What is a field, what is a particle?

...what about algebras?

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Much of the structure of quantum field theory (QFT) is predicated on the principle of locality. Adherence to locality is achieved in Algebraic QFT (AQFT) by the association of algebras of observables with regions of spacetime. Although, by construction, the observables of QFT are local objects, one may consider characterizing the spatial or spacetime features of a *state*. For example, if we have a single-particle state in QFT, how can we say that the particle is localized in a certain region of space? It turns out that such a characterization is obstructed by a collection of no-go theorems that we will review in the first two chapters, which imply the absence of any suitable position operator or local number operator in the local algebra of observables. These difficulties, along with other considerations that involve acceleration, gravity and interactions, suggest that relativistic QFT cannot support a particle ontology. The common factor of all these reasons is the theory of relativity, which is commonly blamed for the inappropriateness of the particle notion in relativistic quantum theories.

Looking towards low energies, one finds the widespread applicability of non-relativistic quantum mechanics (NRQM), a theory in which particle states are localizable by means of their wavefunction. This seems to imply that NRQM can support a particle ontology, so it is natural to ask whether one can make contact between the NRQM description of particles and some appropriate notion in the latent QFT. Admittedly QFT and NRQM are very different theories, both at the dynamical and kinematical level, and recovering one from the other cannot come with no cost. The main undertaking of this thesis will be to illuminate this connection, by starting with a relativistic QFT and making suitable approximations to recover features of NRQM. Furthermore, it has been suggested that the existence of vacuum entanglement in a relativistic QFT is further obscuring the localizability of states. This is why we are investigating the behaviour of vacuum entanglement under the non relativistic approximation to ask whether vacuum entanglement is a relativistic effect.

The title of this thesis is inspired by two readings that have influenced my understanding on fields, particles and algebras for the past two years. The first one is the paper "What is a particle?" by C. Rovelli and D. Colosi [1] and the other one is the Algebraic Quantum Field Theory (AQFT) textbook by R. Haag [2] "Local quantum physics: fields, particles and algebras" that Jason Pye left on my desk one day (this book has been delightfully confusing me ever since). In Chapter 1 we present the reformulation of QFT in the algebraic language, mostly following [2, 3]. Chapter 2 imcludes the main no go theorems that demonstrate how the particle notion is problematic in relativistic quantum theories (not necessarily thought of as relativistic QFT's). In Chapter 3 we are reviewing two relatively recent articles that propose notions of local quanta in QFT, the paper by Colosi and Rovelli [1] and a

follow up paper by Rodriguez-Vazquez et al [4] which is very much related to my research. My attempt for a contribution, in collaboration with the PhD student Jason Pye and my supervisors Achim Kempf and Eduardo Martin Martinez, corresponds to Chapter 4 and the last section of Chapter 2. In Chapter 4 we present the project described just above on the 'Impacts of relativity on localizability and vacuum entanglement'. In Chapter 2 we are characterizing the localizability of one-particle QFT states by means of the Newton Wigner wavefunction as introduced in [5]. In the absence of a relativistically covariant position operator, there cannot be a spatial representation of a state, aka a wavefunction, for a relativistic quantum particle. The 'best possible' notion of a position operator was concocted by Newton and Wigner, and we are making use of it in the context of QFT at section 2.3.2. There we also speculate about how a delocalized 'cat state' gravitates i.e., the gravitational effect of a particle that is superposed in two different locations.

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I also want to thank Jason Pye with whom we have been collaborating on the research project that I present in the last chapter of this thesis, without whom the understanding I tried to develop about these issues would have been impossible (he is solely responsible about my interest and struggles with the algebraic approach). I very much appreciate all the insight that he has developed and has been always eager to share. I want to thank Jose de Ramon (aka Pipo) for the unexpectedly extensive, constructive and honest feedback that he provided in the writing of this thesis, also for always being there to talk (and for being such a character [2]). I would also like to thank the rest of my group mates for all the super interesting discussions, Marco Letizia, Nitica Sakharwade, Ding Jia, Erickson Tjoa, Petar Simidzija, Nicholas Funai, Emma McKay [3], Alison Sachs, the list would be too long because I was lucky to be part of two research groups, but I really want to thank all of the group mates because I have learned from all.

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[1] As in my proper reference frame.

[2] Authentic, opinionated, with a great sense of (dark) humor.

[3] Compiling this thesis would have been impossible without Emma's generous technical support.

[4] With the warmth and the strength of her character.

[5] His stubbornness has been very fruitful I have to admit.

hard to convince. Sitsos, the friend I have been missing the most. My parents Eva and Kostas, my sister Ioli and our dogs Mozart and Hilda, for being so supportive even though they still don't know what I am doing here in Canada. At last but not least, the family that I am so lucky to have here, Lamprini, Charalabos, Maria and Tasos. They helped me so much to adjust, I would have been too homesick without them. I have been very much appreciating this great opportunity of studying here, such a privilege that not many enjoy, but without the people mentioned above it would have been so much harder to deal with the struggles that such an opportunity comes with.

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Chapter 1

Quantum Theory of Fields

In this chapter we present backround material on Quantum Field Theory (QFT) mostly following the book by Wald [3] on "Quantum field theory on curved spacetime and black hole thermodynamics (QFTCS) and black hole thermodynamics". Even though we will not go into the formulation of QFT on curved spacetime, we will be interested in the re formulation of QFT on flat spacetime that is emphasizing the elements needed for extending the theory on curved spacetime. These elements are not interesting only from the gravity point of view, but they can also be very insightful for understanding some features of the theory that otherwise seem puzzling (particle number ambiguity, unitarily inequivalent representations etc). We tried to keep the presentation self contained (and hopefully not pedantic) even though it is not clear if this material makes much sense without having read the references that we are citing here, mostly [3] and [6] the book by Laura Ruetsche "Interpreting quantum theories", which I mostly followed for the presentation of the algebraic approach to QFT (AQFT). The writing of this chapter would have been impossible without the two reading groups that me and my like minded peers and friends run over the summer, with the support of Doreen Fraser and Achim Kempf.

1.1 From Classical to Quantum

In this section we are describing the quantization of a classical system of finitely many degrees of freedom. If we are describing a system of finitely many particles the degrees of freedom correspond to the positions of the particles, but more generally a degree of freedom is defined as an independent physical parameter in the formal description of a physical system. It is a non trivial procedure, how one identifies these independent parameters

of a given physical system, and one could wonder what is meant by "independent". In the case of n particles case for example, their positions are not independent if there are forces between them. Someone could then argue that the degrees of freedom are these parameters that would be independent in the absence of forces/dynamics. This definition suggests that the identification of the degrees of freedom is somehow independent from the particular dynamics that the system can undergo, it is an identification that happens at the kinematical level. In fact, a common definition of kinematics is "a branch of classical mechanics that describes the motion of objects without considering the forces that caused the motion". The degrees of freedom set the "kinematical stage" on which dynamics happens.

The distinction between kinematics and dynamics might be clear cut in classical mechanics, but it is more tricky in quantum theory. Even more so in quantum field theory where Hagg's theorem applies, and the "kinematics gets mixed up with the dynamics" [7]. This was my motivation for reviewing how the kinematics is set in the classical and the quantum theory, through the quantization procedure, and what are the subtleties in the case of infinite degrees of freedom [3, 8]. Another motivation is that the identification of the degrees of freedom might inspire a formal analogy between physical systems for which no physical analogy can be drawn safely. An outstanding formal analogy of this kind that is being used extensively, also in this thesis, is the analogy between a quantum field and an infinite collection of harmonic oscillators.

1.1.1 Classical Mechanics

To describe systems of finitely many degrees of freedom, we introduce n configuration variables or "generalized coordinates" $(q_1, ..., q_n)$ along with the n corresponding momenta $(p_1, ..., p_n)$ define the phase space $\mathcal{P}$, the collection of all possible values of generalized coordinates and momenta $y = (q_1, ..., q_n; p_1, ..., p_n)$. To set the kinematics of the theory we need to specify the possible states and observables. The states of a classical system correspond to points on the phase space $\mathcal{P}$. If the configuration variables live in a manifold $\mathcal{M}$ then it turns out that the phase space has the structure of the cotangent bundle $\mathcal{P} = T^*(\mathcal{M})$.

To describe the dynamics of the system, we need to specify the Hamiltonian which is a function over the phase space $H = H(q_1, ..., q_n; p_1, ..., p_n)$. The last ingredient needed is a "symplectic structure" Ω a non-degenerate, closed, 2-form $\Omega = \sum_m \mathrm{d}p_m \wedge \mathrm{d}q_m$ over the phase space $\mathcal{P}$. This is needed to formulate the dynamical flow on the phase space, and the algebraic structure of the observables as we will describe right below. Overall the phase

space, along with the specification of a Hamiltonian and a symplectic structure $(\mathcal{P}, H, \Omega)$, called a Hamiltonian structure, is all we need to formulate the kinematics and dynamics of a classical system (and to apply the quantisation procedure).

The set of observables of the classical theory consists of real valued smooth functions over the phase space $\mathcal{O} = \{f : \mathcal{P} \to \mathbb{R} : f \in C^\infty(\mathcal{P})\}$ that has the natural structure of an infinite dimensional vector space. We can use the symplectic structure to turn $\mathcal{O}$ into an algebra, and in particular a Poisson algebra. A Poisson algebra is a vector space with two multiplications, an associative one (which makes it an associative algebra) and a non associative one defined by the Lie bracket $\{\cdot, \cdot\}$ (which makes it a Lie algebra). The Lie bracket is a bilinear, antisymmetric map that satisfies the Jacobi identity. In a Poisson algebra the two multiplications are "compatible" if the Leibniz rule $\{fg, h\} = f\{g, h\} + \{f, h\}g$ is satisfied [9]. Then the Lie bracket is also a Poisson bracket, and the algebra is called a Poison algebra. The Poisson bracket is defined through the symplectic structure

$$\{f, g\} \coloneqq \Omega^{ab} \nabla_a f \nabla_b g. \tag{1.1.1}$$

and one can see that it satisfies all the desired properties mentioned above (and most crucially the Leibniz rule) through the properies of Ω and the covariant derivatives.

As we explained above, a general observable of the theory is a smooth function of generalized positions and momenta. Of course this makes positions and momenta a special case of a general observable, namely $f = q_\mu$ and $g = p_\nu$, for which (1.1.1) gives

$$\{q_\mu, p_\nu\} = \delta_{\mu\nu} \tag{1.1.2}$$

$$\{q_\mu, q_\nu\} = \{p_\mu, p_\nu\} = 0. \tag{1.1.3}$$

The generalized positions and momenta are called the fundamental observables of the theory since all other observables are functions of those. Nevertheless, the phase space is an abstract manifold whose elements we represent as $y = (q_1, ..., q_n; p_1, ...p_n)$ only in particular coordinates. We would like to have a coordinate independent way to express the fundamental observables and the relations (1.1.3). This we can do through the symplectic structure Ω [1] in the case that the configuration space $\mathcal{M}$ is a vector space, rather than a general (curved) manifold.

First let us notice that if $\mathcal{M}$ is a vector space, then the cotangent bundle of it $\mathcal{P} = T^*(\mathcal{M})$, the phase space, is also a vector space. The symplectic structure Ω is a two-form

[1] One of the difficulties in reading the book by Wald [3] is that all familiar (coordinate dependent) quantities and expressions are re expressed abstractly in terms of Ω.

so we would have to input two tangent vectors of $\mathcal{P}$ to get a number at each point. But if $\mathcal{P}$ is a vector space, we can identify it with its tangent space, and think of Ω as a map

$$\Omega : \mathcal{P} \times \mathcal{P} \to \mathbb{R}, \tag{1.1.4}$$

where we input directly points of $\mathcal{P}$ rather than its tangent space. This map is explicitly given by the formula $\Omega(y_1, y_2) = \sum_{\mu=1}^{n} [p_{1_\mu} q_{2_\mu} - p_{2_\mu} q_{1_\mu}]$. So we can view Ω as a symplectic structure of the vector space $\mathcal{P}$. Overall, a *linear* system is described by a symplectic *vector space* $(\mathcal{P}, \Omega)$, rather than a general symplectic manifold.

Now we can use Ω to write the linear coordinates of $\mathcal{P}$ in a rather abstract way. One would have to convince themselves that $\Omega(y, \cdot)$ is a general linear function on $\mathcal{M}$. Of course for a fixed y we get a linear function, since $\Omega(\cdot, \cdot)$ is a biliear map. For example, if we fix $y = (0, ...0; 1, ..., 0)$ then $\Omega(y, \cdot) = q_1$. The claim is that if we let y to run over $\mathcal{P}$ we get any possible linear function, and so any possible redefinition of the linear coordinates. So Wald suggests that we need to think of the "fundamental observables" as $\Omega(y, \cdot)$ and rewrite (1.1.3) abstractly as

$$\{\Omega(y_1, \cdot), \Omega(y_2, \cdot)\} = -\Omega(y_1, y_2) \tag{1.1.5}$$

Again there are two reasons for rewriting the familiar equations (1.1.3) as above, one being the coordinate independence of (1.1.5), rather than refering to particular q's and p's as the fundamental observables of the system. The second and most important reason is that (1.1.5) generalizes much more naturally for systems of infinite degrees of freedom. Crucially, in (1.1.3) the index μ is discrete, counting the degrees of freedom of the system. If we direcly try to generalize these relations for μ being a rather continuously infinite index, we run into ill-defined quanities. Relation (1.1.5) is much more straight forward to generalize in the infinite dimensional case, since it is only making use of the symplectic structure of the phase space (independently of the dimensionality of the configuration space). As we will see in the next section, also in the case that the configuration space consists of field configurations rather than generalized positions of n particles, relations (1.1.5) are well defined. As Laura Ruetsche points out [6] from the perspective of the quantisation procedure, the Poisson algebra structure of the fundamental observables written as in (1.1.5) is much cleaner for the purpose of quantising an infinite dimensional system, like a field.

There is one last step we need to do from the perspective of the classical theory, to end up with the structure that facilitates the quantisation procedure. This step only works for the case that the equations of motion are also linear, which means a quadratic Hamiltonian.

Let us consider the solutions of the equations of motion

$$\frac{\mathrm{d}y^\mu}{\mathrm{d}t} = \sum_{\nu=1}^{2n} \Omega^{\mu\nu} \frac{\partial H}{\partial y^\nu}. \tag{1.1.6}$$

For any couple of solutions $y_1(t), y_2(t)$ let us define the symplectic product

$$s(t) = \Omega(y_1(t), y_2(t)). \tag{1.1.7}$$

Using the equations of motion for a quadratic Hamiltonian it is easy to see that $\frac{\mathrm{d}s(t)}{\mathrm{d}t} = 0$ i.e. the symplectic product is preserved under time evolution, which makes the following trick possible. We can identify the space of solutions $\mathcal{S}$ with the phase space $\mathcal{P}$, viewing its elements as initial data to the equations of motion (since the choice of the initial time does not matter). In this way, the symplectic structure of the phase space naturally gives rise to a symplectic structure of the space of solutions

$$\Omega(y_1(t), y_2(t)) = \sum_{\mu=1}^{n} \left[\frac{\mathrm{d}q_{1_\mu}}{\mathrm{d}t} q_{2_\mu} - \frac{\mathrm{d}q_{2_\mu}}{\mathrm{d}t} q_{1\mu} \right] \tag{1.1.8}$$

where we used the equations of motion to eliminate the p's. As Wald puts it in [3] "the symplectic vector space structure of $(\mathcal{S}, \Omega)$ of the manifold of solutions for a linear dynamical system, is the fundamental classical structure that underlies the construction of the quantum theory of a linear field". It is important to realize what a huge benefit this is for linear systems, to appreciate the difficulties in quantizing non linear systems like field theories with interactions [7].

1.1.2 Quantum Mechanics

Moving from classical mechanics to quantum mechanics, the kinematical scenery changes drastically. Rather than points in a phase space, states will be represented by vectors in a Hilbert space $\mathcal{H}$, and rather than smooth functions over the phase space, observables are represented by self-adjoint operators acting on $\mathcal{H}$. At first glance this seems rather obscure, how would someone start from a function over the phase space manifold to define an operator over some Hilbert space? This can be succeeded by means of what is called a correspondence map

$$\hat{} : \mathcal{O} \to \hat{\mathcal{O}} \tag{1.1.9}$$

that maps classical observables $f \in \mathcal{O}$ to quantum observables $\hat{f} \in \hat{\mathcal{O}}$. As we will see there is a set of requirements that such a map should satisfy, and so it is generally non trivial to "put hats" on classical observables. The main structure that needs to be preserved is the algebra structure of the observables i.e. the Poisson bracket. In most textbooks or lecture notes of quantum mechanics we find something like

$$\{x, p\} = 1 \rightarrow [\hat{x}, \hat{p}] = \mathrm{i}\hbar. \tag{1.1.10}$$

This is a rather schematic way to represent the quantization procedure, but does not provide us with much insight per se. Let us try to shed some light to (1.1.10) with a mathematical remark found in [10, 9]. As we explained, classical observables form a commutative Poisson algebra, which is a vector space with two products, the commutative product and the Poisson bracket. If we consider a Poisson algebra that is *not* commutative we can easily see that

$$\{u_1 u_2, v_1 v_2\} = \{u_1, v_1\}[v_2, u_2] = [v_1, u_1]\{u_2, v_2\} \tag{1.1.11}$$

where u_1, v_1, u_2, v_2 are four elements of the algebra (in this case polynomials of x and p). We can see that by evaluating the very left hand side, following the properties of the Poisson bracket, being careful not to commute things around since the algebra is not commutative i.e. $u_1 u_2 \neq u_2 u_1$. If we stare at (1.1.1) we see that one way (turns out the only way! [10]) that it can be satisfied is that the commutator is proportional to the Poisson bracket

$$[v, u] = k\{u, v\} \tag{1.1.12}$$

We can see this as a compatibility condition between the two products of a non commutative Poisson algebra. If it were commutative $[v, u] = 0$ and so (1.1.1) would be trivially satisfied. The fundamental observables of quantum mechanics are non commutative objects [2] that need to satisfy

$$[\hat{x}, \hat{p}] = \mathrm{i}\hbar\{\hat{x}, \hat{p}\} = \mathrm{i}\hbar \tag{1.1.13}$$

where we have identified the proportionality constant k with the Planck constant (multiplied by i for hermiticity reasons).

But what guarantees that the Poisson algebra structure is respected for a pair of general observables, i.e., arbitrary polynomials of $\hat{x}$ and $\hat{p}$? Given three observables of the classical theory that satisy $\{f, g\} = h$ can we have something like $\frac{1}{\mathrm{i}\hbar}[\hat{f}, \hat{g}] = \hat{h}$? Let us denote the

[2] here the hats on x, p are only to denote that they are not number valued objects, no particular representation is implied.

quantisation map as Q in what follows, namely $Q(f) = \hat{f}$. There is a no-go theorem by Groenewold [8] which states that there is no map

$$Q : \mathcal{P}_{\leq 4} \longrightarrow \mathcal{D}(\mathbb{R}^n) \tag{1.1.14}$$

from the polynomials at least order 4 to the space $\mathcal{D}$ of the differential operators with polynomial coefficients (which is what we build out of powers of the "usual" quantum mechanical $\hat{x}$ and $\hat{p}$, represented as multiplication and differentiation operators) such that

1. $Q(1) = \mathbb{I}$
2. $Q(x) = \hat{x}$ and $Q(p) = \hat{p}$
3. $Q(\{f, g\}) = \frac{1}{\mathrm{i}\hbar}[Q(f), Q(g)] \qquad \forall f, g \in \mathcal{P}_{\leq 3}$

From the perspective of (1.1.1) we can see requirement 3 as

$$Q(\{f, g\}) = \Big\{Q(f), Q(g)\Big\} \tag{1.1.15}$$

where I have used bigger Poisson bracket on the right hand side to denote that it is defined over a different space than the Poisson bracket on the left. So one can rephrase the theorem as that there is no such morphism (structure preserving map) of the full Poisson algebra.

Luckily, if we choose for Q the so called Weyl quantisation procedure we can satisfy the requirements of the theorem, including

$$[Q_{\text{Weyl}}(f), Q_{\text{Weyl}}(g)] = \mathrm{i}\hbar\, Q_{\text{Weyl}}(\{f, g\}) \tag{1.1.16}$$

if at least one of the observables is not higher than second order (for a second order Hamiltonian for example, we can have the Poisson bracket with any other observable of the theory). Briefly, Weyl quantisation works as follows. If $f(x, p)$ is a classical observable then

$$Q_{\text{Weyl}}(f) = \int \tilde{f}(a, b) e^{\mathrm{i}(a\hat{x} + b\hat{p})} \mathrm{d}a\, \mathrm{d}b \tag{1.1.17}$$

where $\tilde{f}$ are the Fourier components of f. If $f \in \mathcal{L}^2(\mathbb{R}^{2n})$, i.e. an $\mathcal{L}^2$ function over the phase space, then $Q_{\text{Weyl}}(f)$ is guaranteed to be a Hilbert Schmidt operator over $\mathcal{L}^2(\mathbb{R}^n)$, the Hilbert space of the quantum theory. So we see that once we know how to represent the fundamental observables of the theory so that (1.1.13) is satisfied, we know how to represent a general observable (up to the restrictions of the theorem) through the Weyl

formula (1.1.22). Nevertheless, one has to carefully define the exponentiation $e^{\mathrm{i}(a\hat{x}+b\hat{p})}$ since $\hat{x}, \hat{p}$ are generally unbounded operators.

The fact that the operators $\hat{x}$ and/or $\hat{p}$ have to be unbounded, one can see as follows. First we see that the Hilbert space where the operators act has to be infinite dimensional, because if it was finite dimensional we would be able to take the trace of the commutation relation $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$ to take

$$\mathrm{tr}(\hat{x}\hat{p} - \hat{p}\hat{x}) = \mathrm{i}\hbar\,\mathrm{tr}\mathbb{1} \Rightarrow 0 = \mathrm{i}\hbar N. \tag{1.1.18}$$

This shows that the space cannot be finite dimensional, which allows for the possibility that $\hat{x}$,$\hat{p}$ are unbounded (there cannot be unbounded operators in a finite dimensional space). Nevertheless, one can construct bounded operators in an infinite dimensional space. We can see that $\hat{x}$ and $\hat{p}$ cannot both be bounded, as follows. Since $[\hat{x}^n, \hat{p}] = \mathrm{i}\hbar\hat{x}^{n-1}$ from the triangle inequality for the operator norm we have that

$$2|\hat{p}||\hat{x}|^n \geq n\hbar|\hat{x}|^{n-1} \Rightarrow 2|\hat{p}||\hat{x}| \geq n\hbar \;\; \forall n. \tag{1.1.19}$$

Since n can be arbitrarily large, at least either $\hat{x}$ or $\hat{p}$ ought to be unbounded. More rigorous arguments about this one can find in [11].

We saw that just because of the commutation relations (1.1.13) we have to represent position and/or momentum with unbounded operators. One of the problematic features of unbounded operators, is that they do not form an algebra naturally because multiplication e.g. $\hat{p}\hat{x}$ seen as composition of unbounded maps, is not guaranteed to work. What can go wrong is that once the first operator, $\hat{x}$, acts on a state $|\Psi\rangle$ the image might not be in the domain of the second operator,$\hat{p}$, and as a result we cannot make sense of the expression $\hat{p}\hat{x}\,|\Psi\rangle$ unless we suitably define the domain (self-adjoint unbounded operators are only densly defined).

This is not just some mathematical subtlety, and can have physical consequences. Imagine for example that someone claims that uncertainty principle is violated for a "physical" state, represented by a square integrable wavefunction over the configuration space. This could be considered paradoxical, but it could be that the state fails to be in the intersection of the relevant domains $D(\hat{x}\hat{p})\cap D(\hat{p}\hat{x})$, so the xp commutation relation $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = \mathrm{i}\hbar$ does not apply and as a result the xp uncertainty principle simply does not hold for such a state. There is a way out from this domain nightmare, which is to define the exponentiated version of the commutation relation, known as Weyl relations. In most cases the Weyl relations are equivalent to the usual commutation relations, but there are also simple examples where they are not (for a particle in a box only the Weyl relations are satified!

see [8]). First, we can suitably define the unitaries

$$U(a) = e^{-\mathrm{i}a\hat{x}} \tag{1.1.20}$$

$$V(b) = e^{-\mathrm{i}b\hat{p}} \tag{1.1.21}$$

since unitaries do not suffer the domain issues mentioned above [8, 6]. Then the xp commutation relation can be equivalently written as

$$U(a)V(b) = e^{\mathrm{i}ab}U(a)V(b). \tag{1.1.22}$$

More generally we can define a map from the phase space $\mathcal{P}$ to unitaries acting on $\mathcal{H}$ as

$$W(\gamma) := e^{\mathrm{i}\frac{ab}{2}}U(a)V(b) \tag{1.1.23}$$

where $\gamma = (a, b) \in \mathcal{P}$. This definition is such as we can associate the symplectic structure of the phase space $\mathcal{P}$ to a general form of the Weyl relations (1.1.22) as follows

$$\begin{aligned} W(\gamma_1)W(\gamma_2) &= e^{\mathrm{i}\frac{a_1b_1}{2}}U(a_1)V(b_1)e^{\mathrm{i}\frac{a_2b_2}{2}}U(a_2)V(b_2) \\ &= e^{\frac{\mathrm{i}}{2}[a_1b_1+a_2b_2+a_1b_2-a_1b_2+a_2b_1-a_2b_1]}U(a_1)e^{\mathrm{i}a_2b_1}U(a_2)V(b_1)V(b_2) \\ &= e^{\frac{\mathrm{i}}{2}(a_2b_1-a_1b_2)}e^{\frac{\mathrm{i}}{2}(a_1+a_2)(b_1+b_2)}U(a_1+a_2)V(b_1+b_2) \\ &= e^{\frac{\mathrm{i}}{2}\Omega(\gamma_1,\gamma_2)}W(\gamma_1+\gamma_2). \end{aligned}$$

In the abstract language that we introduced in the previous section, we think of $W(\gamma) = e^{\mathrm{i}\hat{\Omega}(\gamma,\cdot)}$ as the exponentiated version of a generalised fundamental observable. As explained in [6] it turns out that if a quantisation map satisfies

$$W(\gamma_1)W(\gamma_2) = e^{\frac{\mathrm{i}}{2}\Omega(\gamma_1,\gamma_2)}W(\gamma_1+\gamma_2) \tag{1.1.24}$$

$$W^\dagger(\gamma) = W(-\gamma) \tag{1.1.25}$$

this uniquely determines the kinematical pair $(\mathcal{H}, W(\gamma))$ i.e. the states and observables of the quantum theory.

One could worry that given a classical theory the quantization procedure can yield many non-equivalent quantum theories, which would mean that the quantisation method is ambiguous and many "different" quantum theories would correspond to the same classical theory. The Stone von Neumann theorem guarantees that this is not the case, by demonstrating the uniqueness of the quantisation procedure. Apart from avoiding the domain issues, another reason why we introduced the exponentiated version of the observables and

the commutation relations, is that the Stone von Neumann theorem is formulated in the language of the Weyl relations [8, 3, 6].

Concretely, the theorem states that if $(\mathcal{P}, \Omega)$ a finite dimensional symplectic vector space and $(\mathcal{H}, \hat{W}(\gamma))$, $(\mathcal{H}', \hat{W}'(\gamma))$ two strongly continuous, irreducible, unitary representations of the Weyl relations then there is a unitary U such that

$$U : \mathcal{H} \to \mathcal{H}' \tag{1.1.26}$$

$$U^{-1}\hat{W}'(\gamma)U = W(\gamma) \quad \forall \gamma \in \mathcal{P}. \tag{1.1.27}$$

This means that the two kinematical pairs of the quantum theory are unitarily equivalent. We will elaborate on the meaning and significance of this point, but first let us give a couple of definitions.

The irreducibility of the representation $\hat{W}(\gamma)$ means that there is no proper subspace of $\mathcal{H}$ that is invariant under the action of $\hat{W}(\gamma)$ for all $\gamma \in \mathcal{P}$. Given any proper subspace of the Hilbert space, there is y such that $W(y)$ maps some elements outside the space. For example, in the case of a particle over the real line, a subspace could be the wavefunctions of compact support over an interval. If we use the usual Schrodinger representation of the momentum operator we see that the Weyl operator $e^{\mathrm{i}a\partial_x}$ will displace the functions outside the interval.

Irreducibilty has an interesting consequence, that one can extend in the QFT case. When the Weyl relations are irreducibly represented in a Hilbert space $\mathcal{H}$ then for any state $|\Psi\rangle \in \mathcal{H}$ the span of $\{W(y)\,|\Psi\rangle\,, \forall y\}$ in dense in $\mathcal{H}$. This means that we can reach any state in the Hilbert by acting linear combinations of the Weyl operators on a given a state $|\Psi\rangle$, which makes any state cyclic under the action of the Weyl algebra.

Then let us clarify what is meant by unitary equivalence of two representations, as implied by the Stone Von Neumann theorem, and what is the connection to the physical equivalence. Given a Hilbert space $\mathcal{H}$ equipped with a set of unitaries, say V_a, that maps the space to itself, and similarly another space $\mathcal{H}'$ with a set $V'_a : \mathcal{H}' \to \mathcal{H}'$, the two are unitarily equivalent if there is a unitary $U : \mathcal{H} \to \mathcal{H}'$ such that $U^{-1}V'_aU = V_a \ \forall a$. Then it follows that the two states $|\Psi\rangle$ and $|\Psi'\rangle = U\,|\Psi\rangle$ are physically equivalent in terms of expectation values of any observables encoded in the operators V_a, V'_a, simply because

$$\langle\Psi'|V'_a|\Psi'\rangle = \langle\Psi'|U^{-1}UV_aU^{-1}U|\Psi\rangle = \langle\Psi|V_a|\Psi\rangle$$

From this point of view, unitary equivalence is sufficient for physical equivalence, but the question is whether it is also necessary. This becomes important for QFT, since crucially

the theorem applies for quantum theories for which the underlying symplectic vector space (phase space) is finite dimensional, and of course the phase space of a classical field theory is infinite dimensional. With the Stone von Neumann theorem being not applicable, one has reasons to worry about physical inequivalence of unitarily inequivalent representations.

1.2 Harmonic Oscillators

In this section, we apply the tools presented in section 1.1 to the quantization of a single harmonic oscillator, one more confirmation of the famous quote by Sidney Coleman that "The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction". In fact, the abstract reformulation of the quantization of a finite collection of harmonic oscillators is helpful when one wants to consider an infinite collection of harmonic oscillators, or a quantum field. The main goal of this section is to contrast the two methods that one can follow for quantizing a collection of harmonic oscillators, as presented in [3]. When extended to QFT, features of the two strategies are linked to the subtelties associated with the particle notion.

1.2.1 One harmonic oscillator, revisited

A single classical harmonic oscillator in one dimension is characterized by the Lagrangian $\mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2}q^2$ or the Hamiltonian $H = \frac{p^2}{2m} + \frac{\omega^2}{2}q^2$. The canonical quantisation of the system yields $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ and the usual representations of position and momentum as multiplication and differentiation operators in $\mathcal{L}^2(\mathbb{R})$. Given those, the Hamiltonian operator is also uniquely determined since it is quadratic and there are no ordering ambiguities. It is also very common to define the creation and annihilation operators

$$a := \sqrt{\frac{\omega}{2}}q + \mathrm{i}\sqrt{\frac{1}{2\omega}}p \tag{1.2.1}$$

in terms of which the Hamiltonian takes the form $H = \omega(a^\dagger a + \frac{1}{2}\mathbb{1})$. If q, p satisfy the Heisenberg algebra $[q, p] = \mathrm{i}\hbar$, then the creation and annihilation operators satisfy $[a, a^\dagger] = \mathbb{1}$, and their commutator with the Hamiltonian is $[H, a] = -\omega a$. In the Heisenberg picture the equations of motion are

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} = \mathrm{i}[H, a] = -\mathrm{i}\omega a(t) \tag{1.2.2}$$

and from inverting (1.2.1) we get that

$$q(t) = \sqrt{\frac{1}{2\omega}} \left(e^{-\mathrm{i}\omega t} a(0) + e^{-\mathrm{i}\omega t} a^\dagger(0) \right) \tag{1.2.3}$$

which is an expression that resembles the usual field decomposition that we will write down in the next section, only lacking the x dependence since q here is not a field operator.

