-1}$ and $\psi^*(a) = \psi(T^2 a)$. Define $e = e_* \circ \psi^*$, where e_* is the additive character on $\mathbb{F}_q$ defined above. One can show that e satisfies $\{x \in F_\infty : e(x\mathcal{O}) = 1\} = \mathcal{O}$.

Let $m, c \in \mathcal{O}$ and $t \in \mathbb{Z}$. Define the Gauss sum

$$g(m, c; \epsilon^t) = g_t(m, c) := \sum_{y \mod c} \epsilon\left(\left(\frac{y}{c}\right)_n\right)^t e(my/c), \tag{2.3}$$

where the sum is only over $y \not\equiv 0 \mod c$. We denote $g_1(1, c)$ by $g(1, c)$ or simply $g(c)$. In general $g(cc') \neq g(c)g(c')$, but Gauss sums do satisfy a twisted multiplicativity and several other identities that are useful for computation. As mentioned earlier, the properties below follow directly from the corresponding properties of finite field Gauss sums, cf. [19, Section 8.2].

1. If $(c, c') = 1$, we have
$$g_t(m, cc') = \left(\frac{c}{c'}\right)_n^t \left(\frac{c'}{c}\right)_n^t g_t(m, c) g_t(m, c').$$

2. If $(a, c) = 1$, we have
$$g_t(am, c) = \left(\frac{a}{c}\right)_n^{-t} g_t(m, c).$$

3. Let $k, l \in \mathbb{Z}_{\geq 0}$, and let $\phi(P^l) =$ be the number of elements of $(\mathcal{O}/P^l\mathcal{O})^\times$. Then
$$g_t(P^l, P^k) = \begin{cases} p^l g_{tk}(1, P) & \text{if } k = l + 1; \\ \phi(P^k) & \text{if } n | tk \text{ and } l \geq k; \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

4. If $(t, n) = 1$, then
$$g_t(1, P) g_{-t}(1, P) = |P| = p.$$

2.1.3 Root Systems

The p-parts of Weyl group multiple Dirichlet series are built using data from an irreducible, reduced root system Φ of rank r, cf. [17, Chapter 9]. The term "reduced" means that if $\alpha \in \Phi$, the only multiples we allow are $\pm\alpha \in \Phi$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots. Any root of Φ can be written as a $\mathbb{Z}$-linear combination of the α_i. We say $\alpha \in \Phi$ is positive (negative) if when written as a sum of simple roots $\alpha = \sum k_i \alpha_i$, all k_i are nonnegative (nonpositive). We have a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. Let Λ be the root lattice of Φ, i.e. the $\mathbb{Z}$-span of the simple roots. Define the generalized height function $d : \Lambda \to \mathbb{Z}$ by

$$d : \lambda = \sum_{i=1}^{r} \lambda_i \alpha_i \mapsto \sum \lambda_i.$$

Let W be the Weyl group of Φ. Let $(\cdot, \cdot)$ be a W-invariant symmetric, bilinear, positive definite inner product on $\Lambda \otimes \mathbb{R}$, normalized such that the short roots all have length one — this implies that for any $\alpha, \beta \in \Lambda$, we have $(\alpha, \beta) \in \frac{1}{2}\mathbb{Z}$. We have

$$\|\alpha\|^2 = \begin{cases} 1 & \text{for all } \alpha \text{ in types } A, D, E, \\ 1 & \text{for } \alpha \text{ a short root in types } B, C, F_4, G_2, \\ 2 & \text{for } \alpha \text{ a long root in types } B, C, F_4, \\ 3 & \text{for } \alpha \text{ a long root in type } G_2. \end{cases}$$

Let $\langle \alpha_i, \alpha_j \rangle = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ be the ij-entry of the Cartan matrix $C_{ij} = \big(c(i,j)\big)$ of Φ. The Weyl group W is generated by the simple reflections

$$\sigma_j(\alpha_i) = \alpha_i - \langle \alpha_i, \alpha_j \rangle \alpha_j. \tag{2.5}$$

For $w \in W$, let $l(w)$ be the number of σ_j in any reduced expression for w, and let $\mathrm{sgn}(w) = (-1)^{l(w)}$. Let $\Phi(w) = \{\alpha \in \Phi^+ : w\alpha \in \Phi^-\}$ be the set of positive roots made negative by w. One can show that $\#\Phi(w) = l(w)$. If $l(\sigma_i w) = l(w) + 1$, then [18, 5.6]

$$\Phi(\sigma_i w) = \Phi(w) \cup \{w^{-1}\alpha_i\}.$$

In particular, if $l(w\sigma_i) = l(w) + 1$, then

$$\Phi(\sigma_i w^{-1}) = \Phi(w^{-1}) \cup \{w\alpha_i\}. \tag{2.6}$$

Let $\{\check{\alpha}_i = \frac{2\alpha_i}{(\alpha_i,\alpha_i)} : i = 1, \ldots, r\}$ be the set of simple coroots. The fundamental weights $\{\varpi_1, \ldots, \varpi_r\}$ of Φ are the corresponding dual basis, with respect to $(\cdot,\cdot)$. In particular, we have $\langle \varpi_i, \alpha_j \rangle = \delta_{ij}$, which means that we can write $\alpha_j = \sum c(j,i)\varpi_i$ in terms of the fundamental weights using the Cartan matrix. Let L be the weight lattice, generated by the fundamental weights. There is a partial order on L: we say $\mu \succeq \xi$ if $\mu - \xi = \sum k_i \varpi_i$ with all k_i nonnegative. We say $\mu \in L$ is dominant if $\langle \mu, \alpha_i \rangle \geq 0$ for all $i = 1, \ldots, r$ and regular dominant if the inequality is strict. Let $\rho = \sum_{i=1}^{r} \varpi_i$ be the sum of the fundamental weights.

We associate a root system Φ with a complex semisimple Lie algebra $\mathfrak{g}$ in the following way: let $\mathfrak{g}$ be a semisimple complex Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ (a maximal abelian subalgebra) with dual $\mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$, let

$$g_\lambda := \{a \in \mathfrak{g} : [h, a] = \lambda(h)a \text{ for all } h \in \mathfrak{h}\}.$$

Here $[\cdot,\cdot]$ is the Lie bracket. The roots of $\mathfrak{g}$ are the set of all λ such that g_λ is non trivial. Together with the Killing form (cf. [17, Chapter 5]) as the invariant symmetric bilinear form on $\mathfrak{g}$, the roots of $\mathfrak{g}$ form a root system. A well known result states that for any Φ, there exists a semisimple Lie algebra $\mathfrak{g}$ whose root system is isomorphic to Φ. In this case, we say that $\mathfrak{g}$ corresponds to Φ. It is a fact that two semisimple Lie algebras that correspond to isomorphic root systems are isomorphic.

From now on, let $\mathfrak{g}$ be a semisimple Lie algbera corresponding to Φ. Characters of representations of $\mathfrak{g}$ can be considered as functions supported on L. For example, consider the group ring $\mathbb{Z}[L]$ with basis elements $\mathbf{y}^\lambda$, for each $\lambda \in L$. In other words, let $\mathbf{y} = (y_1, \ldots, y_n)$ and for $\mu = \sum_{i=1}^{r} k_i \varpi_i$, put $\mathbf{y}^\mu = y_1^{k_1} \cdots y_r^{k_r}$. Let μ be a dominant, integral weight, and let $V = V(\mu)$ be the finite dimensional, irreducible, complex representation of $\mathfrak{g}$ with highest weight μ (cf. [17, Chapter 21]). Denote by $\Pi(\mu)$ the set of weights of V and by $m_\xi(\lambda)$ the dimension of the weight space V_μ. The formal character ch_μ of V is the sum

$$ch_\mu = \sum_{\xi \in \Pi(\mu)} m_\mu(\xi) \mathbf{y}^\xi.$$

In fact, ch_μ is given by the Weyl character formula (2.16). For example, when $\Phi = A_2$, we have

$$\begin{aligned} ch_0 &= 1 \\ ch_\rho &= y_1^{-2} y_2 + y_1^{-1} y_2^{-1} + y_1^{-1} y_2^2 + 2 + y_1 y_2^{-2} + y_1 y_2 + y_1^2 y_2^{-1} \\ ch_{3\varpi_1} &= y_1^3 + y_1 y_2 + y_1^2 y_2^{-1} + y_1^{-1} y_2^2 + 1 + y_1^{-3} y_2^3 + y_1 y_2^{-2} + y_1^{-2} y_2 + y_1^{-1} y_2^{-1} + y_2^{-3}. \end{aligned}$$

2.2 Chinta and Gunnells p-Part Construction

As we mentioned in the introduction, the coefficients of Weyl group multiple Dirichlet series are complicated expressions involving Gauss sums that are built using combinatorial data from Φ. Recall that P is a prime of $\mathbb{F}_q(T)$ and $p = q^{\deg P}$ is its norm. This section describes the Chinta–Gunnells method [10] to construct, for each prime P, generating functions in $p^{-s_1}, \ldots, p^{-s_r}$ for the P-power coefficients of $Z(\mathbf{s}; \mathbf{m}, \mathbb{F}_q(T))$. In Section 2.3, we explain how to build the global series from these local factors using a twisted analogue of an Euler product.

2.2.1 A Weyl Group Action

To construct the p-parts, we follow Chinta and Gunnells [10]. This construction involves defining a Weyl group action on rational functions to construct an invariant rational function by averaging. To define this action, fix an r-tuple of nonnegative integers $\ell = (l_1, \ldots, l_r) \in (\mathbb{Z}_{\geq 0})^r$. The tuple ℓ is related to $\mathbf{m}$ and is also called a twisting parameter. It determines a dominant weight

$$\theta := \theta(\ell) = \sum_{i=1}^{r} (l_i + 1)\varpi_i. \tag{2.7}$$

The weight θ in turn determines an action of W on Λ by

$$w \bullet \lambda = w(\lambda - \theta) + \theta. \tag{2.8}$$

In particular, when $w = \sigma_j$ is a simple reflection, we have

$$\sigma_j \bullet \lambda = \sigma_j \lambda + (l_j + 1)\alpha_j.$$

Consider $A = \mathbb{C}[\Lambda]$, the ring of Laurent polynomials on Λ. The ring A consists of all expressions f of the form $f = \sum_{\beta \in \Lambda} c_\beta \mathbf{x}^\beta$ with $c_\beta \in \mathbb{C}$ almost all zero. Multiplication in A is defined using addition in $\Lambda : \mathbf{x}^\beta \mathbf{x}^\lambda = \mathbf{x}^{\lambda+\beta}$. For such an f, define the support of f by $\operatorname{supp} f = \{\beta : c_\beta \neq 0\}$. We identify A with $\mathbb{C}[x_1, x_1^{-1} \ldots, x_r, x_r^{-1}]$ via $\mathbf{x}^{\alpha_i} \mapsto x_i$. Then, define a change of variables action on A by

$$\left(\sigma_j(\mathbf{x})\right)_i = p^{-c(i,j)} x_i x_j^{-c(i,j)}. \tag{2.9}$$

Comparing with (2.5), one sees that (2.9) is essentially a reformulation of the standard action of W on Λ (after the introduction of p-powers). If $f_\beta(\mathbf{x}) = \mathbf{x}^\beta$ is a monomial, then

$$f_\beta(w\mathbf{x}) = p^{d(w^{-1}\beta - \beta)} \mathbf{x}^{w^{-1}\beta}.$$

The action we want actually takes place on index n sublattice cosets of the field of fractions $\tilde{A}$ of A. Let

$$n(\alpha) = \frac{n}{\gcd(n, \|\alpha\|^2)}, \qquad \alpha \in \Phi, \tag{2.10}$$

and define $\Lambda' \subset \Lambda$ to be the sublattice generated by the set $\{n(\alpha)\alpha\}_{\alpha\in\Phi}$. We have a decomposition of $\tilde{A}$

$$\tilde{A} = \bigoplus_{\lambda\in\Lambda/\Lambda'} \tilde{A}_\lambda,$$

where $\tilde{A}_\lambda$ is the set of functions $f/g \in \tilde{A}$ such that g lies in the kernel of the map $\nu : \Lambda \to \Lambda/\Lambda'$. For positive integers a and m, let $(a)_m := a - m\lfloor a/m \rfloor$ be the remainder of a upon division by m. Note that

$$(-a)_m = \begin{cases} 0 & \text{if } (a,m) \neq 1, \\ m - (a)_m & \text{otherwise.} \end{cases}$$

Let $k \in \{1,\ldots,r\}$ and define $\delta_{k,\ell} = \delta_k(\lambda) = d(\sigma_k \bullet \lambda - \lambda)$. Let $g_t^*(1,P)$ be the normalized Gauss sum

$$g_t^*(1,P) = \begin{cases} -1 & \text{if } t \equiv 0 \pmod{n}(\alpha_k), \\ g_t(1,P)/p & \text{otherwise,} \end{cases}$$

and define polynomials

$$\begin{aligned} \mathcal{P}_{\beta,\ell,k}(x_k) &= (px_k)^{l_k+1-(\delta_k(\beta))_{n(\alpha_k)}} \frac{1-1/p}{1-(px_k)^{n(\alpha_k)}/p}, \\ \mathcal{Q}_{\beta,\ell,k}(x_k) &= -g^*_{-\|\alpha_k\|^2\delta_k(\beta)}(1,P)(px_k)^{l_k+1-n(\alpha_k)} \frac{1-(px_k)^{n(\alpha_k)}}{1-(px_k)^{n(\alpha_k)}/p}. \end{aligned} \tag{2.11}$$

We finally state the Chinta–Gunnells action on $\tilde{A}$: the simple reflections σ_k takes $f(\mathbf{x}) \in A_\beta$ to $f|_\ell\sigma_k$, where

$$(f|_\ell\sigma_k)(\mathbf{x}) = (\mathcal{P}_{\beta,\ell,k}(x_k) + \mathcal{Q}_{\sigma\bullet\beta,\ell,k}(x_k))f(\sigma_k\mathbf{x}). \tag{2.12}$$

This action satisfies the defining relations for W:

Theorem 2.2.1. *[10, Theorem 3.2] The action of the generators* (2.12) *extends to give an action of W on $\tilde{A}$.* □

2.2.2 The p-Parts and their Properties

We now define the p-parts of Weyl group multiple Dirichlet series as in [10] using (2.12). First, let

$$\Delta(\mathbf{x}) = \prod_{\alpha>0}(1 - p^{n(\alpha)}\mathbf{x}^{n(\alpha)\alpha}),$$
$$D(\mathbf{x}) = \prod_{\alpha>0}(1 - p^{n(\alpha)-1}\mathbf{x}^{n(\alpha)\alpha}),$$
$$j(w, \mathbf{x}) = \operatorname{sgn}(w) \prod_{\alpha\in\Phi(w)} p^{n(\alpha)d(\alpha)}\mathbf{x}^{n(\alpha)\alpha}.$$

Let $x_i = p^{-s_i}$, and recall that we have identified x_i with $\mathbf{x}^{\alpha_i}$, for $i = 1, \ldots, r$. The p-parts are defined in terms of the following W-invariant rational function [10, Theorem 3.5]:

$$F(\mathbf{x}, \ell) := \frac{1}{\Delta(x)} \sum_{w\in W} j(w, \mathbf{x})(1|_w)(\mathbf{x}). \tag{2.13}$$

The invariance of $F(\mathbf{x}; \ell)$ under the action (2.12) yields a "functional equation". Writing $F(\mathbf{x}, \ell) = \sum_{\beta\in\Lambda/\Lambda'} f_\beta(\mathbf{x})$, where each $f_\beta(\mathbf{x}) \in \tilde{A}_\beta$, we have

$$F(x; \ell) = \sum_{\beta\in\Lambda/\Lambda'} (\mathcal{P}_{\beta,\ell,k}(x_k) + \mathcal{Q}_{\sigma\bullet\beta,\ell,k}(x_k)) f_\beta(\sigma_k\mathbf{x}).$$

It is simple to check that $\mathcal{P}_{\beta,\ell,k}(x_k) f_\beta(\sigma_k\mathbf{x}) \in \tilde{A}_\beta$ and $\mathcal{Q}_{\sigma\bullet\beta,\ell,k}(x_k) f_\beta(\sigma_k\mathbf{x}) \in \tilde{A}_{\sigma_k\bullet\beta}$. Since $\sigma_k\bullet$ is an involution, it follows from [10, Theorem 3.5] that

$$f_\beta(\mathbf{x}) = \mathcal{P}_{\beta,\ell,k}(x_k) f_\beta(\sigma_k\mathbf{x}) + \mathcal{Q}_{\beta,\ell,k}(x_k) f_{\sigma_k\bullet\beta}(\sigma_k\mathbf{x}). \tag{2.14}$$

