

# Decompositions of stochastic processes based on irreducible group representations

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## Abstract

Let  $G$  be a topological compact group acting on some space  $Y$ . We study a decomposition of  $Y$ -indexed stochastic processes, based on the orthogonality relations between the characters of the irreducible representations of  $G$ . In the particular case of a Gaussian process with a  $G$ -invariant law, such a decomposition gives a very general explanation of a classic identity in law – between quadratic functionals of a Brownian bridge – due to Watson (1961). Relations with Karhunen-Loève expansions are also discussed, and some further applications and extensions are given – in particular related to Gaussian processes indexed by a torus.

**Key words:** Double Wiener-Itô integrals; Flat Torus; Irreducible representations; Karhunen-Loève expansions; Quadratic functionals; Stochastic processes; Topological compact groups; Watson's duplication identity.

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## 1 Introduction

This paper deals with the study of a class of decompositions for stochastic processes, based on the theory of group representations. As clarified below, the initial impetus for our investigation was provided by the following *duplication identity* due to Watson (see [24]): if  $b$  is a standard Brownian bridge on  $[0, 1]$ , from 0 to 0, then

$$(1) \quad \int_0^1 \left( b(s) - \int_0^1 b(u) du \right)^2 ds \stackrel{\text{law}}{=} \frac{1}{4} \left\{ \int_0^1 b(s)^2 ds + \int_0^1 b_*(s)^2 ds \right\},$$

where  $b_*$  is an independent copy of  $b$ . The reader is referred to [21, p. 220] for a proof of (1) using Karhunen-Loève expansions, and to [20] for a probabilistic discussion based on several identities in law between Brownian functionals. Recently (see [14]), the second author gave a very short proof of (1), suggesting that Watson's result hides indeed a simple algebraic structure. In what follows, we shall bring this structure into light, by showing that (1) is an instance of a quite general phenomenon, related to the invariance properties of the law of  $b$ . In particular, we will provide an exhaustive answer to a question raised Z. Shi and M. Yor in [20]: *what is the probabilistic interpretation of the factor 1/4 in (1)?* Our general framework is roughly as follows.

Let  $G$  be a topological compact group acting on a set  $Y$ , and let  $Z(\omega, y) = Z(y)$  be a stochastic process indexed by the elements of  $Y$ . We will consider a decomposition of the paths of  $Z$ , realized by means of the orthogonality relations between the characters of the irreducible representations of  $G$  (see

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[7] or [19] for any unexplained notion about representations). More specifically, we define  $L^2(G)$  to be the space of complex-valued functions on  $G$  that are square integrable w.r.t. the Haar measure, and we denote by  $\widehat{G}$  the dual of  $G$  (i.e.,  $\widehat{G}$  is the collection of the equivalence classes of the irreducible representations of  $G$ ). Then, a classic result of representation theory states that  $L^2(G)$  can be decomposed into an orthogonal sum of finite dimensional spaces, indexed by the elements  $[\pi]$  of  $\widehat{G}$  and known as the *spaces of matrix coefficients* of irreducible representations. The projection operators on such orthogonal spaces have the form of convolutions with respect to the corresponding characters. Now write  $(g, y) \mapsto g \cdot y$  to indicate the action of  $G$  on  $Y$ , and consider a stochastic process  $Z(y)$ ,  $y \in Y$ , such that, for every fixed  $y \in Y$ , the application  $g \mapsto Z(g \cdot y)$  is in  $L^2(G)$ . One of our main results states that, if the law of  $Z$  is *invariant* with respect to the action of  $G$ , then the above described decomposition of  $L^2(G)$  translates into a (unique) decomposition of  $Z$  into the sum of simpler stochastic processes, each indexed by a distinct element of  $\widehat{G}$ . We write  $Z = \sum_{[\pi] \in \widehat{G}} Z^\pi$  for such a decomposition. In Section 3, we shall prove that, if  $Z$  has a  $G$ -invariant law, then, for distinct  $[\pi], [\sigma] \in \widehat{G}$ , the processes  $Z^\pi$  and  $Z^\sigma$  are non-correlated (in a probabilistic sense), and such that their paths are orthogonal with respect to *any*  $G$ -invariant measure on the parameter space  $Y$ . In particular, when  $Z$  is Gaussian and  $[\pi]$  and  $[\sigma]$  have real characters,  $Z^\pi$  and  $Z^\sigma$  are also Gaussian, and therefore stochastically independent. In the last section we shall discuss some connections between our decomposition and the Karhunen-Loève expansion (see for instance [1]) of suitably regular Gaussian processes.

As a by-product of our analysis, we will show that (1) derives from a very particular case of the decomposition described above. In particular, our results will make clear that there are two crucial elements behind (1), namely: (i) since  $b_t \stackrel{law}{=} b_{1-t}$  (as stochastic processes), the law of  $b(\cdot) - \int_0^1 b(u) du$  is invariant with respect to the elementary action, of  $G = \{1, g\} \simeq \mathbb{Z}/2\mathbb{Z}$  on  $[0, 1]$ , given by  $1 \cdot t = t$  and  $g \cdot t = 1 - t$ , and (ii) Lebesgue measure is invariant with respect to the same action of  $G$ . It follows that the above described theory can be applied, and (1) turns out to be the result of an orthogonal decomposition of the paths of  $b(\cdot) - \int_0^1 b(u) du$  into two independent components. More to the point, the factor  $1/4$  on the right hand side of (1) will appear as the square of a *normalization factor* ( $1/4 = 1/|G|^2$ ), which enters quite naturally into the expression of the projection operators associated to matrix coefficients. The generalizations of (1) given in [13] have similar interpretations in terms of group representations (see Section 4 below for the analysis of a *quadruplication identity*).

Note that, although we are mainly motivated by finite groups, in the first part of the paper we will work in the general framework of topological compact groups. See also [10, 11, 12] for some related results concerning processes defined on the sphere or on a compact commutative group.

The paper is organized as follows. In Section 2 we recall some basic facts about group representations and related orthogonality relations. Section 3 deals with decompositions of stochastic processes in the general case. Section 4 studies the specific setup of Gaussian processes, and contains an extended discussion of generalized Volterra processes. In particular, we establish some necessary and sufficient conditions (based on the method of cumulants) to have that such processes verify a relation analogous to (1). In the last section, we prove several refinements and applications, mainly related to Karhunen-Loève expansions and to Gaussian processes indexed by a torus.

## 2 Preliminaries and main results from group representation theory

### 2.1 Representations of compact groups and orthogonal decompositions

In this section, we shall present several definitions and results from the theory of representations of topological compact groups. Our use of this theory is mainly inspired by the discussion contained in [7, Chapter IV], where a strong accent is placed on the so-called **Peter-Weyl theorem** (see [7, Theorem

4.6.1], as well as the discussion below), and its consequences in terms of the decomposition of the  $L^2$  space associated with a topological compact group (when endowed with its Haar measure). The reader is referred to [7] for any unexplained definition or result. Other classic references for group representations are the monographs [19] and [9].

A **topological group** is a pair  $(G, \mathbb{G})$ , where  $G$  is a group and  $\mathbb{G}$  is a topology such that the following three conditions are satisfied: (i)  $G$  is a Hausdorff topological space, (ii) the multiplication  $G \times G \mapsto G : (g, h) \mapsto gh$  is continuous, (iii) the inversion  $G \mapsto G : g \mapsto g^{-1}$  is continuous. In what follows, when no further specification is given,  $G$  will always denote a topological group (the topology  $\mathbb{G}$  being implicitly defined) which is also **compact** (see e.g. [6, p. 34]) and such that  $\mathbb{G}$  has a countable basis. For such a  $G$ , we will denote by  $C(G)$  the class of continuous, complex-valued functions on  $G$ ;  $\mathcal{G}$  is the (Borel)  $\sigma$ -field generated by  $\mathbb{G}$ . An immediate consequence (see [7, Section 10.3]) of the structure imposed on  $G$ , is that  $G$  always carries a (unique) positive Borel measure, noted  $dg$  and known as the **Haar measure**, such that  $\int_G dg = 1$ , and  $\forall f \in C(G)$  and  $\forall h \in G$

$$\begin{aligned} \int_G f(g) dg &= \int_G f(g^{-1}) dg \\ \int_G f(hg) dg &= \int_G f(gh) dg = \int_G f(g) dg \quad (\text{left and right invariance}); \end{aligned}$$

we shall note  $L^2(G, dg) = L^2(G)$  the Hilbert space of complex valued functions on  $G$  that are square integrable with respect to  $dg$ , endowed with the usual inner product  $\langle f_1, f_2 \rangle_G = \int_G f_1(g) \overline{f_2(g)} dg$ . We note  $\|\cdot\|_G$  the norm associated with  $\langle \cdot, \cdot \rangle_G$ , and we observe that  $L^2(G)$  is the completion of  $C(G)$  with respect to  $\|\cdot\|_G$ .

**Remark** – When  $G$  is finite, then  $\mathbb{G}$  is necessarily the discrete topology, and  $dg$  coincides with the **normalized counting measure** associated with  $G$ , that is

$$dg = \frac{1}{|G|} \sum_{h \in G} \delta_h(dg),$$

where  $\delta_h(\cdot)$  stands for the Dirac mass concentrated at  $h$ , and  $|G|$  is the cardinality of  $G$ .

Let  $V$  be a topological vector space over  $\mathbb{C}$ . A **representation** of  $G$  in  $V$  is an homomorphism  $\pi$ , from  $G$  into  $\mathbf{GL}(V)$  (the set of complex isomorphisms of  $V$  into itself), such that the mapping  $G \times V \mapsto V : (g, v) \mapsto \pi(g)(v)$  is continuous. The **dimension**  $d_\pi$  of a representation  $\pi$  is defined to be the dimension of  $V$ . A representation  $\pi$  of  $G$  in  $V$  is **irreducible**, if the only closed  $\pi(G)$ -invariant subspaces of  $V$  are  $\{0\}$  and  $V$ . It is well known that irreducible representations are defined up to equivalence classes (see [7, p. 210]). Following [7], we will denote by  $[\pi]$  the equivalence class of a given irreducible representation  $\pi$ ; the set of equivalence classes of irreducible representations of  $G$  is noted  $\widehat{G}$ , and it is called the **dual** of  $G$ . Note that, in our setting, irreducible representations *are always finite dimensional*. Moreover, we will systematically assume (without loss of generality, see [7, Corollary 4.2.2]) that every irreducible representation is also unitary. Finally, we recall that, according e.g. to [7, Theorem 4.3.4 (v)], since  $G$  is second countable (and therefore metrizable)  $\widehat{G}$  is necessarily countable.

To every finite dimensional representation  $\pi : G \mapsto \mathbf{GL}(V)$  we associate the mapping

$$\chi_\pi : G \mapsto \mathbb{C} : g \mapsto \text{Trace } \pi(g),$$

called the **character** of  $\pi$ . Two finite-dimensional representations are equivalent if and only if they have the same character. Moreover, it is easily seen that characters are *central*<sup>1</sup> and *continuous* functions on  $G$ .

<sup>1</sup>That is, for every  $x, g \in G$ ,  $\chi_\pi(x^{-1}gx) = \chi_\pi(g)$ .

In this paper, we shall develop some Hilbert space techniques that are directly based on the orthogonality relations between the characters of distinct irreducible representations. To this end, with every  $[\pi] \in \widehat{G}$  we associate a finite dimensional subspace  $M_\pi \subseteq L^2(G)$  in the following way. Select an element  $\pi : G \mapsto \mathbf{GL}(V)$  in  $[\pi]$ , as well as a basis  $\mathbf{e} = \{e_1, \dots, e_n\}$  of  $V$  (plainly,  $n = d_\pi$ ) with respect to which  $\pi$  is unitary; the space  $M_\pi$  is defined as the set of the (complex) linear combinations of the **matrix coefficients** associated with  $\pi$  and with the basis  $\mathbf{e}$ , that is,  $M_\pi$  is composed of the linear combinations of the functions

$$g \mapsto \pi(g)_k^j, \quad j, k = 1, \dots, n,$$

where, for each  $g \in G$ ,  $\{\pi(g)_k^j : j, k = 1, \dots, n\}$  is the matrix representation of  $\pi(g)$  with respect to the basis  $\mathbf{e}$ . Note that such a definition is well given, as  $M_\pi$  does not depend on the choices of the representative element of  $[\pi]$  and of the basis of  $V$ . Of course,  $M_\pi$  is finite dimensional (and therefore closed; more precisely:  $\dim M_\pi = d_\pi^2$ , see [7, Theorem 4.3.4], as well as the discussion below) and  $M_\pi \subseteq C(G)$ , for every  $[\pi] \in \widehat{G}$ .

Before stating one of the crucial results for our analysis, we introduce a **convolution** operation on  $L^2(G)$ , which is defined, for  $f, k \in L^2(G)$ , by the formula

$$(2) \quad (f * k)(u) = \int_G f(g) k(g^{-1}u) dg = \int_G f(ug^{-1}) k(g) dg, \quad u \in G.$$

The following result summarizes all the orthogonality relations – associated with the notion of character – that are relevant to our discussion (for proofs and further analysis in this direction, the reader is referred to [7, paragraphs 4.2–4.6]).

**Theorem 1** *Let the above notation and assumptions prevail. Then,*

1. *if  $[\pi], [\sigma] \in \widehat{G}$ , and  $[\pi] \neq [\sigma]$ , the spaces  $M_\pi$  and  $M_\sigma$  are orthogonal in  $L^2(G)$ ;*
2. *for every  $[\pi] \in \widehat{G}$  the orthogonal projection operator, from  $L^2(G)$  to  $M_\pi$ , is given by*

$$(3) \quad E_\pi : L^2(G) \rightarrow M_\pi : f \mapsto d_\pi (f * \chi_\pi) := E_\pi f;$$

3. *the class  $\{M_\pi : [\pi] \in \widehat{G}\}$  is total in  $L^2(G)$ , and therefore*

$$(4) \quad L^2(G) = \bigoplus_{[\pi] \in \widehat{G}} M_\pi$$

where  $\bigoplus$  stands for a direct Hilbert space (orthogonal) sum;

4. *for every  $[\pi] \in \widehat{G}$ ,*

$$(5) \quad \langle \chi_\pi, \chi_\pi \rangle_G = 1, \quad E_\pi \chi_\pi = \chi_\pi \quad \text{and} \quad E_\sigma \chi_\pi = 0 \quad \text{if } [\sigma] \in \widehat{G} \text{ and } [\pi] \neq [\sigma],$$

and consequently  $\{\chi_\pi : [\pi] \in \widehat{G}\}$  is an orthonormal system in  $L^2(G)$ .

