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NUMERICAL SOLUTIONS OF NON-LINEAR FRACTIONAL TRANSPORT MODELS IN UNCONVENTIONAL HYDROCARBONS RESERVOIRS USING VARIATIONAL ITERATION METOHD

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ABSTRACT

Unconventional hydrocarbon reservoirs, such as, shale gas deposits, offer a new source of energy resources. These reservoirs consist of tight porous rocks which are characterized by nano-scale size porous networks with ultra-low permeability. The mathematical modeling of phenomena through such tight porous media provides its own challenges. In this study, we consider a relatively new approach by modeling the transport of gas by the time-fractional advection-diffusion equation,

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(K\frac{\partial p}{\partial x}\right) - U\left(\frac{\partial p}{\partial x}\right), \quad t > 0,\ 0 \le x \le 1;$$

(with suitable initial and boundary conditions) in order to study the pressure distribution $p(x,t)$ in unconventional reservoirs. In our model, $K = K(p)$ is the diffusivity (which is related to the rock permeability) and $U = U(p, p_x)$ is a convection velocity; and both K and U are highly non-linear. This model is derived from mass balance and momentum balance equations. An approximate series solution is found by using variational iteration method (VIM). Power series solutions are expressed in a form that is easy to implement on computers where symbolic platforms are available. We will analyze the sensitivity of the solutions to different values of the fractional order α, and also compare the solutions with other models and against the data where available.

INTRODUCTION

The flow of gas through shale rocks is described by mathematical models that are based on the following considerations: the amount of gas that is transported through the reservoir and the amount of gas that is retained in it. The geomechanics of the reservoirs play important role in the model development. The intrinsic rock permeability and the rock porosity are among the principal parameters of reservoirs whose accurate determination is major objective of any investigation related to describing the transport processes of gas through the tight porous media.

Previous studies show that the use of the mathematical models based on the traditional Darcy's law is inadequate because a variety of flow regimes occur in tight porous media other than the continuous (viscous) flow [16]. The results obtained by models based on the conventional Darcy's law have significant deviations from experimental data. Many attempts have been made by different researchers to explain such deviations and it was suggested that different approaches (other than Darcy's law) have to be employed in order to describe the gas flow through tight porous media. Different flow regimes have been observed when fluid is present in the interconnected pores and such flow regimes are caused by different physical effects. Among these are the prevailing pore structure and inter-connectivity (network) between the pores, pore size distribution, the flow conditions, and the mean free path. Interconnected pores can be classified into several groups, and different flow regimes occur in these groups based on the Knudsen number [16].

1 The Mathematical Model

The time evolution processes, such as, conduction of heat (temperature distribution) or flow of fluids (pressure distribution) are often modeled by partial differential equations (PDEs). These PDEs are derived from mass balance, momentum balance and energy balance equations. In a single-phase, single-component system, we denote the pressure by $p(x,t)$ at the position x at the time t. If $u(x,t)$ is the velocity flow field in which the fluid is transported, then the time-fractional advection-diffusion equation can be written as follows, see [11],

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(K(p)\frac{\partial p}{\partial x}\right) - U\frac{\partial p}{\partial x}. \qquad (1)$$

The mathematical models, equation (1), are used to study the complex systems, such as, crowded system (protein diffusion within cells) [14], anomalous diffusion through tight porous media [3], tumor invasion [1].

The article is organized as follows: Section (2) contains some basic definitions and results from fractional calculus; Section (3) describes the variational iteration method to obtain the solution of the problem (1) subject to initial and boundary conditions; Section (4) presents a case study of exponential uploading; Section (5) includes the numerical results and provides graphs of the solutions along with error analysis. Finally, the conclusions of the study are stated.

2 Preliminaries

The theory of fractional calculus is rich and deep which was originated by the question "what does it mean by non-integer order derivative?" Many great mathematicians have contributed toward the development of the subject. The definitions of fractional order derivatives and integrals were introduced by Riemann, Liouville and Ries and further extended by many others [8]. In the recent years, the interest in the subject has risen because of its ability to explain the history dependent processes. A great number of books and articles have already been written on the theory of fractional calculus. The basic definitions of fractional calculus are provided here, [4,10].

Riemann-Liouville fractional integral of an absolutely integrable function $f(t)$ is defined by

$$\left({}_0I_t^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau, \quad t>0, \quad \alpha \in R^+.$$

For $\alpha = 1$, it reduces to ordinary integral and for $\alpha = 0$, it becomes identity operator.

