Bounds on Maximum Matchings in 1-Planar Graphs

by

John Wittnebel

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Computer Science

Waterloo, Ontario, Canada, 2019

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my readers.

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Abstract

In this thesis, we study lower bounds on maximum matchings in 1-planar graphs. We expand upon the tools used for proofs of matching bounds in other classes of graphs as well as some original ideas in order to find these bounds.

The first novel results we provide are lower bounds on maximum matchings in 1-planar graphs as a function of their minimum degree. We show that for sufficiently large n, 1-planar graphs with minimum degree 3 have a maximum matching of size at least $\frac{n+12}{7}$, 1-planar graphs with minimum degree 4 have a maximum matching of size at least $\frac{n+4}{3}$, and 1-planar graphs with minimum degree 5 have a maximum matching of size at least $\frac{2n+3}{5}$. All of these bounds are tight. We also give examples of 1-planar graphs with small maximum matching and minimum degree 6 and 7. We conjecture that the 1-planar graph of minimum degree 6 presented has the smallest maximum matching over all 1-planar graphs of minimum degree 6, but it is unclear if the method used for the cases of minimum degree 3, 4, and 5 would work for minimum degree 6.

We also study lower bounds in the class of maximal 1-plane graphs, and 3-connected maximal 1-plane graphs. We find that 3-connected, maximal 1-plane graphs have a maximum matching of size at least $\frac{n+4}{3}$, and that maximal 1-plane graphs have a maximum matching of size at least $\frac{n+6}{4}$. Again, we present examples of such a graph to show this bound is tight. We also show that every simple 3-connected maximum 1-planar graph has a matching of size at least $\frac{2n+6}{5}$, and provide some evidence that this is tight.

Acknowledgements

I would like to thank all the people that made this thesis possible. First, I want to thank my advisor, Therese Biedl, who helped me learn a great deal. I would like to thank my readers Anna Lubiw and Bruce Richter. Finally, I would like to thank my family for the enormous amount encouragement and support they provided.

Dedication

This is dedicated to John and Sue, my father and mother.

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Chapter 1

Preliminaries

1.1 Introduction

A graph is an abstract mathematical structure that is used to model many real-life applications. In fact, almost any binary relationship between objects can be perceived as a set of objects with lines between them representing the relationship of interest. One simple example would be a family tree. Similarly, a map can be viewed as a set of intersections that are connected via roads, or lines.

Because of this, some physical problems can easily be mapped onto problems in graph theory. Consider the problem where a group of students is tasked with pairing up for a group project. We can consider each student as a single node, connected to other nodes (or students) who they are friends with. The question of whether or not there is an arrangement of groups such that every group is comprised of two friends is of practical interest, and mathematically expressed in terms of a problem in graph theory. One could expand on such a problem, by wondering what occurs if there is no arrangement of groups that pleases everyone. In this case, it may be interesting to include everyone in a group and minimize the number of groups in which the two students are non-friends. We could expand this problem even further so that every person has an ordering of preferences of who they would like to be with and we wish to match people up in a way that maximizes overall utility.

These sorts of problem would also apply to roommate allocation or pairing up significant others. Such problems are known as *matching problems*, and will be the focus of this thesis.

Our example of a map being a set of intersections that are connected by roads has an interesting property: the graph can be drawn such that no two edges are crossing. The minimum number of crossings that a graph can have is important for many problems, and graphs that can be drawn without any crossings have many properties that make working with them efficient. However, we note that our example of viewing a roadmap as a graph is a little bit simplistic; there can be roads that cross each other without there being an intersection, such as when there is a tunnel that goes underneath a road. In such cases, it is not possible for a car to move from the tunnel to road and vice-versa, so it makes sense to not represent this as a vertex, but as edges that cross. This thesis will also be exploring properties of graphs that have such crossings.

We will begin with a more formal introduction to the standard definitions and notation before some of the more obscure concepts. Because of this, readers that have a basic knowledge of graph theory can skip Section 1.2.1.

1.2 Definitions

1.2.1 Basic Definitions and Standard Notation

A *graph* $G = (V, E)$ is defined as a pair of sets V and E, where V is a set of labels that are said to be *vertices* and E is a set of unordered pairs of vertices that are called *edges*. That is to say that the edges are not directed and the edge (v, w) is the same as the edge (w, v). We will use $V(G)$ and $E(G)$ (respectively) instead of V and E to avoid confusion when there is more than one graph being discussed. A vertex $v \in V(G)$ is *adjacent* to $w \in V(G)$, or w is a *neighbour* of v, if there exists $e \in E(G)$ such that $e = (u, v)$. A vertex $v \in V(G)$ is *incident* to $e \in E(G)$ if there exists some $w \in V(G)$ such that $e = (v, w)$. If $e = (u, v)$ is an edge, then u and v are the *endpoints* of e. An edge $e_1 \in E(G)$ is *incident* to $e_2 \in E(G)$ (or e_1 and e_2 are incident edges) if e_1 and e_2 share a common endpoint. We say that a graph is *bipartite* if we can split the vertices into a pair of disjoint sets (A, B) (called a *bipartition*) such that no vertex in A is adjacent to any other vertex in A, and no vertex in B is adjacent to any other vertex in B. The *size* of a set T is the number of elements in the set, denoted $|T|$.

It is standard to denote the number of vertices in G by n, and the number of edges in a graph G by m. We will follow this convention, but when there are several graphs being discussed, we will often be more explicit by using $|V(G)|$ and $|E(G)|$, respectively.

If several edges have the same set of endpoints, they are referred to as *multiedges*. A graph is said to be *simple* if there are no multiedges and there are no edges of the form (v, v), known as *loops*. Loops can never be helpful for matchings, and so we assume throughout this thesis that the given graph has no loops. Likewise multiedges can never be helpful for matchings, but we will sometimes permit them since they can be helpful for arguments.

An *independent set of vertices* in G is a set of vertices such that no two vertices in S are adjacent. An *independent set of edges* in G is a set of edges such that no two edges in S are incident. Such a set is also called a *matching*, we will discuss these in more detail in Section 1.2.4.

If G is a graph then H is a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. For any $T \subset V(G)$, the graph *induced* by T is (T, A), where A is the set of edges in G with both endpoints in T. The graph induced by T is denoted by $G[T]$. See Figure 1.1 for an example. We use $G - T$ as a convenient shortcut for $G[V - T]$.

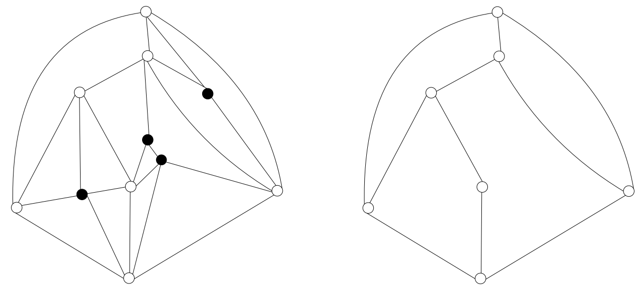


Figure 1.1: A planar graph G (left) and the corresponding $G[T]$ (right). The vertices not in T are filled in.

The *degree of a vertex* $v \in V(G)$ is defined as the number of edges in G that are incident to v. The degree of a vertex v is denoted by $deg(v)$ when the graph we are discussing is clear, or $deg_G(v)$, to clarify that we are discussing the vertex v with regards to the graph G. A graph G is said to be *k-regular* if every vertex of G has degree k. $\delta(G)$ denotes the minimum degree over all vertices in G.

A *complete* graph is a simple graph where every vertex is adjacent to every other vertex. The complete graph on n vertices is denoted K_n.

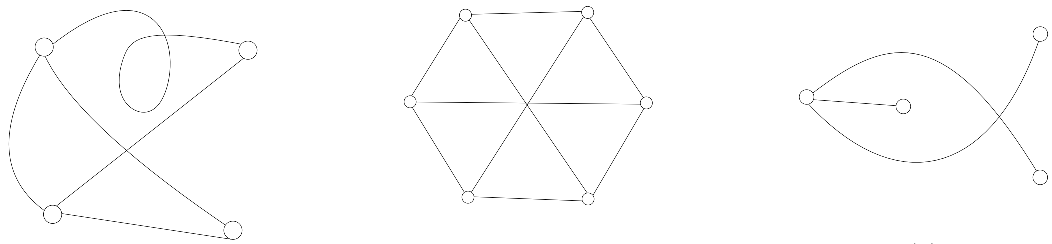
A *walk* in a graph G is a sequence of vertices $\langle v_1, v_2, \ldots, v_k \rangle$ such that for every $1 \leq i < k$, $\langle v_{i-1}, v_i \rangle \in E(G)$. A *path* in a graph G is a walk with the additional constraint that it has no repeated vertices. The *length* of a path (or a walk) $\langle v_1, \ldots, v_k \rangle$ is equal to k. A *closed walk*, or *cycle*, is a walk where the last vertex in the walk is equal to the first vertex in the walk, so the walk is of the form $\langle v_1, v_2, \ldots, v_k, v_1 \rangle$. In the case of closed walks, the last v_1 is often omitted as a closed walk implies that the walk ends where it began. A *simple cycle* is a cycle where every vertex in the cycle appears only one time. The *length* of a cycle $\langle v_1, v_2, \ldots, v_k \rangle$ is equal to k.

A graph is said to be *connected* if there is a walk from any vertex in G to any other vertex in G. Otherwise, it is said to be *disconnected*. A graph is *k-connected* if there does not exist a set of vertices S of size $k-1$ such that $G - S$ is disconnected. A set of vertices T that causes $G - T$ to be disconnected is a *cutting set*. If there is a single vertex v that causes $G - v$ to be disconnected, then v is a *cut vertex*. A *component* of a graph G is a maximal connected subgraph H of G. A component that has size 1 is a *singleton component*.

The following is a useful observation about the components of a graph that will be exploited later on.

Observation 1. *Let $T \subset V(G)$ and s_i be the number of components of size i in $G - T$. Then we have $n = |T| + \sum_{i \geq 1} i \cdot s_i$.*

Proof. There are is_i vertices in components of size i, and since no vertex can exist in two distinct components, there is no double-counting. Thus, the total number of vertices in



(a) (b) (c)

Figure 1.2: Graph drawings that are not good drawings; (a) violates Rule 1, (b) violates Rule 2, and (c) violates Rule 3.

components of $G - T$ is $\sum_{i \geq 1} i \cdot s_i$. The remaining $|T|$ comes from the vertices of T; they are not a part of any component of $G - T$. ■

1.2.2 Planar Graphs

A graph G can be represented as a set of points on the plane, where every vertex has a corresponding unique point on the plane, with a Jordan curve connecting any two points that exist in a tuple in the set $E(G)$. This representation is said to be a *drawing* of the graph G. There are certain rules that we will apply when drawing graphs. In particular, a graph drawing is *good* if it has the following properties:

1. No edge crosses itself.
2. No three edges cross at the same point.
3. Incident edges do not cross each other.
4. No two edges cross each other more than once.

In particular, the graph drawings in Figure 1.2 are not permitted, because they violate the above rules. We only consider good graph drawings in this thesis.

When a drawing is fixed, we say that a *rotation system* around a vertex v is the circular ordering of edges that are incident to v, as encountered when travelling around v in clockwise direction. An *embedding* of a graph G is a collection of rotation systems for every vertex $v \in G$. We note that if H is a subgraph of a graph G that has an embedding, then H inherits the embedding of G.

A *planar drawing* of a graph G is a drawing of G with the added restriction that no edges cross (see Figure 1.3). We use the term *planar embedding* for the set of rotation systems implied by a planar drawing. A graph G is *planar* if there exists a planar embedding for G. A planar graph equipped with a planar embedding is a *plane* graph. The edges in the

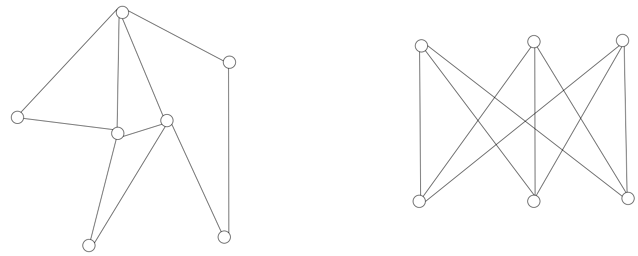


Figure 1.3: A drawing of a simple planar graph (left) and a simple non-planar graph (right).

drawing of a plane graph separate the plane into disjoint regions called *faces*. The set of faces of G is denoted by $F(G)$. The face that extends out infinitely is the *outer face*. An edge e is *incident* to a face f if e is one of the edges that surround f. An edge e is on the *boundary* of a face f if e is incident to f. A vertex v is *incident* to a face f if v is incident to an edge that is incident to f.

Note that the boundary of a face is not necessarily connected if a graph is not connected. A *facial circuit* of face f is a minimal closed walk that includes every vertex incident to f in a single component of G. If f is a face whose boundary is not connected, then for the purposes of defining the degree of f we make its boundary connected arbitrarily. More precisely, define the *augmented facial circuit* by arbitrarily adding a minimum number of edges through the face f so that all vertices incident to f are in the same component, and then takes its facial circuit using the virtual edges.

The *degree of a face* f, $deg(f)$, is the length of the augmented facial circuit of f. See Figure 1.6 for an example, the face f_1 has two facial circuits. A *bigon* is a face of degree 2, necessarily formed by a multiedge. We note that a graph can have multiedges without having bigons, see Figure 1.4. We say that a face f in a planar graph is a *triangle* if f has degree exactly 3 and is made up of 3 distinct edges, so the face f_1 in Figure 1.6 is not a triangle. A *triangulated* graph is a connected plane graph where every face is a triangle. It is well-known that triangulated graphs with at least 4 vertices are 3-connected if they are simple.

A *simple edge* is an edge that is not a multiedge (as always, we assume that none of the edges are loops). A *maximal, simple, planar graph* G is a simple planar graph where no simple edges can be added without causing G to become non-planar. It is well-known that a maximal simple planar graph with at least 3 vertices must be triangulated.

The following result is well-known (see e.g. [4]), and holds for graphs even with multiedges.

Lemma 2 (Euler's Formula)**.** *For any planar connected graph,* $|V(G)|-|E(G)|+|F(G)| = 2$.

The following result is also well-known; we repeat its proof here to convince the reader that it works even for graphs that contain multiedges.

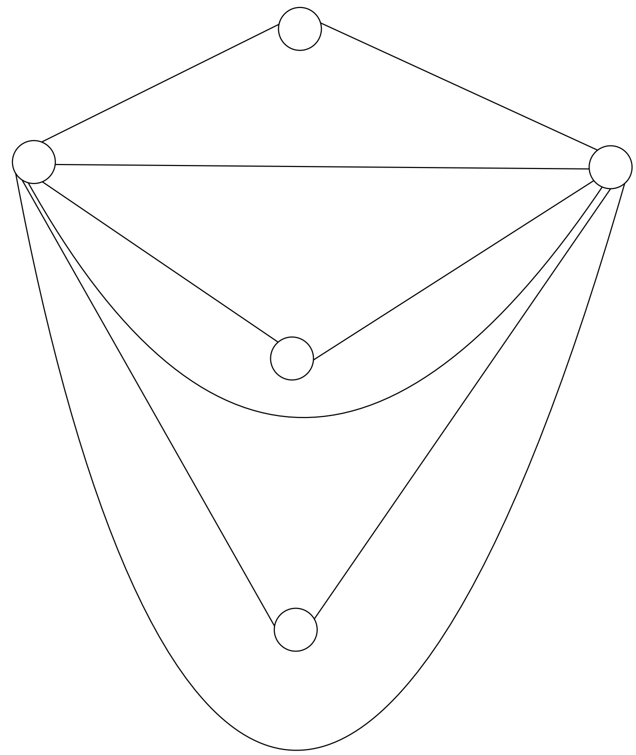


Figure 1.4: A graph that has multiedges but no bigons. This counts as a triangulated graph.

Lemma 3. *A triangulated graph without loops has exactly* $2|V(G)| - 4$ *faces. Furthermore, a planar graph has at most* $2|V(G)| - 4$ *faces of degree 3.*

Proof. Let G be a triangulated graph. Every face of G is incident to exactly three edges, and every edge of G can only be incident to two distinct faces. Thus, $2|E(G)| = 3|F(G)|$. Substituting $|E(G)| = \frac{3}{2}|F(G)|$ into Euler's Formula, we get that $|F(G)| = 2|V(G)| - 4$, as required.

Let H be a planar graph. For every pair of multiedges (u, v) that form a bigon of H, remove one copy of (u, v), repeating until no bigons remain. Proceed to add additional edges to faces of degree at least 4 so that every face has degree 3. The resulting graph is a triangulated graph, and thus has $2|V(H)| - 4$ faces. Since no faces of degree 3 were removed in this process, H has at most $2|V(H)| - 4$ faces of degree 3. ■

1.2.3 1-Planar Graphs

A *1-planar* graph is a graph that has a good drawing in the plane such that every edge is crossed at most once. See Figure 1.5 for an example. Recall that since we do not allow several edges to cross at a single point in graph drawings, the drawing in Figure 1.2a is not considered to be 1-planar. Planar graphs are a very well-studied topic in graph theory,

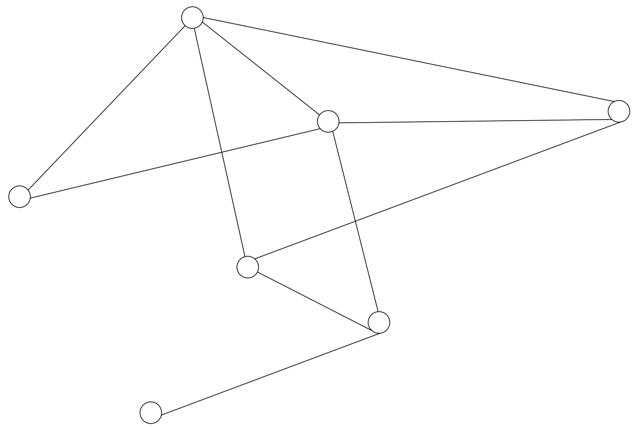


Figure 1.5: A 1-planar drawing of a simple graph.

whereas their 1-planar counterparts have been studied much less and only recently. 1-planar graphs were initially studied by Ringel in 1965 [12], who showed that every 1-planar graph has a 7-coloring. His motivation for studying 1-planar was to study colourings in planar graphs, because the union of a graph and its dual is always a 1-planar graph. But because 1-planar graphs are "close" to planar graphs without being planar, they have become of interest to many researchers; refer to [8] for an extensive overview of existing results for 1-planar graphs.

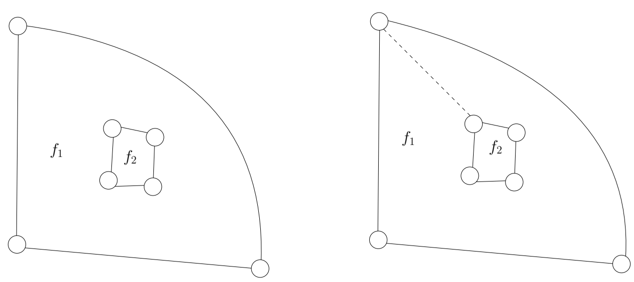


Figure 1.6: Two faces in a 1-planar graph. We have that $deg(f_1) = 9$, $deg(f_2) = 4$.

A *1-planar drawing* of a graph G is a good drawing of G in the plane such that no edge is crossed more than once. A *1-planar embedding* is a set of rotation systems implied by a 1-planar drawing. A *1-planar* graph is, as before, a graph that has a valid 1-planar drawing, and a *1-plane* graph is a 1-planar graph together with a 1-planar drawing. Note that in a 1-planar drawing of a graph, the edges separate the plane into disjoint regions, and we refer to these regions again as *faces*. However, it is important to note that a face may have a crossing on the boundary. Because of this, we refer to the vertices and crossings that lie on the boundary of the face f as the *corners* of f. If all corners of a face f are vertices, then we say that f is an *uncrossed face*, otherwise we say that f is a *crossed face*. An edge of a 1-plane graph is a *planar edge* if it is not crossed, otherwise it is a *crossed edge*. If G is 1-plane, then G^P is the *planarization* of G: the plane graph achieved by replacing every

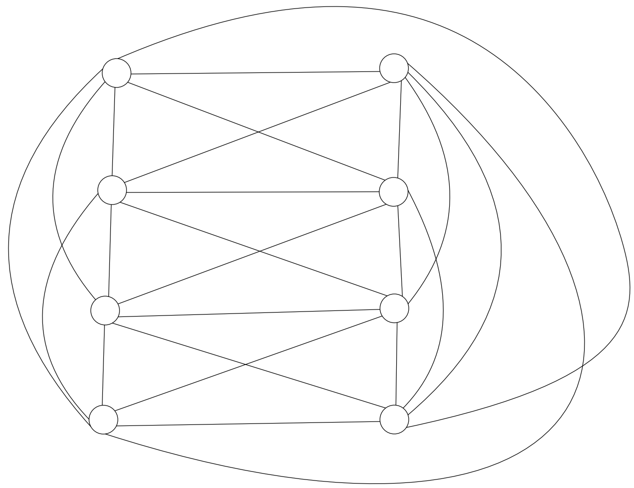


Figure 1.7: An optimal 1-planar graph, with $4n-8$ edges.

crossing in G with a vertex. The degree of a face in a 1-plane graph G is the degree of the corresponding face in G^P. As for planar graphs, a *facial circuit* incident to a face f is a minimal closed walk that includes every corner incident to f that is part of a single component of G^P. A face of G can have several facial circuits if G is disconnected.

A *maximal 1-plane graph* G is a 1-plane graph without bigons or loops where no edges can be added without causing G to become not 1-planar or introducing a bigon or loop. A *maximal 1-planar graph* G is a 1-planar graph without bigons or loops where no edges can be added without causing G to become not 1-planar or introducing a bigon or a loop, regardless of drawing. A *simple, maximal 1-planar graph* G is a simple 1-planar graph where no edges can be added without causing G to become not 1-planar. One interesting fact of 1-planar graphs is that unlike planar graphs, not every maximal 1-planar graph on n vertices has the same number of edges. In 1986, Schumacher proved that every simple, 1-planar graph with $n \geq 5$ contains at most $4n-8$ edges [13]. Simple 1-planar graphs with exactly $4n-8$ edges are referred to as *optimal 1-planar graphs*. See Figure 1.7. However, there are maximal 1-planar graphs with far fewer edges than $4n-8$. This is different from planar graphs, where every maximal planar graph with $n \geq 3$ has exactly $3n-6$ edges. See Figure 1.8 for a simple maximal 1-planar graph that has fewer than $4n-8$ edges.