**Remarks** – (i) According e.g. to [7, p. 221], the **right regular representation**  $R^*$  of  $G$  on  $L^2(G)$  is defined as

$$(R^*(g)f)(x) = f(xg),$$

where  $g, x \in G$ , and  $f \in L^2(G)$ . Then, for every  $[\pi] \in \widehat{G}$ , the space  $M_\pi$  coincides with the  $\pi$ -**isotypical** subspace associated with  $R^*$  (see [7, Theorem 4.4.5]). More to the point (see [7, Corollary 4.3.6]), the restriction of  $R^*$  on  $M_\pi$  is equal to the direct sum of  $d_\pi$  copies of  $\pi$ .

(ii) Since, for every  $[\pi] \in \widehat{G}$ , the function  $G \ni g \mapsto \chi_\pi(g)$  is conjugacy-invariant, the projection  $E_\pi f$ , as defined in (3), is also equal to  $d_\pi(\chi_\pi * f)$ .

(iii) Point 3 of Theorem 1 can be seen as a direct consequence of the **Peter-Weyl theorem** (see [7, Theorem 4.6.1]), stating that the characters of the irreducible representations of  $G$  form an orthonormal basis of the subspace of  $L^2(G)$  composed of central and square integrable functions.

(iv) For future reference, we recall that the following four conditions are equivalent (see [7, p. 235]):

(a)  $G$  is Abelian, (b)  $d_\pi = 1$  for every  $[\pi] \in \widehat{G}$ , (c) every  $f \in C(G)$  is conjugacy-invariant, (d) the convolution operation defined in (2) is commutative. In particular, if  $G$  is Abelian, then the system  $\{\chi_\pi : [\pi] \in \widehat{G}\}$  is orthonormal *and complete* in  $L^2(G)$ . If  $G$  is Abelian and finite, then  $|G| = |\widehat{G}|$ .

## 2.2 Actions and decompositions of complex-valued functions

Consider a measurable space  $(X, \mathcal{X})$ . In this paper, a **left action**  $A$  of  $G$  on  $X$  is a  $\mathcal{G} \otimes \mathcal{X}$ -measurable function, from  $G \times X$  to  $X$  (recall that  $\mathcal{G}$  is the Borel  $\sigma$ -field of  $G$ ), such that, for every  $g, h \in G$  and  $x \in X$ ,

$$A(gh, x) = A(g, A(h, x)).$$

A right action can be defined in a similar way, but we will deal only with left actions; for the sake of simplicity, in the sequel left actions are simply called **actions**. When there is no ambiguity on the action  $A$ , we will sometimes use the customary abbreviation

$$A(g, x) = g \cdot x \quad (g \in G, x \in X).$$

A  $\sigma$ -finite, positive measure  $\nu$  on  $(X, \mathcal{X})$ , is said to be **invariant with respect to the action  $A$  of  $G$**  (or simply, again when there is no ambiguity on the action  $A$ ,  **$G$ -invariant**) if, for every complex valued function  $f \in L^1(\nu)$ ,

$$\int_X f(x) \nu(dx) = \int_X f(A(g, x)) \nu(dx) = \int_X f(g \cdot x) \nu(dx),$$

for every  $g \in G$ .

Now fix an action  $A$  of  $G$  on  $X$ , and consider a measurable function  $Z : X \rightarrow \mathbb{C}$ . We associate to  $Z$  the function

$$(6) \quad Z_* : G \times X \rightarrow \mathbb{C} : (g, x) \mapsto Z_*(g, x) = Z(A(g, x)) = Z(g \cdot x),$$

which is of course  $\mathcal{G} \otimes \mathcal{X}$ -measurable. For each fixed  $x \in X$ , we define the  $\mathcal{G}$ -measurable function

$$(7) \quad Z_G[x] : G \rightarrow \mathbb{C} : g \mapsto Z_*(g, x) ;$$

analogously, for each fixed  $g \in G$ , we note

$$(8) \quad Z_X[g] : X \rightarrow \mathbb{C} : x \mapsto Z_*(g, x),$$

which defines in turn a  $\mathcal{X}$ -measurable mapping. If, for some fixed  $x \in X$ , the above introduced function  $Z_G[x]$  is an element of  $L^2(G)$ , we set, for each  $[\pi] \in \widehat{G}$  and  $g \in G$ ,

$$(9) \quad Z_*^\pi(g, x) = E_\pi Z_G[x](g),$$

where, by using Theorem 1 and (2),

$$(10) \quad E_\pi Z_G[x](g) = d_\pi \int_G \chi_\pi(h) Z_G[x](h^{-1}g) dh = d_\pi \int_G \chi_\pi(h) Z(h^{-1}g \cdot x) dh, \quad g \in G.$$

As usual, we write  $e$  to indicate the **identity** element of the group  $G$ . If, for a measurable mapping  $Z : X \rightarrow \mathbb{C}$ ,  $Z_G[x] \in L^2(G)$  for every  $x \in X$ , we introduce the two functions, defined respectively for a fixed  $x \in X$  and for a fixed  $g \in G$ ,

$$(11) \quad \begin{aligned} Z_G^\pi[x] &: G \rightarrow \mathbb{C} &: g \mapsto Z_*^\pi(g, x) \\ Z_X^\pi[g] &: X \rightarrow \mathbb{C} &: x \mapsto Z_*^\pi(g, x); \end{aligned}$$

finally, for such a  $Z$ , we set

$$(12) \quad Z^\pi(x) = Z_X^\pi[e](x) = Z_*^\pi(e, x) = E_\pi Z_G[x](e) = d_\pi \int_G \chi_\pi(g) Z(g^{-1} \cdot x) dg, \quad x \in X.$$

Note that, since  $A$  is a left action,

$$(13) \quad Z_X^\pi[g](x) = Z^\pi(g \cdot x), \quad g \in G, \quad x \in X.$$

**Remark** – If the function  $Z$  is such that  $Z_G[x] \in L^2(G)$  for every  $x \in X$ , then the mapping  $(g, x) \mapsto Z_*^\pi(g, x)$  is  $\mathcal{G} \otimes \mathcal{X}$ -measurable. It follows that the two mappings  $Z_G^\pi[x]$  and  $Z_X^\pi[g]$  defined in (11) are, respectively,  $\mathcal{G}$ -measurable and  $\mathcal{X}$ -measurable. In particular, the application  $x \mapsto Z^\pi(x)$  (as defined in (12)) is a  $\mathcal{X}$ -measurable mapping.

The following result turns out to be the key tool of our analysis.

**Theorem 2** *Under the above notation and assumptions, fix an action  $A$  of the group  $G$  on  $X$ . Consider moreover two measurable functions  $S, Z : X \rightarrow \mathbb{C}$ , such that for each  $x \in X$ ,  $S_G[x], Z_G[x] \in L^2(G)$ . Then,*

1. *for any  $[\pi], [\sigma] \in \widehat{G}$  such that  $[\pi] \neq [\sigma]$ , and for arbitrary points  $x_1, x_2 \in X$ , the following orthogonality relation is satisfied:*

$$(14) \quad \langle S_G^\pi[x_1], Z_G^\sigma[x_2] \rangle_G = 0;$$

2. *for every  $x \in X$ ,*

$$(15) \quad Z_G[x] = \sum_{[\pi] \in \widehat{G}} Z_G^\pi[x] \quad \text{and} \quad S_G[x] = \sum_{[\pi] \in \widehat{G}} S_G^\pi[x],$$

*where the convergence of the (possibly infinite) series takes place in  $L^2(G)$ , and for any  $x_1, x_2 \in X$*

$$(16) \quad \langle S_G[x_1], Z_G[x_2] \rangle_G = \sum_{[\pi] \in \widehat{G}} \langle S_G^\pi[x_1], Z_G^\pi[x_2] \rangle_G,$$

*with convergence in  $\ell^2$ ;*

3. *in addition to the previous assumptions, suppose there exists a  $G$ -invariant measure  $\nu$  on  $(X, \mathcal{X})$ , such that the functions  $Z_*$  and  $S_*$ , defined according to (6), are elements of*

$$L^2(G \times X, \mathcal{G} \otimes \mathcal{X}, dg \times \nu(dx)) := L^2(dg \times \nu(dx))$$

*and also, for every  $g \in G$ ,  $Z_X[g], S_X[g] \in L^2(X, \mathcal{X}, \nu(dx)) := L^2(\nu(dx))$ ; then, for every  $[\pi] \in \widehat{G}$  and every  $g \in G$ ,  $Z_*^\pi, S_*^\pi \in L^2(dg \times \nu(dx))$ ,  $Z_X^\pi[g], S_X^\pi[g] \in L^2(\nu(dx))$ , and moreover*

$$(17) \quad \int_X S_X^\pi[g](x) \overline{Z_X^\pi[g](x)} \nu(dx) = 0$$

*for every  $[\pi], [\sigma] \in \widehat{G}$ , such that  $[\pi] \neq [\sigma]$ ;*

4. under the assumptions and notation of point 3.,

$$(18) \quad Z_* = \sum_{[\pi] \in \widehat{G}} Z_*^\pi \quad \text{and} \quad S_* = \sum_{[\pi] \in \widehat{G}} S_*^\pi$$

where the series are orthogonal and convergent in  $L^2(dg \times \nu(dx))$ , and therefore

$$(19) \quad \langle Z_*, S_* \rangle_{L^2(dg \times \nu(dx))} = \sum_{[\pi] \in \widehat{G}} \langle Z_*^\pi, S_*^\pi \rangle_{L^2(dg \times \nu(dx))};$$

5. under the assumptions and notation of point 3., for every  $g \in G$ ,

$$(20) \quad Z_X[g](x) = \sum_{[\pi] \in \widehat{G}} Z_X^\pi[g](x) \quad \text{and} \quad S_X[g](x) = \sum_{[\pi] \in \widehat{G}} S_X^\pi[g](x)$$

where the series are orthogonal and convergent in  $L^2(\nu(dx))$ , and

$$(21) \quad \langle Z_X[g], S_X[g] \rangle_{L^2(\nu(dx))} = \sum_{[\pi] \in \widehat{G}} \langle Z_X^\pi[g], S_X^\pi[g] \rangle_{L^2(\nu(dx))}.$$

**Remark** – When  $G$  is finite,  $\widehat{G}$  is also finite (since  $|\widehat{G}|$  coincides with the number of conjugacy classes in  $G$ ). In this case, Theorem 2-2 gives a decomposition of the function  $Z : X \rightarrow \mathbb{C}$ . As a matter of fact, for every  $x \in X$ ,

$$(22) \quad Z(x) = \sum_{[\pi] \in \widehat{G}} Z^\pi(x),$$

where the sum is finite, and on the right-hand side we use the notation introduced in (12).

**Proof of Theorem 2.** (1.) By definition,  $S_G^\pi[x_1]$  and  $Z_G^\sigma[x_2]$  equal the orthogonal projections, respectively of  $S_G[x_1]$  and  $Z_G[x_2]$ , on the finite dimensional spaces  $M_\pi$  and  $M_\sigma$ . Since, according to Theorem 1-1,  $M_\pi$  and  $M_\sigma$  are orthogonal in  $L^2(G)$ , relation (14) follows.

(2.) Relation (15) is an immediate consequence of (4), whereas (16) is a standard formula of the Parseval-Plancherel type.

(3.) Observe first that, by assumption,

$$\int_X \left[ \int_G |Z_*(h, x)|^2 dh \right] \nu(dx) < +\infty.$$

Moreover, we observe that, for every  $[\pi] \in \widehat{G}$ , the continuous function  $|\chi_\pi| : G \rightarrow \mathbb{R}_+$  is bounded by a constant  $\alpha_\pi \in (0, +\infty)$  (since  $G$  is compact), and therefore, thanks to the right invariance of the Haar measure  $dh$  and Jensen inequality,

$$\begin{aligned} \int_X |Z_X^\pi[g](x)|^2 \nu(dx) &= d_\pi^2 \int_X \left| \int_G \chi_\pi(h) Z_*(h^{-1}g, x) dh \right|^2 \nu(dx) \\ &\leq d_\pi^2 \alpha_\pi^2 \int_X \left[ \int_G |Z_*(h^{-1}g, x)|^2 dh \right] \nu(dx) \\ &= d_\pi^2 \alpha_\pi^2 \int_X \left[ \int_G |Z_*(h, x)|^2 dh \right] \nu(dx) < +\infty. \end{aligned}$$

Also,

$$\begin{aligned}
\int_G \int_X |Z_*^\pi(g, x)|^2 \nu(dx) dg &= \int_G \int_X |Z_G^\pi[g](x)|^2 \nu(dx) dg \\
&\leq d_\pi^2 \alpha_\pi^2 \int_G \int_X \left[ \int_G |Z_*(h, x)|^2 dh \right] \nu(dx) dg \\
&= d_\pi^2 \alpha_\pi^2 \int_X \left[ \int_G |Z_*(h, x)|^2 dh \right] \nu(dx) < +\infty,
\end{aligned}$$

since  $\int_G dg = 1$ . It follows that  $Z_*^\pi \in L^2(dg \times \nu(dx))$  and  $Z_X^\pi[g] \in L^2(\nu(dx))$  for any  $g \in G$ , and an analogous conclusion holds for  $S$ . We may prove (17) by using an easy declination of the ‘‘averaging’’ technique (see for instance [7, paragraph 4.2]). Indeed, for  $[\pi], [\sigma] \in \widehat{G}$  such that  $[\pi] \neq [\sigma]$ , thanks to formula (13) and the  $G$ -invariance of  $\nu$ ,

$$\begin{aligned}
\int_X S_X^\pi[g](x) \overline{Z_X^\sigma[g](x)} \nu(dx) &= \int_X S^\pi(g \cdot x) \overline{Z^\sigma(g \cdot x)} \nu(dx) \\
&= \int_G \int_X S^\pi(h \cdot x) \overline{Z^\sigma(h \cdot x)} \nu(dx) dh \\
&= \int_X \left[ \int_G S^\pi(h \cdot x) \overline{Z^\sigma(h \cdot x)} dh \right] \nu(dx) \\
&= \int_X \langle S_G^\pi[x], Z_G^\sigma[x] \rangle_G \nu(dx) = 0,
\end{aligned}$$

where we have used a standard Fubini theorem, as well as Theorem 2-1.

(4.) The first part derives immediately from points 1. and 2., as well as the fact that  $Z_*, S_* \in L^2(dg \times \nu(dx))$  by assumption. Formula (19) is again of the Parseval-Plancherel type.