Rationality of $F(\mathbf{x}; \ell)$ will be key to our future arguments. In particular, we have the following theorem:

Theorem 2.2.2. *[10, Theorem 3.5] The product $N(\mathbf{x},\ell) := F(\mathbf{x},\ell)D(\mathbf{x})$ is polynomial in the x_i.* $\square$

Here are some examples of these polynomials for $n = 3$ and $\Phi = A_2$.

$$\begin{aligned}
N_{A_2}(\mathbf{x};0,0) &= 1 + g_1(p)x_1 + g_1(p)x_2 + p^2x_1^2x_2 + p^2x_1x_2^2 + g_1(p)p^2x_1^2x_2^2 \qquad (2.15)\\
N_{A_2}(\mathbf{x};1,1) &= 1 + g_2(p)px_1^2 + g_2(p)px_2^2 - g_2(p)p^3x_1^3x_2^2 + g_2(p)p^4x_1^3x_2^2 + p^5x_1^4x_2^2\\
&\quad - g_2(p)p^3x_1^2x_2^3 + g_2(p)p^4x_1^2x_2^3 - p^4x_1^3x_2^3 + p^5x_1^3x_2^3 + p^5x_1^2x_2^4 + g_2(p)p^6x_1^4x_2^4\\
N_{A_2}(\mathbf{x};3,0) &= 1 - p^2x_1^3 + p^3x_1^3 + g_1(p)p^3x_1^4 + g_1(p)x_2 - g_1(p)p^2x_1^3x_2 + g_1(p)p^3x_1^3x_2\\
&\quad + p^5x_1^5x_2 - p^5x_1^3x_2^3 + p^6x_1^3x_2^3 - g_1(p)p^5x_1^4x_2^3 + g_1(p)p^6x_1^4x_2^3 - g_1(p)p^5x_1^3x_2^4\\
&\quad + g_1(p)p^6x_1^3x_2^4 - p^7x_1^5x_2^4 + p^8x_1^5x_2^4 + p^8x_1^4x_2^5 + g_1(p)p^8x_1^5x_2^5
\end{aligned}$$

The polynomials $N(\mathbf{x};\ell)$ are the p-parts of $Z(\mathbf{s};\mathbf{m},\mathbb{F}_q(T))$. This means that

$$N(\mathbf{x};\ell) = \sum_{k_1,\dots,k_r \geq 0} H(P^{k_1},\dots,P^{k_r};P^{l_1},\dots,P^{l_r})x_1^{k_1}\cdots x_r^{k_r}.$$

In Section 2.3, we explain in more detail how to build $Z(\mathbf{s};\mathbf{m},\mathbb{F}_q(T))$ from its p-parts.

Remark. In fact, for K any global field, to construct $Z(\mathbf{s};\mathbf{m},K)$, the p-parts $N(\mathbf{x};\ell)$ are defined in essentially the same way that we have outlined in this section. What does change is the definitions of the Gauss sums and power residue symbols (that must be modified to make sense over K) and the rule for building the global series from its p-parts. See, for example, [10, Section 4] for the construction of $Z(\mathbf{s};\mathbf{m},K)$ over any number field K.

There is analogy between the p-parts of Weyl group multiple Dirichlet series and characters of irreducible representations of $\mathfrak{g}$. Recall that the character ch_μ for the irreducible representation V_μ of $\mathfrak{g}$ of highest weight μ is given by the Weyl character formula (cf. [17, Chapter 24]):

$$ch_{\mu} = \frac{\sum w \in W(-1)^{l(w)}\mathbf{y}^{w(\mu+\rho)}}{\sum_{w\in W}(-1)^{l(w)}\mathbf{y}^{w\rho}}. \tag{2.16}$$

Then, we claim it is possible to write

$$F(\mathbf{x},\ell) := \frac{\sum_{w\in W} j(w,\mathbf{x})(1|_{\ell}w)(\mathbf{x})}{\sum_{w\in W} j(w,\mathbf{x})},$$

which is clearly analogous to the Weyl character formula. We remark that the $\mathbf{y}^{\mu}$, $\mu \in L$ form a basis for $\mathbb{Z}[L]$, whereas the $\mathbf{x}^{\lambda}$, $\lambda \in \Lambda$ form a basis for $\mathbb{Z}[\Lambda]$. To compare, one writes elements of the root lattice in terms of the fundamental weights using the Cartan matrix. Under this identification, we have $\mathbb{Z}[L] \simeq \mathbb{Q}[\Lambda] \supset \mathbb{Z}[\Lambda]$.

To prove the claim we must show that $\Delta(\mathbf{x}) = \sum_{w\in W} j(w,\mathbf{x})$. For each $w \in W$, the set $\Phi(w)$ is uniquely determined w; if $w_1 \neq w_2$, then $\Phi(w_1) \neq \Phi(w_2)$. Since $l(w) = \#\Phi(w)$, to form each term of $j(w,\mathbf{x})$ in the sum, we simply choose the (-1) term from all factors of $D(\mathbf{x})$ corresponding to $\alpha \in \Phi(w)$ and the 1 term from each of the factors of $D(\mathbf{x})$ with $\alpha \notin \Phi(w)$.

In general, $F(\mathbf{x};\ell)$ is a rational function in the x_i. But, in some cases the p-parts are true characters. When $n = 1$, the function $F(\mathbf{x},\ell)$ is polynomial in the x_i and can be identified with the character of the irreducible representation of $\mathfrak{g}$ with lowest weight $-\theta$. This follows from the fact that when $n = 1$, the product of $j(w,\mathbf{x})$ and $(1|_{\ell}w)(\mathbf{x})$ corresponds (after shifting and absorption of p-powers) to the standard action of w on the monomial $\mathbf{x}^{-\theta}$. When $n > 1$, the function $F(\mathbf{x};\ell)$ is not in general polynomial in the x_i, but we will still think of it as a "metaplectic" symmetric function.

As noted in the introduction, Brubaker, Bump, and Friedberg [4–6] define the p-parts using crystal graphs, which is different from [10]. This crystal graph definition (see also [1, 3]) is outside the scope of this thesis, but we make a few comments.

When Φ is type A and $n = 1$, the approaches of [4–6] and [10] are equivalent by Tokuyama's formula [27]. We can view the left hand-side of $N(\mathbf{x};\ell) = D(\mathbf{x})F(\mathbf{x};\ell)$ as a p-deformed Weyl denominator — corresponding $D(\mathbf{x})$ — times a p-deformed character (the two deformations are not the same). By Tokuyama's formula, the numerator $N(\mathbf{x};\ell)$ can be written as a sum over elements of a crystal graph. At present, we do not have a metaplectic ($n > 1$) version of Tokuyama's formula for all Φ, although the existence of such a formula is conjectured. However, our Theorem 4.2.1 shows directly that the "stable" coefficients of the p-parts of [4, 5] and [10] do indeed agree.

2.3 Construction of the Global Series over $\mathbb{F}_q(T)$

This section describes how to form the general coefficients of Weyl group multiple Dirichlet series from the p-parts. Recall that for P prime, the coefficient $H(P^{k_1}, \ldots, P^{k_r}; P^{l_1}, \ldots, P^{l_r})$ of $Z(\mathbf{s}; \mathbf{m}, \mathbb{F}_q(T))$ is the $x_1^{k_1} \cdots x_r^{k_r}$ coefficient of $N(\mathbf{x};\ell)$. The remaining coefficients are determined using the following twisted multiplicativity: for fixed $(c_1 \cdots c_r, c_1' \cdots c_r') = 1$, we put

$$H(c_1c_1', \ldots, c_rc_r'; \mathbf{m}) = H(c_1, \ldots, c_r; \mathbf{m})H(c_1', \ldots, c_r'; \mathbf{m})\varphi(\mathbf{c}; \mathbf{c}'),$$

where

$$\varphi(\mathbf{c}; \mathbf{c}') = \prod_{i=1}^{r} \left(\frac{c_i}{c_i'}\right)^{\|\alpha_i\|^2} \left(\frac{c_i'}{c_i}\right)^{\|\alpha_i\|^2} \prod_{i<j} \left(\frac{c_i}{c_j'}\right)^{2c(i,j)} \prod_{i<j} \left(\frac{c_i'}{c_j}\right)^{2c(i,j)}.$$

We also impose the relation that if $(c_1 \cdots c_r; m_1' \cdots m_r') = 1$, then

$$H(c_1, \ldots, c_r; m_1m_1', \ldots, m_rm_r') = \prod_{j=1}^{r} \left(\frac{m_j'}{c_j}\right)^{-\|\alpha_j\|^2} H(c_1, \ldots, c_r; m_1, \ldots, m_r).$$

Using these properties, we see that up to a product of residue symbols, $H(\mathbf{c}; \mathbf{m})$ equals the product over all P dividing the c_i of $H(P^{k_1}, \ldots P^{k_r}; P^{l_1}, \ldots, P^{l_r})$, where

k_i and l_i are determined by the conditions $P^{k_i}||c_i$ and $P^{l_i}||m_i$.

With this definition of $H(\mathbf{c};\mathbf{m})$, we can now make explicit (1.11). Define degree n Weyl group multiple Dirichlet series of type Φ over $\mathbb{F}_q(T)$ with twisting parameter $\mathbf{m}$ by

$$Z(\mathbf{s};\mathbf{m},\mathbb{F}_q(T)) = Z(\mathbf{s};\mathbf{m}) := \sum_{\mathbf{c}\in\mathcal{O}_{mon}^r} \frac{H(\mathbf{c};\mathbf{m})}{|c_1|^{s_1}\cdots|c_r|^{s_r}}. \tag{2.17}$$

We remind the reader that both $Z(\mathbf{s};\mathbf{m})$ and its p-parts depend on Φ and n. Also, because the coefficients are not, in general, multiplicative, $Z(\mathbf{s};\mathbf{m})$ does not have an Euler product.

Define the normalized series

$$Z^*(\mathbf{s};\mathbf{m}) = \Xi(\mathbf{s})Z(\mathbf{s};\mathbf{m}),$$

where

$$\Xi(\mathbf{s},\Phi,n) = \Xi(\mathbf{s}) = \prod_{\alpha=\sum k_i\alpha_i>0} \zeta_{\mathcal{O}}(1+n(\alpha)\sum_{i=1}^{r} k_i(s_i-1)), \tag{2.18}$$

and $\zeta_{\mathcal{O}}$ denotes the zeta function $\zeta_{\mathcal{O}}(s) := \sum_{c\in\mathcal{O}_{mon}} |c|^{-s} = (1-q^{1-s})^{-1}$.

Chinta and Gunnells [10, Theorem 6.1] show that for all Φ and n, the function $Z^*(\mathbf{s};\mathbf{m})$ has analytic continuation to $\mathbb{C}^r$ and satisfies a group of functional equations isomorphic to W. Actually, [10] treats Weyl group multiple Dirichlet series over number fields, but it is clear that the arguments apply to function fields as well.

CHAPTER 3

FUNCTIONAL EQUATIONS

In this chapter, we derive the functional equations of $Z^*(\mathbf{s};\mathbf{m})$. Our contribution is expository. Our main reference is [10], which derives the functional equations of $Z(\mathbf{s};\mathbf{m},K)$ for K any number field. It is clear that the arguments of [10] apply to function fields as well, and the goal of this chapter explain the simplest case $K=\mathbb{F}_q(T)$. In particular, we obtain functional equations similar to those appearing in [8], which treats the case $K=\mathbb{F}_q(T)$ when $\Phi=A_2$.

To state the functional equations of $Z^*(\mathbf{s};\mathbf{m})$, we first define a slightly more general class of series. Let $I=(I_1,\ldots,I_r)$ be an r-tuple of integers such that $I_j\in\{0,\ldots,n(\alpha_j)-1\}$. Then define

$$Z(\mathbf{s};\mathbf{m},I) := \sum_{\substack{\mathbf{c}\in\mathcal{O}_{mon} \\ \deg c_j\equiv I_j \mod n(\alpha_j)}} \frac{H(\mathbf{c};\mathbf{m})}{|c_1|^{s_1}\cdots|c_r|^{s_r}}$$

and

$$Z^*(\mathbf{s};\mathbf{m},I) = \Xi(\mathbf{s})Z(\mathbf{s};\mathbf{m},I),$$

where $\Xi(\mathbf{s})$ is defined in (2.18) and $n(\alpha_j)$ is defined in (2.10). Fix a simple reflection $\sigma_i\in W$. We define an action of σ_i on the r-tuple of complex variables $\mathbf{s}$ by

$$(\sigma_i\mathbf{s})_j = s_j - c(j,i)(s_i-1),$$

where $c(j,i)$ is the Cartan integer. This action is consistent with the change of variables action on monomials (2.9).

For $\mathbf{m}$ and I fixed, let $J_i(m,I) = \deg m_i - \sum_{j\neq i} c(j,i)I_j$, and let $\sigma_i \bullet I$ be the tuple whose jth entry is the α_j coefficient of $\sigma_i \bullet \left(\sum_{i=1}^r I_j\alpha_j\right)$. We will show that

$$Z^*(\mathbf{s};\mathbf{m},I) = |m_i|^{1-s_i} P^{\|\alpha_i\|^2}_{I_i,J_i(m,I)}(s_i) Z^*(\sigma_i\mathbf{s};\mathbf{m},I) \\ + |m_i|^{1-s_i} Q^{\|\alpha_i\|^2}_{I_i,J_i(m,I)}(s_i) Z^*(\sigma_i\mathbf{s};\mathbf{m},\sigma_i \bullet I), \tag{3.1}$$

where $P_{i,j}(s)$ and $Q_{i,j}(s)$ are defined in (3.4).

Verifying (3.1) requires several steps. The crux of the argument uses the functional equations of Kubota's Dirichlet series, i.e. the rank one Weyl group multiple Dirichlet series, which we state in (3.5). To derive (3.1), we fix c_j for $j \neq i$ and define a new single variable Dirichlet series $\mathcal{E}(s_i;\hat{\mathbf{c}}_i;\mathbf{m},I_i)$. Theorem 3.1.2 shows that $\mathcal{E}$ satisfies functional equations of the same form as Kubtota's Dirichlet series. Writing $Z^*(\mathbf{s};\mathbf{m},I)$ in terms of $\mathcal{E}(s_i;\hat{\mathbf{c}}_i;\mathbf{m},I_i)$, we obtain (3.1).

3.1 Kubota's Dirichlet Series

In this section, we define Kubota's Dirichlet series over $\mathbb{F}_q(T)$ and state its functional equations. We also define the series $\mathcal{E}(s_i;\mathbf{a};\mathbf{m},I_i)$ and derive its functional equations.

We previously defined Kubota's Dirichlet series over $\mathbb{Q}$ in (1.4). The analogue of (1.4) over $\mathbb{F}_q(T)$ is

$$D(s,m;\epsilon^t) = \sum_{c\in\mathcal{O}_{mon}} \frac{g(m,c;\epsilon^t)}{|c|^s}. \tag{3.2}$$

To state the functional equations of (3.2), we also define a slightly more general class of series. Let $0 \le i \le n-1$ and $m \in \mathcal{O}_{mon}$ be fixed. Define

$$D(s, m; \epsilon^t, i) = \sum_{\substack{c \in \mathcal{O}_{mon} \\ \deg c \equiv i \mod n}} \frac{g(m, c; \epsilon^t)}{|c|^s}, \tag{3.3}$$

where the sum is taken over $c \in \mathcal{O}_{mon}$ and where s is a complex variable. Also, define the normalized series

$$D^*(s, m; \epsilon^k, i) = (1 - q^{n-ns})^{-1} D(s, m; \epsilon^k, i).$$

For integers i and j define

$$\begin{aligned} P^t_{i,j}(s) &= -(qx)^{1-(j+1-2i)_n} \frac{q-1}{1-q^{n+1}x^n} \\ Q^t_{i,j}(s) &= -\tau(\epsilon^{t(2i-j-1)})(qx)^{1-n} \frac{1-q^n x^n}{1-q^{n+1}x^n}. \end{aligned} \tag{3.4}$$

Note that both $P_{i,j}(s)$ and $Q_{i,j}(s)$ depend only on the value of $2i - j \mod n$. It is shown in [16, Proposition 2.1] (see also [8, 24]) that D^* satisfies the following functional equation:

$$\begin{aligned} D^*(s, m; \epsilon^t, i) &= |m|^{1-s} P^t_{i,\deg m}(s) D^*(2-s, m; \epsilon^t, i) \\ &\quad + |m|^{1-s} Q^t_{i,\deg m}(s) D^*(2-s, m; \epsilon^t, \deg m + 1 - i). \end{aligned} \tag{3.5}$$

In order to compare our notation with that of [10], we now express the $D(s, m; \epsilon^t, i)$ in a slightly different way, namely, as sums over c in fixed equivalence classes of $K_\infty^\times/(K_\infty^\times)^n$. For this we follow [8,24]. For $c, \eta \in K_\infty^\times$, we say that $c \sim \eta$ if and only if $c/\eta \in (K_\infty^\times)^n$. For such an η, define

$$\mathcal{D}(s, m; \epsilon^t, \eta) = \sum_{c \sim \eta} \frac{g_t(m, c)}{|c|^s}. \tag{3.6}$$

The following lemma shows $\mathcal{D}(s, m; \epsilon^t, \pi_\infty^{-i}) = D(s, m; \epsilon^t, i)$:

Lemma 3.1.1. *Let $i \in \{0, \ldots, n-1\}$. A monic polynomial $c = c(T) \in \mathcal{O}_{mon} \subset K_\infty^\times$ satisfies $c \sim \pi_\infty^{-i}$ if and only if $\deg c \equiv i \pmod{n}$.*

Proof. The proof is an application of Hensel's lemma. Suppose that $c/\pi_\infty^{-i} \in (K_\infty^\times)^n$. Then $c/\pi_\infty^{-i} = (L(\pi_\infty))^n$, where we assume

$$L(\pi_\infty) = a_{-k}\pi_\infty^{-k} + a_{-k+1}\pi_\infty^{-k+1} + \cdots + a_0 + \sum_{j=1}^{\infty} a_j \pi_\infty^j,$$

has degree $-k$. Clearing the denominator, we have $c = \pi_\infty^{-i}(L(\pi_\infty))^n$; therefore, $\deg c = -kn + i \equiv i \pmod{n}$.