In a 1-plane graph, a *kite* is a cycle of 4 vertices (v_1, v_2, v_3, v_4) where (v_i, v_{i+1}) are uncrossed with the additional edges (v_1, v_3) and (v_2, v_4) that cross one another.

Let c be a crossing of the edges (v_0, v_2) and (v_1, v_3). We say that an edge (v_i, v_{i+1}) (for $i = 0, \ldots, 3$ and additional modulo 4) is a *kite-edge* if (v_i, v_{i+1}, c) forms a face of a graph. A *kite* is a crossing where all four possible kite-edges exist. [1] We say that the uncrossed edges (v_i, v_{i+1}) of a kite are the *kite-edges* of the kite. For a crossing c, there is a unique kite that

[1] This is not the standard definition that requires that all vertices that make up the kite lie on the outer face. As such, we consider the graph in Figure 1.9 to be a kite, but it would not be considered a kite in other literature.

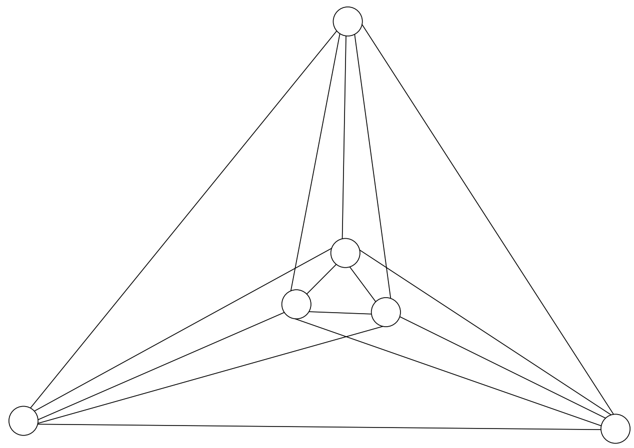


Figure 1.8: K_6 is a simple maximal 1-planar graph with $15 < 4n - 8$ edges.

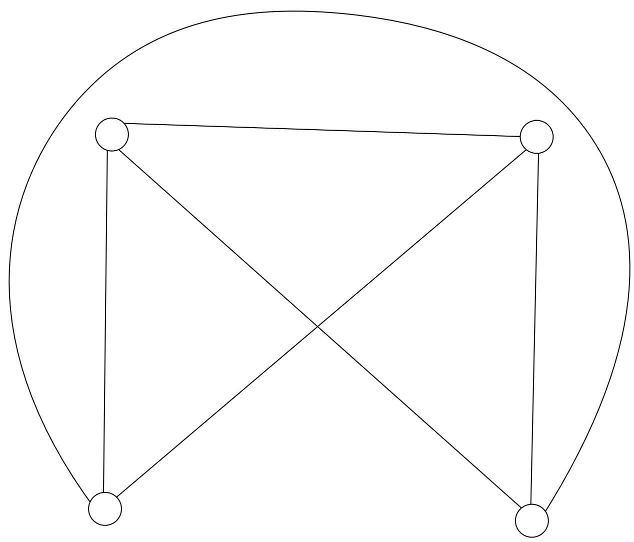


Figure 1.9: A kite, the vertices of the kite need not be incident to the outer face.

contains this crossing, and we refer to the *kite-edges* of the crossing c as the kite-edges of the kite that contains c.

The following lemmas are structural properties of 1-planar graphs that are fairly trivial but will be of importance later.

Lemma 4. *If G is a 1-plane graph that has a face f with at least 4 corners, then we can add a planar edge inside f without creating a bigon or loop.*

Proof. Assume first that f has more than more than one facial circuit on its boundary, say C_1 and C_2. Since every facial circuit of a 1-planar graph must contain a vertex, we can add an edge between a vertex in C_1 and a vertex in C_2. This edge can cross through f, and has

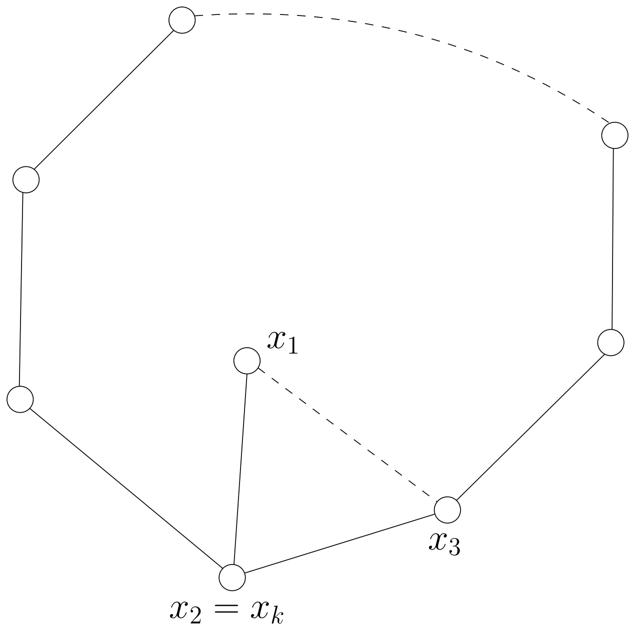


Figure 1.10: Adding the edge (x_1, x_3) when $x_2 = x_k$ in the facial circuit of a face.

not decreased the degree of f since the added edge could be a virtual edge that was added to compute the degree. Thus, no bigons were created during this process.

Now assume that the facial boundary is a single circuit, say $\langle x_1, x_2, \ldots, x_k \rangle$. We have $k = deg(f) \geq 4$. Since G is 1-planar there cannot be 2 consecutive incident crossings. If x_1 is a crossing, then x_2 and x_k are vertices, and $x_2 \neq x_k$ since they are the ends of the crossing edges and the drawing is good. Thus, the edge (x_2, x_k) can be introduced through f, splitting it into two faces: one that has the facial circuit $\langle x_1, x_2, x_k \rangle$ and another that has the facial circuit $\langle x_2, x_3, \ldots, x_k \rangle$. Thus, both of the new faces have degree larger than 2. A symmetric argument holds for if any x_i is a crossing. If none of the corners are crossings, then x_2 and x_k are still both vertices, so the same argument can be applied unless $x_2 = x_k$, as we are not allowed to add loops. If $x_2 = x_k$, then x_1 is a vertex of degree 1. Hence, since $k \geq 4$, $x_1 \neq x_3$, so the edge (x_1, x_3) can be added, splitting f into two faces: one that has the facial circuit $(x_1, x_2 = x_4, x_3\rangle$ and one that has the facial circuit $\langle x_1, x_3, \ldots, x_k \rangle$. See Figure 1.10. ■

Corollary 5. *If G is a maximal 1-plane graph, then all faces of G have at most 3 corners.*

Note that the reverse is not true: a planar triangulated graph with $n \geq 4$ is not maximal because we can pick two adjacent triangles f_1, f_2 and add an edge between the two vertices that belong to only one of them.

Claim 6. *If G is a 1-plane graph where every face has degree 3, then for every crossing c in G, the kite-edges of c are in $E(G)$.*

Proof. If c is a crossing between the edges (a_1, b_1) and (a_2, b_2), the faces that are incident to the paths $(a_1, c, b_2), (a_1, c, a_2), (b_1, c, a_2), (b_1, c, b_2)$ must all have degree 3. It follows that the kite edges $(a_1, b_2), (a_1, a_2), (b_1, a_2), (b_1, b_2)$ are edges in $E(G)$. ■

Lemma 4 and Claim 6 immediately give us the following results.

Corollary 7. *If G is a maximal 1-plane graph with a crossing between the edges ab and $a'b'$, then the kite-edges aa', ab', ba', bb' exist in G.*

Corollary 8. *If every face of a 1-plane graph has degree 3, then every face that is incident to a crossing is part of a kite.*

Lemma 9. *If G is a 1-plane graph with $n \geq 3$ where every face has degree 3, and X is the set of crossings in G, then G has $4|X|$ crossed faces and $2|V(G)| - 2|X| - 4$ uncrossed faces.*

Proof. Recall that G^P is the planarization of G. Observe that since every face of G has degree 3, G^P is a triangulated planar graph. It follows that G^P has $2|V(G^P)| - 4 = 2|V(G)| + 2|X| - 4$ faces. Further, the number of faces in G^P is equal to the number of faces in G, so G has $2|V(G)| + 2|X| - 4$ faces.

Since every face of G has degree 3, the kite-edges around any crossing must exist. Thus, every crossing has 4 incident crossed faces. Also note that every crossed face must be incident to exactly one crossing, by Corollary 8.

It follows that G has $4|X|$ crossed faces, and as a result has $2|V(G)| - 2|X| - 4$ uncrossed faces. ■

1.2.4 Graph Matchings and Existing Results

A *matching* in a graph G is set of edges E such that no two edges in E have the same endpoint, also known as an independent set of edges in G. The vertices that are incident to the edges in E are known as *matched vertices*, or *vertices in the matching*, when the matching being discussed is clear. Since every edge has 2 endpoints, it follows that the largest possible number of edges in a matching of G is $\lfloor \frac{|V(G)|}{2} \rfloor$. A matching in a graph without any unmatched vertices is known as a *perfect matching* and a matching in a graph with exactly one unmatched vertex is known as a *near-perfect matching.* A *maximal matching* is a matching in a graph where no further edges can be added to the matching without removing any edges that already exist within the matching. A *maximum matching* in a graph G is the largest possible matching that can occur in G.

Matchings in graphs have a long and rich history. We only cover a few results here; for more details see for example [3] and [16].

One of the oldest matching results is from 1891; Petersen proved that every 3-regular 2-connected graph has a perfect matching by using the structure of 3-regular graphs to produce a smaller 3-regular, 2-connected graphs where a perfect matching in the smaller, modified graph can be expanded to become a perfect matching in the initial graph [11].

Later, in 1931, König proved that the size of a maximum matching is equal to the number of vertices in a minimum vertex cover of any bipartite graph [9]. In particular, any k-regular bipartite graph has a perfect matching. It was shown later how to find it in $O(n\log(n))$ time [7].

Tutte's Theorem and Berge's Generalization

In 1947, Tutte discovered a characterization for the size of a maximum matching. For an arbitrary subset T of vertices, we let $odd(T)$ denote the number of distinct odd components that exist in the graph $G - T$. Tutte showed that a graph has a perfect matching if and only if for every T, the number of odd components in $G-T$ is at most the size of T [14]. In 1958, Berge generalized this into what is known as the Tutte-Berge Formula [1].

Theorem 10 (The Tutte-Berge Formula)**.** *The number of unmatched vertices in the maximum matching of a graph G is equal to the minimum value of $odd(T) - |T|$, over all possible $T \subset V(G)$.*

The Tutte-Berge formula is a vital tool for many proofs of general matching bounds, and will be used extensively in this paper, but this characterization is of little practical use in finding the size of a maximum matching in any specific graph. The reason for this is that there are an exponential number of subsets T that must be checked to find which subset T maximizes the value of $odd(T) - |T|$.

Algorithms to Find Matchings

In 1965, Edmonds published the Blossom Algorithm, an efficient algorithm for finding and constructing maximum matchings in graphs, which involves repeatedly creating a larger matching from a given matching using a 'blossom' structure [6]. This algorithm has a runtime in $O(mn^2)$, significantly better than brute-forcing using the Tutte-Berge formula.

This was later improved upon by the work of Micali Vazirani, who created an algorithm to find a maximum matching with a runtime of $O(mn^{1/2})$. (This was later corrected to $O(mn^{1/2}\alpha(m,n))$ [15].)

Matching Bounds in Graphs Without Perfect Matchings

A lot of work has been done in regards to perfect matchings, and what graphs contain perfect matchings. However, relatively little work has been done in finding matching bounds for graphs that do not have perfect matchings.

The most prominent work in this field was done by Nishizeki and Baybars in 1979. They proved several lower bounds on maximal matchings in planar graphs relative to their minimum degree and connectivity [10]. We will be reviewing below the proof for the case where the minimum degree of a graph is 3, as the techniques heavily inspired the approach we used to getting bounds on maximum matchings in 1-planar graphs with large minimum degree.

Another paper is by Biedl, Demaine, Duncan, Fleischer, and Kobourov, where, among other things, they proved bounds on maximum matchings in 3-connected planar graph [2]. The proof used by Biedl et al. inspired the ones used in this thesis for the maximal 1-planar case, so we will also be reviewing part of their paper later on.

1.2.5 Our Results

We will prove several results on the size of matchings in 1-planar graphs with large minimum degree and maximal 1-planar graphs. The results are summarized in Table 1.1. For each graph type, there exists a matching with a number of edges in the column 'Matching Lower Bound', and the 'Graph With This Bound' column are graphs with no larger maximum matching. These are existence proofs, we have no algorithm to find them other than the generic matching algorithms discussed in Section 1.2.4.

The thesis will be structured as follows. After reviewing two strongly relevant papers in Section 1.3, we first study 1-planar graphs with large minimum degree and prove the bounds of the first three rows of Table 1.1 in Chapter 2. We then turn to maximal 1-planar graph and prove the bounds of the remaining rows of Table 1.1 in Chapter 3.

1.3 Reviews of Related Papers

We now review two previous papers that proved matching bounds. The main idea that is utilized by these papers is the Tutte-Berge formula, Theorem 10. By Theorem 10, the number of unmatched vertices in any matching of a graph is $max_T(odd(T)-|T|)$. For a graph G of interest, the papers both fix a T and go on to prove an upper bound on $odd(T)-|T|$. Since T was chosen arbitrarily it follows that $max_T(odd(T)-|T|)$ has the same upper bound. This upper bound on the number of unmatched vertices in a maximum matching of G gives us a lower bound on the size of the maximum matching in G.

Graph Properties	Matching Lower Bound	Graph With This Bound
$\delta(G) \geq 3$, 1-planar	$\frac{n+12}{7}$	Figure 2.12
$\delta(G) \geq 4$, 1-planar	$\frac{n+4}{3}$	Figure 2.13
$\delta(G) \geq 5$, 1-planar	$\frac{2n+3}{5}$	Figure 2.14
3-connected, Maximal 1-Plane	$\frac{n+4}{3}$	Figure 3.6
Maximal 1-Plane	$\frac{n+6}{4}$	Figure 3.12
3-connected, Simple, Maximal 1-Planar	$\frac{2n+6}{5}$	Unknown (but see Figure 3.17)

Table 1.1: Summary of our Results: For each graph class, any sufficiently large graph in the class a matching that is at least as large as the bound in the second column. These bounds are tight for the graphs listed in the third column.

1.3.1 Review of [2]

We will now review one of the proofs from Biedl, Demaine, Duncan, Fleischer, and Kobourov [2], that establishes a bound on maximum matchings in triangulated planar graphs. In particular, they prove that every 3-connected planar graph with $n \geq 10$ has a matching of size $\frac{n+4}{3}$. We will later use similar techniques to prove the size of matchings in the maximal 1-planar case.

As always, we wish to bound $odd(T) - |T|$. To do this, we study faces of $G[T]$, the graph induced by T. It helps to modify $G[T]$ so that every face of $G[T]$ contains at most one odd component of $G - T$. Let $T \subset V(G)$, and let H be the graph created by adding a maximal number of planar edges between vertices in T to G without creating a bigon. The following is known from [5].

Lemma 11. *Let G be a planar triangulated graph and let $T \subset V$ be a set of vertices. Then for every face f of $G[T]$ there is at most one component of $G - T$ that has vertices in the interior of f.*

Let f_3 be the number of odd components that exist in a face of degree 3 in $G[T]$, and let f_4 be the number of odd components that exist in a face of degree at least 4 in $G[T]$.

Claim 12. $odd(T) = f_3 + f_4$

Proof. $odd(T) \geq f_3 + f_4$ follows immediately from the definitions: every odd component of $G - T$ must exist in some face f of $G[T]$, and $G[T]$ does not contain any bigons so f has degree at least 3. $odd(T) - |T| \geq f_3 + f_4$ because $odd(T)$ is all components of $G - T$ and $f_3 + f_4$ are all odd components that exist in a face of degree 3 or greater. ∎

Claim 13. *If $|T| \geq 3$, then $f_3 + 2f_4 \leq 2|T| - 4$*

Proof. Let $G[T]'$ be a graph that is created by adding planar edges to $G[T]$ until every face is a triangle. Since $|V(G[T]')| \geq 3$ and $G[T]'$ is planar, it follows that $G[T]'$ has $2|T| - 4$ faces by Lemma 9. Notice that any face of $G[T]$ that had degree 4 or higher was made into at least 2 new faces by Lemma 4, and every face of $H[T]$ that had degree 3 was already a triangle. Thus, any component that was in a face of $G[T]$ of degree 3 has a corresponding face in $G[T]'$. Similarly, any component that was in a face of $G[T]$ of degree 4 or higher has at least 2 corresponding faces in $G[T]'$. Furthermore, no components of $G - T$ share corresponding faces of $G[T]'$ by Lemma 11. It follows that $f_3 + 2f_4 \leq |F(G[T]')| = 2|T| - 4$. ∎

Now we can get a bound on $odd(T) - |T|$ by substituting the values for T and $odd(T)$ from Claim 12 and Claim 13.

Lemma 14. *If $|T| \geq 3$, then $odd(T) - |T| \leq \frac{f_3 - 4}{2}$.*

Proof.

$$\begin{aligned} odd(T) - |T| &= f_3 + f_4 - |T| && \text{(From Claim 12)} \\ &\leq f_3 + f_4 - \frac{f_3 + 2f_4 + 4}{2} && \text{(From Claim 13)} \\ &= \frac{f_3 - 4}{2} \end{aligned}$$

■

Now we can bound $odd(T) - |T|$ based on f_3. Biedl et al. [2] shows a bound on f_3 for 3-connected, planar graphs by using the so-called 4-block tree. Since our techniques used will be different and not related to the 4-block tree, we will simply note the result, and encourage readers to see [2] for further details.

Claim 15. *If G is a 3-connected planar graph, then $f_3 \leq \frac{2n-4}{3}$.*

We can now use these tools to get a bound on the number of unmatched vertices in G.

Theorem 16. *If G is a 3-connected planar graph with $n \geq 10$, then G has a matching of size $\frac{n+4}{3}$.*

Proof. The case $|T| \leq 2$ is easily covered through case analysis, see [2] for details.

The interesting case is when $|T| \geq 3$, and in this case:

$$\begin{aligned} odd(T) - |T| &\leq \frac{f_3 - 4}{2} && \text{(By Claim 13)} \\ &\leq \frac{(2n-4)/3 - 4}{2} && \text{(By Claim 15)} \\ &= \frac{n-8}{3} \end{aligned}$$

By Theorem 10 G has a matching with at most $\frac{n-8}{3}$ unmatched vertices. It follows that the number of vertices in this matching is at least $\frac{2n+8}{3}$, or that the size of the matching is at least $\frac{n+4}{3}$. ■

1.3.2 Review of [10]

We will be using techniques similar to the paper 'Lower Bounds on the Cardinality of the Maximum Matchings of Planar Graphs' by Nishizeki and Baybars [10]. Therefore we give a brief overview of their results for matchings in connected planar graphs when the minimum degree is at least 3. We note that [10] proves several bounds for cases of minimum degree and connectivity; we only review one of them here since the techniques of the other cases are different and not relevant to this thesis. In particular, we will show that any planar graph with $\delta(G) \geq 3$ and $n \geq 14$ has a matching of size at least $\frac{n+4}{3}$.

Let G be a planar graph with $|V(G)| \geq 4$ and let $T \subset V(G)$ with $|T| \geq 3$. The cases when $|V(G)| < 4$ or $|T| < 3$ are trivial and discussed further in [10].

Claim 17. *Let G be a planar graph, with $T \subset V(G)$, such that $G - T$ contains the singleton components $x_1, x_2, \dots, x_{s_1}$. If $|T| \geq 3$, then $2|T| \geq \sum_{i=1}^{s_1}(deg_G(x_i) - 2) + 4$*

Proof. Define G' to be the graph created after the following operations have been performed on G:

1. Delete all components of $G-T$ except for the components of size 1. These components are $x_1, x_2, \dots, x_{s_1}$.
2. Delete all edges in $G[T]$.

Let G' have n' vertices and m' edges. Since G' is a bipartite planar graph and $n' \geq |T| \geq 3$, it is known that $m' \leq 2n' - 4$. It follows from the bipartition $(T, \{x_1, x_2 \dots, x_{s_1}\})$ of G' that $m' = \sum_{i=1}^{s_1}(deg_G(x_i))$. Since $n' = s_1 + |T|$, it follows that

$$\begin{aligned}\sum_{i=1}^{s_1} deg_G(x_i) &= m' \\ &\leq 2n' - 4 \\ &= 2s_1 + 2|T| - 4\end{aligned}$$

as desired. ■

Claim 18. *If G is planar and $T \subset V(G)$, then $3(odd(T) - |T|) \leq n + 2s_1 - 4|T|$, where s_1 is the number of components of size 1 in $G - T$.*

Proof. Let $T \subset V(G)$, and let the odd components of $G - T$ of size 1 be $x_1, x_2, \dots, x_{s_1}$ and the odd components of size greater than 1 be $B_1, B_2, \dots, B_\ell$. We have

$$n \geq s_1 + |T| + \sum_{i=1}^{\ell} |B_i| \geq s_1 + |T| + 3\ell$$

so we get

$$\begin{aligned}3(odd(T) - |T|) &= 3(s_1 + \ell - |T|) \\ &\leq (n - s_1 - |T| - 3\ell) + 3s_1 + 3\ell - 3|T| \\ &= n + 2s_1 - 4|T|\end{aligned}$$

as desired. ■

Lemma 19. *Let G be a planar graph. If $|T| \geq 3$, then*

$$3(odd(T) - |T|) \leq n + 2s_1 - 2\sum_{i=1}^{s_1}(deg_G(x_i) - 2) - 8,$$

where $x_1, \ldots, x_{s_i}$ are the singleton components of $G \setminus T$.

