THREE-DIMENSIONAL STRESS PROBLEM OF A FINITE THICK PLATE WITH A THROUGH-CRACK UNDER TENSION

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ABSTRACT

The three dimensional state of stress in a finite thick rectangular plate with a through-crack under tension is investigated. It is found that the in-plane stresses and the transverse normal stress are singular while the transverse shear stresses are of the order of unity. The only type of singularity encountered is that of inverse square root all through the plate thickness including the corner points at the plate faces. The stress intensity factor which is found to vary with z and the thickness ratio h/a drops rapidly in a thin boundary layer near the plate faces, but without actually vanishing there. The stress intensity factor reduces exactly to that for the plane strain case when $h/a \to \infty$. All the three components of the displacement field are finite at the crack front.

KEY WORDS

Fracture; through-crack; stress intensity factor; thick plate; elasticity; three-dimensional analysis.

INTRODUCTION

Recent studies (Hartranft and Sih, 1969; Sih, Williams and Swedlow, 1966; Folias, 1975, 1980) in three dimensional (3-D) cracked-plate problems with cracks emerging at the free surfaces have raised some questions such as, (1) the type of singularities involved at the crack front, in particular, at the corner points where the crack front penetrates the free plate faces, (2) the type of variation of the stress intensity factor (SIF) across the plate thickness, in particular, near the plate faces, (3) the finiteness or otherwise of the displacement components at the crack front. The studies carried out in the above mentioned references point to the existence of the inverse square root singularity $1/(\bar{r})^{1/2}$ in the stress field interior to the plate thickness with a predominantly plane strain type of deformation; no numerical results are prsented there. However, near the plate faces, while the first two references do not throw any light as to either the type of stress singularity or the deformation character, the studies in next two references reveal a Possion's ratio-dependent stress singularity, viz, $1/(\bar{r})^{1/2+2\mu}$; indicating a displacement singularity at the crack front for $\mu > 1/4$; in contrast, in the first two references the displacement finiteness condition at the crack front is enforced on the solution. The results of (Benthem, 1976; Kawai, 1975) for the stresses near the corner point of a quarter plane crack exhibit $1/(\bar{r})^{\alpha}$ singularity with α varying from 0 to 1/2 in the former's case and 1/2 to 1 in latter's investigation. The experimental studies of reference (Villarreal, Sih and Hartranft, 1975) serve to confirm the qualitative results of (Hartranft, 1969; Sih, 1966; Folias, 1975,1980) regarding the character of the singular deformation field interior to the plate thickness; but, near the plate faces, a rapid decrease in SIF values is indicated, suggesting a reduction in the strength of singularity in the region.

In this paper, the 3-dimensional state of stress in a thick rectangular plate with a through-crack under tension is investigated. The mathematical formulation of the problem is based on the 3-dimensional elasticity equations, (Lure, 1964). The solution obtained satisfies, exactly, the stress-free conditions at the crack surfaces as also the plate faces; and the boundary conditions (B.C) at the exterior edges of the plate, including those of the applied tensile loading, are satisfied in the least square sense. It is found that the in-plane stresses and the transverse normal stress are singular at the crack front while the transverse shear stresses are of the order of unity.

During the early stages of the formulation of the present problem some interesting features concerning the singular stress field at the crack front were observed. Some of these results were reported in (Bapu Rao, 1981). Subsequently, further formulation and numerical studies were carried out. In this paper, only essential features of the formulation of the problem and numerical results for two typical problems of a thick plate with a through-crack under tension are presented. These results and their comparison with existing results lead to some important conclusions.

FORMULATION OF THE PROBLEM

In view of the cylindrical coordinate system chosen $(\bar{r}, \bar{\theta}, \bar{z})$ in association with the Cartesian coordinate system $(\bar{x}, \bar{y}, \bar{z})$, See Fig. 1, it is necessary to consider for analysis only the region defined by $(-a) \leq \bar{x} \leq (L-a)$, $(-B) \leq \bar{y} \leq B$ and $(-h) \leq \bar{z} \leq h$, with appropriate continuity conditions at $\bar{x}=-a$ and boundary conditions at other edges, which are given as