This is all standard textbook presentation of the single harmonic oscillator, but let us comment on some features that will become important in the QFT case, by contrasting the formulation in terms of the observables q, p versus the formulation in terms of the (non hermitian) $a, a^\dagger$. In equation (2.2.13) we defined $a, a^\dagger$ in terms of q, p, but for the so called mode quantisation of QFT we will have to do the inverse. Notice that the representation of q, p as multiplication and differentiation operators has nothing to do with the Hamiltonian of the system, only with the commutation relations (i.e. the symplectic structure), but the very definition of $a, a^\dagger$ takes into account the form of the Hamiltonian. If someone gives us q as a multiplication operator in the Schrodinger picture, there is no way to figure out the Hamiltonian of the system, while if one gives us q as in (1.2.3) above, the Hamiltonian is determined up to a multiple of the identity [3]. This might sound trivial since q in the Heisenberg picture is time dependent and of course the Hamiltonian is determining the time evolution. But looking on the right hand side of (1.2.3), this means that the same "ansatz" in terms of creation and annihilaiton operators cannot be the solution to two different Hamiltonians.

To see this explicitly, let us assume a given $q(t)$ that satisfies the Hamilton equations of motion for two different Hamiltonians H, H'

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = \mathrm{i}[H, q(t)] = \mathrm{i}[H', q(t)] \tag{1.2.4}$$

which implies that $[H - H', q(t)] = 0$. The same holds for $p(t)$, which implies that the difference $H - H'$ commutes with all the Weyl operators (since they are exponentiated linear combinations of q and p). From the Schur's lemma [8] we have that the only operator that commutes with all Weyl operators has to be a multiple of the identity

$$[W(y), H - H'] = 0 \ \forall y \Rightarrow H - H' = a\mathbb{1} \tag{1.2.5}$$

so it is physically the same Hamiltonian up to a redefinition of the ground state energy.

1.2.2 From one to many

In this section we discuss the two possible strategies that one can follow to move from one to many harmonic oscillators. This is because one of the two strategies seems to be favourable when we will try to move from finitely many to infinite harmonic oscillators.

First we try to quantise a classical system of n harmonic oscillators. The classical configuration space of the system is given by the cartesian product $\mathcal{M}_1 \times ... \times \mathcal{M}_n$. It is a functional analytic property [12] that the Hilbert space that corresponds to this composite phase space, is the tensor product of the individual ones

$$\mathcal{H}[\mathcal{M}_1 \times ... \times \mathcal{M}_n] = \mathcal{H}[\mathcal{M}_1] \otimes \mathcal{H}[\mathcal{M}_2] \otimes ... \otimes \mathcal{H}[\mathcal{M}_n] \tag{1.2.6}$$

Given this decomposition for the state space we can extend the operators q_i, p_i in the obvious way, for example q_1 can be thought of as $q_1 \otimes \mathbb{1}_2 \otimes ... \otimes \mathbb{1}_n$ acting on the "big" Hilbert space (1.2.6). We also have the corresponding $a_i, a_i^\dagger$'s for each of the Hilbert spaces defining the energy basis as $a_i \Psi_0^i = 0$ and $H_i \Psi_{n_i}^i = \sqrt{\frac{1}{n_i!}}(a_i^\dagger)^{n_i} \Psi_0^i$. The total vacuum is defined as $a_i \Psi_0 = 0 \ \forall i \Rightarrow \Psi_0 = \otimes_{i=1}^n \Psi_0^i$ and $\{\Psi_{n_1}^1 \otimes \Psi_{n_2}^2 ... \otimes \Psi_{n_n}^n \ \forall n_1, ..., n_n\}$ is a basis for the total Hilbert space, and we are basically done. Once we have the quantum theory of one harmonic oscillator, we take the theory of many harmonic oscillators simply by the tensor product (1.2.6).

One could adapt this strategy also in the case that $n = \infty$. The problem is that in the case of infinite harmonic oscillators is that (1.2.6) would have to be an infinite tensor product which is hard to define. Most commonly infinite tensor products are defined as non separable Hilbert spaces [3] and this causes many mathematical issues (reducibilty of the representations) and physical issues (naively, measurement outcomes of "discrete" observables cannot be represented in the space, because this would correspond to a countable basis). This is why we might want to make use of an alternative way of quantizing the system of n harmonic oscillators that makes explicit use of the symplectic vector space structure of the space of solutions of the equations of motion, and which extends more naturally in the $n = \infty$ case.

In this section we review this method [3] for n being finite. First we need to complexify the space of solutions of the classical equations of motion

$$\mathcal{S}^c = \mathcal{S} \oplus \mathrm{i}\mathcal{S} \tag{1.2.7}$$

[3] even though there are alternative definitions in [13, 14] one of which is adopted in this thesis, and is reviewed in Appendix B.

The symplectic structure is inducing an "inner product" $(,) : \mathcal{S}^c \times \mathcal{S}^c \to \mathbb{C}$ defined as

$$(y_1.y_2) = -\mathrm{i}\Omega(y_1^*, y_2) \tag{1.2.8}$$

This is not quite an inner product in $\mathcal{S}^c$ because it fails to be positive definite. The most crucial step of this method is to find a subspace of $\mathcal{S}^c$ where this map is positive definite and can define an inner product. For $n = 1$ a harmonic oscilator of frequency ω, $\mathcal{S}^c$ is the span of $\{e^{\mathrm{i}\omega t}, e^{-\mathrm{i}\omega t}\}$ and we can easily see that (1.2.8) is positive for

$$(e^{-\mathrm{i}\omega t}, e^{-\mathrm{i}\omega t}) = -i\left((i\omega)e^{i\omega t}e^{-i\omega t}) - e^{i\omega t}(-i\omega)e^{-i\omega t}\right) = -i(2i\omega) = 2\omega > 0. \tag{1.2.9}$$

For n harmonic oscillators of frequencies $\omega_i, i = 1...n$ we define

$$\mathcal{H} = \mathrm{span}\{e^{-\mathrm{i}\omega_i t}\} \tag{1.2.10}$$

since for a general element of $\mathcal{H}$ the map $(,)$ is positive definite and it defines an inner product. We can use this inner product to complete the space, and turn $\mathcal{H}$ into the "one particle" Hilbert space that we are going to use for the Fock space construction. Note that here I am calling $\mathcal{H}$ a one particle space following the usual way of thinking as a Fock space as a "multi particle" Hilbert space, but Wald [3] refrains from using this term until he introduces detector models through which one could justify such an interpretation.

The Fock space is defined as the following tensor sum of symmetrized copies of $\mathcal{H}$

$$\mathcal{F}_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty}\left(\bigotimes^n \mathcal{H}\right), \tag{1.2.11}$$

and a general element of $\mathcal{F}_s(\mathcal{H})$ takes the form $\Psi = (\psi, \psi^{a_1}, \psi^{(a_1a_2)}, \dots)$, where we use the index notation to denote the symmetrization in each of the fixed particle sectors. Note that even though n goes all the way to infinity, each of the sectors is a finite tensor product and so we avoid the problems of the infinite tensor product. If $\mathcal{H}$ is separable, then $\mathcal{F}_s(\mathcal{H})$ is also separable.

Each Fock space construction comes with a set of creation and annihilation operators. To define those, we need to introduce an orthonormal basis $\{\xi_i(t)\}$ for $\mathcal{H}$

$$(\xi_i, \xi_i) = 1 \Rightarrow \xi_i(t) = \frac{e^{-\mathrm{i}\omega_i t}}{\sqrt{2\omega_i}}. \tag{1.2.12}$$

We associate creation and annihilation operators $a_i^\dagger = a^\dagger[\xi_i]$ to the elements of this basis as follows. The creation operator associated with ξ_i is the map $\hat{a}^\dagger(\xi_i) : \mathcal{F}_s(\mathcal{H}) \to \mathcal{F}_s(\mathcal{H})$ such that

$$\hat{a}^\dagger(\xi)\Psi = \left(0, \psi\xi^{a_1}, \sqrt{2}\xi^{(a_1}\psi^{a_2)}, \sqrt{3}\xi^{(a_1}\psi^{a_2,a_3)}, \dots\right). \tag{1.2.13}$$

It follows that the commutation relations are:

$$[a(\xi_i^*), a^\dagger(\xi_j)] = \xi_i^* \xi^j \mathbb{1} = \delta_{ij} \tag{1.2.14}$$

where $\xi^* \in \mathcal{H}^*$, the vector space with the same underlying set as $\mathcal{H}$ except with scalar multiplication modified to $c\overline{\times}\psi = \bar{c}\psi$ where $\psi \in \mathcal{H}$. [4]

Once we declare the Hilbert space to be $\mathcal{H}' = \mathcal{F}_s(\mathcal{H})$ (rather than (1.2.6)) we can ask what are the fundamental observables of the theory. Given a set of creation and annihilation operators, we *define* the position and momentum operators to be

$$q_i' \equiv \xi_i(t) a_i + \xi_i^*(t) a_i^\dagger \tag{1.2.15}$$

$$p_i' \equiv \frac{\mathrm{d}q_i}{\mathrm{d}t} \tag{1.2.16}$$

Let us recall from the previous section that such an expression for q', p' also determines the Hamiltonian H'. Overall, following the alternative quantisation procedure, we end up with the following kinematical structure $(\mathcal{H}'; q', p', H')$, where $\mathcal{H}' = \mathcal{F}_s(\mathcal{H})$ is the state space and q', p', H' are the observables that are associated to this construction. Then one can ask what guarantees the equivalence of this construction with the first one $(\mathcal{H}; q, p, H)$, where we simply formed the tensor product (1.2.6) of n harmonic oscillators.

For n being finite, the answer comes from the Stone von Neumann theorem. Through the definitions (1.2.21), (1.2.22) and the commutation relations (1.2.14) one can see that

$$[q_i', p_j'] = [\xi_i(t) a_i + \xi_i^*(t) a_i^\dagger, \dot{\xi}_j(t) a_j + \dot{\xi}_j^*(t) a_j^\dagger] \tag{1.2.17}$$

$$= \left(\xi_i(t)\dot{\xi}_j(t) - \xi_i^*(t)\xi_j(t)\right)[a_i, a_j^\dagger] = \mathrm{i}\hbar(\xi_i, \xi_j)[a_i, a_j^\dagger] = \mathrm{i}\hbar\delta_{ij}. \tag{1.2.18}$$

Since the q', p' satisfy the same canonical commutation relations as q, p, the unitary equivalence of the two constructions follows from the theorem.

1.2.3 From many to infinite

Since the distinction between the two strategies will become important for the next chapters, let us briefly summarize them before we move on.

[4] This notation is a bit confusing, but it turns out that $\mathcal{H}^*$ defined as above is isomorphic to the dual space of $\mathcal{H}$. Below we will use the fact that $\mathcal{H} \oplus \mathcal{H}^* = \mathcal{S}^c$.

1. If the state space of each oscillator is a Hilbert space $\mathcal{H}_j$, $j = 1...n$, then the Hilbert space of the total system is $\mathcal{H} = \otimes_j \mathcal{H}_j$. From the positions and momenta of each of the oscillators we can define the creation and annihilation operators $a_j := \sqrt{\omega_j/2} q_j + \mathrm{i}\sqrt{1/2\omega_j} p_j$ in terms of which we can solve for the fundamental observables

$$q_j(t) = \sqrt{\frac{1}{2\omega_j}} \left(e^{-\mathrm{i}\omega_j t} a_j + e^{-\mathrm{i}\omega_j t} a_j^\dagger \right) \tag{1.2.19}$$

$$p_j(t) = \frac{\mathrm{d}q_j(t)}{\mathrm{d}t} \tag{1.2.20}$$

In this constructions we know what are the fundamental observables of each party, and the creation and annihilation operators serve as tools for a convenient rewriting of the Hamiltonian and the equations of motion. Following Wald [3] we denote this construction as $(\mathcal{H}; q_j, p_j, H)$.

2. Here the important ingredients were the space of solutions of the classical equations of motion and its symplectic structure through which we defined the Hilbert space $\mathcal{H}$ out of which we built the Fock space $\mathcal{H}' = \mathcal{F}_s(\mathcal{H})$. For a given basis $\{\xi_i\}$ of $\mathcal{H}$ we defined creation and annihilation operators $a_i, a_i^\dagger$, $i = 1, ..., n$, in terms of which we defined the 'fundamental' observables to be

$$q'_i \equiv \xi_i(t) a_i + \xi_i^*(t) a_i^\dagger \tag{1.2.21}$$

$$p'_i \equiv \frac{\mathrm{d}q_i}{\mathrm{d}t} \tag{1.2.22}$$

Even though the $a_i, a_i^\dagger$ are neither fundamental nor observables, they are the primary objects of this construction in terms of which we define the observables of the theory. This construction we denote as $(\mathcal{H}'; q'_i, p'_i, H')$.

At a first look the constructions look very different even though the underlying classical structure is the same, and it is unclear how equations (1.2.19) and (1.2.20) relate to (1.2.21) and (1.2.22). Do they define "the same" observables? And what about the Hilbert spaces, are they "the same" space? After checking that q'_i, p'_i satisfy the Heisenberg algebra, we could be sure that there is a unitary that maps construction $(\mathcal{H}; q_j, p_j, H)$ to $(\mathcal{H}'; q'_i, p'_i, H')$. For the two state spaces this concretely means that $\exists\, U : \mathcal{H} \to \mathcal{H}'$ that maps one space to the other. So the tensor product structure and the Fock space structure are isomoprhic, which in what follows we denote as

$$\mathcal{F}_s(\mathcal{H}) \simeq \otimes_j \mathcal{H}_j \tag{1.2.23}$$

To appreciate the differences (and the equivalences) of the two constructions one has to keep in mind the physical and mathematical meaning of the indices i and j that are being used. In the first strategy, the index j counts the parties, the individual harmonic oscillators, and so the corresponding observables q_j, p_j are "local" having the interpretation of position and momentum of each individual oscillator. The second construction is more abstract and somehow more "global", since the index i is counting the basis elements of the subspace $\mathcal{H}$ and has nothing to do directly with the initial local parties.

The observables of the construction $(\mathcal{H}'; q'_i, p'_i, H')$ are generalized "positions" and "momenta" associated to "modes" of the classical solution space, and each different basis of the solution space corresponds to a different reshuffling of the initial local degrees of freedom. If the initial composition of the total system implied in strategy one is to be given physical significance, with j counting the local parties, then it is unclear how to physically access the $i-$ degrees of freedom defined in strategy two. One should be able to measure the positions and momenta of the individual harmonic oscillators q_j, p_j locally, but one would need access to all of them to measure q'_i, p'_i. One could even argue that the second method provides us with a mathematically equivalent but totally unphysical description of the system.

It is exactly this discussion that underlies the debate about the local versus the global nature of particle states in QFT, which is why I am trying to be over explicit about the two strategies. The first strategy is not so naturally generalizable for infinte degrees of freedom, but could capture the local features of particle states. The second strategy is usually pursued because it is mathematically well defined, but it is responsible for the somehow global features of particle states in QFT that have been proven puzzling and quite unphysical, as best discussed in [1].

Finally, we will rewrite the observables q', p' of the second strategy in a yet more abstract way that the neglects the choice of basis in $\mathcal{H}$. We have been using implicitly the following two decompositions of the complexified space of solutions. One comes from the definition of complexification

$$\mathcal{S}^c = \mathcal{S} \oplus \mathrm{i}\mathcal{S} \tag{1.2.24}$$

The other one comes from the fact that each element of the complexified solution space has a unique decomposition $z = z^+ + z^-$ where $z^+ \in \mathcal{H}$ and $z^- \in \mathcal{H}^*$, namely

$$\mathcal{S}^c = \mathcal{H} \oplus \mathcal{H}^* \tag{1.2.25}$$

This allows the definition of a projection map $K : \mathcal{S} \to \mathcal{H}$ which extracts the positive frequency part of the complexified solution. Then one can define the operator that represents

the classical observable $\Omega(y,\cdot)$ as

$$\hat{\Omega}(y,\cdot) := \mathrm{i}a(Ky^*) - \mathrm{i}a^\dagger(Ky) \tag{1.2.26}$$

This is well defined through (1.2.13) since $Ky \in \mathcal{H}$. Again, we define the observables of the theory in terms of the creation and annihilation operators that one associates to *any* element of the Hilbert space $\mathcal{H}$. In this sense, equation (1.2.26) is the basis independent version of (1.2.21) since it is not referring to a particular basis of $\mathcal{H}$.

All that this definition requires is the map K, a split of the complexified solution space as in (1.2.25). Above we defined this split through the positive frequency solutions (1.2.10) but this split is not unique, and we need to know what specifies this choice for a couple of reasons. First, the split guided by the sign of the frequencies is not available for QFT in a general non stationary spacetime, and second, since the split is not unique also in flat spacetime, we might want to make use of this freedom to solve a particular problem. As we will review in the third chapter the authors of [4] are making use of this ambiguity to define a notion of local quanta, since Hegerfeld's theorem (that will be reviewed below) shows that defining the "one particle" Hilbert space $\mathcal{H}$ as the space of purely positive frequencies/energies creates the problem of instantaneous spreading. In general, the freedom of choosing $\mathcal{H}$ is exactly what is creating the particle number ambiguity in QFT and is potentially the source of unitarily inequivalent representations. This is why we would like to have a characterization of the freedom available in this choice.

Given $(\mathcal{S}^c, \Omega)$ we can choose any $\mathcal{H}$ as long as it satisfies this list of properties:

1. The map $(y_1.y_2) = -\mathrm{i}\Omega(y_1^*, y_2)$ is positive definite in $\mathcal{H}$.
2. The complexified space of solution is the union of the span of $\mathcal{H}$ and the span of $\mathcal{H}^*$.
3. $(z^+, z^-) = 0,\ \forall z^+ \in \mathcal{H}, z^- \in \mathcal{H}^*$

Condition 1 emphasizes that all we need is the positive definiteness of that map to compete the space under the norm induced by this inner product. In that sense all we need to ensure is the positive norm in $\mathcal{H}$, which is not only the case for the positive frequencies solutions. Conditions 2 and 3 together define the split $\mathcal{S}^c = \mathcal{H} \oplus \mathcal{H}^*$.

Given two Hilbert spaces $\mathcal{H}, \mathcal{H}'$ that satisfy this list of properties we would form the kinematical pairs of states and observables

$$(\mathcal{F}[\mathcal{H}]; \hat{\Omega}(y,\cdot)) \tag{1.2.27}$$

$$\hat{\Omega}(y,\cdot) = \mathrm{i}a(Ky^*) - \mathrm{i}a^\dagger(Ky) \tag{1.2.28}$$

and

$$(\mathcal{F}[\mathcal{H}']; \hat{\Omega}'(y, \cdot)) \tag{1.2.29}$$

$$\hat{\Omega}'(y, \cdot) = \mathrm{i}a(K'y^*) - \mathrm{i}a^\dagger(K'y) \tag{1.2.30}$$

where $K : \mathcal{S}^c \to \mathcal{H}$ and $K' : \mathcal{S}^c \to \mathcal{H}'$, since each valid choice of the single particle space defines a different projection map. Through the previous discussion we established that each of those construction would be unitarily equivalent to the tensor product construction (strategy 1) because of the Stone von Neumann theorem, and so they are unitarily equivalent to each other.

Given $\mathcal{H}$ we denote as $(\cdot, \cdot)_\mathcal{H}$ the same map as realised in $\mathcal{H}$, and this is indeed positive definite. Using the definition above for $z_1 = Ky_1, z_2 = Ky_2 \in \mathcal{H}$ we get

$$(Ky_1, Ky_2)_\mathcal{H} = -\mathrm{i}\Omega(\overline{Ky_1}, Ky_2) \tag{1.2.31}$$

which if we brake in real and imaginary parts we easily see that

$$\mathrm{Re}(Ky_1, Ky_2)_\mathcal{H} = \mathrm{Im}\Omega(\overline{Ky_1}, Ky_2) \tag{1.2.32}$$

$$\mathrm{Im}(Ky_1, Ky_2)_\mathcal{H} = -\mathrm{Re}\Omega(\overline{Ky_1}, Ky_2) = -\frac{\Omega(y_1, y_2)}{2} \tag{1.2.33}$$

Now we define a real inner product $\mu : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ in the uncomplexified space of solutions $\mathcal{S}$ through the real part as follows

$$\mu(y_1, y_2) := \mathrm{Re}(Ky_1, Ky_2)_\mathcal{H} = \mathrm{Im}\Omega(\overline{Ky_1}, Ky_2). \tag{1.2.34}$$

The definition (1.2.34) above depends on the map K since we define μ through the real part of the inner product (1.2.33) which has explicit K dependence. On the other hand, the imaginary part does not depend explicitly on K (it depends only on Ω!). Overall we have that

$$(Ky_1, Ky_2)_\mathcal{H} = \mu(y_1, y_2) - \frac{\mathrm{i}}{2}\Omega(y_1, y_2) \tag{1.2.35}$$

Given Ω, all allowed choices for μ imply a different projection map K such that the equation above is satisfied.

As discussed in [3], by suitably saturating the Schwartz inequality that one can write from (1.2.35) we get the following compatibilty condition between Ω and μ

$$\mu(y_1, y_1) = \frac{1}{4}\mathrm{max}_{y_2 \neq 0}\frac{[\Omega(y_1, y_2)]^2}{\mu(y_2, y_2)} \tag{1.2.36}$$

We see that given the symplectic structure Ω the inner product μ is not uniquely determined. We need to keep in mind that (1.2.34) is the definition of μ *given* $\mathcal{H}$, but conversely any choice of μ that satisfies the compatibility condition (1.2.36) induces a valid choice of $\mathcal{H}$.

To summarize, we started from the complexified space of solutions $(\mathcal{S}^c, \Omega)$ and we end up with the space of real solutions $(\mathcal{S}, \mu)$ where μ needs to be compatible with Ω as in (1.2.36), but yet not uniquely defined. When the choice of the single particle Hilbert space $\mathcal{H}$ is not guided by the time translation symmetry of the system, it is possible and preferable to specify $\mathcal{H}$ through the choice of μ. In fact, this is what one does in QFTCS and in the algebraic approach.

1.3 Quantum Field Theory reformulated

In this section we make use of the tools we inroduced above, to quantize a classical field theory. In the first subsection we briefly review the standard textbook presentation of QFT which does not provide us with well defined observables. In the second subsection we follow [3] in a reformulation of QFT that avoids ill defined quantities and expressions, and builds towards the algebraic approach.

1.3.1 A quantum field, take1

The action of a Klein Gordon field in Minkowski space is

$$S = -\frac{1}{2}\int \left(\partial_a\phi\partial^a\phi + m^2\phi^2\right) \mathrm{d}^4x \tag{1.3.1}$$

If we put the field in a box and use the Fourier transform $\phi(t,x) = \frac{1}{\sqrt{L^3}}\sum_k \phi_k(t)e^{\mathrm{i}kx}$ we can rewrite the Lagrangian density as

$$\mathcal{L} = \sum_k \frac{1}{2}|\dot{\phi}(k)|^2 - \frac{1}{2}\omega_k^2|\phi(k)|^2 \tag{1.3.2}$$

where $\omega_k^2 = k^2 + m^2$. This looks like the Langrangian of a system of infinitely many harmonic oscillators. Since $\phi_k^\dagger = \phi_{-k}$ the $\phi(k)$'s cannot be directly thought of as the generalized positions of these oscillators. Nevertheless, we can define creation and annihilation

operators such that

$$\Phi_k = \frac{1}{\sqrt{2\omega_k}}(a_k + a^\dagger_{-k}) \tag{1.3.3}$$

Following the second strategy, we can define $\mathcal{H}$ to be spanned by

$$\psi_k = \frac{e^{\mathrm{i}kx - \mathrm{i}\omega_k t}}{\sqrt{2\omega_k}\sqrt{L}^3} \tag{1.3.4}$$

and the Hilbert space is the Fock space $\mathcal{F}_s(\mathcal{H})$ and is well defined despite $\mathcal{H}$ being infinite dimensional. Nevertheless, the operator

$$\phi(x,t) = \sum_k \psi_k(x,t)a_k + \psi^*_k(x,t)a^\dagger_k \tag{1.3.5}$$

which is the analogue of (1.2.3) is not well defined because the sum does not converge. The theory does not admit a well defined observable that represents the field amplitude at each point. So strategy two succeeded in giving us a well defined state space in the form of the Fock space (rather than the infinite tensor product of strategy one) but failed to give us directly well defined observables.

1.3.2 A quantum field, take2

To make use of the reformulation that we introduced in section 1.2 in the QFT case, we first need to define the classical phase space of a field theory. From the Langrangian (1.3.2) we can specify the conjugate momentum

$$\pi(x) = \frac{\delta S}{\delta \dot{\phi}} = \dot{\phi} \tag{1.3.6}$$

A point in the phase space is a specification of $(\phi(x), \pi(x))$ on a three-dimensional hypersurface Σ_0 that corresponds to some global time $t = 0$. To fully specify this we need to decide what is the function space where the classical field configurations live in. Since the configurations are specified at some time $t = 0$ we should really think of the points of the phase space as the initial data that we input to the equations of motion to get the solution space $\mathcal{S}$. The usual choice is to pick initial data that are smooth and of compact support on Σ_0, namely

$$\mathcal{P} = \{[\phi(x), \pi(x)] \text{ s.t. } \phi, \pi : \Sigma_0 \to \mathbb{R}; \phi, \pi \in C^\infty_0(\Sigma_0)\} \tag{1.3.7}$$

The phase space is equipped with a symplectic structure

$$\Omega\{[\phi_1,\pi_1],[\phi_2,\pi_2]\} = \int_{\Sigma_0} \mathrm{d}^3x(\pi_1\phi_2 - \pi_2\phi_1)$$

which is well defined since ϕ, π's are compactly supported (another choice that guarantees that the integral above converges in the Schwartz space but this turns out to be hard to define in curved spacetimes [3]). The symplectic structure also provides us with a definition of the Poisson bracket

$$\{\Omega([\phi_1,\pi_1],\cdot),\Omega([\phi_2,\pi_2],\cdot)\} = -\Omega([\phi_1,\pi_1],[\phi_2,\pi_2]) \tag{1.3.8}$$

where $\Omega([\phi_1,\pi_1],\cdot),\Omega([\phi_2,\pi_2],\cdot)$ are linear functions over the phase space. How does (2.3.40) relate to the usual $\{\phi(x_1),\pi(x_2)\} = \delta(x_1 - x_2)$? Let us choose in particular $[\phi_1,\pi_1] = [0, f_1]$ and $[\phi_2,\pi_2] = [f_2, 0]$. Then the function $\Omega([0, f_1],\cdot)$ evaluated at a point $[\phi,\pi]$ yields $\Omega([0, f_1],[\phi,\pi]) = \int \mathrm{d}x f_1\phi$ and (2.3.40) becomes

$$\Big\{\int \mathrm{d}x f_1\phi, \int \mathrm{d}x f_2\pi\Big\} = \int \mathrm{d}x f_1 f_2 \tag{1.3.9}$$

This already looks more familiar, but if we choose $f_1 = \delta(x - x_1)$ and $f_2 = \delta(x - x_2)$ (which strictly we cannot, since delta functions do not belong in C_0^∞) we naively retrieve the familiar Poisson bracket

$$\{\phi(x_1),\pi(x_2)\} = \delta(x_1 - x_2) \tag{1.3.10}$$

The moral drawn by the abstract notation is that once we define the observables as linear functions of the form $\Omega\{[\phi_0,\pi_0],\cdot\}$, where $[\phi_0,\pi_0]$ a reference point in phase space, the evaluation map at a point is not included in this space. We see tat even at a classical level, strictly we cannot think of the field amplitudes (at points) as the observables of the theory. This goes through once we quantize the theory, where we will see that well defined observables have to be "smeared over regions". For the quantized theory we want to define observables $\hat{\Omega}$ that obey

$$\Big[\hat{\Omega}\{[\phi_1,\pi_1],\cdot\},\hat{\Omega}\{[\phi_2,\pi_2],\cdot\}\Big] = -\mathrm{i}\Omega([\phi_1,\pi_1],[\phi_2,\pi_2])\mathbb{1} \tag{1.3.11}$$

This substitutes the familiar commutation relations $[\hat{\phi}(x_1,t),\hat{\phi}(x_2,t)] = \mathrm{i}\delta(x_1 - x_2)$ since as we saw in the previous section $\hat{\phi}(x,t)$ is not well defined. By the end of this section, we will view (1.3.11) as a smeared version of the latter.

As we saw previously, the symplectic product is conserved under time evolution so we can freely input solutions to $\hat{\Omega}$ rather than phase space points (initial data). The usual strategy is that we split the complexified space of solutions in the positive and negative frequency parts. Let as call $\mathcal{S}^{c+}$ the span of the positive frequency solutions then we define the Klein Gordon inner product

$$(\phi^+, \psi^+)_{KG} = -\mathrm{i}\Omega(\phi^{+*}, \psi^+) \tag{1.3.12}$$

which is positive definite in $\mathcal{S}^{c+}$, so we can use it to complete it, to end up with the Hilbert space $\mathcal{H}$, and the Fock space $\mathcal{F}_s(\mathcal{H})$. Then the map $\psi \to \psi^+$ defines the map K in terms of which we define the observables

$$\hat{\Omega}(\psi, \cdot) := \mathrm{i}a(K^*\psi^*) - \mathrm{i}a^\dagger(K\psi). \tag{1.3.13}$$

Next we discuss what specifies the map K i.e. the possible choices for $\mathcal{H}$ in QFT, following the lecture notes by Luis Garay [15]. In the case that the complexified space of solution is infinite dimensional what we need to define is the so called complex structure. The complex structure is a linear map $\mathcal{S}_c \times \mathcal{S}_c \to \mathcal{C}$ that has the following properties:

1. Defines a positive definite inner product in $\mathcal{S}_c$ through the Klein Gordon product[5] $(\cdot, \cdot) \equiv \mathrm{i}(J\cdot, \cdot)_{KG}$.
2. The map J is a symplectomorphism i.e. $(J\psi_1, J\psi_2) = (\psi_1, \psi_2)$.
3. $J^2 = -\mathbb{1}$.
4. $J^\dagger = -J$.

We can make sense of each of these properties in view of the rest ones. For example, given the definition of the map $(\cdot, \cdot)$ as in 1, and demanding that J is a symlectomorphism, we derive property 3 as follows

$$(J\psi_1, J\psi_2) = \mathrm{i}(J^2\psi_1, J\psi_2)_{KG} = -\mathrm{i}(J\psi_2, J^2\psi_1)_{KG} = (\psi_2, -J^2\psi_1) = (\psi_1, -J^2\psi_2) \tag{1.3.14}$$

where we have used the antisymmetry of the Klein Gordon inner product and the symmetry of the inner product $(\cdot, \cdot)$. So requiring that J is a symplectomorphism means

$$(J\psi_1, J\psi_2) = (\psi_1, -J^2\psi_2) = (\psi_1, \psi_2) \Rightarrow J^2 = -\mathbb{1}. \tag{1.3.15}$$

[5] which we defined above through the symplectic structure as $(y_1.y_2)_{KG} = -\mathrm{i}\Omega(y_1^*, y_2)$.