Proof. Let G be a planar graph and $T \subset V(G)$. Then by Claim 18

$$3(odd(T) - |T|) \leq n + 2s_1 - 4|T|$$

and by substituting in $|T|$ from Claim 17 into the above equation, we get that

$$3(odd(T) - |T|) \leq n + 2s_1 - 2\sum_{i=1}^{s_1}(deg_G(x_i) - 2) - 8$$

■

Corollary 20. *If G is planar, $T \subset V(G)$ with $|T| \geq 3$, and $\delta(G) \geq 3$, then $odd(T) - |T| \leq \frac{n-8}{3}$.*

Proof. From Lemma 19, and since every vertex has degree at least 3, we get that

$$3(odd(T) - |T|) \leq n + 2s_1 - 2\sum_{i=1}^{s_1} 1 - 8 = n - 8$$

where s_1 is the number of singleton components. Thus, $odd(T) - |T| \leq \frac{n-8}{3}$, as required. ■

Theorem 21. *If G is planar graph with $\delta(G) \geq 3$ and $n \geq 14$, then G has a matching of size at least $\frac{n+4}{3}$.*

Proof. The case where $|T| \leq 2$ is easily covered through case analysis, see [10] for details.

The interesting case is when $|T| \geq 3$, and in this case, $odd(T) - |T| \leq \frac{n-8}{3}$ by Corollary 1.3.2. By Theorem 10 G has a matching with at most $\frac{n-8}{3}$ unmatched vertices, or a matching with at least $\frac{2n+8}{3}$ matched vertices.

When $n \geq 14$, we have that $\frac{n+4}{3} \leq \frac{n-1}{2}$. Thus, the condition that $n \geq 14$ is there because planar graphs with sufficiently small n, $\delta(G) \geq 3$ and a near-perfect matching have a matching of size $\frac{n-1}{2}$ and that is less than $\frac{n+4}{3}$. ■

Chapter 2

Matchings in 1-Planar Graphs with Known Minimum Degree

2.1 A General Method for Finding Lower Bounds in Maximal Matchings of 1-Planar Graphs

We will now prove some bounds on the matching size of 1-planar graphs as a function of their minimum degree. Note that if G is allowed parallel edges then no good bound can be proven as a function of the minimum degree of G. That is to say, for any $k \geq 1$, there is a multigraph with minimum degree k and a maximum matching of size 1. In particular, we can add an arbitrary number of multiedges to the star graph on n vertices until every edge occurs at least k times. See Figure 2.1. Because of this, we will be limiting the scope of this chapter to simple 1-planar graphs.

As always we prove matching bounds using the Tutte-Berge formula, Theorem 10, so finding a lower bound on maximal matchings is equivalent to finding an upper bound on

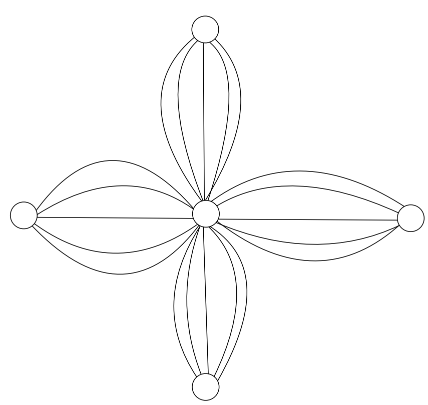


Figure 2.1: A star graph on 5 vertices that can have unbounded minimum degree.

$odd(T) - |T|$ over all possible subsets $T \subset V(G)$. Thus, for G we will fix a $T \subset V(G)$ and show an upper bound on $odd(T) - |T|$. Since T was arbitrary, this will give us a lower bound on the size of maximal matchings.

The main idea will be to give a total weight to the graph based on the number of faces, and then distribute that weight to the odd components of $G - T$. Since we can bound the number of faces in G, this will allow us to bound the number of odd components of $G - T$. For a vertex v in $G - T$, define $deg_T(v)$ to be the degree of v in the graph induced by $T \cup \{v\}$.

The main tool that we will be using for this chapter is the following lemma:

Lemma 22. *If G is a simple 1-planar graph, $T \subset V(G)$ with $|T| \geq 3$ and we assign the following weight to vertices:*

$$w(v) = \begin{cases} 7/3, & \text{if } deg_T(v) = 3,\ v \in V(G-T) \\ 3, & \text{if } deg_T(v) = 4,\ v \in V(G-T) \\ 7/2, & \text{if } deg_T(v) \geq 5,\ v \in V(G-T) \\ 0, & \text{otherwise} \end{cases}$$

then $\sum_{v \in V(G)} w(v) \leq 2n - 4$.

The proof of this lemma will take the rest of this subsection. In order to prove this lemma, we will be creating an auxiliary graph G' to assist us, but it should be noted that we are using this to prove a bound on the original graph.

We need some notation. For any set $X, Y \subset V(G)$, an X*-vertex* is a vertex that is in X. Let an X*-edge* be defined as an edge that is incident to two X-vertices. An XY*-edge* is an edge that is incident to an X-vertex and a Y-vertex. Let an X-neighbour of v be defined as any vertex adjacent to v in G that is in X.

Let $T \subset V(G), S = V(G) \setminus T$.

Define the auxiliary graph G' to be the graph G after the following operations have been performed, in order:

1. Remove all S-edges and T-edges from G.
2. Arbitrarily add a maximal set of planar edges (possibly multiedges), without creating any bigons. Note that we may re-insert many edges that were deleted in Operation 1, but no crossings.

See Figure 2.2 and Figure 2.3 for an example of a 1-plane graph G being converted to the associated graph G'. Note that G' is not unique, but we a G' produced by these operations has been fixed for G.

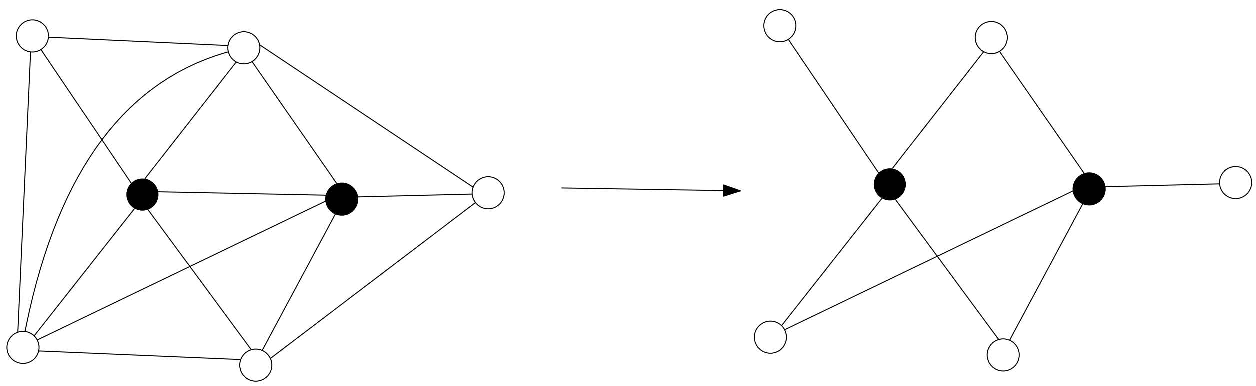


Figure 2.2: Operation 1 of the transformation from G to G'. As always, S-vertices are filled in.

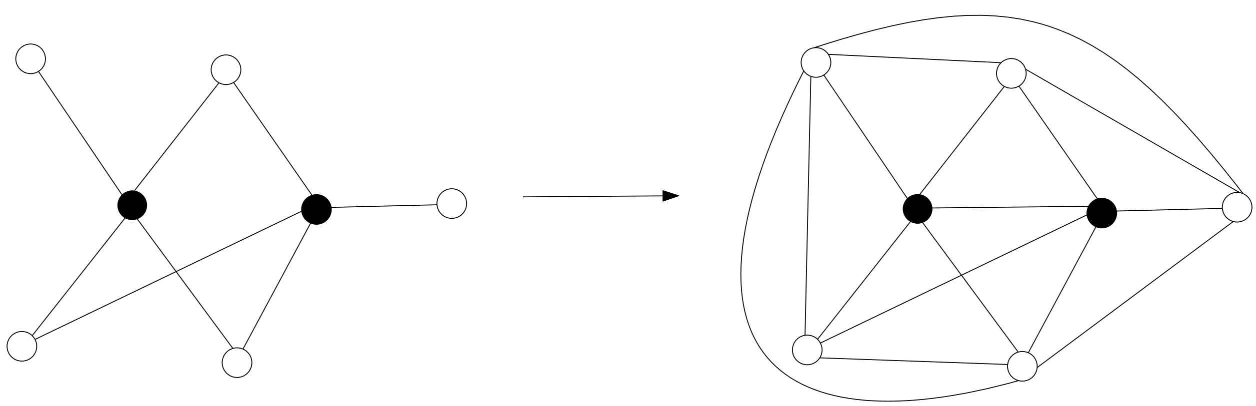


Figure 2.3: Operation 2 of the transformation from G to G'

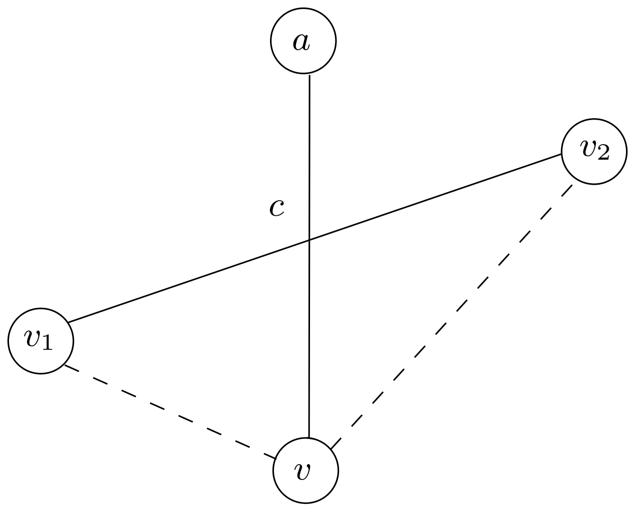


Figure 2.4: Clockwise and counterclockwise vertices around the crossing c.

2.1.1 Some Observations

We will begin by proving some simple properties of G' that will be useful later on. Since G' has a maximal set of planar edges, we have from Lemma 4 and Claim 6:

Claim 23. *Any face f of G' has exactly three corners, and every crossing has all its kite-edges.*

Claim 24. *Every crossing of G' is between two ST-edges. In consequence, every kite in G' contains two S-vertices that are adjacent to each other and two T-vertices that are adjacent to each other.*

Proof. After Operation 1 of the transformation from G to G', the only crossed edges that remain are ST-edges. No crossings are added in the transformation from G to G', so any crossing in G' must be between ST-edges. Thus, any kite in G' will contain two S-vertices and two T-vertices, and all these vertices are pairwise adjacent since all kite-edges exist. ■

Claim 25. *Let $v, a \in V(G')$, v be an S-vertex, and (v, a) be a crossed edge of G'. If the clockwise ordering of neighbours of v has the contiguous subsequence $\langle x, a, y \rangle$, then either x or y is an S-vertex.*

Proof. Let the edge (v, a) be crossed with the edge (v_1, v_2), and let c be the crossing. By Claim 24, one of v_1, v_2, say v_1, is in S. By Claim 23, the face that is incident to the path (v, c, v_1) must be the face $\{v, c, v_1\}$, and the face that is incident to the path (v, c, v_2) must be the face $\{v, c, v_2\}$, see Figure 2.4. Thus, the clockwise ordering of neighbours of v contains the contiguous subsequence $\langle v_1, a, v_2 \rangle$ or the contiguous subsequence $\langle v_2, a, v_1 \rangle$. Since v_1 or v_2 is an S-vertex, the result follows. ■

2.1.2 Assigning Weights

We will now assign weights to the faces of G'. The idea is to assign weights to faces in order to get a bound on the total weight of faces of G', and then distribute that weight to the vertices of G' (which are also the vertices of G) such that every vertex gets the weight desired in Lemma 22.

Let $w : F(G') \to \mathbb{R}$ be the weight function that assigns a weight of $1/2$ to every crossed face and a weight of 1 to every uncrossed face.

Claim 26.

$$\sum_{f \in F(G')} w(f) = 2|V(G')| - 4$$

Proof. Let X be the set of crossings. By Lemma 9, we have $2|V(G')| - 2|X| - 4$ uncrossed faces and $4|X|$ crossed faces.

Thus,

$$\begin{aligned}\sum_{f \in F(G')} w(f) &= 2|V(G')| - 2|X| - 4 + \frac{1}{2}(4|X|) \\ &= 2|V(G')| - 4\end{aligned}$$

■

Now we will distribute the weight from the faces of G' to the S-vertices of G' by the following method:

1. The weight of an uncrossed face f gets evenly distributed between the S-vertices that are corners of f. If none of the corners of f are S-vertices, then the weight of the face is lost.

2. The weight of the faces that form a kite gets evenly distributed between the S-vertices that lie on the kite.

Note that this redistributes the weight of all faces to S-vertices, since every crossed face is part of a kite by Lemma 8. Some of the weight of the faces may be lost in this process, but the total weight of the S-vertices is no larger than the total weight of the faces, i.e. $2n - 4$.

To prove Lemma 22, we will give a lower bound on the total weight that an S-vertex v can receive based on the number of adjacent T-vertices, i.e., $deg_T(v)$.

If v is adjacent to a T-vertex, t, and the next neighbour of v in clockwise order of neighbours around v after t is an S-vertex, we call this *a transition from a T-neighbour to an S-neighbour*. A transition from an S-neighbour to a T-neighbour is defined symmetrically,

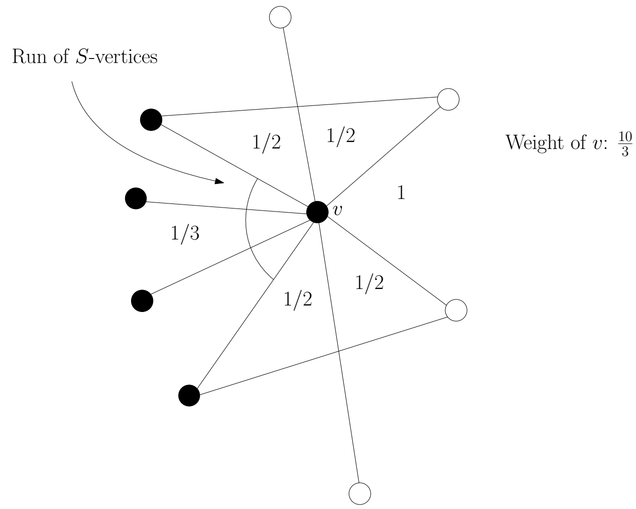


Figure 2.5: The weight of a vertex from examining the clockwise ordering of neighbours

where v is adjacent to an S-vertex and the next vertex adjacent to v in the clockwise order of neighbours is a T-vertex. If v is adjacent to a T-vertex, t, and the next vertex adjacent to v in clockwise order after t is another T-vertex, we say that these vertices are *consecutive T-neighbours* of v. *Consecutive S-neighbours* are defined symmetrically. We define a *run* of S-neighbours to be a maximal sequence of consecutive S-neighbours of v. Let $run_S(v)$ be the number of runs of S-neighbours in the clockwise ordering of vertices adjacent to v.

Claim 27. *Let v be an S-vertex of G'. Then we have the following:*

(D1) For every run of S-neighbours of length at least 2, v receives a weight of at least $1/3$.

(D2) For every two consecutive T-neighbours with uncrossed edges, v receives a weight of 1.

(D3) For every other pair of consecutive neighbours, v receives a weight of $1/2$.

See Figure 2.5 for an example of how the weight of a vertex is determined through examining the clockwise ordering of neighbours of v.

Proof. For (D1), notice that if the clockwise ordering of vertices around v contains two consecutive S-neighbours, say s_1, s_2, then the edges (v, s_1) and (v, s_2) are S-edges, and thus uncrossed by Claim 24. This means that, by Claim 23, $\{v, s_1, s_2\}$ is a face and has weight 1. Thus, the face $\{v, s_1, s_2\}$ will distribute a weight of $1/3$ to v.

For (D2), suppose that the clockwise ordering of neighbours around v contains two consecutive uncrossed edges to T-neighbours, say v_1, v_2. By Claim 23, $\{v, v_1, v_2\}$ is a face. Since $v_1, v_2 \in T$, the weight of the face will be given to v, so v receives a weight of 1.

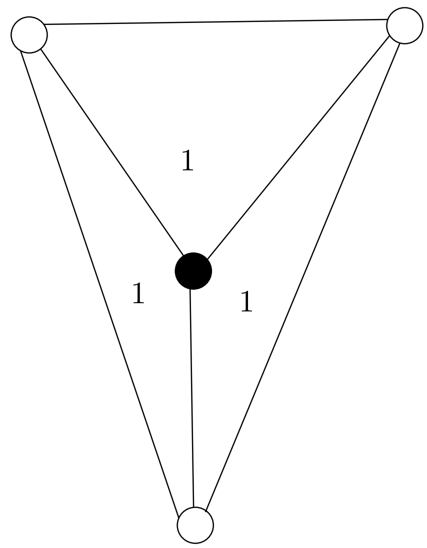


Figure 2.6: The case when $deg_T(v) = 3$, $run_S(v) = 0$

For (D3), suppose that the clockwise ordering of neighbours around v contains the consecutive neighbours v_1 and v_2. There are a few subcases to consider. If neither of the edges (v, v_1) or (v, v_2) are crossed, then by Claim 23, $\{v, v_1, v_2\}$ is a face. Since at least one of v_1 or v_2 is in T (otherwise this would be covered by D1), the face $\{v, v_1, v_2\}$ contributes a weight of at least $1/2$ to v. If the edge (v, v_1) is crossed by the edge (a, b) at c, then by Claim 24, the edges (v, a) and (v, b) are in $E(G')$. It follows that the clockwise ordering around v has the contiguous subsequence $\langle a, v_1, b\rangle$ or $\langle b, v_1, a\rangle$, i.e. $v_2 = b$ or $v_2 = a$. The kite around this crossing assigned weight 1 to v, so we can consider the crossed faces $\{v_1, a, c\}$ and $\{v_1, b, c\}$ to each contribute a weight of $1/2$ to v.

■

2.1.3 Weights of S-vertices

We will now determine the minimum weight that each S-vertex v can receive, based on $deg_T(v)$.

Claim 28. *If $v \in S$ and $deg_T(v) = 3$, then v receives a weight of at least $\frac{7}{3}$.*

Proof. Let v_1, v_2, v_3 be the T-neighbours of v, in clockwise order.

If $run_S(v) = 0$, then none of the edges incident to v are crossed, by Claim 25. It follows from Claim 23 that the faces incident to v are $\{v, v_1, v_2\}, \{v, v_1, v_3\}$, and $\{v, v_2, v_3\}$. Since v_1, v_2, v_3 are all vertices in T, each of these faces contributes a weight of 1 to v. Thus in this subcase v has a weight of 3, which is greater than the claimed $\frac{7}{3}$. See Figure 2.6.

If $run_S(v) = 1$, then let the run of S-neighbours lie clockwise between v_3 and v_1 in the embedding of G'. It follows from Claim 25 that the edge (v, v_2) is not crossed. We now break this down into subcases based on the number of crossed ST-edges incident to v, see Figure 2.7.

1. If none of the ST-edges are crossed, then the faces (v, v_1, v_2), (v, v_2, v_3), each contribute a weight of 1 to v. Furthermore, there are 2 transitions from an S-neighbour to a T-neighbour or vice versa, and each of these transitions contribute a weight of $1/2$. Thus, if none of the ST-edges incident to v are crossed, then v will have a weight of at least 3, greater than the desired $\frac{7}{3}$.

2. If exactly one of the ST-edges is crossed, say (v, v_1), then v has 2 consecutive uncrossed T-neighbours, v_2 and v_3, and that contributes a weight of 1. Furthermore, there are 2 transitions from S-neighbour to T-neighbour or vice versa, that each contribute a weight of $\frac{1}{2}$. Finally, there are the consecutive T-neighbours v_1 and v_2, and that contributes an additional weight of $\frac{1}{2}$. In total, v will have a weight of at least $\frac{5}{2}$. See Figure 2.7.

3. If, say, both (v, v_1) and (v, v_3) are crossed, then there must be at least two S-neighbours in the run of S-neighbours. We will briefly consider the case where v only has one S-neighbour for a contradiction. Let the S-vertex that v is adjacenu to be w. Note that v is incident to two kite-edges of the crossing of (v, v_1) and two kite-edges the crossing of (v, v_3). Thus, since $deg(v) = 4$, the kite-edges of the crossing (v, v_1) and the kite-edges of the crossing (v, v_3) must be the same edges; in particular they must be the edges (v, v_2) and (v, w). However, this means that the edge (w, v_2) is the edge that crosses (v, v_1) and the edge that crosses (v, v_2). See Figure 2.8. This contradicts that G was a simple graph, so v has at least 2 S-neighbours. Thus in this case there are 2 pairs of consecutive T-neighbours, and 2 transitions from S-neighbour to T-neighbour or vice versa, each contributing a weight of $\frac{1}{2}$ to v. Finally, there is at least one pair of consecutive S-neighbours, which gives v an additional weight of $\frac{1}{3}$. Thus, v will have a total weight of at least $\frac{7}{3}$ in this subcase, as required. See Figure 2.7.

If $run_S(v) = 2$, then there are 4 transitions from S-neighbour to T-neighbour or vice versa, each contributing a weight of $1/2$. Furthermore, since $deg_T(v) = 3$, there are two consecutive T-neighbours which contributes a weight of $1/2$. Thus, v will have a total weight of at least $\frac{5}{2}$, which is greater than the claimed $\frac{7}{3}$.

If $run_S(v) = 3$, then there are 6 transitions from S-neighbour to T-neighbour or vice versa, and each contributes a weight of $1/2$. Thus, v will have a total weight of at least 3, which is greater than the claimed $\frac{7}{3}$.

Since $run_S(v)$ cannot be greater than 3 as $deg_T(v) = 3$, we have covered all possible cases. Thus, if $deg_T(v) = 3$, then v has a weight of at least $\frac{7}{3}$. ∎

Claim 29. *If $v \in S$ and $deg_T(v) = 4$, then v receives a weight of at least* 3.

