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An Approximate solution of Fokker-Planck equation for one-dimensional groundwater recharge through porous media

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ABSTRACT
The nonlinear Fokker-Planck equation arising for one-dimensional groundwater recharge through porous media, is in the form of partial differential equation which has been solved by using Homotopy Analysis method with the help of first approximate solution $\theta_0(Z,T)$ for given auxiliary parameter $q=0$. The solution is physically interpreted and concluded that during one-dimensional recharge through unsaturated porous media, the moisture content of the soil is parabolically increasing as depth Z increases for given $T>0$. The graph of the solution is given by using Maple coding.

INTRODUCTION
The uses of analytical techniques for groundwater flow and mass transport in the unsaturated porous media has significant increasingly from last few years, and it has a great important for hydrologist, agriculturists and people related with water resources sciences. Analytical solution provides better insight into the physics behind the transport phenomenon and efficient to use. Analytical approaches are for the most limited to situations of simple geometry domains, linear governing equation and homogeneous porous media. Analytical solutions of the partial differential equation for unsaturated flow under various boundary and initial condition are difficult to obtain because of the nonlinearity in soil hydraulic parameters. Exact analytical solution typically requires specialized forms of the hydraulic conductivity and diffusivity functions for nonlinear diffusion-advection equation. Several investigators have described different relation between the diffusivity coefficient and volumetric water content problems in unsaturated porous media. In 1958 Gradner [6] model provides a relationship between the pressure head 'h' and the volumetric water content θ as, $h(\theta)=a\cdot\theta^{-b}$ where a & b are empirical constants. The exponential hydraulic conductivity function has been widely used, but it is known to have a limited range of application to many real soils. The Brooks-Corey model [4] is a relationship between the reduced water content θ^* and the soil suction. The reduced water content θ^* is defined as a function of two values of moisture: the saturation of the moisture θ_s and the residual of the moisture θ_r; $\theta^*=\dfrac{\theta-\theta_r}{\theta_s-\theta_r}$

where $\theta_r<\theta<\theta_s$, $0<\theta^*<1$. According to Brooks-Corey model, the reduced water content θ^* has the following expression:

$$\theta^*=S^*=\begin{cases}\left(\dfrac{h}{h_{ae}}\right)^{-\lambda} & if\ \ h>h_{ae}\\ 1 & if\ \ h\le h_{ae}\end{cases}$$

where h is the pressure head, h_{ae} is the air entry suction of the reduce water content and S^* is effective degree of saturation between zero and one [4]. The Van Genuchten model [17] provides a relationship between the saturation degree and the soil suction using three empirical constant α, n and m as,

$$\theta^*=S^*=\begin{cases}\left(1+(\alpha\psi)^n\right)^{-m} & if\ \ \psi>0\\ 1 & if\ \ \psi\le 0\end{cases}$$

Where parameter m and n are related by the relation: $m=1-\dfrac{1}{n}$. Other functions developed by Van Genuchten [17] are firmly established for practical applications. Such special forms of the hydraulic functions make it possible to linearize the governing flow equations, and hence solve them analytically. Solutions to the linearized unsaturated flow equations are limited to the steady flow in semi-infinite, homogenous soils (Broadbridge and White [3]; Warrick [19]) and to transient flow in homogeneous and layered soils (Srivastava and Yeh) [16]. The Broadbridge and White Model (1988) [3] has adopted a function form for the diffusivity given by Philip and Knight (1974) [6] which allow for the transformation of the soil water diffusivity $D(\theta)$ as function has the form:

$$D(\theta)=\frac{a}{(b-\theta)^2} \tag{1}$$

Where a and b are constant. As a second step in the solution of the nonlinear flow problem, Broadbridge and White (1988) [3] developed an expression for $K(\theta)$ that in conjunction with the assumed function for $D(\theta)$ transforms equation (8) to the weakly nonlinear Burger's equation. This expression for $K(\theta)$ is given as:

$$K(\theta)=\beta+\gamma(b-\theta)+\frac{\lambda}{2(b-\theta)} \quad (2)$$

where β, γ *and* λ are constants. With the suggested analytical forms for $K(\theta), D(\theta)$ and the imposed boundary conditions, the Hopf-Cole transformations are applied to reduce a nonlinear equation to a linear form that possesses an exact parametric solution [3].