Riemann-Liouville fractional derivative of an absolutely continuous function $f(t)$ is defined by

$${}_0D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}}\,d\tau, \quad 0<\alpha<1.$$

For $\alpha = 1$, it reduces to ordinary derivative and for $\alpha = 0$, it becomes identity operator.

Caputo fractional derivative of a function $f(t)$, whose derivative is absolutely integrable, is defined by

$${}_o^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}}\,d\tau, \quad 0<\alpha<1.$$

Relationship between Riemann-Liouville and Caputo fractional derivative is given by

$$\begin{aligned}\left({}_0^C D_t^\alpha f\right)(t) &= \left({}_0D_t^\alpha f\right)(t) - f\left(0^+\right)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \\ &= {}_0D_t^\alpha\left[f(t) - f\left(0^+\right)\right].\end{aligned}$$

Mittag-Leffler Function is the generalization of the exponential function $e^z = \sum_{k=0}^\infty \frac{z^k}{k!}$.

1-parameter Mittag-Leffler Function

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha>0.$$

2-parameter Mittag-Leffler Function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha>0, \beta>0.$$

3 Variational Iteration Method

In this section, we describe the variational iteration method (VIM) and provide an outline for its implementation [13]. The VIM has been extensively used by many authors [7,12,13,15] to obtain the series solutions of problems arising in different areas of applied mathematics and engineering [2]. Ji-Huan He [6s] proposed the VIM to obtain the solutions of nonlinear differential equations. The method provides the solution in the form of a successive approximations that may converge to the exact solution if such a solution exists. In case, where a closed form of exact solution is not achievable, we use the truncated series as the approximate solution, for instance, the n th partial sum of

the series. The VIM has certain advantages over the other proposed methods, such as, Adomian decomposition method (ADM), and homotopy perturbation method (HPM) [9]. In case of ADM, a lot of work has to be done to compute the Adomian polynomials for nonlinear terms and in case of HPM, a huge amount of calculations has to be done when degree of nonlinearity increases. On the other hand, no specific requirements are needed for nonlinear operators in order to use VIM [13].
The basic concepts and main steps for the implementation of the VIM are explained here. Consider the problem (1),

$$\frac{\partial^\alpha p}{\partial t^\alpha}=\frac{\partial}{\partial x}\left(K(p)\frac{\partial p}{\partial x}\right)-U\frac{\partial p}{\partial x}.$$

The VIM presents a correctional functional in *t*-direction for the equation (3) in the form (assuming p_n is known),

$$p_{n+1}(x,t)=p_n(x,t)+{}_0I_t^\alpha\left[\lambda(t)\left\{\frac{\partial^\alpha p_n}{\partial t^\alpha}-\frac{\partial}{\partial x}\left(K\frac{\partial p}{\partial x}\right)+U\frac{\partial p}{\partial x}\right\}\right],$$

where λ is called Lagrange multiplier and its value is found by variational theory [6] and it is $\lambda=-1$. Thus the above equation becomes

$$p_{n+1}(x,t)=p_n(x,0)+{}_0I_t^\alpha\left[\frac{\partial}{\partial x}\left(K\frac{\partial p}{\partial x}\right)-U\frac{\partial p}{\partial x}\right]. \quad (2)$$

Equation (2) is a recursive relation and we can set the zeroth approximation as the initial condition, $p_0(x,t)=p(x,0)=f(x)$. By using p_0, we obtain a sequence of successive approximations, and the exact solution is obtained by taking the limit $n\to\infty$ of the n^{th} approximation, that is, $p(x,t)=\lim_{n\to\infty}p_n(x,t)$.

4 A Case Study: Exponential Uploading

We consider the case of exponential uploading, that is, initial condition is given as $p(x,0)=e^{-cx}$, ($c>0$). The boundary condition is $p(x,t)\to 0$ as $x\to\infty$. In order to further investigate the problem (1), we subdivide the problem into two subcases: (i) *K* and *U* are constants, and (ii) *K* and *U* depend on *p*.

Subcase I We consider *K* and *U* are one, then the problem (1) becomes

$$\frac{\partial^\alpha p}{\partial t^\alpha}=\frac{\partial^2 p}{\partial x^2}-\frac{\partial p}{\partial x}. \quad (3)$$

The zeroth approximation is taken as $p_0(x,t)=e^{-cx}$ and the subsequent approximations are obtained by using the formula (2). The n^{th} approximation is written as follows:

$$p_n(x,t)=\sum_{k=0}^{n}a_k(x)\frac{t^{\alpha k}}{\Gamma(\alpha k+1)}, \quad (4)$$

where $a_n(x)=\left(c^2+c\right)^n e^{-cx}$. By taking the limit $n\to\infty$ of equation 4, we obtain the exact solution,

$$p(x,t)=\sum_{k=0}^{\infty}a_k(x)\frac{t^{\alpha k}}{\Gamma(\alpha k+1)}. \quad (5)$$