The fact that J squares to minus the identity operator is the most defining property of this map, and this is actually why it is called a complex structure. We can also see easily how property 4 follows

$$(J^\dagger\phi_1, \phi_2) = (\phi_1, J\phi_2) = (J\phi_1, J^2\phi_2) = (J\phi_1, -\phi_2) = (-J\phi_1, \phi_2) \Rightarrow J^\dagger = -J \tag{1.3.16}$$

This implies that the operator $\mathrm{i}J$ is self-adjoint and so splits the space in two orthogonal subspaces $\mathcal{H}, \mathcal{H}^*$, which was the point of the whole construction. Now, we can write the projector K that projects a complex a solution into $\mathcal{H}$ as

$$K = \frac{1}{2}(\mathbb{1} - \mathrm{i}J) \tag{1.3.17}$$

One can show the following relation between the Klein Gordon product and the maps K, J

$$(\psi_1, \psi_2)_{KG} = 2(K\psi_1, K\psi_2) - (\psi_1, \psi_2) \tag{1.3.18}$$

where J is hidden in the definition of K and the very definition of $(\cdot, \cdot)$. Note the the left hand side has nothing to do with the choice of J since the Klein Gordon inner product only depends on the symplectic structure Ω. Through the relation above we can see that the Klein Gordon inner product is in fact positive in $\mathcal{H}$. Given a $\phi \in \mathcal{H}$ we know that $K\phi = \phi$ since K is defined as a projector that takes as in $\mathcal{H}$, and so

$$(\phi, \phi)_{KG} = 2(K\phi, K\phi) - (\phi, \phi) = (\phi, \phi) > 0$$

In other words, within $\mathcal{H}$ the Klein Gordon product coincides with the inner product $(\cdot, \cdot)$ and as a result it is positive definite. Note though that one can find element that do not belong in $\mathcal{H}$ and still enjoy positive Klein Gordon norm, which implies that the map J can be defined in multiple ways, each of which would define a diferent $\mathcal{H}$.

In any case, once we are given a particular complex structure $\mathcal{J}$ we have all the ingredients to build the states and observables of the QFT. Since J specifies K and $\mathcal{H}$, we use $\mathcal{H}$ to build the Fock space $\mathcal{F}_s(\mathcal{H})$, and K to define our fundamental field observables as in (1.3.13).

To conclude this part, I will just mention that there is a relation between J and μ the real inner product defined in section 1.2.3. It holds that

$$(\psi_1, \psi_2)_{KG} = -\mathrm{i}(\psi_1, J\psi_2) = 2\mathrm{i}\mu(\psi_1, J\psi_2) \tag{1.3.19}$$

which can be seen as a compatibility condition between μ, J and Ω. By following this path, one can reduce the choice of J to a choice of an inner product μ, in the spirit of

the section 1.2.2. Overall, both in the finite and infinite dimensional case the choice of the single particle sector $\mathcal{H}$ boils down to specifying a real inner product μ for the space of solutions.

Finally, I would like to show in what sense the fundamental observables as defined abstractly in (1.3.13) can be interpreted as smeared field observables i.e. the spacetime average of the Heisenberg operator that would represent the field amplitude at a point. Then we can make sense of the field amplitudes as operator valued distributions, since they yield an operator only under integration. For this we need to establish a map between the space of test functions on Minkowski space

$$\mathcal{T} = C_0^\infty(\mathbb{R}^4) \tag{1.3.20}$$

and the classical solutions that go in the defintion (1.3.13). Recall that the space of solutions is defined as solutions of initial data $[\phi, \pi]$ that live in $C_0^\infty(\Sigma_0)$, so we wish to establish a map from $C_0^\infty(\mathbb{R}^4)$ to $C_0^\infty(\Sigma_0)$. This we can do quite naturally through the equations of motion, using the following result from the theory of partial differential equations.

Let $f \in \mathcal{T}$ be the source of the Klein Gordon equations, then the advanced and retarted Green's functions satisfy

$$(\Box - m^2)Af = f \tag{1.3.21}$$
$$(\Box - m^2)Rf = f \tag{1.3.22}$$

where Rf vanishes in the causal future, and Af vanishes in the causal past of the support of f. Then the difference $Ef \equiv Af - Rf$ satisfies the homogeneous Klein Gordon equation, and so belongs to $\mathcal{S}$. The initial data of the solution Ef lies in $C_0^\infty(\Sigma_0)$ where $\Sigma_0 \in \text{supp}f$. This establishes the map

$$E : \mathcal{T} \to \mathcal{S} \tag{1.3.23}$$

namely $\psi = Ef$ as we described just above. One can see that this is an onto map, that has the following property

$$\int \mathrm{d}^4x \, \psi f = \Omega(Ef, \psi) \tag{1.3.24}$$

where Ω the symplectic structure of $\mathcal{S}$. This is exactly the property that will allow us the "smeared field operator" interpretation in the quantum case where Ω will be represented by $\hat{\Omega}$.

To see how (1.3.24) comes about let us consider a test function f such that $\text{supp} f = 0$ for $t \notin [t_1, t_2]$. Using (1.3.21) and integrating by parts twice, we get

$$\int \mathrm{d}^4x \psi f = \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}^3x \, \psi(\Box - m^2)Af = \left(\int \mathrm{d}^3x (\partial_t \psi) Af - \psi \partial_t (Af) \right)_{t_1}^{t_2}$$

which corresponds to the boundary terms from the integration by parts over time. The rest of the boundary terms vanish, as well as the integral $\int \mathrm{d}^4x Af(\Box - m^2)\psi$ since $\psi \in \mathcal{S}$. Next we use the fact that Af vanishes at $t = t_2$ to write the above as

$$\int \mathrm{d}^4x \psi f = \int_{\Sigma_{t_1}} \mathrm{d}^3x \big(\psi \partial_t (Af) - (\partial_t \psi) Af \big) = \Omega(Af, \psi). \tag{1.3.25}$$

Last step is to use that Rf vanishes at $t = t_1$ and so $Af = Ef$ which gives us (1.3.24).

At the classical level we see that the classical observable $\Omega(Ef, \cdot)$ corresponds to taking the spacetime average with weight f. Motivated by that we define

$$\hat{\phi}(f) \equiv \hat{\Omega}(Ef, \cdot). \tag{1.3.26}$$

If we want to use the corresponding $a, a^\dagger$ we know that

$$\hat{\Omega}(Ef, \cdot) = \mathrm{i}a(K^*(Ef)^*) - \mathrm{i}a^\dagger(K(Ef)). \tag{1.3.27}$$

Equation (1.3.26) is the definition of the smeared field operators over the spacetime region that the test function f is compactly supported.

1.4 Fields *and* Particles

The strategy that we followed so far from [3], is that one needs to extend the methods for quantizing mechanical systems (meaning systems of finitely many degrees of freedom) to field systems (of infinitely many degrees of freedom). This is very different in spirit than the usual narrative that a quantum mechanical description fails for a relativistic *particle*, and somehow relativity enforces a *field* theoretic description. As Wald puts it "quantum field theory is the quantum theory of a field, not a theory about 'particles'.." and in fact we have not appealed to the particle notion so far. Even the Fock space construction was not motivated by allowing multi-particle states, and $\mathcal{H}$ is not called a single particle Hilbert space by Wald. The Fock space is only presented as the mathematical construction that can facilitate the mathematical subtleties of systems of infinite degrees of freedom (for free

theories) and any interpretation in terms of the Fock states as "particle" states needs to be justified.

The proposed justification is rather operational, and it comes from the necessity to describe the particle like phenomenology of detection experiments. For this one needs to suggest how to couple the field to another (detector) system i.e., propose a detector model. The detector model that is most commonly used in Relativistic Quantum Information (RQI) was first proposed by Unruh in [16] where he considers two cases, a field coupled to another field and a field coupled to a quantum mechanical particle in a box. In the second case, we want to couple a (relativistic) field system, which is described by field operators $\hat{\Phi}(x)$, to a (non-relativistic) particle that is described by a position operator $\hat{Q}$. How can we couple the position operator $\hat{Q}$ of a particle to a quantum field? In a follow up paper by Wald and Unruh [17] they state that this can be achieved through the interaction Hamiltonian

$$H_{int} = \epsilon(t) \int_{\tilde{\Sigma}_t} \hat{\Phi}(x)\delta(\tilde{x} - \hat{Q})\mathrm{d}\tilde{V} \tag{1.4.1}$$

The field is defined over the same physical space where the particle lives, or mathematically the spectrum of $\hat{Q}$ takes the values x that are in the input of the field operator. But then we need a notion of where and when this interaction happens. "When" is explicitly stated in (1.4.1) through the function $\epsilon(t)$ whose compact support defines the duration of the interaction. "Where" the interaction happens is hidden in what is meant by the mysterious δ function in (1.4.1), as we will explain just below. Also note that the x in the field operator corresponds to any point of the manifold, while $\tilde{x} \in \tilde{\Sigma}_t$, the spatial hypersurface defined at time t.

Now we need to make sense of the δ function that appears in the interaction Hamiltonian. First, notice that in the input we are subtracting an operator from a number, so we rather mean $\delta(\tilde{x}\mathbb{1} - \hat{Q})$. This object can be defined as mapping real valued functions to operator valued functions $f(x) \to f(\hat{x})$ in the distributional sense

$$f(\hat{Q}) = \int \mathrm{d}x f(x)\delta(x - \hat{Q}). \tag{1.4.2}$$

We know how to make sense of a function of an operator through the spectral theorem

$$g(\hat{Q}) = \int \mathrm{d}q\, g(q)\, |q\rangle \langle q|\,. \tag{1.4.3}$$

Just by comparing the two last equations someone could think of this δ function as representing a projector for every q as in $\delta(q - \hat{Q}) = |q\rangle \langle q|$. In any case, equation (1.4.4)

generalizes more naturally than (1.4.3) in the case that f is operator valued. If we want to write (1.4.3) with another hat on f we take

$$\hat{f}(\hat{Q}) = \int \mathrm{d}q \hat{f}(q) \otimes |q\rangle \langle q| \,. \tag{1.4.4}$$

The tensor product in the middle is necessary so that this is an operator that acts on the composite system $\mathcal{H}_D \otimes \mathcal{H}_F$ of the detector and the field. One can think of this as an operator that correlates the "field amplitude at each point q" with the corresponding projector $|q\rangle \langle q|$ for the quantum mechanical system. If (1.4.4) seems puzzling, we can equivalently write (1.4.4) simply with another hat on the function

$$\hat{f}(\hat{Q}) = \int \mathrm{d}x \hat{f}(x) \delta(x - \hat{Q}) \tag{1.4.5}$$

Then we can compactly write the interaction Hamiltonian (1.4.1) as

$$H_{int} = \epsilon(t) \hat{\Phi}(\hat{Q}). \tag{1.4.6}$$

This looks different that the usual interaction Hamiltonian [3]

$$H_{int} = \epsilon(t) \int \mathrm{d}^3 x \, \hat{\Phi}(x) \left(F(x) \hat{A} + \text{h.c.} \right) \tag{1.4.7}$$

where $\hat{A}$ some observable of the detector and $\epsilon(t) \in C_0^\infty(\mathbb{R})$ and $F(x) \in C_0^\infty(\mathbb{R}^3)$, the smearing functions that are non zero only when/where the interaction happens. Now we would like to bring the interaction Hamiltonian (1.4.1) in the form (1.4.7). For that we need to represent the δ function in the Hilbert space of the detector using, for example, the energy eigenbasis $\{|E_i\rangle\}$ as follows

$$\delta(x - \hat{Q}) = \sum_{ij} \langle E_i | \delta(x - \hat{Q}) | E_j \rangle \, |E_i\rangle \langle E_j| \tag{1.4.8}$$

$$= \sum_{ij} \langle E_i | \delta(x - \hat{Q}) \left(\int \mathrm{d}q \, |q\rangle \langle q| \right) |E_j\rangle \, |E_i\rangle \langle E_j| \tag{1.4.9}$$

$$= \sum_{ij} \langle E_i | \int \mathrm{d}q \, \delta(x - q) \, |q\rangle \langle q | E_j \rangle \, |E_i\rangle \langle E_j| \tag{1.4.10}$$

$$= \sum_{ij} \psi_i^*(x) \psi_j(x) \, |E_i\rangle \langle E_j| \tag{1.4.11}$$

So we can rewrite the interaction Hamiltonian (1.4.1) as

$$H_{int} = \epsilon(t) \int_{\tilde{\Sigma}} \mathrm{d}\tilde{V} \hat{\Phi}(x) \left(\sum_{ij} \psi_i^*(\tilde{x}) \psi_j(\tilde{x}) \right) |E_i\rangle \langle E_j| \tag{1.4.12}$$

which in fact looks like (1.4.7). If we restrict in the two lowest energy levels of the detector we take exactly (1.4.7) for $\hat{A} = |E_0\rangle \langle E_1|$ and $F(x) = \psi_0^*(x)\psi_1(x)$. Note that the spatial smearing F now has a physical meaning in terms of the wavefuntions of the energy eigenstates of the detector. Along these lines, in [18, 19] the authors write an interaction Hamiltonian of the dipole moment of a Hydrogen atom with the electromagnetic field and the spatial smearing is associated to the orbitals of the Hydrogen atom. Mathematically this creates a bit of a subtlety, since F is defined to be compactly supported in (1.4.7), which is not the case for energy eigenfunctions of a system under a realistic potential. Maybe this is one of the reasons why the initial model [16] involved a particle in a box, of which the wavefunction has to be compactly supported by construction[6].

Next we want to consider the time evolution of the composite system under the total Hamiltonian

$$H = \left(\int \mathrm{d}x \hat{\Pi}^2(x) + \hat{\Phi}^2(x)(\Box - m^2)\hat{\Phi}(x) \right) \otimes \mathbb{1}_D \tag{1.4.13}$$

$$+ \epsilon(t) \int_{\tilde{\Sigma}_t} \hat{\Phi}(x) \otimes (F(\tilde{x})\hat{A} + F^*(\tilde{x})\hat{A}^\dagger) \mathrm{d}\tilde{V} + \mathbb{1}_F \otimes \sigma \hat{A}^\dagger \hat{A} \tag{1.4.14}$$

The equations of motion for the field (in the Heisenberg picture) becomes

$$(\Box - m^2)\hat{\Phi}(x,t) = \epsilon(t) \int_{\tilde{\Sigma}_t} (F(\tilde{x})\hat{A}(t) + F^*(\tilde{x})\hat{A}^\dagger(t)) \mathrm{d}\tilde{V} \tag{1.4.15}$$

so the linear coupling with the detector acts like a source for the field and it is zero outside the support of the smearings, where/when the field evolves freely.

For the detector the equation of motion is

$$\frac{\mathrm{d}\hat{A}(t)}{\mathrm{d}t} = -\mathrm{i}\sigma \hat{A}(t) + \mathrm{i}[\hat{A}^\dagger, \hat{A}]\epsilon(t) \int_{\tilde{\Sigma}} \hat{\Phi}(t, \tilde{x}) \psi^*(\tilde{x}) \mathrm{d}\tilde{V}. \tag{1.4.16}$$

[6] Another reason why it might be preferential to couple the field to a particle in a box comes from the use of the interaction picture. As Doreen Fraser pointed out in private communication: Haag's theorem entails that free and interacting Hamiltonians are only well-defined on unitarily inequivalent Hilbert spaces. The interaction picture uses both the free Hamiltonian and the interacting part of the Hamiltonian to define time evolutions (for states and operators), which cannot be defined on the same Hilbert space. Haag's theorem can be evaded by putting the system in a box [7].

The full time evolution (1.4.15) and (1.4.16) is usually hard to solve and so we apply time dependent perturbation theory with respect to $\epsilon(t)$. For this purpose, it is more natural to write the observables in the interaction picture, which for the detector means $A^I(t) = e^{-\mathrm{i}\sigma t}A$. The interaction Hamiltonian becomes

$$H^I_{int} = \int \mathrm{d}^3x \big(\epsilon(t)e^{-\mathrm{i}\sigma t}F(x)\phi_I(t,x) + \mathrm{h.c.}\big). \tag{1.4.17}$$

In first order the time evolution yields

$$|\Psi_f\rangle = \big(\mathbb{1} - \mathrm{i}\int \mathrm{d}t H^I_{int}\big)\,|\Psi_i\rangle\,. \tag{1.4.18}$$

Through the integration over time we can perform a spacetime smearing of the field operator

$$\int \mathrm{d}t\, H^I_{int} = \int \mathrm{d}t \int \mathrm{d}^3x \big(\epsilon(t)e^{-\mathrm{i}\sigma t}F(x)\phi_I(t,x) + \mathrm{h.c.}\big) = \phi_I[f]\hat{A} + \mathrm{h.c.} \tag{1.4.19}$$

where $f(t,x) = \epsilon(t)e^{-\mathrm{i}\sigma t}F(x)$ depends on the temporal and spatial smearings, and the energy gap of the detector σ. The time evolution operator in first order is simply given by $\Big(\mathbb{1} - \phi_I[f]\hat{A} + \mathrm{h.c.}\Big)$. We see that the detector observable $\hat{A}$ is coupled to a smeared field operator $\phi_I[f]$, with a smearing that is specified by the parameters of the interaction Hamiltonian.

It is quite common to attribute the smearings to the detector system, as a switching function and spatial profile of the detector. Mathematically it is clear that they correspond to choices that one has to make to specify the interaction between the detector and the field [7], so one could attribute the smearings to neither, or both. Through (1.4.19) it makes more sense to me that the smearings are used to turn the operator valued distribution $\phi_I(t,x)$ into the well defined operator $\phi_I[f]$. One step further, we can use (1.3.13) to write the smeared field operator as

$$\phi_I[f] = \mathrm{i}a(K^*(Ef)^*) - \mathrm{i}a^\dagger(KEf) \tag{1.4.20}$$

This expression is quite abstract, but it simplifies in particular cases in a very intuitive way. Remember that K is the map that distills the positive frequency part of a solution. In the

[7] Note that these choices need to be made at the "rest frame" of the detector, because for example a smearing that is spatial in that frame, is not purely spatial in a boosted frame. See [20] if you are worried about the relativistic covariance of the interaction Hamiltonian.

case that the time scale that defines $\epsilon(t)$ is much slower than σ, namely the switching of the interaction is slow as compares to the energy scale of the detector, then the smearing function $f(t,x) = \epsilon(t)e^{-\mathrm{i}\sigma t}F(x)$ is going to be basically of positive frequency, and the same for Ef. As a result $KEf \approx Ef$ and $KEf^* \approx 0$, and so the interaction Hamiltonian simplifies to

$$\phi_I(f) \approx -\mathrm{i}a^\dagger(Ef) = \mathrm{i}a^\dagger(\xi) \tag{1.4.21}$$

where we just denote $\xi = -Ef$. Then the time evolution looks like

$$|\Psi_f\rangle = (\mathbb{1} - \mathrm{i}\phi_I(f)A + \mathrm{h.c.})\,|\Psi_I\rangle \tag{1.4.22}$$

$$= \left(\mathbb{1} + a^\dagger(\xi)A - a(\xi^*)A^\dagger\right)|\Psi_I\rangle\,. \tag{1.4.23}$$

If we choose the initial state to be $|\Psi_I\rangle = |0\rangle\,|n_\Psi\rangle$, we drop the second term since $A\,|0\rangle = 0$. One can see through the definition (1.2.13) that what remains is

$$|\Psi_f\rangle = |\Psi_I\rangle - \sqrt{n}(\xi,\psi)\,|(n-1)_\psi\rangle\,|\sigma\rangle\,. \tag{1.4.24}$$

We see that the transition of the detector from the ground state to the first excited state denoted as $|\sigma\rangle$ is accompanied with a "downward transition" for the field state. This process would occur with probability that is propotional to n, as someone might expect. As discussed in [3] this could justify the narration that the detector "absorbing one particle of the field". As Wald puts it "One views the quantum field theory as actually being a theory of 'particles'..one views the interaction of the field with a quantum mechanical system as enabling the quantum mechanical system to absorb or emit these particles". Nevertheless this story is very particular to this case and cannot be supported in the general case where for example we would have terms that correspond to both the field and the detector being excited.

1.5 Algebras

In this section we present some well known results of the algebraic approach to quantum field theory, since some of them will concern us in the next chapters. These results are negative, or puzzling at the very least, and it is non trivial to demystify them mostly due to the high level of mathematical abstraction. Here we will not delve into the mathematical background necessary to examine the technical aspects of these results, nevertheless we will state and discuss the statements, because in the project presented in Chapter 4 we

attempt to make sense of their implications in the concrete context of ordinary quantum field theory (as sketched in section 1.3.1).

In some sense algebras are the most basic mathematical structures that one could require to do physics. Quoting Laura Ruetsche [6] "Intuitively an algebra is just a collecton of elements along with a way of taking their products and linear combinations. Insofar that it's the business of natural science to weave physical magnitudes into functional relationships, organizing physical magnitudes into an algebra undewrites that business...it is an austere response to the surfeit of unitary inequivalent Hilbert space realizations of the CCRs characterizing a QM_∞[8] might be to interpret the theory in terms of a structure that all Hilbert space realizations share. That structure turns out to be the structure of an abstract C*-algebra". In the approach that we followed above [3, 6] we were starting from the classical space of solutions of a field theory to define the quantum field theory by determining states and observables $\{\mathcal{F}_s(\mathcal{H}), \hat{\Omega}(\psi, \cdot)\}$. We discussed what are the choices that may lead to two such constructions that are unitarily inequivalent to each other. The algebraic approach treats these different constructions somehow more democratically, based on the observation that the algebraic relations satisfied by the observables remain the same.

Given the symplectic vector space $(\mathcal{S}, \Omega)$ of a free theory, equipped with an inner product $\mu : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ we defined the Weyl operators $\hat{W}(\psi) = e^{i\hat{\Omega}(\psi,\cdot)} \ \forall \psi \in \mathcal{S}$ that satisfy the Weyl relations

$$\hat{W}(\psi_1)\hat{W}(\psi_2) = e^{\frac{i\Omega(\psi_1,\psi_2)}{2}}\hat{W}(\psi_1 + \psi_2) \tag{1.5.1}$$

and $\hat{W}^\dagger(\psi) = \hat{W}(-\psi)$. The span of the Weyl operators forms a subalgebra of the space $\mathcal{L}(\mathcal{F})$ of bounded operators on the Fock space, which has the natural structure of a $^*-$ algebra.[9] If we use the norm that is defined in $\mathcal{L}(\mathcal{F})$ to close this subalgebra, we end up with a C^*- subalgebra $\mathcal{A}$ called the Weyl algebra. Two different choices for the inner product μ_1, μ_2 might define unitarily inequivalent Fock spaces, but the corresponding Weyl algebras are guaranteed to be isomorphic [6]. This is why in the algebraic approach the

[8] A term introduced by Laura Ruetsche to denote the quantum theory of systems of infinite degrees of freedom i.e. quantum field theories and the thermodynamic limit of statistical mechanics.

[9] Just a quick reminder of all the definitions used here. An algebra $\mathcal{A}$ is a vector space with an additional bilinear map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. An algebra that has a norm is called Banach algebra (if complete) and has the property that $||A_1 A_2|| \leq ||A_1||\,||A_2||$. A $^*-$ algebra is an algebra that posses an antilinear map $^* : \mathcal{A} \to \mathcal{A}$ such that $A^{**} = A$ and $(A_1 A_2)^* = A_2^* A_1^*$. A Banach $^*-$ algebra has both structures, plus the property $||A^*|| = ||A||$. Finally, a C^*- algebra is a Banach $^*-$ algebra which also satisfies $||A^* A|| = ||A||^2$ (rather than $\leq$). The space of *bounded* linear operators over a Hilbert space has the natural structure of a C^*- algebra because of the norm induced by the inner product in the Hilbert space and the adjoint map.

fundamental observables are defined as the elements of (any possible) Weyl algebra $\mathcal{A}$, and the states are defined abstractly as functionals over the algebra $\omega : \mathcal{A} \to \mathbb{R}$. Given an observable $A \in \mathcal{A}$, the result of the action of the state $\omega(A)$ is to be thought of as the expectation value of this observable. Given a representation of the Weyl algebra, that is a map $\Pi : \mathcal{A} \to \mathcal{L}(\mathcal{F})$, and a density matrix $\rho : \mathcal{F} \to \mathcal{F}$ we define an algebraic state

$$\omega(A) := \mathrm{tr}[\rho\Pi(A)] \tag{1.5.2}$$

which is the familiar expression for the expectation value.

Alternatively, this functional can be defined via the inner product μ, for example as

$$\omega[W(\psi)] = e^{-\frac{\mu(\psi,\psi)}{2}} \tag{1.5.3}$$

without using any representation of the algebra. In general, the specification of this functional is equivalent to knowing the $n-$point functions [3], while the definition above corresponds to Gaussian states (or the specification of the $2-$point function).

Overall, the algebraic approach amounts to specifying the pair $(\mathcal{A}, \omega)$. To see how this connects to the possible concrete representations $\{\mathcal{F}_s(\mathcal{H}), \hat{\Omega}(\psi, \cdot)\}$ we need to invoke the GNS construction. We stated just above that given a state ρ and a representation Π we can define an algebraic state ω as in (1.5.2). The GNS construction is basically doing the converse, given an abstract algebraic state, defined as in (1.5.3) for example, we can find a concrete state ρ over some Hilbert space. Explicitly the theorem states the following, let $\mathcal{A}$ be a unital C^*-algebra (that contains an identity element) and a state $\omega : \mathcal{A} \to \mathbb{C}$, then there exists a Hilbert space $\mathcal{F}$ and a representation Π such that

$$\omega(A) = \langle\Psi|\Pi(A)|\Psi\rangle \tag{1.5.4}$$

where $|\Psi\rangle$ a state in $\mathcal{F}$, that has the extra property to be cyclic. This means that the set $\{\Pi(A)\,|\Psi\rangle \;\; \forall A \in \mathcal{A}\}$ is dense in $\mathcal{F}$.

We see that the GNS construction is specifying $(\mathcal{F}, \Pi, |\Psi\rangle)$ from $(\mathcal{A}, \omega)$. At first, this specification seems impossible. Starting with $(\mathcal{A}, \omega)$, how could we possibly end up with some Hilbert space, a representation and a state? The idea of the proof is rather simple, so I will try to sketch it here. The idea is that we can use the state ω to define a bilinear map over the algebra as $< A, B >_{\mathcal{A}} := \omega(A^*B)$. Then we can suitably turn this into an inner product, to complete under the induced norm, and turn the algebra itself into a Hilbert space. This naturally gives us a representation, and a state that corresponds to the identity element of the algebra, so that (1.5.4) is satisfied.

How does that connect with the QFT as constructed in the previous section? If the state ω is defined as in (1.5.3) (for some μ that is compatible with Ω as in (1.2.36)) the GNS construction will yield $\mathcal{F}$ to be the Fock space, and the representation will map the algebra to the observables $\Omega(\psi, \cdot)$, while the state Ψ will correspond to the vaccum state. So if we do the GNS construction, from (1.5.4) we would get that

$$\omega[W(\psi)] := e^{-\frac{\mu(\psi,\psi)}{2}} = \langle 0|\Pi(W)|0\rangle = \langle 0|\Omega(\psi, \cdot)|0\rangle. \tag{1.5.5}$$

With the algebraic approach we can go around the issue of unitary inequivalence, in the sense that it captures the algebraic structure that all representations share, but one could still worry that different representations could give physically inequivalent results (meaning different expectation values as defined in (1.5.4)). In the standard Hilbert space approach for finite degrees of freedom unitary equivalence is ensured by the Stone von Neumann theorem, but in the algebraic approach there should be a representation independent notion of physical equivalence, which is not tied to unitary equivalence of the representations. In fact, this is the role of Fell's theorem [3, 6] which states the following. Let $(\mathcal{F}_1, \Pi_1)$ and $(\mathcal{F}_2, \Pi_2)$ two representations of the same Weyl algebra $\mathcal{A}$ (that could be unitary inequivalent). Let $A_1, ..., A_n$ elements of the algebra and $\epsilon_1, ..., \epsilon_n$ real numbers. If ω_1 is an algebraic state that corresponds to a density matrix in $\mathcal{F}_1$ then there exists an ω_2 that corresponds to a density matrix in $\mathcal{F}_2$ such that

$$|\omega_1(A_i) - \omega_2(A_i)| < \epsilon_i \tag{1.5.6}$$

This is good news, since it means that with no finite number of expectation values of finitely many observables one can distinguish the two states from each other (even if unitarily inequivalent!). Nevertheless, as pointed out by Doreen Fraser and further discussed in [6], the theorem only applies to the algebraic observables in representations of the Weyl algebra, which do not include observables such as the total number operator in the QFT. Furthermore, (1.5.6) states physical equivalence in terms of *expectation values*, while the question becomes more subtle if one wants to ask about *probabilities* of successive measurements. Probabilities are again expectation values of projectors, but projectors are not guaranteed to be elements of the algebra [6].

In the Hilbert space approach, observables are represented by self-adjoint operators that are associated to a set of projectors through the spectral theorem. Then one can find the probability of successive measurements of $A_1, ..., A_n$ to yield certain outcomes using expressions of the form

$$\text{Prob} = \text{tr}[P_n...P_1\rho P_1...P_n] \tag{1.5.7}$$

To extend measurement theory in the algebraic approach we should be able to calculate such expressions. For this we would have to impose extra conditions on $\mathcal{A}$ to guarantee that it includes projectors that are associated to its self-adjoint elements. This could be the structure of von Neumann algebras for example with all their different types (in fact the classification of von Neumann algebras [6] is based on the existence/properties of projectors in the algebra). For example the algebraic approach to ordinary QM yields von Neumann algebra of type I which includes the usual rank 1 projectors and one can apply the usual rules for probabilities Prob $= ||P|\Psi\rangle||^2$ for the corresponding outcome. We will not go through the definitions of von Neumann algebras here, but I would like to mention that Wald considers this to be a step back, from the algebraic approach towards the Hilbert space approach, because von Neumann algebras are rather concrete C^*- algebras, tied to their representaton in a Hilbert space [3]. As we will discuss in the end of this section, when one applies the algebraic approach to quantum field theory projectors are not very well behaved objects (and the type of von Neumann algebra that one would associate to a QFT is the most exotic one).

Next we will sketch the algebraic approach to QFT (AQFT), the main idea being the association of *local* algebras of observables to regions of spacetime. Given a globally hypebolic spacetime $(\mathcal{M}, g_{ab})$ for any open subset $\mathcal{O} \subset \mathcal{M}$ we can define the space of test functions $\mathcal{T}_\mathcal{O}$ that are compactly supported in $\mathcal{O}$. In the previous section we saw how to associate the space of test functions with the space of solutions $\mathcal{S}$. We can suitably define $\mathcal{S}_\mathcal{O}$ the space of solutions with initial data that are compactly supported in $\mathcal{O}$, as a closed subspace of $\mathcal{S}$. Then the Weyl algebra spanned by $\hat{W}(\psi) = e^{\mathrm{i}\hat{\Omega}(\psi,\cdot)}$ for $\psi \in \mathcal{S}_\mathcal{O}$ is a $C^* - -$ subalgebra of the full algebra $\mathcal{A}$. Along these lines the construction yields a net of C^*- algebras $\{\mathcal{A}_\mathcal{O}\}$ indexed by open spacetime regions $\mathcal{O}$, where it is built in that if $\mathcal{O} \subset \mathcal{O}'$ then $\mathcal{A}_\mathcal{O} \subset \mathcal{A}_{\mathcal{O}'}$. This is usually postulated in the axiomatic approach to QFT as the isotony postulate, since it has the interpretation that local observables/operations over a region should be contained in observables/operations over a bigger region that contains the first one. Along with other axioms, the most famous one is the microcausality $[A, B] = 0$ is $A \in \mathcal{O}_1, B \in \mathcal{O}_2$ if the two regions $\mathcal{O}_1, \mathcal{O}_2$ are spacelike separated. The covariance of the algebra elements under the action of the Poincare group (in flat spacetime) is required, and the existence of a unique Poincare invariant vacuum state $|0\rangle$ is also postulated. The list of axioms can be found in [2, 6].