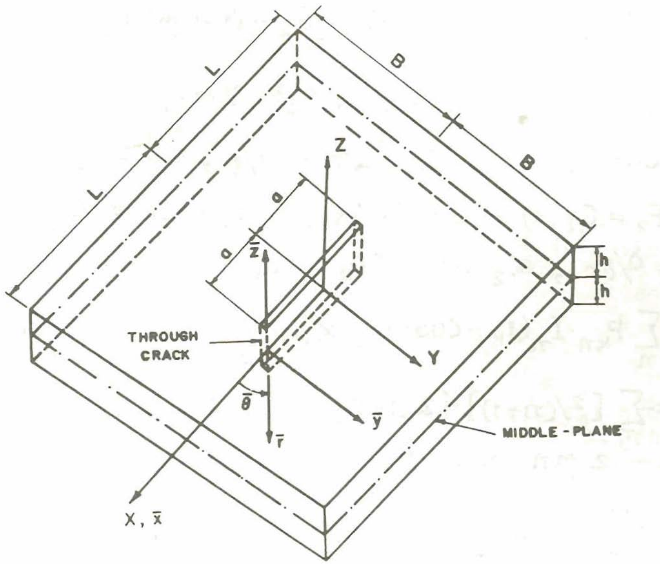

Fig. 1. Coordinate system and plate dimensions

at $\bar{x}=-a$, $u = \sigma_{xz} = \sigma_{xy} = 0$; at $\bar{x}=(L-a)$, $\sigma_x = \sigma_{xy} = \sigma_{xz} = 0$ (1)

at $\bar{y}=\pm B$, $\sigma_y = \sigma$ (applied tension), $\sigma_{yz} = \sigma_{xy} \equiv 0$ (2)

The governing equations (G.E) of the problem are the 3-D elasticity equations in terms of displacement components (u,v,w). These equations as also those for the stress components ($\sigma_r, \sigma_\theta, \sigma_{r\theta}, \sigma_z, \sigma_{rz}, \sigma_{\theta z}$) are available in (Hartranft, 1969; Lure, 1964). It is convenient to nondimensionalize ($\bar{r}$, $\bar{\theta}$, $\bar{z}$) with respect to a : $r=\bar{r}/a$, $\theta=\bar{\theta}$, and $z=\bar{z}/a$ where $2a$ is the crack length. The solution to G.E must satisfy the following boundary conditions :

(i) at the plate faces ($z = \pm h/a = \pm \zeta$), $\sigma_{rz} = \sigma_{\theta z} = \sigma_z = 0$;

(ii) at the crack surfaces ($\theta = \pm \pi$), $\sigma_{r\theta} = \sigma_\theta = \sigma_{\theta z} = 0$

and also those defined by eqs. 1 and 2. The solution to G.E is given as

$$u_i = Re \sum_{K=1}^{\infty} \Psi_i(z,\lambda_K) \cdot \alpha_i X_a(r,\theta,\lambda_K) + X_i(r,\theta) + \quad (3)$$

$$+\sum_{\ell=1}^{\infty} F_i(z)\cdot \alpha_i X_b(r,\theta,\ell) \quad ; \quad i = 1-3 \tag{3}$$

$$u_1 = u/a \quad , \quad u_2 = v/a \quad , \quad u_3 = W/a \tag{4}$$

$$\Psi_1 = \Psi_2 = [-C_K\cdot \bar{C}_K - (z/\rho)\cdot S_K\cdot \bar{S}_K + (1-2\mu)\cdot C_K\cdot \bar{S}_K/\lambda_K]$$

$$\Psi_3 = [S_K\cdot \bar{C}_K\cdot \lambda_K/\rho - C_K\cdot \bar{S}_K\cdot \lambda_K\cdot z/\rho^2 + 2(1-\mu)\cdot S_K\cdot \bar{S}_K/\rho\,] \tag{5}$$

$$C_K = \cos\lambda_K\cdot z/\rho \ , \ S_K = \sin\lambda_K\cdot z/\rho \ , \ F_3 = -S_\ell(z)\cdot \ell\pi/\rho$$

$$F_1 = F_2 = C_\ell(z) = \cos \ell\pi z/\rho \ , \ S_\ell(z) = \sin \ell\pi z/\rho \ , \ X_3 = 0 \tag{6}$$

$$\alpha_1 = \partial/\partial r \ , \ \alpha_2 = 1/r\cdot \partial/\partial\theta \ , \ \alpha_3 = 1 \ ; \ r_k = \lambda_K\cdot r/\rho$$

