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LOCALLY INJECTIVE MODULES AND LOCALLY PROJECTIVE MODULES

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ABSTRACT. Our dual notions "locally injective" and "locally projective" modules in Mod-R are good tools to study the relations between the singular, respectively cosingular, submodule of $\mathrm{Hom}_R(M,W)$ and the total $\mathrm{Tot}(M,W)$. These notions have further interesting properties.

1. Introduction. For a ring R with $1 \in R$ we denote by Mod-R the category of all unitary R-right modules. If A is a submodule of the module M, then $A \subseteq^0 M$, respectively $A \subseteq^* M$, denotes that A is a small, or superfluous, respectively a large or essential, submodule of M. Further $A \subseteq^\oplus M$ means that A is a direct summand of M. We have to use the following fundamental lemma.

Lemma 1.1. *For $f \in \mathrm{Hom}_R(W,M)$ the following conditions are equivalent*

(i) *There exists $g \in \mathrm{Hom}_R(W,M)$ such that $e := gf = e^2 \neq 0$ (e is an idempotent in $\mathrm{End}(M)$).*

(ii) *There exists $h \in \mathrm{Hom}_R(W,M)$ such that $d := fh = d^2 \neq 0$ (d is an idempotent in $\mathrm{End}(W)$).*

(iii) *There exist direct summands $0 \neq A \subseteq^\oplus M$, $B \subseteq^\oplus W$, such that the mapping $A \ni a \mapsto f(a) \in B$ is an isomorphism.*

For the proof, and for the proof of the following lemma, see [**4**]. If the conditions of the lemma are satisfied for f, we say that f is *partially invertible* (abbreviated 'pi'). The *total* of M, W, denoted by

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When I gave the talk at the Honolulu Conference I did not know that the notation "locally projective" was already used by Birge Zimmermann in her paper, *Pure submodules of direct products of free modules*, Math. Ann. **224** (1976), 233–245. I regret this very much and hope that this remark will help to avoid confusion.

Tot (M, W), is

$$\text{Tot}\,(M, W) := \{f \in \text{Hom}_R(M, W) \mid f \text{ is not pi}\}.$$

Lemma 1.2. *If a product of homomorphisms in* Mod -R *is* pi*, then each of its factors is* pi*.*

This implies for arbitrary $M, W, X, Y \in \text{Mod-}R$:

$$\text{Hom}_R(W, Y)\text{Tot}\,(M, W)\text{Hom}_R(X, M) \subseteq \text{Tot}\,(X, Y).$$

This means that the total is a semi-idea in Mod-R. In general, Tot (M, W) is not additively closed. But there are interesting conditions for M and W such that Tot (M, W) is additively closed, see [**1**–**4**]. The total Tot (M, W) contains the radical, the singular submodule and the cosingular submodule of $\text{Hom}_R(M, W)$. If it is equal to one (or all) of these, it is additively closed. Later we come back to this situation. First we consider the strongest restriction for the total which is possible, that is: the total is equal to zero. In a joint paper with Beidar [**1**], we proved the following theorem.

Theorem 1.3. 1) *For a module V the following are equivalent*:

(i) Tot $(V, M) = 0$ *for all* $M \in \text{Mod-}R$.

(ii) Tot $(V, C) = 0$ *for some cogenerator C of* Mod -R.

(iii) *V is a direct sum of simple injective submodules.*

2) *For a module W the following are equivalent*:

(i) Tot $(M, W) = 0$ *for all* $M \in \text{Mod-}R$.

(ii) Tot $(G, W) = 0$ *for some generator G of* Mod -R.

(iii) *Every nonzero submodule of W contains a nonzero projective submodule, which is a direct summand of W.*

If V, respectively W, satisfies the conditions in 1), respectively 2), we called it a left-TOTO-module, respectively a right-TOTO-module (for modules M, W with Tot $(M, W) = 0$, see also [**3**]). The conditions (i) and (ii) in 1) and 2) are dual. But what about the duality of V and W?

Here we consider the similar question but under less strong restrictions for the totals (then $\mathrm{Tot}\,(V,M)=0$, respectively $\mathrm{Tot}\,(W,M)=0$). This leads to the notions of locally injective and locally projective, for which the duality is obvious. Locally injective and locally projective modules have further interesting properties which we also present here.

Definitions and main theorem.