One of the most striking results of AQFT is the Reeh Schlieder theorem [21, 6]. It simply states that the unique vacuum state $|0\rangle$ is cyclic under the action of a local algebra $\mathcal{A}_\mathcal{O}$. This means that the span$\{A|0\rangle, \forall A \in \mathcal{A}_\mathcal{O}\}$ is dense in space of states. Depending on the interpretation that one assigs to the elements of the local algebras, this suggests that one can get arbitrarily close to *any* (global) state of the field by means of local operations

over a bounded spacetime region. For example, one could worry that by manipulating observables inside a lab which is finitely extended in space and time, we could create a "tardigrado [10] (something like a beer, or a bear)" in spacelike separation [11]. The intuition is that because the vacuum state is entangled, one can create any global state only by means of local operations. Redhead has formulated a baby version of the Reeh Schlieder theorem in [21] by means of local operations on a singlet state, and in [22] Halvorson elaborates on why this is not paradoxical.

An important corollary to the Reeh Schlieder theorem is that the vacuum is not only a cyclic, but also a separating state in the local algebra [21, 23]. This has the important implication that there cannot be local number operators in the local algebra, which is a negative result for the particle ontology of QFT if one expects to be able to count particles locally. A state is separating if it can 'separate' the elements of the local algebra from each other, namely two elements are the same if their action on the vacuum gives the same result $A|0\rangle = B|0\rangle \Rightarrow A = B$. If we take both operators on the same side calling $\alpha = A - B$ we see that this implies

$$\alpha|0\rangle = 0 \Rightarrow \alpha = 0. \tag{1.5.8}$$

As a result any local annihilation operator ought to be trivially zero, which goes through for creation operators and number operators (since the algebra is closed under the $^*-$ operation) to yield that we cannot have number operators that are local in a relativistic QFT.

Now let us return to the question of what kind of projectors one can find in such a local algebra. As shown in [21] they all have to be infinite dimensional, as a consequence of the exotic type of von Neumann algebra that one would have to introduce (type III). This means that rank one projectors $|\Psi\rangle\langle\Psi|$ in terms of which we can formulate "yes-no" kind of questions do not belong in the algebra. Finally, in view of (1.5.8) we conclude that if the probability of a measurement is zero $p = ||P|0\rangle||^2 \Rightarrow P = 0$ the corresponding projector has to vanish. This implies that any non zero projector is associated with a non zero probablity, so literally anything is possible in the vacuum.

Let us pause for a second to summarize what kind of operators one can/cannot find in the so called local algebras of observables. We cannot find a position operator, or

[10] Here tardigrade are recruited for the sake of the argument, more information about these creatures you can find at 'Antarctic expedition yields remains of tiny, ancient 'water bears" The Guardian (2019, January 18).

[11] Talk by Juan Leon at the Relativistic Quantum Information Conference North 2016 at the IQC https://www.youtube.com/watch?v=zm2bl7yYcJ0

a local number operator and neither rank one projectors that we would like to use for measurement theory. What about observables of the theory, like the stress energy tensor for example? The way we defined the local Weyl algebra, it contains only exponentiated linear combinations of the (smeared) field operators, so the stress energy tensor (which is at least quadratic) is not naturally included [3]. Overall the local algebras seem to be rather dystopian places where one cannot find most of the things they would wish for. Nevertheless, it seems that this formalism provides a natural platform in which one can formalize the concept of local measurements, as was recently done in [24] thanks to the built in locality. Ordinary measurement theory is rarely explored in a spacetime context, where one has to define a measurement theory that is compatible with relativistic causality. Despite the recent progress, the discussion about local measurements in QFT has been controversial from the 90's until today [24, 25, 26, 27, 28, 29].

I would like to close this section with some references about the historical/philosophical aspects of the algebraic approach. The axiomatization of QFT naturally followed the axiomatization of QM in an attempt to clarify the conceptual and mathematical structure of quantum field theories. For a historical introduction one can look at [30]. Because of the heavy mathematical machinery it is not as popular in our days when it comes to practical purposes, but it is much more prominent for interpretative purposes and so it is more well spread amongst the philosophers of QFT or mathematical physicists. In [31] one can find the proceedings of a conference that happened at Boston Univeristy in 1996 on the "Conceptual foundations of quantum field theory". In this conference, great physicists and philosophers like Weinberg, DeWitt, Wightman, Rovelli, Redhead and Teller, gathered to discuss what QFT is about. Many different views were expressed about fields, particles and algebras, and much of the discussion was around contrasting AQFT with the ordinary QFT that is most commonly recruited in high energy particle physics. Interestingly, some people from the quantum gravity community like Carlo Rovelli defended the standard over the algebraic approach, arguing that the very starting point of AQFT, that is associating algebras of observables to regions of spacetime, is not compatible with how we should think of spacetime according to general relativity [12] [31].

Today we are in the paradoxical situation of having QFT's 'that work' (indeed very successfully) by means of perturbation methods, while the axiomatic approach has not yet succeeded to describe realistic particle interactions. The two variants of QFT can be thought of as complementary to each other [33] but hopefully "...the 'complementarity' in the formulations of quantum field theory is only a historical accident. We have to live with it now, as we had to live with it for the past half-century or so, but it is not here to

[12] Rovelli's argument against AQFT is based on the discussion about the ontology of spacetime as presented in his article "The disappearance of space and time" [32].

stay. Either through more effort on the foundations of quantum field theory, or through a new mathematical language, or through a yet unsuspected physical principle, we hope that this paradoxical situation will be resolved..."[33]. My opinion is that one would have to work at both fronts to resolve this situation, both trying to make better sense of why perturbation theory works but also investigating how far one can go without quitting on mathematical rigor of AQFT, the most formal way of combing the principles of quantum theory and special relativity. There are lessons to be learned both from the 'success' of standard QFT and the 'failures' of AQFT, especially from the perspective of the yet to be found theory of quantum gravity. One could argue, why bother axiomatizing QFT since we know it is not the end of the story? For me the reason is that a good understanding of *why* (rather than how) a theory works, can provide guidance in the regime where it fails to work.

The interpretational debates about (A)QFT are open ajoint today, mostly amongst philosophers [34, 35] and (surprisingly) not so much amongst physicists. Maybe the incredible success of the standard model of particle physics for the past decades has undermined the pragmatic relevance of this debate, but these issues are becoming increasingly important for low energy considerations in Relativistic Quantum Information (RQI). This is because in RQI we wish to apply notions of information theory in the yet more fundamental framework of QFT when relativity cannot be ignored [36]. This requires a different understanding of QFT which is irrelevant for high energy physics and so not very well developed. For example, the very notion of a subsystem or a 'party' and the rules for combing subsystems with tensor product structures are postulated in the axiomatic formulation of quantum mechanics, but are much harder to establish in QFT (as we discuss in the last chapter and in Appendix B). Clarifying the conceptual and mathematical basis of QFT, is hopefully becoming one of the goals that RQI can contribute to and benefit from.

Chapter 2

What About Particles

2.1 Theorem by Malament

In this section we will review and discuss the theorem by Malament, a no-go theorem that demonstrates "Why there cannot be a relativistic quantum mechanics of (localizable) particles" as it is the title of the corresponding paper [37]. The term quantum mechanics, suggests that the systems we are describing are localizable in space. In classical mechanics for example we are describing the motion of either pointlike or finitely extended objects, in contrast to classical field theory where we are describing fields i.e. entities that are assigned a value for every point in space and time. Quantum mechanics, is called mechanics because it is describing systems that are localizable by means of a wavefunction, but as the title of the paper suggests, this cannot be if one encorporates relativity. If we cannot have relativistic quantum *mechanics*, it seems that we can only have relativistic quantum *field theories*, or at least historically we did not come up with any other way to consistently unify quantum theory and special relativity. As it is quoted in [37] "although it is not a theorem, it is widely believed that it is impossible to reconcile quantum mechanics and relativity, except in the context of a quantum field theory. A quantum field theory is a theory in which the fundamental ingredients are fields rather than particles; the particles are little bundles of energy in the field." (Weinberg 1987, 78-79).

In this spirit, Malaments theorem could appeal to someone who wants to remain agnostic as to whether a relativistic quantum theory shall be mechanics or field theory, or anything else. Nevertheless, they would still have to make a minimal set of assumptions to describe the system with a structure that is consistent both with quantum theory and special relativity. Under this set of assumptions, Malament shows that such a system cannot

be localizable, and in this sense any attempt of a relativistic quantum theory cannot give mechanics. In particular, the theorem points out the friction between localizability and act-outcome correlation in spacelike separation which would violate relativistic causality. Concretely, the theorem states that any candidate for a relativistic quantum mechanical representation of a single (localizable) particle, presumably, will include the following elements :

- A Hilbert space $\mathcal{H}$, the rays of which represent the pure states of the particle.
- A strongly continuous unitary representation $U(\alpha)$ in $\mathcal{H}$ of the translation group α in Minkowski space $\mathcal{M}$.
$$\alpha \to U(\alpha) \tag{2.1.1}$$
- Considering the foliation of Minkowski spacetime by a family of spacelike hypersurfaces Σ_t in a fixed frame, we require an assignment of a projection operator P_Δ on $\mathcal{H}$ for every open bounded spatial subregions $\Delta \subset \Sigma$, that is
$$\Delta \to P_\Delta \tag{2.1.2}$$

These elements put together, comprise what is called a localization structure $(\mathcal{M}, \mathcal{H}, \Delta \to P_\Delta, \alpha \to U(\alpha))$ and amounts to specifying states, dynamics and measurement tools, along with the symmetries of the underlying spacetime that we wish to represent in the Hilbert space. Note that the theorem is only concerned with translations, and not boosts. Now let us impose the following constrains

1. Translation Covariance:
$$P_{\Delta+\alpha} = U(\alpha) P_\Delta U(-\alpha) \tag{2.1.3}$$
for all vectors α and spatial sets Δ in $\mathcal{M}$.
2. Energy Condition:
$$U(t\alpha) = e^{-itH(\alpha)} \tag{2.1.4}$$
for all future directed unit timelike vectors in $\mathcal{M}$. The spectrum of the self-adjoint Hamitonian operator is bounded from below.

3. Localizability:

$$P_{\Delta_1} P_{\Delta_2} = P_{\Delta_2} P_{\Delta_1} = 0 \tag{2.1.5}$$

if Δ_1,$\Delta_2 \subset \Sigma_t$ are disjoint spatial sets in the same hyperplane Σ_t.

4. Locality:

$$[P_{\Delta_1}, P_{\Delta_2}] = 0 \tag{2.1.6}$$

if $\Delta_1 \subset \Sigma_{t_1}$,$\Delta_2 \subset \Sigma_{t_2}$ for $t_1 \neq t_2$ and such that the two spatial intervals are spacelike separated.

Localizability is introduced in condition 3, stating that projector that correspond to disjoint spatial intervals at the same time should be orthogonal, so that if the particle is detected in say Δ_1 it has zero probability to be detected in Δ_2 if a position measurement happens at t. The only condition that appeals to relativity is condition 4 (locality), in a sense that we will explain below. First note that the locality condition is weaker than localizability, since two projectors need not be orthogonal to commute. Someone could find it reasonable to demand orthogonality, rather than commutativity, since it should be impossible to detect the particle in spacelike separated regions because intuitively this would mean superluminal propagation. But instead, condition 4 only imposes statistical independence of position measurements in spacelike separation. The difference between the two requirements is the difference between micro and macro causality, and Malament's theorem requires macro causality to be respected. To see this we will demonstrate that condition 4 is equivalent to statistical independence by recruiting Luder's rule for conditional probabilities

$$[P_{\Delta_1}, P_{\Delta_2}] = 0 \Leftrightarrow \mathrm{tr}[\rho P_{\Delta_2}] = \mathrm{tr}[\rho' P_{\Delta_2}] \tag{2.1.7}$$

where $\rho' = P_{\Delta_1} \rho P_{\Delta_1} + (1 - P_{\Delta_1})\rho(1 - P_{\Delta_1})$ the statistical mixture of the particle being and not being in region Δ_1. Equation (2.1.7) means that if statistics for position measurements is collected in Δ_2, it should be independent from whether a position measurement has happened in Δ_1 of which the outcome is not known. One way of (2.1.7) is straight forward (from left to right) assuming commutativity we see that

$$\begin{aligned}
\mathrm{tr}[\rho' P_{\Delta_2}] &= \mathrm{tr}[P_{\Delta_1} \rho P_{\Delta_1} P_{\Delta_2} + (1 - P_{\Delta_1})\rho(1 - P_{\Delta_1}) P_{\Delta_2}] && (2.1.8)\\
&= \mathrm{tr}[P_{\Delta_1}^2 \rho P_{\Delta_2} + (1 - P_{\Delta_1})^2 \rho P_{\Delta_2}] && (2.1.9)\\
&= \mathrm{tr}[P_{\Delta_1} \rho P_{\Delta_2} + (1 - P_{\Delta_1})\rho P_{\Delta_2}] && (2.1.10)\\
&= \mathrm{tr}[\rho P_{\Delta_2}] && (2.1.11)
\end{aligned}$$

The converse is a bit more tedious, but overall macrocausality and commutativity of the projectors are equivalent in this context.

To make sense of condition 4, the only controversial condition that goes into the theorem, we had to make two choices. We have to pick a notion of causality (here macrocausality) and the rules for probablities and state update from measurement theory (here Luder's rule). This demonstrates how notions of localisation, causality and measurement theory come together to form such arguments. In particular, one cannot really make a connection between notions of localizations and notions of causality without using tools from measurement theory. Even worse, in such arguments someone has to measure "locally" to test if causality is violated, so a specific notion of "local measurement" is needed, and this tends to be at odds with relativistic causality. In the context of Malament's theorem this is manifested through the trivialization of the spatial projectors P_Δ. In particular, the theorem states that if a localization structure $(\mathcal{M}, \mathcal{H}, \Delta \to P_\Delta, \alpha \to U(\alpha))$ satifies conditions 1-4 then

$$P_\Delta = 0 \quad \forall \Delta. \tag{2.1.12}$$

The theorem states that the projectors that one would wish to associate with bounded spatial regions are trivially zero. As a result, the probability of detecting a particle within any bounded regions Δ is zero

$$\text{Prob}_\rho(\Delta) = \text{tr}[\rho P_\Delta] = 0 \quad \forall \Delta \tag{2.1.13}$$

because of (2.1.12). Note that we have made no statement about the "size" of Δ, it can be an arbitrarily small or large interval as long as it is bounded. This is to draw a distinction between Malament's theorem and arguments that set the Compton length or the Planck length (in gravitational scenaria) as a limit to the localizability of a particle. Nevertheless, one can imagine that the very definition of localizability as expressed in condition 3, our ability to sharply "tell apart" disjoint spatial intervals is what should break down at smaller scales. A generalization of Malament's theorem in the case of unsharp boundaries can be found in [38].

It is common that the theorem by Malament is phrased as "there is no position operator for a relativistic quantum particle" or "there is no position operator in QFT" and in fact the particle ontology of a relativistic QFT is debatable till today. The non existence of a position operator follows since the projectors P_Δ would have to form its projection valued measures. Or conversely if we had a position operator, we would write down these projectors as

$$P_\Delta = \int_\Delta \mathrm{d}x \, |x\rangle \langle x| \tag{2.1.14}$$

which is obviously not trivially zero. A position operator provides us with a (rigged) basis over which we can expand the state to get the particle's wavefunction $\Psi(x) = \langle x|\Psi\rangle$. Then in fact the probability of detecting a particle in a given region is given by the usual expression

$$\text{Prob}_\Psi(\Delta) = \text{tr}[|\Psi\rangle\langle\Psi| P_\Delta] = \text{tr}[|\Psi\rangle\langle\Psi| \int_\Delta \mathrm{d}x\, |x\rangle\langle x|] = \int_\Delta \mathrm{d}x |\Psi(x)|^2 \tag{2.1.15}$$

which of course is generally non zero in non relativistic quantum mechanics, but zero in (2.1.13). We conclude that if there is no position operator there cannot be a spatial representation of a pure state of the particle i.e. a wavefunction with the usual probabilistic interpretation, and in this sense "there cannot be relativistic quantum mechanics of (localizable) particles". We conclude this section with the proof of the theorem, since it is quite insightful to see what enforces (2.1.12). Crucially, the proof makes use of the following lemma.

Lemma

Let $U(t) = e^{\imath t H}$ a family of strongly continuous one-parameter group of unitary operators, where the spectrum of the generator H is bounded from below. Let P_1, P_2 be two projector operators such that (i) $P_1 P_2 = 0$, and (ii) there is $\epsilon > 0$ such that $[P_1, U(-t)P_2U(t)] = 0 \ \forall t \in (-\epsilon, \epsilon)$. Then it follows that

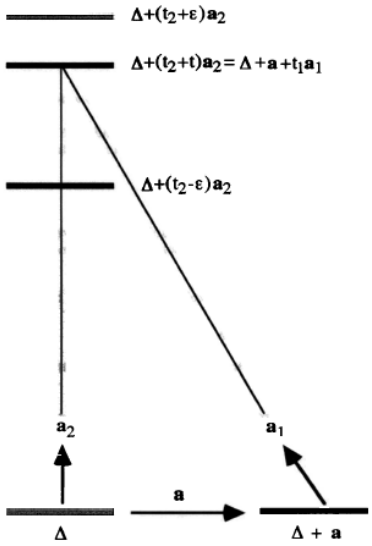
$$P_1 U(-t) P_2 U(t) = 0 \ \forall t. \tag{2.1.16}$$

Before we proceed with the proof of the theorem we can already see how this technical lemma becomes relevant in such considerations. This is because it provides a link between the requirements 3 and 4, localizability and locality. If the two projectors of the lemma are associated to disjoint spatial subsets of the same hypersurface, then (i) is satistied because of localizability. Then there should exist an ϵ that specifies the time period for which P_1 commutes with the time evolved version of P_2 (condition (ii) of the lemma) so that the locality condition is met as long as they stay spacelike separated. Then (2.1.16) means that P_1 and $U(-t)P_2U(t)$, the time evolved version of P_2, will be orthogonal for all times.

Proof

The proof consists of two applications of the lemma. First we apply the lemma for $P_1 = P_\Delta$, $P_2 = P_{\Delta+\alpha}$ and $U(t) = U(t\alpha_1)$ where α a purely spacelike displacement along the hyper surface Σ where Δ belongs, and α_1 a timelike vector (see figure below).

Condition (i) is satisfied from the localizability condition. Condition (ii) is also satisfied since α_1 is timelike there is an $\epsilon > 0$ such that for sufficiently small times $\Delta + \alpha + \alpha_1 t$



remains spacelike separated from Δ and so

$$[P_\Delta, U(t\alpha_1)P_{\Delta+\alpha}U(-\alpha_1 t)] = 0 \tag{2.1.17}$$

form the locality condition. Then it follows from the lemma that

$$P_\Delta P_{\Delta+\alpha+\alpha_1 t} = 0 \ \ \forall t \tag{2.1.18}$$

where we have used the covariance condition. Next we consider a purely time like vector α_2, and t_2 large enough so that $\Delta + t_2\alpha_2$ is in the timelike future of $\Delta + \alpha$. So we can find t_1 such that

$$\Delta + (t + t_2)\alpha_2 = \Delta + \alpha + \alpha_1 t_1 \tag{2.1.19}$$

for any $t \in (-\epsilon, \epsilon)$ so that the region remains in the spacelike future of $\Delta + \alpha$. From (2.1.18) for $t = t_1$ we have that

$$P_\Delta P_{\Delta+\alpha+\alpha_1 t_1} = 0 \tag{2.1.20}$$

so from (2.1.19)

$$P_\Delta P_{\Delta+(t+t_2)\alpha_2} = 0 \tag{2.1.21}$$

which can be rewritten as $[P_\Delta, P_{\Delta+(t+t_2)\alpha_2}] = [P_\Delta, U(-t\alpha_2)P_{\Delta+t_2\alpha_2}U(t\alpha_2)] = 0$ which is condition (ii) of the lemma. So we can use the lemma once more for $P_1 = P_\Delta$, $P_2 = P_{\Delta+\alpha_2 t_2}$

and $U(t) = U(t\alpha_2)$. In fact condition (i) is also satisfied $P_\Delta P_{\Delta+\alpha_2 t_2} = 0$ from (2.1.21) for $t = 0$. From the lemma we conclude that

$$P_\Delta U(-t\alpha_2) P_{\Delta+\alpha_2 t_2} U(t\alpha_2) = P_\Delta U(\alpha_2(t+t_2)) P_\Delta U(-\alpha_2(t+t_2)) = 0 \tag{2.1.22}$$

for every t. So picking $t = -t_2$ we get that

$$P_\Delta^2 = P_\Delta = 0 \tag{2.1.23}$$

which completes the proof.

2.2 Theorems by Hegerfeld and Halvorson

Following a yet more minimal approach, Hegerfeld formulated another no-go theorem which emphasizes the role of positivity of energy in the instantaneous spreading of a wavefunction of a particle. The original version of the proof [39, 40, 41] assumes the existence of a positive operator $N(\Delta)$ (not necessarily a projector) that we can use to calculate the probability of a particle being in the region Δ as

$$P_{\psi_t}(\Delta) = \langle \psi_t, N(\Delta)\psi_t \rangle. \tag{2.2.1}$$

We assume that the particle was initially confined in the region Δ

$$P_{\psi_0} = \langle \psi_0, N(\Delta)\psi_0 \rangle = 1 \tag{2.2.2}$$

Since the state is normalized $\langle \psi_0, \psi_0 \rangle = 1$ subtracting this from the equation above we can see that $N(\Delta)\psi_0 = \psi_0$. If the particle is not propagating with infinite velocity we might require that for every t there is a constant R_t such that for all displacements $\alpha > R_t$ we have that

$$P_{\psi_t}(\Delta + \alpha) = \langle \psi_t, N(\Delta+\alpha)\psi_t \rangle = 0 \tag{2.2.3}$$

i.e. the particle cannot be detected arbitrarily far away in a finite time. The probability (2.2.3) we can write as

$$P_{\psi_t}(\Delta+\alpha) = \langle \psi_t, U_\alpha N(\Delta) U_{-\alpha}\psi_t \rangle = \langle \psi_t U_\alpha^\dagger \sqrt{N(\Delta)}, \sqrt{N(\Delta)} U_\alpha^\dagger \psi_t \rangle = || \sqrt{N(\Delta)} U_\alpha^\dagger \psi_t || = 0$$

for $\alpha > R_t$. We have used time translation invariance and the positivity of the operator N to define the square root and write this probability as a norm of a vector, to argue that the vector it self has to vanish for these values of α. Multiplying with the square root

$$N(\Delta) U^\dagger(\alpha)\psi_t = 0, \quad \alpha > R_t \tag{2.2.4}$$

Taking an inner product with ψ_0 we have that

$$\langle \psi_0, N(\Delta)U^\dagger(\alpha)\psi_t\rangle = 0 \Rightarrow \langle \psi_0 N(\Delta), U^\dagger(\alpha)\psi_t\rangle = 0 \Rightarrow \langle \psi_0, U^\dagger(\alpha)U(t)\psi_0\rangle = 0 \tag{2.2.5}$$

where we used that $N(\Delta)\psi_0 = \psi_0$. We can define this last expression as

$$f_t(\alpha) = \langle \psi_0, U^\dagger(\alpha)U(t)\psi_0\rangle = \int \frac{\mathrm{d}p}{\sqrt{p^2+m^2}}|\psi_0(p)|^2 e^{-\imath\sqrt{p^2+m^2}t}e^{\imath p\alpha} \tag{2.2.6}$$

which we require to be zero for $\alpha > R_t$, so $f_t(\alpha)$ is compactly supported. Then the Fourier transform $f_t(p) = \frac{|\psi_0(p)|^2}{\sqrt{p^2+m^2}}e^{-\imath\sqrt{p^2+m^2}t}$ has to be entire (analytic everywhere) and it is not because of the square root. The only alternative is that f_t identically vanishes $\forall\alpha$. But then for $t = 0$ and $\alpha = 0$ we get

$$f_0(0) = \langle \psi_0, N(\Delta)\psi_0\rangle = 0 \tag{2.2.7}$$

which contradicts our initial assumption. This demonstrates that because of the positivity of energy the particle cannot propagate causally.

In [42] Halvorson reformulated the theorem more abstractly as follows. The theorem by Hegerfeld only requires time-translation invariance, since it is not making use of a representation of the translation group. So the localization system we have been restricting above is

$$(\mathcal{H}, \Delta \to E_\Delta, t \to U_t). \tag{2.2.8}$$

The operators E_Δ that we associate to spacial regions, are positive operators. Projectors are only a special case, which allows to generalise Malament's result for unsharp localization. The restriction that the theorem imposes to this localization structure are the following.

1. Time translation covariance with energy bounded from below

$$U(-t\alpha)E_\Delta U(t\alpha) = E_{\Delta+\alpha} \tag{2.2.9}$$

   for every timelike vector α.

2. Monotonicity: If $\{\Delta_n\}$ is a downword nesting of the hypersurface Σ such that $\bigcup_n \Delta_n = \Delta$, then $\bigcup_n E_{\Delta_n} = E_\Delta$.

3. No instantaneous wavepacket spreading: If $\Delta \subset \Delta'$ there is $\epsilon > 0$ such that $E_\Delta \leq E_{\Delta'+t}$ for $0 \leq t < \epsilon$.

The theorem states that if we impose the following constrains to the localisation structure (3.2.4) it follows that

$$U_t E_\Delta U_{-t} = E_\Delta. \tag{2.2.10}$$

Note that from time translation covariance it should be that $U_t E_\Delta U_{-t} = E_{\Delta+t}$ rather than (2.2.10), which means that the time evolution of the positive operators becomes trivial. The theorem by Hegerfeld is usually stated as "a wavepacket that is initially localized will propagate superluminally because of the positivity of energy". Here we see that imposing finite propagation (condition 4) the wavepacket is doomed to remain in the initial regions for all times, because (2.2.10) is equivalent to

$$E_\Delta \psi = \psi \Rightarrow E_\Delta \psi_t = \psi_t \ \ \forall t. \tag{2.2.11}$$

We can see the equivalence as follows

$$E_\Delta \psi = \psi \Rightarrow U_t E_\Delta \psi = U_t \psi \Rightarrow U_t E_\Delta U_{-t} U_t \psi = U_t \psi \Rightarrow E_\Delta \psi_t = \psi_t \tag{2.2.12}$$

The proof makes use of the following lemma.

Lemma

Let a family of unitary operators $U_t = e^{\imath t H}$ where H is bounded from below and A is a positive operator. Then for any state ψ either (i) $\langle U_t\psi, AU_t\psi\rangle = 0 \ \forall t$, or (ii) $\langle U_t\psi, AU_t\psi\rangle \neq 0$ for almost all t.

Proof

Considering this family of intervals Δ_n bigger than Δ, from NIWS (condition 4) it follows that for every Δ_n there is an ϵ_n such that

$$\langle U_t\psi, E_{\Delta_n} U_t\psi\rangle = 1 \tag{2.2.13}$$

for $t < \epsilon_n$. The intuition is that due to finite propagation speed if the particle is initially localized at Δ it should be with certainty within a bigger Δ_n up to sufficiently small time ϵ_n. If we combine (2.2.13) with the unitarity of time evolution $\langle U_t\psi, U_t\psi\rangle = 1$ we get that

$$\langle U_t\psi, (1 - E_{\Delta_n}) U_t\psi\rangle = 1 \tag{2.2.14}$$

for $t < \epsilon_n$. Since $(1 - E_{\Delta_n})$ is a positive operator we can use the lemma above. Since this holds in a region of t's (ii) of the lemma cannot hold so we have that it holds for all t i.e.

$$E_{\Delta_n} U_t\psi = U_t\psi \tag{2.2.15}$$

for all t and for all n. From the monotonicity condition it follows that

$$E_\Delta U_t \psi = U_t \psi \tag{2.2.16}$$

which as we showed above it is equivalent to

$$U_t E_\Delta U_{-t} = E_\Delta, \tag{2.2.17}$$

which concludes the proof. Comparing to Malament's theorem we see that $E_\Delta = 0$ is a special case of (2.2.17). In this sense Hegerfeld's theorem is more general, it trivializes the time evolution of E_Δ rather than requiring that they are trivially zero.

Contrasting the two proofs, one can reflect on the role that relativity plays in the conclusion. My understanding is that the proof by Hegerfeld seems to be using the relaivistic symmentry explicitly (since the analyticity argument is based on the square root of the dispersion relation) while Halvorson's proof seems to be using only the positivity of energy indeed. To clarify the role of relativity in that, Halvorson moves on to formulate a Malament/Hegerfeld kind of theorem in his thesis [42] as an attempt to investigate the (in)compatibilities of the requirements that go into either/both theorems.

First he introduces a notion of probability conservation abstractly as follows, if $\{\Delta_n,\ n \in \mathbb{N}\}$ is a covering of Σ and $\{\Pi_n,\ n \in \mathbb{N}\}$ a covering of $\Sigma + t$, then $\cup_n \Delta_n = \cup_n \Pi_n$. This is to separate this condition form the condition of unitary time evolution, which implies this kind of probablity conservation to find the particle somewhere only if a position basis exists. Halvorson came up with the following indespensable set of conditions that one might wish to impose to a localization structure $(\mathcal{H}, \Delta \to E_\Delta, t \to U_t)$.

1. Probablity conservation
2. Localizability
3. Time-translation covariance
4. Energy bounded from below
5. Microcausality

From (all) these five conditions it follows $U_t E_\Delta U_{-t} = E_\Delta$. These conditions are indispensable in the sense that no one of them can be dropped without sucrificing the conclusion. For example if someone drops microcausality, then then E_Δ's correspond to the usual position operator of non relativistic quantum mechanics and so we get that $U_t E_\Delta U_{-t} = E_{\Delta+t}$

instead of (2.2.17). Also, if we only require 1-4 we get Malament's result $E_\Delta = 0$ instead of (2.2.17), and so on. In this way, Halvorson's version of the Malament/Hegerfeld type of theorems closes this circle of no go results by spelling out and separating all the ingredients that are necessary for the negative conclusion.

2.3 Newton Wigner position operator?

It might seem quite contrary to the spirit of the no-go theorems presented above that Newton and Wigner introduced a position operator by means of which they can define "localized states for elementary systems" in [5]. As we will discuss later, this operator fails to be fully relativistically covariant, but it is "good enough" under certain conditions. One might notice that the title of the paper refers to "elementary systems" rather than "elementary particles". Usually the term elementary particle means that it is not possible/purposeful to assign further structure, so an elementary particle does not consist of others. The notion of an elementary system is quite more abstract, as it refers to a system that is described by an irreducible representation of the Poincare group. Again, for the next subsection we remain agnostic as to whether a field theoretic decription is needed for an elementary system. Without going into the complications of a QFT, Newton and Wigner start from the Poincare group to find a way to represent the state $|\Psi\rangle$ of an elementary system in position space. We are only making quantum field theoretic considerations in subsection 2.3.2, where I present some results from an on going project for which we associate the Newton Wigner wavefunction with the stress energy density of one particle states in QFT.

2.3.1 Particle interpretation versus covariance

To start with, one might wonder why the irreducibility of the representation is characterized as "elementary". As Newton and Wigner explain in [5] this is because "there must be no relativistically invariant distinction between the various states of the system which would allow for the principle of superposition". The superposition principle follows from the fact that the state space of the system is a vector space, and for an elementary system there should not be any subspace of the state space that is invariant under the action of the Poincare group (which would be the case if the representation was reducible). In other words, for an elementary system any "brunch" of a superposed state should be accessible through the action of the Poincare group. Newton and Wigner undertook the task of defining localized states such systems, since this is not naturally provided by the Poincare

group. The observables of the system, given by the generators of the Poincare group, are the energy, the momentum and the angular momentum, while there is no position operator conjugate to the momentum operator i.e. satisfying the Heisenberg algebra $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$. As we described in the previous chapter, for a quantum mechanical system the commutator between the position and momentum operators is the very starting point of the quantisation procedure. It is this commutation relation that allows to translate between the spatial and the momentum space wavefunction through the Fourier transform

$$\Psi(p) = \langle p|\Psi\rangle = \int \mathrm{d}x \langle p|x\rangle\langle x|\Psi\rangle = \int \mathrm{d}x e^{-\mathrm{i}px}\Psi(x) \tag{2.3.1}$$

For an elementary system what we naturally have from the Poincare group is the components $\Psi(p_0, p)$ of its state $|\Psi\rangle$ over the energy and momentum spectrum. Then the question is how we translate this to position space since there is no position operator to provide us with a spatial representation of the state $\Psi(x)$, and overall the machinery described just above does not apply.