$$X_a = \sum_n p_{kn}\cdot I_n(r_k)\cdot \cos n\theta \ , \ X_b = \sum_n t_{\ell n}\cdot I_n(\ell\pi r/\rho)\cdot \cos n\theta \tag{7,8}$$

$$X_1 = -\sum_n [2/(n+1)]\cdot[2(1-2\mu)-n]\cdot A_n\cdot r^{n+1}\cos n\theta - 2\mu n\cdot N_n\cdot r^{n-1}\cdot \cos n\theta \tag{9}$$

$$X_2 = -\sum_n [2/(n+1)]\cdot[4(1-\mu)+n]\cdot A_n\cdot r^{n+1}\sin n\theta + 2\mu n\cdot N_n\cdot r^{n-1}\cdot \sin n\theta \tag{10}$$

In eqs. 3-10, $I_n(\lambda_K r/\rho)$ and $I_n(l\pi r/\rho)$ are modified Bessel functions of the first kind, and λ_K are the complex roots of the equation $\sin 2\lambda_K + 2\lambda_K = 0$, (this equation is obtained by satisfying the stress free B.C's at plate faces). Here $\bar{C}_K$ and $\bar{S}_K$ are the values of C_K and S_K at $z=\rho$; p_{Kn} and $(N_n, t_{\ell n}, A_n)$ are the unknown complex and real constants respectively.

The first terms in eq. 3 automatically satisfies the B.C. at the plate faces by virtue of the equation $\sin 2\lambda_K + 2\lambda_K = 0$, (k=1,2,3,...); these conditions are also satisfied independently by the second and third terms of eq. 3 when considered together, leading to the elimination of some of the associated unknown constants. In order to satisfy the B.C at the crack surfaces (these B.C.'s are to be satisfied by all the three terms in eq. 3 considered together) it is useful to express the stresses in the form of a power series in r, Sine and Cosine functions of θ, and Fourier Series in z. This can be accomplished by expressing $(C_K, S_K z/\rho)$ and $(S_K, C_K z/\rho)$ in eq. 5 in Fourier series of the form $C_\ell(z)$ and $S_\ell(z)$, l=1,2,3,..., as also terms of the order of unity $O(z^0)$, by taking advantage of the power series form in r of Bessel functions and biharmonic functions, and by virtue of the existing Sine and Cosine terms in θ present in eqs. 7-10. It is now appropriate to consider the range of n occurring in the expressions for the stresses and displacements, eqs. 7-10. The range of n is determined from the satisfaction of the finiteness condition of displacement components at the crack front. It is found that there are four groups of n whose ranges are : $n=2j\pm 1/2$, $n=2j+1$ and $n=2j$, $j=0,1,2,...$; significantly the limit placed is on the lower value of n, ($n=-1/2$). Satisfaction of the B.C.'s at the crack surfaces (by setting $\Theta=\pm\pi$ in the expressions for σ_θ, $\sigma_{\theta z}$, and $\sigma_{r\theta}$) with respect to each power of r for each range of n lead to singular stresses. This method of satisfying B.C's has been employed for a two dimensional bending problem of a cracked plate, (Murthy, 1981). The B.C's at the exterior edges of the plate as defined by eqs. 1 and 2 are satisfied with respect to stress resultants (rather than unit stresses) and average u over the thickness, in the least square sense. The satisfaction of all B.C.'s mentioned above lead to the determination of all unknown constants for given plate dimensions and h/a ratios.

RESULTS AND DISCUSSIONS

It is found that the in-plane stresses ($\sigma_r, \sigma_{r\theta}, \sigma_\theta$) and σ_z are singular with a $1/(\bar{r})^{1/2}$ singularity; the stresses σ_{rz} and $\sigma_{\theta z}$ are of the order of unity. The corresponding expressions are given as

$$\sigma_r = K(z)\cdot[1/4(2\bar{r})^{1/2}]\cdot(5\cos\theta/2 - \cos 3\theta/2) + O(\bar{r}^0)$$

$$\sigma_\theta = K(z)\cdot[1/4(2\bar{r})^{1/2}]\cdot(3\cos\theta/2 + \cos 3\theta/2) + O(\bar{r}^0) \qquad (11)$$

$$\sigma_{r\theta} = K(z)\cdot[1/4(2\bar{r})^{1/2}]\cdot(\sin\theta/2 + \sin 3\theta/2) + O(\bar{r}^0)$$

$$\sigma_z = K^*(z)\cdot[4\mu/2(2\bar{r})^{1/2}]\cdot\cos\theta/2 + O(\bar{r}^0)$$

$$\sigma_{rz} = O(\bar{r}^0), \quad \sigma_{\theta z} = O(\bar{r}^0) \qquad (12)$$