Ground water recharge problem has discussed by many researchers with different viewpoints. Swartzendruber uses Philip's [14] method to get graphical illustration of a mathematical solution for horizontal water function. Verma and Mishra [18] have obtained solution by similarity transformation of a one-dimensional vertical ground water recharges through porous media. Using singular perturbation technique, Mehta and coworkers [12] provided an approximate solution where the change in average diffusivity coefficient being very small; it has treated as constant. Hari Prasad et al. [7] had provided a numerical model to simulate water flow through unsaturated zones and studied the effect of unsaturated soil parameters on water movement during different processes such as gravity drainage and infiltration. Parikh, Mehta and Pardhan have obtained transcended solution of Fokker-Planck equation of vertical groundwater recharge in a dry region [13].

NOMENCLATURE

V	=	Volume flux of moisture
S	=	Saturation of the soil

Greek Symbols

ϕ	=	Porosity of soil
ρ	=	Density
θ	=	Moisture Content of Soil
ψ	=	Soil suction

Subscripts

s	=	bulk density

Assumption and Mathematical Statement of the Problem

The purpose of this investigated model is to discuss the approximate analytical solution of non-linear partial differential equation arising in ground water recharge phenomenon, to examine the moisture content in homogeneous porous media. The change in moisture content in porous media and distribution of pore pressure can be calculated using Homotopy Analysis method. Its solution provides the moisture content of the porous media (soil) at any depth Z at time T > 0. At any depth Z > 0, the graph of moisture content versus time T shows that the moisture content increases as time T increases.

1 Mathematical Structure

When water flow through unsaturated porous media in vertically downward direction, hydraulic conductivity is varies nonlinearly with the volumetric water content; $K=K(\theta)$.The variation of the hydraulic conductivity with the volumetric water content θ in unsaturated homogeneous porous media for small Reynolds number the volume of flow of water described by Darcy's law as [5],

$$\vec{V}=-K(\theta)\nabla H \quad (3)$$

Where $\vec{V}$ = volume flux of moisture

$K(\theta)$ = coefficient of the volumetric water content,

∇H = gradient of the whole moisture potential

Such groundwater flow satisfies the equation of continuity as follows,

$$\frac{\partial}{\partial t}(\rho_s \phi S)=-\nabla M \quad (4)$$

Where ρ_s is the bulk density of the soil on dry weight basis, M is the mass of flux of the water at any time $t \geq 0$.

Considering that water is incompressible, $M=\rho\vec{V}$ and the water content of the soil is given by standard relation with saturation of soil S as $\theta=\phi S$ [3].

Where ϕ is porosity and S is a saturation of the soil.

Equation (4) reduces to,

$$\frac{\partial}{\partial t}(\rho_s \theta)=-\nabla(\rho\vec{V}) \quad (5)$$

Where ρ is the flux density.

Using equation (3) in (5), we get

$$\frac{\partial}{\partial t}(\rho_s \theta)=-\nabla(\rho(-K(\theta)\nabla H)) \quad (6)$$

It is also considered here as that the flow takes place only in vertical downward direction [17], equation (6) reduced to,

$$\rho_s \frac{\partial \theta}{\partial t}=\rho\frac{\partial}{\partial z}\left(K(\theta)\frac{\partial H}{\partial z}\right) \quad (7)$$

In unsaturated soil instead of pressure head h one introduces, the soil suction ψ by negative forces of capillary and pressure head is negative $\psi=|h|$. For reduced water content, only the soil suction matter. The water thus moves inside the unsaturated soil from a point having a greater pressure head (or a lower suction value $\psi=|h|$) to another point by a smaller pressure head (or a greater suction), until these values become equal. For unsaturated porous media, H is total soil moisture potential: $H=\psi-gz$, where ψ is the pressure potential (soil matric suction), z is the elevation in the vertical downward direction of flow, and g is gravitational constant. Hence equation (7) will be,

$$\frac{\partial \theta}{\partial t}=\frac{\rho}{\rho_s}\frac{\partial}{\partial z}\left(K(\theta)\frac{\partial \psi}{\partial z}\right)-\frac{\rho}{\rho_s}g\frac{\partial K(\theta)}{\partial z} \quad (8)$$

The equation (8) can be written as,

$$\frac{\partial \theta}{\partial t}=\frac{\partial}{\partial z}\left(D(\theta)\frac{\partial \theta}{\partial z}\right)-\frac{\rho g}{\rho_s}K'(\theta)\frac{\partial \theta}{\partial z} \quad (9)$$

Where z is depth in vertical downward direction, t is time, $\theta(z,t)$ is volumetric soil water content, $D(\theta)=\frac{\rho K}{\rho_s}\frac{\partial\psi}{\partial\theta}$ is called the diffusivity coefficient, $K(\theta)$ is the coefficient of the volumetric water content and $K'(\theta)=\frac{dK}{d\theta}$.