A natural guess would be to take the relativistically covariant version of a Fourier transform as follows. Lets denote the position and momentum four-vectors as x, p, and their spatial components as $\boldsymbol{x}, \boldsymbol{p}$. Given the function $\Psi(\mathsf{p})$ we can define

$$\Psi(\mathsf{x}) = \int \mathrm{d}\mu(\mathsf{p})\, \Psi(\mathsf{p}) e^{i\mathsf{p}\mathsf{x}} \tag{2.3.2}$$

where $\mathsf{p}\mathsf{x} = p_\mu x^\mu$ and $\mathrm{d}\mu(\mathsf{p})$ the Lorentz- invariant measure

$$\mathrm{d}\mu(\mathsf{p}) = \frac{\mathrm{d}p_0 \mathrm{d}\boldsymbol{p}}{\sqrt{2\pi}^3}\delta(p^\mu p_\mu - m^2)\Theta(p_0) = \frac{\mathrm{d}p_0 \mathrm{d}\boldsymbol{p}}{\sqrt{2\pi}^3 2p_0}\delta(p_0 - \sqrt{\boldsymbol{p}^2 + m^2}). \tag{2.3.3}$$

If we call $\omega_p = \sqrt{\boldsymbol{p}^2 + m^2}$ we have that

$$\int \mathrm{d}\mu(\mathsf{p}) f(\mathsf{p}) = \int \frac{\mathrm{d}p_0 \mathrm{d}\boldsymbol{p}}{\sqrt{2\pi}^3 2p_0}\delta(p_0 - \sqrt{\boldsymbol{p}^2 + m^2}) f(p_0, \boldsymbol{p}) = \frac{1}{\sqrt{2\pi}^3}\int \frac{\mathrm{d}\boldsymbol{p}}{2\omega_p} f(\omega_p, \boldsymbol{p}) \tag{2.3.4}$$

To justify calling the Fourier transform (2.3.2) let us quickly check how it transforms under a Lorentz transformation $\mathsf{x}' = \Lambda\mathsf{x}$.

$$\Psi(\mathsf{x}') = \Psi(\Lambda\mathsf{x}) = \int \mathrm{d}\mu(\mathsf{p})\Psi(\mathsf{p}) e^{i\mathsf{p}(\Lambda\mathsf{x})} = \Psi'(\mathsf{x}). \tag{2.3.5}$$

Next wee notice that $\mathsf{p}(\Lambda\mathsf{x}) = p_\mu x'^\mu = p_\mu \tilde{\Lambda}^\mu{}_\nu x^\nu = (\Lambda^{-1})_\nu{}^\mu p_\mu x^\nu = (\Lambda^{-1}\mathsf{p})\mathsf{x}$. So by defining $\mathsf{p}' = \Lambda^{-1}\mathsf{p} \Rightarrow \mathsf{p} = \Lambda\mathsf{p}'$ equation (2.3.5) becomes

$$\Psi'(\mathsf{x}) = \int \mathrm{d}\mu(\mathsf{p})\Psi(\mathsf{p})e^{i\mathsf{p}(\Lambda\mathsf{x})} = \int \mathrm{d}\mu(\mathsf{p})\Psi(\mathsf{p})e^{i(\Lambda^{-1}\mathsf{p})\mathsf{x}} \tag{2.3.6}$$

$$= \int \mathrm{d}\mu(\Lambda\mathsf{p}')\Psi(\Lambda\mathsf{p}')e^{i\mathsf{p}'\mathsf{x}} = \int \mathrm{d}\mu(\mathsf{p}')\Psi'(\mathsf{p}')e^{i\mathsf{p}'\mathsf{x}} \tag{2.3.7}$$

where $\Psi' = \Psi \circ \Lambda$. Since p, p' are dummy indices we can write

$$\Psi'(x) = \int \mathrm{d}\mu(\mathsf{p})\Psi(\mathsf{p})e^{i\mathsf{p}(\Lambda\mathsf{x})} = \int \mathrm{d}\mu(\mathsf{p})\Psi(\Lambda\mathsf{p})e^{i\mathsf{p}\mathsf{x}}. \tag{2.3.8}$$

Written as a spatial Fourier transform this means that

$$\Psi(t', x') = \int \frac{\mathrm{d}\boldsymbol{p}}{\sqrt{2\pi}^3}\frac{\Psi(\omega_p, \boldsymbol{p})e^{i\omega_p t'}}{2\omega_p}e^{-i\boldsymbol{p}\boldsymbol{x}} \tag{2.3.9}$$

$$= \int \frac{\mathrm{d}\boldsymbol{p}}{\sqrt{2\pi}^3}\frac{\Psi(\omega'_p, \boldsymbol{p}')e^{i\omega_p t}}{2\omega_p}e^{-i\boldsymbol{p}\boldsymbol{x}} = \Psi'(t, x) \tag{2.3.10}$$

which demonstrates the invariance of the position space wavefunction defined as in (2.3.2).

One last property of the transformation (2.3.2) that we will use below is that we can invert it as follows

$$\Psi(t, x) = \int \frac{\mathrm{d}\boldsymbol{p}}{\sqrt{2\pi}^3}\frac{\Psi(\omega_p, \boldsymbol{p})e^{i\omega_p t}}{2\omega_p}e^{-i\boldsymbol{p}\boldsymbol{x}} \Rightarrow \Psi(\omega_p, \boldsymbol{p}) = 2\omega_p \int \mathrm{d}\boldsymbol{x}\Psi(t, \boldsymbol{x})e^{i(-\omega_p t + \boldsymbol{p}\boldsymbol{x})} \tag{2.3.11}$$

Now we want to make sense of (2.3.2) as a solution of the Klein Gordon equation $(\Box + m^2)\Psi(\mathsf{x}) = 0$. This is a second order differential equation, and so to solve it we need to input the values of the function and its time derivative at some initial time. It follows from definition of (1.1.12) and its inverse form, that Ψ and $\partial_t\Psi$ are not independent [2]. We can see this as follows

$$\frac{\partial\Psi}{\partial t}(t, \boldsymbol{x}) = \frac{1}{\sqrt{2\pi}^3}\int \frac{\mathrm{d}^3 p}{2\omega_p}\Psi_p(-i\omega_p)e^{i(\boldsymbol{p}\boldsymbol{x} - \omega_p t)} \tag{2.3.12}$$

$$= \frac{1}{\sqrt{2\pi}^3}\int \frac{\mathrm{d}^3 p}{2\omega_p}2\omega_p\Big(\int \mathrm{d}^3 x'\Psi(t, \boldsymbol{x}')e^{-i\boldsymbol{p}\boldsymbol{x}'}\Big)(-i\omega_p)e^{i\boldsymbol{p}\boldsymbol{x}} \tag{2.3.13}$$

$$= \int \mathrm{d}^3 x'\Psi(t, \boldsymbol{x}')\frac{1}{\sqrt{2\pi}^3}\int \mathrm{d}^3 p(-i\omega_p)e^{i\boldsymbol{p}(\boldsymbol{x} - \boldsymbol{x}')} \tag{2.3.14}$$

$$= \int \mathrm{d}^3 x'\Psi(t, \boldsymbol{x}')\mathcal{E}(\boldsymbol{x} - \boldsymbol{x}') \tag{2.3.15}$$

where $\mathcal{E}$ the integral kernel $\mathcal{E}(\boldsymbol{x}-\boldsymbol{x'}) = -\frac{i}{\sqrt{2\pi}^3}\int \sqrt{\boldsymbol{p}^2+m^2}e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x'})}\mathrm{d}^3p$.

This is a case in which someone really needs to keep track of the factors ω_p, since it was exactly these "leftovers" that gave us the integral kernel above that non locally associates the value of the derivative at a point with the value of the function everywhere. Much of the study of non local effects in relativistic quantum theories requires examining the behaviour of kernels like the one above, since they directly follow from imposing the relativistic symmetries to the system. Another instant of this, has to do with the inner product as written in position space. In momentum space $\langle\Psi|\Psi'\rangle = \int \frac{\mathrm{d}^3p}{2\omega_p}\Psi_p\Psi'_p$ and it is easy to see that for $\Psi=\Psi'$ we take a positive Lorentz invariant number as a norm of the state. Now we will use the inverse transform (2.3.11) to write the inner product in position space

$$\langle\Psi|\Psi'\rangle = \int \frac{\mathrm{d}^3p}{2\omega_p}\left(2\omega_p\int \mathrm{d}^3x\Psi(t,\boldsymbol{x})e^{i\boldsymbol{p}\boldsymbol{x}}\right)2\omega_p\int \mathrm{d}^3x'\Psi'(t,\boldsymbol{x}')e^{-i\boldsymbol{p}\boldsymbol{x}'} \tag{2.3.16}$$

$$= \int \mathrm{d}^3x\int \mathrm{d}^3x'\Psi(t,\boldsymbol{x})\Psi'(t,\boldsymbol{x}')\int \mathrm{d}^3p\,2\omega_p e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')} \tag{2.3.17}$$

$$= -2i\int \mathrm{d}^3x\int \mathrm{d}^3x'\Psi(t,\boldsymbol{x})\mathcal{E}(\boldsymbol{x}-\boldsymbol{x}')\Psi(t,\boldsymbol{x}'). \tag{2.3.18}$$

Or to make it explicitly (anti)symmetric

$$\langle\Psi|\Psi'\rangle = i\int \mathrm{d}^3x\int \mathrm{d}^3x'\Psi(t,\boldsymbol{x})\Psi'(t,\boldsymbol{x}')\Big(\int \mathrm{d}^3p\,\omega_p e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')} - \int \mathrm{d}^3p'\,\omega'_p e^{i\boldsymbol{p}(\boldsymbol{x}'-\boldsymbol{x})}\Big) \tag{2.3.19}$$

$$= i\int \mathrm{d}^3x\Big(\Psi'\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi'}{\partial t}\Big) \tag{2.3.20}$$

where we see the symplectic structure of the space of solution appearing. This inner product is not positive definite, so we see why we would fail to interpret $\Psi(x)$ as a wavefunction that gives us the probablity amplitude of finding the particle at x. This demonstrates that the straightforward generalizaton of non relativistic wave mechanics is rather problematic. One can have a relativistically covariant "wavefunction" $\Psi(\mathbf{x})$ that satisfies the relativistic wave equation, but it cannot carry the interpretation of ordinary quantum mechanical wavefunction. To fix this, Newton and Wigner introduced the definition

$$\Psi_{\text{NW}}(\boldsymbol{p}) = \frac{\Psi(\omega_p,\boldsymbol{p})}{\sqrt{2\omega_P}} \qquad \Psi_{\text{NW}}(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3p}{\sqrt{2\pi}^3}\Psi_{\text{NW}}(\boldsymbol{p})e^{i(\omega_p t-\boldsymbol{p}\boldsymbol{x})} \tag{2.3.21}$$

The definition is exactly such that we get rid of the annoying factors of $\sqrt{\omega_p}$, to eliminate the non locality in the inner product (2.3.18). Indeed, the inner product in position space

becomes

$$\langle\Psi|\Psi'\rangle = \int \frac{\mathrm{d}^3p}{2\omega_p}\Psi_p\Psi'_p = \int \mathrm{d}^3p\,\Psi_{\text{NW}}(\boldsymbol{p})\Psi'_{\text{NW}}(\boldsymbol{p}) \tag{2.3.22}$$

$$= \int \mathrm{d}^3x\,\Psi_{\text{NW}}(t,\boldsymbol{x})\Psi'_{\text{NW}}(t,\boldsymbol{x}')\int \mathrm{d}^3p\,e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')} = \int \mathrm{d}^3x|\Psi_{\text{NW}}(t,\boldsymbol{x})|^2 \tag{2.3.23}$$

and could be interpreted as a probability of finding the particle at x. Of course $\Psi_{\text{NW}}(x)$ still satisfies the Klein Gordon equation but it fails to be covariant (as we demonstrated (2.3.2) is the covariant one). The Newton Wigner wavefunction relates to (2.3.2) nonlocally

$$\Psi_{\text{NW}}(t,\boldsymbol{x}) = \int \mathrm{d}^3p\Psi_{\text{NW}}(\boldsymbol{p})e^{i(\omega_pt-\boldsymbol{p}\boldsymbol{x})} = \int \mathrm{d}^3p\frac{\Psi_p}{2\omega_p}\sqrt{2\omega_p}e^{i(-\omega_pt+\boldsymbol{p}\boldsymbol{x})} \tag{2.3.24}$$

$$= \int \frac{\mathrm{d}^3p}{2\omega_p}\int \mathrm{d}^3x'2\omega_p\Psi(t,\boldsymbol{x}')e^{i(\omega_pt-\boldsymbol{p}\boldsymbol{x}')}\sqrt{2\omega_p}e^{i(-\omega_pt+\boldsymbol{p}\boldsymbol{x})} \tag{2.3.25}$$

$$= \int \mathrm{d}^3x'\Psi(t,\boldsymbol{x}')\int \mathrm{d}^3p\sqrt{2\omega_p}e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')} \tag{2.3.26}$$

So we see that

$$\Psi_{\text{NW}}(t,\boldsymbol{x}) = \int \mathrm{d}^3x'\Psi(t,\boldsymbol{x}')K(\boldsymbol{x}-\boldsymbol{x}') \tag{2.3.27}$$

where the integral kernel is the Fourier transform of $\sqrt{2\omega_p}$, $K(\boldsymbol{x}-\boldsymbol{x}') = \int \mathrm{d}^3p\sqrt{2\omega_p}e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')}$.

Overall, we see that trying to come up with a spatial characterization of the state of a relativistic system we are facing the following trade off, covariance versus the probability interpretation. By choosing the Newton Wigner definition, we rather sacrifice covariance, but this could be justified for particular purposes (for example if one only cares about a particle description in a fixed frame).

Closing this section, I would like to mention that the literature is quite messy on this topic. This is mostly because of different conventions regarding the distribution of these ω_k factors, as it is also very common in QFT, only that in this kind of discussion it becomes particularly important to keep track. The narration above is mostly based on Hagg's book [2] but if you look at the initial paper by Newton and Wigner [5] you will see that their definition of the NW wavefunction is $\Psi_{\text{NW}}(t,x) = \int \frac{\mathrm{d}p}{2\omega_p}\tilde{\Psi}(p)e^{\mathsf{i}\mathsf{x}\mathsf{p}}$ which looks more than the covariant definition (2.3.2). Later in the paper they show that translation covariance of this state imposes that $\Psi(p)$ has to be proportional to $\sqrt{\omega_p}$, say $\tilde{\Psi}(p) = \sqrt{\omega_p}\Psi(p)$ and so we can rewrite their definition as

$$\Psi_{\text{NW}}(t,x) = \int \mathrm{d}p\frac{\Psi(p)}{\sqrt{2\omega_p}}e^{\mathsf{i}\mathsf{x}\mathsf{p}} \tag{2.3.28}$$

where we can call $\Psi_{\text{NW}}(\boldsymbol{p}) := \frac{\Psi(\omega_p,\boldsymbol{p})}{\sqrt{2\omega_P}}$ which matches exactly the definition given above in (2.3.21) which clarifies how the two definitions are consistent.

2.3.2 A field theoretic twist

First, some comments on the kind of arguments that have been presented so far regarding the particle concept, the field concept and how they relate to each other. The spirit of the the first chapter, following Wald [3], was that quantum field theory is the quantum theory of fields and there is no reference whatsoever to the notion of particle. One can easily accept this view for a QFT that comes from a classical field theory, like Maxwell's electromagnetism. Quantizing a classical field theory, one should end up with a quantum field theory for light. But what about the classical (non-relativistic) mechanics of massive particles? Once we quantize that, we end up with non relativistic quantum mechanics. It is because of relativity, that we need to describe a massive relativistic quantum particle with a massive Klein Gordon field, which in fact has no classical analogue (we need both c and $\hbar$ to form the Compton scale so that the Klein Gordon equation makes sense dimensionally, see Appendix A1). In the end, it seems that we need a field description for something that started out as a particle and a particle description for light that started out as a field (the famous photons). These two notions are so interwined that one can keep chasing ones tail indefinitely.

Even if we consider the field description as fundamental, we still need to account for particle-like phenomenology and this is the role of the detector models that we discussed at the end of the first chapter. But then one can ask what the detector is, a particle or a field? And in fact, both scenarios have been investigated in the literature [16, 3, 24]. A view that takes field theory seriously (complemented with models that can explain experience) seems to be supported by the no go theorems of chapter 3, which seriously challenge the fundamental status of the particle concept. So up to section 3.4 of this thesis, one has every reason to believe that fields are fundamental, particles don't make any sense in a relativistic setup and all we can hope for is empirically successful detector models.

The spirit of the Newton Wigner attempt is quite the converse of that. They tried to rescue the particle concept for a relativistic quantum theory, without introducing any field concepts. The connection to field theoretic concepts was made later by Fulling [12]. Coincidentally, this also came up in my research when my supervisors asked me to calculate the components of the stress energy tensor for single particle states. Their intuition was that if we want to ask "where a particle is" in quantum field theory, a theory in which observables are local, we should better look at the spatial profile of observables when the

state of the field is what we call a one particle state. As we saw, in the absence of a position operator in QFT there is no wave function to tell us where the particle might be. But one can still refer to the Newton Wigner wavefunction, to which we managed to associate the energy density of a 'particle' in QFT.

We consider one particle states that are formed by acting a smeared field operator on the ground state

$$|\Psi\rangle = \int \mathrm{d}x \Psi(x) \hat{\Phi}(x,0) |0\rangle = \int \mathrm{d}x \Psi(x) \int \frac{\mathrm{d}k}{\sqrt{2k_0}} e^{-ikx} a_k^\dagger |0\rangle \tag{2.3.29}$$

$$= \int \frac{\mathrm{d}k}{\sqrt{2k_0}} \int \mathrm{d}x \Psi(x) e^{-ikx} |k\rangle = \int \frac{\mathrm{d}k}{\sqrt{2k_0}} F(k) |k\rangle \tag{2.3.30}$$

where $F(k) = \mathcal{F}[\Psi(x)]$ the Fourier transform of the spatial smearing $\Psi(x)$. We see that this is in fact a one particle state since the negative frequency part of the field operator annihilates the vacuum.

It easy to see that for such states the expectation value of the field operator is zero and as a result

$$(\Delta\Phi)^2 = \langle\Psi|(\hat{\Phi} - \bar{\Phi})^2|\Psi\rangle = \langle\Psi|\hat{\Phi}^2|\Psi\rangle. \tag{2.3.31}$$

We need to calculate the expectation value of the field squared over such states

$$\langle\Psi|\hat{\Phi}^2(x,t)|\Psi\rangle = \left(\int \frac{dp}{\sqrt{2p_0}} F^*(p) \langle 0| \hat{a}_p\right) \hat{\Phi}^2(x,t) \left(\int \frac{dp'}{\sqrt{2p'_0}} F(p') \hat{a}^\dagger_{p'} |0\rangle\right) \tag{2.3.32}$$

Using the usual field decomposition $\hat{\Phi}(x,t) = \int \mathrm{d}k u_k(x,t)\hat{a}_k + u_k^*(x,t)\hat{a}_k^\dagger$ the non zero terms this expectation value are

$$\int \frac{dp}{\sqrt{2p_0}} \frac{dp'}{\sqrt{2p'_0}} dk dk' \big(F^*(p) u_k u^*_{k'} F(p') \langle 0|\hat{a}_p \hat{a}_k \hat{a}^\dagger_{k'} \hat{a}^\dagger_{p'}|0\rangle + F^*(p) u_k^* u_{k'} F(p') \langle 0|\hat{a}_p \hat{a}_k^\dagger \hat{a}_{k'} \hat{a}^\dagger_{p'}|0\rangle\big).$$

From the commutation relation $[a_k^\dagger, a_p] = \delta(k-p)$ we have that

$$\langle 0|\hat{a}_p \hat{a}_k \hat{a}^\dagger_{k'} \hat{a}^\dagger_{p'}|0\rangle = \delta(k'-p)\delta(p'-k) + \delta(k'-k)\delta(p'-p) \tag{2.3.33}$$

$$\langle 0|\hat{a}_p \hat{a}_k^\dagger \hat{a}_{k'} \hat{a}^\dagger_{p'}|0\rangle = \delta(k-p)\delta(k'-p') \tag{2.3.34}$$

and finally we get

$$\langle\Psi|\hat{\Phi}^2|\Psi\rangle = 2\left|\int \frac{dk}{\sqrt{2k_0}} F(k) u_k(x,t)\right|^2 + \int \frac{dp}{2p_0} |F(p)|^2 \int dk |u_k(x,t)|^2 \tag{2.3.35}$$

If the smearing $\Psi(x)$ is square integrable, so is its Fourier transform $F(p)$. The divergent second term goes away by normal ordering since $\langle 0|\hat{\Phi}^2|0\rangle = \int \mathrm{d}k|u_k(x,t)|^2$. Subtracting the vacuum expectation value, the renormalized field fluctuations are

$$(\Delta\Phi)^2 = \langle\Psi| : \hat{\Phi}^2(x,t) : |\Psi\rangle = 2\left|\int \frac{\mathrm{d}k}{\sqrt{2\omega_k}} F(k)u_k(x,t)\right|^2 \tag{2.3.36}$$

$$= 2\left|\int \frac{\mathrm{d}k}{2\omega_k} F(k)e^{i(kx-\omega_k t)}\right|^2 = |\Psi_{\text{NW}}(x,t)|^2 \tag{2.3.37}$$

So we see that the spatial profile of the expectation value of the square field operator (and so the field fluctuations) are given by the Newton Wigner wavefunction of the one particle state. What other quantities of a QFT could one associate to the Newton Wigner wavefunction of a one particle state? In every quantum theory we can project the abstract state to the basis that corresponds to the classical configuration space.[1] Similarly to how we get a wavefunction for a particle through $\langle x|\Psi\rangle = \Psi(x)$, we get a wavefunctional for a quantum field theory $\Psi[f(x)] = \langle f(x)|\Psi\rangle$, which gives the probability amplitude for the field to be found in a particular field configuration. One then can ask, if the field is in a one particle state, can we associate the possible field configuration with the Newton Wigner wavefunction of that state? We find that

$$\Psi[f(x)] = \langle f(x)|0\rangle \int dx f(x)\Psi_{\text{NW}}(x). \tag{2.3.38}$$

This means that the probability of finding a particular field configuration is basically that over the vacuum state multiplied with the inner product of the field configuration with the NW wavefunction. For example a field configuration that is orthogonal to the Newton Wigner wavefunction will have zero probability to be found.

Next we find that similarly to (2.3.37) we can associate the components of the stress energy tensor to the Newton Wigner wavefunction for one particle states, exactly because it is quadratic in the field operators. For example, the stress energy density is given by

$$\hat{T}_{00} = \frac{1}{2}[(\partial_t\hat{\Phi})^2 + (\nabla\hat{\Phi})^2 + m^2\hat{\Phi}^2] \tag{2.3.39}$$

Taking the expectation value over $|\Psi\rangle$ and renormalizing as before we get

$$\begin{aligned}\langle\Psi| : \hat{T}_{00} : |\Psi\rangle &= \left|\int \frac{\mathrm{d}k}{\sqrt{2k_0}} F(k)\partial_t u_k(x,t)\right|^2 + \left|\int \frac{\mathrm{d}k}{\sqrt{2k_0}} F(k)\nabla u_k(x,t)\right|^2 + m^2\left|\int \frac{\mathrm{d}k}{\sqrt{2k_0}} F(k)u_k(x,t)\right|^2 \\ &= |\nabla\Psi_{\text{NW}}(x,t)|^2 + |\partial_t\Psi_{\text{NW}}(x,t)|^2 + m^2|\Psi_{\text{NW}}(x,t)|^2\end{aligned}$$

[1] Schematically one can see the quantisation procedure as assigning a ket to every possible configuration, which would be a position for a particle $x \to |x\rangle$ and a field configuration for a field $f(x) \to |f(x)\rangle$.

We saw in the previous section that the Newton Wigner wavefunction is defined in a way that it could maintain the interpretation of probability density to find the particle in position space, similarly to non relativistic quantum mechanics. Nevertheless, in QM the wave function is not an observable and cannot be associated with densities of observables as above because this can only happen in a field theory, where observables are locally defined. It is in a QFT that we managed to associate the probablity density with the energy density of the particle, and yet in a quite non expected way. One would expect that the energy density shall be higher were the amplitude of the probability is higher, but we see in (2.3.40) how the derivatives also contribute. Overall we see that the energy density for one particle states is shaped by the Newton Wigner wavefunction and its derivatives. Given this, one could try to answer where/how such a particle gravitates using semiclassical gravity and the semicalssical Einstein equation. To the extent of the applicability of semiclassical gravity for such states, one can argue that gravitational field of such a particle is not only sensitive to the "center of mass" of the particle but also the profile of the derivatives of its wavefuntion. One can try to go one step further, and consider a superposition of such a particle in two different locations i.e. the superposition $|\Psi\rangle = \alpha\,|\Psi_1\rangle + \beta\,|\Psi_2\rangle$ of two states with different momentum smearings $F_{1,2}(k)$ and Newton Wigner wavefunctions $\Psi_{1,2}(x,t)$. We get that

$$\langle\Psi| : T_{00} : |\Psi\rangle = |\alpha|^2\langle\Psi_1| : T_{00} : |\Psi_1\rangle + |\beta|^2\langle\Psi_2| : T_{00} : |\Psi_2\rangle \tag{2.3.40}$$

$$+ 2\mathrm{Re}[\alpha\beta^*\big((\nabla\Psi_2^*)(\nabla\Psi_1) + (\partial_t\Psi_2^*)(\partial_t\Psi_1) + \Psi_2^*\Psi_1\big)] \tag{2.3.41}$$

If the branches of the superposition are orthogonal, say for example $\Psi_{1,2}$ are compactly supported in disjoint regions, then the third term in the expression above vanishes. But if the two branches are interfering it seems that semiclassical gravity will see the interference. Even further, notice that the two wavefunction could be such that the third term is negative. This is quite an interesting feature because it means that one single particle can be characterized by negative energy density due to its delocalization in space. There are more interesting questions that one could ask in this setup, and this is work in progress.

Chapter 3

Local versus global

One of the most common problems that a student of quantum mechanics learns how to solve, is the so called particle in a box. By box we usually mean an infinite square well potential, so that the particle has zero probability to be found outside the box i.e. the particle's wavefunction is compactly supported inside the box. So in this set up we know that there is one particle definitely inside the box, maybe with higher probability to be found on the right or the left depending on what its wavefunction looks like. Then in most quantum field theory textbooks one finds the argument that if we make the box smaller and smaller we are decreasing the uncertainty in position so much that the uncertainty in momentum or energy is large enough to allow for particle creation. Due to the special theory of relativity we enter the regime where we cannot know how many particles are in the box, and somehow we have to switch to a quantum field theoretic description to accommodate the phenomenon of particle creation and annihilation.

On top of the physical intuition described above, we just saw some formal mathematical results like the Malament's theorem, only one of the results that support the common knowledge that there is no position operator in quantum field theory and" x is just a label of the field operator, as it should, since the theory is supposed to be treating space and time on equal footing due to relativity". After all, instead of position operators we should better be looking for local number operators in a QFT. But as we discussed the Reeh Schlieder theorem implies that such objects cannot exist either. In the absence of well defined position operators and local number operators it becomes unclear what are the spatial features that one can attribute to a QFT state.

Due to all these subtleties associated with position space, momentum space is somehow preferable in the standard quantisation procedure. This is the reason why the so called

quanta are not particle like entities in any sense of the word, they are rather global. This rather awkward situation has motivated the operational view that "particles are whatever the detectors detect". As Paul Davies puts it "there are quantum states and there are particle detectors. Quantum field theory enables us to predict probabilistically how a particular detector will respond to that state. That is all. That is all there can ever be in physics, because physics is about the observations and measurements that we can make in the world. We can't talk meaningfully about whether such-and-such a state contains particles except in the context of a specified particle detector measurement" [43].

Of course physical theories need to account for particle phenomenology, but this does not shed light on the ontological question "what do the detectors detect". This question is irrelevant if we adopt Davies' opinion that physics is about the observations and measurements that we can make in the world, as quoted above. From a realist's perspective physics is about what kind of things and processes the world is made out of, independently of us making measurements and observations. As long as we agree that there is some kind of reality out there with which we interact, the details of our interaction with it and the available theories through which we describe these interactions, should provide us with some clues about the nature of reality. Of course not all elements of the mathematical description correspond to elements of reality. It is highly non trivial how one identifies these elements of the 'best' working theories that we have and of those that will succeed them, converging towards 'how things are'. Succeeding in this task we should be able to successfully describe phenomenology without involving any 'miracles', but a successful phenomenological description cannot provide answers to ontological questions. To put it simply, ontology can explain the phenomenology but not the other way around.

In the two articles that we will review below [4, 1] the authors are not necessarily defending an ontological status of local particles but nevertheless they are introducing mathematical notions of local quanta. After all an ontological interpretation, or any interpretation, boils down to establishing a map between the mathematical formalism of the theory and elements of reality [44]. If one wishes to defend the ontological status of particles in QFT, they should better identify the corresponding mathematical structure that would allow for such an interpretation. In the two articles, the authors investigate whether there is an alternative quantisation procedure, or any alternative way of looking at the usual quantisation of a field theory, that would allow for a notion of quanta that are local. Nevertheless, there remains some hesitation in making the interpretative step as we will discuss in the last section of this chapter.

3.1 Local quanta, unitary inequivalence and vacuum entanglement

I was particularly happy to run into this article, first because it is very closely related to my research interests (chapter 4) and second because it serves as a very concrete example of the abstract QFT construction that we reviewed above [3] (chapter 1). The authors apply the alternative quantization of section 1.3.2 for a QFT in a box. As we saw, there are certain ambiguities in this quantisation procedure which challenge the applicability/usefulness of the particle notion in QFT. Nevertheless, the authors of this paper suggest that it is exactly this ambiguity that we can make use of, to construct strictly localised and quasi-localised one particle states. Their alternative quantisation procedure turns out to be unitarily inequivalent to the ordinary one, but both constructions are argued to be "legal" and physically relevant.

First let us ask, what does it mean to define local quanta for a QFT in a box? It means that one can answer whether such a particle is likely to be found on the left or right side of the box, without having to split the box into two boxes. Without going into the discussion of what a box represents in QFT, we just comment that this question cannot be answered in the standard quantization that uses global stationary modes. We might have this intuition that doing QFT in a box plays some role of confinement, but everything remains global within the box. Quanta in a box inherit all the side effects of the standard quantisation procedure, and the only way to localize them is to define them in a yet smaller box, which is equivalent with splitting the initial one into many. So let's see how they go around this issue in this paper.