In eqs. 11 and 12, $K(z)$ is the stress intensity factor and $K^*(z)$ is the factor associated with the singular term of σ_z. The corresponding expressions are given as

$$K(z) = -2(2a)^{1/2}\cdot G[\{Re\sum_{k=1}^{\infty}(\mu/\beta^2)\cdot P_{k(-1/2)} + 4A_{-1/2}\} + (1/\beta^2)\cdot\sum_{\ell=1}^{\infty}2(-1)^\ell\cdot C_\ell(z)\times\{Re\sum_{k=1}^{\infty}\lambda_k^2\cdot P_{k(-1/2)}\times (\mu\cdot A_{k\ell} - (1+\mu)\cdot G_{k\ell}) - (-1)^\ell\cdot(\ell\pi/2)^2\cdot t_{\ell(-1/2)}\}] \qquad (13)$$

$$K^{*}(z) = -2(2a)^{1/2}\cdot G\Big[\Big\{\mathrm{Re}\sum_{k=1}^{\infty}(1/\mu\xi^{2})\cdot p_{k(-1/2)} + 4A_{-1/2}\Big\} + \sum_{\ell=1}^{\infty}(-1)^{\ell}\cdot(2/\mu\xi^{2})\cdot C_{\ell}(z)\times\Big\{\mathrm{Re}\sum_{k=1}^{\infty}\lambda_{k}^{2}\cdot A_{k\ell}\cdot p_{k(-1/2)} + (-1)^{\ell}\cdot(\ell\pi/2)^{2}\cdot t_{\ell(-1/2)}\Big\}\Big] \qquad (14)$$

In eqs. 13 and 14, $p_{k(-1/2)}$, $A_{-1/2}$ and $t_{\ell(-1/2)}$ are the unknown constants referred to earlier with respect to n=-1/2. Here, $A_{k\ell}$ and $G_{k\ell}$ are known functions of λ_k and $l\pi$; G is the shear modulus. It can be observed from eqs. 11 that the only type of singularity encountered is that of inverse square root all through the plate thickness including the corner points where the crack penetrates the plate faces. In reference (Folias, 1975) a Poisson's ratio-dependant $1/(\bar{r})^{1/2+2\mu}$ type of singularity was exhibited; in the present analysis, it is noted, no such dependence on μ is imposed. The angular variation of singular stresses are identical to those of (Hartranft, 1969). The SIF is found to depend on z and h/a ratio. It can be noted from eq. 53 that it does not vanish at the plate faces and, therefore, the in-plane stresses preserve their $1/(\bar{r})^{1/2}$ singularity all through the plats thickness including the corner points at the plate faces. It should also be observed from eq. 14 that $K^{*}(z)$ vanishes at the plate faces, thereby leading to vanishing of σ_z here. It is interesting to note that the expressions for $K(z)$ and $K(z)^{*}$ reduce exactly to those for the plane strain case when $2h\rightarrow\infty$ ($\xi\rightarrow\infty$); thus plane strain results aare recovered from eqs. 11 and 12. It is noteworthy that all the three displacement components are finite ah the crack front. In contrast, in (Folias, 1975), they were found to be singular for certain values of μ ($\mu > 1/4$) in the vicinity of the corner points.