(5.) Formula (20) derives from the elementary relation

$$\int_X \left| Z_X[g](x) - \sum_{[\pi] \in \widehat{G}} Z_X^\pi[g](x) \right|^2 \nu(dx) = \int_G \int_X \left| Z_*(h, x) - \sum_{[\pi] \in \widehat{G}} Z_*^\pi(h, x) \right|^2 \nu(dx) dh = 0,$$

where the first equality is due to the  $G$ -invariance of  $\nu$ , and the second comes from point 4. Relation (21) is straightforward. ■

### 3 Decompositions of stochastic processes: general results

Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a probability space, and let  $(Y, \mathcal{Y})$  be a measurable space. A  $Y$ -indexed stochastic process  $Z$  is a  $\mathcal{F} \otimes \mathcal{Y}$ -measurable application  $Z : \Omega \times Y \rightarrow \mathbb{C} : (\omega, y) \mapsto Z(\omega, y)$ <sup>2</sup>. To simplify some arguments, we shall systematically suppose that the  $\sigma$ -field  $\mathcal{F}$  contains the singletons, that is,  $\{\omega\} \in \mathcal{F}$  for every  $\omega \in \Omega$ .

In this section, the product space  $\Omega \times Y$  will play roughly the same role as the space  $(X, \mathcal{X})$  in Section 2. As a consequence, we shall sometimes use the compact notation

$$(23) \quad \Omega \times Y = X_0, \quad \mathcal{F} \otimes \mathcal{Y} = \mathcal{X}_0,$$

and write  $x_0$  to indicate the generic element  $(\omega, y)$  of  $X_0$ . Given a topological compact group  $G$  and an action  $A$  of  $G$  on  $X_0$ , for fixed  $y \in Y$  and  $g \in G$ , we write  $Z(g \cdot y)$  to indicate the random variable

$$\Omega \ni \omega \mapsto Z(g \cdot (\omega, y)).$$

<sup>2</sup>We write  $Z(y)$ , for fixed  $y \in Y$ , to indicate the random variable  $\omega \mapsto Z(\omega, y)$ .



We say that the law of a family  $\mathbf{Z} = \{Z_i : i \in I\}$  of stochastic processes is **invariant with respect to the action**  $A$  of  $G$  on  $X_0$  (or, simply,  **$G$ -invariant**) if, for every  $n \geq 1$  and every continuous, bounded function  $f$  on  $\mathbb{C}^n$

$$\mathbf{E}[f(Z_{i_1}(y_1), \dots, Z_{i_n}(y_n))] = \mathbf{E}[f(Z_{i_1}(g \cdot y_1), \dots, Z_{i_n}(g \cdot y_n))]$$

for every  $g \in G$ , every  $(y_1, \dots, y_n) \in Y^n$ , and every  $(i_1, \dots, i_n) \in I^n$ .

**Remark** – Every action  $A'$  of  $G$  on  $Y$  always defines an action  $A$  on  $X_0$ , through the relation: for every  $x_0 = (\omega, y) \in X_0$ ,

$$(24) \quad A(g, x_0) = g \cdot x_0 = (\omega, A'(g, y)).$$

Analogously, every action  $\underline{A}'$  of  $G$  on  $\Omega$  defines an action  $\underline{A}$  on  $X_0$ : for every  $x_0 = (\omega, y) \in X_0$ ,

$$(25) \quad \underline{A}(g, x_0) = g \cdot x_0 = (\underline{A}'(g, \omega), y);$$

note that, if  $A$  has the form (24) and  $Y$  is **homogeneous**, then every  $Y$ -indexed process  $Z$  with a  $G$ -invariant law is also **stationary**, in the sense that  $Z(y) \stackrel{\text{law}}{=} Z(y')$  for every  $y, y' \in Y$ .

In the sequel, whenever it is given an action  $A' : (g, y) \mapsto g \cdot y$  of  $G$  on  $Y$ , we will write  $g \cdot x_0$ ,  $x_0 \in X_0$ , to indicate the image of the action  $A$  on  $X_0$  defined in (24); a similar convention, based on (25), holds for actions  $\underline{A}'$  on  $\Omega$ . Moreover, we will systematically work under the following assumption.

**Assumption A** – Every  $Y$ -indexed stochastic process  $Z$  considered in the following (not necessarily with a  $G$ -invariant law) is such that, for every  $x_0 = (\omega, y) \in \Omega \times Y$ , the mapping

$$g \mapsto Z(g \cdot x_0)$$

is an element of  $L^2(G)$ .

**Remark** – Assumption A can be relaxed in several directions: for instance, at the cost of some heavier notation, most of the subsequent results can be immediately extended to stochastic processes  $Z$  such that, for every fixed  $y \in Y$ , the mapping  $g \mapsto Z(g \cdot y)$  is in  $L^2(G)$  a.s.- $\mathbf{P}$ . Note that when  $G$  is finite Assumption A becomes immaterial.

Now fix an action  $A$  of  $G$  on  $X_0$ . To every  $Y$ -indexed stochastic process  $Z$  we associate: the mapping  $Z_* : G \times X_0 \mapsto \mathbb{C}$ , according to (6), and the mappings  $Z_G[x_0] : G \mapsto \mathbb{C}$  and  $Z_{X_0}[g] : X_0 \mapsto \mathbb{C}$  as given, respectively, by (7) for fixed  $x_0 = (\omega, y) \in X_0$ , and by (8) for fixed  $g \in G$ . Analogously, for every  $[\pi] \in \widehat{G}$ , the mapping  $Z_*^\pi : G \times X_0 \mapsto \mathbb{C}$ , is defined according to (9), whereas, for fixed  $x_0 \in X_0$  and for fixed  $g \in G$ , respectively,  $Z_G^\pi[x_0] : G \mapsto \mathbb{C}$  and  $Z_{X_0}^\pi[g] : X_0 \mapsto \mathbb{C}$ , are defined through (11). Finally, the mapping  $Z^\pi : X_0 \mapsto \mathbb{C}$  is given by (12).

**Proposition 3** *Under the above notation and assumptions:*

1. for every fixed  $x_0 \in X_0$  and for every  $[\pi] \in \widehat{G}$ ,  $Z_G[x_0]$  and  $Z_G^\pi[x_0]$  are  $(G, \mathcal{G})$ -measurable functions;
2. for every fixed  $g \in G$  and for every  $[\pi] \in \widehat{G}$ ,  $Z_{X_0}[g]$  and  $Z_{X_0}^\pi[g]$  are  $Y$ -indexed stochastic processes;
3. if  $Z$  has a  $G$ -invariant law the following three statements hold: (3-i) for every  $g \in G$ , the law of  $Z_{X_0}[g]$  is  $G$ -invariant and equal to the law of  $Z$ ; (3-ii) for every  $[\pi] \in \widehat{G}$  and  $g \in G$ , the law of  $Z_{X_0}^\pi[g]$  is  $G$ -invariant and equal to the law of  $Z^\pi$ ; (3-iii) the set of stochastic processes  $\{Z, Z^\pi : [\pi] \in \widehat{G}\}$  has a  $G$ -invariant law.

**Proof.** Points 1. and 2. are straightforward. Point (3-i) derives immediately from the relation: for every  $x_0 \in X_0$

$$Z_{X_0} [g] (h \cdot x_0) = Z (gh \cdot x_0), \quad \forall h \in G,$$

and the fact that the law of  $Z$  is  $G$ -invariant. To prove point (3-ii), we can first use the invariance properties of  $dg$ , as well as the fact that  $\chi_\pi(\cdot)$  is central for every  $[\pi] \in \widehat{G}$ , to obtain that for any  $h \in G$

$$(26) \quad \begin{aligned} Z^\pi (h \cdot x) &= \int_G Z (gh \cdot x) \chi_\pi (g^{-1}) dg = \int_G Z (g \cdot x) \chi_\pi (hg^{-1}) dg \\ &= \int_G Z (g \cdot x) \chi_\pi (g^{-1}h) dg = \int_G Z (h \cdot (g \cdot x)) \chi_\pi (g^{-1}) dg \\ &= \int_G Z_X [h] (g \cdot x) \chi_\pi (g^{-1}) dg, \end{aligned}$$

from which deduce that  $Z^\pi$  has a  $G$ -invariant law since, thanks to point (3-i),  $Z_X [h]$  has the same law as  $Z$ . To conclude, just use relation (13), and again point (3-i) applied to the process  $Z^\pi$ . Point (3-iii) derives immediately from formula (26). ■

The following result translates the first part of Theorem 2 into the context of this section. It shows, in particular, that any  $G$ -invariant stochastic process admits a pointwise  $L^2$ -decomposition in terms of simpler  $G$ -invariant stochastic processes, indexed by the elements of  $\widehat{G}$ .

**Theorem 4** *Let the above notation prevail, and consider an action  $A$  of  $G$  on  $X_0 = \Omega \times Y$ . Let  $S$  and  $Z$  be two  $Y$ -indexed stochastic processes verifying Assumption A, and fix  $[\pi], [\sigma] \in \widehat{G}$  such that  $[\pi] \neq [\sigma]$ . Then,*

1. for any  $(\omega_1, y_1), (\omega_2, y_2) \in X_0$ ,  $\langle S_G^\pi [(\omega_1, y_1)], Z_G^\sigma [(\omega_2, y_2)] \rangle_G = 0$ ;
2. if, for some  $y_1, y_2 \in Y$ ,  $S(y_1), Z(y_2) \in L^2(\mathbf{P})$ , then  $S^\pi(y_1), Z^\sigma(y_2) \in L^2(\mathbf{P})$ ;
3. if the vector  $(S, Z)$  has a  $G$ -invariant law and  $S(y_1), Z(y_2) \in L^2(\mathbf{P})$ , then

$$(27) \quad \mathbf{E} \left[ S^\pi(y_1) \overline{Z^\sigma(y_2)} \right] = 0;$$

4. if  $S$  has a  $G$ -invariant law and  $S(y_1) \in L^2(\mathbf{P})$ , then

$$(28) \quad S(y_1) = \sum_{[\pi] \in \widehat{G}} S^\pi(y_1),$$

where the series on the right hand side is orthogonal and convergent in  $L^2(\mathbf{P})$ .

**Proof.** Point 1. is a direct consequence of Theorem 2-1, whereas point 2. derives from the inequality

$$\begin{aligned} \mathbf{E} \left[ |S^\pi(y_1)|^2 \right] &= d_\pi^2 \int_\Omega \mathbf{P}(d\omega) \left| \int_G \chi_\pi(g) S(g^{-1} \cdot (\omega, y_1)) dg \right|^2 \\ &\leq d_\pi^2 \alpha_\pi^2 \int_\Omega \mathbf{P}(d\omega) \int_G |S(g^{-1} \cdot (\omega, y_1))|^2 dg \\ &= d_\pi^2 \alpha_\pi^2 \int_G \mathbf{E} \left[ |S(g^{-1} \cdot y_1)|^2 \right] dg = d_\pi^2 \alpha_\pi^2 \mathbf{E} \left[ |S(y_1)|^2 \right] < +\infty, \end{aligned}$$

and a similar calculation for  $Z^\pi$ . To see point 3., just write, due to the  $G$ -invariance of  $(S, Z)$  and the fact that  $\int_G dg = 1$ ,

$$\begin{aligned} \mathbf{E} \left[ S^\pi(y_1) \overline{Z^\sigma(y_2)} \right] &= \int_G \mathbf{E} \left[ S^\pi(g \cdot y_1) \overline{Z^\sigma(g \cdot y_2)} \right] dg \\ &= \int_\Omega \left\{ \int_G S^\pi [(\omega, y_1)](g) \overline{Z^\sigma [(\omega, y_2)](g)} dg \right\} \mathbf{P}(d\omega) \\ &= \int_\Omega \langle S_G^\pi [(\omega, y_1)], Z_G^\sigma [(\omega, y_2)] \rangle_G \mathbf{P}(d\omega) = 0, \end{aligned}$$

where we have used a Fubini theorem, as well as point 1., with  $\omega_1 = \omega_2 = \omega$ . To prove point 4., let  $[\pi(i)]$ ,  $i = 1, 2, \dots$ , be an enumeration of the elements of  $\widehat{G}$ , and observe that, according to (15), for every  $x_0 = (\omega, y_1) \in X_0$

$$\lim_{N \rightarrow +\infty} \int_G \left| S_G[x_0](g) - \sum_{i=1}^N S_G^{\pi(i)}[x_0](g) \right|^2 dg = 0,$$

and also, thanks to (16),

$$\int_G \left| S_G[x_0](g) - \sum_{i=1}^N S_G^{\pi(i)}[x_0](g) \right|^2 dg = \int_G \left| \sum_{i=N+1}^{\infty} S_G^{\pi(i)}[x_0](g) \right|^2 dg \leq \int_G |S_G[x_0](g)|^2 dg.$$

Now observe that, for fixed  $y_1 \in Y$ , the random variable  $\omega \mapsto \int_G |S(g \cdot (\omega, y_1))|^2 dg$  is in  $L^1(\mathbf{P})$ , since, due to the  $G$ -invariance of the law of  $S$ ,

$$\int_{\Omega} \left[ \int_G |S(g \cdot (\omega, y_1))|^2 dg \right] \mathbf{P}(d\omega) = \int_G \mathbf{E} \left[ |S(g \cdot y_1)|^2 \right] dg = \mathbf{E} \left[ |S(y_1)|^2 \right] < +\infty.$$

Finally, according to Proposition 3-3-iii, the class  $\left\{ S, S^{\pi} : [\pi] \in \widehat{G} \right\}$  has a  $G$ -invariant law, and therefore

$$\begin{aligned} \mathbf{E} \left[ \left| S(y_1) - \sum_{i=1}^N S^{\pi(i)}(y_1) \right|^2 \right] &= \int_G \mathbf{E} \left[ \left| S(g \cdot y_1) - \sum_{i=1}^N S^{\pi(i)}(g \cdot y_1) \right|^2 \right] dg \\ &= \mathbf{E} \left[ \int_G \left| S_G[x_0](g) - \sum_{i=1}^N S_G^{\pi(i)}[x_0](g) \right|^2 dg \right] \xrightarrow{N \rightarrow +\infty} 0, \end{aligned}$$

due to an application of the dominated convergence theorem. ■

When  $G$  is finite, formula (28) holds even if the law of  $S$  is not  $G$ -invariant (but, in this case, the series is not necessarily orthogonal in  $L^2(\mathbf{P})$ ). We now apply Theorem 2 to further characterize actions of the specific form (24). Observe that the following Theorem applies to processes whose laws are not necessarily  $G$ -invariant.