For the converse, suppose that $\deg c = k \equiv i \pmod{n}$, and write

$$\begin{aligned} c &= T^{i+kn} + a_1 T^{i+kn-1} + \cdots + a_{i+kn} \qquad a_j \in \mathbb{F}_q, \\ &= \pi_\infty^{-i-kn} + a_1 \pi_\infty^{-i-kn+1} \cdot + a_{i+kn}. \end{aligned}$$

Then

$$c/\pi_\infty^{-i} = \pi_\infty^{-kn} + a_1\pi_\infty^{-kn+1} + \cdots + a_{i+kn}\pi_\infty^{i} = \pi_\infty^{-kn}(1+X),$$

where $X = a_1\pi_\infty + \cdots + a_{i+kn}\pi_\infty^{i+kn} \in (\pi_\infty)$. Define $f(u) = u^n - (1+X) \in K_\infty^\times[u]$. Then $(1+X) \in (K_\infty^\times)^n$ if and only if $f(u) = 0$ has a solution in $K_\infty^\times$. Note that $u = 1$ is a solution modulo (π_∞), and our assumption $q \equiv 1 \pmod{2n}$ implies $f'(u)$ is a unit. By Hensel's Lemma, there is a unique $K_\infty^\times$ solution to $f(u) = 0$. $\square$

There is yet another way to express the $\mathcal{D}(s, m; \epsilon^t, \pi_\infty^{-i})$ that is useful for comparing [8, 24] with [10]. Essentially, one replaces the coefficients $g_t(m, c)$ in (1.4) with $g_t(m, c)\Psi_i(c)$, where Ψ_i is a complex-valued function whose effect is to restrict the sum to the equivalence classes $[\pi_\infty^{-i}] \in K_\infty^\times/(K_\infty^\times)^n$. For $0 \le i \le n-1$, define

$$\Psi_i(c) = \begin{cases} 1 & \text{if } c \sim \pi_\infty^{-i} \quad \text{and } c \in \mathcal{O}_{mon}, \\ (c, \epsilon)_\infty^{-t} & \text{if } c \sim \pi_\infty^{-i} \quad \text{and } c \text{ has leading coefficient } \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

The set $\{\Psi_i : 0 \le i \le n-1\}$ forms a basis for a more general space of functions $\mathcal{M}_t(\Omega)$, defined in [2, 10]. Let $\Omega = \mathbb{F}_q^\times K_\infty^{\times n}$. Note that Ω is maximal isotropic for the Hilbert symbol (2.2), in the sense that for any $\varepsilon_1, \varepsilon_2 \in \Omega$, we have $(\varepsilon_1, \varepsilon_2)_\infty = 1$. Define $\mathcal{M}_t(\Omega)$ as the space of functions $\Psi : K_\infty^\times \to \mathbb{C}$ that satisfy

$$\Psi(\varepsilon c) = (c, \varepsilon)_\infty^{-t} \Psi(c),$$

for all $\varepsilon \in \Omega$. To see that the Ψ_i form a basis for $\mathcal{M}_t(\Omega)$, note that any $\Psi \in \mathcal{M}_t(\Omega)$ is completely determined by its values on a set of representatives for $K_\infty^\times/\Omega$. Thus, $\dim \mathcal{M}(\Omega) = \dim K_\infty^\times/\Omega$. The claim now follows from Lemma 3.1.1, which shows that the representatives of $K_\infty^\times/\Omega$ are exactly π_∞^{-i}, for $i \in \{0, \ldots, n-1\}$.

As mentioned in this chapter's introduction, to obtain the functional equations for $Z^*(\mathbf{s}; \mathbf{m})$ we consider a new series of one variable. Let $i \in \{0, \ldots, r\}$, and fix $\mathbf{a} = (a_1, \ldots, \hat{a}_i, \ldots, a_r) \in (\mathcal{O}_{mon})^{r-1}$, where the hat means omit a_i. For $0 \le j \le (n(\alpha_i) - 1)$, define

$$\mathcal{E}(s_i, \mathbf{a}; \mathbf{m}, \pi_\infty^{-j}) := \sum_{\substack{c_i \in \mathcal{O}_{mon} \\ c_i \sim \pi_\infty^{-j}}} \frac{H(a_1, \ldots, c_i, \ldots, a_r; \mathbf{m})}{|c_i|^{s_i}},$$

and let

$$\mathcal{E}^*(s_i, \mathbf{a}; \mathbf{m}, \pi_\infty^{-j}) = (1 - q^{n(\alpha_i)(1-s_i)})^{-1} \mathcal{E}(s_i, \mathbf{a}; \mathbf{m}, \pi_\infty^{-j}).$$

The key step in proving the functional equations of $Z^*(\mathbf{s}; \mathbf{m})$ is to show that $\mathcal{E}^*$ satisfies functional equations of the same form as the Kubota series (3.3). This is the content of [10, Theorem 5.8]. To prove this, we rewrite $\mathcal{E}^*$ in terms of Kubota series and rational functions in q^{-s_i}; these rational functions are closely related to the p-parts of $Z^*(\mathbf{s}; \mathbf{m})$ and satisfy their own functional equations.

To define the rational functions mentioned above, let P be prime and let $\mathbf{k} = (k_1, \ldots, k_r)$ be an r-tuple of nonnegative integers. Recall that $i \in \{0, \ldots, r\}$ is fixed, and let $l_i = \operatorname{ord}_P m_i$ and $n = n(\alpha_i)$. Writing $\beta = \sum k_j \alpha_j$, define a new tuple

$\mathbf{k}'$ by setting $\sum k_j' \alpha_j = \sigma_i \bullet \beta$. Then, define

$$N^{(P;\mathbf{k})}(x;\mathbf{m},\alpha_i) = \sum_{j\geq 0} H(P^{k_1},\ldots,P^{jn+(k_i)_n},\ldots,P^{k_r};\mathbf{m})x^{jn+(k_i)_n},$$

where as before $(a)_n = a - n\lfloor a/n \rfloor$. The rational functions we are interested in are

$$\begin{aligned} f^{(P;\mathbf{k})}(x;\mathbf{m},\alpha_i) &= \frac{N^{(P;\mathbf{k})}(x;\mathbf{m},\alpha_i)}{1-p^{n-1}x^n} \\ &- \delta^m_{k_i,k_i'} g(m_i^{-1}P^{l_i},P;\epsilon^{\|\alpha_i\|^2(k_i-k_i')})p^{(k_i-k_i'-1)_n}x^{(k_i-k_i')_n}\frac{N^{(P;\mathbf{k}')}(x;\mathbf{m},\alpha_i)}{1-p^{n-1}x^n}, \end{aligned}$$

where $\delta^m_{i,j}$ is 0 if $i \equiv j \mod m$ and 1 otherwise. These functions satisfy the following functional equations [10, Theorem 4.1]:

$$\frac{f^{(P;\mathbf{k})}(x;\mathbf{m},\alpha_i)}{f^{(P;\mathbf{k})}(1/(p^2x);\mathbf{m},\alpha_i)} = \begin{cases} (px)^{l_i+1-(k_i'-k_i)_n} & \text{if } (k_i'-k_i)_n \neq 0, \\ (px)^{l_i+1-n} & \text{otherwise.} \end{cases} \tag{3.7}$$

We are now in a position to sketch a proof of the following special case of [10, Theorem 5.8].

Theorem 3.1.2 ([10, Theorem 5.8]). *Let $\mathbf{a} \in (\mathcal{O}_{mon})^{r-1}$ be fixed, and let $A = \prod_{j\neq i} a_j^{-c(j,i)}$, where $c(j,i)$ is the ji-entry of the Cartan matrix for Φ. Put $\mathcal{E}(s_i,\mathbf{a};\mathbf{m},j) := \mathcal{E}(s_i,\mathbf{a};\mathbf{m},\pi_\infty^{-j})$. Then*

$$\begin{aligned} \mathcal{E}^*(s_i,\mathbf{a};\mathbf{m},j) &= |Am_i|^{1-s_i}P^k_{j,\deg m_i}\mathcal{E}^*(2-s_i,\mathbf{a};\mathbf{m},j) \\ &+ |Am_i|^{1-s_i}Q^k_{j,\deg m_i}\mathcal{E}^*(2-s_i,\mathbf{a};\mathbf{m},\deg m_i+1-j). \end{aligned} \tag{3.8}$$

Proof. Our proof is the same as that of [10, Theorem 5.8]. The general argument is to write $\mathcal{E}^*(s_i,\mathbf{a};\mathbf{m},j)$ in terms of Kubota series $D(s_i,m;\epsilon^t,j)$ (for various j) and the $f^{(P;\mathbf{k})}(x;\mathbf{m},\alpha_i)$. The functional equations (3.8) then follow directly from the functional equations (3.5) and (3.7).

To simplify notation, we assume that $i=1$. Let $P_1,\ldots,P_v$ be the prime divisors of $a_2\cdots a_r m_1\cdots m_r$, with $p_j = |P_j|$. Let $S = \{P_1,\ldots,P_v\}$. Write $a_j = P^{\beta_{j1}}\cdots P^{\beta_{jv}}$

for $j = 2, \ldots, r$ and $Am_1 = P^{l_1} \cdots P^{l_r}$. Using the exact same argument as in [10], one can show that

$$\mathcal{E}(s_1, \mathbf{a}; \mathbf{m}, j) = \xi \sum_{k_1, \ldots, k_v = 0}^{m-1} \mathcal{D}(s_1, P^{(l_1 - 2k_1)_n} \cdots P_v^{(l_v - 2k_v)_n}, \epsilon^{\|\alpha_1\|}, P_1^{-k_1} \cdots P_v^{-k_v} \pi_\infty^{-j})$$

$$\times C(k_1, \ldots, k_r) \prod_{i=1}^{v} f^{(P_i; k_i, b_{21}, \ldots, b_{ri})}(p_i^{-s_1}; \mathbf{m}, \alpha_1), \tag{3.9}$$

where ξ and $C(k_1, \ldots, k_r)$ are products of residue symbols. In particular, for $\eta' \sim P^{2k_1 - l_1 - 1}$ and $K_j = l_j - 2k_j$, we have [10, Lemma 5.9]

$$\frac{C(k_1, \ldots, k_r) \Psi_{I_1}^{(P_1^{k_1} \cdots P_v^{k_v})}}{C(l_1 - k_1 + 1, \ldots, l_r - k_r + 1) \hat{\Psi}_{I_{1\eta'}}^{(P_1^{(l_1+1-k_1)_n} \cdots P_v^{(l_v+1-k_v)_n})}} = \left(\frac{m_1 P_2^{K_2} \cdots P_v^{K_v}}{P_1^{2k_1 - l_1 - 1}} \right)^{-\|\alpha_1\|^2}, \tag{3.10}$$

where we define $\Psi^{(a)}(c) = \Psi(ac)$ and $\hat{\Psi}_\eta(c) = (\eta, c)_\infty^t \Psi(\eta c)$. It is clear that $\hat{\Psi}_\eta \in \mathcal{M}_t(\Omega)$ and depends only on the class of $\eta \in K_\infty^\times / (K_\infty^\times)^n$.

The proof of (3.9) is a lengthy, but straightforward, computation with residue symbols. The idea is to use twisted multiplicativity to rewrite the coefficients $H(c_1, a_2, \ldots, a_r)$ in terms of Gauss sums and prime power coefficients. This follows from considering $c_1 = cc'$, where we assume that $(c, a_2 \cdots a_r m_1 \cdots m_r) = 1$. Summing all relevant c, up to a product of residue symbols, we can write $\mathcal{E}(s_1, \mathbf{a}; \mathbf{m}, j)$ as the sum of the product of Kubota series of the form $\mathcal{D}_S(s_1; m; \epsilon^{\|\alpha_1\|^2}, \eta)$ and polynomials $N^{(P;\mathbf{k})}(p_i^{-s_1}; \mathbf{m}, \alpha_1)$. Here $\mathcal{D}_S(s; m; \epsilon^t, \eta)$ is a generalization of (3.6); for a finite set of primes S, it is defined exactly the same as (3.6), except that in the sum, we restrict to only those c relatively prime to the elements of S.

To obtain $\mathcal{D}(s_1; m, \epsilon^{\|\alpha_1\|^2}, \eta)$ from $\mathcal{D}_S(s_1; m, \epsilon^{\|\alpha_1\|^2}, \eta)$, we use the following result of [24] (see also [10, Lemma 5.4]) to “remove” primes from S one by one:

$$D_{S\cup\{P\}}(s,mP^i,\epsilon^t,\pi_\infty^{-j}) = \frac{D_S(s,mP^i,\epsilon^t,\pi_\infty^{-j})}{1-|P|^{n-1-ns}} \\ - \frac{g(mP^i,P^{i+1};\epsilon^t)}{|P|^{(i+1)s}} \frac{D_S(s,mP^{n-i-2},\epsilon^t,\pi_\infty^{-i-j-1})}{1-|P|^{n-1-ns}}, \tag{3.11}$$

Making a change of variables and applying (3.10), we put the two terms on the left-hand side of (3.11) together to obtain (after v iterations) (3.9).