The convergence of Solution $p(x,t)$ is shown by ratio test. We denote the n^{th} term of equation (5) by

$$s_n(x)=a_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}.$$

Taking the ratio of the n^{th} term, we obtain

$$\frac{s_{n+1}(x)}{s_n(x)}=(c^2+c)e^{-cx}t^\alpha\left|\frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)}\right|. \quad (6)$$

Note that $(c^2+c)e^{-cx}$ is bounded above by (c^2+c) on $x>0$, t^α is bounded above by T on $0<t\le T$, and by Wendel's double inequality [5], we have $\lim_{n\to\infty}\left|\frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)}\right|=0$. Hence, equation (6) gives $\lim_{n\to\infty}\left|\frac{s_{n+1}(x)}{s_n(x)}\right|=0.$ Thus, the series solution obtained in equation (5) converges (absolutely) for all *x* and *t*.

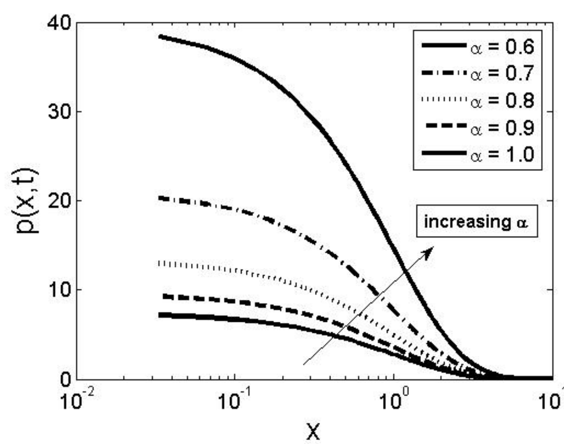

Figure 1: Plots of the solution $p(x,t)$ at the time instant $t=1$, on $x>0$, for the different values of α as shown in the figure**.**

By choosing $c=1$, we obtain the following: $a_n(x)=2^n e^{-x}$, and hence the series solution can be written in the form

$$p(x,t)=e^{-x}\sum_{k=0}^{\infty}\frac{\left(2t^\alpha\right)^k}{\Gamma(\alpha k+1)}. \quad (7)$$

The closed form of the solution is $p(x,t)=e^{-x}E_\alpha(2t^\alpha)$. Figure 1 shows the plots of the solution $p(x,t)$ for different values of α. It is noticed that the magnitude of the solution decreases with the decrease in the values of α.

Figure 2 shows the behavior of the solution with respect to time. It is observed that the magnitude of the solution increases as we progress in the time. Numerical solution is illustrated for $\alpha = 0.6$ and at different time instances.

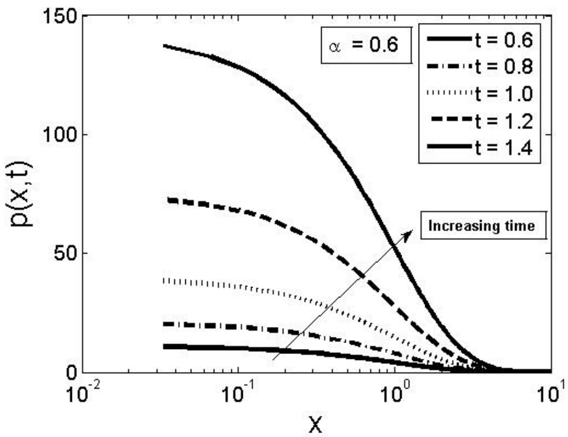

Figure 2: The behavior of the solution is shown as the time passes off. The magnitude of the solution increases as the time increases

Figure 3 shows the behavior of the pressure distribution at the different positions *x* in the spatial domain against the time *t*.

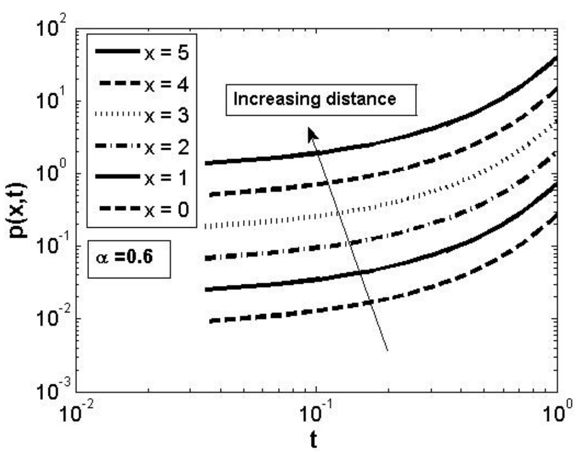

Figure 3: The behavior of the solution is plotted at different positions in the spatial domain. It is noted that the solution values increases as the time increases.