The expression in equation (9) is θ dependent equation. Generally θ dependent equation is called one dimension Fokker-Planck equation. This equation (9) is a model based on Darcy-Buckingham approach in vertical downward direction flow of water in unsaturated porous media.

For the sake of convenience, the moisture contents consider in uniform soil take place in positively downward direction from z = 0 top of the bottom z = L where water table is saturated as shown in figure 1. We consider following new independent variables $Z=\frac{z}{L}, and\, T=\frac{\rho g}{\rho_s L}t$ has been introduced to simplify the equation (9) as,

$$\frac{\partial\theta}{\partial T}=\varepsilon\frac{\partial}{\partial Z}\left(D(\theta)\frac{\partial\theta}{\partial Z}\right)-K'(\theta)\frac{\partial\theta}{\partial Z} \tag{10}$$

Where $\varepsilon=\frac{\rho_s L}{\rho g}$ is an auxiliary parameter. As given by Broadridge and White model (1988) [3]; the soil water diffusivity $D(\theta)$ and hydraulic conductivity $K(\theta)$ from equation (1) and (2) are written as

$$D(\theta)=\frac{a}{b^2}\left(1-\frac{\theta}{b}\right)^{-2}=\frac{a}{b^2}\left(1+\frac{2\theta}{b}\right) \tag{11}$$

and

$$K(\theta)=\beta+\gamma b+\frac{\delta}{2b}+\left(\frac{\delta}{2b^2}-\gamma\right)\theta \tag{12}$$

Where a, b, $\beta,\gamma\, and\, \lambda$ are constants.

Using equation (11) and (12) in (10), we get

$$\frac{\partial\theta}{\partial T}=\varepsilon\frac{a}{b^3}\frac{\partial}{\partial Z}\left((b+2\theta)\frac{\partial\theta}{\partial Z}\right)-\left(\frac{\delta}{2b^2}-\gamma\right)\frac{\partial\theta}{\partial Z} \tag{13}$$

The equation (13) is nonlinear second order partial differential equation which governs moisture content of soils for the one-dimensional unsteady flow in unsaturated porous medium in a downward direction. For one-dimensional Fokker-Planck diffusion-convection equation in vertical groundwater recharge problem, let's assumed that the moisture content of the soil at top a large basin is θ_c. It is necessary to choose appropriate boundary and initial condition to solve the equation (13). Hence we choose the appropriate boundary and initial conditions,

$$\theta(0,T)=\theta_c, \text{ for any } T>0 \tag{14}$$

At the top of a dry region, the moisture content of homogeneous soil is θ_c

The initial condition as,

$$\theta(Z,0)=\theta_c e^Z, \text{ for any } Z>0 \tag{15}$$

Since initial moisture content of the homogeneous soil is $\theta_c e^Z$ (very small) for Z > 0 and T = 0 is very near to top of matrix.

Equation (13) rewritten as,

$$\frac{\partial\theta}{\partial T}=\varepsilon\left[A\left(\frac{\partial\theta}{\partial Z}\right)^2+B\frac{\partial^2\theta}{\partial Z^2}+A\theta\frac{\partial^2\theta}{\partial Z^2}\right]-C\frac{\partial\theta}{\partial Z} \tag{16}$$