As we know already, to quantise a classical real valued field obeying the equations of motion $(\Box + m^2)\phi(x,t) = 0$ we have to introduce the complexified vector space of solutions $\mathcal{S}^{\mathbb{C}}$ a suitable subspace of which will serve as the one particle Hilbert space. This space is equipped with the usual Klein-Gordon (pseudo) inner product which fails to be positive definite. Then we introduce a complete orthonormal basis of the complexified space of solutions $\{f_m(x,t), f_m^*(x,t)\}$ where m is a discrete index when we are quantising in a box, such that $(f_m, f_n) = \delta_{mn}$, $(f_m^*, f_n^*) = -\delta_{mn}$, $(f_m^*, f_n) = 0$ and as a result any complex valued solution can be written as $\Phi(x,t) = \sum_m a_m f_m(x,t) + a_m^* f_m^*(x,t)$ where $a_m^* = (f_m^*, \Phi)$. The one particle space is defined as the span of the positive norm modes, so that we can complete to obtain a Hilbert space

$$\mathcal{H} = \text{span}\{f_m\} \tag{3.1.1}$$

The complex structure that we introduced in section.. in this context is simply $J = \mathrm{i}\sum_m |f_m\rangle\langle f_m| + |f_m^*\rangle\langle f_m^*|$. The definition of the one particle Hilbert space, or equivalently

the definition of the map J, depends explicitly on the choice of basis for $\mathcal{S}^{\mathbb{C}}$ which leads to the well-known particle number ambiguity in QFT, and is also the source of unitarily inequivalent representations.

In this paper they consider a quantum field in an one-dimensional box of size L, where the usual choice of modes is

$$U_n(x,t) = \frac{e^{-\mathrm{i}\omega_n t}}{\sqrt{L\omega_n}} \sin\frac{n\pi x}{L}, \quad \omega_n = \sqrt{\frac{\pi^2 n^2}{L^2} + m^2} \tag{3.1.2}$$

First we need to clarify why the quantisation that follows from this choice is global. There is no state in the one particle Hilbert space that can be confined in a *subregion* of $[0, L]$ for any arbitrarily small time interval Δt. This is guaranteed by Hegerfeld's theorem [41] and as he says for this behaviour "no field theory, no relativity is needed, only a Hilbert space $\mathcal{H}$ and positivity of energy". It can be argued to be an overstatement that relativity plays no role in this, but indeed instantaneous spreading is well known in the context of non relativistic quantum mechanics. It is not a problem that the wavefunction of the quantum mechanical particle $\Psi(x,t)$ spreads superluminally because of the non relativistic nature of the theory. This is why the first attemt for relativistic quantum mechanics has been to solve relativistic wave equations where a propagation velocity for $\Psi(x,t)$ is built in. But as we saw in this case Ψ cannot be interpreted as a probablity amplitude, and there is no clear connection of the variable x and the position of the particle to start with.

In the context of a relativistic field theory the field amplitude $\Phi(x,t)$ satisfies a relativistic wave equation, and it does not carry any interpretation of a wavefunction, despite what the term "second quantization" might suggest. If we were to restrict only to the positive frequency part, the field expectation value needs to have support in the entire cavity for almost all times (even if we specify initial conditions of compact support in a subregion) which for example implies that observables like the Hamiltonian density will also be generally supported in the whole box. Since Φ does not have the interpretation of a wavefunction, we can only think of superluminal propagation in terms of the local observable quantities of the theory. It is the very definition of the one particle sector in terms of only positive frequencies that makes Hegerfeld's theorem applicable, so here they propose an alternative definition of $\mathcal{H}$ that suitably mixes positive and negative frequencies so that superluminal propagation is avoided. Another choice of local non-stationary modes, allows for localisable one particle states i.e. local excitations that can be localised instantly and then causally spread out in the rest of the box.

To motivate these local modes, the authors are considering what happens if we place a perfect mirror at a location $x = r < R$ which imposes the boundary condition $\phi(r,t) = 0\ \forall t$.

Then we would consider the usual stationary modes $\{u(x,t), u^*(x,t)\}$ and $\{\bar{u}(x,t), \bar{u}^*(x,t)\}$ at each side of the cavity

$$u_l(x,t) = \frac{1}{\sqrt{r\omega_l}} e^{-\mathrm{i}\omega_l t} \sin\frac{l\pi x}{r}, \quad \bar{u}_l = \frac{1}{\sqrt{r\bar{\omega}_l}} e^{-\mathrm{i}\bar{\omega}_l t} \sin\frac{l\pi(x-r)}{R-r} \tag{3.1.3}$$

where $\omega, \bar{\omega}$ the corresponding frequencies as in (3.2.2). But this leaves us with two distinct cavities $[0,r]$ and $[r,R]$ which is a different physical problem (remember, we don't want to split the box into two boxes!). Mathematically, these modes are in fact not suitable for quantising the whole cavity $[0,R]$ since they do not form a basis for the complexified space of solutions $\mathcal{S}^{\mathbb{C}}$ over $[0,R]$.

The remedy that the authors are suggesting is that we can use the standard modes (3.1.3) at some $t = 0$, only as initial conditions, while for later times they are free to spread out over the entire box without satisfying the boundary condition at r. We can convince ourselves that this would define another basis of $\mathcal{S}^{\mathbb{C}}$ for the entire cavity. First it is noted that at $t = 0$ these modes comprise the Fourier basis in $[0,r]$ and $[r,L]$ and by Fourier analysis we have pointwise convergence in $\mathcal{L}^2([0,L])$ norm which means we can generate any initial condition and thus any solution of $\mathcal{S}^{\mathbb{C}}$. To solve for all times, they impose initial conditions

$$u_l(x,t=0) = \frac{1}{\sqrt{r\omega_l}} \sin\frac{l\pi x}{r}\Theta(r-x), \quad \dot{u}_l(x,t=0) = -\mathrm{i}\omega_l u_l(x,t=0)$$

and similarly for $\bar{u}$. A we described in words, the initial profile of the modes is given by the stationaty ones on each side of the box, and the choice of the first derivative is such that these non stationary modes "imitate" the stationary ones. Instead of solving explicitly, we can calculate the inner product of the local modes with the global ones at $t = 0$, and since the inner product is time independent, we can expand on the global basis to solve for all times

$$u_m(x,t) = \sum_n (U_n, u_m)U_n(x,t) - (U_n^*, u_m)U_n^*(x,t) \tag{3.1.4}$$

$$\bar{u}_m(x,t) = \sum_n (U_n, \bar{u}_m)U_n(x,t) - (U_n^*, \bar{u}_m)U_n^*(x,t) \tag{3.1.5}$$

Now we can use the basis $\{u(x,t), u^*(x,t), \bar{u}(x,t), \bar{u}^*(x,t)\}$ to quantise the field in the full cavity which suggests the following expansion for the field operator

$$\Phi(x,t) = \sum_m a_m u_m(x,t) + \bar{a}_m \bar{u}_m(x,t) + a_m^\dagger u_m^*(x,t) + \bar{a}_m^\dagger \bar{u}_m^*(x,t) \tag{3.1.6}$$

Note that this decomposition need to have four terms, rather than the usual two terms when we expand over the global modes $\Phi(x,t) = \sum_m A_m U_m(x,t) + A_m^\dagger U_m^*(x,t)$.

Next we notice that by construction the $a, a^\dagger$ and $\bar{a}, \bar{a}^\dagger$ associated to the non-overlapping initial conditions commute, which implies the split

$$\mathcal{F}^L = f \otimes \bar{f} \tag{3.1.7}$$

where $f, \bar{f}$ the Fock spaces associated with the corresponding creation and annihilation operators $\{a, a^\dagger\}$ and $\{\bar{a}, \bar{a}^\dagger\}$, and L stands for is local. This means that with this quantisation scheme we end up with a tensor product structure between the left and right side of the box, which is not a feature of the standard quantisation procedure.

Let us pause for a second, to compare with the scenario where instead we do have two boxes to start with. In this case of course the Hilbert space of the composite system is given by a tensor product. By virtue of (3.1.7) one might worry (as I did when I first read this) that the authors have split the big box into two boxes, even though they did not mean to. To clarify this one has to be careful with the role of time evolution, and the translation between solutions and initial data that we established in the previous chapter. The decomposition (3.1.7) was argued to follow from the commutativity of $a, a^\dagger$ and $\bar{a}, \bar{a}^\dagger$, which here is guaranteed by $\{u(x,t), u^*(x,t), \bar{u}(x,t), \bar{u}^*(x,t)\}$ being an orthonormal basis for the full solution space. These modes were chosen to initially compactly supported on the disjoint regions $(0,r)$ and (r,L) but exactly after they will start fuzzing in a way that they span the full solution space, so the commutativity of the creation and annihilation operators happens non trivially, comparing to the case of two boxes where operators of different systems trivially commute to give a tensor product structure.

We can also see how the tensor product comes about without appealing to the commutativity of the operators, through the construction that we presented in the previous chapter. We have used the identification between solutions and initial data to split *only initially* the support of the solutions. If we denote as $\mathcal{S}_L$ the solutions that are initially compactly supported in $(0,r)$ and $\mathcal{S}_R$ the solutions that are initially supported in (r,L), the full space of solutions can be decomposed as $\mathcal{S} = \mathcal{S}_L \oplus \mathcal{S}_R$ by linearity. One can see that this decomposition translates into the choice of the single particle space $\mathcal{H}_L \oplus \mathcal{H}_R$ and so the Fock space of the full theory is

$$\mathcal{F}_L = \mathcal{F}[\mathcal{H}_L \oplus \mathcal{H}_R] = \mathcal{F}[\mathcal{H}_L] \otimes \mathcal{F}[\mathcal{H}_R] = f \otimes \bar{f} \tag{3.1.8}$$

as in (3.1.7). In this sense the commutativity of the operators follows from the underlying tensor product structure, rather than the other way round, but always dealing with one box.

This tensor product structure is maybe the most important feature of the local Fock space. It implies that the corresponding ground state is unentangled

$$|0\rangle_L = |0\rangle \otimes |\bar{0}\rangle \tag{3.1.9}$$

and so in this way of quantizing a QFT in a box, there is no vacuum entanglement. Furthermore, this is exactly what allows for the local quanta to be strictly localized.

To see this, first let us define the notion of strict localisation as introduced by Knight [45]. A state $|\Psi\rangle$ is strictly localised in a region $\mathcal{R}$ if the expectation value of every local observable $\mathcal{O}(x)$ outside that region is the vacuum expectation value, namely

$$\langle\Psi|\mathcal{O}(x)|\Psi\rangle = \langle 0|\mathcal{O}(x)|0\rangle, \;\; \forall x \notin \mathcal{R} \tag{3.1.10}$$

Knight had shown that the usual global Fock states cannot be strictly localized. Nevertheless, the local quanta defined here can be, thanks to the separability of the vaccum. Ecplicitly, let us consider a local single excitation $|\Psi\rangle = a_n^\dagger |0\rangle_L = |1_n, \bar{0}\rangle$ on the left side of the box, noting that any observable on the right side of the box will be function of $\bar{a}_m, \bar{a}_m^\dagger$. So we see that

$$\langle\Psi|\mathcal{O}(\bar{a}_m, \bar{a}_m^\dagger)|\Psi\rangle = \langle\bar{0}, 1_n|\mathcal{O}(\bar{a}_m, \bar{a}_m^\dagger)|1_n, \bar{0}\rangle = \langle\bar{0}|\mathcal{O}(\bar{a}_m, \bar{a}_m^\dagger)|\bar{0}\rangle \tag{3.1.11}$$

the expectation value of any observable on the right hand side of the box is vacuum expectation value when we create a local excitation on the left side of the box, which happens trivially because of separability of the ground state.

Next we need to see how these local excitations propagate causally, and how microcausality is built into this construction. To see this the authors introduce a third subregion of the box $[\tilde{r}, R]$ where $\tilde{r} > r$ and built the corresponding Fock space $\tilde{f}$ out of the creation and annihilation operators $\tilde{a}, \tilde{a}^\dagger$ defined in terms of local modes that are compactly supported at a later time $\tau > 0$. Knowing the expressions for $a, a^\dagger$ and $\tilde{a}, \tilde{a}^\dagger$ in terms of the global $A, A^\dagger$ we can calculate that

$$[\tilde{a}_\nu, a_m^\dagger] = (\tilde{u}_n, u_m) \tag{3.1.12}$$

which is zero for $\tau < |r - \tilde{r}|$ which is when the two regions are spacelike separated. This imples that the operators commute under this conditions so we can show strict locality

$$\langle 1_n, \bar{0}|\mathcal{O}(\tilde{a}, \tilde{a}^\dagger)|1_n, \bar{0}\rangle = \langle 0_L|a_n\mathcal{O}(\tilde{a}, \tilde{a}^\dagger)a_n^\dagger|0_L\rangle = \langle 0_L|\mathcal{O}(\tilde{a}, \tilde{a}^\dagger)a_n a_n^\dagger|0_L\rangle = \langle 0_L|\mathcal{O}(\tilde{a}, \tilde{a}^\dagger)|0_L\rangle \tag{3.1.13}$$

which demonstrates that the local excitation propagates causally. Note that it is not clear if this commutativity implies some tensor product structure for $\mathcal{F}_L$ or to put it differently it is not explained in the paper if we should think of the Fock space $\tilde{f}$ as a factor of $\mathcal{F}_L$ as we did before, since the modes we used to construct those were localised at a different time. My understanding is that the reasoning I tried to scketch for (3.1.8) doesn't go through, exactly because the solutions are compactly supported at different times (but not very sure about that). At least we cannot so straightforwardly decompose the space of solutions with this choice of initial data at different times.

Finally, the authors demonstrate how the local quantisation described above is unitarily inequivalent to the usual quantisation. For this they are using the following criterion (that can be traced back to Fulling [12]), given a Bogoliubov tranformation between two Fock space representations i.e. two sets of creation and annihilation operators here $\{a_n, a_n^\dagger, \tilde{a}_n, \tilde{a}_n^\dagger\}$ the local ones and $\{A_n, A_n^\dagger\}$ the global ones, the two representations are unitarily inequivalent iff the expectation value of the number operator as defined in one representation over the vacuum state defined in the other, is divergent. In our case this means that the expectation value local number operator $N = \sum_n \left(a_n^\dagger a_n + \tilde{a}_n^\dagger \tilde{a}_n\right)$ over the global vacuum diverges

$$\langle 0_G | N | 0_G \rangle = \infty \tag{3.1.14}$$

Laura Ruetsche in [6] takes the symptom of unitary inequivalence (3.1.14) as implying incommersurable particle notions. In the case where there exists a Bogoliubov transformation but (3.1.14) is not divergent in fact the two theorists that adopt the different descriptions will disagree on the content of particle states, but like two rulers that use different units of length, the measurements of one can be translated in the measurements of the other. As Laura Ruetsche explains "In short, they agree about what states are in the extension of the particle concept, and about what discriminations, even modal ones, that concept makes possible. This suggests to me that the theorists disagree not about what particles are, but about how to enumerate them. Insofar as their enumerations are intertranslatable, their particle notions are notational variants on one another." On the contrary, unitary inequivalence is rather more puzzling, like two rulers that cannot be compared to each other.

This sounds quite discouraging, why would one take the local quanta seriously if they end up being unitarily inequivalent to the usual global ones? Crucially, despite the unitary inequivalence, the authors point out the following mathematical asymmetry. Even though the global number operators are not well-defined in $\mathcal{F}_L$, the local number operators are well-defined in $\mathcal{F}_G$. For example we can use the local number operators to excite the global

vacuum which gives what they call quasi-localised states. Of course we should not expect strictly localised states by locally exciting the global vacuum, since strict localisation was based on the separability of the local vacuum which the global does not posses. As the authors explain, the failure of these "quasi localized" states to be strictly localised is directly related to the vacuum entanglement and the Reeh-Schlieder theorem.

Finally, one of the important points of the paper is the convergence of the local modes to the global ones which is also the theme of the Rovelli paper that we will discuss next. We need to keep in mind that this only well defined because the local modes converge to the global ones in the strong operator topology in $\mathcal{F}_G$ where they are both well-defined despite the inequivalence of the representations. So we have to make use of the mathematical asymmetry that we discussed above to define the limit in which the local becomes global for $r \to R$. Concretely, we have that

$$a_l = \sum_N (u_l, U_N) A_N + (u_l, U_N^*) A_N^\dagger \tag{3.1.15}$$

where

$$(u_l, U_N) = \frac{1}{\sqrt{Rr\Omega_N \omega_l}} \frac{\frac{l\pi}{r}(-1)^l \sin\frac{N\pi r}{R}}{\Omega_N - \omega_l} \tag{3.1.16}$$

$$(u_l, U_N^*) = -\frac{1}{\sqrt{Rr\Omega_N \omega_l}} \frac{\frac{l\pi}{r}(-1)^l \cos\frac{N\pi r}{R}}{\Omega_N + \omega_l} \tag{3.1.17}$$

where $\Omega_N = m\sqrt{1 + (\frac{N\pi}{mR})^2}$ and $\omega_l = m\sqrt{1 + (\frac{l\pi}{mr})^2}$. For $r \to R$ we get that $(u_l, U_N) \to \delta_{Nl}$ and $(u_l, U_N^*) \to 0$.. Another interesting limit is the one in which we are trying to localise the particle beyond its Compton wavelength, namely $r \ll \lambda_c$ where $\lambda_c = \frac{1}{m}$. In this case the authors claim that the β coefficients of the Bogoliubov transformation have to vanish and to see that I wrote them explicitly in terms of the quantities $r/\lambda_c = mr$ and $R/\lambda_c = mR$

$$(u_l, U_N^*) = \frac{l\pi(-1)^l \sin\frac{N\pi r}{R}}{mr(\pi^2 N^2 + m^2 R^2)^{1/4}(\pi^2 l^2 + m^2 r^2)^{1/4}} \frac{1}{\sqrt{1 + (\frac{\pi N}{mR})^2} + \sqrt{1 + (\frac{\pi l}{mr})^2}} \tag{3.1.18}$$

One can convince themselves that this goes to zero for $mr \to 0$. What exactly changes when we approach the Compton wavelength? In that limit, we have that

$$\langle 0_G | n_l | 0_G \rangle = \sum_N |(u_l, U_N)^*|^2 \to 0 \tag{3.1.19}$$

so the expectation value of number of local quanta over the global vacuum vanishes. This is another manifestation of the fact that β coefficients vanish in this limit we have that the Bogoliubov transformation (4.1.9) reduces to

$$a_l = \sum_N (u_l, U_N) A_N \tag{3.1.20}$$

In the limit, the local annihilation operators are just a linear combination of the global annihilation operators (and not the creation operators). This means that the two sets of operators, local and global, ought to annihilate the same vaccum in that limit $|0\rangle_L \to |0\rangle_G$. Does this imply that the global vacuum would adopt features of the local vacuum in the limit, like the separability between left and right of the box and accordingly the failure of Reeh Schlieder theorem, existence of local number operators on the global vacuum etc. This is a question that also came up in my research, on a different but yet similar set up. We attempt to answer some of these questions in the next chapter.

3.2 What is a particle?

The motivation for this paper [1] mostly comes from the challenges of the particle notion in classical and quantum gravity. On curved spacetime there is no preferred notion of particle states, but still one has to account for the particle like phenomenology which is that "particles" are entities that are detected locally. This is why the authors ask if particles are global or local entities and quantify how these two notions converge to each other in the limit in which particle detectors are sufficiently large. The main idea is that QFT should be interpreted through local observable quantities, rather than a theory about particles, and it is local observable quantities and correlators through which the authors are defining the convergence of the local to global particle notions in the weak topology.

This is easier to see first in the example of a chain of n coupled harmonic oscillators q_i, $i = 1...n$ described by the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n} \left(q_i^2 + p_i^2\right) + \lambda \sum_{i=1}^{n-1} q_i q_{i+1} \tag{3.2.1}$$

By diagonalising the Hamiltonian we obtain the normal modes of the system Q_a, $a = 1...n$ in terms of which we define our notion of global quanta as usual. The eigenstates of the Hamiltonian are of the form $|n_1, ..., n_n\rangle$ where n_a the number of excitations of the a-th

normal mode. So from now on the index i is to indicate the local degrees of freedom while the index a indicates the global normal modes. Now let us consider the state

$$|i\rangle = \sum_{a=1}^{n} U_{ia} |1_a\rangle \tag{3.2.2}$$

where $|1_a\rangle = |0...1_a...0\rangle$ a single excitation of the a-th global mode. This is a one- particle state, in the global sense, which we could think of as concentrated mostly on the i-th local oscillator. But crucially this is *not* what we would call a local excitation of the i-th degree of freedom, denoted as $|i\rangle_{\text{loc}}$. The main point of this paper is demonstrated by exactly contrasting the two states $|i\rangle$ and $|i\rangle_{\text{loc}}$.

Let us now define $|i\rangle_{\text{loc}}$ by splitting the chain of oscillators in two regions R_1 and R_2 where region R_1 consists of the first n_1 degrees of freedom that act on a Hilbert space $\mathcal{H}_1$ and R_2 consisting of the remaining degrees of freedom acting on a Hilbert space $\mathcal{H}_2$ so that the total Hilbert space is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. We can also split the Hamiltonian in the local terms of the two regions plus the interaction terms $H = H_1 + H_2 + V$. Now we want to define local excitations in region R_1, keeping in mind that the state (3.2.2) with $i < n_1$ is rather a *global* state in R_1. If we make a local measurement in region R_1 we cannot end up with a state like (3.2.2) since it cannot be an eigenstate of any local observable. Since this state is constructed out of normal modes it is generally a state in which the two regions are correlated.

Now if we restrict to observables in region R_1 like the local Hamiltonian H_1 and find the normal modes that diagonalize *only* H_1 then we can define the local vacuum $|0\rangle_1$ and similarly to (3.2.2) we can define

$$|i\rangle_1 = \sum_{b=1}^{n_1} U_{ib} |b\rangle_1 \tag{3.2.3}$$

where $b = 1...n_1$ is indexing the normal modes in the region R_1. By repeating this procedure for the region R_2 we can define the local vacuum $|0\rangle_2$ as the lowest eigenstate of H_2 alone. Then we can define the local vacuum $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$ and the state

$$|i\rangle_{\text{loc}} = |i\rangle_1 \otimes |0\rangle_2 \tag{3.2.4}$$

The claim is that this state, rather than (3.2.2), is what can be measured by a local detector in region R_1. Then the question is whether $|i\rangle_{\text{loc}} \to |i\rangle$ in the limit where R_1 is sufficiently large. Turns out that this is not the case since $|i\rangle$ captures correlations between the two regions that are not captured by $|i\rangle_{\text{loc}}$. Nevertheless, what is converging

is the expectation value of local measurements. One of the main claim of this paper is that local observables cannot distunguish the local from the global states in the limit that the subregion is sufficiently large.

To demonstrate this point the authors consider the measurement of a local excitation in region R_1 as represented by the projector $P_i^{\text{loc}} = |i\rangle_1 \langle i|_1$ when the system is in the global state $|i\rangle$. The quantity

$$\langle i|P_i^{\text{loc}}|i\rangle = 1 - \frac{\lambda^2}{16} \tag{3.2.5}$$

captures the error between the local and global states, since if we replace $|i\rangle$ with $|i\rangle_{\text{loc}}$ this probability would be equal to 1. We see that the correction is second order in the coupling strength λ in the Hamiltonian, which makes sense since it is exactly the correlations between the two regions induced by the interacting Hamiltonian that creates this difference between local and global. We also observe that this quantity is independent of n_1 i.e. the size of the region which shows that 'smaller' local excitations do not differ from the global excitation $|i\rangle$ more than the 'bigger' ones as one might have expected.

Overall they conclude that global and local do not converge in the Hilbert space norm since $|i\rangle_{\text{loc}} \not\to |i\rangle$ but they do converge in the weak operator topology. To put it simply, for the purpose of describing particles as somehow local entities we should better refer to local observables rather than local states. Quantum field theory is naturally providing us with local observable quantities, since observables are functions of the field operators, but the notion of local states is more tricky.

The local object that the authors examine for a scalar quantum field theory is the two point function. They show how the two point function over the local vacuum converges to the one over the global Minkowski vacuum. For that they consider a quantum field theory in a box of size L and a local detector of size R, and for a massive quantum field theory the Compton wavelength $\lambda_c = 1/m$ is the lengthscale to which we can compare both the size of the box and the detector. Note that similar to the previous paper we have three characteristic length scales, but now "the size of the detector" plays the role of the subregion in the box. The subregion considered in the previous paper was crucial in how the local quantisation lead to unitarily inequivalent Hilbert spaces which seems not to be an issue in this paper.

The authors calculate the two point function of the field in the box over the local vacuum, and after a lengthy calculation it turns out to be

$$\langle 0_L|\phi(x,t)\phi(x',t)|0_L\rangle = \frac{2}{\pi}\sum_n [K_0(m|x-x'|+2nL) - K_0(m|x+x'|+2nL)] \tag{3.2.6}$$

where K_0 is the so called Mac Donald function. In the case that the Compton wavelength is much larger than the size of the box i.e. $mL \gg 1$ this function becomes

$$K_0(m|x \pm x'| + 2nL) \approx \sqrt{\frac{\pi}{2(m|x \pm x'| + 2nL|)}} e^{-m|x \pm x' + 2nL|} \tag{3.2.7}$$

so we see that it falls exponentially in mL i.e. the ratio of the Compton wavelength with the size of the box. In this limit $K_0(m|x - x'|) \gg K_0(m|x + x'|)$ which for $0 \ll x, x' \ll L$ gives

$$\lim_{x \to x'} \langle 0_L|\phi(x,t)\phi(x',t)|0_L\rangle = \frac{2}{\pi} K_0(m|x - x'|) \sim \langle 0|\phi(x,t)\phi(x',t)|0\rangle \tag{3.2.8}$$

so away from the boundaries of the box the local two point function converges to the one calculated over the global vacuum.

Returning to the initial motivation about local observables, note that for example the stress energy tensor components can be written as functions of this coincidence limit of the two point function as we learn in Wald [3]. So the convergence of the two point function (3.2.8) should go through all the way to stress energy tensor components, but this would be something to check explicitly. Then indeed we would be justified in describing particles through their local energy momentum since the energy tensor components would be insensitive to the difference between local and global states. Nevertheless the authors discuss that the distinction between global and local remains important at a conceptual level. If we ask ourselves what kind of entities is a quantum field theory describing, we might be tempted to say that it doesn't really matter given the analysis above, they could be local or global as long as any local measurement cannot tell the two apart. We need to keep in mind that the global particle notion tight to the global symmetries of spacetime which does not survive on curved spacetime snd this is privileging the local notion. The global notions are not generalisable on curved spacetime so we have any reason to distrust them. As the authors put it "local particle detectors detect particles also on curved spacetime".

As to whether a quantum field theory is a theory about particles, the authors argue that the answer is partially yes and partially no. Fock particles do not correspond to real physical objects since they carry 'unpalatable physical properties' and they cannot be detected by a localized detector. The elements of the Fock basis, providing us with the particle content of a QFT state, forms this picture of a QFT state consisting of a superposition of fixed particle sectors. This is mathematically precise but it is not to be taken literally since with no experiment we can measure the global number operators which would reduce the state to one of the fixed particle sectors. The Fock basis is a useful mathematical construction, but it is an unphysical one.

On the other hand, eigenstates of local observables are somehow more physical. In contrast to Fock states that are defined once and for all, the notion of a local particle depends on the observable we want to consider and as the authors say '"no observable is more real than others". Since local observables in a given region in general will not commute there is no unique local particle basis. In the absense of local number operators (as an implication of the Reeh Schlieder theorem) really all we have is local observables providing us with complementary characterizations of the same local object that we call a particle. The authors close the paper with the phrase that "the world is much more subtle than a bunch of particles that interact".

3.3 What do particle detectors detect?

It is a common view amongst the philosophers of quantum field theory that QFT is not a theory about particles [46, 47, 34, 6]. So far we mostly presented negative arguments about localizabiilty and local countability, but there are many more reasons why QFT's cannot support a particle ontology. These reason are acceleration on flat spacetime (Unruh effect), the curvature of a general spacetime and the existence of interactions [34]. In all these reasons it is relativity that undermines the particle interpretation. If QFT's are not about particles, then what are they about? A natural guess would be fields, but this is also depatable for similar reasons [48]. If one comes up with some ontology for QFT that does not involve particles, then it seems mysterious why the detectors respond in the way they do. The mystery can be resolved only by grounding the detector models on the theory and the ontology that it admits. As Doreen Fraser puts it in [49] "Davie's slogan is 'particles are what particle detectors detect'. On its own, this brand of operationalism about particles addresses the first motivation for a non-fundamental particle notion, namely to account for the phenomenology of particle detection, but does not adress the other motivations. The other motivations require a connection of the experimental context of particle detectors and the theory. How does QFT enable us to predict the response of particle detectors?".

The spirit of both papers is that the global features of the particle states (quanta) is rather an artifact of the quantisation procedure, and so it does not provide the basis on which one can doubt the fundamental status of particles in QFT thought of as local entities. In both papers the authors propose candidate states that can play the role of local particle states and then they examine the limit in which these local notion converge to the global ones. In this sense both papers express the view that the global particle notion (quanta) is approximate. It provides an approximately accurate description of the states that each approach identifies as representing the system."The particulate properties

of global Fock space are an artifact of the simplification taken by approximating a truly observed local particle with easier-to-deal-with Fock particles" [34].

Nevertheless in neither approaches the authors are making strong ontological statements about the nature of the proposed local particle states. In the Colosi and Rovelli paper (I) emphasis is given in the local nature of observables through which they define their local particle notion, but with this they rather justify the operational view that "whether a particle exists or not depends on what I decide to measure". In the paper by M. Rodriguez-Vazquez et al (II) the ontological status of the proposed local states is unclear to me due to the unitary inequivalence. As we discussed the authors are proposing a local notion of particle states that live in a representation which is unitary inequivalent to the usual one. Since the two representations are unitarily inequivalent, this raises the question of which one should we take seriously at an ontological level. One can wonder why the issue of unitary inequivalence does not come up in I. In this spirit one has to carefully identify the similarities and the differences between the two constructions. My understanding is that the local particle notion proposed in I is within the context of the standard quantisation procedure. The remedy they are proposing is to remove attention form the global nature of the Fock states to the local nature of observables through which they can define particle states locally but nevertheless living in the *same* Hilbert space, namely the Fock space spanned by the global particle states.

Even though II is thought of as a follow up to I, the proposed remedy is of a different type. The authors introduce a set of local creation and annihilation operators in terms of which they construct an alternative, unitarily inequivalent Fock space representation that can support what they call "local quanta". The term might even sound paradoxical since the term quanta was originally adopted to emphasize the global features of the Fock particle states, but in fact this term captures the essense of their proposal. Fock space constructions provide us with a notion of *quanta*, but nothing prevents us from using an alternative Fock space that has the desired *local* features built in by construction. Overall the alternative construction introduces features that are not existent in the usual global Fock space construction, neither in I. One of these features is the separability of the local vacuum state, which allows for the existence of local number operators since the Reeh Schlieder theorem does not apply. This was also the reason why they can create strictly localised states by locally exciting the local vacuum, while in the usual quantisation we know that no state of fixed particle content can be strictly localised [45]. In fact in II there cannot be a notion of local number operator, which is the reason why they emphasize that it is not the states that we should be looking at but rather the locally defined observables. On the contrary, in I the observables on the two sides of the box as somehow trivially localised thought of as functions of the corresponding creation and annihilation operators

that are by construction associated to each side.

The existense or absense of unitary inequivalence also changes the notion of convergence that the authors of either paper are using. As we saw above, in II it is emphasized that eventhough the local states do not convrege to the global ones, expectation values of observables converge to each other so the limit is taken in the weak operator topology. In I they work out the Bogoliubov transformation between the local and global creation and annihilation operators, first to demonstate unitary inequivalence. For example if one were to ask whether the local vacuum converges to the global vacuum in some limit, this would be an ill posed question since the two vacua live in unitarily inequivalent Hilbert space representations. This is where the mathematical asymmetry played an important role, because the local operators are well-defined in the global Fock space eventhough the inverse is not true. This allowed to define the local to global convergence in the strong operator topology with respect to the global Hilbert space $\mathcal{F}_G$ where both objects are well defined. This might sound like a technical remark but I think it is important for the following reason. This mathematical asymmetry is exactly the reason why it is "legal" to use the local excitation operators to excite the global vacuum which led to the so called quasi localised states.