Figure 4 shows that the higher order accuracy in the numerical solution can be obtained just by using first few numerical approximations.

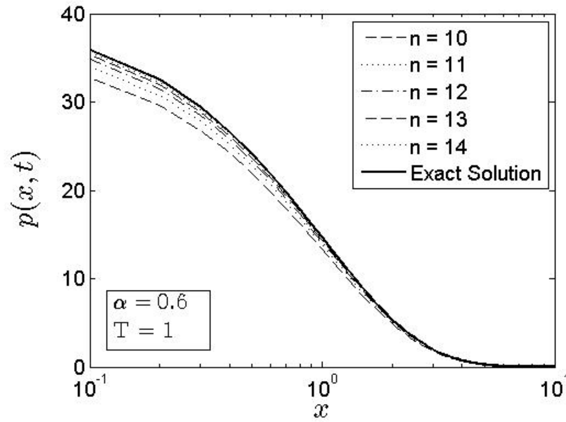

Figure 4: Numerical accuracy of the solution is illustrated. Different curves correspond to the different truncated series solutions.

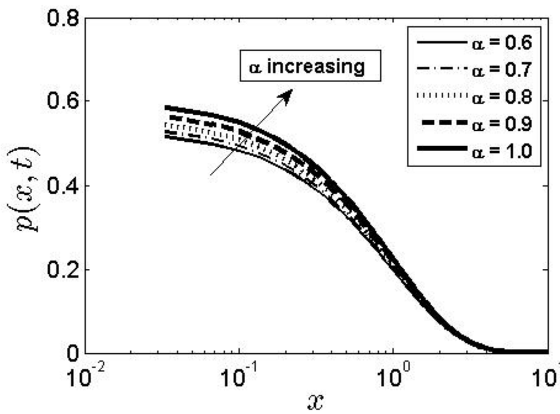

Figure 5: Plots of the solution $p(x,t)$ at the time instant $t = 1$, on $x > 0$, for the different values of α as shown in the figure.

Subcase II

We consider $K(p) = \frac{1}{p}$ and $U(p, p_x) = K\frac{\partial p}{\partial x}$, then the problem (1) becomes

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(\frac{1}{p}\frac{\partial p}{\partial x}\right) - \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)^2. \quad (8)$$

We find the series solution of the above problem by using VIM with the following initial $p(x,0) = e^{-x}$, and the boundary condition is $p(x,t) \to 0$ as $x \to \infty$.

The zeroth approximation is taken as $p_0(x,t) = e^{-x}$ and then by substituting this into the iterated integral we obtain the first approximation

$$p_1(x,t) = e^{-x} + {}_0I_t^\alpha\left[\frac{\partial}{\partial x}\left(\frac{1}{e^{-x}}\frac{\partial}{\partial x}\left(e^{-x}\right)\right) - \frac{1}{e^{-x}}\left(\frac{\partial}{\partial x}\left(e^{-x}\right)\right)^2\right],$$

which simplifies to

$p_1(x,t) = e^{-x}\left[1 - \frac{t^\alpha}{\Gamma(\alpha+1)}\right].$

On the same pattern, we obtain the n^{th} order approximate solution

$$p_n(x,t) = e^{-x}\left[1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \dots + \frac{\left(-t^\alpha\right)^n}{\Gamma(n\alpha+1)}\right].$$

By taking the limit $n \to \infty$, we obtain

$$p(x,t) = e^{-x} E_\alpha(-t^\alpha).$$

Figures 5, 6 and 7 show the behavior of the solution.

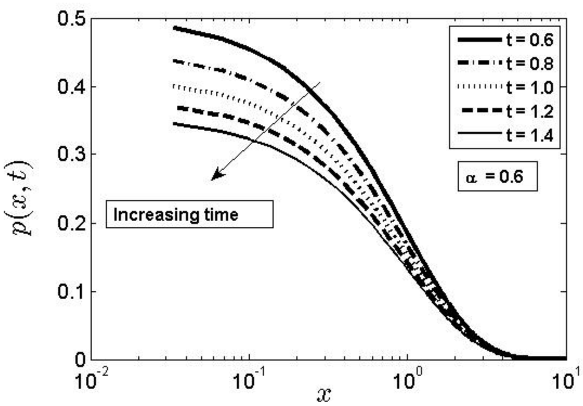

Figure 6: The behavior of the solution is shown as the time passes off. The magnitude of the solution increases as the time increases