**Theorem 5** *Let the action  $A : G \times X_0 \mapsto X_0$  be such that,  $\forall (\omega, y) \in X_0$ ,  $A(g, (\omega, y)) = (\omega, A'(g, y))$ , where  $A'$  is an action on  $(Y, \mathcal{Y})$ . Consider moreover two  $Y$ -indexed stochastic processes  $S, Z$  (not necessarily with  $G$ -invariant laws), as well as a  $\sigma$ -finite positive measure  $\mu$  on  $(Y, \mathcal{Y})$ , which is invariant with respect to the action  $A'$  of  $G$  on  $Y$ . Suppose that, for every fixed  $\omega^* \in \Omega$ , the applications  $(g, y) \mapsto Z(\omega^*, A'(g, y))$  and  $(g, y) \mapsto S(\omega^*, A'(g, y))$  are elements of  $L^2(dg \times \mu(dy))$ , and also that, for every fixed  $(\omega^*, g^*) \in \Omega \times G$ , the mappings  $y \mapsto Z(\omega^*, A'(g^*, y))$  and  $y \mapsto S(\omega^*, A'(g^*, y))$  are in  $L^2(\mu(dy))$ . Then,*

1. for every fixed  $(\omega^*, g^*) \in \Omega \times G$ , and for every  $[\pi], [\sigma] \in \widehat{G}$  such that  $[\pi] \neq [\sigma]$ ,

$$(29) \quad \int_Y S_{X_0}^{\pi}[g^*](\omega^*, y) \overline{Z_{X_0}^{\sigma}[g^*](\omega^*, y)} \mu(dy) = 0;$$

2. for every fixed  $\omega^* \in \Omega$ ,

$$(30) \quad S(\omega^*, y) = \sum_{[\pi] \in \widehat{G}} S^{\pi}(\omega^*, y) \quad \text{and} \quad Z(\omega^*, y) = \sum_{[\pi] \in \widehat{G}} Z^{\pi}(\omega^*, y),$$

where the two series are orthogonal and convergent in  $L^2(\mu(dy))$ , and therefore

$$(31) \quad \langle S(\omega^*, \cdot), Z(\omega^*, \cdot) \rangle_{L^2(\mu(dy))} = \sum_{[\pi] \in \widehat{G}} \langle S^\pi(\omega^*, \cdot), Z^\pi(\omega^*, \cdot) \rangle_{L^2(\mu(dy))};$$

3. if moreover  $Z(\omega, y) \in L^2(\mathbf{P}(d\omega) \times \mu(dy))$ , then,

$$Z^\pi(\omega, y) \in L^2(\mathbf{P}(d\omega) \times \mu(dy)), \text{ for every } [\pi] \in \widehat{G},$$

and

$$(32) \quad Z(\omega, y) = \sum_{[\pi] \in \widehat{G}} Z^\pi(\omega, y),$$

where the series is orthogonal and convergent in  $L^2(\mathbf{P}(d\omega) \times \mu(dy))$ .

**Proof. (1.)** For every  $\omega^* \in \Omega$ , the measure  $\nu^*$  on  $(X_0, \mathcal{X}_0) = (\Omega \times Y, \mathcal{F} \otimes \mathcal{Y})$ , defined by  $\nu^*(d\omega, dy) = \delta_{\omega^*}(d\omega) \times \mu(dy)$ , where  $\delta_{\omega^*}$  is the Dirac mass at  $\omega^*$ , is invariant with respect to the action  $A$  of  $G$  on  $X_0$ . Moreover, it is easily seen that the assumptions in the statement imply that  $\nu^*$  satisfies all the hypotheses of Theorem 2-3, so that formula (29) follows immediately, by observing that, for every  $g \in G$ ,

$$\int_Y S_{X_0}^\pi[g](\omega^*, y) \overline{Z_{X_0}^\sigma[g](\omega^*, y)} \mu(dy) = \int_{X_0} S_{X_0}^\pi[g](x_0) \overline{Z_{X_0}^\sigma[g](x_0)} \nu^*(dx_0).$$

(2.) This is a direct consequence of Theorem 2-5 (in the case  $g = e$ ).

(3.) First observe that  $Z^\pi(\omega, y) \in L^2(\mathbf{P}(d\omega) \times \mu(dy))$ , since, thanks to the  $G$ -invariance of  $\mu$ ,

$$\begin{aligned} \mathbf{E} \left[ \int_Y |Z^\pi(y)|^2 \mu(dy) \right] &\leq d_\pi^2 \alpha_\pi^2 \int_Y \mu(dy) \int_\Omega \mathbf{P}(d\omega) \int_G dg |Z(\omega, g^{-1} \cdot y)|^2 \\ &= d_\pi^2 \alpha_\pi^2 \int_G \int_Y \mathbf{E} \left[ |Z(g^{-1} \cdot y)|^2 \right] \mu(dy) dg \\ &= d_\pi^2 \alpha_\pi^2 \int_Y \mathbf{E} \left[ |Z(y)|^2 \right] \mu(dy) < +\infty. \end{aligned}$$

The rest of the proof is similar to the proof of Theorem 3-4. Let indeed  $[\pi(i)]$ ,  $i = 1, 2, \dots$ , be an enumeration of  $\widehat{G}$ , and observe that Theorem 2-4 (formula (18)) implies that, for every  $\omega^* \in \Omega$ ,

$$\begin{aligned} &\lim_{N \rightarrow +\infty} \int_Y \int_G \left| Z(\omega^*, g \cdot y) - \sum_{i=1}^N Z^{\pi(i)}(\omega^*, g \cdot y) \right|^2 dg \mu(dy) \\ &= \lim_{N \rightarrow +\infty} \int_{X_0} \int_G \left| Z(g \cdot x_0) - \sum_{i=1}^N Z^{\pi(i)}(g \cdot x_0) \right|^2 dg \nu^*(dx_0) = 0, \end{aligned}$$

and (19) yields also

$$\begin{aligned} \int_Y \int_G \left| Z(\omega^*, g \cdot y) - \sum_{i=1}^N Z^{\pi(i)}(\omega^*, g \cdot y) \right|^2 dg \mu(dy) &\leq \int_Y \int_G |Z(g \cdot y)|^2 dg \mu(dy) \\ &= \int_Y |Z(y)|^2 \mu(dy) \in L^1(\mathbf{P}), \end{aligned}$$

since  $Z \in L^2(\mathbf{P}(d\omega) \times \mu(dy))$  by assumption. Since  $\mu$  is  $G$ -invariant, and by dominated convergence,

$$\begin{aligned} \mathbf{E} \left[ \int_Y \left| Z(y) - \sum_{i=1}^N Z^{\pi(i)}(y) \right|^2 \mu(dy) \right] &= \int_G \mathbf{E} \left[ \int_Y \left| Z(g \cdot y) - \sum_{i=1}^N Z^{\pi(i)}(g \cdot y) \right|^2 \mu(dy) \right] dg \\ &= \mathbf{E} \left[ \int_Y \int_G \left| Z(g \cdot y) - \sum_{i=1}^N Z^{\pi(i)}(g \cdot y) \right|^2 dg \mu(dy) \right] \xrightarrow{N \rightarrow +\infty} 0. \end{aligned}$$

■

## 4 Applications to Gaussian processes

### 4.1 Decomposition of real-valued Gaussian processes

Keep the previous notation and assumptions (in particular, Assumption A holds throughout the following). In this paragraph, we apply the above established results to the case of a **two-dimensional real-valued Gaussian process** of the type

$$(Z_1, Z_2) : \Omega \times Y \rightarrow \mathbb{R}^2 : (\omega, y) \mapsto (Z_1(\omega, y), Z_2(\omega, y))$$

with a **covariance structure** given by

$$(33) \quad R_{i,j}(y_1, y_2) = \mathbf{E}[Z_i(y_1) Z_j(y_2)], \quad i, j = 1, 2, \quad y_1, y_2 \in Y.$$

Note that our definition of two-dimensional Gaussian process also covers the case  $Z_1 = Z_2$ . In this paragraph, we will consider exclusively actions of the type (24), where  $A'$  is an action of the topological compact group  $G$  on  $Y$ . Note that, under such assumptions,  $(Z_1, Z_2)$  has a  $G$ -invariant law if, and only if,

$$(34) \quad R_{i,j}(g \cdot y_1, g \cdot y_2) = R_{i,j}(y_1, y_2), \quad \text{for every } g \in G, i, j = 1, 2, \text{ and } y_1, y_2 \in Y.$$

When the function  $R_{i,j}$  satisfies (34), we say that  $R_{i,j}$  is a  **$G$ -invariant covariance function**.

In the sequel, the Cartesian product  $G \times G = G^2$  is systematically endowed with the **product group** structure, as described e.g. in [19, Section 3.2]. The generic element of  $G^2$  is noted  $(g_1, g_2)$ ;  $G^2$  is again a topological and compact group, with Haar measure given by  $dg_1 \times dg_2$ . Recall (see again [19, Section 1.5 and 3.2]) that  $[\rho] \in \widehat{G^2}$  if, and only if,  $[\rho] = [\pi_1] \otimes [\pi_2]$ , where  $([\pi_1], [\pi_2]) \in \widehat{G} \times \widehat{G}$ , and  $\otimes$  stands for the (tensor) product between representations. The following assumption will hold for the rest of the section.

**Assumption B** – For every two-dimensional Gaussian process  $(Z_1, Z_2)$  considered in the sequel, and for every fixed  $y_1, y_2 \in Y$ , the application

$$(35) \quad (R_{i,j})_{G^2}[y_1, y_2] : G \times G \rightarrow \mathbb{R} : (g_1, g_2) \mapsto R_{i,j}(g_1 \cdot y_1, g_2 \cdot y_2)$$

(see (34), and observe that (35) is consistent with the notation introduced in (7)) is an element of  $L^2(G^2)$ , for every  $i, j = 1, 2$ .

Again, if  $G$  is finite, Assumption B is redundant. Given  $\widehat{G^2} \ni [\rho] = [\pi_1] \otimes [\pi_2]$ , we define, for fixed  $y_1, y_2 \in Y$ ,

$$(36) \quad (R_{i,j})_{G^2}^\rho[y_1, y_2] = (R_{i,j})_{G^2}^{\pi_1 \otimes \pi_2}[y_1, y_2]$$

according to (11). The following result, which is a consequence of Theorem 4, will lead to a very general version of Watson's duplication identity.

**Proposition 6** Let  $(Z_1, Z_2)$  be a two dimensional real-valued Gaussian process with a  $G$ -invariant law. Then,

1. the collection of (possibly complex-valued) stochastic processes  $\{Z_1, Z_2, Z_1^\pi, Z_2^\sigma : [\pi], [\sigma] \in \widehat{G}\}$  is jointly Gaussian;
2. for every  $[\pi], [\sigma] \in \widehat{G}$  such that  $[\pi] \neq [\sigma]$  and  $\chi_\pi$  is real valued, the two processes  $Z_i^\pi$  and  $Z_j^\sigma$  are independent for every  $i, j = 1, 2$ ;
3. for every  $[\pi], [\sigma]$  as at point 2.,  $(R_{i,j})_{G^2}^{\pi \otimes \sigma} [y_1, y_2] = 0$ , for every  $i, j = 1, 2$ ;
4. for  $i = 1, 2$ ,  $Z_i(y) = \sum_{[\pi] \in \widehat{G}} Z_i^\pi(y)$  in  $L^2(\mathbf{P})$ .

**Proof.** Point 1. is immediate, since the action  $A$  of  $G$  on  $X_0$  has the form (24). Since  $\chi_\pi$  is real-valued,  $Z_i^\pi$  is also real-valued, and moreover, for every  $y_1, y_2 \in Y$ , according to Theorem 4-3,  $\mathbf{E} \left[ Z_i^\pi(y_1) \overline{Z_j^\sigma(y_2)} \right] = 0$ , thus implying that  $Z_i^\pi(y_1)$  is independent of both the real and imaginary parts of  $Z_j^\sigma(y_2)$ . This concludes the proof of point 2. To see point 3., just write, for  $h_1, h_2 \in G$

$$\begin{aligned} (R_{i,j})_{G^2}^{\pi \otimes \sigma} [y_1, y_2] (h_1, h_2) &= \int_G dg_1 \int_G dg_2 \chi_\pi(g_1) \chi_\sigma(g_2) (R_{i,j})_{G^2} [y_1, y_2] (g_1^{-1} h_1, g_2^{-1} h_2) \\ &= \mathbf{E} [Z_i^\pi(h_1 \cdot y_1) Z_j^\sigma(h_2 \cdot y_2)] = 0, \end{aligned}$$

due to point 2.. Point 4. comes immediately from Theorem 4-4. ■

Of course, point 1. of Proposition 3 still holds when the law of the Gaussian process  $(Z_1, Z_2)$  is not  $G$ -invariant. The combination of Theorem 5 and Proposition 6 yields immediately the following

**Proposition 7** Let  $G$  be such that  $\chi_\pi$  is real-valued for every  $[\pi] \in \widehat{G}$ . Let  $(Z_1, Z_2)$  be a two dimensional real-valued Gaussian process with a  $G$ -invariant law, and consider a  $G$ -invariant,  $\sigma$ -finite and positive measure  $\mu$  on  $(Y, \mathcal{Y})$ . Suppose that, for any fixed  $\omega^* \in \Omega$  and  $i = 1, 2$ , the mapping  $(g, y) \mapsto Z_i(\omega^*, g \cdot y)$  is in  $L^2(dg \times \mu(dy))$ , and also that, for every fixed  $(\omega^*, g^*) \in \Omega \times G$ , the mapping  $y \mapsto Z_i(\omega^*, g^* \cdot y)$  is an element of  $L^2(\mu(dy))$ . Then, for every  $i, j = 1, 2$ ,

1. the Gaussian processes  $Z_i^\pi(\omega, y)$  and  $Z_j^\sigma(\omega, y)$  are independent for every  $[\pi] \neq [\sigma]$ , and orthogonal in  $L^2(\mu(dy))$  for every  $\omega \in \Omega$ ;
2. for every  $[\pi] \in \widehat{G}$ ,

$$(37) \quad \mathbf{E} [Z_i^\pi(y_1) Z_j^\pi(y_2)] = (R_{i,j})_{G^2}^{\pi \otimes \pi} [y_1, y_2] (e, e) = R_{i,j}^{\pi \otimes \pi} (y_1, y_2);$$