Where $A=\frac{2a}{b^3}, B=\frac{a}{b^2}$ and $C=\left(\frac{\delta}{2b^2}-\gamma\right)$. Since moisture content θ is increasing as depth Z increase for T > 0. It appropriate to choose guess value of moisture content of the solution as, [10]

$$\theta(Z,T)=\left(\theta_c e^Z+\frac{1}{4}ZT\right) \tag{17}$$

2 The Solution with HAM

Let $\mathrm{N}\left[\Theta(Z,T;q)\right]=0$ denote a nonlinear equation, Θ be a function of homotopy parameter q, whose Maclarian series

$$\Theta(Z,T;q)=\sum_{m=0}^{+\infty}\theta_m(Z,T)q^m \tag{18}$$

Where N is a nonlinear operator, $\Theta(Z,T;q)$ is considered as unknown function that represent moisture content θ at any depth Z for given time $T\geq 0$ for $0\leq q\leq 1$. We use auxiliary linear operator $\Im=\partial/\partial T$ and initial approximation of moisture content of the soil is $\theta_0(Z,T)=\left(1-e^{-Z}\right)T$ to construct the corresponding zeroth-order deformation equation. As the auxiliary linear operator which satisfies $\Im[C_1]=0$, where C_1 is arbitrary constant. We construct a homotopy as [10],

$$\mathrm{H}\left[\Theta(Z,T;q);\theta_0(Z,T),H(Z,T),\hbar,q\right] \\ =(1-q)\left\{\Im\left[\Theta(Z,T;q)-\theta_0(Z,T)\right]-q\hbar H(Z,T)\mathrm{N}\left[\Theta(Z,T;q)\right]\right\} \tag{19}$$

Enforcing the Homotpy (19) to be zero [10],

$$\mathrm{H}\left[\Theta(Z,T;q);\theta_0(Z,T),H(Z,T),\hbar,q\right]=0$$

Establish the zero-order deformation equation of moisture content as [10],

$$(1-q)\Im\left[\Theta(Z,T;q)-\theta_0(Z,T)\right]=q\hbar H(Z,T)\mathrm{N}\left[\Theta(Z,T;q)\right] \tag{20}$$

Where $\theta_0(Z,T)$ denote a guess value of the exact solution $\theta(Z,T)$ which we want to find, $\hbar\neq 0$ is an auxiliary parameter, $H(Z,T)\neq 0$ is an auxiliary function, $q\in[0,1]$ is an embedding parameter and $\Im$ is an auxiliary linear operator with the property

$\Im\left[\Theta(Z,T;q)\right]=0$ when $\theta(Z,T;q)=0$

When $q=0$, the zero-order deformation equation (20) becomes

$$\Im\left[\Theta(Z,T;0)-\theta_0(Z,T)\right]=0 \tag{21}$$

This gives,

$$\Theta(Z,T;0)=\theta_0(Z,T) \tag{22}$$

When $q=1$, since $\hbar\neq 0$, $H(Z,T)\neq 0$ the zero-order deformation equation (20) is equivalent to

$$\mathrm{N}\left[\Theta(Z,T;1)\right]=0 \tag{23}$$

This is exactly the same as the original equation, provided

$$\Theta(Z,T;1)=\theta(Z,T) \tag{24}$$

According to (22) and (24) as the embedding parameter q increases from 0 to 1, solution $\Theta(Z,T;q)$ varies continuously from the initial guess $\theta_0(Z,T)$ of the moisture content of soil to the solution $\theta(Z,T)$, and its solution is assumed as,

$$\Theta(Z,T;q)=\Theta(Z,T;0)+\sum_{m=1}^{\infty}\theta_m(Z,T)q^m \tag{25}$$

Where $\theta_m(Z,T)=\frac{1}{m!}\frac{\partial^m\Theta(Z,T;q)}{\partial q^m}\Big|_{q=0}$ (26)

i.e. the moisture content of the soil is a function of depth Z, and time T for any parametric value q . The moisture content of the soil at top of large basin $\Theta(Z,T;0)$ and sum of moisture contents of the soil at different depth layer for different value of parameter q is expressed as, moisture content at time $T=0$, $\theta_0(Z,T)$ and sum of moisture content $\theta_1(Z,T)$, $\theta_2(Z,T)$,… at different time T. Here, the series (25) is called homotopy-series and $\theta_m(Z,T)$ is called the mth-order derivative of Θ. Auxiliary parameter $\hbar$ in homotopy-series (25) can be regard as iteration factor and is widely used in numerical computations. It is well known that the properly chosen iteration factor can ensure the convergence of homotopy series (25) depends upon the value of $\hbar$, one can ensure that convergent of homotopy series, solution simply by means of choosing the proper value of $\hbar$ as shown by Liao [8, 9, 10, 11]. If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, the auxiliary function $H(X,T)$ are so properly chosen, the series (25) converges at $q=1$. If at q = 1 it is equivalent to the original equation $N[\Theta(Z,T;1)]=0$.