On the question of physical equivalence, in a seminar given at Perimeter Institute by one of the authors of II, the speaker elaborated on this point when the question was raised[1]. What they claimed is that in a practical situation in which we only have access to the field by coupling a detector to it, the coupling will effectively introduce an ultraviolet regularization for the field. If the field is in the box (I am assuming, not explicit in their argument) we also have an IR cutoff which makes the degrees of freedom finite. For finite degrees of freedom the Stones Von Neumann theorem applies and there cannot be inequivalent representations. Along these lines the speaker claimed that unitary inequivalence is an interesting mathematical feature, but from this perspective "it does not seem to be an issue for physics". One of the most important results on the issue of unitary versus physical equivalence is Fell's theorem. As we briefly saw in the previous chapter, the theorem states that any finite set of measurements of finite accuracy cannot distinguish two inequivalent representations. From the operational point of view this theorem settles the issue in the sense that unitary inequivalence can be argued to be physically irrelevant which resembles a lot the spirit of the answer presented above. Indeed introducing detectors one can argue that they should not be able to tell inequivalent representations apart on the basis of Fell's theorem, even without involving regularisations that the detectors shall introduce to get rid of the unitary inequivalence at the first place. The motivation for this would be that Fell's theorem is a general mathematical result that should hold in any concrete

[1] http://pirsa.org/14050117/

realization, while the detector coupling along with the regularisation that it would introduce is model dependent. I would like to investigate these two aspects in the context of the Unruh-DeWitt detector model that is broadly used in Relativistic Quantum Information, and as introduced by Wald [3]. How do we specify a detector model that couples to the local rather than the global quanta of the QFT? Can we apply Fell's theorem to see the two notions are physically equivalent (under a given precision)? Such questions about inequivalent particle notions have been investigated in [50] by means of Fell's theorem to argue that Rindler quanta are in fact "real", and could be applied in the context of II.

Chapter 4

Impacts of relativity on localizability and vacuum entanglement

As we discussed so far, in relativistic quantum theories there are many subtleties associated to position space which is why momentum space is somehow preferred in the quantisation procedure, blurring the particle interpretation of a QFT. In the quest of answering whether one can describe localized particle-like entities in a quantum field theory, it is helpful to clarify how one creates local excitations i.e. how to excite the ground state of a quantum field locally over a bounded spatial region. To form what we call "local excitations" we have to recruit local creation and annihilation operators $\{a_x, a_x^\dagger\}$ [1] indexed by the position space variable x, the same x as in the field operator, to excite the ground state $|0\rangle$. From the perspective of the Reeh Schlieder theorem this would seem as a hopeless expectation, since it implies the non existence of local creation operator (section 1.5, equation (1.5.8)). Nevertheless, it is worth attempting in the context of ordinary quantum field theory to see concretely what is problematic about local excitations, but also what tools are available.

To define local creation and annihilation operators, we will recruit the analogy between a free massive Klein Gordon field and an infinite collection of harmonic oscillators. This analogy is usually made in momentum space (through the normal modes decomposition) but it can also be pursued in position space where one can define an infinite collection of coupled harmonic oscillators. This way we can compare the local and the global description in the fully relativistic theory, but also under a non relativistic approximation. The reason for studying the non relativistic regime of the theory, is that most of the features

[1] Just to clarify that the $\{,\}$ notation is used to bracket these objects and not to denote their anticommutator.

or problems of a relativistic quantum field theory, are usually blamed to relativity. Malament's and Hegerfeld's theorems shows that a position operator cannot be compatible with relativistic causality, and the Reeh Schlieder theorem is formulated for relativistic QFT's [2]. By "undoing" what relativity is doing in the non relativistic limit we can ask, do we recover notions of localizabiltiy? Looking towards low energies, one finds the widespread applicability of non-relativistic quantum mechanics (NRQM), a theory in which particle states are localizable by means of their wavefunction, which seems to imply that NRQM can support a particle ontology. So it is natural to ask whether one can make contact between the NRQM description of particles and some appropriate notion in the latent QFT.

What precisely is the difficulty with localized particle states in a relativistic QFT? The problem can be seen as a result of competing requirements that one would like to attribute to such states. Perhaps the most basic requirement is that particles should be entities that can be counted (there should be an observable number operator acting on a corresponding Fock space), that particles should persist in time (at least in free theories) and that the particle excitations should exhibit the appropriate relativistic dispersion relation between mass, momentum, and energy [47, 44]. What are the obstructions to localizing particle states that satisfy these requirements? We wish to investigate whether the incompatibility of these requirements is smoothed out in the non relativistic regime of the theory, where one would wish to recover localizable and locally countable particle states. Finally, it has been argued [4, 52] that localizability of particle states is further obscured by the existence of vacuum entanglement, which has been argued to be a relativistic effect. We wish to investigate whether a diminishing behaviour of vacuum entanglement allows for the recovery of localizability of particle states in the non relativistic limit.

Most of the material presented in this chapter is a result of my collaboration with Jason Pye, and is included in a paper that is under completion.

4.1 An infinite collection of harmonic oscillators

As we explained in section 1.2.3 to move from finitely many to infinite harmonic oscillators, we had two strategies to choose from. The most straight forward one would be to work with the infinite tensor product, and the second one is to use the structure of the classical space of solutions to define a Fock space as copies of what is called the one particle Hilbert

[2] Even though one can find both positive and negative arguments in the literature as to whether the Reeh Schlieder theorem applies in Galilean QFT's [51, 46].

space

$$\mathcal{F}[\mathcal{H}] := \bigoplus_{n=0}^{\infty} (\mathcal{H}^{\otimes n})_S \tag{4.1.1}$$

The Fock space is the main Hilbert space structure that one wishes to associate to a quantum field theory, because it naturally represents the quanta interpretation of a QFT as a theory that can describe a system with indefinite number of quanta, that is a theory that can describe "particle creation and annihilation".

Nevertheless, the second strategy establishes the formal analogy between a quantum field theory and an infinite collection of harmonic oscillators, say indexed by i. The Hilbert space of such a system corresponds to an infinite tensor product structure $\otimes_i^{\infty}\mathcal{H}_i$ that I will refer to as $i-$TPS. Most commonly the analogy is motivated by the form of the Hamiltonian in position and momentum space, in which cases $i = x$ or k accordingly. Then the Hilbert space would be a continuously infinite tensor product, unless cutoffs are imposed. The second strategy is not usually undertaken explicitly because infinite tensor products are tricky to define. Nevertheless, this is the structure we are appealing to in this project since infinte tensor products can be suitably defined (as I reproduce in Appendix B) establishing an explicit relation between the Fock space and the infinite tensor product.

Here I would like to sketch how the Fock space structure of the Hilbert space relates to different $i-$TPS structures, and then we will address the particular cases in which i represents position or momentum. Each Fock space construction (for a given basis of $\mathcal{H}$) is associated to a set of creation and annihilation operators, and a vacuum state defined as the state that is being annihilated by all the annihilation operators. If $\{e_i\}$ a basis for $\mathcal{H}$ then

$$\mathcal{F}[\mathcal{H}] \simeq \otimes_i \mathcal{H}_i \tag{4.1.2}$$

where $\mathcal{H}_i$ is the space where the creation and annihilation operators $a_i := a[e_i]$ act. As we saw in section 1.2.2 (equations (1.2.13)-(1.2.14)) if the basis $\{e_i\}$ is orthonormal the corresponding creation and annihilation operators satisfy the algebra $[a_i, a_j^\dagger] = \delta_{ij}$ and the corresponding vacuum is defined as

$$a_i \left|0\right\rangle = 0 \ \forall i \Rightarrow \left|0\right\rangle = \otimes_i \left|0\right\rangle_i \tag{4.1.3}$$

Equation (4.1.2) provides the relation between the two ways of viewing the Hilbert space of a QFT, the Fock space and the infinite tensor product. Every change of basis in the single particle sector of the Fock space, corresponds to a different tensor product structure

of the Hilbert space, a "rearrangement" of the degrees of freedom or a different infinite collection of harmonic oscillators through which we are describing the quantum field theory. For another orthonormal basis of $\mathcal{H}$ say $\{f_j\}$ we have another $j-$TPS of the Hilbert space and the corresponding operators $a[f_j] := \tilde{a}_j$ are related to the previous ones via $\tilde{a}_j = \sum_i (e_i^*, f_j)\, a_i$. Since there is no mixing (zero β coefficient) the two annihilate the same vacuum (4.1.3), that is $|0\rangle = \otimes_i |0\rangle_i = \otimes_j |0\rangle_j$.

I would like to comment that it is not clear what is the physics that one can associate to these different tensor product structures and (4.1.2) can only establish a purely mathematical (formal) analogy. Nevertheless, a physical analogy is usually motivated in the case that i represents position or momentum, interpreting the QFT as an infinite collection of uncoupled harmonic oscillators in momentum space, or an infinite collection of coupled harmonic oscillators in position space. In [53] it has been proposed that tensor product structures are observable induced, in the sense that the observables associated to each of the "parties" of the tensor product should better be physically accessible. Otherwise we are dealing with a mathematical decomposition of the system that can be argued to be physically irrelevant. Here, appealing to the position space $x-$TPS can be physically motivated in terms of "local access" to the observables of the theory that are locally defined in a QFT. Moreover, a well defined notion of $\otimes_x \mathcal{H}_x$ can be recruited to define localized subsystems.

4.1.1 From global to local: The two localization schemes in QFT

In this section we will define the local degrees of freedom and we will consider how one moves from position to momentum space, which in NRQM of a single particle we can simply do with the Fourier transform. In analogy here, can we use the Fourier basis in the one particle sector $\mathcal{H}$ of the QFT to switch between the two tensor product structures ($x, k-$TPS) as we described in the previous section? It turns out that to go from momentum space to what we would naturally call position space in a QFT we need to do something more radical, which is to move to a different Hilbert space and yet a unitarily inequivalent one.

To define local creation and annihilation operators, we will recruit the analogy between a free massive Klein Gordon field and an infinite collection of coupled harmonic oscillators in position space. This analogy is usually made in momentum space, inspired by the form

of the Hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}x[c^2\Pi(x)^2 + (\nabla\Phi(x))^2 + k_c^2\Phi(x)^2] \tag{4.1.4}$$

$$= -\frac{1}{2}\int \mathrm{d}x\left[mc^2\left(a_x^\dagger a_x + a_x a_x^\dagger\right) - \frac{\hbar^2}{2m}\left(a_x^\dagger + a_x\right)\nabla^2(a_x^\dagger + a_x)\right] \tag{4.1.5}$$

where the local creation and annihilation operators are defined through the field operators $\Phi(x), \Pi(x)$ as

$$a_x = \sqrt{\frac{m}{2}}\Phi(x) + \frac{i}{\sqrt{2m}}\Pi(x). \tag{4.1.6}$$

We see that these oscillators are coupled through the Laplacian operator, which is exactly what we "switch off" with the Fourier transform to define an infinite collection of uncoupled harmonic oscillators in momentum space. We also note that we can desinde the local oscillators (4.2.4) only for a massive field, and the mass of the fields specifies their frequency.

As we saw, the Reeh Schlieder theorem asserts that there are no local creation and annihilation operators which seems to be at odds with definition (4.2.4). This is because we could define non-zero annihilation operators over a region by smearing (4.2.4), which would contradict the implication (1.5.8) of the theorem. The reason we do not run into this contradiction is that the theorem does not apply because the local annihilation operators do not annihilate the global vacuum (as the theorem requires). Of course they define their own "local vacuum" through

$$a_x \left|0\right\rangle_L = 0, \quad \forall x \tag{4.1.7}$$

but this is not the ground state of the Hamiltonian (4.1.5). The ground state $|0\rangle$ is the 'global vacuum' defined by the annihilation operators in momentum space $a_k \left|0\right\rangle_G = 0\,\forall k$. To summarize

$$|0\rangle = \left|0\right\rangle_G \neq \left|0\right\rangle_L . \tag{4.1.8}$$

One can see this through the Bogoliubov transformation between the two sets of creation and annihilation operators $\{a_x, a_x^\dagger\}$ and $\{a_k, a_k^\dagger\}$ (derived in Appendix A). We get that [3]

$$a_x = \frac{1}{2}\int \frac{\mathrm{d}k}{\sqrt{2\pi}}\left[c_+(k)e^{ikx}a_k + c_-(k)e^{-ikx}a_k^\dagger\right]. \tag{4.1.9}$$

[3] We are writing everything in $1+1$ dimensions only to avoid heavier notation.

The coefficients are given by $c_\pm(k) = [1 + (k/k_c)^2]^{-1/4} \pm [1 + (k/k_c)^2]^{1/4}$ where $k_c = mc/\hbar$ the Compton wavelength.

We see that the local annihilation operators are a mixture of the global creation and annihilation operators, which is why the two sets of operators do not define the same vacuum. The fact that the ground state is not the local vacuum

$$|0\rangle \neq |0\rangle_L = \otimes_x |0\rangle_x \tag{4.1.10}$$

also demonstrates that the ground state is not separable with respect to the local degrees of freedom and so the ground state is entangled. As discussed in [4] it is the entanglement of the ground state that prevents the existence of local creation and annihilation operators over the global vacuum $|0\rangle_G$. On the other hand, the local vacuum $|0\rangle_L$ can be annihilated locally by (4.2.4) since it is separable and the Reeh Schlieder theorem does not apply.

From the perspective of the local harmonic oscillators, the field amplitude $\Phi(x)$ is the amplitude of the oscillator at each point x with $\mathcal{L}_2(\mathbb{R}, \mathrm{d}\Phi(x))$ being the corresponding Hilbert space. Then one can think of the total Hilbert space of the collection of oscillators as

$$\otimes_x \mathcal{L}_2(\mathbb{R}, \mathrm{d}\Phi(x)) \tag{4.1.11}$$

i.e. the continuously infinite tensor product of $\mathcal{H}_x := \mathcal{L}_2(\mathbb{R}, \mathrm{d}\Phi(x))$, the space where the local operators (4.2.4) act. Following Appendix B we can establish that

$$\otimes_x L_2(\mathbb{R}, \mathrm{d}\Phi(x)) \cong \mathcal{F}[L_2(\mathbb{R}, \mathrm{d}x)] := \mathcal{F}_L \tag{4.1.12}$$

to define the continuous tensor product as a separable Fock space, that we denote as the local Fock space $\mathcal{F}_L$.

Distinct from the local quantization, the most popular quantization scheme [4], as seen in most particle physics textbooks, is to quantize the normal modes of the free Hamiltonian of the theory. In flat spacetime, the normal modes for the free Klein-Gordon theory are simply plane waves. In Fourier space, the Hamiltonian is

$$H = \frac{1}{2} \int \frac{\mathrm{d}k}{\sqrt{2\pi}} \left[c^2 |\Pi_k|^2 + \frac{\omega_k^2}{c^2} |\Phi_k|^2 \right] \tag{4.1.13}$$

$$= \frac{1}{2} \int \frac{\mathrm{d}k}{\sqrt{2\pi}} \hbar\omega_k (a_k^\dagger a_k + a_k a_k^\dagger). \tag{4.1.14}$$

[4] The main appeal of the momentum space quantization scheme is that the "global" excitations $a_k^\dagger |0\rangle_G$ exhibit the appropriate relativistic dispersion relation between mass, momentum and energy.

where $\omega_k := c\sqrt{k^2 + k_c^2}$ and the momentum space creation and annihilation operators defined as

$$a_k := \sqrt{\frac{\omega_k}{2\hbar c^2}}\Phi_k + i\sqrt{\frac{c^2}{2\hbar\omega_k}}\Pi_k. \tag{4.1.15}$$

The corresponding Hilbert space is the usual "global" Fock space as defined in chapter 1 and will be denoted $\mathcal{F}_G$ as opposed to the local Fock space $\mathcal{F}_L$ defined above. The Bogoliubov transformation (4.1.9) is associating operators that act on different spaces, namely $\{a_x, a_x^\dagger\}$ acting on $\mathcal{F}_L$ and $\{a_k, a_k^\dagger\}$ acting on $\mathcal{F}_G$. In Appendix B we demonstrate that $\mathcal{F}_L$ and $\mathcal{F}_G$ are unitarily inequivalent unless further conditions are imposed. When two Hilbert spaces are unitarily inequivalent, it basically means that the operators that are acting on one of them cannot act on the other. The unitary inequivalence of $\mathcal{F}_L$ and $\mathcal{F}_G$ can be seen as the concrete Hilbert space version of the abstract algebraic statement that "there is no local number operators" meaning there is no local number operators *in* $\mathcal{F}_G$.

As we discussed in the previous section, changing a basis in the one particle Hilbert space $\mathcal{H}$ can be seen as a rearrangement of the degrees of freedom, or a different infinite collection of harmonic oscillators through which we are describing the quantum field theory. In this spirit, one can start from the global Fock space $\mathcal{F}_G$ and define a change of basis through the Fourier transform of the global creation and annihilation operators

$$\tilde{a}_y := \int \frac{\mathrm{d}k}{\sqrt{2\pi}} e^{iky} a_k. \tag{4.1.16}$$

This definition is inspired by NRQM where we switch between position and momentum space with the Fourier transform. In fact, these operators are commonly introduced in the literature as non-relativistic field operators [54, 55]. In the following section we will comment on the relevance of these operators in the non-relativistic limit. However, first we must clarify that perhaps counter to expectations, the index y is not what we would naturally identify as corresponding to a point in space. One can use the Bogoliubov transformation between the local and global creation and annhilation operators (4.1.9) and the definition (4.1.16) to derive the transformation

$$\tilde{a}_y = \int \mathrm{d}x [F_+(y-x)a_x + F_-(y-x)a_x^\dagger], \tag{4.1.17}$$

where $F_\pm(x) = \int \frac{\mathrm{d}k}{\sqrt{2\pi}} e^{ikx} c_\pm(k)$, the Fourier transform of the Bogoliubov coefficients (4.1.9). The operators $\{a_x, a_x^\dagger\}$ we defined in terms of the field operators $\Phi(x), \Pi(x)$ so the are the ones that we would naturally associate to the local degrees of freedom. We see that the

relation (4.1.17) between $\{a_x, a_x^\dagger\}$ and $\{\tilde{a}_y, \tilde{a}_y^\dagger\}$ is non-local, and in that sense y does not represent the local degrees of freedom faithfully. Although the functions $F_\pm$ are non-local, they decay asymptotically as $F_\pm(x) \sim e^{-k_c|x|}$ and hence the non-locality of the operators $\tilde{a}_y$ is suppressed at distances much larger than the Compton wavelength of the field.

At the kinematical level, the difference between the two localization schemes $\{a_x, a_x^\dagger\}$ and $\{\tilde{a}_y, \tilde{a}_y^\dagger\}$ is the difference between changing a basis (of $\mathcal{F}_G$) and changing a Hilbert space (moving to $\mathcal{F}_L$). Again, because of the unitary inequivalence $\{a_x, a_x^\dagger\}$ are not well defined in $\mathcal{F}_G$. The best we can naturally define in $\mathcal{F}_G$ is the "non relativistic" ones $\{\tilde{a}_y, \tilde{a}_y^\dagger\}$. In the next section we investigate if in any sense the two localization schemes coincide in the non relativistic limit. Finally, from the definition (4.1.16) it follows that $|0\rangle_G = \otimes_y |0_y\rangle$ since the two sets of annihilation operators ought to annihilate the same vacuum. We see that the ground state of the Hamiltonian is separable with respect to the y degrees of freedom, which faithfully represent the local degrees of freedom x only up to the Compton scale. This observation motivates asking what is the fate of the ground state entanglement towards the non relativistic regime of the theory, which we investigate in the last section. We close this section with a table that summarizes the differences between the two localization schemes as defined in (4.1.9) and (4.1.16).

Scheme 1: $\{a_x, a_x^\dagger\}$	**Scheme 2**: $\{\tilde{a}_y, \tilde{a}_y^\dagger\}$
locally related to $\Phi, \Pi(x)$	non locally related to $\Phi, \Pi(x)$
Bogoliubov mixing of $\{a_k, a_k^\dagger\}$	Fourier transform of $\{a_k, a_k^\dagger\}$
$\|0\rangle \neq \otimes_x \|0\rangle_x$	$\|0\rangle = \otimes_y \|0\rangle_y$ (!)
live in $\mathcal{F}_{\text{L}} \not\supset \mathcal{F}_{\text{G}}$	live in $\mathcal{F}_{\text{G}}$

4.2 The non-relativistic approximation

The non relativistic limit for a classical particle, is the limit of small velocities as compared to the speed of light $v/c \ll 1$. The velocities in question are typically the velocities of each of the individual particles in the system (and not, for example, the total momentum, which could be small for, e.g., two particles moving near the speed of light in opposite directions). It is less obvious how to take this limit for a quantum particle, and even more so in the context of QFT, a theory that can accomodate multi-particle states. A QFT naturally provides us with a momentum operator that corresponds to the *total* momentum operator. To take the non relativitic limit, we need to impose a restriction that guarantees that also

for multi particle states the momentum of each of the particles is sufficiently low, since this is not a priori guaranteed by cutting of the spectrum of the total momentum operator.

For more general single-particle wavepacket states of the form $\int \frac{\mathrm{d}k}{\sqrt{2\pi}}\psi(k)a_k^\dagger \left|0\right\rangle_{\mathrm{G}}$, one may consider declaring that such particles are "slow" if the expectation value and the variance of the momentum is small. For example, this could be achieved by restricting the set of allowable wavepackets to those with a suitably quick decay at large momenta. However, this space fails to be a closed linear space, which is crucial for the superposition principle to hold, since as we will see it is these wavepacket functions which become the wavefunctions of particles in the non-relativistic limit. The only means to ensure this is to impose a hard cutoff on the set of allowable wavenumbers, i.e. the support of these wavepackets should be restricted to $\mathrm{supp}(\psi) \subset \{|k| < \Lambda\}$, where Λ is a cutoff such that $\Lambda/k_c \ll 1$. The corresponding single particle Hilbert space (bandlimited wavefunctions) will be denoted $B(\Lambda)$. Notice that imposing such a cutoff picks out a preferred frame (that is, we break the Lorentz-invariance of the theory at this stage) but of course there cannot be a frame-independent notion of "slow".

To build towards multi particle states, let us now consider a general two-particle state $\int \frac{\mathrm{d}k_1\mathrm{d}k_2}{2\pi}\psi(k_1,k_2)a_{k_1}^\dagger a_{k_2}^\dagger \left|0\right\rangle_{\mathrm{G}}$. The appropriate restriction would be to limit the support of ψ in both variables k_1 and k_2 to the region $\{|k_1|, |k_2| < \Lambda\}$. Hence, the two-particle subspace of the theory should be identified as $(B(\Lambda)^{\otimes 2})_S$, and similarly for higher particle-number subspaces. Overall, our construction will be to create the Fock space out of symmetrised copies of $B(\Lambda)$, namely define it as $\mathcal{F}[B(\Lambda)]$. This is a subspace of the global Fock space

$$B(\Lambda) \subset L_2(\mathbb{C}, \mathrm{d}k) \implies \mathcal{F}[B(\Lambda)] \subset \mathcal{F}[L_2(\mathbb{R}^n, \mathrm{d}k)] := \mathcal{F}_G \tag{4.2.1}$$

and it is where the non relativistic approximation holds. Because this restricted Fock space is a subspace of the global Fock space, it is straightforward to define the restriction of operators to this subspace, for example, the total momentum operator becomes

$$\hat{P}\big|_{\mathcal{F}[B(\Lambda)]} = \int_{|k|<\Lambda} \frac{\mathrm{d}k}{\sqrt{2\pi}} \hbar k \, a_k^\dagger a_k, \tag{4.2.2}$$

which we see is simply a restriction of the integration range of the k values. As we mentioned above, this is different than a restriction of the spectrum of $\hat{P}$, which consists of elements of the form $\int \mathrm{d}k \hbar k n_k$ where $k \in \mathbb{R}$ and $n_k \in \mathbb{N}_0$. Hence, a restriction on the spectrum of $\hat{P}$ would also entail a restriction on the admissible values n_k, which is not the aim here since a state of many slow particles would have large total momentum, but should be admitted to the non-relativistic regime.

Note that one can also think of the bandlimited Fock space as obtained by removing (tracing out) the set of degrees of freedom associated with wavenumbers above the cutoff, i.e.,

$$\mathcal{F}[B(\Lambda)] \cong \otimes_{|k|<\Lambda} L_2(\mathbb{C}, \mathrm{d}\Phi_k) \subset \mathcal{F}[L_2(\mathbb{R}^n, \mathrm{d}k)] \cong \otimes_k L_2(\mathbb{C}, \mathrm{d}\Phi_k). \tag{4.2.3}$$

Then an operational means through which one could motivate this kind of cutoff necessary for the non-relativstic limit is through an interface with a probing system which only couples to this subset of modes. This could occur, for example, in a detector model which couples to the field via a bandlimited smearing function, which intuitively would correspond to a large detector (compared to the Compton wavelength of the field). We can consistently describe the physics restricted to $\mathcal{F}[B(\Lambda)] \cong \otimes_{|k|<\Lambda} L_2(\mathbb{C}, \mathrm{d}\Phi_k)$, since in the free theory each k sector is decoupled.

In analyzing the behavior of the QFT in the non-relativistic limit, we will need to understand how various operators behave in the regime where $\Lambda/k_c \ll 1$. For our purposes, we expand various operators in powers of Λ/k_c and characterizing them in the non-relativistic regime by keeping terms up to second order (the kinetic term of the Hamiltonian). For example, we can expand the Bogoliubov transformation between the local and global creation/annihilation operators (4.1.9) by expanding the coefficients $c_\pm(k)$ in powers of Λ/k_c. To second order, these coefficients become $c_+(k) \approx 1$ and $c_-(k) \approx \frac{1}{4}(k/k_c)^2$, hence the Bogoliubov transformation is

$$a_x = \int_{|k|<\Lambda} \frac{\mathrm{d}k}{\sqrt{2\pi}} e^{ik\cdot x} \left[a_k - \frac{1}{4} \left(\frac{k}{k_c} \right)^2 a^\dagger_{-k} \right] \tag{4.2.4}$$

We wish to understand the impact of relativity on localizability by taking the non relativistic limit, to see what features of NRQM one recovers in the limit. The features we are interested in are both dynamical and kinematical. Do we recover a Schrodinger type of equation and, starting from QFT, what is the object that can play the role of a wavefunction in the limit? Do we recover a position operator and the Heisenberg algebra accordingly? As we will describe, the recovery of non-relativistic QM from QFT is only approximate, and we will discuss how it differs from standard NRQM.

The key difficulty comes from the comparison of the two localization schemes $\{a_x, a_x^\dagger\}$ and $\{\tilde{a}_y, \tilde{a}_y^\dagger\}$ that we introduced in the previous section. Features of the two schemes have been investigated in the literature [51, 56]. Here we are comparing the two localization schemes in the non relativistic limit. As we are finding, they do not coincide in the limit as someone would hope, so we are still in the awkward situation of having to choose between the two. We make this choice by demonstrating which one can better imitate the quantum mechanical behaviour in the limit.

The subspace in which we wish to recover the quanum mechanical description is $\mathcal{F}[B(\Lambda)]$ in which we can define $\tilde{a}_y = \int_{|k|<\Lambda} \frac{dk}{\sqrt{2\pi}} e^{ik\cdot y} a_k$ and we can write (4.2.4) as

$$a_x = \tilde{a}_x - \frac{1}{4}\int_{|k|<\Lambda} \frac{dk}{\sqrt{2\pi}} \left(\frac{k}{k_c}\right)^2 a^\dagger_{\mathbf{-k}} \tag{4.2.5}$$

We see that the two operators differ at second order of the expansions. We observe that the difference does not matter for quasi-localized one particle states but it does matter for higher particle states. To see this let us consider the state $|\Psi(t)\rangle = \int dx \Psi(x,t) a^\dagger_x |0\rangle$ where $|0\rangle$ is the global vacuum. This is called a quasi-localized state because we are acting with the local creation operators on the global vacuum, which we are allowed to do because of the cuct off (see Appendix B). Through (4.2.5) we see that

$$|\Psi(t)\rangle = \int dx \Psi(x,t) a^\dagger_x |0\rangle = \int dx \Psi(x,t) \tilde{a}^\dagger_x |0\rangle \tag{4.2.6}$$

which means that the two schemes coincide for these states. Nevertheless, it is easy to convince ourselves that for higher particle states the two schemes differ. For example, for a two particle state we find that

$$|\Psi\rangle = \int dx_1\, dx_2 \Psi(x_1,x_2) a^\dagger_{x_1} a^\dagger_{x_2} |0\rangle \tag{4.2.7}$$

$$= \int dx_1\, dx_2 \Psi(x_1,x_2) \tilde{a}^\dagger_{x_1} \tilde{a}^\dagger_{x_2} |0\rangle - \frac{1}{4}\int dx_1 dx_2 \Psi(x_1,x_2) g(x_1,x_2) |0\rangle \tag{4.2.8}$$

where $g(x_1,x_2) = \int_{|k|<\Lambda} \frac{dk}{\sqrt{2\pi}} \left(\frac{k}{k_c}\right)^2 e^{ik(x_1-x_2)}$.

We see that $\tilde{a}^\dagger_{x_1}...\tilde{a}^\dagger_{x_N}|0\rangle$ and $a^\dagger_{x_1}...a^\dagger_{x_N}|0\rangle$ are different states for $N \neq 1$. It is the wavepackets of the first kind

$$|\Psi(t)\rangle = \int dx_1...dx_N \Psi(x_1,...,x_N,t) \tilde{a}^\dagger_{x_1}...\tilde{a}^\dagger_{x_N} |0\rangle \tag{4.2.9}$$

that evolve under a multi-particle Schroedinger equation in the non relativistic limit. To consider the time evolution of these states we need to restrict the Hamiltonian operator (4.1.5) in the subspace $\mathcal{F}[B(\Lambda)]$ where our approximation holds

$$H|_{\mathcal{F}[B(\Lambda)]} = \frac{1}{2}\int_{|k|<\Lambda} \frac{dk}{\sqrt{2\pi}} \left[mc^2 - \frac{\hbar^2 k^2}{2m}\right] a^\dagger_k a_k \tag{4.2.10}$$

$$= \int dy \left[mc^2 \tilde{a}^\dagger_y \tilde{a}_y - \frac{\hbar^2}{2m} \tilde{a}^\dagger_y \nabla^2 \tilde{a}_y\right] \tag{4.2.11}$$

We see that in fact the restricted Hamiltonian in terms of the $\{\tilde{a}_y^\dagger, \tilde{a}_y\}$ operators takes the form of a non relativistic Hamiltonian as we would expect in the limit.

Note that if we were to recruit the $\{\tilde{a}_y^\dagger, \tilde{a}_y\}$ operators for the full relativistic Hamiltonian (4.1.5) we get

$$H = \frac{1}{2}\int_{|k|<\Lambda} \frac{dk}{\sqrt{2\pi}} \omega_k a_k^\dagger a_k = \int dy dy' f(y-y') \tilde{a}_y^\dagger \tilde{a}_y' \tag{4.2.12}$$

where $f(y-y') = \int_{|k|<\Lambda} \frac{dk}{\sqrt{2\pi}} \omega_k e^{ik(y-y')}$ a non local integral Kernel. Comparing (4.2.11) with (4.2.12) we see that what the non relativistic approximation is doing is "switching off" the non locality that is introduced through this kernel, when it comes to the Hamiltonian operator. Of course the Hamiltonian is always local with respect to the $\{a_x, a_x^\dagger\}$'s as in (4.1.5), but in the limit it becomes local with respect to the $\{\tilde{a}_y, \tilde{a}_y^\dagger\}$'s, even though the two sets of operators still differ (4.2.5).