It remains to see that (3.9) satisfies (3.8). For this, we first apply the functional equations of D^*. Assume $\mathbf{k} = (k_1,\ldots,k_r)$, where each $k_j \in \{0,\ldots,n(\alpha_1)-1\}$, and let

$$E = E(\mathbf{k}) = P^{(l_1-2k_1)_n}\cdots P_v^{(l_v-2k_v)_m}$$
$$F = F(\mathbf{k}) = P^{-k_1}\cdots P_v^{-k_v}.$$

Set $e = \deg E$ and $f = \deg F$. It follows from (3.5) that

$$D^*(s_1,E;\epsilon^{\|\alpha_1\|^2},i-f) = |E|^{1-s_1}P_{i-f,e}^{\|\alpha_1\|^2}(s_1)D^*(2-s_1,E;\epsilon^{\|\alpha_1\|^2},i-f) \\ + Q_{i-f,e}^{\|\alpha_1\|^2}(s_1)D^*(2-s_1,E;\epsilon^{\|\alpha_1\|^2},e+1-(i-f)).$$

Recall that $P_{i,j}^{\|\alpha_1\|^2}$ and $Q_{i,j}^{\|\alpha_1\|^2}$ depend only on the value of $2i-j$ modulo $n = n(\alpha_1)$. We have

$$\begin{aligned} 2i-2f-e &= 2i - \sum_{j=1}^{v} \deg P_j(-2k_j-(l_j-2k_j)_n) \\ &\equiv 2i - n + \sum_{j=1}^{v} \deg P_j (l_j)_n \\ &\equiv 2i - \deg Am_1 \mod n. \end{aligned}$$

Thus, we see $P_{i-f,e}(s_1) = P_{i,\deg Am_1}(s_1)$ and $Q_{i-f,e}(s_1) = Q_{i,\deg Am_1}(s_1)$ do not depend on $\mathbf{k}$. The result now follows from [10] using (3.7). □

3.2 Functional Equations for $Z^*(\mathbf{s};\mathbf{m})$

We now derive the functional equations for $Z^*(\mathbf{s};\mathbf{m})$. Fix a simple reflection $\sigma_i \in W$, and let $n = n(\alpha_i)$. We decompose the series in the following way:

$$\begin{aligned} Z^*(\mathbf{s};\mathbf{m},I) =& \Xi(\mathbf{s}) \sum_{\substack{\mathbf{c}=(\mathcal{O}_{mon})^r \\ c_j \equiv I_j \pmod{n}(\alpha_j)}} \frac{H(c_1,\ldots,c_i,\ldots,c_r;\mathbf{m})}{|c_1|^{s_1}\cdots|c_r|^{s_r}} \\ =& \frac{\Xi(\mathbf{s})}{\zeta(ns_i-n+1)} \sum_{\substack{\mathbf{c}=(\mathcal{O}_{mon})^{r-1} \\ c_j \equiv I_j \pmod{n}(\alpha_j)}} \frac{1}{\prod_{j=1,j\neq i}^r |c_j|^{s_j}} \mathcal{E}^*(s_i,\hat{\mathbf{c}}_i;\mathbf{m},I_i) \end{aligned} \tag{3.12}$$

Let $C = \prod_{j\neq i} c_j^{c(j,i)}$. Applying (3.8) to $\mathcal{E}^*(s_i,\hat{\mathbf{c}}_i;\mathbf{m},I_i,i)$, we have (3.12) equals

$$\begin{aligned} &\frac{\Xi(\mathbf{s})}{\zeta(ns_i-n+1)} \sum_{\substack{\mathbf{c}=(\mathcal{O}_{mon})^{r-1} \\ c_j \equiv I_j \pmod{n}(\alpha_j)}} \frac{1}{\prod_{j=1,j\neq i}^r |c_j|^{s_j}} |Cm_i|^{1-s_i} \\ &\times \left(P^{\|\alpha_i\|^2}_{I_i,\deg Cm_i}(s_i)\mathcal{E}^*(2-s_i,\hat{\mathbf{c}}_i;\mathbf{m},I_i) + Q^{\|\alpha_i\|^2}_{I_i,\deg Cm_i}(s_i)\mathcal{E}^*(2-s_i,\hat{\mathbf{c}}_i;\mathbf{m},\deg Cm_i+1-I_i)\right). \end{aligned} \tag{3.13}$$

Using the definition of C, we compute $\deg Cm_i = \deg m_i - \sum_{j\neq i} c(j,i)\deg c_j$. Substituting this into (3.13), we have

$$\begin{aligned} &\frac{\Xi(\mathbf{s})}{\zeta(ns_i-n+1)} |m_i|^{1-s_i} \sum_{\substack{\mathbf{c}=(\mathcal{O}_{mon})^{r-1} \\ c_j \equiv I_j \pmod{n}(\alpha_j)}} \frac{1}{\prod_{j=1,j\neq i}^r |c_j|^{s_j-c(j,i)(s_i-1)}} \\ &\times \left(P^{\|\alpha_i\|^2}_{I_i,\deg m_i - \sum_{j\neq i} c(j,i)I_j}(s_i)\mathcal{E}^*(2-s_i,\hat{\mathbf{c}}_i;\mathbf{m},I_i) \right. \\ &\left. + Q^{\|\alpha_i\|^2}_{I_i,\deg m_i - \sum_{j\neq i} c(j,i)I_j}(s_i)\mathcal{E}^*(2-s_i,\hat{\mathbf{c}}_i;\mathbf{m},\deg m_i - \sum_{j\neq i} c(j,i)I_j + 1 - I_i)\right). \end{aligned}$$

Recall that σ_i permutes the positive roots of Φ other than α_i. It follows that

$$\frac{\Xi(\mathbf{s})}{\zeta(ns_i-n+1)} = \frac{\Xi(\sigma_i\mathbf{s})}{\zeta(n(2-s_i)-n+1)}.$$

Putting this together, we have

$$\begin{aligned} Z^*(\mathbf{s};\mathbf{m},I) =& |m_i|^{1-s_i} P^{\|\alpha_i\|^2}_{I_i,\deg m_i - \sum_{j\neq i} c(j,i)I_j} Z^*(\sigma_i\mathbf{s};\mathbf{m},I) \\ &+ |m_i|^{1-s_i} Q^{\|\alpha_i\|^2}_{I_i,\deg m_i - \sum_{j\neq i} c(j,i)I_j} Z^*(\sigma_i\mathbf{s};\mathbf{m},\sigma_i \bullet I). \end{aligned}$$

where $\sigma_i \bullet I$ is defined as in this chapter's introduction.

It will be convenient to express this functional equation in a slightly different way. Summing $Z^*(\mathbf{s};\mathbf{m},I)$ over all I, we have

$$Z^*(\mathbf{s};\mathbf{m}) = |m_i|^{1-s_i} \sum_I \left(P^{\|\alpha_i\|^2}_{I_i,\deg m_i - \sum_{j\neq i} c(j,i) I_j} + Q^{\|\alpha_i\|^2}_{(\sigma_i \bullet I)_i,\deg m_i - \sum_{j\neq i} c(j,i) I_j} \right) Z^*(\mathbf{s};\mathbf{m};I).$$

CHAPTER 4

LOCAL RESULTS

This chapter describes our results related to the p-parts of Weyl group multiple Dirichlet series. The p-part construction is purely local. This means that to build Weyl group multiple Dirichlet series over any global field K as in (1.11), one uses the same p-parts. Of course, the definitions for the Gauss sums and power residue symbols must be modified, but after this is done, results of this section apply to the p-parts of Weyl group multiple Dirichlet series defined over any global field K.

4.1 The Support of $N(\mathbf{x};\ell)$

The main goal of this section is to prove Theorem 4.1.2, thereby extending [9, Theorem 3.2] to general n and Φ. The key tool will be Theorem 4.1.1, which gives an explicit recurrence relation on the coefficients of $N(\mathbf{x};\ell)$.

4.1.1 A Recurrence Relation

The invariance of $F(\mathbf{x};\ell)$ under the W-action (2.12) induces a recurrence relation on the coefficients of the numerator, i.e. the p-parts. This relation is summarized in Theorem 4.1.1 below.

Theorem 4.1.1. *Let $N(\mathbf{x};\ell) = \sum a_\lambda \mathbf{x}^\lambda$ be the p-part of* (1.11) *defined as in [10]. The coefficients a_λ satisfy a recurrence relation under the W-action. Fix a simple reflection σ_k. Let $\mu = \sigma_k \bullet \lambda$, $\alpha = \alpha_k$, $n = n(\alpha_k)$, $\delta = \delta_k(\lambda)$, and $\nu = n - (\delta_k(\lambda))_n$. Then, if $\delta \equiv 0 \pmod{n}$, we have*

$$-p^{n+1} a_{\lambda - n\alpha} + a_\lambda = -p^{1-\delta}\left(a_\mu - p^{n+1} a_{\mu+n\alpha}\right). \tag{4.1}$$

Otherwise, we have

$$g_{\|\alpha\|^2\delta}(1,P)^{-1} p^{1-\nu} a_{\lambda-\nu\alpha} + a_\lambda = g_{\|\alpha\|^2\delta}(1,P)^{-1} p^{1-\delta}\left(a_\mu + g_{\|\alpha\|^2\delta}(1,P) p^{-(\nu-1)} a_{\mu+\nu\alpha}\right). \tag{4.2}$$

Proof. Let $l = l_k$. If $f \in \tilde{A}$ and $g(\mathbf{x}) \in \tilde{A}_\beta$ with $\beta \in \Lambda'$, then for all $w \in W$, we have [10, Lemma 3.4]

$$(fg|_\ell w)(\mathbf{x}) = g(w\mathbf{x})(f|_\ell w)(\mathbf{x}). \tag{4.3}$$

In particular, since $D(\mathbf{x}) \in \tilde{A}_0$, (4.3) implies $(F|_\ell\sigma)(\mathbf{x}) = (N|_\ell\sigma)(\mathbf{x})/D(\sigma\mathbf{x})$. The W-invariance of $F(\mathbf{x};\ell)$ yields

$$N(\mathbf{x},\ell) = \frac{D(\mathbf{x})}{D(\sigma\mathbf{x})}(N|_\ell\sigma)(\mathbf{x}). \tag{4.4}$$

It is a simple calculation to check that

$$\frac{D(\mathbf{x})}{D(\sigma\mathbf{x})} = \frac{p^{n+1}\mathbf{x}^{n\alpha}(1 - p^{n-1}\mathbf{x}^{n\alpha})}{p^{n+1}\mathbf{x}^{n\alpha} - 1}. \tag{4.5}$$

To isolate the terms on the right-hand side that contribute to the coefficient of $\mathbf{x}^\lambda$, we use $\mathcal{P}$ and $\mathcal{Q}$ from (2.11) to define new functions

$$P_\lambda(x) = (1 - 1/p)(px)^{l+1-(\delta)_n},$$

$$Q_\lambda(x) = -g_{\|\alpha\|^2\delta}(1,P)^{-1}(px)^{l+1-n},$$

$$R_\lambda(x) = g_{\|\alpha\|^2\delta}(1,P)^{-1}(px)^{l+1},$$

Substituting (4.5) into (4.4) using (2.12), we obtain

$$\sum(a_{\lambda-n\alpha}p^{n+1}-a_\lambda)\mathbf{x}^\lambda=\sum a_\lambda\left[P_\lambda(x)+Q_\lambda(x)+R_\lambda(x)\right]p^{n-l+\delta}\mathbf{x}^{n\alpha+\sigma\lambda}, \tag{4.6}$$

where we have used that the change of variables action under σ takes $\mathbf{x}^\lambda$ to $p^{d(\sigma\lambda-\lambda)}\mathbf{x}^{\sigma\lambda}$, that $d(\sigma\lambda-\lambda)=\delta-l-1$, and that $g^*_{-k}(1,P)=g_k(1,P)^{-1}$. A straightforward calculation shows that the terms on the right-hand side that contribute to the coefficient of $\mathbf{x}^\lambda$ are

$$\begin{aligned}
&a_\gamma P_\gamma(x)p^{n-l+\delta_k(\gamma)}\mathbf{x}^{n\alpha+\sigma\gamma},\ \text{with } \gamma=\sigma\bullet\lambda+\nu\alpha\\
&a_\gamma Q_\gamma(x)p^{n-l+\delta_k(\gamma)}\mathbf{x}^{n\alpha+\sigma\gamma},\ \text{with } \gamma=\sigma\bullet\lambda\\
&a_\gamma R_\gamma(x)p^{n-l+\delta_k(\gamma)}\mathbf{x}^{n\alpha+\sigma\gamma},\ \text{with } \gamma=\sigma\bullet\lambda+n\alpha.
\end{aligned}$$

For convenience, we show the computation for P_γ. For any γ, the monomial contribution from P_γ is

$$\mathbf{x}^{(l+1-\delta(\gamma))\alpha+\sigma\gamma}\mathbf{x}^{n\alpha}=\mathbf{x}^{\sigma\bullet\gamma+(n-(\delta(\gamma))_n)\alpha}.$$

We need only check that the exponent is λ when $\gamma=\sigma\bullet\lambda+\nu\alpha$. This requires simplifying $[\sigma\bullet(\sigma\bullet\lambda+\nu\alpha)]+[(n-(\delta(\sigma\bullet\lambda+\nu\alpha)))_n\alpha]$. The first term is $\lambda-\nu\alpha$, so it will suffice to show that the second term is $\nu\alpha$, or equivalently, that

$$(\delta(\sigma\bullet\lambda+(n-(\delta(\lambda))_n\alpha))_n=(\delta(\lambda))_n.$$

We have $\delta(\sigma\bullet\lambda)_n=n-\delta(\lambda)_n$. It follows that $\sigma\bullet\lambda+(n-\delta_n(\lambda))\alpha=\sigma\bullet\lambda+\delta(\sigma\bullet\lambda)_n\alpha$, and

$$\begin{aligned}
\delta(\sigma\bullet\lambda+\delta(\sigma\bullet\lambda)_n\alpha)&=(d(\lambda-\sigma\bullet\lambda-2\delta(\sigma\bullet\lambda)_n\alpha))_n\\
&=(\delta(\sigma\bullet\lambda)-2\delta(\sigma\bullet\lambda)_n)_n\\
&=(-\delta(\lambda)-2(n-\delta(\lambda)_n)_n\\
&=\delta(\lambda)_n.
\end{aligned}$$

Checking the contributions for Q_γ and R_γ is similar. For the Q_γ term, this reduces to showing that $l+1+\sigma\gamma=\lambda$ when $\gamma=\sigma\bullet\lambda$. For the R_γ term, we must show that $(n+l+1)\alpha+\sigma\gamma=\lambda$ when $\gamma=\sigma\bullet\lambda+n\alpha$. These statements are both clear from definition (2.8).

Collecting the coefficients of $\mathbf{x}^\lambda$ and moving all of the terms of (4.6) to the right-hand side, we obtain a five term recurrence relation:

$$\begin{aligned} 0 &= a_\lambda - p^{n+1}a_{\lambda-n\alpha} - (1-1/p)p^{1+n+\delta(\gamma)-\delta(\gamma)_n}a_{\mu+(n-\delta(\lambda)_n)\alpha} \\ &\quad + g_{\|\alpha\|^2\delta}(1,p)^{-1}p^{1-\delta(\lambda)-n}a_{\mu+n\alpha} - g_{\|\alpha\|^2\delta}(1,P)^{-1}p^{1-\delta(\lambda)}a_\mu, \end{aligned} \tag{4.7}$$

where in (4.7) we put $\gamma=\mu+(n-(\delta(\lambda)_n)\alpha$.

We next apply (4.6) a second time, now with $\mathbf{x}^{\mu+n\alpha}$ as the monomial on the left-hand side. First, we calculate the contributions to the coefficient of $\mathbf{x}^{\mu+n\alpha}$ on the right-hand side. We have

$$\begin{aligned} &a_\gamma P_\gamma(x)p^{n-l+\delta}(\gamma)\mathbf{x}^{n\alpha+\sigma\gamma},\ \text{with } \gamma=\lambda-(n-\delta(\lambda)_n)\alpha \\ &a_\gamma Q_\gamma(x)p^{n-l+\delta}(\gamma)\mathbf{x}^{n\alpha+\sigma\gamma},\ \text{with } \gamma=\lambda-n\alpha \\ &a_\gamma R_\gamma(x)p^{n-l+\delta}(\gamma)\mathbf{x}^{n\alpha+\sigma\gamma},\ \text{with } \gamma=\lambda. \end{aligned}$$

Again, when we collect the coefficients of $\mathbf{x}^{\mu+n\alpha}$, now moving all of the terms to the left-hand side, and we obtain a second five-term recurrence relation. To compare with (4.7), we have multiplied each term by $g_{\|\alpha\|^2\delta(\lambda)}(1,P)^{-1}p^{1-\delta(\lambda)-n}$.

$$\begin{aligned} g_{\|\alpha\|^2\delta(\lambda)}(1,P)^{-1}p^{2-\delta(\lambda)}a_\mu - g_{\|\alpha\|^2\delta(\lambda)}(1,P)^{-1}p^{1-\delta(\lambda)-n}a_{\mu+n\alpha} + p^{1+n}a_{\lambda-n\alpha} \\ -pa_\lambda - (1-1/p)g_{\|\alpha\|^2\delta(\lambda)}(1,P)^{-1}p^{2-\delta(\lambda)+\delta(\gamma)-\delta(\gamma)_n}a_{\lambda-(n-\delta(\lambda)_n)} &= 0 \end{aligned} \tag{4.8}$$

where in (4.8) we put $\gamma=\lambda-(n-\delta(\lambda)_n)\alpha$.

Adding (4.7) and (4.8) and simplifying, we obtain the n recurrence relations (4.1) and (4.2) stated in the theorem. To simplify, note that when $\gamma=\lambda-(n-$

$\delta(\lambda_n))\alpha$, we have

$$1 - \delta(\lambda) + \delta(\gamma) - \delta(\gamma)_n = 1 + n - \delta(\lambda)_n.$$

Using the fact that $(n - \delta(\lambda)_n) = (-\delta(\lambda))_n$, we have

$$\begin{aligned}
1 - \delta(\lambda) + \delta(\gamma) - \delta(\gamma)_n &= 1 - \mu(\lambda) + \delta(\lambda) + 2n - 2\delta(\lambda)_n - (\delta(\lambda) + 2(n - \delta(\lambda)_n)_n \\
&= 1 + 2n - 2\delta(\lambda)_n - (\delta(\lambda) - 2\delta(\lambda))_n \\
&= 1 + 2n - 2\delta(\lambda)_n - n + \delta(\lambda)_n \\
&= 1 + n - \delta(\lambda)_n.
\end{aligned}$$

Similarly, when $\gamma = \mu + (n - \delta(\lambda)_n)\alpha$, we have

$$n + \delta(\gamma) - \delta(\gamma)_n = \delta(\lambda)_n - \delta(\lambda) - n.$$

The result now follows by direct computation. □

We note that a four term recurrence relation on the coefficients of $N(\mathbf{x};\ell)$ equivalent to Theorem 4.1.1 previously appeared in [9] for Φ simply laced and $n = 2$. Also, [23] gives a five term recurrence relation equivalent to (4.7) for $\Phi = A_r$ and $n \gg r$.