Definition 2.1. 1) The module V is called locally injective if, for every submodule $A \subseteq V$, which is not large in V, there exists an injective submodule $Q \subseteq V$, $Q \neq 0$, with $A \cap Q = 0$ (the abbreviation for locally injective is 'li').

2) The module W is called locally projective if, for every submodule $B \subseteq W$, which is not small in W, there exists a projective direct summand $P \subseteq^{\oplus} W$, $P \neq 0$, with $P \subseteq B$ (the abbreviation for locally projective is 'lp').

For these notions we have our main theorem.

Theorem 2.2. 1) *For a module V the following conditions are equivalent*:

(i) *For all $M \in \mathrm{Mod}\text{-}R$,*

$$\mathrm{Tot}\,(V,M) = \{f \in \mathrm{Hom}_R(V,M) \mid \mathrm{Ker}\,(f) \subseteq^* V\}$$
$$(\,= \textit{singular submodule of } \mathrm{Hom}_R(V,W) = \Delta(V,M)).$$

(ii) *V is locally injective.*

2) *For a module W the following conditions are equivalent*:

(i) *For all $M \in \mathrm{Mod}\text{-}R$,*

$$\mathrm{Tot}\,(M,W) = \{f \in \mathrm{Hom}_R(M,W) \mid \mathrm{Im}\,(f) \subseteq^0 W\}$$
$$(\,= \textit{cosingular submodule of } \mathrm{Hom}_R(M,W) = \nabla(V,M)).$$

(ii) *W is locally projective.*

Proof. 1) (i) $\Rightarrow$ (ii). Let $A \subseteq V$ and suppose A is not large in V. Denote by $\nu : V \to V/A$ the natural epimorphism with $\mathrm{Ker}\,(\nu) = A$, and let $\rho : V/A \to I$ be a monomorphism into an injective module I (e.g., I is the injective hull of V/A). Then the kernel of $f := \rho\nu$ is A and by assumption f is pi. By 1.1 (iii) there exist $0 \neq Q \subseteq^{\oplus} V$, $B \subseteq^{\oplus} I$ such that

$$Q \ni q \longmapsto f(q) \in B$$

is an isomorphism. This implies that B and Q are injective and $A \cap Q = 0$.

1) (ii) $\Rightarrow$ (i). Consider $f \in \mathrm{Hom}_R(V, M)$ with $A := \mathrm{Ker}(f)$ is not large in V. By assumption there exists an injective $0 \neq Q \subseteq V$ with $A \cap Q = 0$. Now we restrict f to Q:

$$f_0 : Q \ni q \longmapsto f(q) \in f(Q).$$

Since $A \cap Q = 0$, f_0 is an isomorphism and also $f(Q)$ is injective, hence a direct summand of M. By 1.1 (iii), this means that f is pi.

2) (i) $\Rightarrow$ (ii). Let $B \subseteq W$ and suppose B is not small in W. Denote by $\rho : P \to B$ a projective extension of B (e.g., P can be free) and by $\iota : B \to W$ the inclusion. Then, for $f := \iota\rho :$ $\mathrm{Im}(f) = B$; hence by assumption f is pi. Then by 1.1(iii), there exist $0 \neq P_0 \subseteq^{\oplus} P$ and $B_0 \subseteq^{\oplus} W$ such that

$$P_0 \ni x \longmapsto f(x) \in B_0$$

is an isomorphism. B_0 is then a nonzero projective direct summand of W contained in B.

2) (ii) $\Rightarrow$ (i). Consider $f \in \mathrm{Hom}_R(M, W)$ with $\mathrm{Im}(f)$ not small in W. Then by assumption there exists a projective $0 \neq P \subseteq^{\oplus} W$, $P \subseteq \mathrm{Im}(f)$. By the modular law $P \subseteq^{\oplus} \mathrm{Im}(f)$. Denote by $\pi : \mathrm{Im}(f) \to P$ the projection. Since P is projective, the epimorphism $\pi f : M \to P$ splits, hence $M = A \oplus \mathrm{Ker}(\pi f)$ and

$$A \ni a \longmapsto \pi f(a) \in P$$

is an isomorphism. By 1.1 it follows that πf is pi and then by 1.2 also f is pi. $\square$

3. Further properties of locally injective and locally projective modules.

Property 3.1. 1) *For a module* V *the following are equivalent*:

(i) V *is locally injective and has no proper large submodule.*

(ii) V *is a left*-TOTO-*module in the sense of* [**1**], *which means that* $\mathrm{Tot}\,(V, M) = 0$ *for all* $M \in \mathrm{Mod}\text{-}R$.