3.  $Z_i(\omega, y) = \sum_{[\pi] \in \widehat{G}} Z_i^\pi(\omega, y)$  both in  $L^2(\mu(dy))$  for every  $\omega \in \Omega$  and in  $L^2(\mathbf{P}(d\omega) \times \mu(dy))$ ;
4. for every  $\lambda \in \mathbb{R}$ ,

$$(38) \quad \mathbf{E} \left[ \exp \left( i \lambda \int_Y Z_i(y) Z_j(y) \mu(dy) \right) \right] = \prod_{[\pi] \in \widehat{G}} \mathbf{E} \left[ \exp \left( i \lambda \int_Y Z_i^\pi(y) Z_j^\pi(y) \mu(dy) \right) \right]$$

**Example** (A group-theoretic proof of the (polarized) Watson's identity) – As a first illustration of our techniques, we shall obtain a class of identities in law – between functionals of two correlated Brownian bridges – extending Watson's identity (1). Our method of proof, which is directly based on the discussion contained in this paragraph, generalizes the simple proof of (1) given by the second author in [14], and will motivate the content of the subsequent section. To this end, we consider a two-dimensional

Brownian bridge  $\underline{b} = \{b_1(t), b_2(t) : t \in [0, 1]\}$  with correlation parameter equal to  $\rho \in [0, 1]$ . This means that  $\underline{b}$  is a two-dimensional, real-valued Gaussian process such that, for every  $s, t \in [0, 1]$ ,  $\mathbf{E}[b_i(s)b_i(t)] = s \wedge t - st$ ,  $i = 1, 2$ , and  $\mathbf{E}[b_1(s)b_2(t)] = \rho \times (s \wedge t - st)$ . By  $\underline{b}_* = \{b_{*1}(t), b_{*2}(t) : t \in [0, 1]\}$ , we denote an independent copy of  $\underline{b}$ , and we also write, for  $i = 1, 2$  and  $t \in [0, 1]$ ,

$$(39) \quad v_i(t) = b_i(t) - \int_0^1 b_i(s) ds \quad \text{and} \quad v_{*i}(t) = b_{*i}(t) - \int_0^1 b_{*i}(s) ds.$$

Now consider the group  $G = \{e, g\} \simeq \mathbb{Z}/2\mathbb{Z}$ , where  $e$  stands again for the identity element. It is plain (see e.g. [19, Chapter 2]) that in this case  $\widehat{G} = \{[\pi_u], [\pi_a]\}$ , where  $[\pi_u]$  and  $[\pi_a]$  are the equivalence classes, respectively of the *unity* and of the *alternating* representation; in particular,  $\chi_{\pi_u}(e) = \chi_{\pi_u}(g) = 1$ , and  $\chi_{\pi_a}(e) = 1 = -\chi_{\pi_a}(g)$ . We fix the following elementary action of  $G$  on  $[0, 1]$ :  $e \cdot t = t$  and  $g \cdot t = 1 - t$ ,  $\forall t \in [0, 1]$ . It is well known that  $\underline{b}$ , and therefore the vector  $(v_1, v_2)$ , has a  $G$ -invariant law, so that the content of Proposition 6 can be directly applied. To do this, we first set, according to (12) and for  $i = 1, 2$  and  $t \in [0, 1]$ , and since  $d_{\pi_u} = d_{\pi_a} = 1$ ,

$$\begin{aligned} v_i^{\pi_u}(t) &= \frac{d_{\pi_u}}{|G|} \{ \chi_{\pi_u}(e) v_i(e^{-1} \cdot t) + \chi_{\pi_u}(g) v_i(g^{-1} \cdot x) \} = \frac{1}{2} (b_i(t) + b_i(1-t)) - \int_0^1 b_i(s) ds \\ &= \frac{1}{2} (b_i(t) + b_i(1-t)) - \frac{1}{2} \int_0^1 (b_i(s) + b_i(1-s)) ds \\ v_i^{\pi_a}(t) &= \frac{d_{\pi_a}}{|G|} \{ \chi_{\pi_a}(e) v_i(e^{-1} \cdot t) + \chi_{\pi_a}(g) v_i(g^{-1} \cdot x) \} = \frac{1}{2} (b_i(t) - b_i(1-t)), \end{aligned}$$

and an analogous definition holds for  $v_{*i}^{\pi_u}$  and  $v_{*i}^{\pi_a}$ ,  $i = 1, 2$ . Now observe that Proposition 6-2 (in the case  $(Z_1, Z_2) = (v_1, v_2)$ ) implies that, for any  $i, j = 1, 2$ , the two processes  $v_i^{\pi_u}$  and  $v_j^{\pi_a}$  are independent. Moreover, the restriction of Lebesgue measure to  $[0, 1]$  is trivially  $G$ -invariant, so that all assumptions of Proposition 7 are satisfied (again with  $(Z_1, Z_2) = (v_1, v_2)$  and  $\mu$  equal to Lebesgue measure) and therefore

$$\begin{aligned} & \int_0^1 v_1(t) v_2(t) dt \stackrel{\text{law}}{=} \int_0^1 v_1^{\pi_u}(t) v_2^{\pi_u}(t) dt + \int_0^1 v_{*1}^{\pi_a}(t) v_{*2}^{\pi_a}(t) dt \\ &= \frac{1}{4} \int_0^1 \left( b_1(t) + b_1(1-t) - \int_0^1 (b_1(s) + b_1(1-s)) ds \right) \times \\ & \quad \times \left( b_2(t) + b_2(1-t) - \int_0^1 (b_2(s) + b_2(1-s)) ds \right) dt \\ & \quad + \frac{1}{4} \int_0^1 (b_{*1}(t) - b_{*1}(1-t)) (b_{*2}(t) - b_{*2}(1-t)) dt. \end{aligned}$$

Next, consider a correlated two-dimensional standard Brownian motion  $\underline{W} = \{W_1(t), W_2(t) : t \in [0, 1]\}$  with correlation  $\rho$ ,<sup>3</sup> and independent of  $\underline{b}$ . Routine computations show the following identities in law:

$$\begin{aligned} & \{b_1(t) + b_1(1-t), b_2(t) + b_2(1-t) : t \in [0, 1/2]\} \stackrel{\text{law}}{=} \{W_1(2t), W_2(2t) : t \in [0, 1/2]\} \\ & \{b_1(t) - b_1(1-t), b_2(t) - b_2(1-t) : t \in [0, 1/2]\} \stackrel{\text{law}}{=} \{b_1(2t), b_2(2t) : t \in [0, 1/2]\}, \end{aligned}$$

implying that

$$\int_0^1 v_1(t) v_2(t) dt \stackrel{\text{law}}{=} \frac{1}{4} \int_0^1 \left( W_1(t) - \int_0^1 W_1(s) ds \right) \left( W_2(t) - \int_0^1 W_2(s) ds \right) dt + \frac{1}{4} \int_0^1 b_1(t) b_2(t) dt.$$

We eventually use some standard arguments (see e.g. [2, Lemma 2]) to prove that

$$\int_0^1 \left( W_1(t) - \int_0^1 W_1(s) ds \right) \left( W_2(t) - \int_0^1 W_2(s) ds \right) dt \stackrel{\text{law}}{=} \int_0^1 b_1(t) b_2(t) dt$$

<sup>3</sup>That is,  $\underline{W}$  is a two-dimensional Gaussian process such that, for  $i = 1, 2$  and  $s, t \in [0, 1]$ ,  $\mathbf{E}[W_i(s)W_i(t)] = s \wedge t$  and  $\mathbf{E}[W_1(s)W_2(t)] = \rho \times (s \wedge t)$ .

and therefore

$$\int_0^1 \left( b_1(t) - \int_0^1 b_1(s) ds \right) \left( b_2(t) - \int_0^1 b_2(s) ds \right) dt \stackrel{law}{=} \frac{1}{4} \int_0^1 (b_1(t) b_2(t) + b_{*1}(t) b_{*2}(t)) dt$$

(Watson's identity (1) can be obtained by setting  $\rho = 1$ ).

**Remark** – By using e.g. [2, Proposition 2], we obtain that, for  $\lambda > 0$  sufficiently small and  $\rho \in [0, 1]$

$$\begin{aligned} & \mathbf{E} \left[ \exp \left( \lambda^2 \int_0^1 \left( b_1(t) - \int_0^1 b_1(s) ds \right) \left( b_2(t) - \int_0^1 b_2(s) ds \right) dt \right) \right] \\ &= \frac{(\lambda/2)^2 \sqrt{1-\rho^2}}{\sin \frac{\lambda}{2} \sqrt{1+\rho} \sinh \frac{\lambda}{2} \sqrt{1-\rho}} \end{aligned}$$

Note that the  $G$ -invariant process  $(v_1, v_2)$ , introduced in formula (39) of the previous example, has the remarkable property that

$$(40) \quad \int_0^1 v_1^{\pi_u}(t) v_2^{\pi_u}(t) dt \stackrel{law}{=} \int_0^1 v_{*1}^{\pi_a}(t) v_{*2}^{\pi_a}(t) dt.$$

In the next paragraph we shall establish necessary and sufficient conditions to ensure that, in the case of a finite  $G$ , a  $G$ -invariant Gaussian process  $(Z_1, Z_2)$  (with some special structure) is such that

$$(41) \quad \int_Y Z_1^\pi(y) Z_2^\pi(y) \mu(dy) \stackrel{law}{=} \int_Y Z_1^\sigma(y) Z_2^\sigma(y) \mu(dy), \text{ for every } [\pi], [\sigma] \in \widehat{G}.$$

In the sequel, an identity such as (41) will be called a **Watson's type relation**.

## 4.2 Watson's type relations for Volterra processes

Throughout this section,  $G$  stands for a finite group such that the character  $\chi_\pi(\cdot)$  is real-valued for every  $[\pi] \in \widehat{G}$ . To simplify some technical points of our discussion (in particular, to apply several crucial properties of multiple Wiener-Itô integrals) we will consider a two-dimensional, real-valued Gaussian process  $(Z_1, Z_2)$  such that its components are correlated Volterra processes. To define such objects, take a measurable space  $(T, \mathcal{T}, \tau)$ , where  $\tau$  is positive,  $\sigma$ -finite and non-atomic, and write  $L_{\mathbb{R}}^2(d\tau)$  to indicate the Hilbert space of real-valued, square-integrable functions with respect to  $\tau$ . In what follows, we will write

$$(42) \quad X = \{X(f) : f \in L_{\mathbb{R}}^2(d\tau)\}$$

to indicate an **isonormal Gaussian process** (or a **Gaussian measure**) on  $L_{\mathbb{R}}^2(d\tau)$ . This means that  $X$  is a centered Gaussian family indexed by the elements of  $L_{\mathbb{R}}^2(d\tau)$ , defined on some probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  and such that, for every  $f_1, f_2 \in L_{\mathbb{R}}^2(d\tau)$ ,

$$\mathbf{E}(X(f_1) X(f_2)) = \int_T f_1(t) f_2(t) \tau(dt).$$

Now fix a measurable space  $(Y, \mathcal{Y})$ . A two-dimensional Gaussian process  $\{(Z_1(y), Z_2(y)) : y \in Y\}$  is called a **correlated (generalized) Volterra process**, with respect to  $X$  and with parameter  $\rho \in [0, 1]$ , if there exist two  $\mathcal{Y} \otimes \mathcal{T}$ -measurable applications

$$Y \times T \rightarrow \mathbb{R} : (y, t) \mapsto \phi_i(y, t), \quad i = 1, 2,$$



such that: (a) for every  $y \in Y$  the application  $t \mapsto \phi_i(y, t)$  is an element of  $L^2_{\mathbb{R}}(d\tau)$ , (b) a.s.  $-\mathbb{P}$ ,

$$(43) \quad Z_i(y) = X(\phi_i(y, \cdot)), \quad i = 1, 2,$$

and (c) for every  $y_1, y_2 \in Y$  and by using the notation introduced in (33),

$$(44) \quad \begin{aligned} R_{1,1}(y_1, y_2) &= R_{2,2}(y_1, y_2) \quad \text{and} \\ R_{1,2}(y_1, y_2) &= R_{2,1}(y_1, y_2) = \rho R_{1,1}(y_1, y_2). \end{aligned}$$

Note that, if  $\rho = 1$ , then  $Z_1(y) = Z_2(y)$  p.s.- $\mathbf{P}$ ,  $\forall y \in Y$ ; moreover, the covariance structure of a Gaussian process  $(Z_1, Z_2)$  of the type (43) may be rewritten as

$$(45) \quad R_{i,j}(y_1, y_2) = \mathbf{E}[Z_i(y_1)Z_j(y_2)] = \int_T \phi_i(y_1, t)\phi_j(y_2, t)\tau(dt), \quad i, j = 1, 2;$$

as a consequence, in view of (44) and (45), and given an action  $g \cdot y$  of  $G$  on  $Y$ ,  $(Z_1, Z_2)$  has a  $G$ -invariant law if, and only if, for  $i$  equal to 1 or 2,

$$(46) \quad \int_T \phi_i(g \cdot y_1, t)\phi_i(g \cdot y_2, t)\tau(dt) = \int_T \phi_i(y_1, t)\phi_i(y_2, t)\tau(dt),$$

for every  $y_1, y_2 \in Y$  and every  $g \in G$ . In the sequel, to simplify the notation, we will write

$$(47) \quad R_{1,1}(\cdot, \cdot) = R_{2,2}(\cdot, \cdot) = R(\cdot, \cdot).$$

We now fix an action  $g \cdot y$  of  $G$  on  $Y$ , as well as a  $G$ -invariant, positive and  $\sigma$ -finite measure  $\mu$  on  $(Y, \mathcal{Y})$ . For every real-valued  $\Phi, \Psi \in L^2(Y^2, \mathcal{Y}^2, d\mu \times d\mu) := L^2(d\mu \times d\mu)$ , we define, for  $y_1, y_2 \in Y$ ,

- (i)  $[\Phi \otimes_{(1)} \Phi](y_1, y_2) = \Phi(y_1, y_2)$ ;
- (ii)  $[\Phi \otimes_{(2)} \Psi](y_1, y_2) = \int_Y \Phi(y_1, x)\Psi(y_2, x)\mu(dx)$ ;
- (iii)  $\forall p \geq 3$ ,  $[\Phi \otimes_{(p)} \Phi](y_1, y_2) = [[\Phi \otimes_{(p-1)} \Phi] \otimes_{(2)} \Phi](y_1, y_2)$ ;

Observe that, if  $\Phi \in L^2(d\mu \times d\mu)$ , then the application  $y \mapsto [\Phi \otimes_{(p)} \Phi](y, y)$  is an element of  $L^1(Y, \mathcal{Y}, d\mu)$  for every  $p \geq 2$ . Finally, for  $\rho \in [0, 1]$  as above, we introduce the following set of real constants

$$(48) \quad \begin{aligned} K(1, \rho) &= 2\rho \\ K(n, \rho) &= 2 \sum_{j=0}^{\frac{n}{2}-1} \binom{n-1}{2j} \rho^{2j} + 2 \sum_{j=0}^{\frac{n}{2}-1} \binom{n-1}{2j+1} \rho^{2j+2}, \quad n \text{ even}, n \geq 2, \\ K(n, \rho) &= 2 \sum_{j=0}^{\frac{n-1}{2}} \binom{n-1}{2j} \rho^{2j+1} + 2 \sum_{j=0}^{\frac{n-3}{2}} \binom{n-1}{2j+1} \rho^{2j+1}, \quad n \text{ odd}, n \geq 3. \end{aligned}$$

Note that  $K(n, 1) = 2^n$  for every  $n \geq 1$ ,  $K(2p, \rho) > 0$  for every  $p \geq 1$ , and, for  $p \geq 0$ ,  $K(2p+1, \rho) = 0$  if, and only if,  $\rho = 0$  (since  $\rho$  is real). In the next result, under some additional integrability assumptions, we state necessary and sufficient conditions to have that property (41) is satisfied.