4.1.2 The Support of $N(\mathbf{x};\ell)$

Let θ be defined as in (2.7). A result of Chinta, Friedberg, and Gunnells [9, Theorem 3.2] states that when $n = 2$ and Φ is simply laced, the support of $N(\mathbf{x};\ell)$ is contained in the shifted weight polytope Π_θ, defined as the convex hull of the points $\theta - w\theta$, for $w \in W$. In fact, Π_θ is the weight polytope for the representation of lowest weight $-\theta$, shifted by θ. Our next theorem extends this result to general Φ and n.

Theorem 4.1.2. *The support of $N(\mathbf{x};\ell)$ is contained in Π_θ.*

Remark. Let Θ be the set of dominant weights in the representation of highest weight θ. All points in the convex hull of Π_θ can be written $\theta - w\xi$, for $w \in W$ and $\xi \in \Theta$. To see this, recall that Π_θ corresponds to the weight polytope for the representation of lowest weight $-\theta$, after shifting by θ. Reflection through the origin takes the weight polytope for the representation V_θ of highest weight θ to the weight polytope for the representation of lowest weight $-\theta$. In particular, the dominant weights of V_θ are taken to those of the representation with lowest weight $-\theta$.

Our proof follows [9]. We require a few geometric results whose proof can be found in [9] as they do not require $n = 2$ or Φ simply laced. Note that Π_θ is cut out by the inequalities

$$\langle w\varpi_i, \mathbf{x} - (\theta - w\theta)\rangle \geq 0 \qquad w \in W,\ i = 1, \ldots, r. \tag{4.9}$$

Moreover, these inequalities are redundant in the sense of the following lemma. Before we state the lemma, recall that for $w \in W$, the right and left descent sets of W are

$$\mathcal{R}(w) = \{\sigma_i : l(w\sigma_i) < l(w)\}$$

and

$$\mathcal{L}(w) = \{\sigma_i : l(\sigma_i w) < l(w)\}.$$

Lemma 4.1.3. *[9, Lemma 3.4] Let $\sigma_k \in \mathcal{R}(w)$ and let $u = w\sigma_k$. If $k \neq j$, the inequalities*

$$\langle w\varpi_j, \mathbf{x} - (\theta - w\theta)\rangle \geq 0$$

and

$$\langle u\varpi_j, \mathbf{x} - (\theta - u\theta)\rangle \geq 0$$

are equivalent. □

The second geometric lemma we require describes, for any $\lambda \in \Pi_\theta$, the final point of support in the ray $\lambda + m\alpha_k$, for $m \in \mathbb{Z}$, $k = 1, \ldots, r$.

Lemma 4.1.4 (Lemma 3.5 [9])**.** *Let $\mu = \theta - w\theta$ be a vertex of Π_θ, and suppose $\sigma_k \in \mathcal{L}(w)$. Then any lattice point of the form $\mu + m\alpha_k$, where m is a positive integer, lies outside of Π_θ. Similarly, let $u = \sigma_k w$ and let $\lambda = \theta - u\theta$. Then any point of the form $\lambda - m\alpha_k$, where m is a positive integer, lies outside of Π_θ.* □

We now continue to the proof of Theorem 4.1.2.

Proof of Theorem 4.1.2. The proof requires showing that $a_\lambda = 0$ if λ violates the inequalities (4.9) that cut out Π_θ. We induct on the length of w. If $l(w) = 0$, then $a_\lambda = 0$ if λ violates the inequalities active at the origin (otherwise we would have polar terms, a contradiction since $N(\mathbf{x}; \ell)$ is polynomial). Now suppose $l(w) > 0$ and that we have verified the inequalities at all vertices where $\theta - u\theta$, where $l(u) < l(w)$. Let $\sigma_k \in \mathcal{R}(w)$; since $l(w) > 0$, this set is nonempty. Lemma 4.1.3 then implies that $a_\lambda = 0$ unless λ satisfies

$$\langle w\varpi_j, \mathbf{x} - (\theta - w\theta)\rangle \geq 0,$$

for all $j \neq k$. Here we use induction, so that $l(w\sigma_k) < l(w)$ implies

$$\langle w\sigma_k\varpi_j, \mathbf{x} - (\theta - w\sigma_k\theta)\rangle \geq 0,$$

for all $j \neq k$. If $\#\mathcal{R}(w) > 1$, all desired inequalities must hold for the support of $N(\mathbf{x}; \ell)$ at the vertex $\theta - w\theta$ (we have already shown that they hold for ϖ_j, $j \neq k$

— simply choose $\sigma_j \in \mathcal{R}(w)$ with $j \neq k$ to achieve the last inequality.) Thus, it suffices to assume that $\mathcal{R}(w) = \{\sigma_k\}$. In this case, we must show that $a_\lambda = 0$ if λ violates the inequalities:

$$\langle w\varpi_k, \mathbf{x} - (\theta - w\theta)\rangle, \tag{4.10}$$

for some $w \in W$.

Our proof of (4.10) is by contradiction. Let $\sigma_j \in \mathcal{L}(w)$, and choose $\mu \in \Lambda$ such that μ violates (4.10), but $a_\mu \neq 0$. Further, assume that $a_{\mu'} = 0$ for all $\mu' = \mu + m\alpha_j$ with $m > 0$ (we can find such a μ by Lemma 4.1.4.) If μ violates the inequalities (4.10), so must all points $\mu + m\alpha_j$ with $m > 0$. In other words, δ is the final point of support on the ray $\mu + m\alpha_j$, $m \in \mathbb{Z}$. Now apply Theorem 4.1.1 with $\sigma = \sigma_j$ to a_μ, where a_μ is the first coefficient on the right-hand side of the relation - this means $\mu = \sigma_j \bullet \lambda$ for some $\lambda \in \Lambda$. Here we set $n = n(\alpha_j)$ and $\delta = \delta(\lambda)$. We have $a_{\mu+m\alpha_j} = 0$ for all $m > 0$, and if $\delta \equiv 0 \mod n$ then $g_\delta(1, P) = -1$, so the value of $\delta_m(\sigma \bullet \mu)$, as far as the right-hand side is concerned, doesn't matter.

In the case $\delta \equiv 0 \pmod{n}$, applying σ_j produces the left-hand side

$$-p^{n+1} a_{\lambda - n\alpha_j} + a_\lambda.$$

Otherwise, we have

$$g_{\|\alpha\|^2\delta}(1, P)^{-1} p^{1-\delta_n} a_{\lambda - \delta_n}\alpha_j + a_\lambda.$$

One checks that a_λ vanishes by the induction hypothesis, since $l(\sigma_j w) < l(w)$ implies $\lambda = \theta - \sigma_j w\theta$ violates the inequalities active at $\theta - \sigma_j w\theta$. Again by Lemma 4.1.4, this means $a_{\lambda - n\alpha}$ and $a_{\lambda - \delta_n \alpha_j}$ also vanish. Therefore, the left-hand side is identically zero; thus, a_μ vanishes. It follows that $a_\mu = 0$ unless μ satisfies (4.10). $\square$

Figures 4, 5, and 6 below illustrate Theorem 4.1.2 for the $\Phi = A_2$ and $n = 3$ p-parts, which appeared in (2.15).

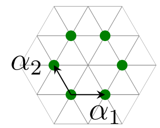

Figure 4: A_2, $N(\mathbf{x}; 0, 0)$

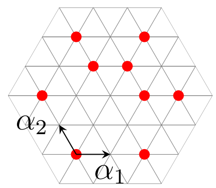

Figure 5: A_2, $N(\mathbf{x}; 1, 1)$

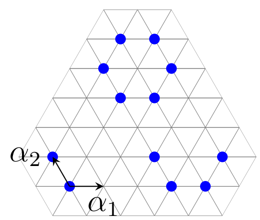

Figure 6: A_2, $N(\mathbf{x}; 3, 0)$

Our next result also concerns the support of the p-parts, but it has a different flavor than Theorem 4.1.2. Instead of the support of $N(\mathbf{x}; \ell)$, we consider the support of the rational function $f(\mathbf{x}; \ell) := \Delta(\mathbf{x})F(\mathbf{x}; \ell) = N(\mathbf{x}; \ell)\Delta(\mathbf{x})/D(\mathbf{x})$. Lemma 4.1.5 and its consequence, Corollary 4.1.6, are key tools in proving Theorem 5.2.1 in the next chapter.

Lemma 4.1.5. *The support of $j(w, \mathbf{x})(1|_\ell w)(\mathbf{x})$ is contained in the cone determined by $\Phi(w)$ and shifted by $\theta - w\theta$.*

Proof. We prove by induction on $l(w)$. When $l(w) = 0$, the statement is clear. Now assume that $l(w\sigma_k) = l(w) + 1$. We wish to show that $j(w\sigma_k, \mathbf{x})(1|_\ell w\sigma_k)(\mathbf{x})$ is supported on the cone determined by $\Phi(w\sigma_k)$ and shifted by $\theta - \sigma_k w\theta$. By our assumption on the relative lengths of w and $w\sigma_k$ one can show (see, for example, [18, Section 5.6]) that $\Phi(w\sigma_k) = \sigma_k(\Phi(w)) \cup \{\alpha_k\}$.

Since $j(w, \mathbf{x}) \in \tilde{A}_0$ for all w, it follows from (4.3) that $[j(w, \mathbf{x})f|_\ell \sigma_k](\mathbf{x}) = j(w, \sigma_k \mathbf{x})[f|_\ell(\mathbf{x})]$. By [10, Lemma 3.3], up to sign we have

$$\begin{aligned} j(w, \sigma_k \mathbf{x}) &= \prod_{\alpha \in \Phi(w)} p^{n(\alpha)d(\alpha)} (\sigma_k \mathbf{x})^{n(\alpha)\alpha} \\ &= \prod_{\alpha \in \sigma_k(\Phi(w))} p^{n(\alpha)d(\alpha)} \mathbf{x}^{n(\alpha)\alpha} \\ &= \prod_{\alpha \in \Phi(w\sigma_k) \setminus \{\alpha_k\}} p^{n(\alpha)d(\alpha)} \mathbf{x}^{n(\alpha)\alpha} \\ &= j(w\sigma_k, \mathbf{x}) / p^{n(\alpha_k)d(\alpha_k)} \mathbf{x}^{n(\alpha_k)\alpha_k} \end{aligned}$$

where we have used that $n(\alpha) = n(\sigma_k \alpha)$. From the above equality, we see

$$\begin{aligned} j(w\sigma_k, \mathbf{x})(1|_l w\sigma_k)(\mathbf{x}) &= j(w\sigma_k, \mathbf{x})[(1|_\ell w)|_l \sigma_k](\mathbf{x}) \\ &= p^{n(\alpha_k)d(\alpha_k)} \mathbf{x}^{n(\alpha_k)\alpha_k} \left(j(w, \sigma_k \mathbf{x})[(1|_\ell w)|_\ell \sigma_k](\mathbf{x}) \right) \\ &= p^{n(\alpha_k)d(\alpha_k)} \mathbf{x}^{n(\alpha_k)\alpha_k} \left([j(w, \mathbf{x})(1|_\ell w)]|_\ell \sigma_k \right)(\mathbf{x}). \end{aligned}$$

Our inductive hypothesis implies that $j(w, \mathbf{x})(1|_\ell w)(\mathbf{x})$ is supported on the cone defined by $\Phi(w)$ and shifted by $\theta - w\theta$. Thus, we can assume that

$$j(w, \mathbf{x})(1|_\ell w)(\mathbf{x}) = \sum f_\beta(\mathbf{x}),$$

where each $f_\beta(\mathbf{x})$ is a monomial supported on $\Phi(w)$ shifted by $\theta - w\theta$.

Using (2.12), we have

$$\left([j(w, \mathbf{x})(1|_\ell w)]|_\ell \sigma_k \right)(\mathbf{x}) = \sum [\mathcal{P}_{\beta,\ell,k}(x_k) + \mathcal{Q}_{\sigma_k \cdot \beta, \ell, k}(x_k)] f_\beta(\sigma_k \mathbf{x}),$$

where, up to constants a_β, $f_\beta(\sigma_k \mathbf{x}) = a_\beta p^{d(\sigma_k \beta - \beta)} \mathbf{x}^{\sigma_k \beta}$. By definition, we see that

$$\mathcal{P}_{\beta,\ell,k}(x_k) f_\beta(\sigma_k \mathbf{x}) = \mathbf{x}^{\sigma_k \bullet \beta} \mathbf{x}^{-\delta_k(\beta)_{n(\alpha_k)}} \tilde{P}(x_k)$$

and

$$\mathcal{Q}_{\sigma_k \bullet \beta, \ell, k}(x_k) f_\beta(\sigma_k \mathbf{x}) = \mathbf{x}^{\sigma_k \bullet \beta} \mathbf{x}^{-n(\alpha_k)} \tilde{Q}(x_k),$$

where $\tilde{P}(x_k)$ and $\tilde{Q}(x_k)$ are rational functions supported on the ray determined by α_k. After multiplying by $x_k^{n(\alpha_k)}$, each of these two terms is supported on $\sigma_k(\Phi(w)) \cup \alpha_k = \Phi(w\sigma_k)$. Moreover, since the original cone had been shifted to the vertex $\mathbf{x}^{w\bullet 0}$, which corresponds to w, the new cone is shifted to the vertex $\mathbf{x}^{\sigma_k \bullet w \bullet 0}$, which corresponds to a reflection under σ_k. $\square$

Using this result, we obtain the following corollary:

Corollary 4.1.6. *The support of* $\Delta(\mathbf{x})F(\mathbf{x};\ell)$ *lies outside the polytope* Π_θ*.*

Proof. The result follows from the definition of $\Delta(\mathbf{x})F(\mathbf{x};\ell) = \sum_{w\in W} j(w,\mathbf{x})(1|_\ell w)(\mathbf{x})$. $\square$

Figures 7, 8, and 9 below illustrate Corollary 4.1.6 for various ℓ when $n = 3$ and Φ is A_2 and B_2. The plotted points represent the nonzero coefficients of $f(\mathbf{x};\ell) = \Delta(\mathbf{x})F(x;\ell)$, where as before $x_1^{k_1}x_2^{k_2}$ corresponds to $\mathbf{x}^{k_1\alpha_1+k_2\alpha_2}$.

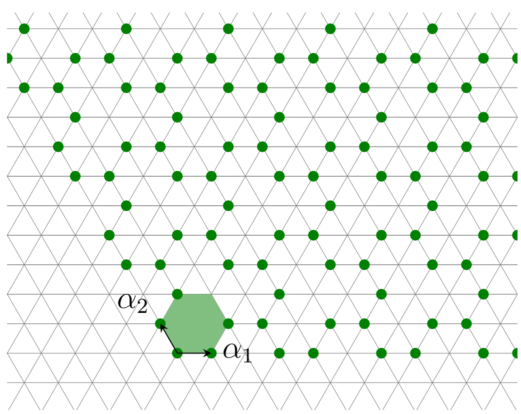

Figure 7: A_2, $f(\mathbf{x};0,0)$

4.2 Stable Coefficients

Recall that there are two main methods to define p-parts of Weyl group multiple Dirichlet series. These are the techniques of [4, 5] and [10]. In this section, we

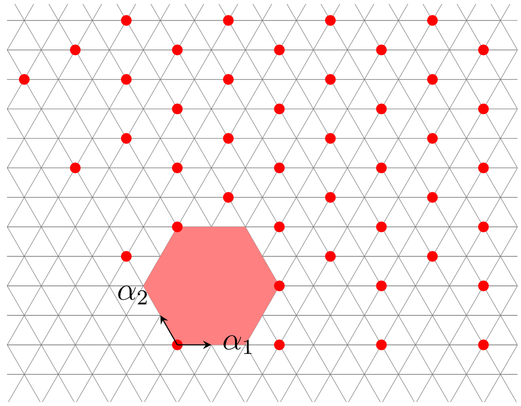

Figure 8: A_2, $f(\mathbf{x}; 1, 1)$

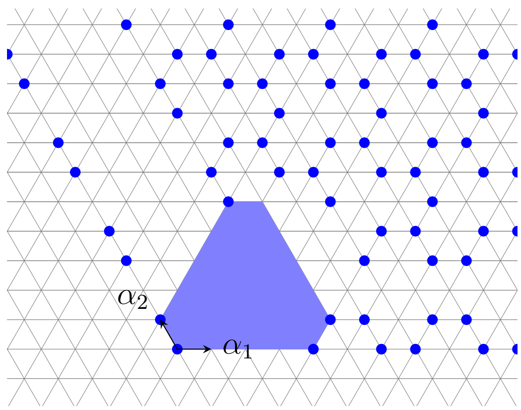

Figure 9: A_2, $f(\mathbf{x}; 3, 0)$

compare these methods in the "stable case". The stable coefficients of the p-parts are those attached to the vertices of the polytope Π_θ. In fact, when $n \gg r$, these are the only nonzero coefficients [4, 5]. In this situation, we say that we are in the stable case. For the precise stability condition, see [5, (20)] (when Φ is type A, it is enough to have $n \geq \sum_{i=1}^r (l_i + 1)$). For each ℓ, [4, 5] defines the stable coefficients by

$$A_\lambda = \prod_{\alpha \in \Phi(w^{-1})} g_{\|\alpha\|^2 d_\theta(\alpha)}(p^{d_\theta(\alpha)-1}, p^{d_\theta(\alpha)}),$$

where $\lambda = \theta - w\theta$ for some $w \in W$, and $d_\ell(\alpha) := \langle \theta, \alpha \rangle$.