Hence the moisture content of soil can be expressed as,

$$\theta(Z,T)=\Theta(Z,T;q)\big|_{q=1}=\sum_{m=0}^{+\infty}\theta_m(Z,T) \tag{27}$$

Equation (27) be one of the solution of the original equation (12) of the moisture content of soil and besides its solution obvious at q = 0. The series (27) is called homotopy series solution of $N[\Theta(Z,T;1)]=\theta(Z,T)$.

According to the definition (26), the governing equation can be deduced from the zero-order deformation equation (20). Define the vector

$$\vec{\theta}_n=\{\theta_0(Z,T),\theta_1(Z,T),\ldots\theta_n(Z,T)\}$$

Differentiating equation (20) m times with respect to the embedding parameter ε and then setting $q=0$ and finally dividing them by $m!$, we have the so-called mth order deformation equation of the moisture content θ will be as,

$$\Im\left[\theta_m(Z,T)-\chi_m\theta_{m-1}(Z,T)\right]=q\hbar H(Z,T)R_m\left(\vec{\theta}_{m-1},Z,T\right) \tag{28}$$

Where

$$R_m\left(\vec{\theta}_{m-1},Z,T\right)=\frac{1}{(m-1)!}\frac{\partial^{m-1}N\left[\Theta(Z,T;q)\right]}{\partial q^{m-1}}\Bigg|_{q=0}$$

And, $\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}$

It should be emphasized that $\theta_m(Z,T)$ for $m\ge 1$ is governed by the linear equation (26) with the linear boundary condition that come from the original problem, which can solve by symbolic computation software Maple as bellow. The rule of solution expression as given by equation (16) and equation (20), the auxiliary function independent of q can be chosen as $H(X,T)=1$ [10].

According to (24) and taking inverse of equation (28) the equation (28) become,

$$\theta_m(Z,T)=\chi_m\theta_{m-1}(Z,T)+\hbar\Im^{-1}\left[R_m\left(\vec{\theta}_{m-1},Z,T\right)\right]$$

$$R_m\left(\vec{\theta}_{m-1},Z,T\right)=\frac{1}{m!}\frac{\partial^{m-1}N\left[\Theta(Z,T;q)\right]}{\partial q^{m-1}}$$

In this way, we get $\theta_m(Z,T)$ for $m=1,2,3\ldots$ successively by using Maple software as,

$$\theta_1(Z,T)=\frac{1}{48}T\hbar\left(\varepsilon T^2+6\varepsilon\theta_c e^Z TZ+12\varepsilon\theta_c e^Z T-12T-12Z+96\varepsilon\theta_c^2 e^{2Z}+48\varepsilon\theta_c e^Z-96\theta_c e^Z\right) \tag{33}$$

$$\theta_2(Z,T)=\frac{1}{384}\begin{pmatrix}-96\hbar T^2\varepsilon\theta_c e^Z Z-256\hbar T^2\varepsilon\theta_c e^Z-384\varepsilon\theta_c e^Z+448\hbar T^2\varepsilon^2\theta_c^2 e^{2Z}+768T\hbar\theta_c e^Z+96\hbar Z\\ +96Z+192T+1536\theta_c e^Z-24\hbar\varepsilon T^2-96\hbar\varepsilon\theta_c e^Z TZ+768\hbar\theta_c e^Z-3072T\hbar\varepsilon\theta_c^2 e^{2Z}\\ -96\varepsilon\theta_c e^Z T-8\varepsilon T^2-48\varepsilon\theta_c e^X TX-768\varepsilon\theta_c^2 e^{2Z}-384\hbar\varepsilon\theta_c e^Z+3456T\hbar\theta_c^3\varepsilon^2 e^{3Z}+192\hbar T\\ -960\hbar\varepsilon\theta_c e^Z T-768\hbar\varepsilon\theta_c^2 e^{2Z}+3\hbar T^3X^2\varepsilon^2\theta_c e^Z+18\hbar T^3 Z\varepsilon^2\theta_c e^Z+320\hbar T^2 Z\varepsilon^2\theta_c^2 e^{2Z}\\ +48\hbar T^2\varepsilon^2\theta_c e^Z Z+20T^3\hbar\theta_c e^Z\varepsilon^2+128\hbar T^2\varepsilon^2\theta_c e^Z+230\hbar T\varepsilon^2\theta_c^2 e^{2Z}+192\hbar T\varepsilon^2\theta_c e^Z\end{pmatrix} \tag{34}$$