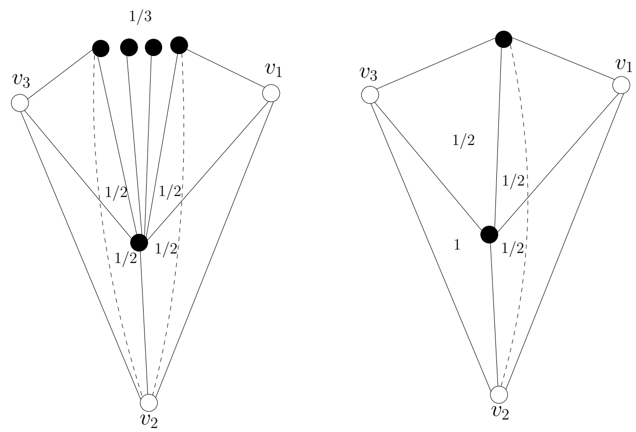


Figure 2.7: The cases when $deg_T(v) = 3$ and $run_S(v) = 1$

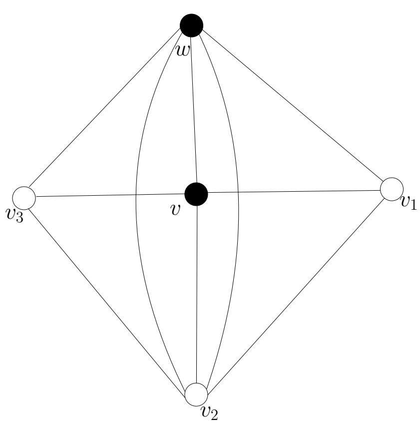


Figure 2.8: Multiedge occuring when both (v, v_3) and (v, v_1) are crossed and v has 1 S-neighbour.

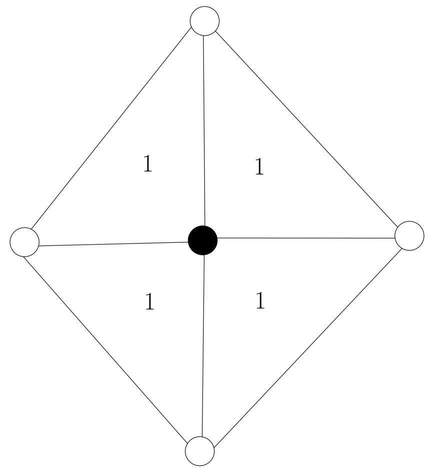


Figure 2.9: The case when $deg_T(v) = 4$, $run_S(v) = 0$

Proof. Let v_1, v_2, v_3, v_4 be the T-neighbours of v, in clockwise order.

If $run_S(v) = 0$, then none of the edges incident to v are crossed by Claim 25. It follows from Claim 23 that there are 4 unique consecutive pairs of uncrossed T-neighbours. Since each of these consecutive pairs gives a weight of 1 to v, v has a total weight of 4, as desired. See Figure 2.9.

If $run_S(v) = 1$, let the run of S-neighbours be between v_4 and v_1, in clockwise order. By Claim 25, the edges $(v, v_2), (v, v_3)$ are uncrossed. Thus there is a consecutive pair of uncrossed T-neighbours, and this contributes a weight of 1 to v. There are 2 other consecutive T-neighbour pairs (v_1, v_2) and (v_3, v_4), and these each contribute a weight of at least 1/2 to v. Finally, there are 2 transitions from S-neighbour to T-neighbour or vice versa, each which contributes a weight of 1/2 to v. In total, v will have a weight of at least 3, as desired. See Figure 2.10.

If $run_S(v) \geq 2$, then the number of S-neighbour to T-neighbour transitions, T-neighbour to S-neighbour transitions, and consecutive T-neighbours is at least 6. Since each of these transitions contributes a weight of 1/2, v will have a total weight of at least 3, as desired.

Thus, if $deg_T(v) = 4$, v will have a weight of at least 3. ∎

Claim 30. *If $deg_T(v) \geq 5$ for some $v \in S$, then v receives a weight of at least $\frac{7}{2}$.*

Proof. Let $v_1, v_2, \ldots, v_k$ be the distinct T-neighbours of v, in clockwise order, where $k \geq 5$. If $run_S(v) = 0$, then none of the edges incident to v are crossed by Claim 25. It follows from Claim 23 that there are at least 5 unique consecutive pairs of uncrossed T-neighbours. Each of these consecutive pairs gives a weight of 1 to v, v has a total weight of 5, greater than the desired $\frac{7}{2}$, so we are done in this subcase.

If $run_S(v) = 1$, let the run of S-neighbours be between v_k and v_1, in clockwise order. By Claim 25, the edges $(v, v_2), (v, v_3), (v, v_4)$ are all uncrossed. Two consecutive uncrossed T-neighbours each produce a weight of 1, so the transitions (v_2, v_3) and (v_3, v_4) each contribute

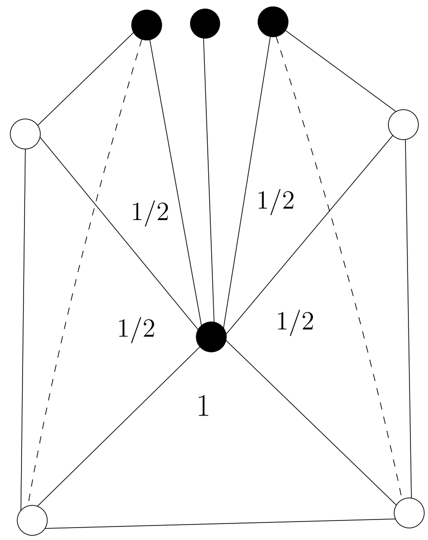


Figure 2.10: The case when $deg_T(v) = 4$, $run_S(v) = 1$

a weight of 1. The consecutive T-neighbours $(v_{k-1}, v_k), (v_1, v_2)$ each contribute a weight of at least 1/2. Thus the weight from consecutive T-neighbours gives v a total weight of at least 3. Finally, there are 2 transitions from S-neighbour to T-neighbour or vice versa, each giving v an additional weight of 1/2. Thus in this case the total weight v will have is at least 4, which is greater than the desired $\frac{7}{2}$, so we are done in this subcase.

If $run_S(v) \geq 2$, then the number of transitions from T-neighbour to S-neighbour, transitions from S-neighbour to T-neighbour, and consecutive T-neighbours is at least 7. Each of these transitions/consecutive T-neighbours contribute a weight of at least 1/2 to v, so the total weight will be at least 7/2, as desired.

Thus, regardless of the number of runs of S-neighbours v has, v will receive a weight of at least 7/2 if v has at least 5 T-neighbours. ∎

2.1.4 Putting it All Together

Now we finally have all the tools necessary to prove Lemma 22 (from Page 20).

Proof. Let G be a simple 1-planar graph, $T \subset V(G)$, and construct G' from G. If we assign a weight of 1 to each uncrossed face of G' and a weight of 1/2 to each crossed face, i.e. using the weight function w', then we can distribute the weight of the faces to each vertex $v \in S$ such that:

1. If $deg_T(v) = 3$, then $w'(v) \geq 7/3$ (by Claim 28)
2. If $deg_T(v) = 4$, then $w'(v) \geq 3$ (by Claim 29)

3. If $deg_T(v) \geq 5$, then $w'(v) \geq 7/2$ (By Claim 30)

Furthermore, from Claim 26, $\sum_{f \in F(G')} w'(f) = 2n - 4$. Let w be the weight function as described in the condition of Lemma 22. Consider the vertex v in G and the corresponding vertex v' when G is transformed into G'. Notice that since no ST-edges were removed, the number of T-neighbours of v' is no less than the number of T-neighbours of v, therefore $w(v) \leq w'(v')$.

Thus,

$$\begin{aligned}
\sum_{v \in V(G)} w(v) &\leq \sum_{v \in V(G')} w'(v) \\
&\leq \sum_{f \in F(G')} w'(f) \\
&\leq 2|V(G')| - 4 \\
&= 2|V(G)| - 4
\end{aligned}$$

as required. ■

At this point, we have all the tools necessary to proceed and prove upper bounds for $odd(T) - |T|$ for 1-planar graphs with specified minimum degree.

2.2 Matchings in 1-planar Graphs with Known Minimum Degree

Using Lemma 22, we will now prove tight bounds on matchings in 1-planar graphs with minimum degree 3, 4, and 5. We will also show the results we have discovered for 1-planar graphs of minimum degree 6 and 7, and why our techniques do not work in those cases.

Note that the cases where a 1-planar graph has minimum degree 1 or 2 have no non-trivial bounds. The star graph on n vertices is a 1-planar graph with a maximum matching of size 1 and minimum degree 1. The complete bipartite graph $K_{2,n}$ has minimum degree 2 and a maximum matching of size 2, see Figure 2.11.

2.2.1 Matchings in 1-planar Graphs with Minimum Degree 3

Let G be a simple 1-planar graph. We again note that if G is not simple, then no non-trivial bound can be obtained. We fix a $T \subset V(G)$ for the remainder of this section.

Let $\mathcal{S}_i$ denote the set of components of size i in $G - T$ and let $s_i = |\mathcal{S}_i|$. Let S_i denote the set of vertices in components of size i in $G - T$.

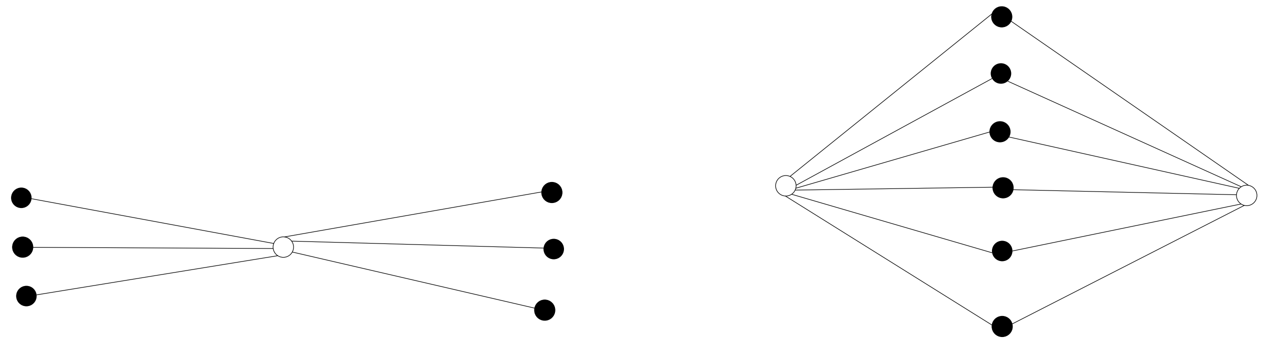


Figure 2.11: 1-planar graphs with minimum degree 1 and 2. These graphs have maximum matchings of size 1 and 2, respectively.

Lemma 31. *Let G be a 1-planar graph with $\delta(G) \geq 3$. Then, for any $T \subset V(G)$ with $|T| \geq 3$, we have $s_1 \leq 6|T| - 12$.*

Proof. Let G' be the graph G after removing all components of $G - T$ of size greater than 1. Observe that every component in $G' - T$ is a singleton, so $V(G') = T \cup S_1$. Also, if $v \in V(G' - T)$, then $deg_T(v) \geq 3$, since $\delta(G) \geq 3$ and all of the vertices adjacent to v in G are vertices in T and also exist in G'.

If we assign the weights from Lemma 22 to the vertices of G', then every vertex v in S gets weight at least $\frac{7}{3}$, since $deg_T(v) \geq 3$. It follows that $\frac{7}{3}|S_1| + 0|T|$ is at most $2(|S_1| + |T|) - 4$.

Therefore, by Lemma 22 applied to G',

$$\begin{aligned}
\frac{7}{3}|S_1| + 0|T| &\leq 2(|S_1| + |T|) - 4 \\
\Leftrightarrow \frac{1}{3}|S_1| &\leq 2|T| - 4 \\
\Leftrightarrow |S_1| &\leq 6|T| - 12 \\
\Leftrightarrow s_1 &\leq 6|T| - 12
\end{aligned}$$

■

Theorem 32. *If G is a 1-plane graph with $|V(G)| \geq 7$, and $\delta(G) \geq 3$, then G has a matching of size at least $\frac{n+12}{7}$.*

Proof. Recall that there exists a matching M in G such that the number of unmatched vertices is $odd(T) - |T|$, for some $T \subset V(G)$, by the Tutte-Berge formula, Theorem 10. We will now bound $odd(T) - |T|$ in two cases; one case where $|T| \geq 3$, and one where $|T| < 3$.

Case 1: $|T| \geq 3$. Then we have

$$\begin{aligned}
odd(T) - |T| &= s_1 + (\sum_{\substack{i \geq 3 \\ i \text{ odd}}} s_i) - |T| && \text{(by definition of } s_i\text{)} \\
&\leq \frac{5}{7}(n - (\sum_{i \geq 1} i s_i) - |T|) + s_1 + (\sum_{\substack{i \geq 3 \\ i \text{ odd}}} s_i) - |T| && \text{(by Observation 1)} \\
&\leq \frac{5n}{7} + \frac{2}{7}s_1 - \frac{12}{7}|T| && \text{(since } \frac{5}{7}\sum_{i \geq 2} i s_i \geq \sum_{\substack{i \geq 3 \\ i \text{ odd}}} s_i\text{)} \\
&\leq \frac{5n}{7} + \frac{12}{7}|T| - \frac{24}{7} - \frac{12}{7}|T| && \text{(by Lemma 31)} \\
&\leq \frac{5n - 24}{7}
\end{aligned}$$

Case 2: $1 \leq |T| \leq 2$
In this case, we know that $s_1 = 0$, since S_1 is an independent set and every vertex in S_1 has degree at least 3. It follows that every odd component in $G - T$ has size at least 3. Thus, $odd(T) - |T| \leq \frac{n-|T|}{3} - |T| \leq \frac{n-4|T|}{3} \leq \frac{n-4}{3} \leq \frac{5n-24}{7}$, since $n \geq 6$.

Case 3: $|T| = 0$
In this case, we know that $s_1 = 0$ as above. Furthermore, $s_3 = 0$ since every vertex in S_3 is adjacent to at most 2 S-vertices, but G has a minimum degree of 3. Thus, we have $odd(T) - |T| \leq \frac{n-|T|}{5} - |T| = \frac{n}{5} \leq \frac{5n-24}{7}$ since $n \geq 7$.

In all cases, we have $odd(T) - |T| \leq \frac{5n-24}{7}$, and thus the number of matched vertices is at least $\frac{2n+24}{7}$, so the maximum matching of G has size at least $\frac{n+12}{7}$, as required. ∎

Theorem 33. *There exists an infinite family of 1-planar graphs with $\delta(G) = 3$ and a maximum matching of size $\frac{n+12}{7}$.*

Proof. Consider a triangulated graph G on k vertices. In each face of G add three vertices of degree three with edges to the corners of the face that they are placed in, constructing a $K_{3,3}$. See Figure 2.12 for an example and observe that this can be done while maintaining 1-planarity.

The number of vertices in this new graph, H, is $n = k + (2k - 4)3 = 7k - 12$, since there are $2k - 4$ faces in G and each face adds 3 more vertices. Notice that $\delta(H) = 3$: every vertex in G has degree at least 3 since G is triangulated. Every vertex that was not originally in G was placed in a face of degree 3 and had edges added to each of the face's corners, and therefore have degree 3.

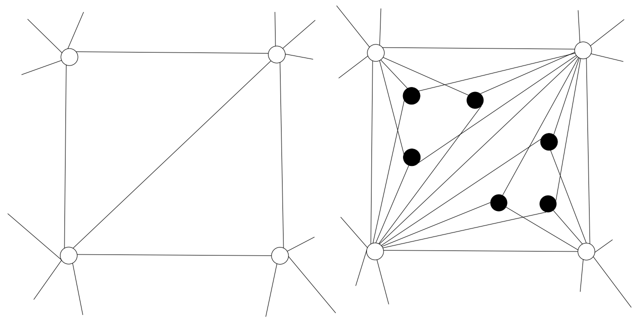


Figure 2.12: A 1-planar graph with minimum degree 3 where any matching has size at most $\frac{n+12}{7}$

Let $T = V(G)$. Notice that the $6k-12$ vertices that were added only had neighbours in $V(G)$, and are singleton components in $H - V(G)$. Thus, $odd(T) = 6k - 12$, which means:

$$\begin{aligned} odd(T) - |T| &= 6k - 12 - k \\ &= 5k - 12 \\ &= 5((n+12)/7) - 12 \\ &= \frac{5n-24}{7} \end{aligned}$$

This means that the number of unmatched vertices in a maximum matching of H is at least $\frac{5n-24}{7}$, by Tutte's theorem. This means that the number of matched vertices is at most $n - \frac{5n-24}{7} = \frac{2n+24}{7}$. Thus, the size of the maximum matching in H is at most $\frac{n+12}{7}$, matching the bound of Theorem 2.2.1. ∎

2.2.2 Matchings in 1-planar Graphs with Minimum Degree 4

This proof will be using the same technique that was used in the section for minimum degree 3.

Lemma 34. *Let G be a 1-planar graph with $\delta(G) \geq 4$. For any $T \subset V(G)$ with $|T| \geq 3$, $s_1 \leq 2|T| - 4$.*

Proof. Let G' be the graph G after removing all components of $G - T$ of size greater than 1. Observe that every component in $G' - T$ is a singleton, say a vertex v, and $deg_T(v) \geq 4$ since $\delta(G) \geq 4$ and all of the vertices adjacent to the singleton are vertices in T. If we assign weights to G from Lemma 22, then every vertex in $G' - T$ gets a weight of at least 3 since $\delta(G) \geq 4$, so $3|V(G' - T)| \leq 2|V(G')| - 4$.

Thus, since $V(G'-T)=S_1$, it follows that $|V(G')|=|V(G'-T)|+|T|=s_1+|T|$. Therefore,

$$\begin{aligned}3s_1 &\le 2s_1+2|T|-4\\ s_1 &\le 2|T|-4\end{aligned}$$

as required. ∎

Theorem 35. *If G is a 1-plane graph with $|V(G)|\ge 20$ and $\delta(G)\ge 4$, then G has a matching of size at least $\frac{n+4}{3}$.*

Proof. Recall that there exists a matching M in G such that the number of unmatched vertices is $odd(T)-|T|$, for some $T\subset V(G)$, by Tutte's Matching Theorem. We will now bound $odd(T)-|T|$ depending on $|T|$.
Case 1: $|T|\ge 3$.
Then we have

$$\begin{aligned}
odd(T)-|T| &\le s_1+(\sum_{\substack{i\ge 3\\ i \text{ odd}}} s_i)-|T| && \text{(by definition of } s_i)\\
&\le \frac{1}{3}(n-(\sum_{i\ge 1} is_i)-|T|)+s_1+(\sum_{\substack{i\ge 3\\ i \text{ odd}}} s_i)-|T| && \text{(by Observation 1)}\\
&\le \frac{n}{3}+\frac{2}{3}s_1-\frac{4}{3}|T| && \text{(since } \frac{1}{3}\sum_{i\ge 2} is_i \ge \sum_{\substack{i\ge 3\\ i \text{ odd}}} s_i)\\
&\le \frac{n}{3}+\frac{2}{3}(2|T|-4)-\frac{4}{3}|T| && \text{(by Lemma 34)}\\
&\le \frac{n-8}{3}
\end{aligned}$$

Case 2: $|T|=2$.

In this case, we know that $s_1=0$, since S_1 is an independent set and every vertex in S_1 has degree at least 4. It follows that every odd component in $G-T$ has size at least 3. Thus, $odd(T)-|T|\le \frac{n-|T|}{3}-|T|\le \frac{n-4|T|}{3}\le \frac{n-8}{3}$.
Case 3: $|T|\le 1$

In this case, $s_1=0$ since S_1 is an independent set and every vertex in S_1 has degree at least 4. Also, $s_3=0$ since every vertex in S_3 has at most 2 neighbours in S and at most one neighbour in T but should have degree at least 4. Thus, every odd component in $G-T$ has size at least 5. Thus, $odd(T)-|T|\le \frac{n-|T|}{5}-|T|\le \frac{n}{5}\le \frac{n-8}{3}$, since $n\ge 20$.

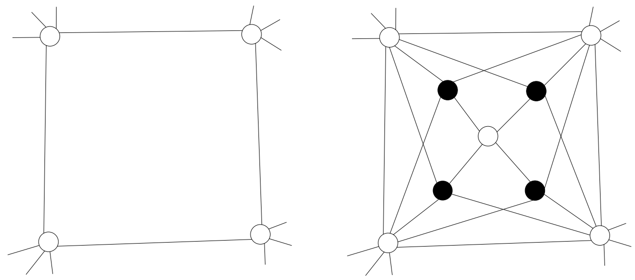


Figure 2.13: 1-planar graph with minimum degree 4 and small maximum matching

So in all cases we have $odd(T) - |T| \leq \frac{n-8}{3}$. Since the number of unmatched vertices in M is $odd(T) - |T|$, the number of matched vertices is at least $n - (odd(T) - |T|)$. Thus, there are at least $n - \frac{n-8}{3} = \frac{2n+8}{3}$ vertices in M. It follows that the size of M is at least $\frac{n+4}{3}$, as required.

We now show that this bound is tight. ■

Theorem 36. *There exists an infinite family of 1-planar graphs with $\delta(G) = 4$ and a maximum matching of size $\frac{n+4}{3}$.*

Proof. Take a 2-connected, planar graph G on $k \geq 4$ vertices where every face has degree 4. In each face of G, insert four vertices that are singleton components and are adjacent to three of the vertices incident to f. Then add one more vertex in the face that will be adjacent to the four inserted vertices. See Figure 2.13 for an illustration of how this can be done while maintaining 1-planarity.

G has k vertices, $k-2$ faces, and in every face 5 vertices were added. Thus, the constructed graph has $6k - 10$ vertices. We can select T to be the vertices of G and the vertex in each face that connected to the other 4 vertices. Since every other vertex is an odd component of size one in $G - T$, we get that $|T| = 2k - 2$ and $odd(T) = 4k - 8$. Thus, $odd(T) - |T| = 2k - 6 = \frac{(6k-10)-8}{3} = \frac{n-8}{3}$. By Theorem 10, a maximum matching M of G has at least $\frac{n-8}{3}$ unmatched vertices, so M has at most $\frac{2n+8}{3}$ matched vertices, or $|M| \leq \frac{n+4}{3}$. By Theorem 35 we have that $|M| \geq \frac{n+4}{3}$, so it follow that $|M| = \frac{n+4}{3}$, as required. ■

2.2.3 Matchings in 1-planar Graphs with Minimum Degree 5

This proof will again be using the same proof techniques that were used in the Section 2.2.1 and Section 2.2.2, with minor modifications since components of size 3 are also necessary to consider in this case.