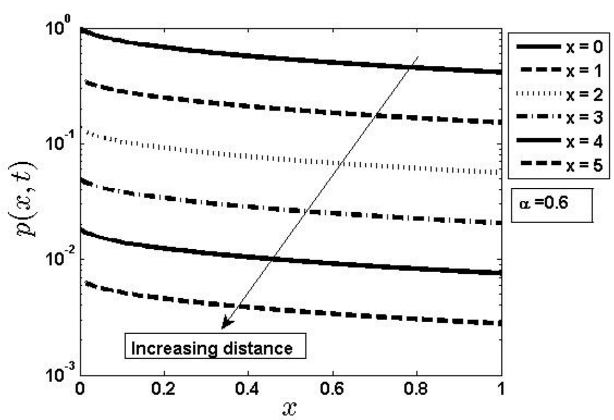

Figure 7: The behavior of the solution is plotted at different positions in the spatial domain. It is noted that the solution values increases as the time increases.

5 Error Analysis

We investigate the truncating error by defining the relative error as follows:

$$E_n(x,t) = \frac{\left|p(x,t) - p_n(x,t)\right|}{\left|p(x,t)\right|}, \tag{9}$$

where $p_n(x,t)$ is the n^{th} order approximate solution given by equation (4). Figure 8 and 9 show the plots of relative error at the point $(x,t) = (1,1)$ against the number of terms n in the truncated VIM solution, for cases $\alpha = 0.3, 0.4, \dots, 0.9$. The vertical axis is scaled as Natural Logarithm. The relative errors decay exponentially fast with n, but with the convergence rates that decrease as α approaches 0. Thus, a higher order approximate solution is required to achieve a given level of accuracy as α decreases.

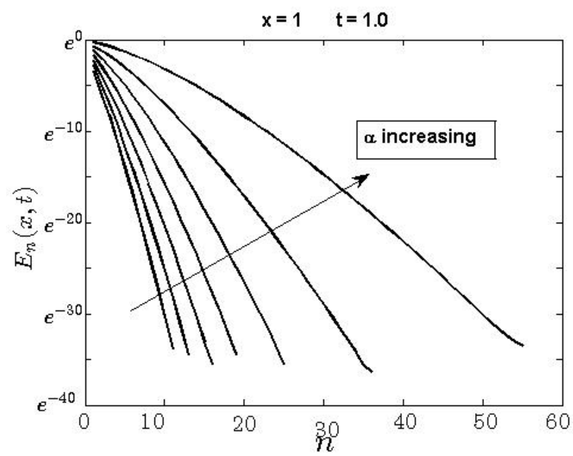

Figure 8: Plots of relative errors are shown for the subcase (i) against the number of terms *n*.

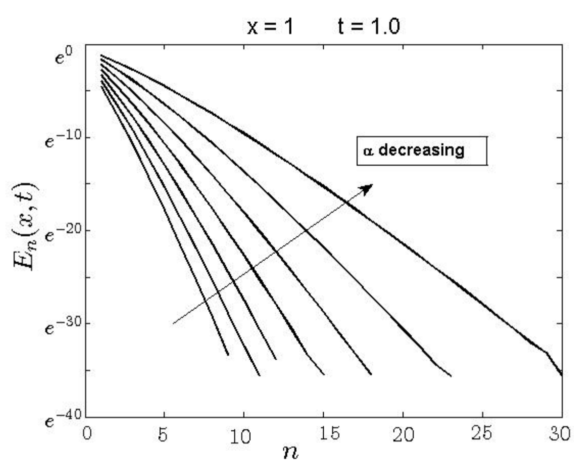

Figure 9: Plots of relative errors are shown for the subcase (ii) against the number of terms *n*.

CONCLUSIONS

We have presented the solutions of nonlinear fractional advection-diffusion equations that often arise in the modeling of pressure distribution in the unconventional hydrocarbon reservoirs. The solutions are found by using variational iteration method against different cases of diffusivity and convective coefficients. The general form of the solutions, obtained by VIM, is expressed in such a way so that it can be implemented on the computer with no difficulty. Validation of the numerical procedure is done for two problems whose exact solutions are known. Results obtained by VIM are in agreement with the exact solution. It is shown that only few successive approximations lead to a very good estimate of the exact solution. The truncation errors decay exponentially fast as *n* increases. VIM proves to be very efficient and fast in finding the solutions of fractional differential equations.

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