**Theorem 8** Consider a finite group  $G$  such that  $\chi_\pi(\cdot) \in \mathbb{R}$ , for every  $[\pi] \in \widehat{G}$ . Let the process  $(Z_1, Z_2)$  be a correlated Volterra process of the type (43), for some correlation coefficient  $\rho \in [0, 1]$ , and assume  $(Z_1, Z_2)$  has a  $G$ -invariant law. Let also  $\mu(\cdot)$  be a  $G$ -invariant, positive measure satisfying the assumptions of Proposition 7, and suppose moreover

$$(49) \quad \mathbf{E} \left( \int_Y Z_1(y)^2 \mu(dy) \right) = \int_Y \int_T \phi_1(y, t)^2 \mu(dy) \tau(dt) = \int_Y R(y, y) \mu(dy) < +\infty.$$

Then,

1. the covariance functions  $R$  and  $R^{\pi \otimes \pi}$ , defined respectively according to (47) and (37), for  $[\pi] \in \widehat{G}$ , satisfy

$$(50) \quad \int_Y \int_Y R(x, y)^2 \mu(dx) \mu(dy) < +\infty \quad \text{and} \quad \int_Y \int_Y R^{\pi \otimes \pi}(x, y)^2 \mu(dx) \mu(dy) < +\infty;$$

2. the random variables

$$\int_Y Z_1^\pi(y) Z_2^\pi(y) \mu(dy), \quad [\pi] \in \widehat{G},$$

are stochastically independent;

3. for every  $[\pi] \in \widehat{G}$ , the process  $(Z_1^\pi, Z_2^\pi)$  is a correlated Volterra process, with parameter  $\rho$ ;

4. the following three conditions are equivalent: (i) for every  $[\pi], [\sigma] \in \widehat{G}$ ,

$$(51) \quad \int_Y Z_1^\pi(y) Z_2^\pi(y) \mu(dy) \stackrel{\text{law}}{=} \int_Y Z_1^\sigma(y) Z_2^\sigma(y) \mu(dy),$$

- (ii) for every  $[\pi] \in \widehat{G}$  and every  $n \geq 1$

$$(52) \quad K(n, \rho) \int_Y [R^{\pi \otimes \pi} \otimes_{(n)} R^{\pi \otimes \pi}](y, y) \mu(dy) = \frac{K(n, \rho)}{|\widehat{G}|} \int_Y [R \otimes_{(n)} R](y, y) \mu(dy),$$

- (iii) for every  $[\pi] \in \widehat{G}$  and every  $n \geq 1$

$$(53) \quad K(n, \rho) \int_Y [R^{\pi \otimes \pi} \otimes_{(n)} R^{\pi \otimes \pi}](y, y) \mu(dy) = K(n, \rho) \int_Y [R^{\sigma \otimes \sigma} \otimes_{(n)} R^{\sigma \otimes \sigma}](y, y) \mu(dy).$$

**Remarks** – (i) In view of (45), both formulae (52) and (53) can be immediately reformulated in terms of the kernels  $\phi_1$  and  $\phi_2$ .

(ii) The role of the constants  $K(n, \rho)$  in (52) and (53) is immaterial for  $\rho \neq 0$ , or for  $n$  even and  $\rho \in [0, 1]$ .

Before proving Theorem 8, we state some interesting consequences of Theorem 8-4.

**Proposition 9** Let  $G = \{e, g\} \simeq \mathbb{Z}/2\mathbb{Z}$ , where  $e$  stands for the identity element. Keep the assumptions and the notation of Theorem 8, and suppose moreover that  $\rho \neq 0$ . Then, condition (51) is verified if, and only if, for every  $n \geq 1$

$$\int_Y [R \otimes_{(n)} R](y, g \cdot y) \mu(dy) = 0.$$

**Proof.** As already pointed out, in this case  $\widehat{G} = \{[\pi_u], [\pi_a]\}$ , where  $[\pi_u]$  and  $[\pi_a]$  are the equivalence classes of the unity and of the alternating representation. Moreover, due to Theorem 8 and the fact that  $K(n, \rho) \neq 0$ , we know that (51) holds if, and only if,

$$(54) \quad \int_Y [R^{\pi_u \otimes \pi_u} \otimes_{(n)} R^{\pi_u \otimes \pi_u}](y, y) \mu(dy) = \frac{1}{2} \int_Y [R \otimes_{(n)} R](y, y) \mu(dy),$$

for any  $n \geq 1$ , where

$$\begin{aligned} R^{\pi_u \otimes \pi_u}(y_1, y_2) &= \frac{1}{4} (R(e \cdot y_1, e \cdot y_2) + R(g \cdot y_1, e \cdot y_2) + R(e \cdot y_1, g \cdot y_2) + R(g \cdot y_1, g \cdot y_2)) \\ &= \frac{1}{2} (R(y_1, y_2) + R(y_1, g \cdot y_2)), \end{aligned}$$

due to the  $G$ -invariance of the law of  $(Z_1, Z_2)$ . Finally, since  $\mu$  is also  $G$ -invariant, one can easily prove that, for  $n \geq 1$ ,

$$\int_Y [R^{\pi_u \otimes \pi_u} \otimes_{(n)} R^{\pi_u \otimes \pi_u}](y, y) \mu(dy) = \frac{1}{2} \int_Y [R \otimes_{(n)} R](y, y) \mu(dy) + \frac{1}{2} \int_Y [R \otimes_{(n)} R](y, g \cdot y) \mu(dy),$$

thus yielding, via (54), the desired conclusion. ■

**Remark** – The process  $(v_1, v_2)$  defined in formula (39) of the previous section can be represented as a correlated Volterra process, with parameter  $\rho \in [0, 1]$  and covariance structure

$$(55) \quad \begin{aligned} \mathbf{E}[v_1(s) v_1(t)] &= \mathbf{E}[v_2(s) v_2(t)] = R_v(s, t) = s \wedge t - \frac{s+t}{2} + \frac{(s-t)^2}{2} + \frac{1}{12} \\ \mathbf{E}[v_1(s) v_2(t)] &= \mathbf{E}[v_2(s) v_1(t)] = \rho R_v(s, t), \end{aligned}$$

where  $s, t \in [0, 1]$ . Moreover, its law is  $G$ -invariant, where  $G = \mathbb{Z}/2\mathbb{Z}$ . Since (40) holds, we deduce from Corollary 9 that for every  $n \geq 1$

$$\int_0^1 [R_v \otimes_{(n)} R_v](t, 1-t) dt = 0.$$

The next result, which is again a consequence of Theorem 8, is very useful to deal with multiparameter processes.

**Proposition 10** Fix  $d \geq 2$ . Let  $(Y^{(i)}, \mathcal{Y}^{(i)}, \mu^{(i)})$ ,  $i = 1, \dots, d$ , be a collection of measure spaces, with  $\mu^{(i)}$  positive and  $\sigma$ -finite, and let  $G^{(i)}, \dots, G^{(d)}$  be finite groups with real-valued characters, such that, for each  $i = 1, \dots, d$ , an action  $g_i \cdot y_i$  of  $G_i$  on  $Y_i$  is well defined. We note

$$\begin{aligned} Y &= Y^{(1)} \times \dots \times Y^{(d)}, \quad G = G^{(1)} \times \dots \times G^{(d)}, \\ \mu &= \mu^{(1)} \times \dots \times \mu^{(d)} \end{aligned}$$

and we endow  $G$  with the product group structure (see [19, Section 3.2]). Let also  $\{(Z_1(y), Z_2(y)) : y \in Y\}$  be a correlated Volterra process with parameter  $\rho \in [0, 1]$ , such that, for every  $y = (y_1, \dots, y_d)$  and  $x = (x_1, \dots, x_d)$  in  $Y$

$$\mathbf{E}[Z_1(x) Z_1(y)] = R(x, y) = \prod_{i=1}^d R_i(x_i, y_i) \quad \text{and} \quad \mathbf{E}[Z_1(x) Z_2(y)] = \rho R(x, y),$$

where for each  $i$ ,  $R_i$  is a  $G^{(i)}$ -invariant covariance function such that

$$\int_{Y^{(i)}} R_i(y, y) \mu^{(i)}(dy) < +\infty.$$

Then,

1. the application

$$(g, y) \mapsto g \cdot y : (g_1, \dots, g_d; y_1, \dots, y_d) \mapsto (g_1 \cdot y_1, \dots, g_d \cdot y_d)$$

is an action of  $G$  on  $Y$ ;

2. the process  $(Z_1, Z_2)$  has a  $G$ -invariant law;

3. for every  $[\pi_i], [\sigma_i] \in \widehat{G}^{(i)}$ ,  $i = 1, \dots, d$ , every  $x, y \in Y$

$$R^\theta(x, y) = \begin{cases} 0 & \text{if there exists } i \text{ such that } [\pi_i] \neq [\sigma_i] \\ \prod_{i=1}^d R^{\pi_i \otimes \sigma_i}(x_i, y_i) & \text{otherwise,} \end{cases}$$

where  $\theta = [\pi_1 \otimes \dots \otimes \pi_d] \otimes [\sigma_1 \otimes \dots \otimes \sigma_d]$  is a generic element of  $\widehat{G}^2$ ;

4. if, for each  $i = 1, \dots, d$ , the function  $R_i$  satisfies either one of conditions (52) and (53), then  $(Z_1, Z_2)$  verifies Watson's relation (51) for every  $[\pi], [\sigma] \in \widehat{G}$ .

**Proof.** Points 1. and 2. are trivial. Point 3 is a consequence of the  $G^{(i)}$  invariance of each  $R_i$ , as well as of Proposition 6-3. To prove point 4., suppose that each  $R_i$  verifies (53), and also  $\rho \neq 0$ . Then,  $K(n, \rho) \neq 0$  for each  $n$ , and

$$\begin{aligned} \frac{1}{|\widehat{G}|} \int_Y [R \otimes_{(n)} R](y, y) \mu(dy) &= \frac{1}{|\widehat{G}|} \prod_{i=1}^d \int_{Y^{(i)}} [R_i \otimes_{(n)} R_i](y_i, y_i) \mu^{(i)}(dy_i) \\ &= \prod_{i=1}^d \int_{Y^{(i)}} [R_i^{[\pi_i] \otimes [\pi_i]} \otimes_{(n)} R_i^{[\pi_i] \otimes [\pi_i]}](y_i, y_i) \mu^{(i)}(dy_i), \end{aligned}$$

for every  $[\pi_i] \in \widehat{G}^{(i)}$ ,  $i = 1, \dots, d$ , since  $|\widehat{G}| = \prod_{i=1, \dots, d} |\widehat{G}^{(i)}|$ . To conclude, just observe that, thanks to point 3.,

$$\prod_{i=1}^d \int_{Y^{(i)}} [R_i^{[\pi_i] \otimes [\pi_i]} \otimes_{(n)} R_i^{[\pi_i] \otimes [\pi_i]}](y_i, y_i) \mu^{(i)}(dy_i) = \int_Y [R^{\eta \otimes \eta} \otimes_{(n)} R^{\eta \otimes \eta}](y, y) \mu(dy),$$

where  $\eta = \pi_1 \otimes \dots \otimes \pi_d$  (to deal with the case  $\rho = 0$ , just perform the same argument for even cumulants).

■

**Example** (A quadruplication identity). Let  $\mathbb{B}_0 = \{\mathbb{B}_{0,1}(t_1, t_2), \mathbb{B}_{0,2}(t_1, t_2) : (t_1, t_2) \in [0, 1]^2\}$  be a correlated tied-down Brownian sheet, that is,  $\mathbb{B}_0$  is a two-dimensional Gaussian process such that

$$\begin{aligned} \mathbf{E}[\mathbb{B}_{0,1}(t_1, t_2), \mathbb{B}_{0,1}(s_1, s_2)] &= \mathbf{E}[\mathbb{B}_{0,2}(t_1, t_2), \mathbb{B}_{0,2}(s_1, s_2)] \\ &= (t_1 \wedge s_1 - s_1 t_1)(t_2 \wedge s_2 - s_2 t_2), \\ \mathbf{E}[\mathbb{B}_{0,1}(t_1, t_2), \mathbb{B}_{0,2}(s_1, s_2)] &= \mathbf{E}[\mathbb{B}_{0,2}(t_1, t_2), \mathbb{B}_{0,1}(s_1, s_2)] \\ &= \rho \times (t_1 \wedge s_1 - s_1 t_1)(t_2 \wedge s_2 - s_2 t_2), \end{aligned}$$

where  $(s_1, s_2), (t_1, t_2) \in [0, 1]^2$  and  $\rho \in [0, 1]$ . Note that  $\mathbb{B}_0$  can be represented as a Volterra process. Moreover,  $\mathbb{B}_0$  has the law of a correlated Brownian sheet  $\mathbb{W}$  (with the same parameter<sup>4</sup>), conditioned to