Lemma 37. *Let G be a 1-planar graph with $\delta(G) \geq 5$. For any $T \subset V(G)$ with $|T| \geq 3$, $\frac{3}{2}s_1 + s_3 \leq 2|T| - 4$.*

Proof. Let G' be the graph G after removing all even components of $G - T$, all components of $G - T$ of size greater than 3, and all edges that are not ST-edges.

Observe that every vertex v in S_1 has $deg_T(v) \geq 5$ in G', since $\delta(G) \geq 5$ and edges incident to a singleton component are ST-edges. Also observe that every vertex v in S_3 has $deg_T(v) \geq 3$ in G'. This follows since $\delta(G) \geq 5$, and the only edges incident to a vertex in S_3 are ST-edges and edges to the other S-vertices in the same component. Since the component has size 3, at most 2 incident edges are removed when constructing G'.

As with the previous sections, we will assign weights using the weight function w from Lemma 22 and be using the result that $\sum_{v \in S} w(v) \leq 2|V(G')| - 4$.

Since every vertex in S_1 has degree at least 5 in G', and every vertex in S_3 has degree at least 3 in G', we assigned a weight of at least $\frac{7}{2}$ to every vertex in S_1 and a weight of at least $\frac{7}{3}$ to every vertex in S_3. Therefore,

$$\frac{7}{2}s_1 + \frac{7}{3}|S_3| \leq 2|V(G')| - 4$$

Recall that $s_3 = |\mathcal{S}_3|$, so there are $3s_3$ vertices in odd components of size 3. Thus, we have

$$\frac{7}{2}s_1 + 7s_3 \leq 2|V(G')| - 4$$

Since $|V(G')| = s_1 + 3s_3 + |T|$, we get:

$$\begin{aligned}\frac{7}{2}s_1 + 7s_3 &\leq 2(s_1 + 3s_3 + |T|) - 4\\ \frac{3}{2}s_1 + s_3 &\leq 2|T| - 4\end{aligned}$$

as required. ∎

Theorem 38. *If G is a 1-plane graph with $\delta(G) \geq 5$ and $|V(G)| \geq 21$, then G has a matching of size at least $\frac{2n+3}{5}$.*

Proof. Recall that there exists a matching M in G such that the number of unmatched vertices is $odd(T) - |T|$, for some $T \subset V(G)$, by Theorem 10. We will now bound $odd(T) - |T|$ in cases; one case where $|T| \geq 3$, and two where $|T| < 3$.

Case 1: $|T| \geq 3$

$$
\begin{aligned}
odd(T) - |T| &\le (s_1 + \sum_{\substack{i\ge 3\\ i \text{ odd}}} s_i) - |T| && \text{(by definition of } s_i\text{)}\\
&\le \frac{1}{7}(n - (\sum_{i\ge 1} is_i) - |T|) + s_1 + s_3 + s_5 + (\sum_{\substack{i\ge 7\\ i \text{ odd}}} s_i) - |T| && \text{(by Observation 1)}\\
&\le \frac{n}{7} + \frac{6}{7}s_1 + \frac{4}{7}s_3 + \frac{2}{7}s_5 - \frac{8}{7}|T| && \text{(since } \frac{1}{7}\sum_{i\ge 4} is_i \ge \sum_{\substack{i\ge 7\\ i \text{ odd}}} s_i\text{)}\\
&\le \frac{n}{7} + \frac{6}{7}s_1 + \frac{4}{7}s_3 + \frac{2n}{35} - \frac{8}{7}|T| && \text{(since } s_5 \le \frac{n}{5}\text{)}\\
&\le \frac{n}{5} + \frac{4}{7}(2|T| - 4) - \frac{8}{7}|T| && \text{(by Lemma 37)}\\
&\le \frac{n}{5} - \frac{16}{7}\\
&< \frac{n-6}{5}
\end{aligned}
$$

Case 2: $1 \le |T| \le 2$
In this case, we know that $s_1 = 0$ and $s_3 = 0$: $s_1 = 0$ since S_1 is an independent set and every vertex in S_1 has degree at least 5, and $s_3 = 0$ since every vertex in S_3 has at most 2 neighbours in S and at most 2 neighbours in T, but should have degree at least 5. It follows that every odd component in $G - T$ has size at least 5. Thus, $odd(T) - |T| \le \frac{n-|T|}{5} - |T| \le \frac{n-6|T|}{5} \le \frac{n-6}{5}$

Case 3: $|T| = 0$
Every vertex in S_i is adjacent to at most $i - 1$ S-vertices. Furthermore, every vertex in G is an S-vertex since $|T| = 0$. It follows that s_1, s_3, s_5 are all equal to 0. Thus, any odd component in $G - T$ must have size at least 7. Thus, we have $odd(T) - |T| \le \frac{n}{7} \le \frac{n-6}{5}$ because $n \ge 21$.

So in all cases we have $odd(T) - |T| \le \frac{n-6}{5}$. Since the number of unmatched vertices in M is $odd(T) - |T| \le \frac{n-6}{5}$, the number of matched vertices in M is $n - (odd(T) - |T|) \ge n - \frac{n-6}{5} = \frac{4n+6}{5}$ and therefore the size of M is at least $\frac{2n+3}{5}$, as required. ■

We now show that this bound is tight.

Theorem 39. *There exists an infinite family of 1-planar graphs with $\delta(G) = 5$ and a maximum matching of size $\frac{2n+3}{5}$.*

Proof. Consider a 1-planar embedding of K_6, see Figure 2.14. Note that there is a vertex that is a corner of the outer face. Now consider k such K_6 embeddings, where one corner on

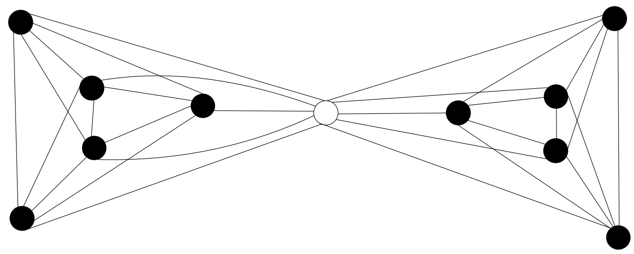


Figure 2.14: 1-Planar graph with minimum degree 5 and small maximum matching

the outside face of every embedding is combined as a single vertex. That combined vertex is the only vertex in T.
This graph has $n = 5k + 1$ vertices, satisfies the condition that $\delta(G) = 5$, and is 1-planar. Observe that with this selection of T, $odd(T) - |T| = \frac{n-1}{5} - 1 = \frac{n-6}{5}$. By Theorem 10, a maximum matching M of G has at least $\frac{n-6}{5}$ unmatched vertices, so M has at most $\frac{4n+6}{5}$ matched vertices, or $|M| \leq \frac{2n+3}{5}$. By Theorem 38 we have that $|M| \geq \frac{2n+3}{5}$, so it follow that $|M| = \frac{2n+3}{5}$, as required. ■

2.2.4 Open Problem in Graphs of Higher Minimum Degree

In this section we will show matching results for 1-planar graphs with minimum degree 6 and 7, and the issues with applying the techniques of the previous sections to these graphs. It should be noted that no simple 1-planar graphs with minimum degree 8 or higher exist since such graphs would have at least $4n$ edges whereas simple 1-planar graphs have at most $4n - 8$ edges.

Matchings in 1-Planar Graphs With Minimum Degree 6

Observation 40. *There is a family of 1-planar graphs G with $\delta(G) = 6$ with a maximum matching of size $\frac{3n+4}{7}$.*

Proof. Consider the 1-planar graph with minimum degree 6 on 8 vertices in Figure 2.15, obtained by inserting a kite into each of the six faces of a cube. We can make an arbitrary number of copies of this graph and identify a vertex v on the outer face in all the graphs as a single vertex. Let G be the resulting graph and $T = \{v\}$. Thus, $G - T$ is a collection of odd components of size 7. It follows that $odd(T) - |T| \geq \frac{n-1}{7} - 1 = \frac{n-8}{7}$. Thus, by Theorem 10, any graph constructed in this manner will have a maximum matching of size at most $\frac{n-(odd(T)-|T|)}{2} = \frac{3n+4}{7}$, as required. One can easily find a matching of this size in G, by taking three edges in each component and for one component adding an edge to v. ■

This naturally leads to the following:

Conjecture 41. *Every 1-planar graph with $\delta(G) \geq 6$ has a matching of size $\frac{3n+4}{7}$.*

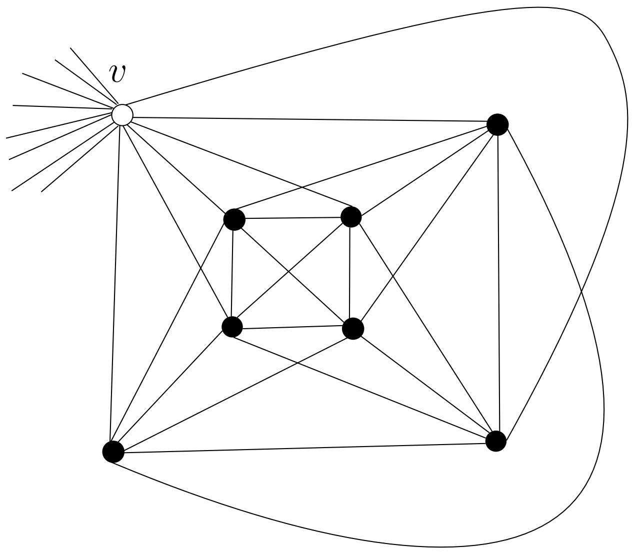


Figure 2.15: A 1-planar graph with minimum degree 6 over 8 vertices. The building block for 1-planar graphs with minimum degree 6 and small maximum matching.

It may be possible to prove Conjecture 41 with similar techniques as we applied earlier. In particular, we could prove this by making minor modifications to Lemma 22 to handle $deg_T(v) = 2$ and if we could argue that $\frac{3}{2}s_1 + s_3 + \frac{s_5}{2} \leq 2|T| - 4$. This in turn would hold if we could argue that every component in $\mathcal{S}_5$ contributes a weight of at least $\frac{21}{2}$. This seems difficult to prove. In particular, if v is a vertex in S_5, then v could have 4 neighbours in the component, which means that possibly $deg_T(v) = 2$. We cannot give weight greater than 2 to a vertex with $deg_T(v) = 2$ and have Lemma 22 continue to hold. So with this approach any component of size 5 contributes weight of only 10.

However, we hypothesize that not all the vertices that belong to a component of size 5 in $G - T$ can actually have $deg_T(v) = 2$, that some must be larger due to how odd components of size 5 are embedded. As such, analyzing the individual vertices of each component seems to fall short for this case, and a different approach would likely be necessary.

Matchings in 1-Planar Graphs With Minimum Degree 7

Observation 42. *There is a family of 1-planar graphs G with $\delta(G) = 7$ with a maximum matching of size $\frac{15n+16}{31}$.*

Proof. We will use the same technique as for Observation 40. That is, we will construct a 1-planar graph with minimum degree 7 over 32 vertices, see Figure 2.16. With this graph we can use an arbitrary number of copies of this graph with one vertex between them all being identified as a single vertex v. This will give us a family of graphs where $T = \{v\}$ and $odd(T) - |T| = \frac{n-1}{31} - 1 = \frac{n-32}{31}$. Thus, any graph constructed by this process will have a maximum matching of size at most $\frac{n-(odd(T)-|T|)}{2} = \frac{15n+16}{31}$. One can easily find a matching

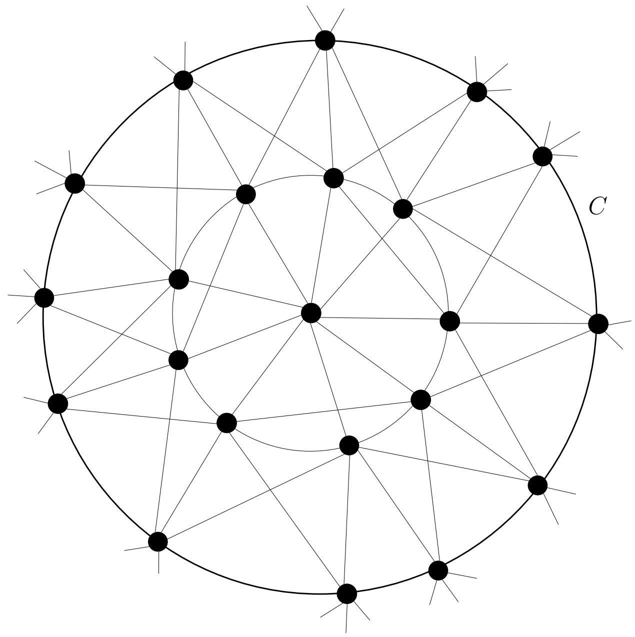


Figure 2.16: A 1-planar graph with minimum degree 7 over 32 vertices. We only show half of the graph inside the cycle C (thick edges); the other half is symmetric after rotating along C.

of this size by taking 15 edges in each component and for one component adding an edge to v in each component and for one component adding an edge to v. ■

We do not even have a good conjecture on what the size of the maximum matching might be for 1-planar graphs of minimum degree 7, but we suspect that the bound in Observation 42 is not tight. To prove a bound with our techniques, we would have to modify Lemma 22 to assign even more weight if $deg_T(v) = 6$ and $deg_T(v) = 7$; again it is not clear what these weights should be. We leave this for future study.

Chapter 3

Matchings in Maximal 1-Planar Graphs

In Chapter 2 we excluded multigraphs, as we showed that no non-trivial matching bound of a 1-planar graph G can be found with respect to the minimum degree $\delta(G)$ if multiedges are permitted. In particular, the graph G shown in Figure 2.1 can be created; G has arbitrarily high $\delta(G)$ and a maximum matching of size 1. Now that we are changing our focus to maximal 1-planar graphs, we will allow graphs to contain multiedges, while maintaining the assumptions of graph drawings from Chapter 1.

There are several graph classes that were defined in the introduction but will be repeated briefly as a refresher. Recall that a *maximal 1-plane graph* is defined as a 1-planar graph G with a fixed 1-planar drawing without bigon such that no edge can be added to G without creating a bigon or having an edge that is crossed twice. A *maximal 1-planar graph* is defined as a 1-planar graph G that has a 1-planar drawing that has no bigon, and no edge can be added to G without creating a bigon or having an edge that is crossed twice, regardless of 1-planar drawing. A *maximal, simple 1-plane graph* is a simple 1-planar graph with a fixed 1-planar drawing that does not have a bigon such that no edge can be added to G without it being a multiedge, or without having an edge that is crossed twice. A *maximal, simple 1-planar graph* is symmetrically defined; regardless of the 1-planar drawing the graph must be a maximal, simple 1-plane graph.

We note that the bounds on maximum matchings in maximal 1-planar graphs and maximal 1-plane graphs could in theory be very different, since a poor drawing of a 1-planar graph G could cause G to become maximal 1-plane, even though G may not be maximal 1-planar. See Figure 3.1. Since more edges would have to be added to G in order to make it maximal 1-planar, there is reason to suppose that maximal 1-plane graphs may have smaller maximum matchings than maximal 1-planar graphs.

In this chapter we will investigate maximum matchings in maximal, simple 1-planar graphs and maximal 1-plane graphs. In particular, we will determine how small they can be

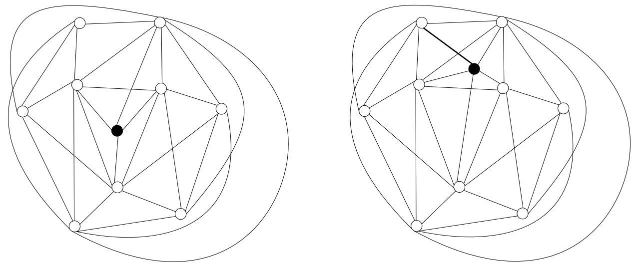


Figure 3.1: A maximal 1-plane graph (left), that is not maximal 1-planar. The figure on the right shows how the drawing can be changed to allow additional edges (added edge is bold, moved vertices are filled in).

and the differences between the matching bounds of these graph classes. Further, many of the bounds we will present are tight.

3.1 Properties of Maximal 1-Plane Graphs

We review a few properties of maximal 1-plane graphs. Clearly by Lemma 4 all faces have degree 3.

Claim 43. *Any 1-plane graph where all faces have degree 3 is 2-connected.*

Proof. Let G be a 1-plane graph where all faces have degree 3 and let G^P be the planarization of G. Since every face of G has degree 3, it follows that G^P is a triangulated graph, and thus G^P is 2-connected.

Suppose by way of contradiction that G is not 2-connected: there exists a vertex $v \in V(G)$ such that $G - v$ is not connected. In particular, suppose that the vertex $a \in V(G)$ and the vertex $b \in V(G)$ are in separate components of $G - v$. Since G^P was 2-connected, $G^P - v$ is still connected, thus there exists paths from a to b in $G^P - v$. Let π be a path from a to b in $G^P - v$ that uses the smallest number of dummy vertices. We know that π must contain at least 1 dummy vertex, or π would be a path from a to b in $G - v$, contradicting that a and b are in separate components of $G - v$. Furthermore, we know that π cannot contain two consecutive dummy vertices since that would mean that G contains an edge that is crossed twice.

Let the subpath (v_1, c, v_2) be in π, where c is a dummy vertex, and v_1, v_2 are vertices in $V(G - v)$. This means that v_1 and v_2 are vertices that are incident to the kite formed by c. Thus, we can construct a path from a to b in $G - v$ that uses fewer dummy vertices than in π, by replacing the subpath (v_1, c, v_2) with the kite-edge (v_1, v_2) of c. This is a contradiction since π was a path in $G - v$ that had the smallest number of dummy vertices. ■

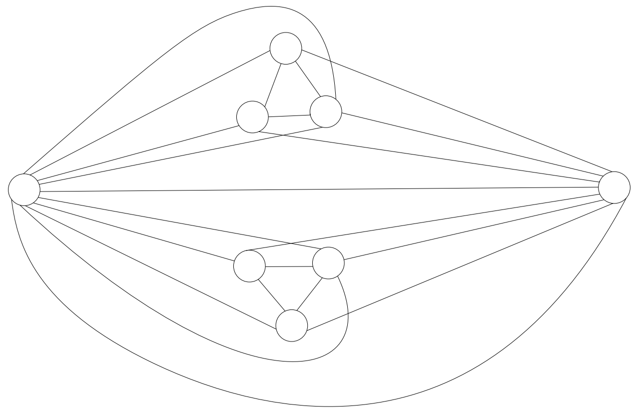


Figure 3.2: A 1-plane graph where all faces have degree 3 but G is not 3-connected.

We remark here that Claim 43 would be false for “3-connected”, see for example the graph in Figure 3.2. 3-connectivity would hold for simple 1-plane graphs, but we do not need this and hence do not prove it here.

3.2 An Auxiliary Graph

As always, we prove our matching bound based on the Tutte-Berge Formula, Theorem 10. Given a subset $T \subset V(G)$, we will bound the number of odd components that can exist in $G - T$. To do this, we will use a method similar to the paper reviewed in Section 1.3.1. In Section 1.3.1, it was essential that every component of $G-T$ was contained in a face of $G[T]$ (see Lemma 11). This is true for planar graphs, but for 1-planar graphs it is possible that a component of $G - T$ spans several faces of $G[T]$, see Figure 3.3.

To resolve this issue, we define a new graph H, that will be useful for several upcoming sections.

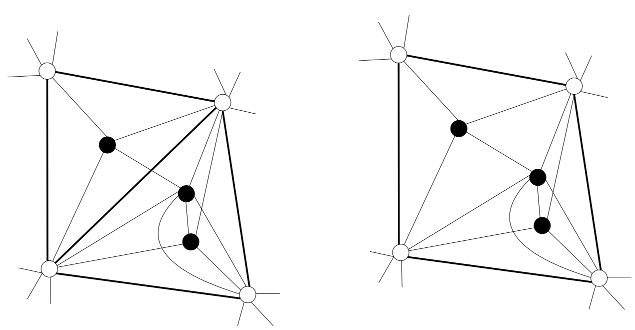


Figure 3.3: A portion of a maximal 1-plane graph G, where a component of $G-T$ exists in two distinct faces of $G[T]$. Edges of $G[T]$ are thickened, and as always, vertices in $S = V(G) - T$ are filled in. One possible corresponding graph H is on the right.

Recall that a T*-vertex* is a vertex in T, a T*-edge* is an edge with two incident T-vertices. S*-vertex* and S*-edge* are defined symmetrically, where $S = V(G) - T$. An ST*-edge* is an edge with one incident T-vertex and one incident S-vertex.

Definition 44. *Given a 1-plane graph G and a set $T \subset V(G)$, define the process to derive a 1-plane graph H as follows:*

1. *For every T-edge e that crosses an ST-edge or S-edge, remove e.*
2. *For every T-edge e_1 that crosses another T-edge e_2, arbitrarily remove one of e_1, e_2.*
3. *The 1-plane drawing of H is inherited from the 1-plane drawing of G.*

Note that H is not unique, and any H that is created by this process will suffice. We will assume that a corresponding H is fixed for any given G.

We note several properties of the graph H that will be useful.

Claim 45. *If G is a 1-plane graph where every face has degree 3, and H is the graph from Definition 44, then H has the following properties:*

1. *Every T-edge of H is a planar edge.*
2. *Every component of $H - T$ is contained in a face of $H[T]$.*
3. *Every face of H has degree 3.*
4. $odd_G(T) = odd_H(T)$
5. *H is 2-connected.*

Proof. For (1), we note that any crossing that was incident to a T-edge in G was removed, so there are no crossings in H that are incident to a T-edge. It follows that every T-edge in H is a planar edge.

For (2), assume that there is a component, C, of $H - T$ that exists in two distinct faces of $H[T]$, say f_1 and f_2. Then there must be an edge, e in C that crosses a closed walk formed by the boundary of f_1. Thus, e crossed an edge in $H[T]$, contradicting that every T-edge of H is a planar edge.

For (3), every uncrossed face in G remains in H. If a crossing c exists between the edges (a_1, a_2) and (b_1, b_2) in G, then the faces $\{a_1, a_2, c\}, \{a_1, b_1, c\}, \{a_2, b_1, c\}, \{a_2, b_2, c\}$ all exist in G because every face of G has degree 3. Thus if the crossed edge, say (a_1, a_2), was removed then the resulting faces are $\{a_1, b_1, b_2\}$ and $\{a_2, b_1, b_2\}$. It follows that the crossed faces in G that are no longer in H merge together to form faces of degree 3 in H.