…………

Using initial guess value of moisture content from equation (17) and successive moisture content form (31) and (26) etc. and using in equation (28), we get

$$\theta(Z,T)=\left\{\begin{matrix}\theta_c e^Z+\frac{1}{4}TZ+\frac{1}{48}T\hbar\begin{pmatrix}\varepsilon T^2+6\varepsilon\theta_c e^Z TZ+12\varepsilon\theta_c e^Z T-12T-\\ 12Z+96\varepsilon\theta_c^2 e^{2Z}+48\varepsilon\theta_c e^Z-96\theta_c e^Z\end{pmatrix}+\\ \frac{1}{384}\begin{pmatrix}-96\hbar T^2\varepsilon\theta_c e^Z Z-256\hbar T^2\varepsilon\theta_c e^Z-384\varepsilon\theta_c e^Z+448\hbar T^2\varepsilon^2\theta_c^2 e^{2Z}\\ +768T\hbar\theta_c e^Z+96\hbar Z+96Z+192T+1536\theta_c e^Z-24\hbar\varepsilon T^2\\ -96\hbar\varepsilon\theta_c e^Z TZ+768\hbar\theta_c e^Z-3072T\hbar\varepsilon\theta_c^2 e^{2Z}-96\varepsilon\theta_c e^Z T\\ -8\varepsilon T^2-48\varepsilon\theta_c e^X TX-768\varepsilon\theta_c^2 e^{2Z}-384\hbar\varepsilon\theta_c e^Z+192\hbar T\\ +3456T\hbar\theta_c^3\varepsilon^2 e^{3Z}-960\hbar\varepsilon\theta_c e^Z T-768\hbar\varepsilon\theta_c^2 e^{2Z}+3\hbar T^3X^2\varepsilon^2\theta_c e^Z\\ +18\hbar T^3 Z\varepsilon^2\theta_c e^Z+320\hbar T^2 Z\varepsilon^2\theta_c^2 e^{2Z}+48\hbar T^2\varepsilon^2\theta_c e^Z Z\\ +20T^3\hbar\theta_c e^Z\varepsilon^2+128\hbar T^2\varepsilon^2\theta_c e^Z+230\hbar T\varepsilon^2\theta_c^2 e^{2Z}+192\hbar T\varepsilon^2\theta_c e^Z+\ldots\end{pmatrix}+\ldots\end{matrix}\right\} \tag{35}$$

Where $\theta_1(Z,T),\theta_2(Z,T),\ldots$ are given by equation (33) and (34) respectively represents moisture content of the soil at any time T for vertical direction depth Z for $T>0$. The solution is an infinite series solution, which represents the approximate value of moisture content for time $T>0$. It is convergent at $q=1$ for auxiliary parameter $\hbar=0.1$

3 Numerical and Graphical Solution

As mentioned by Liao [10], the use of auxiliary parameter $\hbar$ brings a dramatic advantage. It should be noted that the auxiliary parameter $\hbar$ controls the convergence and accuracy of the solution series. The solution represented by (35) contains the auxiliary parameter $\hbar$ which gives the convergence region and rate of approximation for the Homotopy analysis method. In order to define the region such that the solution series is independent of $\hbar$, multiple curves are plotted. The region, where the moisture content of soil $\theta'(Z,T), \theta''(Z,T), \theta'''(Z,T)$ and $\theta^{iv}(Z,T)$ verse $\hbar$ is the horizontal line known as the convergence region for the corresponding function. The common region among $\theta'(Z,T)$, $\theta''(Z,T)$, $\theta'''(Z,T)$ and $\theta^{iv}(Z,T)$ are known as overall convergence region. Figure (2) indicate that the valid region of $\hbar$ is about -5.0 to 1.0. Similarly, it can find the value of the convergent control parameter $\hbar$ for different values of constant parameters [10].