2) *For a module* W *the following are equivalent*

(i) W *is a locally projective and* W *has no nonzero small submodule* (*that is,* $\mathrm{Rad}\,(W) = 0$.)

(ii) W *is a right*-TOTO-*module in the sense of* [**1**], *which means that* $\mathrm{Tot}\,(M, W) = 0$ *for all* $M \in \mathrm{Mod}\text{-}R$.

Proof. 1) (i) $\Rightarrow$ (ii). Since V has no proper large submodule, V is semi-simple. If $A \subseteq V$ is a simple submodule, then $V = A \oplus C$. Since C is not large in V, there exists an injective submodule $0 \neq Q \subseteq V$ with $Q \cap C = 0$. Since $V = A \oplus C$ and A is simple, this implies that C is maximal and then also $V = Q \oplus C$. Then it follows that $Q \cong V/C \cong A$, hence A is also injective.

1) (ii) $\Rightarrow$ (i). If V is a left-TOTO-module, it is semi-simple and therefore has no proper large submodule. If $A \subsetneqq V$, then not all simple submodules of V are contained in A. Let Q be a simple (and injective) submodule not contained in A. Then $A \cap Q = 0$.

2) (i) $\Rightarrow$ (ii). Clear.

2) (ii) $\Rightarrow$ (i). A small submodule cannot contain a nonzero projective direct summand of W, since this is not a small submodule. The rest is clear. $\square$

Property 3.2. *Direct summands of* li*, respectively* lp*, modules are again* li*, respectively* lp.

Proof. Assume V is li and $V = A \oplus B$, $A \neq 0$. Let $A_0 \subseteq A$ and suppose that A_0 is not large in A. Then $A_0 \oplus B$ is not large in V. By assumption there exists an injective $Q \subseteq V$ such that $(A_0 \oplus B) \cap Q = 0$. Denote by π the projection of $V = A \oplus B$ onto A. Then we consider the mapping

$$\mu : Q \ni q \longmapsto \pi(q) \in \pi(Q).$$

Assume $\pi(q) = 0$ and $q = a + b$, $a \in A$, $b \in B$. That means $a = 0$, hence $q = b \in (A_0 \oplus B) \cap Q = 0$, hence $q = 0$. Therefore, μ is an isomorphism and $\pi(Q)$ is also injective. Since $\pi(Q) \subseteq A$, we need only show $A_0 \cap \pi(Q) = 0$. Assume $a \in A_0 \cap \pi(Q)$; then there exists $q \in Q$, $q = a + b$, $b \in B$; but then $q = a + b \in (A_0 + B) \cap Q = 0$, hence $q = 0$. Therefore, $\pi(Q)$ is the injective module we were looking for.

Assume now that $W = A \oplus B$ is lp and $0 \neq A_0 \subseteq A$. Then there exists a projective $0 \neq P \subseteq^{\oplus} W$ with $P \subseteq A_0$. Since $P \subseteq A_0 \subseteq A$, P is also a direct summand of A. □

Property 3.3. 1) *Injective modules are* li. 2) *Projective and semi-perfect modules are* lp.

Proof. 1) If Q is injective and $A \subseteq Q$, A not large in Q, then Q contains an injective hull $I(A)$ of $A : Q = I(A) \oplus Q_1$. Here $Q_1 = 0$ is not possible, since otherwise A would be large in $I(A) = Q$. Then $A \cap Q_1 = 0$.

2) If P is projective and $B \subseteq P$, B not small in P, then we consider $\nu : P \to P/B$. Since P is semi-perfect, there exists a decomposition $P = P_1 \oplus P_2$ where $\nu|P_1$ is a projective cover of P/B, that is, $\mathrm{Ker}\,(\nu \mid P_1) \subseteq^0 P_1$ and $P_2 \subseteq \mathrm{Ker}\,(\nu) = B$. Assume $P_2 = 0$, then $P_1 = P$ and $\mathrm{Ker}\,(\nu \mid P_1) = \mathrm{Ker}\,(\nu) = B \subseteq^0 P$, a contradiction! □

Property 3.4. 1) *Assume V is* li *and satisfies the maximum condition for injective submodules. Then for every $A \subseteq V$, there exists an injective submodule $Q \subseteq V$ such that*

$$A \cap Q = 0, \quad A \oplus Q \subseteq^* V.$$

In particular, for $A = 0$, it follows that $Q = V$.