For (4), only T-edges are removed, thus $G - T = H - T$. It follows that $odd_G(T) = odd_H(T)$.

(5) follows immediately from Claim 43. ■

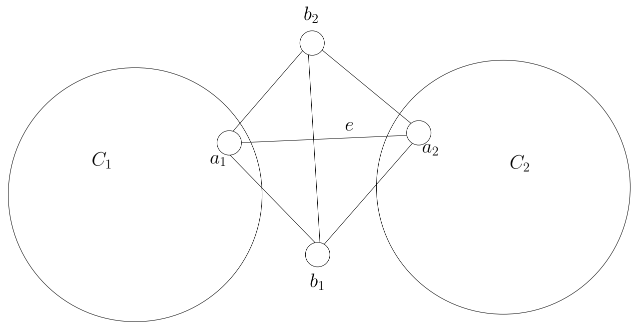


Figure 3.4: The path with the least dummy vertices from C_1 to C_2 and the usable kite-edges to use fewer dummy vertices.

3.2.1 Faces of $H[T]$

Our goal will be to show the following:

Lemma 46. *If G is a 1-plane graph where every face of G has degree 3, $T \subset V(G)$, and H is the graph from Definition 44, then every face of $H[T]$ contains at most one component of $H - T$.*

Proof. Examine H^P, the planarization of H. By Claim 45, H^P is a triangulated graph. Thus, every face of $H^P[T]$ contains at most one component of $H^P - T$ by Lemma 11. All we need to do to apply Lemma 11 is show that no two distinct components C_1, C_2 of $H - T$ are combined into a single component of $H^P - T$ during the process of planarizing H.

Assume for contradiction that C_1 and C_2 are in one component of $H^P - T$, so some path connects a vertex of C_1 with a vertex of C_2 in $H^P - T$. Let π be such a path that uses the smallest number of dummy vertices. π does not exist in $H - T$, so it must contain a dummy-vertex c that replaced a crossings (say between edges (a_1, a_2) and (b_1, b_2)) in H. The neighbours of c in π are connected along the kite-edges $(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_2, b_2)$. If π used subpath (a_i, c, b_j) for some $i, j \in \{1, 2\}$ then let π' be the path obtained by shortcutting along the edge (a_i, b_j). If π used (say) subpath (a_1, c, a_2), then at least one of $\{b_1, b_2\}$ is not in T (because all T-edges are planar). If $b_1 \notin T$, then let π' be the path obtained by replacing (a_1, c, a_2) be (a_1, b_1, a_2), else replace it by (a_1, b_2, a_2). Either way, we obtain a path π' connecting C_1 and C_2 with fewer dummy vertices. But this contradicts that π is the path connecting C_1 and C_2 that uses the fewest dummy vertices. See Figure 3.4. ■

The following claim provides insight into the structure of H that will also be useful later on.

Claim 47. *Let C be a component of $H - T$. If C lies in the face f of $H[T]$, then every vertex that is incident to f is adjacent to a vertex in C.*

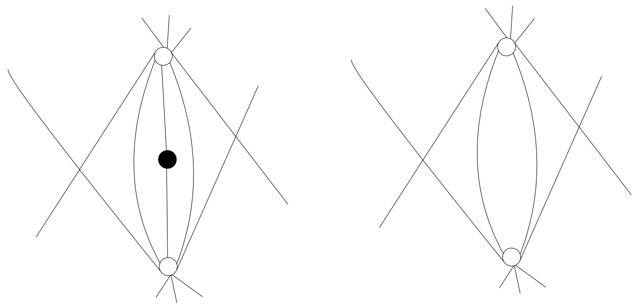


Figure 3.5: A section of $G = H$ (left), without bigon. $H[T]$ (right) contains a bigon since the S-vertex is removed.

Proof. Let $s = (v_1, v_2, \dots, v_k)$ be a facial circuit of f, in counterclockwise order. We have $k \geq 2$, else H has a loop.

Suppose for contradiction that v_i (for some $i \in \{1, \dots, k\}$) is not adjacent to any vertex of C. Consider the rotation system around v_i in H, which includes (v_i, v_{i-1}) and (v_i, v_{i+1}). If (v_i, v_{i-1}) is directly followed by (v_i, v_{i+1}) in this ordering then (since all faces in H have degree 3) we have that $\{v_i, v_{i-1}, v_{i+1}\}$ is a face of H, contradicting that C is in f.

So we know that v_{i-1} (in the CCW ordering of neighbours around v_i) is followed by some vertex a. We know that $a \notin C$ by the assumption that v_i is not adjacent to any vertex of C. It follows from Lemma 46 that a cannot be a vertex of a component inside f. If $a = v_\ell$, then since every face of H has degree 3, $(v_{i-1}, v_i, a = v_\ell)$ is a face, again contradicting that C is in f.

Since all other cases are covered, a must lie outside the face f. Therefore, (v_i, a) must cross the edge (v_j, v_{j+1}), for some $j \in \{1, \dots, k\}$. But (v_j, v_{j+1}) is a T-edge, hence planar, and so we have reached a contradiction.

$\blacksquare$

Using Lemma 46, we can obtain a bound on $odd_H(T)$ by counting the faces of $H[T]$. We will be distinguishing odd components of $H - T$ based on the degree of the face of $H[T]$ that it lies in. For $i = 2, 3$, define f_i to be the number of odd components of $G - T$ that lie inside a face of degree i of $H[T]$. Let f_4 be the number of odd components of $G - T$ that lie inside a face of degree at least 4 of $H[T]$. In particular, $odd(T) = f_2 + f_3 + f_4$, which we assume throughout our analysis.

Note that if $|T| \geq 3$ (which we assume throughout most of our analysis) then f_2 counts the number of odd components inside a bigon of $H[T]$. Observe that $H[T]$ could have a bigon, even if G and H do not, see Figure 3.5. However, for 3-connected graphs this is impossible since the vertices of the bigon would form a cutting pair.

Observation 48. *If G is 3-connected, then $f_2 = 0$, hence $odd_G(T) = f_3 + f_4$.*

We need to generalize Claim 13 to 1-plane graphs.

Claim 49. *If G is a 1-plane graph where every face has degree 3 and $|T| \geq 3$, then $f_3 + 2f_4 \leq 2|T| - 4$.*

Proof. Consider the graph H constructed from G, as in Definition 44. If the planar graph $H[T]$ has any faces of degree 4 or more, then add edges inside such faces to make them triangulated. Every face of $H[T]$ that had degree 3 and contained an odd component of $H - T$ remains a triangle that contains an odd component. Every face of $H[T]$ that had degree at least 4 and an odd component of $H - T$ became at least 2 triangles once the maximal number of planar edges were added. Since every face of $H[T]$ contains at most one component by Lemma 46, and the total number of triangles in the new graph is at most $2|T| - 4$, by Claim 3, the result immediately follows. ∎

3.2.2 Assigning Weights

To obtain a matching bound, we can use a similar technique to Section 1.3.1, by assigning a weight to each face in G and then distributing that to the components of $G - T$. We must first define a weight function and get a relationship between the total weight of the faces in G and the number of vertices in G. Later proofs will use other weight functions w_i ($i \geq 2$) where the weights are modified for some special faces.

Define a weight function $w_1 : F(G) \to \mathbb{R}$ as:

$$w_1(f) = \begin{cases} 2, & \text{if } f \text{ is an uncrossed face} \\ 1, & \text{if } f \text{ is a crossed face} \end{cases}$$

We note that every face of $H[T]$ contains at most a single component of $H - T$ by Lemma 46. Thus, for $i = 1, 2, 3$, we can extend w_i to faces of $H[T]$ and odd components of $H - T$ as follows. For a fixed drawing of G and any face f of $H[T]$, let $\mathscr{F}_f$ be the set of all faces of G contained in the face f of $H[T]$. Set $w_i(f) := \sum_{f' \in \mathscr{F}_f} w_i(f')$. If C is a component of $H - T$ that lies in the face f of $H[T]$, then set $w_i(C) = w_i(f)$. We stress that the weight of a face f of $H[T]$ is based on the weight of the faces of G contained in f.

Note that $w_1(f) = 2w(f)$, from Chapter 2, and so as in Claim 26 one can prove:

Lemma 50. *Let G be a 1-plane graph where every face has degree 3. Then $\sum_{f \in F(G)} w_1(f) = 4n - 8$.*

3.3 3-Connected, Maximal 1-Plane Graphs

Now that we have the necessary tools with our auxiliary graph, we will start by proving a lower bound on maximum matchings in a 1-plane graph G with n vertices that satisfies the following properties:

1. G is 3-connected; and
2. every face of G has degree 3.

Note that every 3-connected, maximal 1-plane graph satisfies these by Lemma 4, but we do not require maximality for the bound we will prove here. We would also like to mention that the bound proved here could easily be obtained by extracting a maximal planar graph inside G; the main point of our proof is to be a warm-up for later sections.

To prove the matching bound for a graph G, we use the corresponding graph H defined in Section 3.2 and the weight function w_1 defined in Section 3.2.2.

Claim 51. *If C is a component of $H - T$ that resides in a face of $H[T]$ of degree 3, then $w_1(C) \geq 6$.*

Proof. Let f' be a face of $H[T]$ that has degree 3 and contains the component C.

Consider the graph that is induced by f' and C. Create an identical copy C' of C on the side of f that does not contain a component, and let the graph induced by $C \cup C' \cup f'$ be K. We note that every face of H has degree 3 by Claim 45, and since f' cannot contain any additional components by Lemma 46, it follows that every face of K also has degree 3. Applying the weight function w_1 to the faces of K, we get that:

$$
\begin{aligned}
w_1(C) &= \frac{w_1(C) + w_1(C')}{2} \\
&= \frac{\sum_{f \in F(K)} w_1(f)}{2} \\
&= \frac{4|V(K)| - 8}{2} && \text{(by Lemma 50)} \\
&= 2(2|C| + 3) - 4 && \text{(since } |V(K)| = 2|C| + 3\text{)}
\end{aligned}
$$

Thus, $w_1(C) = 4|C| + 2 \geq 6$, as desired.

■

Corollary 52. *$6f_3 \leq 4n - 8$.*

Proof. Every odd component in a face of $H[T]$ with degree 3 has a weight of at least 6, and every face of $H[T]$ has non-negative weight, so $6f_3$ is at most the total weight of the faces of G. ■

Claim 53. *If $|T| \geq 3$, then $odd_G(T) - |T| \leq \frac{|V(G)|-8}{3}$.*

Proof. We have

$$\begin{aligned}
12(odd_G(T) - |T|) &\leq 12(f_3 + f_4) - 6(2|T|) && \text{(by Observation 48)} \\
&\leq 12f_3 + 12f_4 - 6(f_3 + 2f_4 + 4) && \text{(by Claim 49)} \\
&\leq 6f_3 - 24 \\
&\leq 4|V(G)| - 32 && \text{(by Lemma 50)}
\end{aligned}$$

Therefore, $odd_G(T) - |T| \leq \frac{|V(G)|-8}{3}$ as desired. ■

Theorem 54. *If G is a 3-connected, 1-plane graph where $n \geq 11$ and every face is a triangle, then G has a matching of size at least $\frac{n+4}{3}$.*

Proof. If $|T| \geq 3$, then from Claim 53, we have $odd_G(T) - |T| \leq \frac{|V(G)|-8}{3}$. Applying the Tutte-Berge Formula (Theorem 10), therefore G has a matching of size at least $\frac{|V(G)|+4}{3}$.

If $|T| \leq 2$, then $odd(T) \leq 1$ by 3-connectivity. Since $odd(T) - |T| \leq 1$ in this case, it follows that G has a perfect or near-perfect matching of size at least $\frac{n-1}{2} \geq \frac{n+4}{3}$ by $n \geq 11$. ■

In particular, we have reproved the bound from [2].

Corollary 55. *Every maximal planar graph with $n \geq 11$ has a matching of size at least $\frac{n+4}{3}$.*

Proof. Since every face of a maximal planar graph is a triangle, Theorem 54 applies to maximal planar graphs as well. ■

Corollary 56. *Every 3-connected, maximal 1-plane graph with $n \geq 11$ has a matching of size at least $\frac{n+4}{3}$.*

Proof. From Corollary 5, every face of a maximal 1-plane graph is a triangle. It follows that a 3-connected, maximal 1-plane graph with $n \geq 11$ satisfies the conditions of Theorem 54, and thus have a matching of size at least $\frac{n+4}{3}$. ■

Now we show that this bound is tight.

Theorem 57. *There is a 3-connected, maximal 1-plane graph with a maximum matching of size $\frac{n+4}{3}$.*

Proof. Consider a cycle on $k \geq 3$ vertices, let these vertices be the set of vertices T_1. Introduce 2 vertices that are adjacent to each of the k vertices of the cycles, let these vertices be the set of vertices T_2. The vertices in T_1 and T_2 will be our T-vertices, note that T induces a triangulated graph. In every face, insert a single vertex, and add edges so that the resulting graph is maximal planar. The single vertices added to each face will be the S-vertices, thus G has $2(k+2) - 4 = 2k$ S-vertices and $k+2$ T-vertices. Thus, we get that

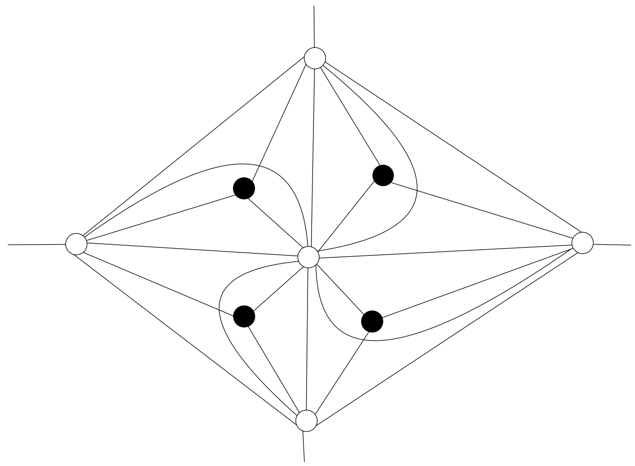


Figure 3.6: Half of a 3-connected, maximal 1-plane graph with a maximum matching of size $\frac{n+4}{3}$. The outside of the cycle is symmetric to the inside of the cycle.

$odd(T) - |T| = (2(k+2) - 4) - (k+2) = k - 2 = \frac{n-8}{3}$. It follows that every matching of G has at least $\frac{n-8}{3}$ unmatched vertices, or G has a maximum matching of size at most $\frac{n+4}{3}$. Since every 3-connected maximal planar graph has a matching of size at least $\frac{n+4}{3}$ by [2], G has a maximum matching of size $\frac{n+4}{3}$.

We must make one adjustment to make this graph maximal 1-plane. For every vertex $v_1 \in T_1$ and every vertex in $v_2 \in T_2$, add a copy of (v_1, v_2) so that it does not form a bigon. This can be done by crossing the ST-edge in the face to the left of the original edge (v_1, v_2). See Figure 3.6. ■

3.4 All Maximal 1-Plane Graphs

In Section 3.3, we found a tight bound for matchings in maximal 1-plane graphs that were 3-connected, and we showed that this was equivalent to the bound in the 3-connected planar case. We will now examine what occurs when we remove the restriction that G is 3-connected. We suspect that we could have graphs with smaller maximum matchings because in this case we allow subgraphs such as in Figure 3.7 where there are odd components of $G - T$ with a single vertex of degree 2. Unlike in planar graphs where having such components can make maximum matchings to be arbitrarily small, in maximal 1-plane graphs such components force there to be nearby crossings, and that requirement in the structure of the graph makes this case non-trivial.

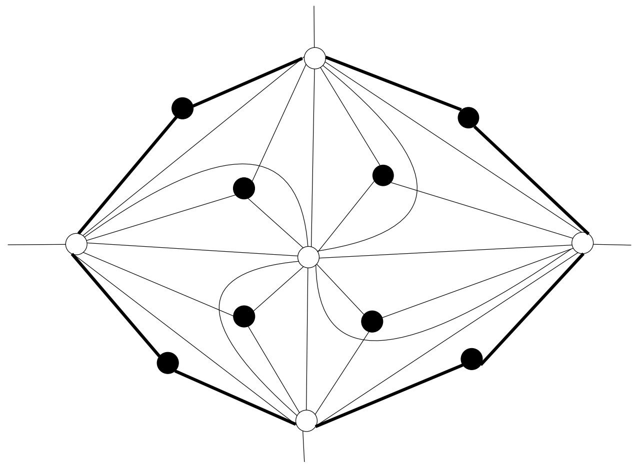


Figure 3.7: Half of a maximal 1-plane graph with singleton components in T of degree 2. The outside of the cycle formed by the bold edges is symmetric to the inside.

3.4.1 Bounding $odd(T) - |T|$ When $G[T]$ has no Crossings

We will take some time to get a bound on $odd(T) - |T|$ over subsets of vertices T where $G[T]$ is a plane graph. This will be essential later on in order to prove bounds on matchings in maximal 1-plane graphs. As we will see later, kites that are formed by T-vertices will become an issue that needs to be addressed.

So let G be a maximal 1-plane graph and let $T \subset V(G)$ be such that $G[T]$ is plane when inheriting the drawing of G.

We again use the graph H from Section 3.2 and its properties from Claim 45 in order to determine which faces of G are associated with which components of $H - T$. As before, all faces of H are triangles, but note that now some faces of $H[T]$ may be bigons, since G need not be 3-connected.

Let B^+ be the set of edges in $H[T]$ that are incident to a bigon or triangular face f in $H[T]$ such that f contains an odd-sized component of $H - T$. Let C_x be the subset of faces of G that are crossed and incident to an edge in B^+. Let C_o be the subset of faces of G that are uncrossed and incident to an edge in B^+. Let S_x be the subset of faces of G that are crossed and not in C_x. Let S_o be the subset of faces of G that are uncrossed and not in C_o. See Figure 3.8 for an example.

Claim 58. *If* $|T| \geq 3$*, then* $|C_o| \leq |B^+| \leq |C_x|$

Proof. Let uv be an edge in B^+. Since $u, v \in T$, uv is planar in G by assumption, and therefore incident to 2 faces of G, say f and f'. Since all faces of G have degree 3, G has no bridges, and thus $f \neq f'$. Let the corner of f that is not u or v be x_1, and let the corner of f' that is not u or v be x_2.

We claim that x_1 and x_2 cannot both be vertices. To see this, we will go into case analysis.

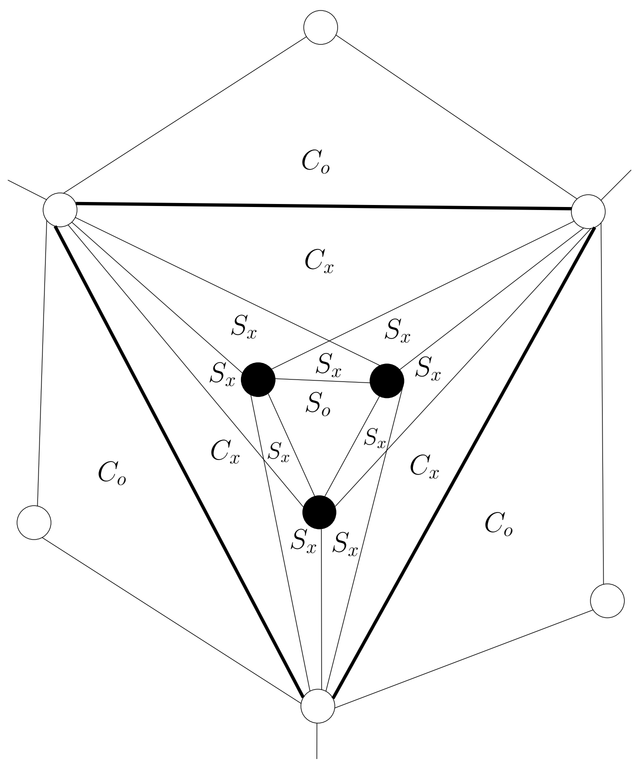


Figure 3.8: We show part of a maximal 1-plane graph to illustrate C_x,C_o,S_x,S_o. The edges in B^+ are bold.

Case 1: $x_1 \in T, x_2 \in T$
In this case, all vertices of f,f' are in T, so f and f' are also faces of $H[T]$ and thus do not contain a component of $H - T$. This contradicts that $uv \in B^+$.

Case 2: x_1, x_2 **are vertices, one of** x_1, x_2 **is in** S
Note that the edge (x_1, x_2) could have been added by going through the faces f and f' without violating 1-planarity. Since G was maximal 1-plane, thus the edge (x_1, x_2) through f and f' must be in G (but was removed in H). However, since one of x_1, x_2 is in S, the edge (x_1, x_2) is not a T-edge, but crossed the T-edge uv. This contradicts the construction of H.

It follows that x_1 or x_2 is a crossing. Thus every edge in B^+ adds at least one face in C_x and at most one face in C_o. Furthermore, every face in C_x can only be incident to a single edge in B^+ since if a face of degree 3 has a crossing as a corner then it can only be incident to a single uncrossed edge and all edges in B^+ are uncrossed. Thus, $|C_o| \leq |B^+| \leq |C_x|$. ■

Let $w_2 : F(G) \to \mathbb{R}$ be the weight function that is defined as:

$$w_2(f) = \begin{cases} 4, & \text{if } f \in C_o \\ -1 & \text{if } f \in C_x \\ w_1(f) & \text{otherwise} \end{cases}$$

Recall that we defined $w_i(f)$ (for faces of $H[T]$) and $w_i(C)$ (for components of $H - T$) in Section 3.2.2; we now use this for $i = 2$.

Claim 59. $\sum\limits_{f \in F(G)} w_2(f) \leq 4n - 8.$

Proof. Every face of C_x is a crossed face, so if $f \in C_x$, then $w_2(f) - w_1(f) = -2$. Every face of C_o is an uncrossed face, so if $f \in C_o$, then $w_2(f) - w_1(f) = 2$. If f is in S_o or S_x, then $w_2(f) = w_1(f)$. Thus, we have

$$\begin{aligned} \sum_{f \in F(G)} w_2(f) &= \sum_{f \in F(G)} w_1(f) + 2|C_o| - 2|C_x| \\ &\leq \sum_{f \in F(G)} w_1(f) && \text{(by Claim 58)} \\ &= 4n - 8 && \text{(by Lemma 50)} \end{aligned}$$

as required. ■

We will now create a general lemma for how much weight each component of $H - T$ is assigned.