The preceding discussion was made without any reference to numerical results but, solely from the nature of the mathematical expressions presented in eqs. 11-14 and also those not presented here to save space. Therefore, it will be interesting to discuss the numerical results obtained. Pig. 2 shows the distribution of SIF across the plate thickness for the case with B/L = 0.875, a/L = 0.5 and h/a = ξ = 1.5 and for applied tensile loading. Corresponding results obtained by finite element method, (FEM), (Raju and Newman, 1977) and the boundary integral equations method, (BIE), (Tan and Fenner, 19791 are also presented for comparison purposes. The K values from the plate middle plane z=0 to about z=0.6 ξ are in good agreement with those of FEM and BIE. At z=0, the K-value of the present method is about 1.438 while that of FEM is 1.401; it is of interest to note that the corresponding plane strain value is equal to about 1.43. In fact, as can be

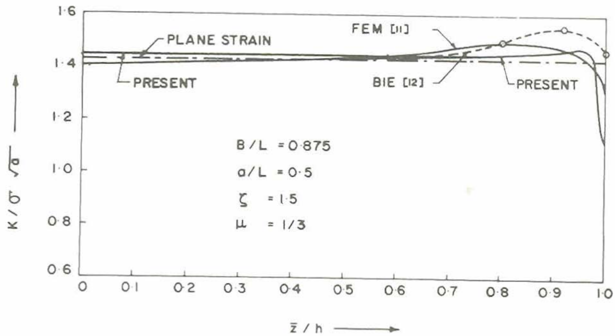

Fig. 2. SIF distribution across plate thickness

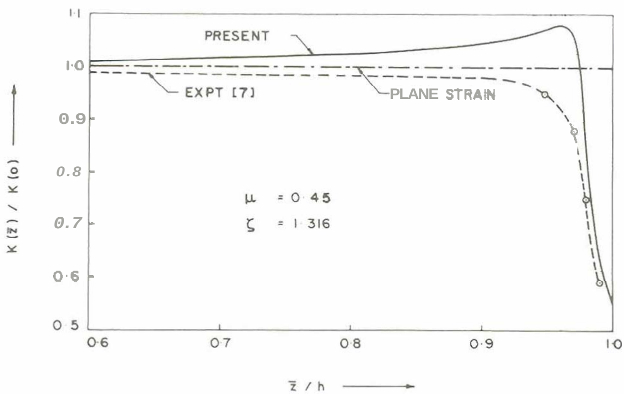

Fig. 3. SIF distribution across plate thickness (normalized by mid-plane value)

observed from Fig. 2, the singular deformation character is more or less equal to that of plane strain in the region interior to the plate thickness $z=0.0-0.6\zeta$; this is in agreement with the qualitative behaviour predicted in (Hartranft, 1969; Sih, 1966; Folias, 1975) for this zone. However, beyond this zone, the K-value increases gradually with increase in Z upto a maximum value and then drops rapidly in a small boundary layer near the plate faces. The location and magnitude of the maximum value of the present method are different from those of (Raju, 1977; Tan, 1979). Also, the K-value at $z=\zeta$ (free plate faces) is much less than those of these references; when compared to the mid-plane value the percentage drop is about 19.86 for the present case. In this connection it should be pointed out that the FEM can not represent accurately drastic changes in slopes such as those realized in present K-distribution in the boundary layer region. However, the slopes encountered at points $z \simeq \xi$ and at $z = \xi$ for the FEM case (see Fig. 2) do suggest a trend which leads to a larger drop than numerically indicated. Fig. 3 shows a comparison of the SIF distributions across the thickness normalised by the $K(o)$ value (mid-plane value) for the case with $\xi = 1.316$ obtained in this paper and by the photoelastic method (Villarreal, 1975). As can be observed from this figure, the agreement between the two distributions is reasonablly good for the region, $z = (0.0-0.85)\xi$, (approximately); the predominantly plane strain type of behaviour observed here confirms the qualitative results of (Hartranft, 1969; Sih, 1966; Polias, 1975) for this region. Although for larger values of z increasing differences are observed, the significant feature noticed here lies in the qualitatively identical distributions of the two methods, all through the thickness except in a small region $z = 0.9$ to 0.97ξ approximately. At the plate faces, the percentage reduction in the SIF value is found to be 45 as against 40 (approximately) of (Villerreal, 1975).

Based on the nature of the mathematical description of the singular field and the physical character of the numerically evaluated SIF distribution, verified by experimental results, it can be concluded that $1/\sqrt{r}$ is the only type of singularity present all through the plate thickness, although the strength of singularity decreases near the plate faces. Secondly, the region interior to the plate thickness experiences a more or less plane strain type of singular deformation; but, near the plate faces, there is a clear departure from this behaviour as suggested by the rapid decrease in SIF values. The displacements are finite at the crack front all through the plate thickness.

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