2) *Assume W is* lp *and satisfies the maximum condition for projective direct summands of W. Then for every $B \subseteq W$ there exist a projective direct summand $P \subseteq^{\oplus} W$ and a $U \subseteq W$ such that*

$$P \cap U = 0, \quad B = P \oplus U.$$

In particular, for $B = W$, it follows that $W = P$.

Proof. 1) Let Q be an injective submodule which is maximal with respect to $A \cap Q = 0$. Assume that $A \oplus Q$ is not large in V. Then there exists an injective submodule $Q_0 \subseteq V$, $Q_0 \neq 0$, with $(A \oplus Q) \cap Q_0 = 0$ which contradicts the maximality of Q. Hence $A \oplus Q \subseteq^* V$. If $A = 0$, then $Q \subseteq^* V$. But since $Q \subseteq^\oplus V$, this implies $Q = V$.

2) Let P be a projective direct summand of W, which is maximal in $P \subseteq^\oplus B$ and suppose $B = P \oplus U$. Assume that U is not small in W. Then there exists a projective direct summand $P_0 \neq 0$ of W with $P_0 \subseteq^\oplus U : U = P_0 \oplus U_0$. But then $B = (P \oplus P_0) \oplus U_0$. To get a contradiction we have still to show that $P \oplus P_0$ is a direct summand of W. Let $W = P \oplus C = P_0 \oplus C_0$. Then, since $P \subseteq B$, it follows that $B = P \oplus (C \cap B)$. Since $P_0 \subseteq C \cap B \subseteq C$, it follows further that $C = P_0 \oplus (C_0 \cap C)$ and then $W = P \oplus P_0 \oplus (C_0 \cap C)$. Since $P \oplus P_0$ is a direct summand of W, we have a contradiction to the maximality of P. Hence $U \subseteq^0 W$.

For $B = W$ it follows that $W = P \oplus U = P$ since $U \subseteq^0 W$. □

In the following we call a set of submodules $\{U_i \mid i \in I\}$, $U_i \subseteq M$ *independent* if the sum of the U_i is direct.

Property 3.5. 1) *Assume V is* li *and let $A \subseteq V$. Then there exists a maximal independent set $\{Q_i \mid i \in I\}$ of injective $Q_i \subseteq V$ with*

$$A \cap \Big(\bigoplus_{i \in I} Q_i\Big) = 0, \quad A \oplus \Big(\bigoplus_{i \in I} Q_i\Big) \subseteq^* V.$$

2) *For a module V the following are equivalent*:

(i) *V is* li.

(ii) *There exists an independent set $\{Q_i \mid i \in I\}$ of injective $Q_i \subseteq V$ with $\oplus_{i \in I} Q_i \subseteq^* V$.*

3) *If V is* li *and R_R is Noetherian, then V is injective.*

Proof. 1) Since the union of an ascending chain of sets $\{Q_i \mid i \in I\}$ with $A \cap (\oplus_{i \in I} Q_i) = 0$ is again such a set, we can apply Zorn's lemma.

Therefore we can assume that $\{Q_i \mid i \in I\}$ is maximal. But then also $A \oplus (\oplus_{i\in I} Q_i)$ is large in V; otherwise an injective direct summand $Q \subseteq^{\oplus} V$ would exist with $A \oplus (\oplus_{i\in I} Q_i) \cap Q =)$, contradicting the maximality of $\{Q_i \mid i \in I\}$.

2) (i) $\Rightarrow$ (ii). By 1) for $A = 0$.

(ii) $\Rightarrow$ (i). Consider $A \subseteq V$, A not large in V. Then there exists $0 \neq B \subseteq V$ with $A \cap B = 0$. Since $\oplus_{i\in I} Q_i$ is large in V,

$$\Big(\bigoplus_{i\in I} Q_i\Big) \cap B \neq 0.$$

Then there exist finitely many Q_i, say $Q_1, \dots, Q_n$ (new indices), such that for $Q := Q_1 \oplus \dots \oplus Q_n$ also $Q \cap B \neq 0$. Since $Q \cap B$ is a submodule of the injective module Q, an injective hull $I(Q \cap B)$ of $Q \cap B$ is contained in Q. Since $Q \cap B \subseteq^* I(Q \cap B)$, by $A \cap (Q \cap B) \subseteq A \cap B = 0$, it follows that $A \cap I(Q \cap B) = 0$. Hence $I(Q \cap B)$ is an injective module we were looking for.