Lemma 60. *Let f be a face of $H[T]$ where the boundary of f is a simple cycle, and f contains the component C. Then $w_2(f) \geq 4|C| - 4$.*

Proof. Consider the graph that is induced by f and C. Since f is a simple cycle, we can create an identical copy of C on the side of f that does not contain C, and let this new graph be K. Let the two copies of C be C' and C'', maintaining that every vertex in C' and C'' is an S-vertex and every vertex of f is a T-vertex. We note that every face of H is a triangle by Claim 45, so it follows that every face of K is also a triangle. Let $C_x^K, C_o^K, S_x^K, S_o^K$ be the corresponding sets C_x, C_o, S_x, and S_o for the graph K. Since every face in C_x^K must be incident to a T-edge, $|C_x^K| \leq 2deg(f)$. Applying the weight function w_2 to the faces of K, we get:

$$\begin{aligned}
w_2(C) &= \frac{\sum\limits_{v \in V(K)} w_2(v)}{2} \\
&\geq \frac{\sum\limits_{v \in V(K)} w_1(v) - 2|C_x^K|}{2} \\
&= \frac{4|V(K)| - 8 - 2|C_x^K|}{2} && \text{(by Lemma 50)} \\
&\geq \frac{4|V(K)| - 8 - 4deg(f)}{2} \\
&= 2(2|C| + deg(f)) - 4 - 2deg(f) && \text{(since } |V(K)| = 2|C| + deg(f)\text{)} \\
&= 4|C| - 4
\end{aligned}$$

as required. ■

Claim 61. *If f is a face of $H[T]$, then $w_2(f) \geq 0$.*

We remark first that this is where it is vital that there are no kites formed by T-edges, as a kite in $G[T]$ could cause there to be faces f of $H[T]$ that have weight less than 0, if every face is in C_x. See Figure 3.9.

Proof. Recall that $w_2(f)$ is defined as the sum of the weights of faces in G that make up f. If f' is an uncrossed face of G that lives in f, then $w_2(f') = 2$ or $w_2(f') = 4$ by definition. We will show that the crossed faces of G that make up f have a sum weight that is non-negative.

Instead of looking at these crossed faces individually, we will group them by the kites that they form. Note that all 4 crossed faces of a kite live in a single face of $H[T]$, since every edge in $H[T]$ is uncrossed.

Since $G[T]$ is planar, every kite in G contains at most 3 vertices in T. Thus in a kite of G there are at most 2 kite-edges that are T-edges, and at most 2 faces that are in C_x. Each face in C_x contributes a weight of -1, and the remaining faces of the kite are in S_x, each contributing a weight of 1. Thus, the total weight of faces that make up a kite in G is at least 0, and thus non-negative. ■

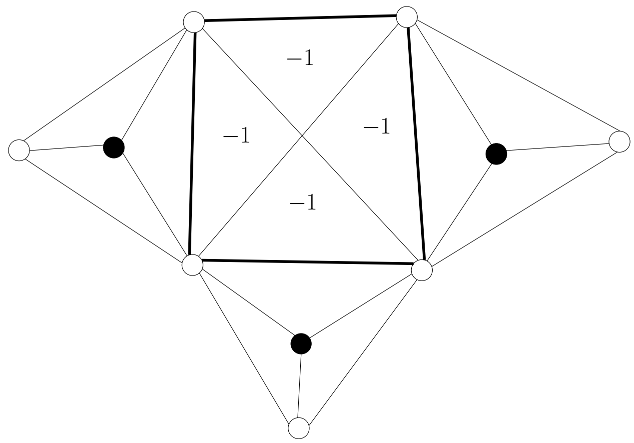


Figure 3.9: A face of $H[T]$ with weight -4 that occurs as a result of crossed T-edges.

Claim 62. *If C is an odd component that lies in a face f of degree 3 of $H[T]$, then $w_2(C) \geq 4$.*

Proof. f has degree 3, so it is a triangle of $H[T]$, and thus the boundary of f is a simple cycle. If $|C| \geq 3$, then by Lemma 60, $w_2(C) = w_2(f) \geq 4|C| - 4 \geq 8$. Otherwise, we have $|C| = 1$. Let $f = \{v_1, v_2, v_3\}$ and let C be the vertex v. By Claim 47, the edges $(v, v_1), (v, v_2), (v, v_3)$ all exist in H. v is not adjacent to any other vertex, since if v was adjacent to another T-vertex w, then (v, w) crosses an edge on the boundary of f, contradicting the construction of H. If v were adjacent to an S-vertex, then $|C| > 1$.

If none of the edges (v, v_i) are crossed in G, then the three faces $\{v, v_1, v_2\}$, $\{v, v_1, v_3\}$, $\{v, v_2, v_3\}$ are all in the set C_o, and thus are given a total weight of 4, so the weight of f is at least 4 and we are done. Suppose (v, v_1) is crossed by the edge e in G. Since every face of G has degree 3, the endpoints of e must be adjacent to v. It follows that e must be the edge (v_2, v_3). See Figure 3.10 for an example of how this could occur. So if (v, v_1) is a crossed edge, then (v, v_2) and (v, v_3) are kite-edges, hence uncrossed. By Corollary 7 the face $\{v, v_2, v_3\}$ is an uncrossed face, so it is in C_o and has weight 4. The faces $\{v, v_2, x\}$ and $\{v, v_3, x\}$ (where x is the crossing of (v, v_1) with (v_2, v_3)) are in the set S_x and contribute a weight of 1, and the faces $\{v_2, v_1, v\}$ and $\{v_3, v_1, v\}$ are in the set C_x and contribute a weight of -1. Thus, in this case C has a total weight of 4 as well. ■

Claim 63. *If $|T| \geq 3$ and C is an odd component that lies in a face f of degree 2 of $H[T]$, then $w_2(C) \geq 8$.*

Proof. Since $|T| \geq 3$, f must be a bigon and not a single edge. Therefore, the boundary of f is a simple cycle so if $|C| \geq 3$ we have $w_2(C) \geq 4|C| - 4 \geq 8$ by Lemma 60.

Otherwise, since C is an odd component, $|C| = 1$. Let C consist of the vertex v that lies in the face of $H[T]$ induced by $\{v_1, v_2\}$. By Claim 47, the edges (v, v_1) and (v, v_2) exist in G. As in the previous proof one argues that v is not adjacent to any other vertex. Thus, the faces $\{v_1, v_2, v\}$ and $\{v_2, v_1, v\}$ are both in C_o and thus each have a weight of 4. Thus, if $|C| = 1$ then $w_3(C) = 8$, as required. ■

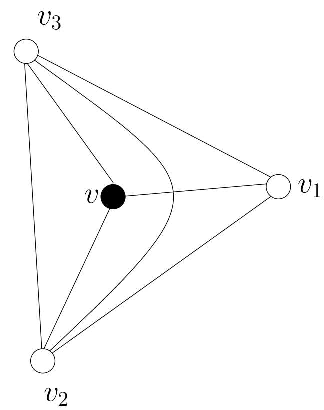


Figure 3.10: The edge (v_2, v_3) crossing the edge (v_1, v), causing (v_2, v) and (v_3, v) to be uncrossed.

Recall that f_i is the number of odd components in faces of $H[T]$ that have degree i.

Corollary 64. *If* $|T| \geq 3$*, then* $8f_2 + 4f_3 \leq 4|V(G)| - 8$

Proof. This follows from Claim 59, Claim 61, Claim 62, and Claim 63. Since every odd component that exists in a bigon of $G[T]$ has a weight of at least 8, every odd component that exists in a face of $G[T]$ of degree 3 has a weight of at least 4, and all other faces of $H[T]$ have non-negative weight, $8f_2 + 4f_3$ is at most the total weight of the faces in G. Since the total weight of the faces is at most $4|V(G)| - 8$, the result follows. ∎

Lemma 65. *Let G be a maximal 1-plane graph, $T \subset V(G)$ such that $|T| \geq 3$ and $G[T]$ is plane when inheriting the drawing of G. Then $odd(T) - |T| \leq \frac{|V(G)|-6}{2}$.*

Proof.

$$\begin{aligned}
8(odd(T) - |T|) &\leq 8(f_2 + f_3 + f_4) - 4(2|T|) \\
&\leq 8f_2 + 8f_3 + 8f_4 - 4(f_3 + 2f_4 + 4) && \text{(By Claim 49)} \\
&= 8f_2 + 4f_3 - 16 \\
&\leq 4|V(G)| - 8 - 16 && \text{(By Corollary 64)} \\
odd(T) - |T| &\leq \frac{|V(G)| - 6}{2}
\end{aligned}$$

∎

This result allows us to find a bound on maximal 1-plane graphs by modifying the graph to meet the restrictions of Lemma 65, as we will discuss in the next section.

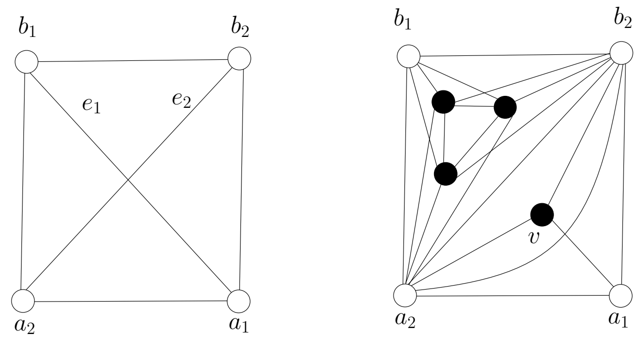


Figure 3.11: Transforming a kite in T.

3.4.2 Removing the Restriction

Lemma 65 requires that $G[T]$ is planar. We will now show how to overcome this restriction to get the desired matching bound.

Let G be a maximal 1-plane graph, and let $T \subset V(G)$. We will create an auxiliary graph K that inherits T from G, but change it so that $K[T]$ does not have any crossings. Then we will use the results of the previous section to get a matching bound on K, and use that to get a matching bound on G.

Let K be the graph G where for every two crossing T-edges $e_1 = (a_1, b_1), e_2 = (a_2, b_2)$, the following operations are performed:

1. Remove e_1.
2. In the planar face $\{a_2, b_1, b_2\}$, add a component of size 3 such that the component and $\{a_2, b_1, b_2\}$ induces a K_6. All the added vertices are S-vertices.
3. In the planar face $\{a_1, b_1, b_2\}$, add a singleton component, v, such that the component and planar face induces a K_4. Then, add a copy of e_2 that crosses the edge (v, a_1).

See Figure 3.11 for an example of these operations being performed.

Observation 66. *If G is a maximal 1-plane graph and $T \subset V(G)$, then K is a maximal 1-plane graph and $K[T]$ is planar when inheriting the drawing of G.*

Theorem 67. *If G is a maximal 1-plane graph with $n \geq 8$, then G has a matching of size at least $\frac{n+6}{4}$.*

Proof. Fix T such that $odd(T) - |T|$ is maximized.

We first consider the case where $|T| \geq 3$. From G and T, we can construct K. Since K is a maximal 1-plane graph and $K[T]$ is plane when inheriting the drawing of K, it follows from Lemma 65 that $odd_K(T) - |T| \leq \frac{|V(K)|-6}{2}$. Let $|X|$ be the number of crossings in $G[T]$.

We note that $|V(K)| = n + 4|X|$, since we added 4 vertices for every crossing in $G[T]$, and did not add vertices at any other point. Further, $odd_K(T) = odd_G(T) + 2|X|$, since we added 2 odd components for every crossing in $G[T]$, and no other odd components were added or removed. Thus, we have

$$\begin{aligned} odd_G(T) - |T| &= odd_K(T) - 2|X| - |T| \\ &\leq \frac{|V(K)| - 6}{2} - 2|X| && \text{(By Lemma 65)} \\ &= \frac{(n + 4|X|) - 6}{2} - 2|X| \\ &= \frac{n-6}{2} \end{aligned}$$

Thus, if $|T| \geq 3$, then since $odd_G(T) - |T| \leq \frac{n-6}{2}$, it follows from the Tutte-Berge formula that G has a matching of size at least $\frac{n+6}{4}$.

If $|T| < 3$, and $H[T]$ has a single face, then by Claim 45, $odd(T) \leq 1$. Thus, $odd(T) - |T| \leq 1$, so G has a perfect or near-perfect matching. Since $n \geq 8$, we get $\frac{n-1}{2} \geq \frac{n+6}{4}$, as required.

Alternatively, it is possible that $|T| < 3$ and $H[T]$ has more than 1 face. This can only occur if $|T| = 2$ and there are several T-edges separating components, see Figure 3.12. Let v_1, v_2 be the T-vertices of G, and let the clockwise ordering of faces of $H[T]$ around v_1 be $f'_1, f'_2, \ldots, f'_k$, noting that each of these faces is a bigon in $H[T]$. Each face f'_i must have a component of $G - T$, otherwise it is a bigon in G, and we do not allow bigons in G. For $i \in \{1, \ldots, k\}$, let the component C_i of $H - T$ be in the face f'_i.

Since G is maximal, if C_i is a singleton component then C_{i-1} and C_{i+1} must have size at least 2, since otherwise an edge could be added by crossing a T-edge to make them into a single component of $H - T$. If k is even, then for every odd i we can group the components C_i and C_{i+1}. Each of these pairs of components will either have 2 odd components and at least 4 S-vertices, or 1 odd component and at least 3 S-vertices. Thus, in this case there are at least twice as many S-vertices as odd components, so we get that $odd(T) \leq \frac{n-2}{2}$. If k is odd, then we group the components in the same way as before, but C_k is not grouped with any other C_i. Since C_k was selected arbitrarily, we can select it to be a component of size at least 2. As before, every pair of components has at least twice as many S-vertices as odd components, and C_k is not a singleton so it is either a single odd component of size at least 3 or not an odd component. In either case, if we consider C_k as a group of components by itself it also has at least twice as many S-vertices as odd components. Thus, even if k is odd, $odd(T) \leq \frac{n-2}{2}$.

It follows that when $|T| = 2$, $odd(T) - |T| \leq odd(T) - 2 \leq \frac{n-6}{2}$. Thus, any matching of G has size at least $\frac{n+6}{4}$, as required. ■

Theorem 68. *There are maximal 1-plane graphs with arbitrarily large n that have a maximum matching of size $\frac{n+6}{4}$.*

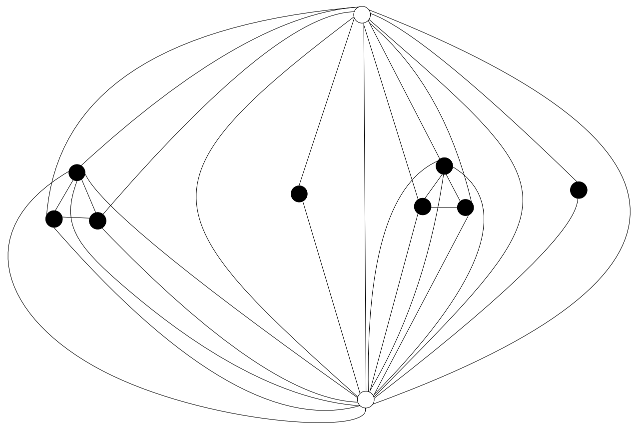


Figure 3.12: A maximal 1-plane graph with a maximum matching of size $\frac{n+6}{4}$.

Proof. Let n be an arbitrarily large even number. Consider a graph with 2 T-vertices v_1, v_2, and $\frac{n-2}{2}$ multiedges (v_1, v_2). Let the clockwise order of faces around v_1 be $f'_1, f'_2, \ldots, f'_{\frac{n-2}{2}}$. In f'_i, where i is odd, introduce an S-vertex v with the edges (v, v_1) and (v, v_2). In f'_i, where i is even, introduce the vertices x_1, x_2, x_3, and add the edge $(x_1, x_2), (x_2, x_3), (x_3, x_1)$ and a maximal number of edges (x_i, v_j) for $i \in \{1, 2, 3\}, j \in \{1, 2\}$. See Figure 3.12. Thus, $odd(T) = \frac{n-2}{2}$, $|V(G)| = 2 + \frac{n-2}{4} + 3\frac{n-2}{4} = n$, and $|T| = 2$. Thus, $odd(T) - |T| = \frac{n-6}{2}$. By Theorem 10, a maximum matching M of G has at least $\frac{n-6}{2}$ unmatched vertices, so M has at most $\frac{n+6}{2}$ matched vertices, or $|M| \leq \frac{n+6}{4}$. By Theorem 67 we have $|M| \geq \frac{n+6}{4}$, so it follows that $|M| = \frac{n+6}{4}$, as required. ■

3.5 3-Connected, Simple, Maximal 1-Planar Graphs

Now we will be removing the restriction that the 1-planar drawing is fixed and examine a graph that is 3-connected, simple, maximal 1-planar.

However, rather than studying simple graphs, we will study a class of graphs that we call *multiedge-uncrossed-maximal 1-plane*. A graph G is *multiedge-uncrossed* if every multiedge in the drawing of the graph is uncrossed. For example, the graph in Figure 3.12 is not multiedge-uncrossed. A graph G is *multiedge-uncrossed-maximal 1-plane* if G is a multiedge-uncrossed 1-plane graph and no edges can be added to the drawing of G without breaking the condition of being 1-plane or being multiedge-uncrossed. This graph class will be useful to us since, as we will see later (in Theorem 78) a simple, 3-connected, maximal 1-planar graph can easily be modified into a multiedge-uncrossed-maximal 1-plane graph by duplicating edges.

3.5.1 Bounding $odd(T) - |T|$ When $G[T]$ is a Plane Graph

Similar to Section 3.4, we will take some time to get a bound on $odd(T) - |T|$ over subsets of vertices T where $G[T]$ is a plane graph.

Let G be a multiedge-uncrossed-maximal 1-plane graph where every face has degree 3. Additionally we assume throughout this subsection that all faces of G have degree 3; this is not necessarily true for all multiedge-uncrossed 1-plane graphs (consider K_4 drawn as a kite). Let $T \subset V(G)$ so that $G[T]$ is plane. The graph H that we construct is the same as in Definition 44, where T-edges of G that are crossed are removed. Since G is a 1-plane graph where every face has degree 3, the properties of H from Claim 45 apply.

Additionally, we will be using the following property of G:

Claim 69. *If G is a 3-connected, multiedge-uncrossed-maximal 1-plane graph, and $T \subset V(G)$ such that $G[T]$ is plane, then there are no multiedges in $H[T]$.*

Proof. Suppose for contradiction that there is a multiedge (v_1, v_2) in $H[T]$, label them e_1 and e_2 to distinguish between them. The edges e_1 and e_2 do not form a bigon in H, since that would also form a bigon in G, so there must be vertices of H inside and outside the cycle (v_1, v_2). Additionally, e_1 and e_2 are not crossed in G, since G is multiedge-uncrossed. Thus, the edges of the cycle (v_1, v_2) are planar in G and there are vertices on both sides of this cycle. This implies that $\{v_1, v_2\}$ is a cutting pair of G, contradicting that G is 3-connected. ■

Face Weights

As in Section 3.4, we will assign weights to the faces of G based on their relation to T-edges.

The following definitions are nearly identical to the corresponding definitions from Section 3.4. The primary difference is that $H[T]$ cannot contain a bigon by Claim 69. Let B^+ be the set of edges in $H[T]$ that are incident to a triangle in $H[T]$ that contains an odd-sized component of $H - T$. Let C_x be the subset of faces of G that are crossed and incident to an edge in B^+. Let C_o be the subset of faces of G that are uncrossed and incident to an edge in B^+. Let S_x be the subset of faces of G that are crossed and not in C_x. Let S_o be the subset of faces of G that are uncrossed and not in C_o. Note that $F(G) = C_x \cup C_o \cup S_x \cup S_o$.

We now assign weights to the faces of G, in order to bound the number of odd components in $G - T$. Let the weight function $w_3 : F(G) \to \mathbb{R}$ be defined as:

$$w_3(f) = \begin{cases} 10/3, & \text{if } f \in C_o \\ -1/3, & \text{if } f \in C_x \\ w_1(f) & \text{otherwise} \end{cases}$$

Recall that we argued in Claim 58 that $|C_o| \leq |B^+| \leq |C_x|$ for maximal 1-plane graphs. This claim unfortunately does not transfer immediately to here because the proof of Case 2 in Claim 58 relied on us being able to add an edge that crossed an existing uncrossed edge. If we repeated the proof exactly then it would break down because adding the same edge could cause the graph to no longer be multiedge-uncrossed 1-plane. Therefore, we repeat the proof with minor modifications here.

Claim 70. *If* $odd(T) \geq 2$*, then* $|C_o| \leq |B^+| \leq |C_x|$.

Proof. Let $uv \in B^+$. Edge uv is incident to 2 faces of $F(G)$, say f and f', and we know that $f \neq f'$ since G has no bridges. Let the corner of f that is not u or v be x_1, and let the corner of f' that is not u or v be x_2. See Figure 3.13.

As in Claim 58, we will argue that x_1 or x_2 is a crossing through case analysis. Assume for contradiction that both x_1 and x_2 are vertices.

Case 1: $x_1 \in T, x_2 \in T$
In this case, all vertices of f,f' are in T, so f and f' are also faces of $H[T]$ and thus do not contain a component of $H - T$. This contradicts that $uv \in B^+$.

Case 2: One of x_1, x_2 **is in** S**,** $(x_1, x_2) \notin E(H)$
Note the edge (x_1, x_2) could have been added by going through the faces f and f' since G was multiedge-uncrossed-maximal 1-plane. Thus the edge (x_1, x_2) through f and f' must be in G (but was removed in H). However, since one of x_1, x_2 is in S, the edge (x_1, x_2) is not a T-edge, but crossed the T-edge uv. This contradicts the construction of H.

Case 3: One of x_1, x_2 **is in** S**,** $(x_1, x_2) \in E(H)$
This case covers the situation where the edge (x_1, x_2) is in H (and thus G) but adding an additional edge (x_1, x_2) through the faces f and f' would cause the graph to no longer be multiedge-uncrossed. Since $uv \in B^+$, there must be a triangle of $H[T]$ that is incident to uv and contains a component of $H - T$, let this face of $H[T]$ be f. We can without loss of generality assume that x_1 is in the face f. Since none of the edges of $H[T]$ are crossed and (x_1, x_2) is an edge, it follows that x_2 is either a corner of f, or inside f.