<sup>4</sup>That is,

$$\mathbb{W} = \{\mathbb{W}_1(t_1, t_2), \mathbb{W}_2(t_1, t_2) : (t_1, t_2) \in [0, 1]^2\}$$

is a two-dimensional Gaussian process such that

$$\begin{aligned} \mathbf{E}[\mathbb{W}_1(t_1, t_2), \mathbb{W}_1(s_1, s_2)] &= \mathbf{E}[\mathbb{W}_2(t_1, t_2), \mathbb{W}_2(s_1, s_2)] = (t_1 \wedge s_1)(t_2 \wedge s_2), \\ \mathbf{E}[\mathbb{W}_1(t_1, t_2), \mathbb{W}_2(s_1, s_2)] &= \mathbf{E}[\mathbb{W}_2(t_1, t_2), \mathbb{W}_1(s_1, s_2)] = \rho \times (t_1 \wedge s_1)(t_2 \wedge s_2). \end{aligned}$$

vanish on the edges of the square  $[0, 1]^2$ . Now define, for  $i = 1, 2$ , the compensated processes

$$\mathbb{U}_i(t_1, t_2) = \mathbb{B}_{0,i}(t_1, t_2) - \int_0^1 \mathbb{B}_{0,i}(t_1, u_2) du_2 - \int_0^1 \mathbb{B}_{0,i}(u_1, t_2) du_1 + \int_{[0,1]^2} \mathbb{B}_{0,i}(u_1, u_2) du_1 du_2,$$

where  $(t_1, t_2) \in [0, 1]^2$ . We claim that the following identity in law holds

$$(56) \quad \int_{[0,1]^2} \mathbb{U}_1(t_1, t_2) \mathbb{U}_2(t_1, t_2) dt_1 dt_2 \stackrel{\text{law}}{=} \frac{1}{16} \sum_{i=1}^4 \int_{[0,1]^2} \mathbb{B}_{0,1}^{(i)}(t_1, t_2) \mathbb{B}_{0,2}^{(i)}(t_1, t_2) (t_1, t_2)^2 dt_1 dt_2,$$

where  $\mathbb{B}_0^{(i)} = (\mathbb{B}_{0,1}^{(i)}, \mathbb{B}_{0,2}^{(i)})$ ,  $i = 1, \dots, 4$ , are four independent copies of  $\mathbb{B}_0$ . As a matter of fact, standard calculations show that

$$\begin{aligned} \mathbf{E}[\mathbb{U}_1(t_1, t_2) \mathbb{U}_1(s_1, s_2)] &= \mathbf{E}[\mathbb{U}_2(t_1, t_2) \mathbb{U}_2(s_1, s_2)] = R_v(s_1, t_1) R_v(s_2, t_2), \\ \mathbf{E}[\mathbb{U}_1(t_1, t_2) \mathbb{U}_2(s_1, s_2)] &= \mathbf{E}[\mathbb{U}_2(t_1, t_2) \mathbb{U}_1(s_1, s_2)] = \rho \times R_v(s_1, t_1) R_v(s_2, t_2), \end{aligned}$$

where  $R_v$  is defined as in (55). Since  $R_v$  is invariant with respect to the action of  $\{e, g\} \simeq (\mathbb{Z}/2\mathbb{Z})$  on  $[0, 1]$  given by  $e \cdot t = t$  and  $g \cdot t = 1 - t$ , Proposition 10-2 entails that the law of the vector  $(\mathbb{U}_1, \mathbb{U}_2)$  is invariant with respect to the action of the product group  $G = \{e, g\} \times \{e, g\} \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$  on  $[0, 1]^2$  defined as

$$\begin{aligned} (e, e) \cdot (t_1, t_2) &= (t_1, t_2), & (e, g) \cdot (t_1, t_2) &= (t_1, 1 - t_2) \\ (g, e) \cdot (1 - t_1, t_2) &= (t_1, t_2), & (g, g) \cdot (t_1, t_2) &= (1 - t_1, 1 - t_2). \end{aligned}$$

Now recall that  $\widehat{G} = \{[\pi_u] \otimes [\pi_u], [\pi_a] \otimes [\pi_u], [\pi_u] \otimes [\pi_a], [\pi_a] \otimes [\pi_a]\}$ , where  $\pi_u$  and  $\pi_a$  are the unity and *alternating* representation of  $\mathbb{Z}/2\mathbb{Z}$ . According to Proposition 7-4 (since Lebesgue measure on  $[0, 1]^2$  is also  $G$ -invariant) and Proposition 10-4, for every  $\lambda \in \mathbb{R}$ ,

$$\begin{aligned} &\mathbf{E} \left[ \exp \left( i\lambda \int_{[0,1]^2} \mathbb{U}_1(t_1, t_2) \mathbb{U}_2(t_1, t_2) dt_1 dt_2 \right) \right] \\ &= \mathbf{E} \left[ \exp \left( i\lambda \int_{[0,1]^2} \mathbb{U}_1^{\pi_a \otimes \pi_a}(t_1, t_2) \mathbb{U}_2^{\pi_a \otimes \pi_a}(t_1, t_2) dt_1 dt_2 \right) \right]^4. \end{aligned}$$

To conclude, we use Proposition 7-2 to show that

$$\begin{aligned} \mathbf{E}[\mathbb{U}_1^{\pi_a \otimes \pi_a}(t_1, t_2) \mathbb{U}_1^{\pi_a \otimes \pi_a}(s_1, s_2)] &= \mathbf{E}[\mathbb{U}_2^{\pi_a \otimes \pi_a}(t_1, t_2) \mathbb{U}_2^{\pi_a \otimes \pi_a}(s_1, s_2)] \\ &= R_v^{\pi_a \otimes \pi_a}(s_1, t_1) R_v^{\pi_a \otimes \pi_a}(s_2, t_2), \\ \mathbf{E}[\mathbb{U}_1^{\pi_a \otimes \pi_a}(t_1, t_2) \mathbb{U}_2^{\pi_a \otimes \pi_a}(s_1, s_2)] &= \mathbf{E}[\mathbb{U}_2^{\pi_a \otimes \pi_a}(t_1, t_2) \mathbb{U}_1^{\pi_a \otimes \pi_a}(s_1, s_2)] \\ &= \rho \times R_v^{\pi_a \otimes \pi_a}(s_1, t_1) R_v^{\pi_a \otimes \pi_a}(s_2, t_2), \end{aligned}$$

thus implying that

$$\begin{aligned} &\left\{ \mathbb{U}_1(t_1, t_2), \mathbb{U}_2(t_1, t_2) : (t_1, t_2) \in [0, 1/2]^2 \right\} \\ &\stackrel{\text{law}}{=} \left\{ 4^{-1} \mathbb{B}_{0,1}(2t_1, 2t_2), 4^{-1} \mathbb{B}_{0,2}(2t_1, 2t_2) : (t_1, t_2) \in [0, 1/2]^2 \right\}, \end{aligned}$$

and therefore

$$\int_{[0,1]^2} \mathbb{U}_1^{\pi_a \otimes \pi_a}(t_1, t_2) \mathbb{U}_2^{\pi_a \otimes \pi_a}(t_1, t_2) dt_1 dt_2 \stackrel{\text{law}}{=} \frac{1}{4} \int_{[0,1/2]^2} \mathbb{B}_{0,1}(2t_1, 2t_2) \mathbb{B}_{0,2}(2t_1, 2t_2) dt_1 dt_2,$$

so that (56) is obtained by a standard change of variables on the right hand side of the previous expression. The reader is referred to [13] for other two-parameters generalizations of Watson identity.

### 4.3 Proof of Theorem 8

(1.) Since  $G$  is finite, to prove both inequalities in formula (50) it is sufficient to show that, for every  $g, h \in G$ ,

$$\int_Y \int_Y R(h \cdot y, g \cdot z)^2 \mu(dz) \mu(dy) < +\infty.$$

But, since  $\mu$  is  $G$ -invariant, and taking into account (45),

$$\begin{aligned} \int_Y \int_Y R(h \cdot y, g \cdot z)^2 \mu(dz) \mu(dy) &= \int_Y \int_Y R(y, z)^2 \mu(dz) \mu(dy) \\ &= \int_Y \int_Y \left( \int_T \phi_1(z, t) \phi_1(y, t) \tau(dt) \right)^2 \mu(dz) \mu(dy) \\ &< +\infty, \end{aligned}$$

due to (49), as well as to an application of the Cauchy-Schwarz inequality.

(2.) This is a direct consequence of Proposition 7-1..

(3.) By additivity of Gaussian measures, for every  $y \in Y$ ,  $i = 1, 2$  and  $[\pi] \in \widehat{G}$ ,

$$Z_i^\pi(y) = \frac{1}{|G|} \sum_{g \in G} Z(g \cdot y) \chi_\pi(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} X(\phi_i(g \cdot y, \cdot)) \chi_\pi(g^{-1}) = X(\phi_i^{(\pi)}(y, \cdot))$$

where

$$(57) \quad \phi_i^{(\pi)}(y, t) := \frac{1}{|G|} \sum_{g \in G} \phi_i(g \cdot y, t) \chi_\pi(g^{-1}), \quad (y, t) \in Y \times T.$$

(note that  $\phi_i^{(\pi)} \in L^2(d\mu \times d\tau)$ ). Moreover, for any  $y_1, y_2 \in Y$ ,

$$\begin{aligned} \mathbf{E}[Z_1^\pi(y_1) Z_1^\pi(y_2)] &= \mathbf{E}[Z_2^\pi(y_1) Z_2^\pi(y_2)] = R^{\pi \otimes \pi}(y_1, y_2), \quad \text{and} \\ \mathbf{E}[Z_1^\pi(y_1) Z_2^\pi(y_2)] &= \rho R^{\pi \otimes \pi}(y_1, y_2), \end{aligned}$$

due to formula (37), thus yielding the desired result.

(4.) Fix  $[\pi] \in \widehat{G}$ . Since  $Z_1^\pi$  and  $Z_2^\pi$  are Volterra processes with respect to the Gaussian measure  $X$ , we may apply a standard version of the multiplication formula for Wiener-Itô integrals (see e.g. [4, p. 211]) to obtain

$$\int_Y Z_1^\pi(y) Z_2^\pi(y) \mu(dy) = \int_Y \rho R^{\pi \otimes \pi}(y, y) \mu(dy) + I_2^X(\Phi^{(\pi)}),$$

where  $I_2^X$  stands for a double Wiener-Itô integral with respect to  $X$  (see again [4]), and  $\Phi^{(\pi)}$  is the symmetrized kernel

$$\Phi^{(\pi)}(t_1, t_2) = \frac{1}{2} \int_Y \left[ \phi_1^{(\pi)}(y, t_1) \phi_2^{(\pi)}(y, t_2) + \phi_1^{(\pi)}(y, t_2) \phi_2^{(\pi)}(y, t_1) \right] \mu(dy),$$

where  $\phi_i^{(\pi)}$ ,  $i = 1, 2$ , is defined as in (57). On the other hand,

$$\begin{aligned} \int_Y Z_1(y) Z_2(y) \mu(dy) &= \int_Y \rho R(y, y) \mu(dy) + I_2^X(\Phi), \quad \text{where} \\ \Phi(t_1, t_2) &= \frac{1}{2} \int_Y [\phi_1(y, t_1) \phi_2(y, t_2) + \phi_1(y, t_2) \phi_2(y, t_1)] \mu(dy). \end{aligned}$$

Now, it is well known that the law of a double Wiener-Itô integral is determined by its cumulants (see [22]). We therefore note  $\kappa_n(J)$ ,  $n \geq 1$ , the  $n$ th cumulant of a given random variable  $J$ , and use a version of the *diagram formulae* for cumulants of multiple stochastic integrals (as presented, for instance, in [23],

[17, Proposition 9 and Corollary 1] or [8, Section 2]) to obtain that, for every  $n \geq 2$ , there exists a universal combinatorial coefficient  $c_n > 0$  such that, for any  $[\pi] \in \widehat{G}$ ,

$$\kappa_n \left( I_2^X \left( \Phi^{(\pi)} \right) \right) = c_n \int_{T^n} \Phi^{(\pi)}(t_1, t_2) \Phi^{(\pi)}(t_2, t_3) \cdots \Phi^{(\pi)}(t_n, t_1) \tau(dt_1) \cdots \tau(dt_n)$$

and also

$$\kappa_n \left( I_2^X (\Phi) \right) = c_n \int_{T^n} \Phi(t_1, t_2) \Phi(t_2, t_3) \cdots \Phi(t_n, t_1) \tau(dt_1) \cdots \tau(dt_n).$$

By using the relations

$$\begin{aligned} \int_T \phi_1(y_1, t) \phi_1(y_2, t) \tau(dt) &= \int_T \phi_2(y_1, t) \phi_2(y_2, t) \tau(dt) = R(y_1, y_2) \\ \int_T \phi_1(y_1, t) \phi_2(y_2, t) \tau(dt) &= \int_T \phi_2(y_1, t) \phi_1(y_2, t) \tau(dt) = \rho R(y_1, y_2), \end{aligned}$$

as well as a combinatorial argument, we finally obtain, for  $n \geq 2$ ,

$$\begin{aligned} \kappa_n \left( I_2^X \left( \Phi^{(\pi)} \right) \right) &= \frac{c_n \times K(n, \rho)}{2^n} \int_Y [R^{\pi \otimes \pi} \otimes_{(n)} R^{\pi \otimes \pi}] (y, y) \mu(dy) \\ \kappa_n \left( I_2^X (\Phi) \right) &= \frac{c_n \times K(n, \rho)}{2^n} \int_Y [R \otimes_{(n)} R] (y, y) \mu(dy). \end{aligned}$$

To conclude, use independence to write, for  $n \geq 1$ ,

$$\kappa_n \left( \int_Y Z_1(y) Z_2(y) \mu(dy) \right) = \sum_{[\pi] \in \widehat{G}} \kappa_n \left( \int_Y Z_1^\pi(y) Z_2^\pi(y) \mu(dy) \right),$$

and observe that, thanks to the translation invariance property of cumulants (see e.g. [16, Corollary 4.1]), for any  $n \geq 2$ ,

$$\begin{aligned} \kappa_n \left( \int_Y Z_1(y) Z_2(y) \mu(dy) \right) &= \kappa_n \left( I_2^X (\Phi) \right) \\ \kappa_n \left( \int_Y Z_1^\pi(y) Z_2^\pi(y) \mu(dy) \right) &= \kappa_n \left( I_2^X \left( \Phi^{(\pi)} \right) \right), \quad [\pi] \in \widehat{G}, \end{aligned}$$

and consequently

$$\kappa_n \left( I_2^X (\Phi) \right) = \sum_{[\pi] \in \widehat{G}} \kappa_n \left( I_2^X \left( \Phi^{(\pi)} \right) \right).$$

The proof is completed by standard arguments. ■

## 5 Refinements and further applications

### 5.1 Connections with Karhunen-Loève expansions

In this paragraph, we elucidate some of the connections between our decomposition of stochastic processes, and Karhunen-Loève (KL) expansions of Gaussian processes indexed by the elements of a measurable space  $(T, \mathcal{T})$  (for fundamental facts about KL expansions, see e.g. [1], [21, Chapter 5], as well as [3] and the references therein). In what follows,  $G$  is a topological compact group, acting on  $T$  through the application  $(g, t) \mapsto g \cdot t$ ,  $t \in T$ . We write  $m(dt)$  to indicate a  $G$  invariant measure on  $(T, \mathcal{T})$ . We also consider a *positive definite* kernel  $R(s, t)$ ,  $s, t \in T$ , such that  $R$  is the covariance function of a real valued, centered Gaussian process  $\mathbb{X} = \{\mathbb{X}(t) : t \in T\}$ , defined on some probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , and such

that, for every  $\omega \in \Omega$ , the function  $t \mapsto \mathbb{X}(t)$  is in  $L^2(T, dm)$ . We note  $\lambda_1 > \lambda_2 > \dots > 0$  the sequence of the eigenvalues of  $R$  (with respect to  $m(\cdot)$ ), whereas  $E_1, E_2, \dots$  indicate the associated eigenspaces. For every  $j \geq 1$ ,  $n_j$  is the (finite) dimension of  $E_j$ . The next assumption, ensuring that  $\mathbb{X}$  can be represented (in some weak sense) as a KL series, will be in order throughout the paragraph.