3) Again by 1) for $A = 0$, we have $\oplus_{i\in I} Q_i \subseteq^* V$. If R_R is Noetherian, then the sum $\oplus_{i\in I} Q_i$ is injective, hence a direct summand of V. But a large direct summand must be the whole module. □

Property 3.6. *Assume that W is* lp *and $B \subseteq W$. Then there exists a maximal independent set $\{P_i \mid i \in I\}$ or projective $P_i \subseteq^{\oplus} W$, $P_i \subseteq B$ with the following property: If $C \subseteq B$ and*

(1) $$\Big(\bigoplus_{i\in I} P_i\Big) \cap C = 0,$$

then $C \subseteq^0 W$. That means that $P := \oplus_{i\in I} P_i$ is "nearly large" in B.

Proof. By Zorn's lemma there exists a maximal independent set $\{P_i \mid i \in I\}$. Assume for $C \subseteq B$ that (1) is satisfied, but C is not small in W. Then there must exist a $0 \neq P \subseteq^{\oplus} W$, P projective and $P \subseteq C$, contradicting the maximality of $\{P_i \mid i \in I\}$. □

Property 3.7. *If W is* lp, *then for every $x \in W$, $x \notin \mathrm{Rad}\,(W)$, there exists $f \in W^* := \mathrm{Hom}_R(W, R)$ such that $xf(x) \neq 0$.*

Proof. Since $x \notin \operatorname{Rad}(W)$, xR is not small in W. Hence there exists a projective module $0 \neq P \subseteq xR$ which is a direct summand of W: $W = P \oplus C$. Then by the modular law $xR = P \oplus (C \cap xR)$. Now there exist $p, c \in R$ such that

$$x = xp + xc, \quad xp \in P, \quad xc \in (C \cap xR) \tag{2}$$

and $xpR \subseteq P$. If $xb \in P$, then by (2)

$$xb = xpb + xcb.$$

Since the sum is direct, we have

$$xb = xpb, \quad xcb = 0,$$

hence $xpR = P$. Now we consider the epimorphism

$$R \ni r \longmapsto xpr \in P.$$

Since P is projective, this epimorphism splits. This means that there exists an idempotent $e \in R$, $e \neq 0$, such that

$$g : eR \ni er \ni er \longmapsto xper \in P$$

is an isomorphism and $xp(1-e) = xp - xpe = 0$. By (2) it follows that also $xe = xpe + xce$ and, since $xpe \neq 0$ and the sum is direct, also $xe \neq 0$. Then define $f \in W^*$ by

$$f \mid P := g^{-1}, f \mid C := 0.$$

This implies $xf(xpe) = xg^{-1}(xpe) = xe \neq 0$ and, since $xf(xpe) = xf(x)pe$, also $xf(x) \neq 0$. $\square$

Note that the last property is a "weak form" of W being a semi-prime module. It is also a "weak form" of W being torsionless.

4. What is contained in the total? Consider arbitrary modules M, W and $f \in \operatorname{Hom}_R(M, W)$. By 1.1 (iii) we know that f is partially invertible if and only if there exist $A \subseteq^{\oplus} M$, $A \neq 0$, $B \subseteq^{\oplus} W$ such that

$$f_0 : A \ni a \longmapsto f(a) \in B$$

is an isomorphism. Then the kernel of $f = \mathrm{Ker}\,(F)$ cannot be large, since then $A \cap \mathrm{Ker}\,(f) \neq 0$ and f could not be an isomorphism. Hence $f \in \mathrm{Tot}\,(M,W)$, that is, $\Delta(M,W) \subseteq \mathrm{Tot}\,(M,W)$. Similarly, if f is partially invertible, then it could not have a small image since $B \subseteq \mathrm{Im}\,(f)$. Hence also $\nabla(M,W) \subseteq \mathrm{Tot}\,(M,W)$. Now using our Theorem 2.2, we get the following.

Remark 4.1. 1) If V is li and if $f \in \mathrm{Hom}_R(V,M)$ and $\mathrm{Im}\,(f) \subseteq^0 M$, then $\mathrm{Ker}\,(f) \subseteq^* V$.

2) If W is lp and if $f \in \mathrm{Hom}_R(M,W)$ and $\mathrm{Ker}\,(f) \subseteq^* M$, then $\mathrm{Im}\,(f) \subseteq^0 W$.

It is easy to also give a direct proof of these facts.