If x_2 is a corner of f, then since f has degree 3, f is the face $\{u, v, x_2\}$. Thus, in $H[T]$ one side of the edge uv is the face $\{u, v, x_2\}$, and no vertices lie in that face, and the other side is the face $\{u, v, x_2\}$ that contains the component with x_1. Since $(u, x_2), (v, x_2), (u, v)$ are all simple edges in $H[T]$ by Claim 69, it follows that x_1 is the only S-vertex of H, so $odd(T) = 1$. This contradicts the assumption that $odd(T) \geq 2$.

If x_2 lies in f, then f is on both sides of the edge uv, so uv appears twice in the facial walk of f, contradicting that f has degree at most 3, or contradicting that $|T| \geq 4$ if $f = \{u, v\}$. Thus, x_2 cannot lie in f or be a corner of f, so this case cannot occur.

It follows that x_1 or x_2 is a crossing. Thus every edge in B^+ adds at least one face in C_x and at most one face in C_o. Furthermore, every face in C_x can only be incident to a single edge in B^+ since if a face of degree 3 has a crossing as a corner then it can only be incident to a single uncrossed edge and all edges in B^+ are uncrossed. Thus, $|C_o| \leq |B^+| \leq |C_x|$. ■

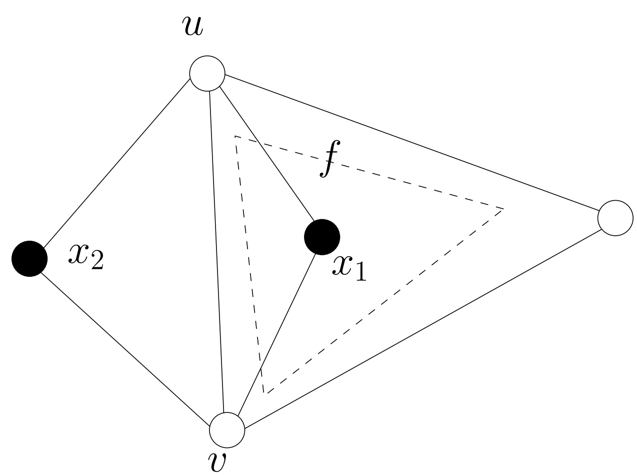


Figure 3.13: The face that x_1 exists in within H.

Lemma 71. *If $odd(T) \geq 2$, then* $\sum_{f \in F(G)} w_3(f) \leq 4n - 8$.

Proof. Recall the weight function w_1 defined earlier. Notice that any face f in C_x has $w_3(f) - w_1(f) = -4/3$, any face $f \in C_o$ has $w_3(f) - w_1(f) = 4/3$, and any other face has $w_3(f) - w_1(f) = 0$. Thus, by Claim 70,

$$\sum_{f \in F(G)} w_3(f) = \sum_{f \in F(G)} w_1(f) - \frac{4}{3}|C_x| + \frac{4}{3}|C_o| \leq \sum_{f \in F(G)} w_1(f) = 4n - 8$$

as desired. ∎

Recall that we defined $w_i(C)$ and $w_i(f)$ for a component C and a face f on Page 47, and we now apply this for $i = 3$.

Claim 72. *If C is an odd component that lies in a triangle of $H[T]$, then $w_3(C) \geq 10$.*

Proof. Let $f = (a_1, a_2, a_3)$ be a triangle of $H[T]$ that contains the component C. By Lemma 46, there exists only one component inside f. We distinguish cases, noting that we do not need to consider the case $|C| = 2$ since C is an odd component:

Case 1: $|C| = 1$
If C is a single vertex v, then since G is 3-connected, the edges $(v, a_1), (v, a_2), (v, a_3)$ are all edges in G, see Figure 3.14.

Note that if (v, a_1) was crossed at the crossing c, then v would be incident to all the other vertices that form the kite associated with c. That is, the kite must be formed by the vertices $\{v, a_1, a_2, a_3\}$, and the edge (v, a_1) would be crossed by the edge (a_2, a_3), similar to Figure 3.10. However, the edge (a_2, a_3) is part of the triangle that contains f and thus

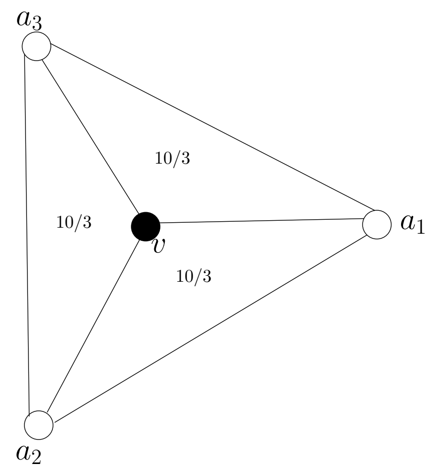


Figure 3.14: The case where $|C| = 1$ in a 3-connected, multiedge-uncrossed, maximal 1-plane graph.

uncrossed, so an additional edge (a_2, a_3) that is crossed would not be allowed without violating the property of G being multiedge-uncrossed. A symmetric argument holds for a_2 and a_3, so each of the faces $\{a_1, a_2, v\}$,$\{a_2, a_3, v\}$,$\{a_1, a_3, v\}$ is in C_o and thus has weight 10/3. Therefore, $w_3(f)$ is at least 10 in this case.

Case 2: $|C| \geq 3$
Consider the graph that is induced by f and C. Since f is a triangle, we can create an identical copy of C on the side of f that does not contain a component, and let this new graph be K. Let the two copies of C be C' and C'', maintaining that every vertex in C' and C'' is an S-vertex and every vertex of f is a T-vertex. We note that every face of H is a triangle by Claim 45, so it follows that every face of K is also a triangle. Let $C_x^K, C_o^K, S_x^K, S_o^K$ be the corresponding sets C_x, C_o, S_x, and S_o for the graph K. Since every face in C_x^K must be incident to a T-edge, $|C_x^K| \leq 2deg(f) = 6$. Applying the weight function w_3 to the faces of K, we get:

$$\begin{aligned}
w_3(C) &\geq \frac{\sum\limits_{f \in F(K)} w_1(f) - \frac{4}{3}|C_x^K|}{2} \\
&\geq \frac{4|V(K)| - 8 - \frac{24}{3}}{2} \\
&= 2|V(K)| - 8 \\
&= 2(2|C| + 3) - 8 \\
&\geq 2(9) - 8 \\
&= 10
\end{aligned}$$

Thus, regardless of the size of the component, if C exists in a triangle of $H[T]$, then $w_3(C) \geq 10$. ■

Claim 73. *If f is a face of $H[T]$, then* $\sum_{f \in \mathscr{F}_f} w_3(f) \geq 0$.

Proof. The argument is as for Claim 61, except that a kite now has a weight of at least $2(\frac{-1}{3}) + 2(\frac{10}{3}) \geq 0$. ■

Corollary 74. *If $odd(T) \geq 2$, then $f_3 \leq \frac{2n-4}{5}$*

Proof. By Claim 72, every component in a face of degree 3 of $H[T]$ has weight at least 10. Furthermore, by Lemma 71, the total weight of faces in $F(G)$ is at most $4n-8$. Since all other faces of H have non-negative weight by Claim 73, it follows that $f_3 \leq \frac{4n-8}{10} = \frac{2n-4}{5}$. ■

Lemma 75. *If G is a 3-connected, multiedge-uncrossed-maximal 1-plane graph where every face has degree 3, and $T \subset V(G)$ is a set such that $|T| \geq 3$, $odd(T) \geq 2$ and $G[T]$ is plane, then $odd(T) - |T| \leq \frac{n-12}{5}$.*

Proof. We have

$$
\begin{aligned}
10(odd(T) - |T|) &\leq 10odd(T) - 5(2|T|) \\
&\leq 10(f_3 + f_4) - 5(f_3 + 2f_4 + 4) && \text{(By Claim 49)} \\
&\leq 5f_3 - 20 \\
&\leq 5(\frac{2n-4}{5}) - 20 && \text{(By Corollary 74)} \\
&\leq 2n - 24.
\end{aligned}
$$

■

3.5.2 Removing the Restriction

Lemma 75 requires that $G[T]$ is plane. Similarly as in Section 3.4.2 we can overcome this restriction by modifying G, but the construction is slightly different if we want the result to be multiedge-uncrossed-maximal 1-plane.

Let G be a multiedge-uncrossed-maximal 1-plane graph, and let $T \subset V(G)$. We will create an auxiliary graph K that inherits T from G, but change it so that $K[T]$ does not have any crossings.

Let K be the graph G where for every two crossing T-edges $e_1 = (a_1, b_1)$,$e_2 = (a_2, b_2)$, the following operations are performed:

1. Remove e_1.

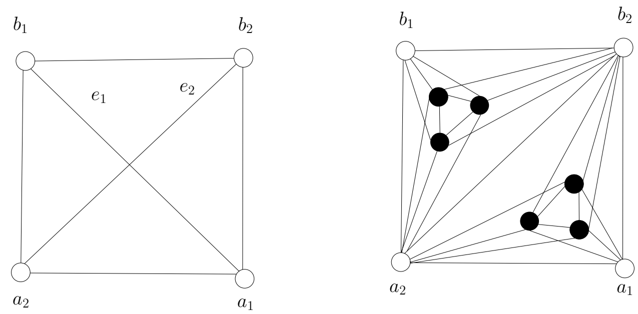


Figure 3.15: Transforming a kite in $G[T]$ so that no kites formed by T-edges.

2. In the planar faces $\{a_1, a_2, b_2\}$ and $\{a_2, b_1, b_2\}$ add a component of size 3 such that the component together with the face vertices induces a K_6. All the added vertices are S-vertices.

See Figure 3.15 for an example of these operations being performed. Since all crossed edges that were introduced are simple edges, we have the following.

Observation 76. *If G is multiedge-uncrossed-maximal 1-plane, $T \subset V(G)$, then K is multiedge-uncrossed-maximal 1-plane and $K[T]$ is planar when inheriting the drawing of G.*

Lemma 77. *If G is a 3-connected, multiedge-uncrossed-maximal 1-plane graph where every face is a triangle, $T \subset V(G)$, $|T| \geq 3$, then $odd(T) - |T| \leq \frac{|V(G)|-12}{5}$.*

Proof. Let $T \subset V(G)$, and construct K from G. We will now go into cases based on $odd_G(T)$.

If $odd_G(T) \leq 1$, then the left-hand side is negative by $|T| \geq 3$ and the inequality holds.

If $odd_G(T) \geq 2$ and $|T| \geq 3$, then $odd_K(T) \geq 2$, $K[T]$ is plane, and every face of K has degree 3. Thus, by Lemma 75 $odd_K(T) - |T| \leq \frac{|V(K)|-12}{5}$.

Let X be the set of crossings between T-edges in G. Thus, $odd_K(T) = odd_G(T) + 2|X|$, since 2 odd components of size 3 were added for every crossing between two T-edges in G. Furthermore, $|V(G)| = |V(K)| - 6|X|$, since 6 vertices were added for every crossing in X.

Thus, $odd_G(T) - |T| \leq (odd_K(T) - 2|X|) - |T| \leq \frac{|V(K)|-12}{5} - 2|X| \leq \frac{|V(G)|+6|X|-12}{5} - 2|X| \leq \frac{|V(G)|-12}{5}$, as required. ∎

Theorem 78. *Every simple, 3-connected, maximal 1-planar graph with $|V(G)| \geq 17$ has a matching of size at least $\frac{2n+6}{5}$.*

Proof. Let G be a simple, 3-connected, maximal 1-planar graph and fix a drawing of G that minimizes the total number of crossings.

Fix $T \subset V(G)$. If $|T| \leq 2$, then $odd(T) \leq 1$, since otherwise the vertices of T form a cutting set of size 2 contradicting that G is 3-connected. Thus, $odd(T) - |T| \leq 1$, meaning G has a perfect or near-perfect matching. Thus, since $n \geq 17$, we have that $\frac{n-1}{2} \geq \frac{2n+6}{5}$, so G has a matching of size $\frac{2n+6}{5}$, as required.

If $|T| \geq 3$, add a maximal number of edges to G without creating a bigon, breaking the condition of being multiedge-uncrossed, or introducing any additional crossings. Let this new graph be G'. We note immediately that G' is multiedge-uncrossed-maximal 1-plane by construction. Additionally, if a simple edge was added to G', then G was not simple, 3-connected, maximal 1-planar, so every edge that was added to G in the construction of G' must be a multiedge. Recall that in a multiedge-uncrossed 1-plane graph not all faces necessarily have degree 3, but we can argue here that G' is special.

Claim 79. *Every face of G' has degree 3.*

Proof. Suppose for contradiction that the face f of G' has degree at least 4. Then by Lemma 4, we can add a planar edge (say (v_1, v_2)) to G' inside f without creating any bigons. Since G' is multiedge-uncrossed-maximal 1-plane, it follows that adding the edge (v_1, v_2) in f would have made G' no longer multiedge-uncrossed, meaning that the edge (v_1, v_2) exists in G' as a crossed edge. However, the drawing of G was chosen to minimize the number of crossings, and (v_1, v_2) could have been drawn to go through the face f and not be crossed. This is a contradiction. ■

Since G' is multiedge-uncrossed-maximal 1-plane and every face of G' has degree 3, we can apply Lemma 77. Thus $odd_{G'}(T) - |T| \leq \frac{|V(G')|-12}{5}$ so by the Tutte-Berge formula, G has a matching of size at least $\frac{2|V(G')|+6}{5}$. Since the only changes that were made to obtain G' from G were adding multiedges and changing the drawing, any matching of G' can be inherited by G, and thus G also has a matching of size $\frac{2|V(G)|+6}{5}$, as required. ■

Every lower bound on maximum matching that we proved so far has been tight, with the possible exception of the bound in Theorem 78. We believe that the bound is tight, and present a graph that we believe makes the bound tight, but are currently limited by a lack of tools to explore all possible 1-planar embeddings.

Conjecture 80. *There exists a simple, 3-connected, maximal 1-planar graph with a maximum matching of size $\frac{2n+6}{5}$.*

Theorem 81. *There exists a simple, 3-connected, maximal 1-plane graph with a maximum matching of size $\frac{2n+6}{5}$.*

Proof. Consider a cycle on k vertices, $a_1, a_2, \ldots, a_k$, with $k > 2$ and k even. Add two vertices, v_1, v_2 that are incident to every vertex a_i of the cycle. This creates a triangulated graph and these $k+2$ vertices will make up our T-vertices. Let the clockwise orientation of faces around v_1 be $f_1, f_2, \ldots, f_k$, where f_i is incident to (a_i, a_{i+1}). For every odd i, insert a single

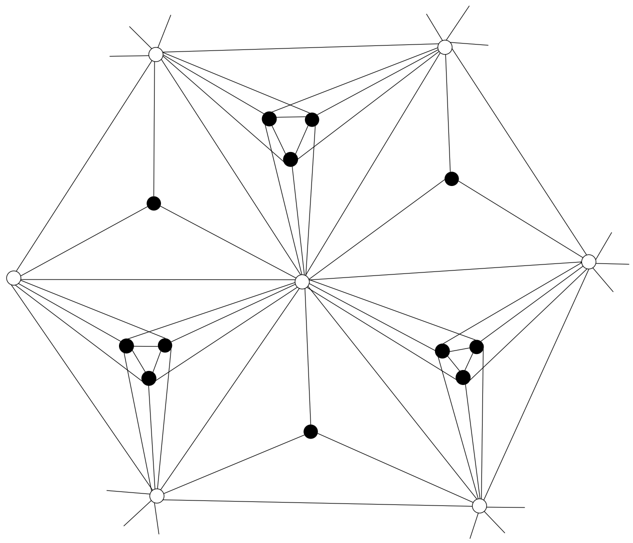


Figure 3.16: A potentially simple, 3-connected, maximal 1-planar graph with small maximum matching. The outerside of the hexagon is symmetric to the inside, rotated so that no two singleton components are adjacent to the same T-edge.

S-vertex into f_i that is incident to the T-vertices that are incident to f_i. For every even i, insert 3 S-vertices so that the S-vertices inserted and the vertices incident to f_i forms a K_6.

Let the clockwise orientation of faces around v_2 be $f'_1, f'_2, \ldots, f'_k$, where f_i and f'_i are both incident to (a_i, a_{i+1}). For every odd i, insert 3 S-vertices that are incident to f'_i. For every even i, insert a single S-vertex into f'_i. See Figure 3.16 for an example of this graph.

Let the resulting graph be G. We notice that the T-vertices form a triangulated graph on $k+2$ vertices, so there are $2|T|-4 = 2(k+2)-4 = 2k$ faces that S-vertices were inserted into. In half of the faces, 1 S-vertex was inserted, and in the other half 3 S-vertices were inserted. Thus, the total number of S-vertices in G is $k(3) + k(1) = 4k$. Therefore, the total number of vertices in G is $4k + k + 2 = 5k + 2$. Since $odd(T) - |T| = 2k - (k+2) = k - 2 = \frac{n-12}{5}$, any matching M of G has at least $\frac{n-12}{5}$ unmatched vertices, so M has at most $\frac{4n+12}{5}$ matched vertices, or $|M| \leq \frac{2n+6}{5}$. ■

3.6 Open Problems for Matching Bounds in Simple Maximal 1-Planar Graphs

Similar to how removing the requirement of 3-connectivity allowed graphs to have a lower maximum matching in the 1-plane case, this may also be true in the 1-planar case.

Conjecture 82. *Every simple, maximal 1-planar graph with sufficiently large n has a matching of size at least $\frac{3n+14}{10}$, and this bound is tight.*

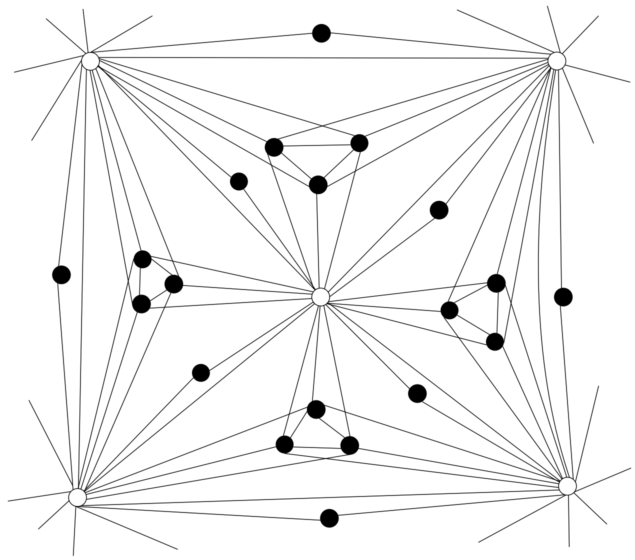


Figure 3.17: A potentially simple maximal 1-planar graph with small maximum matching. The outerside is symmetric to the inside.

This conjecture arose because we can construct a simple maximal 1-plane graph that matches it, Figure 3.17, but as before we do not know whether this graph is maximal in all possible 1-planar embeddings.

Theorem 83. *There is a simple maximal 1-plane that has a maximum matching of size* $\frac{3n+14}{10}$.

Proof. Consider a triangulated graph on k vertices. In each face f, insert 3 S-vertices so that the S-vertices and the vertices incident to f form K_6. For each T-edge uv, insert an S-vertex that is only incident to u and v. See Figure 3.17 for an example of this construction. Since the T-vertices formed a triangulated graph on k vertices, there are $3k-6$ S-vertices that have degree 2 and $(2k-4)3$ S-vertices that form part of a K_6 subgraph. In total, there are $10k-18$ vertices in the constructed graph, G. One odd component was added for every face of $G[T]$, and one odd component was added for every edge of $G[T]$. It follows that $odd(T) - |T| = (3k-6) + (2k-4) - k = 4k-10$. Computation as before shows that therefore any matching has size at most $\frac{3n+14}{10}$. We can easily find a matching in G of this size. ■

It is possible that considering graphs where $G[T]$ is planar and assigning weights to the faces such as in the previous sections may work. We tried proving Conjecture 82 by modifying the weight function further. The bound would hold if we could find a weight function such that $w_4(C) = 10$ for an odd component inside a bigon of $H[T]$ and $w_4(C) = 5$ for an odd component inside a triangular face of $H[T]$ while $w_4(C) \geq 0$ for all other components. The natural way to do this would be to assign faces in C_x a weight of -2, assign faces in C_o a weight of 5, and maintaining $w_4 = w_1$ on faces of S_x, S_o. However, when doing the four faces that form a kite that has 3 T-vertices and 1 S-vertex will have a negative weight, so the arguments used in Claim 61 and Claim 73 will not work. Thus, in order to maintain that $w_4(C) \geq 0$ for every component of $H - T$, our auxiliary graph must also modify kites of G that have a single S-vertex while staying multiedge-uncrossed, and without increasing the size of the matching in the graph. Finding such a construction remains for future study.

Chapter 4

Conclusion

By expanding on the proof ideas of Tutte ([14]), Nishizeki et al. ([10]), Biedl et al. ([2]), and many others, we have been able to establish lower bounds on the size of matchings in several classes of 1-planar graphs. We began by reviewing their works, highlighting tools that helped us to bound the maximum number of unmatched vertices in any matchings.

In Chapter 1, we laid the groundwork by introducing graph terminology. We went on to briefly discuss the history of matchings and established results. Finally, we reviewed some papers that used similar tools to familiarize the reader with the methods we will use to prove our results.

In Chapter 2, we proved tight lower bounds on maximum matchings of 1-planar graphs as a function of their minimum degree, showing tight bounds for minimum degree at least 3, 4, and 5 (see Theorem 2.2.1, Theorem 35, and Theorem 38). We did this by distributing a weight to every vertex of minimum degree, and giving an upper bound on the total weight of vertices by using this assignment. We provided conjectures for lower bounds in maximal 1-planar graphs of minimum degree 6 and 7, but using the same tools used in the cases of minimum degree 3, 4, and 5 does not appear to immediately yield any results for these cases.

In Chapter 3, we proved tight lower bounds on maximum matchings in 3-connected, maximal 1-plane graphs and maximal 1-plane graphs. Finally, we showed a lower bound in simple, 3-connected, maximal 1-planar graphs, but were unable to show it is tight. We conjecture a graph that has the matching bound with the required properties, but the tools to prove that such a graph is maximum 1-planar are currently lacking. It is unclear what bounds can be obtained for simple maximal 1-planar graphs, using similar techniques to the 3-connected case seems like it may work but would require more research and there may be more effective approaches.

Finally, as for open problems, we are interested in matching bounds for other graph classes that are close to planar graphs, such as k-planar graphs (for constant k) or graphs that are embedded on surfaces of small genus.

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