**Assumption C** – For every  $j$ , note  $(f_{j,1}, \dots, f_{j,n_j})$  an orthonormal basis of  $E_j$  (with respect to the inner product of the space  $L^2(T, dm)$ ). **(C-i)** The process  $\mathbb{X}$  admits the following KL expansion: there exists an array of i.i.d.  $N(0, 1)$  random variables  $\{\xi_{j,l} : j \geq 1, l = 1, \dots, n_j\}$  such that, as  $N \rightarrow +\infty$  and p.s. - **P**, the process

$$\mathbb{X}_N(t) = \sum_{j=1}^N \sqrt{\lambda_j} \{ \xi_{j,1} \times f_{j,1}(t) + \xi_{j,2} \times f_{j,2}(t) + \dots + \xi_{j,n_j} \times f_{j,n_j}(t) \}, \quad t \in T,$$

converges to  $\mathbb{X}$  in  $L^2(T, dm)$ . **(C-ii)** The processes  $\mathbb{X}, \mathbb{X}_N$  ( $N \geq 1$ ) and  $f_{j,l}$  ( $j \geq 1, l = 1, \dots, n_j$ ) satisfy Assumption A of section 3.1, with  $Y = T$ .

The reader is referred once again to [1] or [21] for (rather general) sufficient conditions, ensuring the validity of Assumption **(C-i)** in the case  $[0, 1]^d$ ,  $d \geq 1$ . Assumption **(C-ii)** is redundant for  $G$  finite.

According to (12), for every  $[\pi] \in \widehat{G}$  and every  $N \geq 1$ , we define

$$\begin{aligned} (58) \quad f^\pi(t) &= d_\pi \int_G \chi_\pi(g) f(g^{-1} \cdot t) dg, \quad f \in L^2(T, dm), \\ \mathbb{X}_N^\pi(t) &= d_\pi \int_G \chi_\pi(g) \mathbb{X}_N(g^{-1} \cdot t) dg \\ &= \sum_{j=1}^N \sqrt{\lambda_j} \{ \xi_{j,1} \times f_{j,1}^\pi(t) + \xi_{j,2} \times f_{j,2}^\pi(t) + \dots + \xi_{j,n_j} \times f_{j,n_j}^\pi(t) \}, \\ \mathbb{X}^\pi(t) &= d_\pi \int_G \chi_\pi(g) \mathbb{X}(g^{-1} \cdot t) dg. \end{aligned}$$

Note that, according to Proposition 7-3, a.s.-**P**,

$$(59) \quad \mathbb{X}(t) = \sum_{[\pi] \in \widehat{G}} \mathbb{X}^\pi(t),$$

with convergence in  $L^2(T, dm)$ . The following Proposition states some remarkable relations between (59) and KL expansions, in the case of Gaussian processes with a  $G$  invariant law. Note that, given an irreducible representation  $\pi$ , we note  $\bar{\pi}$  the irreducible representation defined by the homomorphism  $g \mapsto \bar{\pi}(g)$ ,  $g \in G$ , where  $\bar{\pi}(g)$  is the matrix whose entries are the complex conjugate of the entries of  $\pi(g)$  (this is also known as the **contragredient representation** associated with  $\pi$ , see e.g. the discussion contained in [7, Ch. 4]).

**Proposition 11** *Let the notation and assumptions of this paragraph prevail. Then,*

1. for each  $[\pi] \in \widehat{G}$ ,  $\mathbb{X}_N^\pi(t) \rightarrow \mathbb{X}^\pi(t)$ , as  $N \rightarrow +\infty$ , a.s.-**P** in  $L^2(T, dm)$ ;
2. suppose  $\mathbb{X}$  has a  $G$ -invariant law; then, for each  $j \geq 1$ , the application

$$g \mapsto \{ f(t) \mapsto f(g^{-1} \cdot t) : f \in E_j \}$$

is a finite dimensional representation of  $G$ ;



3. for  $j \geq 1$ , write

$$E_j = E_j^1 \oplus \cdots \oplus E_j^{h_j},$$

with  $1 \leq h_j \leq n_j$ , to indicate the canonical decomposition of  $E_j$ , where  $E_j^l$  ( $l = 1, \dots, h_j$ ) is the direct sum of the irreducible representations contained in  $E_j$  that are equivalent to the same  $[\pi_{j,l}] \in \widehat{G}$  (see [19, Section 2.6]); then, for every  $\pi \in \widehat{G}$ ,  $f^\pi$  (as defined in (58)) is equal to zero if  $[\pi_{j,l}] \neq [\bar{\pi}]$  for every  $l = 1, \dots, h_j$ , and equal to the projection of  $f$  on  $E_j^l$  if  $[\pi_{j,l}] = [\bar{\pi}]$  for some  $l = 1, \dots, h_j$ .

**Proof. (1.)** Just write

$$\begin{aligned} \int_T (\mathbb{X}_N^\pi(t) - \mathbb{X}^\pi(t))^2 m(dt) &= d_\pi^2 \int_T \left( \int_G \chi_\pi(g) (\mathbb{X}_N(g^{-1} \cdot t) - \mathbb{X}(g^{-1} \cdot t)) dg \right)^2 m(dt) \\ &\leq d_\pi^2 \alpha_\pi^2 \int_T \left[ \int_G \chi_\pi(g) ((\mathbb{X}_N(g^{-1} \cdot t) - \mathbb{X}(g^{-1} \cdot t)))^2 dg \right] m(dt) \\ &= d_\pi^2 \alpha_\pi^2 \int_T ((\mathbb{X}_N(t) - \mathbb{X}(t)))^2 m(dt) \rightarrow 0, \end{aligned}$$

thanks to Assumption C, as well as the  $G$ -invariance of  $m$ .

(2.) A function  $f$  is in  $E_j$  if, and only if,

$$\lambda_j f(t) = \int_T R(t, s) f(s) m(ds).$$

Now suppose  $\mathbb{X}$  has a  $G$ -invariant law. Then,  $R$  is also  $G$ -invariant, and moreover, for every  $g \in G$  and  $f \in E_j$ ,

$$\begin{aligned} f(g^{-1} \cdot t) &= \int_T R(g^{-1} \cdot t, s) f(s) m(ds) \\ &= \int_T R(g^{-1} \cdot t, g^{-1} \cdot s) f(g^{-1} \cdot s) m(ds) \quad (G\text{-invariance of } m) \\ &= \int_T R(t, s) f(g^{-1} \cdot s) m(ds) \quad (G\text{-invariance of } R), \end{aligned}$$

and therefore  $f \in E_j$ . This concludes the proof.

(3.) This point is a direct application of Theorem 8 in [19]. ■

## 5.2 Watson's identity on the $n$ -dimensional flat tori

Watson's identity concerns processes defined on  $[0, 1]$  and taking the same values at  $t = 0$  and  $t = 1$ , in other words on a circle. Among the various geometrical sets arising as generalizations of the circle in higher dimensions, we will consider the  $n$ -dimensional torus. Recall that an  $n$ -dimensional lattice is a set

$$\Gamma := \left\{ \sum_{i=1}^n a_i v_i : a_1, \dots, a_n \in \mathbb{Z} \right\}$$

where  $v_1, \dots, v_n$  are  $n$  independent vectors in  $\mathbb{R}^n$ . The dual lattice  $\Gamma^*$  is defined to be the set of  $v^* \in \mathbb{R}^n$  such that

$$\langle v | v^* \rangle \in \mathbb{Z}, \text{ for all } v \in \Gamma.$$

The quotient space  $T_\Gamma := \mathbb{R}^n / \Gamma$  is the  $n$ -dimensional torus associated with  $\Gamma$ , and it is endowed with the measure  $dm$  inherited from the Lebesgue measure on  $\mathbb{R}^n$ . Consider a centered Gaussian process  $X := \{X(t) : t \in T_\Gamma\}$ , with covariance function  $K$ . For  $n = 1$ ,  $\Gamma = \mathbb{Z}$ , one has  $T^1 = \mathbb{R}/\mathbb{Z}$  and  $X$  is a centered Gaussian process defined on  $[0, 1]$  such that  $X(0) = X(1)$ . In this case  $X$  can be a Brownian

bridge or the Watson process. These processes are involved in Watson's identity (1). We propose an assumption on  $X$ , implying that this process satisfies an identity analogue of Watson's duplication identity (1) in higher dimensions. The techniques we adopt represent a  $n$ -dimensional generalization of the line of reasoning that the second author used in [14].

**Assumption D** – There exists a function  $k : T_\Gamma \rightarrow \mathbb{R}$  such that

$$(60) \quad K(s, t) = k(t - s) \quad (s, t \in T_\Gamma)$$

Note that this assumption is equivalent to the hypothesis that  $K$  is invariant under the isometry group of  $T_\Gamma$  which is composed of all translations of vector  $v \in \{\sum_{i=1}^n a_i v_i : 0 \leq a_i \leq 1, 1 \leq i \leq n\}$ . Let us first check that the covariance function of Watson's process given by (55) can be put in the form (60). If for  $s \in \mathbb{R}$  we denote by  $\bar{s} \in [0, 1]$  the corresponding class in  $\mathbb{R}/\mathbb{Z}$ , we have, for  $s, t \in [0, 1]$ ,

$$|s - t| - \frac{1}{2} = \begin{cases} t - s - \frac{1}{2} = \frac{1}{2} - (1 + s - t) = \frac{1}{2} - \overline{s - t} & \text{if } s < t, \\ s - t - \frac{1}{2} = (\overline{s - t}) - \frac{1}{2} & \text{if } s \geq t, \end{cases}$$

hence

$$|s - t| - \frac{1}{2} = \pm \left( \overline{(s - t)} - \frac{1}{2} \right) \quad (s, t \in [0, 1]).$$

This allows us to obtain the expression

$$(61) \quad s \wedge t - \frac{s + t}{2} + \frac{(s - t)^2}{2} + \frac{1}{12} = \frac{1}{2} \left( |s - t| - \frac{1}{2} \right)^2 - \frac{1}{24} = \frac{1}{2} \left( \overline{s - t} - \frac{1}{2} \right)^2 - \frac{1}{24} =: k(u)$$

where  $k(u) = (\bar{u} - 1/2)^2/2 - 1/24$  for  $u \in T^1$ .

**Lemma 12** *If the centered Gaussian process  $X$  satisfies Assumption D, then  $K$  admits a Karhunen-Loève expansion of the form*

$$(62) \quad K(s, t) = \sum_{v \in \Gamma^*} \lambda_v \{ \alpha_v \cos(2\pi \langle v | s \rangle) \cos(2\pi \langle v | t \rangle) + \alpha_v \sin(2\pi \langle v | s \rangle) \sin(2\pi \langle v | t \rangle) \}$$

where  $\lambda_v \in [0, \infty)$  for each  $v \in \Gamma^*$ , and  $\alpha_v > 0$  is chosen such that

$$\int_{T_\Gamma} \alpha_v^2 \cos^2(2\pi \langle v | s \rangle) dm(s) = \int_{T_\Gamma} \alpha_v^2 \sin^2(2\pi \langle v | s \rangle) dm(s) = 1.$$

**Proof.** The functions  $\{u \mapsto \cos(2\pi \langle v | u \rangle), u \mapsto \sin(2\pi \langle v | u \rangle) : v \in \Gamma^*\}$  form a complete set of orthogonal functions in  $L^2(T_\Gamma)$ . The Fourier series of  $k$  in this basis has the form

$$k(u) = \sum_{v \in \Gamma^*} \{ a_v \cos(2\pi \langle v | u \rangle) + b_v \sin(2\pi \langle v | u \rangle) \}.$$

Since  $k(u) = k(x - y) = K(x, y) = K(y, x) = k(y - x) = k(-u)$ , one has  $b_v = 0$  for each  $v \in \Gamma^*$ . If we replace  $u$  by  $x - y$  and use the identity  $\cos(a - b) = \cos a \cos b + \sin a \sin b$  we obtain the desired K-L expansion of  $K$ . ■

**Theorem 13** *If the centered Gaussian process  $X$  satisfies the assumption D, then*

$$(63) \quad \int_T X^2(t) dm(t) = \frac{1}{4} \int_T X_1^2(t) dm(t) + \frac{1}{4} \int_T X_2^2(t) dm(t)$$

where  $X_1(t) := \frac{X(t) - X(-t)}{2}$  and  $X_2(t) := \frac{X(t) + X(-t)}{2}$  are two independent centered Gaussian processes such that

$$\int_T X_1^2(t) dm(t) \stackrel{(law)}{=} \int_T X_2^2(t) dm(t)$$

and  $X_1(0, \dots, 0) = X_1(\frac{v_1}{2}, \dots, \frac{v_n}{2}) = 0$ .

**Proof.** From the preceding Lemma,  $X$  has a K-L expansion of the form

$$X(t) = \sum_{v \in \Gamma^*} \lambda_v \{ \xi_v \alpha_v \cos(2\pi \langle v|t \rangle) + \xi'_v \alpha_v \sin(2\pi \langle v|t \rangle) \}$$

(where the  $\xi_v$  and the  $\xi'_v$  are independent standard Gaussian random variables) and the claimed identity is clearly fulfilled with

$$X_1(t) = \sum_{v \in \Gamma^*} \lambda_v \xi_v \alpha_v \sin(2\pi \langle v|t \rangle), \quad X_2(t) = \sum_{v \in \Gamma^*} \lambda_v \xi'_v \alpha_v \sin(2\pi \langle v|t \rangle)$$

■

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