We call $\mathrm{Tot}\,(M,W)$ *minimal* if

$$\mathrm{Tot}\,(M,W) = \{f \in \mathrm{Hom}_R(M,W) \mid \mathrm{Ker}\,(f) \subseteq^* M \vee \mathrm{Im}\,(f) \subseteq^0 W\}.$$

Then all totals $\mathrm{Tot}\,(V,M)$ and $\mathrm{Tot}\,(M,W)$ which occur in Theorem 2.2 are minimal.

If M and W are modules which have no nonzero direct summands, then (by (1.1)(iii)) $\mathrm{Tot}\,(M,W) = \mathrm{Hom}_R(M,W)$. By this fact, it is easy to give examples of modules M, W such that $\mathrm{Tot}\,(M,W)$ is very far from being minimal.

In the following we denote

$$H := \mathrm{Hom}_R(M,W), \quad S := \mathrm{End}\,(W_R), \quad T := \mathrm{End}\,(M_R).$$

Then H is an S-T-bimodule. We now consider radicals of H. First we have the module radicals $\mathrm{Rad}\,({}_SH)$, $\mathrm{Rad}\,(H_T)$. For example,

$$f \in \mathrm{Rad}\,({}_SH) \Longleftrightarrow Sf \subseteq^0 {}_SH.$$

Moreover, there is the following radical of H which we denote by capital letters:

$$\mathrm{RAD}\,(H) := \{f \in \mathrm{Hom}_R(M,W) \mid \text{for all } g \in \mathrm{Hom}_R(W,M) : fg \in \mathrm{Rad}\,(S)\}.$$

We get the same radical by changing the sides, since there is the following well-known lemma.

Lemma 4.2. *If* $f \in \operatorname{Hom}_R(M,W)$, $g \in \operatorname{Hom}_R(W,M)$, *then*

$$fg \in \operatorname{Rad}(S) \Longleftrightarrow gf \in \operatorname{Rad}(T).$$

For $M = W$ *all the radicals are the same.*

Remark 4.3. $\operatorname{Rad}({}_SH) + \operatorname{Rad}(H_T) \subseteq \operatorname{RAD}(H)$.

Proof. If $Sf \subseteq^0 {}_SH$, then for all $g \in \operatorname{Hom}_R(W,M)$ we have $Sfg \subseteq^0 {}_SS$, since the image of a small submodule is small. Then $fg \in \operatorname{Rad}(S)$. Similarly for the other case. ◻

In [**4**] we proved the following additive closure property:

$$\operatorname{Rad}({}_SH) + \operatorname{Rad}(H_T) + \operatorname{Tot}(M,W) = \operatorname{Tot}(M,W),$$

which for $0 \in \operatorname{Tot}(M,W)$ includes

$$\operatorname{Rad}({}_SH) + \operatorname{Rad}(H_T) \subseteq \operatorname{Tot}(M,W).$$

Now we show

Lemma 4.4. $\operatorname{RAD}(H) + \operatorname{Tot}(M,W) = \operatorname{Tot}(M,W)$.

Proof. Let $f \in \operatorname{RAD}(H)$, $t \in \operatorname{Tot}(M,W)$, and assume $f+t$ is pi. Then there exists $g \in \operatorname{Hom}_R(W,M)$ such that

$$e := (f+t)g = fg + tg = e^2 \neq 0.$$

By assumption $fg \in \operatorname{Rad}(S)$ and $tg \in \operatorname{Tot}(S)$. Thus

$$1_S - fg = 1_S - e + tg$$

is invertible. Now it follows for $h := (1_S - fg)^{-1}$ that

$$h(1_S - fg) = 1_S = h(1_S - e + tg).$$

Multiplication with e implies

$$e = htge \in \operatorname{Tot}(S).$$

This is a contradiction since $e \neq 0$ cannot be in the total. □

If V is injective, then for all M

$$\mathrm{Tot}\,(V, M) = \Delta(V, M) = \mathrm{RAD}\,(\mathrm{Hom}_R(V, M)).$$

If W is projective, then for all M

$$\mathrm{Tot}\,(M, W) = \nabla(M, W) = \mathrm{RAD}\,(\mathrm{Hom}_R(M, W))$$

(see [**4**]).

We would like to know if the same is true if V is li, respectively W is lp. It would be enough to show that V cannot be isomorphic to a large submodule, respectively W cannot have a homomorphism onto W with a small kernel.

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