Our next theorem compares the stable coefficients above with those constructed in [10]. As in [9], we caution the reader that we now make a slight change in notation: the element w now corresponds to the coefficient of $\lambda = \theta - w^{-1}\theta$. This change is to aid the comparison of the sets $\Phi(w^{-1})$ and $\Phi(\sigma_k w^{-1})$.

Theorem 4.2.1. *Assume $n \gg r$, i.e. that we are in the stable case. Let $\theta - w^{-1}\theta = \sum k_i \alpha_i$, and let $N(\mathbf{x};\ell) = \sum a_\lambda \mathbf{x}^\lambda$ be the p-part constructed via [10]. Then $a_\lambda = A_\lambda$. In other words, the stable coefficients of [4, 5] and [10] agree.*

Proof. Our proof, as in [9], is by induction on $l(w)$. Assume that $a_0 = 1$. If $l(w) = 0$, then we have an empty product and the statement holds trivially. Now, suppose that the coefficients of [4, 5] and [10] agree on all $v \in W$ with $l(v) \leq l(w)$. Let $\mu = \theta - \sigma_j w^{-1}\theta$, with $l(\sigma_j w^{-1}) = l(w^{-1}) + 1$, i.e. $\mu = \sigma_j \bullet \lambda$ and $\mu < \lambda$. Applying (4.1) or (4.2) with a_μ on the right-hand side, the outer terms vanish by Lemma 4.1.4. We have,

$$a_\mu = g_{\|\alpha_j\|^2\delta(\lambda)}(1, P)p^{\delta(\lambda)-1}a_\lambda,$$

where we have used that when $\delta(\lambda) \equiv 0 \pmod{n}(\alpha_j)$, we have $g_{\|\alpha_j\|^2\delta(\lambda)}(1, P) = -1$. Comparing with $\prod_{\alpha\in\Phi(\sigma_j w^{-1})} g_{\|\alpha\|^2\delta(\alpha)}(P^{d_\theta(\alpha)-1}, P^{d_\theta(\alpha)})$, it follows from (2.6) and our induction hypothesis that the product over $\alpha \in \Phi(w^{-1})$ is just the coefficient of a_λ. It remains to show

$$g_{\|w\alpha_j\|^2 d_\theta(w\alpha_j)}(P^{d_\theta(w\alpha_j)-1}, P^{d_\theta(w\alpha_j)}) = g_{\|\alpha_j\|^2\delta(\lambda)}(1, P)p^{\delta(\lambda)-1}.$$

In the case when $\delta \equiv 0 \pmod{n}(\alpha_j)$ we are done by property (2.4). In the case $\delta \not\equiv 0 \pmod{n(\alpha_j)}$, property (2.4) implies it is sufficient to show that $\delta_j(\lambda) = d_\theta(w\alpha_j)$. By definition, we have

$$\begin{aligned}\delta(\lambda) &= d(\mu - \lambda)\\ &= d(\left(\sum_i k_i c(i,j)\right)\alpha_j - (l_j+1)\alpha_j)\\ &= \sum k_i c(i,j) - l_j - 1\end{aligned}$$

On the other hand, we have

$$\begin{aligned} d_\theta(w\alpha_i) &= \langle \theta, w\alpha_j \rangle \\ &= \langle \sigma_j w^{-1}\theta, -\alpha_j \rangle \\ &= \langle \theta - \sum k_i\alpha_i, -\alpha_j \rangle \\ &= -l_j - 1 + \langle \sum k_i\alpha_i, \alpha_j \rangle \\ &= \sum k_i c(i,j) - l_j - 1 \end{aligned}$$

This implies the result. □

4.3 Unstable Coefficients

In the final section of this chapter, we address the extent to which $N(\mathbf{x};\ell) = \sum a_\lambda \mathbf{x}^\lambda$ is completely determined by Theorem 4.1.1. Our main result is Theorem 4.3.3, which states that the set of all such $N(\mathbf{x};\ell)$ form a $\mathbb{C}$-vector space with dimension at most the number of regular dominant weights in the representation V_θ of highest weight θ. This result extends [9, Theorem 5.7] to general n and Φ, and shows that when $\ell = (0,\dots,0)$, the p-part $N(\mathbf{x};\ell)$ is completely determined by Theorem 4.1.1 after setting $a_0 = 1$.

To prove Theorem 4.3.3, we require a few more geometric facts. Again we follow [9]. Recall that by Theorem 4.1.2, the support of $N(\mathbf{x};\ell) = \sum a_\lambda \mathbf{x}^\lambda$ consists of all $\lambda = \theta - w\xi$ such that $w \in W$ and $\xi \in \Theta$. For any $\xi \in \Theta$, define $O_\xi := \{\theta - w\xi : w \in W\}$ to be the W-orbit of the coefficient $a_{\theta-\xi}$ under the $\bullet$ action. Let $\mathcal{O} = \{O_\xi : \xi \in \Theta\}$ be the set of all such orbits. There is a natural partial order on $\mathcal{O}$ given by the poset relation on the weights: we say $O_\xi \leq O_{\xi'}$ if and only if $\xi \preceq \xi'$. Under this identification of ξ with $\theta - \xi$, the condition $\xi \preceq \xi'$ becomes

$(\theta - \xi) \succeq (\theta - \xi')$.

Lemma 4.3.1. *[9, Lemma 5.3] Let $\lambda = \theta - uP\xi$ be a vertex of Π_ξ, where $P \subset W$ is the stabilizer of ξ, and suppose $u \in uP$ is the unique maximal element in this coset. If $w = \sigma_k u$ and $l(w) > l(u)$, then $\mu - \theta - wP\xi$ is a different vertex of Π_ξ. Moreover, if any point of the form $\mu + m\alpha_k$, $m \geq 1$ lies in an orbit $O \in \mathcal{O}$, we have $O > O_\xi$. Similarly, if any point of the form $\lambda - m\alpha_k$, $m \geq 1$ lies in an orbit $O \in \mathcal{O}$, we have $O > O_\xi$.* □

A similar statement is true if σ_k fixes a vertex of Π_ξ.

Lemma 4.3.2. *[9, Lemma 5.5] Let $\lambda = \theta - uP\xi$ be a vertex of Π_ξ. Let $w = \sigma_k u$ with $l(w) > l(u)$, and suppose $\mu - \theta - wP\xi$ equals λ. Then if any point of the form $\mu + m\alpha_k$, $m \geq 1$ lies in an orbit $O \in \mathcal{O}$, we have $O > O_\xi$. Similarly, if any point of the form $\lambda - m\alpha_k$, $m \geq 1$, lies in an orbit O, we hve $O > O_\xi$.* □

We now come to the main result of this section:

Theorem 4.3.3. *Let $\ell \in (\mathbb{Z}_{\geq 0})^r$. Define Θ^+ to be the set of all regular dominant weights in the representation of highest weight θ. Suppose that the coefficients $a_{\theta-\xi}$ of $N(\mathbf{x}; \ell)$ are known for all $\xi \in \Theta^+$. Then, $N(\mathbf{x}; \ell)$ is completely determined by the relations* (4.1), (4.2) *after setting $a_0 = 1$.*

Proof. We induct on the poset $\mathcal{O}$. First, we know all the points in the orbit O_θ by Theorem 4.2.1. Let $\xi \in \Theta$ with $\xi \neq \theta$, and assume we know all of the coefficients for orbits $O > O_\xi$. There are two possibilities, either ξ is regular (meaning $\#O_\xi = \#W$), or not.

Suppose first that ξ is regular, and let $\lambda = \theta - \xi$. By assumption the value of a_λ, the coefficient associated with ξ, is known. We must show that we can determine a_δ for all $\delta \in O_\xi$. Any such δ is of the form $\theta - w\xi = w \bullet \lambda$, $w \in W$ and thus can be obtained by successively applying simple reflections σ_j. From relation (4.1) or (4.2), we see that when we apply σ_j to a_λ, the outer term on the left-hand side is either $\theta-(\xi+\mu_n\alpha_j)$, or $\theta-(\xi+n\alpha_j)$, where $n=n(\alpha_j)$ and $\nu=n-\mu(\lambda)_n$. In both cases, these terms come from a previously determined orbit by Lemma 4.3.1, $O_{\xi'}$, where $O_{\xi'}>O_\xi$. Applying $\sigma_j \bullet (\theta-\xi')$ we get the outer term on the right-hand side, so this term also belongs to $O_{\xi'}$ and hence is previously determined. Since we know three out of four terms of the relation, a_δ is determined.

Now, assume that ξ is not regular. Since $\#O_\xi < \#W$, there exists a simple reflection σ_j such that a_λ is taken to itself under relation (4.1) or (4.2). By Lemma 4.3.2, all other $a_{\lambda'}$ involved in the recurrence are predetermined; therefore we know a_λ. We may now successively apply (4.1) or (4.2) to determine the remaining coefficients $\theta-w\xi$ in the orbit O_ξ. □

Corollary 4.3.4. *When $\ell=(0,\ldots,0)$, $N(\mathbf{x};\theta)$ is completely determined by the relations from (1) after setting the constant term to 1.*

Proof. The proof is identical to that of [9, Corollary 5.8]. □

CHAPTER 5

GLOBAL RESULTS

In this chapter, we state our global results. These results describe the relationship between Weyl group multiple Dirichlet series over the rational function field and their p-parts. Section 5.1 discusses the untwisted case, and Section 5.2 considers the twisted case. Combining the results of these two sections gives an analogy between Weyl group multiple Dirichlet series over the rational function field and characters of representations of $\mathfrak{g}$, which we will discuss in more detail in Section 5.2. We end this chapter with an example.

In what follows, we abuse notation and refer to $F(\mathbf{x}; \ell)$, defined in (2.13), instead of $N(\mathbf{x}; \ell)$, as the p-parts (cf. Section 2.3). Let $X_i = q^{-s_i}$. Under this identification, we put $\mathcal{Z}^*(\mathbf{X}; \mathbf{m}) = Z^*(\mathbf{s}; \mathbf{m})$.

5.1 The Untwisted Case

To prove Theorem 5.2.1, we exploit the similarity between untwisted Weyl group multiple Dirichlet series over $\mathbb{F}_q(T)$ and their untwisted p-parts. Recall that we say $Z(\mathbf{s}; \mathbf{m})$ is untwisted when $\mathbf{m} = (1, \ldots, 1)$. Similarly, the untwisted p-part corresponds to $\ell = (0, \ldots, 0)$. A simple variable change transforms one into the other. Chinta [8] previously noted this fact for arbitrary n and $\Phi = A_2$, as did

Mohler [23] for $n \gg r$ and $\Phi = A_r$. Proposition 5.1.1 gives the exact relationship between the untwisted p-parts and global series for general Φ and n.

Proposition 5.1.1. *Let $\tau(\epsilon^k)$ and $g_k(1,P)$ be the Gauss sums defined by* (2.1) *and* (2.3)*, respectively. Let $\tilde{F}(\mathbf{X};0,\dots,0)$ denote $F(\mathbf{x};0,\dots,0)$ after the variable change*

$$\begin{cases} s_i & \mapsto \quad 2-s_i \\ p=|P| & \mapsto \quad 1/q \\ g_k^*(1,P) & \mapsto \quad \tau(\epsilon^k) \end{cases}, \tag{5.1}$$

Then, we have

$$\mathcal{Z}^*(\mathbf{X};1,\dots,1) = \tilde{F}(\mathbf{X};0,\dots,0).$$

In other words, (5.1) *transforms the untwisted p-part to the rational function corresponding to the untwisted Weyl group multiple Dirichlet series over $\mathbb{F}_q(T)$.*

Proof. Notice that (5.1) takes the functional equations of the p-parts (2.14) to the functional equations of $\mathcal{Z}^*(\mathbf{X};\mathbf{m})$ (3.1). To see this, let $\beta = \sum_{j=1}^r \beta_j\alpha_j$, and let $(\sigma_i \bullet \beta)_j$ denote the α_j coefficient of $\sigma_i \bullet \beta$. A simple computation shows that

$$(\sigma_i \bullet \beta)_j = \begin{cases} \beta_j & \text{if } i \neq j, \\ 1-\beta_i-\sum_{j\neq i}\beta_j c(j,i) & \text{if } i=j. \end{cases}$$

It follows that $\delta_i(\beta) = 1-\sum_{j\neq i} c(j,i)\beta_j - 2\beta_i$. Put $I=\beta_i$ and $J = -\sum_{j\neq i} c(j,i)\beta_j$. Then, (5.1) takes $\mathcal{P}_{\beta,\mathbf{0},i}(x_i)$ and $\mathcal{Q}_{\beta,\mathbf{0},i}(x_i)$ of (2.11) to $\tilde{P}_{I,J}^{\|\alpha_i\|^2}(s_i)$ and $\tilde{Q}_{I,J}^{\|\alpha_i\|^2}(s_i)$ of (3.4), respectively.

Furthermore, these two functions have the same polar behavior. Notice that (5.1) takes $\Delta(\mathbf{x})^{-1}$ to $\Xi(\mathbf{X})$. By definition, we have

$$\Delta(\mathbf{x}) = \prod_{\alpha>0}(1-p^{n(\alpha)d(\alpha)}\mathbf{x}^{n(\alpha)}),$$

and

$$\Xi(\mathbf{X}) = \prod_{\alpha=\sum k_i\alpha_i>0} \zeta_{\mathcal{O}}(1+n(\alpha)\sum_{i=1}^{r} k_i(s_i-1)).$$

Using that $\zeta_{\mathcal{O}}(s) = \frac{1}{1-q^{1-s}}$ we rewrite

$$\Xi(\mathbf{X}) = \prod_{\alpha=\sum k_i\alpha_i>0} \frac{1}{1-q^{1-(1-n(\alpha)d(\alpha)+n(\alpha)\sum_{i=1}^{r} k_i s_i)}} = \prod_{\alpha>0} \frac{1}{1-q^{n(\alpha)d(\alpha)}\mathbf{X}^{n(\alpha)}}.$$

The claim now follows from the fact that (5.1) sends $p^n x_i^n$ to $q^n X_i^n$.

We have shown that both $\tilde{F}(\mathbf{X};0,\ldots,0)$ and $\mathcal{Z}^*(\mathbf{X};1,\ldots,1)$ satisfy the same functional equations and have the same polar behavior. In addition, both have constant term equal to one. Applying Theorem 4.3.3 and Corollary 4.3.3, both functions are uniquely determined by their functional equations; thus, they must be equal. □

If we replace $l_i = \deg m_i$ for $i = 1,\ldots,r$ in the proof above, we see that even in the twisted case, Weyl group multiple Dirichlet series over the rational function field and their p-parts satisfy (up to (5.1)) the same functional equations. It follows that many of the results of Chapter 4 apply to the global series. In particular, Theorem 4.3.3 shows that, up to a finite number of coefficients, $\mathcal{Z}^*(\mathbf{X};\mathbf{m})$ is completely determined by its functional equations.

5.2 The Twisted Case

We are now able to state the main contribution of this chapter, Theorem 5.2.1. This result gives an analogy between Weyl group multiple Dirichlet series over the rational function field and characters of representations of $\mathfrak{g}$. In particular, Theorem 5.2.1 shows that we can uniquely write $\mathcal{Z}^*(\mathbf{X};\mathbf{m})$ as a weighted sum of p-parts $\tilde{F}(\mathbf{X};\ell)$.