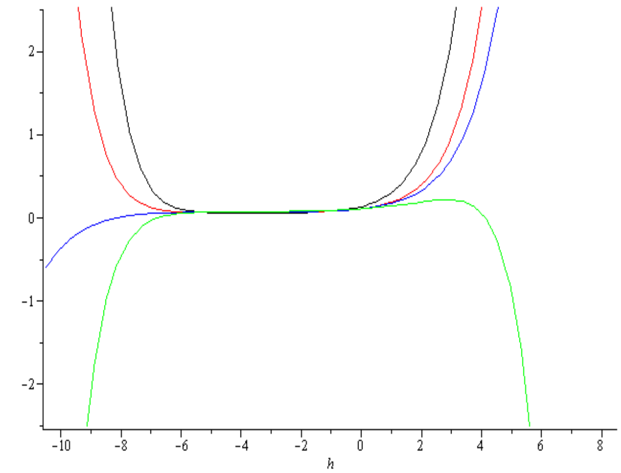

Figure 2: the $\hbar$-curve of $\theta'(Z,T)$, $\theta''(Z,T)$ $\theta'''(Z,T)$ and $\theta^{iv}(Z,T)$ given by (35) when H(Z, T) =1.

The numerical and graphical presentation of equation (35) in the present work has been carried out using Maple coding. Figure 3 represents the graphs of moisture content $\theta(Z,T)$ vs. depth Z, for T = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0 for fixed value $\theta_c = 0.1$ and $\varepsilon = \frac{\rho_s L}{\rho g} = \frac{0.2 \times 2}{0.1 \times 9.8} = 0.408163 \approx 0.4$ are fixed, and Table I indicates the numerical values.

TABLE I: Moisture content $\theta(Z,T)$ for different depth Z for fixed time T = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0.

Depth Z	Moisture Content of the soil									
	T=0.1	T=0.2	T=0.3	T=0.4	T=0.5	T=0.6	T=0.7	T=0.8	T=0.9	T=1.0
0.1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.3178	0.3477	[illegible]	[illegible]
0.5	[illegible]	[illegible]	[illegible]	0.2672	[illegible]	[illegible]	0.3391	[illegible]	[illegible]	[illegible]
0.6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
0.8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.496	0.5407	[illegible]	[illegible]
0.9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

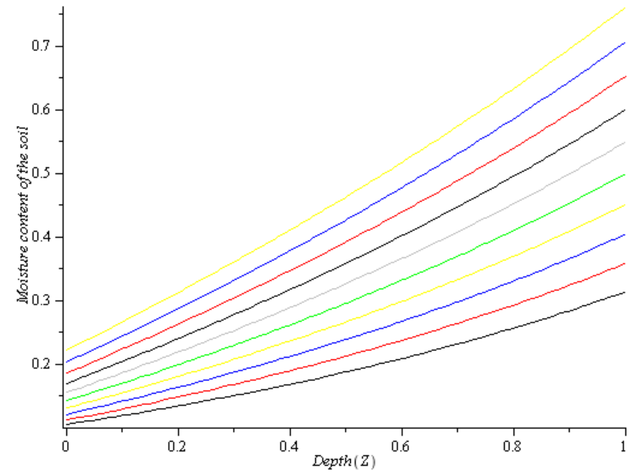

Figure 3: Represents moisture content $\theta(Z,T)$ vs. depth Z for auxiliary parameter $\hbar = 0.1$ and auxiliary function $H(Z,T) = 1$ when depth $0 \le Z \le 1$, time $0 \le T \le 1$, $\theta_c = 0.1$ and $\varepsilon \approx 0.4$ are considered.

CONCLUSIONS

The equation (35) represents moisture content of the soil for any depth Z for any time $T > 0$. It is converges for embedded parameter $q = 1$ and for auxiliary parameter $\hbar = 0.1$ which is expressed in term of exponential terms of Z and time $T > 0$. The moisture content θ find out from guess value of the exact solution for $Z = 0$, T=0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. The figure number 3 of solution for moisture content θ vs. depth Z and time T shows that the moisture content of the soil is increasing as depth Z increasing from 0 to 1 and $T > 0$. From figure 2, it can conclude that for T=0.1 moisture content of soil is linearly increasing as depth Z increasing but when time T is increasing and due to different deformation added to θ, the moisture content of soil is successively increasing parabolically. Hence solution is graphically as well as physically consistent with the phenomenon. From figures 3 and analytical result (35), It's concluded that the moisture content of soil is increasing when depth as well as time increases.

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