Theorem 5.2.1. *Let $\mathbf{m} \in \mathcal{O}^r$, and put $\ell = (\deg m_1, \ldots, \deg m_r)$. For ease of notation, let $\tilde{F}(\mathbf{X};\theta)$ denote $F(\mathbf{x};\ell)$ after applying* (5.1)*, where θ is given by* (2.7)*. Let Θ be the set of dominant weights in the representation of highest weight θ, and let $\Theta^+ \subset \Theta$ be the subset of regular dominant weights. Then, we have*

$$\mathcal{Z}^*(\mathbf{X};\mathbf{m}) = \sum_{\xi\in\Theta^+} M_{\theta-\xi}\tilde{F}(\mathbf{X};\xi)\mathbf{X}^{\theta-\xi},$$

where for $\lambda = \sum_{i=1}^r \lambda_i\alpha_i$, the coefficients M_λ are the character sums

$$M_\lambda = \sum_{\substack{\mathbf{c}\in(\mathcal{O}_{mon})^r \\ \deg c_i=\lambda_i}} H(\mathbf{c};\mathbf{m}).$$

Remark. The M_λ are the $\mathbf{X}^\lambda$ coefficients of $Z(\mathbf{s};\mathbf{m})$ — the original series, without the normalizing factors — expressed as a power series in $X_i = q^{-s_i}$.

Proof. The proof is in three steps. The first two show that the set $\{\tilde{F}(\mathbf{X};\lambda) : \lambda \in \Theta^+\}$ forms $\mathbb{C}$-basis for $\mathcal{Z}^*(\mathbf{X};\mathbf{m})$. In the third step, we use Lemma 4.1.5 to compute the coefficients M_λ.

We begin by showing that any rational function whose denominator equals the that of $\mathcal{Z}^*(\mathbf{X};\mathbf{m})$ and that satisfies functional equations of the form (3.1), can be written as

$$\sum_{\xi\in\Theta^+} m_\xi\tilde{F}(\mathbf{X};\xi)\mathbf{X}^{\theta-\xi}, \qquad m_\xi \in \mathbb{C}. \tag{5.2}$$

Let $\mathcal{F}(\mathbf{X}) = \mathcal{N}(\mathbf{X})/\mathcal{D}(\mathbf{X})$ be a function satisfying these hypotheses, and write $\mathcal{N}(\mathbf{X}) = \sum_\lambda b_\lambda\mathbf{X}^\lambda$. By proposition 5.1.1 and Theorem 4.3.3, the polynomial $\mathcal{N}(\mathbf{X})$ is completely determined by the values $\{b_{\theta-\xi} : \xi \in \Theta^+\}$ and the recurrence relations (4.1) and (4.2).

Recall that all p-parts have constant coefficient equal to one. Let $m_\theta := b_0$ be the constant coefficient of $\mathcal{N}(\mathbf{X})$. By Theorem 4.2.1, the stable coefficients of

$\tilde{N}(\mathbf{X};\theta)$ and $m_\theta\mathcal{N}(\mathbf{X})$ agree. Write

$$\mathcal{N}(\mathbf{X}) = m_\theta\tilde{N}(\mathbf{X};\theta) + E_\theta(\mathbf{X}),$$

where $E_\theta(\mathbf{X})$ is a polynomial supported on the orbits $O_{\xi'} = \{\mathbf{X}^{\theta - w\xi'} : w \in W\}$. All such ξ' satisfy $\xi' \prec \theta$, since θ is the unique maximal element of Θ. Let $\mathcal{S}_\theta$ be the set of all such ξ'. Choose a maximal element $\xi \in \mathcal{S}_\theta$ with respect to the partial order on L. If ξ is not regular, then there exists a simple reflection taking ξ to itself. It follows that the $\mathbf{X}^\xi$ coefficients of both $\mathcal{N}(\mathbf{X})$ and $\tilde{N}(\mathbf{X};\theta)$ are completely determined by the (3.1) together with the coefficients associated to orbits $O_{\xi''}$, where ξ'' is such that $\xi'' \succ \xi$. If the $\mathbf{X}^\xi$ coefficient of $E_\theta(\mathbf{X})$ is nonzero, we contradict the maximality of ξ. Thus, ξ is regular. In addition, we insist that $\xi \in \Theta$, which implies ξ is the unique maximal element of $\mathcal{S}_\theta$.

Maximality of $\xi \in \mathcal{S}_\theta$ implies that $E_\theta(\mathbf{X})\mathbf{X}^{\xi-\theta}$ satisfies a "global version" of the relations (4.1) and (4.2) corresponding to taking $\theta = \xi$. Let m_ξ be the constant coefficient of $E_\theta(\mathbf{X})\mathbf{X}^{\xi-\theta}$, and write

$$E_\theta(\mathbf{X}) = m_\xi\tilde{N}(\mathbf{X};\xi)\mathbf{X}^{\theta-\xi} + E_\xi(\mathbf{X}),$$

where $E_\xi(\mathbf{X})$ is supported on the orbits $O_{\xi'}$ with $\xi' \prec \xi$. By the same argument as above, the maximal such ξ' is regular. Continuing in this fashion, after finitely many ($\#\Theta^+$) iterations, our error term will vanish. This proves the first statement.

Now we claim that any rational function of the form (5.2) with denominator equal to that of $\mathcal{Z}^*(\mathbf{X};\mathbf{m})$ satisfies the functional equations (3.1). It is enough to show that for all $\xi \in \Theta^+$, the p-parts $\tilde{F}(\mathbf{X};\xi)$ satisfy the same functional equations as $\tilde{F}(\mathbf{X};\theta)$. This claim follows immediately from [12] and [22]. For Φ type A, Chinta and Offen [12] define a Weyl group action on rational functions that is independent of θ. Equation [12, (9.2)] shows that this θ-independent action is equivalent to the action (2.12) of [10]. In [22], McNamara generalizes the action of [12] to all Φ. For the reader's convenience, we also provide a direct proof.

To simplify notation, we work with the p-parts directly. It is enough to show that $F(\mathbf{x};\xi)\mathbf{x}^{\theta-\xi}$ is invariant under the $|_\theta w$ action. Note by definition, $F(\mathbf{x};\xi)$ is invariant under the $|_\xi w$ action. By definition (2.8), for $w \in W$, we have

$$w \bullet_\xi (\lambda) + (\theta - \xi) = w \bullet_\theta \lambda + (\theta - \xi). \tag{5.3}$$

Let $f_\beta(\mathbf{x}) = \mathbf{x}^\beta \in A_\beta$. We claim (5.3) implies

$$(f|_\xi w)(\mathbf{x})\mathbf{x}^{\theta-\xi} = (f\mathbf{x}^{\theta-\xi}|_\theta w)(\mathbf{x}).$$

To see this, let σ_k be a simple reflection. Writing $\xi = \sum(l_i+1)\varpi_i$, we put $(\xi)_k = l_k$. Recall that $\mathcal{P}_{\beta,\xi,k}$ and $\mathcal{Q}_{\beta,\xi,k}$ depend only on $\delta_{\xi,k}(\beta)$ and l_k. It follows from (5.3) that $\delta_{\theta,k}(\beta+\theta-\xi) = \delta_{\xi,k}(\beta)$. Let $C = -\sum_i c(i,k)$. A simple calculation shows that $(qx_k)^C = (f_{\beta+\theta-\xi}(\sigma_k\mathbf{x}))/(f_\beta(\sigma_k\mathbf{x})\mathbf{x}^{\theta-\xi})$.

Now write $\theta = \sum(a_j+1)\varpi_j$, $\xi = \sum(b_j+1)\varpi_j$ and $\theta - \xi = \sum k_i\alpha_i$. (All weights of the representation of $\mathfrak{g}$ of highest weight θ are an integral root distance from θ.) Using the Cartan matrix to base change, we have $(\theta)_k - (\xi)_k = \sum k_i c(i,k) = -C$. The claim now follows from the definition of the action in (2.12).

Using Theorem 2.2.1, for all $u \in W$, we have

$$\left(\left((1|_\xi w)(\mathbf{x})\mathbf{x}^{\theta-\xi}\right)|_\theta u\right)(\mathbf{x}) = (1|_\xi wu)(\mathbf{x})\mathbf{x}^{\theta-\xi}.$$

This proves the second statement.

It remains to identify the coefficients M_λ. We now show that $m_\xi = M_{\theta-\xi}$, where M_λ is the $\mathbf{X}^\lambda$ coefficient of $Z(\mathbf{s};\mathbf{m})$, written as a power series in the $X_i = q^{-s_i}$.

For each $\xi \in \Theta^+$, we have $\tilde{F}(\mathbf{X};\xi) = \tilde{f}(\mathbf{X};\xi)/\tilde{\Delta}(\mathbf{X})$. It follows that

$$\sum_{\xi\in\Theta^+} m_\xi \tilde{F}(\mathbf{X};\xi) = \frac{1}{\tilde{\Delta}(\mathbf{X})}\sum_\xi m_\xi \tilde{f}(\mathbf{X};\xi)\mathbf{X}^{\theta-\xi}. \tag{5.4}$$

Since $\mathcal{Z}^*(\mathbf{X};\mathbf{m}) = \Xi(\mathbf{s})Z(\mathbf{s};\mathbf{m})$ and $\Xi(\mathbf{s}) = \tilde{\Delta}(\mathbf{X})^{-1}$, from (5.4) we have

$$Z(\mathbf{s};\mathbf{m}) := \sum_{\lambda} M_\lambda \mathbf{x}^\lambda = \sum_{\xi\in\Theta+} m_\xi \tilde{f}(\mathbf{X};\xi)\mathbf{X}^{\theta-\xi}.$$

Writing both sides as power series in the X_i, the constant coefficient of every series on the right-hand side is one. To equate m_ξ with $M_{\theta-\xi}$, we induct on the weight of ξ. It is clear that $m_\theta = M_0 = 1$. Choose $\xi \prec \theta$, and suppose that $m'_\xi = M_{\theta-\xi'}$ for all $\xi' \succ \xi$. By Lemma 4.1.5, the $\tilde{f}(\mathbf{X};\xi')\mathbf{X}^{\theta-\xi'}$ are supported outside or on the boundary of the Π'_ξ. This means that scaling $\tilde{f}(\mathbf{X};\xi')\mathbf{X}^{\theta-\xi'}$ by $M_{\theta-\xi'}$ does not affect the coefficients of $\tilde{f}(\mathbf{X};\xi)\mathbf{X}^{\theta-\xi}$. Since the constant coefficient of $\tilde{f}(\mathbf{X};\xi)\mathbf{X}^{\theta-\xi}$ is one, we must have $m_\xi = M_{\theta-\xi'}$. Combining this statement with the first two claims concludes the proof. $\square$

Theorem 5.2.1 gives an analogy between Weyl group multiple Dirichlet series over the rational function field and characters of representations of the semisimple complex Lie algebra $\mathfrak{g}$ associated to Φ. This analogy makes use of the analogy discussed in Section 2.2 between the p-parts of Weyl group multiple Dirichlet series and characters of highest weight representations of $\mathfrak{g}$. We note that unlike usual characters of representations of $\mathfrak{g}$, whose expression in terms of character of highest weight representations usually involves a sum over all dominant weights, here we need only look at the "metaplectic" symmetric functions (i.e. the p-parts) corresponding to the *regular* dominant weights.

Theorem 5.2.1 gives a way to compute the rational functions $\mathcal{Z}^*(\mathbf{s};\mathbf{m})$ by computing finitely many series coefficients of $Z(\mathbf{s};\mathbf{m})$ and finitely many p-parts. Of course, computing the p-parts is implicit in computing the series coefficients, as these must be known in order to determine the $H(\mathbf{c};\mathbf{m})$. We hope that this result will prove useful for computing examples, which is important to both formulating and testing conjectures. At present, there are several techniques for computing the p-parts, and there is active work to show that these different methods coincide with the approach of [10] that we have outlined here. A computationally efficient technique to compute the p-parts would give, via our theorem, a relatively fast way to compute examples of Weyl group multiple Dirichlet series over the rational function field.

5.3 An Example

In this section, we apply Theorem 5.2.1 to an example for $\Phi = A_2$. We take $q = 7$ and $n = 3$. Note that $q \equiv 1 \pmod{2n}$. We choose $\mathbf{m} = (T^3 + 5T + 2, 1)$, where $T^3 + 5T + 2$ is irreducible over $\mathbb{F}_7$.

To apply Theorem 5.2.1, we put $l_i = \deg m_i$ and $\theta = 4\varpi_1 + \varpi_2$. Note that $\alpha_1 = 2\varpi_1 - \varpi_2$ and $\alpha_2 = 2\varpi_2 - \varpi_1$. One can show

$$\Theta^+ = \{\theta, 2(\varpi_1 + \varpi_2), \varpi_1 + \varpi_2\},$$

and

$$\{\theta - \xi : \xi \in \Theta^+\} = \{0, \alpha_1, 2\alpha_1 + \alpha_2\}.$$

Let $\mathbf{X} = (X_1, X_2)$ where $X_i = q^{-s_i}$. Theorem 5.2.1 implies that

$$\mathcal{Z}^*(\mathbf{X}; \mathbf{m}) = M_0 \tilde{F}(\mathbf{X}; 3, 0) + M_{\alpha_1} \tilde{F}(\mathbf{X}; 1, 1) X_1 + M_{2\alpha_1+\alpha_2} \tilde{F}(\mathbf{X}; 0, 0) X_1^2 X_2, \quad (5.5)$$

where

$$M_0 = H(1, 1; \mathbf{m}) = 1,$$

$$M_{\alpha_1} = \sum_{c_1 \deg 1} H(c_1, 1; \mathbf{m}),$$

$$M_{2\alpha_1+\alpha_2} = \sum_{c_1 \deg 2} \sum_{c_2 \deg 1} H(c_1, c_2; \mathbf{m}).$$

The M_λ can now be computed directly (preferably, using a computer algebra system).

To compute M_{α_1}, note that by degree constraints, all c_1 in the sum are necessarily relatively prime to $m_1 = T^3+5T+2$ and $m_2 = 1$. Using twisted multiplicativity, we have $H(c_1, 1; T^3 + 5T + 2, 1) = H(c_1, 1; 1, 1)\left(\frac{T^3+5T+2}{c_1}\right)^{-1}$. Since $\deg c_1 = 1$ implies c_1 is irreducible, the coefficient $H(c_1, 1; 1, 1)$ is simply the x_1 coefficient of the p-part $N(\mathbf{x}; (0, 0))$ — this is $g(1, p)$. It follows

$$M_{\alpha_1} = \sum_{\deg c_1=1} g(1, c_1)\left(\frac{T^3 + 5T + 2}{c_1}\right)^{-1}.$$

Computing this term in Magma gives $M_{\alpha_1} = \tau(\epsilon)(-0.5 + 2.598i)$.

Computing $M_{2\alpha_1+\alpha_2}$ is similar. Again, by degree constraints, all c_1, c_2 in the sum are relatively prime to (m_1, m_2), so we have $H(c_1, c_2; T^3 + 5T + 2, 1) = H(c_1, 1; 1, 1)\left(\frac{T^3+5T+2}{c_1}\right)^{-1}$. Moreover, $\deg c_2 = 1$ implies c_2 is irreducible. To compute $H(c_1, c_2; \mathbf{m})$, we decompose c_1 into irreducibles and consider the corresponding p-parts. Table 1 below shows the computation of $H(c_1, c_2; 1, 1)$ in each of the different cases. We denote by N_P the p-part $N(\mathbf{x}; (0, 0)$ corresponding to the prime P.

Table 1: Computing $H(c_1, c_2; 1, 1)$

c_1, such that	$H(c_1, c_2; 1, 1)$
c_1, c_1 is irreducible	$[x_1]N_{c_1} * [x_2]N_{c_2} = g(c_1)g(c_2)\left(\frac{c_1}{c_2}\right)^{-1}$
$(c_2)^2$, c_2 is irreducible	$[x_1^2x_2]N_{c_2} = \lvert c_2\rvert^2$
P^2, $P \neq c_1$	$[x_1^2]N_P * [x_2]N_{c_2} = 0$
P_1P_2, $P_1 \neq P_2, P_i \neq c_1$	$[x_1]N_{P_1} * [x_1]N_{P_2} * [x_2]N_{c_2} = g(P_1)g(P_2)g(c_2)\left(\frac{P_1}{P_2}\right)^2\left(\frac{P_1}{c_2}\right)^{-1}\left(\frac{P_2}{c_2}\right)^{-1}$
Pc_2, $P \neq c_2$	$[x_1]N_P * [x_1x_2]N_{c_2} = 0$

Using Magma we compute $M_{2\alpha+\alpha_2} \approx \tau(\epsilon)^3(6.5 + 2.598i)$.

Remark. To simplify the computation of the Gauss sums, we used the following very useful fact: for $c \in A$, let $\mu(c)$ denote the usual mobius function, which takes value 1 if c is square free and has an even number of irreducible factors, -1 if c is square free and has an odd number of irreducible factors, and 0 otherwise. Then [24, Theorem 2.1]

$$g(1,c) = \mu(c)\left(\left(\frac{c'}{c}\right)_3\right)(-\tau(\epsilon))^{\deg c}.$$

Remark. We note that M_{α_1} and $M_{2\alpha_1+\alpha_2}$ are algebraic. In fact, they live in the compositum of $\mathbb{Q}(\zeta_3)$ and $\mathbb{Q}(\zeta_7+\zeta_7^{-1})$, where ζ_a denotes a primitive ath root of unity.

Let $\tau_1 = \tau(\epsilon)$ and $\tau_2 = \tau(\epsilon^2)$. Applying (5.1) to the p-parts of (2.15), we have

$$\begin{aligned}
\tilde{N}(\mathbf{X};0,0) &= 1 + q\tau_1 X_1 + q\tau_1 X_2 + q^4 X_1^2 X_2 + q^4 X_1 X_2^2 + q^5\tau_1 X_1^2 X_2^2\\
\tilde{N}(\mathbf{X};1,1) &= 1 + q^2\tau_2 X_1^2 + q^2\tau_2 X_2^2 + q^5\tau_2 X_1^3 X_2^2 - q^6\tau_2 X_1^3 X_2^2 + q^7 X_1^4 X_2^2 + q^5\tau_2 X_1^2 X_2^3\\
&\quad - q^6\tau_2 X_1^2 X_2^3 + q^7 X_1^3 X_2^3 - q^8 X_1^3 X_2^3 + q^7 X_1^2 X_2^4 + q^9\tau_2 X_1^4 X_2^4\\
\tilde{N}(\mathbf{X};3,0) &= 1 + q^3 X_1^3 - q^4 X_1^3 + q^4\tau_1 X_1^4 + q\tau_1 X_2 + q^4\tau_1 X_1^3 X_2 - q^5\tau_1 X_1^3 X_2 + q^7 X_1^5 X_2\\
&\quad + q^6 X_1^3 X_2^3 - q^7 X_1^3 X_2^3 + q^7\tau_1 X_1^4 X_2^3 - q^8\tau_1 X_1^4 X_2^3 + q^7\tau_1 X_1^3 X_2^4 - q^8\tau_1 X_1^3 X_2^4\\
&\quad + q^{10} X_1^5 X_2^4 - q^{11} X_1^5 X_2^4 + q^{10} X_1^4 X_2^5 + q^{11}\tau_1 X_1^5 X_2^5
\end{aligned}$$

For a monomial $\mathbf{X}^\lambda$ and polynomial $f(\mathbf{X})$, let $[\mathbf{X}^\lambda]f(\mathbf{X})$ denote the corresponding coefficient of $f(\mathbf{X})$. As a check, we also compute independently the coefficients $[X_1X_2^2]Z$ and $[X_1^2X_2^2]Z$. Using similar methods as outline above, we use Magma to compute that $[X_1X_2^2]Z = -171.5-891.140i$ and $[X_1^2X_2^2]Z = -1453.720+1910.890i$.

The rational function $D(\mathbf{X})$ and power series expansion of $\Xi(\mathbf{X})$ involve all higher order terms, so $[X_1X_2^2]Z$ and $[X_1^2X_2^2]Z$ should agree with the corresponding coefficients on the right hand side of (5.5). It follows from the theorem that we should have

$$[X_1X_2^2]Z = [X_1X_2^2]\tilde{N}(\mathbf{X};3,0) + [X_2^2]M_{\alpha_1}\tilde{N}(\mathbf{X};1,1)X_1,$$

or equivalently, that $[X_1 X_2^2]Z = q^2 \tau_2 M_{\alpha_1}$. Similarly, we should have

$$[X_1^2 X_2^2]Z = [X_1^2 X_2^2]\tilde{N}(\mathbf{X}; 3, 0) + [X_1 X_2^2] M_{\alpha_1} \tilde{N}(\mathbf{X}; 1, 1) X_1 + [X_2] M_{2\alpha_1+\alpha_2} \tilde{N}(\mathbf{X}; 0, 0) X_1^2 X_2,$$

or equivalently, that $[X_1^2 X_2^2]Z = q\tau_1 M_{2\alpha_1+\alpha_2}$. It's easy to check that these identities hold.

Figure 10 below shows the support of the numerator of the global series $Z^*(\mathbf{X}; \mathbf{m})$, expressed in terms of the p-parts. The support of $\tilde{N}(\mathbf{X}; 3, 0)$ is shown in blue, the support of $M_{\alpha_1} \tilde{N}(\mathbf{X}; 1, 1) X_1$ is shown in red, and the support of $M_{2\alpha_1+\alpha_2} \tilde{N}(\mathbf{X}; 0, 0) X_1^2 X_2$ is shown in green.

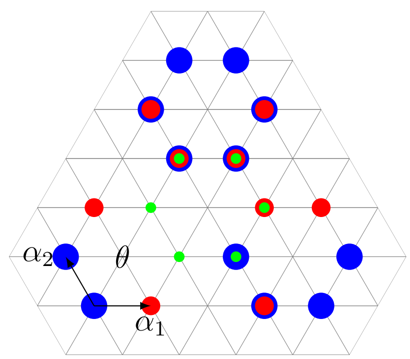

Figure 10: The support of the numerator of $Z^*(\mathbf{X}; \mathbf{m})$

A P P E N D I X

MAGMA CODE

```
/*Program to compute coefficients of Weyl group mutliple Dirichlet series.
 Input required: q=p^n*/

/*Defines the complex field*/
C<i>:=ComplexField();
/*Defines the finite field with q elements*/
k<t>:=GF(q);
/*Defines \pi*/
PI:=Pi(C);
/*Define the integers*/
Z:=IntegerRing();
/*Defines the rational fucntion field over F_q*/
r<X>:=FunctionField(k);
/*Defines the polynomial ring over F_q*/
A<y>:=PolynomialRing(r);
/*Defines the polynomial ring over the rational function field over F_q*/
RA<Y>:=PolynomialRing(k);
/*Defines the rational function field as an algebraic function field
(in the language of Magma), i.e. as a finite extension of the rational
function field*/
R<x>:=FunctionField(y-X);
/*Sets a primitive cube root of unity of F_q*/
CRU:=RootOfUnity(3,k);
/*Our choice of twisting parameter*/
m1:=x^3+5*x+2;
Pl1:=Places(R,1);
pl1:=#Pl1;
Pl2:=Places(R,2);
pl2:=#Pl2;

N:=function(P);
```

```
/*Given a place P of F_q(T), output the norm of the corresponding monic,
 irreducible polynomial*/
        return q^(Degree(P));
end function;

/*Given a place P of F_q(T), output the corresponding monic, irreducible
 polynomial*/
ul:=function(P);
        return r!UniformizingElement(P);
end function;


epsilon:=function(a);
/*embeds the cube roots of unity mu_3 of F_q into C*/
        if a eq CRU then
                return Exp(4*PI*i/3);
                else if a eq CRU^2 then
                        return Exp(2*PI*i/3);
                        else if a eq CRU^3 then
                                return 1;
                                else
                                return 0;
                        end if;
                end if;
        end if;
end function;

chicube:=function(a);
/*multiplicative character from F_q->mu_3*/
        if a ne 0 then
                return a^(Z!((q-1)/3));
        else
                return 0;
        end if;
end function;

cr:=function(a);
/*mbeds chi(F_q)->C*/
        if chicube(a) ne 0 then
                return epsilon(chicube(a));
        else
                return 0;
        end if;
```

```
end function;

cres:=function(a,P);
/*Given a place P and a polynomial a, computes the cubic residue a on P*/
F<y>,m:=ResidueClassField(P);
        if m(a) ne 0 then
                b:=m(a^(Z!((N(P)-1)/3)));
                return epsilon(b);
        else
                return 0;
        end if;
end function;

cres2:=function(a,P);
/*Given a place P and a polynomial a, computes the cubic residue a on P,
 specific to rational function field --- agrees with cres*/
        u:=RA!UniformizingElement(P);
        n:=RA!a;
        b:=Resultant(u,n);
        return epsilon(chicube(b));
end function;

cresd:=function(P);
/*Given a place P, computes the cubic residue of P' on P, where P' is the
 derivative of the polynomial corresponding to P*/
F<y>,m:=ResidueClassField(P);
        b:=m(Derivative(ul(P))^(Z!((N(P)-1)/3)));
        return epsilon(b);
end function;

cresd2:=function(P);
/*Given a place P, computes the cubic residue of P' on P, where P' is the
 derivative of the polynomial corresponding to P, specific to rational
 function field --- agrees with cresd*/
        u:=RA!UniformizingElement(P);
        b:=Resultant(u,Derivative(u));
        return epsilon(chicube(b));
end function;

tau:=function(FD);
/*Computes the finite field Gauss sum for F_q, using the embedding epsilon*/
        t:=0;
        for j in FD do
```

```
                        t+:=cr(j)*Exp(2*PI*i*Z!TraceAbs(j)/Characteristic
(FD));
        end for;
        return t;
end function;

taupow:=function(FD,a);
/*Computes the finite field Gauss sum for F_q, using the embedding
 epsilon^a*/
        t:=0;
        for j in FD do
                if j ne 0 then
                        t+:=(cr(j)^a)*Exp(2*PI*i*Z!TraceAbs(j)/
Characteristic(FD));
                end if;
        end for;
        return t;
end function;

gaupr:=function(P);
/*Computes the Gauss sum for a place P using Patterson's Theorem 2.1*/
        return -cresd(P)*(-tau(k))^(Degree(P));
end function;

M10:=function(m);
/*Given a twisting paremeter m=(m1,1), m1 irreducible, compute the x
 series coefficient of Z(s,m)*/
s:=0;
        for j in [2..#Pl1] do
                if cres(m,Pl1[j]) ne 0 then
                        s+:=gaupr(Pl1[j])*cres(m,Pl1[j])^(-1);
                end if;
        end for;
        return s;
end function;

M21:=function(m);
/*Given a twisting parameter m=(m1,1), m1 irreducible, compute the x^2y
 series coefficient of Z(s,m)*/
s:=0;
        /*Run over all 2|1*/
        for j1 in [1..#Pl2] do
                for j2 in [2..#Pl1] do
```

```
                        if cres(m,Pl2[j1]) ne 0 then
                                s+:=gaupr(Pl2[j1])*gaupr(Pl1[j2])*cres(ul
(Pl2[j1]),Pl1[j2])^(-1)*cres(m,Pl2[j1])^(-1);
                        end if;
                end for;
        end for;
        /*Run over all (11|1)*/
        for j1 in [2..#Pl1] do
                        if cres(m,Pl1[j1]) ne 0 then
                                s+:=q^(2)*cres(m,Pl1[j1])^(-2);
                        end if;
        end for;
        /*Run over all 11|1*/
        for j1 in [2..#Pl1] do
                for j2 in [j1+1..#Pl1] do
                        for j3 in [2..#Pl1] do
                                if j2 ne j3 and j1 ne j3 then
                                if cres(m,Pl1[j1])*cres(m,Pl1[j2]) ne 0 then
                                        s+:=gaupr(Pl1[j1])*gaupr(Pl1[j2])
*gaupr(Pl1[j3])*cres(ul(Pl1[j1]),Pl1[j2])^2*cres(ul(Pl1[j1]),Pl1[j3])^(-1)
*cres(ul(Pl1[j2]),Pl1[j3])^(-1)*cres(m,Pl1[j1])^(-1)*cres(m,Pl1[j2])^(-1);
                                end if;
                                end if;
                        end for;
                end for;
        end for;
        return s;
end function;

M12:=function(m);
/*Given a twisting parameter m=(m1,1), m1 irreducible, compute the xy^2
 coefficient of Z(s,m)*/
s:=0;
        /*Run over all 1|2*/
        for j1 in [1..#Pl2] do
                for j2 in [2..#Pl1] do
                        if cres(m,Pl1[j2]) ne 0 then
                                s+:=gaupr(Pl2[j1])*gaupr(Pl1[j2])*cres(ul
(Pl1[j2]),Pl2[j1])^(-1)*cres(m,Pl1[j2])^(-1);
                        end if;
                end for;
        end for;
        /*Run over all (1|11)*/
```

```
        for j1 in [2..#Pl1] do
                        if cres(m,Pl1[j1]) ne 0 then
                                s+:=q^(2)*cres(m,Pl1[j1])^(-1);
                        end if;
        end for;
        /*Run over all 1|11*/
        for j1 in [2..#Pl1] do
                for j2 in [j1+1..#Pl1] do
                        for j3 in [2..#Pl1] do
                                if j2 ne j3 and j1 ne j3 then
                                if cres(m,Pl1[j3]) ne 0 then
                                        s+:=gaupr(Pl1[j1])*gaupr(Pl1[j2])
*gaupr(Pl1[j3])*cres(ul(Pl1[j1]),Pl1[j2])^2*cres(ul(Pl1[j1]),Pl1[j3])^(-1)
*cres(ul(Pl1[j2]),Pl1[j3])^(-1)*cres(m,Pl1[j3])^(-1);
                                end if;
                                end if;
                        end for;
                end for;
        end for;
        return s;
end function;

M22:=function(m);
/*Given a twisting parameter m=(m1,1), m1 irreducible, compute the x^2y^2
series coefficient of Z(s,m)*/
s:=0;
        /*Run over all 2|2*/
        for j1 in [1..pl2] do
                for j2 in [1..pl2] do
                        if j1 ne j2 then
                                s+:=gaupr(Pl2[j1])*gaupr(Pl2[j2])*cres(ul
(Pl2[j1]),Pl2[j2])^(-1)*cres(m,Pl2[j1])^(-1);
                        end if;
                end for;
        end for;
        /*Run over all (11|11) */
        for i1 in [2..pl1] do
                s+:=gaupr(Pl1[i1])*q^2*cres(m,Pl1[i1])^(-2);
        end for;
        /*Run over all (11|1)1 */
        for i1 in [2..pl1] do
                for i2 in [2..pl1] do
                        if i1 ne i2 then
```

```
                                s+:=gaupr(Pl1[i1])*q^2*cres(ul(Pl1[i1]),Pl1
[i2])^2*cres(ul(Pl1[i2]),Pl1[i1])^(-2)*cres(m,Pl1[i1])^(-2);
                        end if;
                end for;
        /*Run over all 1(1|11) */
        end for;
        for i1 in [2..pl1] do
                for i2 in [2..pl1] do
                        if i1 ne i2 then
                                s+:=gaupr(Pl1[i1])*q^2*cres(ul(Pl1[i1]),Pl1
[i2])^2*cres(ul(Pl1[i2]),Pl1[i1])^(-2)*cres(m,Pl1[i1])^(-1)*cres(m,Pl1[i2])
^(-1);
                        end if;
                end for;
        end for;
        /*Run over all 2|11*/
        for i1 in [2..pl1] do
                for i2 in [i1+1..pl1] do
                        for j1 in [1..pl2] do
                                s+:=gaupr(Pl1[i1])*gaupr(Pl1[i2])*gaupr(Pl2
[j1])*cres(ul(Pl1[i1]),Pl1[i2])^(2)*cres(ul(Pl1[i1]),Pl2[j1])^(-1)*cres(ul
(Pl1[i2]),Pl2[j1])^(-1)*cres(m,Pl2[j1])^(-1);
                        end for;
                end for;
        end for;
        /*Run over all 11|2*/
        for i1 in [2..pl1] do
                for i2 in [i1+1..pl1] do
                        for j1 in [1..pl2] do
                                s+:=gaupr(Pl1[i1])*gaupr(Pl1[i2])*gaupr(Pl2
[j1])*cres(ul(Pl1[i1]),Pl1[i2])^(2)*cres(ul(Pl1[i1]),Pl2[j1])^(-1)*cres(ul
(Pl1[i2]),Pl2[j1])^(-1)*cres(m,Pl1[i1])^(-1)*cres(m,Pl1[i2])^(-1);
                        end for;
                end for;
        end for;
        /*Run over all 11|11*/
        for i1 in [2..pl1] do
                for i2 in [i1+1..pl1] do
                        for j1 in [2..pl1] do
                                for j2 in [j1+1..pl1] do
                                        if i1 ne j1 and i1 ne j2 and i2 ne
j1 and i2 ne j2 then
                                                s+:=gaupr(Pl1[i1])*gaupr
```

```
(Pl1[i2])*gaupr(Pl1[j1])*gaupr(Pl1[j2])*cres(ul(Pl1[i1]),Pl1[i2])^2*cres(ul
(Pl1[j1]),Pl1[j2])^(2)*cres(ul(Pl1[i1]),Pl1[j2])^(-1)*cres(ul(Pl1[i2]),Pl1
[j1])^(-1)*cres(ul(Pl1[i1]),Pl1[j1])^(-1)*cres(ul(Pl1[i2]),Pl1[j2])^(-1)
*cres(m,Pl1[i1])^(-1)*cres(m,Pl1[i2])^(-1);
                                        end if;
                                end for;
                        end for;
                end for;
        end for;
        return s;
end function;
```

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