We can translate the abstract time evolution of the state (4.2.9) under the Hamiltonian (4.1.5) $i\hbar\partial_t |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle$ into a differential equation for the smearing function $\Psi(x_1, ..., x_N, t)$. Note that we can absorb all the time dependence in the smearing function because the state (4.2.9) is a $N-$particle state in $\mathcal{F}_G$ and the Hamiltonian preserves the global number operator, but not the local one. This is why the quasi-local states do not retain the same "form" over time. It is easy to see that the smearing function indeed satisfies the Schroedinger equation

$$i\hbar\partial_t \Psi(x_1, ..., x_N, t) = \left(mc^2 N - \frac{\hbar^2}{2m} \sum_{i=1}^{N} \nabla_i^2 \right) \Psi(x_1, ..., x_N, t) \tag{4.2.13}$$

Viewing QFT as "second quantisation" implies that what we are quantizing is the wavefunction of the first quantised theory, and the field operator represents the operator valued wave function. We see that this is not the case, since it is rather the smearing functions that obey the Schroedinger equation in the non relativistic limit and play the role of the wavefunction in NRQM. To justify that the smearing function imitates the wavefunction in the limit, it does not suffice to recover the right (non relativistic) equations of motion, but we also need to make sense of the inputs $x_1, ..., x_N$ of the smearing function. A quantum mechanical wave function is defined as $\psi(x_1, ..., x_N, t) = \langle x_1, ..., x_N | \psi(t) \rangle$ the coefficients of the state over the position basis provided by the position operators $\hat{X}_1 ... \hat{X}_N$ of the N particles, that satisfy the Heisenberg algebra with the conjugate momentum operators $[\hat{X}_i, \hat{P}_j] = i\hbar\delta_{ij}$. In QFT we do not have the position operators to provide us with such basis, and the inputs $x_1, ..., x_N$ rather correspond to N possible labels of the infinite degrees of freedom as represented by the $\{\tilde{a}_y, \tilde{a}_y^\dagger\}$'s.

Starting from a QFT of infinite degrees of freedom, and taking the non relativistic limit, we cannot end up with N degrees of freedom satisfying the Heisenberg algebra, that is N pairs of position and momentum operators such that $[\hat{x}_i, \hat{p}_i] = i\hbar,\ i = 1...n$, which is a consequence of the fact that quanta are aggreagable, can be counted, but cannot carry labels [44]. Since quanta cannot carry labels, the corresponding observables cannot carry labels, they are rather collective. Even when restricted to a fixed particle sector of the Fock space, an operator that carries a label, say $\hat{x}_1$ the position of 'the first' particle would not be well defined in this space (because of the antisymmetrization in the very definition of the Fock space). In a QFT we naturally have a total momentum operator, and we can build the conjugate total position operator for the "centre of mass" of the N-particles

$$\hat{X} := \hat{N}^+ \int dx x \tilde{a}_x^\dagger \tilde{a}_x \tag{4.2.14}$$

where $\hat{N}^+$ is the pseudo-inverse of $\hat{N}$. The eigenstates of the position operator are

$$|x_1, \dots, x_n\rangle := \tilde{a}_{x_1}^\dagger \cdots \tilde{a}_{x_n}^\dagger |0\rangle_G \tag{4.2.15}$$

with eigenvalues $\frac{1}{N}\sum_{i=1}^N x_i$. This operator satisfies the Heisenberg algebra with the total momentum operator

$$\left[\hat{X}, \hat{P}\right] = \left[\hat{N}^+ \int dx x \tilde{a}_x^\dagger \tilde{a}_x, \int_{|k|<\Lambda} \frac{dk}{\sqrt{2\pi}} \hbar k a_k^\dagger a_k\right] = i\hbar(\mathbb{1}_{\mathcal{F}[B(\Lambda)]} - |0\rangle\langle 0|) \tag{4.2.16}$$

except in the zero particle subspace, which is annihilated by both $\hat{X}$ and $\hat{P}$.

In this section we saw by what means and in which sense we recover the non relativistic equations of motion and the Heisenberg algebra in the non relativistic limit. We have the Schroedinger equation (4.2.13) for the smearing function of a N particle state and a total position operator that satisfies the Heisenberg algebra with the total momentum operator as restricted to the subspace where the non relativistic approximation holds. We saw the negative result that the two localization schemes do not coincide under the non relativistic approximation, but yet the operators $\{\tilde{a}_x, \tilde{a}_x^\dagger\}$ are justified as 'non relativistic' because if we define (4.2.9) and (4.2.14) in terms of $\{a_x, a_x^\dagger\}$ we do not recover these results. On top of the discrepancy between the two localization schemes in the limit, imposing a cutoff in momentum space non trivially affects both schemes since $[\hat{\Phi}(x), \hat{\Pi}(x')] = i\hbar \int_{|\mathbf{k}|<\Lambda} \mathrm{d}k e^{ik(\mathbf{x}-\mathbf{x'})} \neq \delta(x - x')$. The effect of an ultraviolet cutoff to the local degrees of freedom has been studied in [57]. We elaborate more on this point in our upcoming paper [58].

What we described above is not really the full structure of NRQM, but only an approximation. It has been argued in [59] that the full structure cannot be recovered under any approximation methods. In ordinary NRQM arbitrarily large momenta are allowed, while starting from the relativistic QFT we had to restrict the allowed momenta to take the non relativistic limit and as a result we only recover a bandlimited version of NRQM. This is to be expected, similar to how arbitrarily large speeds are allowed in classical Newtonian mechanics, since there is no c and no Compton wavelength built in NRQM to set the limitations of the theory. Only going back words, from the relativistic theory to the non relativistic theory, we can concretely set the limitations of the latter.

4.3 Vacuum entanglement/frustration in the limit

The usual intuition about why there is ground state entanglement in a quantum field theory is that the local degrees of freedom are coupled through the d'Alembertian operator in the Hamiltonian. In [60] for example, they calculate the entanglement entropy of the ground state making the analogy with a system of coupled harmonic oscillators. As we showed in the previous section the global vacuum is separable with respect to the $y-$degrees of freedom, despite the fact that the non relativistic Hamiltonian couples the $\{\tilde{a}_y, \tilde{a}_y^\dagger\}$ as in (4.2.11). So we see that eventhough the Hamiltonian is non-local, the ground state is not entangled in the limit. Of course any local Hamiltonian possess a separable ground state, but not all interactions are inducing entanglement in the ground state. As we will discuss, it depends on the type of the interaction terms whether the ground state is separable or not.

The question then is what kinds of interactions in the Hamiltonian are inducing entanglement in the ground state. One intuitive criterion is the commutativity of the interaction terms with the local terms. If the interaction terms commute with the local ones then the ground state is separable. This is because if they do commute, there is a common set of eigenvectors and since we know that the eigenstates of the local Hamiltonian are separable, this is still the case for the full Hamiltonian. But this is not conclusive, since the local and non-local terms might only share the ground state in which case they might not commute. To demonstrate this let us consider the example of N uncoupled harmonic oscillators. The Hilbert space of such a composite system is given by the tensor product $\mathcal{H} = \otimes_i \mathcal{H}_i$ and the Hamiltonian is $H = \sum_i \omega_i a_i^\dagger a_i$. Since the oscillators are uncoupled the ground state is clearly $|0\rangle = \otimes_i |0_i\rangle$. Now let us introduce the operators

$$\tilde{a}_j = \sum_j U_{ji} a_i \tag{4.3.1}$$

where U_{ji} are the coefficients of a $N \times N$ unitary matrix so that the tilted set of creation and annihilation operators also satisfy $[\tilde{a}_j, \tilde{a}_k] = \delta_{jk}$. The tilted operators suggest a different splitting of the Hilbert space [53], namely $\mathcal{H} = \otimes_j \tilde{\mathcal{H}}_j$ where $\tilde{\mathcal{H}}_j$ the space where $\tilde{a}_j, \tilde{a}_j^\dagger$ act. Let us now rewrite the Hamiltonian with respect to the j degrees of freedom

$$H = \sum_{i=1}^{N} \omega_i a_i^\dagger a_i = \sum_{j,k=1}^{N} \tilde{a}_j^\dagger \left(\sum_i \omega_i U_{ij} U_{ik}^* \right) \tilde{a}_k = \sum_{j,k=1}^{N} \tilde{a}_j^\dagger \Delta_{jk} \tilde{a}_k \tag{4.3.2}$$

where $\Delta_{jk} = \sum_i \omega_i U_{ij} U_{ik}^*$. We see that in general the Hamiltonian becomes non-local. Note that the coupling of the tilded harmonic oscillators comes from the off-diagonal elements of the matrix we defined as Δ. In the case that all the harmonic oscillators are of the same frequency ω then $\Delta_{jk} = \omega \delta_{jk}$ are the components of a diagonal matrix and the Hamiltonian remains local. In this case it is intuitive that the ground state of the Hamiltonian is separable with respect to both tensor product structures. In the general case that Δ is not a diagonal matrix, it is still true that

$$|0\rangle = \otimes_i |0\rangle_i = \otimes_j |0\rangle_j \tag{4.3.3}$$

So looking at the Hamiltonian as rewritten in (4.3.2) we have an example of a Hamiltonian with interaction terms of which the ground state is not entangled (4.3.3).

Maybe this would make us expect that the interaction terms have to commute with the local ones. Nevertheless, it is easy to check that this is not the case

$$\begin{aligned} \left[\sum_j \Delta_{jj} \hat{N}_j, \sum_{k \neq l} \Delta_{kl} \tilde{a}_k^\dagger \tilde{a}_l \right] &= \sum_{j,k \neq l} \Delta_{jj} \Delta_{kl} \left[\hat{N}_j, \tilde{a}_k^\dagger \tilde{a}_l \right] \\ &= \sum_{j,k \neq l} \Delta_{jj} \Delta_{kl} \left(\tilde{a}_k^\dagger \tilde{a}_l \delta_{jl} - \delta_{jk} \tilde{a}_k^\dagger \tilde{a}_l \right) = \sum_{j \neq k} \Delta_{kj} (\Delta_{jj} - \Delta_{kk}) \tilde{a}_k^\dagger \tilde{a}_j \neq 0 \end{aligned}$$

So this is a counterexample where the fact that the local terms do not commute with the non local onesnon does not imply entanglement in the ground state.

As has been examined in [61] "ground state entanglement may or may not occur when the interaction terms of the Hamiltonian fail to commute with the local terms". The authors quantify this statement by proving a bound for the ground state entanglement, which depends on the so called frustration energy of the Hamiltonian. Consider a Hamiltonian of the form $H = H_L + H_I$, then the frustration energy is defined as follows

$$E_f = E_0 - E_0^L - E_0^I \tag{4.3.4}$$

where E_0 is the ground state energy of the total Hamiltonian and E_0^L, E_0^I are the ground states energy of the local and the interacting part respectively.

We expect the frustration energy to be relevant to the ground state entanglement since it captures whether a given ground state can "minimize" both the local and the interaction terms of a Hamiltonian. In fact, in the case that the local and interacting terms share a common ground state $E_0 = E_0^L + E_0^I$ which makes the frustration energy vanish. To bound the entanglement $\mathcal{E}[|0\rangle]$ of the ground state by the frustration energy the authors need to introduce another energy scale ΔE (which will be discussed below). Then the so called frustration entanglement bound is

$$\mathcal{E}[|0\rangle] \leq \frac{E_f}{\Delta E} \tag{4.3.5}$$

Given a set of systems that are coupled to each other and without explicitly calculating the ground state, this bound provides a way to quantify how much entanglement can be contained in the ground state. For example, if the interaction is such that the frustration energy is zero the ground state has to be separable.

Indeed in the example we gave above, the frustration energy of the Hamiltonian is zero. To see this let us split the total Hamiltonian (4.3.2) in the local and interacting part

$$H = \sum_j \Delta_{jj} N_j + \sum_{j \neq k} \Delta_{kj} a_k^\dagger a_j \tag{4.3.6}$$

It is easy to see that $E_0^I = 0$ and so the frustration energy (4.3.4) will be

$$E_f = E_0 - E_0^L = \sum_i \frac{\hbar\omega_i}{2} - \sum_j \frac{\hbar\Delta_{jj}}{2} = \frac{\hbar}{2}\sum_i \omega_i (1 - \sum_j |U_{ij}|^2) = \frac{\hbar}{2}\sum_i \omega_i (1 - \delta_{ii}) = 0$$

From the entanglement frustration bound (4.3.5) we can convince ourselves that the interaction terms in the Hamiltonian (4.3.6) are not inducing entanglement in the ground state. This behaviour crucially depends on the type of interaction terms in the Hamiltonian. To apply this bound to the QFT case, we briefly review the derivation of the bound. The first step is to expand the ground state over the eigenstates of the local Hamiltonian $|0\rangle = \alpha \left|0^L\right\rangle + \beta \left|E^\perp\right\rangle$ where $\left|0^L\right\rangle$ the local ground state and $\left|E^\perp\right\rangle$ the expansion over the rest of the orthogonal eigenstates. This means that $\langle E^\perp | H_L | E^\perp \rangle \geq E_1^L$ which we are going to use below as follows

$$\langle 0|H_L|0\rangle = |\alpha|^2 E_0^L + |\beta|^2 \langle E^\perp | H_L | E^\perp \rangle \tag{4.3.7}$$

$$\geq |\alpha|^2 E_0^L + |\beta|^2 (E_1^L \pm E_0^L) \tag{4.3.8}$$

$$= E_0^L + (1 - |\alpha|^2)(E_1^L - E_0^L) \tag{4.3.9}$$

Then it is easy to see from the definition of the frustration energy that

$$E_f \geq \langle 0|H_L|0\rangle - E_0 \geq (1-|\alpha|^2)(E_1^L - E_0^L) \tag{4.3.10}$$

using the inequality above. If we call $\Delta E = E_1^L - E_0^L$ the first excitation gap of the local Hamiltonian we have that

$$\frac{E_f}{\Delta E} \geq (1-|\alpha|^2). \tag{4.3.11}$$

To complete the proof we need to associate the right hand side of the inequality to a measure of ground state entanglement defined as

$$\mathcal{E}(|\phi\rangle) = 1 - \max_{\phi_1,...,\phi_n} |\langle\phi|\phi_1 \otimes ... \otimes \phi_n\rangle|^2. \tag{4.3.12}$$

Since the ground state of the local Hamiltonian is a the separable state $\left|E_0^L\right\rangle = |E_0^1, ..., E_0^n\rangle$ we have that the entanglement of the total ground state satisfies

$$\mathcal{E}(|0\rangle) \leq 1 - |\langle 0|E_0^L\rangle|^2 = 1 - |\alpha|^2 \tag{4.3.13}$$

Combining (4.3.11) with (4.3.13) we get the entanglement frustration bound $\mathcal{E}(|0\rangle) \leq \frac{E_f}{\Delta E}$.

Going back to the QFT case, the Klein-Gordon Hamiltonian as written in terms of local creation and annihilation oscillators

$$H = -\frac{1}{2}\int \mathrm{d}x \left[mc^2\left(a_x^\dagger a_x + a_x a_x^\dagger\right) - \frac{\hbar^2}{2m}\left(a_x^\dagger + a_x\right)\nabla^2\left(a_x^\dagger + a_x\right)\right] \tag{4.3.14}$$

there are also interaction terms of the kind $a_x a_{x'}, a_x^\dagger a_{x'}^\dagger, a_x a_{x'}^\dagger$(since the discrete version of the Laplacian is coupling neighboring points x, x') and so the vacuum state that they define $\otimes_x |0_x\rangle$ is not an eigenstate of the interaction terms. As a result $|0\rangle \neq \otimes_x |0_x\rangle$ and there is ground state entanglement. After the non-relativistic approximation described above the Hamiltonian becomes

$$H = mc^2 \int \mathrm{d}x \hat{N}_x - \frac{\hbar^2}{2m}\int \mathrm{d}x\, \tilde{a}_x^\dagger \nabla^2 \tilde{a}_x \tag{4.3.15}$$

Just from comparison with the Hamiltonian (4.3.6) of the discrete example, we see that the interaction is exactly of the same type (if we would introduce a discretization of the d'Alembertian operator). In analogy we can see that

$$E_f = \int_{|k|<\Lambda} \mathrm{d}k(\hbar\omega_k - mc^2) \to 0 \tag{4.3.16}$$

in the limit $\hbar\omega_k \to mc^2$. The local energy scale ΔE that goes in the proof of the bound can be naturally identified with mc^2. Applying the bound we get that

$$\mathcal{E}(|0\rangle) \leq \int_{|k|<\Lambda} \mathrm{d}k \left(\frac{\hbar\omega_k}{mc^2} - 1 \right) \to 0 \tag{4.3.17}$$

in the ultra non relativistic limit where the energy is dominated by the mass.

To conclude, (4.3.16) this shows that vacuum entanglement vanishes ultimately in the limit where the mass is dominant. The benefit of applying the entanglement frustration bound is that we can argue this only through the behaviour of the frustration energy in the limit, without explicitly calculating any measure of entanglement for the ground state. Also, it establishes the picture that in the ultra non relativistic limit the harmonic oscillators in position space are still coupled but yet not frustrated, and so not entangled. This discussion is particularly relevant for cases in which entanglement of the ground state is treated as a resource, entanglement harvesting for example [62]. Along these lines we are also investigating the "bipartite" two mode entanglement of the vacuum in the presence of the ultraviolet cutoff [58] and in future work it would be interesting to investigate its extractability. So far we find that the ground state is still entangled at second order of the approximation, which settles that not all of vacuum entanglement can be blamed to relativity. All these considerations boil down to the expression for the β coefficient of the Bogoliubov transformation that mixes the local and the global creation and annihilation operators, since this is the reason why the global ground state fails to be separable with respect to the local degrees of freedom. The Bogoliubov transformation as derived in the appendix reads

$$a_x = \frac{1}{2} \int \frac{\mathrm{d}k}{\sqrt{2\pi}} e^{ikx} \left(c_+(k) a_k + c_-(k) a^\dagger_{-k} \right) \tag{4.3.18}$$

$$c_\pm(k) = \sqrt{mc^2/\hbar\omega_k} \pm \sqrt{\hbar\omega_k/mc^2} \tag{4.3.19}$$

for an arbitrary dispersion relation ω_k. Here we considered the non relativistic dispersion relation as an approximation to the relativistic one, but one can apply similar approximation methods for a general (analytic) dispersion relation which can give interesting results for vacuum entanglement in quantum field theories with modified dispersion relations, like in condensed matter systems or modified theories of gravity.

Chapter 5

Epilogue

In this thesis we summarized the mathematical results which demonstrate that the very intuitive picture of particle like entities localized in space does not have a mathematical analogue in relativistic quantum theories. This intuition is formed by how we interact with the world, and in particular the particle likeness of the detection processes and the localized 'clicks' of the detection devices. Overall we find our selves in the paradoxical situation of "particle talk without particle ontology" [42]. Quoting Haag "...it is not possible to assume that an electron has, at a particular instant of time, any position in space; in other words, the concept of position at a given time is not a meaningful attribute of the electron. Rather, position is an attribute of the interaction between the electron and a suitable detection device."[36]

A possible way to resolve this could be a first-principles approach to detector models. One might wonder how far this can go, since models are called as such because they require some phenomenological input that cannot be 'derived' from the theory. Nevertheless, an important step in this direction would be to carefully translate notions from measurement theory for the detector models that we are using in relativistic setups. This is becoming increasingly important for information theoretic considerations, where one can ask how is quantum information localized and how it is transmitted on relativistic spacetimes. For example, can we really think of a qubit 'traveling' on Minkowski spacetime? Can we really 'measure' a quantum field by coupling it to a detector system? When/where and why do the detectors click? These are the kinds of questions that I would like to investigate further.

References

[1] Daniele Colosi and Carlo Rovelli. What is a particle? *Classical and Quantum Gravity*, 26(2):025002, 2008.

[2] Rudolf Haag. *Local quantum physics: Fields, particles, algebras*. Springer Science & Business Media, 2012.

[3] Robert M Wald. *Quantum field theory in curved spacetime and black hole thermodynamics*. University of Chicago Press, 1994.

[4] Matías R Vázquez, Marco del Rey, Hans Westman, and Juan León. Local quanta, unitary inequivalence, and vacuum entanglement. *Annals of Physics*, 351:112–137, 2014.

[5] Theodore Duddell Newton and Eugene P Wigner. Localized states for elementary systems. *Reviews of Modern Physics*, 21(3):400, 1949.

[6] Laura Ruetsche. *Interpreting quantum theories*. Oxford University Press, 2011.

[7] John Earman and Doreen Fraser. Haags theorem and its implications for the foundations of quantum field theory. *Erkenntnis*, 64(3):305, 2006.

[8] C Brian. Hall, Quantum theory for mathematicians. *Graduate Texts in Mathematics*, 267, 2013.

[9] Achim Kempf. Lecture notes on Advanced Quantum Theory AMATH473. 2017.

[10] Paul Adrien Maurice Dirac. *The principles of quantum mechanics*. Number 27. Oxford University Press, 1981.

[11] Michael Reed and Barry Simon. Functional analysis, vol. i, Academic Press, San Diego, 1980.

[12] Stephen A Fulling. *Aspects of quantum field theory in curved spacetime*, volume 17. Cambridge University Press, 1989.

[13] Vern I Paulsen. Functional Analysis for Quantum Information (FAQI) notes. 2016.

[14] C J Isham and N Linden. Continuous histories and the history group in generalized quantum theory A topos foundation for theories of physics: I. Formal languages for physics. *Continuous time and consistent histories, Journal of Mathematical Physics*, 36:44, 1995.

[15] Luis J Garay. Lecture notes: Quantum fields in curved spacetimes, Complutense De Madrid, Universidad. 2016.

[16] William G Unruh. Notes on black-hole evaporation. *Physical Review D*, 14(4):870, 1976.

[17] William G Unruh and Robert M Wald. What happens when an accelerating observer detects a Rindler particle. *Physical Review D*, 29(6):1047, 1984.

[18] Alejandro Pozas-Kerstjens and Eduardo Martín-Martínez. Entanglement harvesting from the electromagnetic vacuum with hydrogenlike atoms. *Physical Review D*, 94(6):064074, 2016.

[19] Eduardo Martín-Martínez, Miguel Montero, and Marco del Rey. Wavepacket detection with the Unruh-DeWitt model. *Physical Review D*, 87(6):064038, 2013.

[20] Eduardo Martín-Martínez and Pablo Rodriguez-Lopez. Relativistic quantum optics: The relativistic invariance of the light-matter interaction models. *Physical Review D*, 97(10):105026, 2018.

[21] Michael Redhead. More ado about nothing. *Foundations of Physics*, 25(1):123–137, 1995.

[22] Rob Clifton and Hans Halvorson. Entanglement and open systems in algebraic quantum field theory. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 32(1):1–31, 2001.

[23] Edward Witten. Notes on some entanglement properties of quantum field theory. *arXiv:1803.04993*, 2018.

[24] Christopher J Fewster and Rainer Verch. Quantum fields and local measurements. *arXiv:1810.06512*, 2018.

[25] Rafael D Sorkin. Impossible measurements on quantum fields. In *Directions in general relativity: Proceedings of the 1993 International Symposium, Maryland*, volume 2, pages 293–305, 1993.

[26] Fay Dowker. Useless qubits in 'relativistic quantum information'. *arXiv:1111.2308*, 2011.

[27] Dionigi MT Benincasa, Leron Borsten, Michel Buck, and Fay Dowker. Quantum information processing and relativistic quantum fields. *Classical and Quantum Gravity*, 31(7):075007, 2014.

[28] Eduardo Martín-Martínez. Causality issues of particle detector models in QFT and quantum optics. *Physical Review D*, 92(10):104019, 2015.

[29] Charis Anastopoulos and Ntina Savvidou. Time of arrival and localization of relativistic particles. *arXiv:1807.06533*, 2018.

[30] Sergey S Horuzhy. *Introduction to algebraic quantum field theory*, volume 19. Springer Science & Business Media, 2012.

[31] Tian Yu Cao. *Conceptual foundations of quantum field theory*. Cambridge University Press, 2004.

[32] Dennis Dieks. *The ontology of spacetime*, volume 1. Elsevier, 2006.

[33] Charis Anastopoulos. *Particle or wave: the evolution of the concept of matter in modern physics*. Princeton University Press, 2008.

[34] Doreen Fraser. Quantum field theory: Underdetermination, inconsistency, and idealization. *Philosophy of Science*, 76(4):536–567, 2009.

[35] David Wallace. In defence of naiveté: The conceptual status of lagrangian quantum field theory. *Synthese*, 151(1):33–80, 2006.

[36] Asher Peres and Daniel R. Terno. Quantum information and relativity theory. *Rev. Mod. Phys.*, 76:93–123, Jan 2004.

[37] David B Malament. In defense of dogma: Why there cannot be a relativistic quantum mechanics of (localizable) particles. In *Perspectives on quantum reality*, pages 1–10. Springer, 1996.

[38] Hans Halvorson and Rob Clifton. No place for particles in relativistic quantum theories? In *Ontological aspects of quantum field theory*, pages 181–213. World Scientific, 2002.

[39] Gerhard C Hegerfeldt. Causality problems for fermis two-atom system. *Physical Review Letters*, 72(5):596, 1994.

[40] Gerhard C Hegerfeldt. Instantaneous spreading and Einstein causality in quantum theory. *Annalen der Physik*, 7(7-8):716–725, 1998.

[41] Gerhard C Hegerfeldt. Particle localization and the notion of Einstein causality. *arXiv preprint quant-ph/0109044*, 2001.

[42] Hans Halvorson. Locality, localization, and the particle concept: Topics in the foundations of quantum field theory. PhD thesis, 2001.

[43] Paul CW Davies. Particles do not exist. *Quantum theory of gravity*, page 66, 1984.

[44] Paul Teller. *An interpretive introduction to quantum field theory*. Princeton University Press, 1997.

[45] AL Licht. Strict localization. *Journal of Mathematical Physics*, 4(11):1443–1447, 1963.

[46] Jonathan Bain. Quantum field theories in classical spacetimes and particles. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 42(2):98–106, 2011.

[47] Doreen Fraser. The fate of particles in quantum field theories with interactions. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 39(4):841–859, 2008.

[48] David John Baker. Against field interpretations of quantum field theory. *The British Journal for the Philosophy of Science*, 60(3):585–609, 2009.

[49] Doreen Fraser. Particles in quantum field theory. In E. Knox and A. Wilson (Eds.), Routledge Companion to Philosophy of Physics. New York: Routledge, forthcoming (draft, 2017).

[50] Rob Clifton and Hans Halvorson. Are Rindler quanta real? Inequivalent particle concepts in quantum field theory. *The British journal for the philosophy of science*, 52(3):417–470, 2001.

[51] Hans Halvorson and Michael Müger. Algebraic quantum field theory. *arXiv preprint math-ph/0602036*, 2006.

[52] Hans Halvorson. Reeh-schlieder defeats newton-wigner: On alternative localization schemes in relativistic quantum field theory. *Philosophy of Science*, 68(1):111–133, 2000.

[53] Paolo Zanardi, Daniel A Lidar, and Seth Lloyd. Quantum tensor product structures are observable induced. *Physical Review Letters*, 92(6):060402, 2004.

[54] David Tong. Quantum field theory, university of cambridge part iii mathematical tripos. *Quantum Field Theory. University of Cambridge*, page 71, 2012.

[55] C Anastopoulos and B L Hu. Problems with the Newton Schroedinger equations. *New Journal of Physics*, 16(8):085007, 2014.

[56] Federico Piazza and Fabio Costa. Volumes of space as subsystems. *arXiv:0711.3048*, 2007.

[57] Jason Pye, William Donnelly, and Achim Kempf. Locality and entanglement in bandlimited quantum field theory. *Physical Review D*, 92(10):105022, 2015.

[58] Maria Papageorgiou and Jason Pye. Impacts of relativity on localizability and vacuum entanglement, manuscript in preparation.

[59] David Wallace. Emergence of particles from bosonic quantum field theory. *arXiv preprint quant-ph/0112149*, 2001.

[60] Mark Srednicki. Entropy and area. *Physical Review Letters*, 71(5):666, 1993.

[61] Christopher M Dawson and Michael A Nielsen. Frustration, interaction strength, and ground-state entanglement in complex quantum systems. *Physical Review A*, 69(5):052316, 2004.

[62] Jose Trevison, Koji Yamaguchi, and Masahiro Hotta. Pure state entanglement harvesting in quantum field theory. *Progress of Theoretical and Experimental Physics*, 2018(10):103A03, 2018.

[63] Dionigi M T Benincasa, Leron Borsten, Michel Buck, and Fay Dowker. Quantum information processing and relativistic quantum fields. *Classical and Quantum Gravity*, 31(7):75007, 2014.

APPENDICES

Appendix A

Derivation of the Bogoliubov transformation

A.1 Position space

In this appendix we are not using natural units so that we can keep track of the constants since the non-relativistic approximation involves comparing to the Compton wavelength $\lambda = \hbar/mc$. One can think of the Klein-Gordon equation for a free massive scalar field

$$\left((\partial_t/c)^2 - \nabla^2 + (1/\lambda)^2\right)\hat{\Phi}(x,t) = 0 \tag{A.1.1}$$

as describing a system of coupled harmonic oscillators in position space, where the coupling is provided by the d'Alembert operator. Then we can suitably define the corresponding creation and annihilation operators in position space as

$$\hat{a}_x = \frac{\alpha}{\sqrt{\hbar}}\hat{\Phi}(x) + \frac{\mathrm{i}}{2\alpha\sqrt{\hbar}}\hat{\Pi}(x) \tag{A.1.2}$$

so that the commutation relation for the field operators

$$[\hat{\Phi}(x,t), \hat{\Pi}(x',t)] = \mathrm{i}\hbar\delta(x-x') \tag{A.1.3}$$

implies that $[\hat{a}_x, \hat{a}_{x'}] = \delta(x-x')$. To fix the coefficient α we need to use the inverse transformations

$$\hat{\Phi}(x) = \frac{\sqrt{\hbar}}{2\alpha}\left(\hat{a}_x^\dagger + \hat{a}_x\right) \tag{A.1.4}$$

$$\hat{\Pi}(x) = \mathrm{i}\alpha\sqrt{\hbar}(\hat{a}_x^\dagger - \hat{\alpha}_x) \tag{A.1.5}$$

to rewrite the Hamiltonian of the Klein-Gordon field in terms of the creation and annihilation operators as follows

$$H = \frac{1}{2}\int \mathrm{d}x \left(c^2\hat{\Pi}^2(x,t) + \frac{1}{\lambda^2}\hat{\Phi}^2(x,t) - \hat{\Phi}(x,t)\nabla^2\hat{\Phi}(x,t) \right)$$
$$= \frac{\hbar}{2}\int \mathrm{d}x \left(-c^2\alpha^2(a_x^\dagger - a_x)^2 + \frac{1}{(2\alpha\lambda)^2}(a_x^\dagger + a_x)^2 - \frac{1}{(2\alpha)^2}(a_x^\dagger + a_x)\nabla^2(a_x^\dagger + a_x) \right)$$
$$= \frac{\hbar}{2}\int \mathrm{d}x \left(-\alpha^2c^2 + \frac{1}{(2\alpha\lambda)^2} \right)\left(a_x^{\dagger 2} + a_x^2\right) + \left(\alpha^2c^2 + \frac{1}{(2\alpha\lambda)^2} \right)(a_x^\dagger a_x + a_x a_x^\dagger) - \frac{1}{(2\alpha)^2}(a_x^\dagger + a_x)\nabla^2(a_x^\dagger + a_x$$

We pick α such that the first term vanishes, namely

$$\alpha = \sqrt{\frac{1}{2c\lambda}} = \sqrt{\frac{m}{2\hbar}} \tag{A.1.6}$$

since $\lambda = \hbar/mc$ the Compton wavelength. So with this choice the Hamiltonian becomes

$$\hat{H} = -\frac{1}{2}\int \mathrm{d}x \left(mc^2(a_x^\dagger a_x + a_x a_x^\dagger) - \frac{\hbar^2}{2m}(a_x^\dagger + a_x)\nabla^2(a_x^\dagger + a_x) \right) \tag{A.1.7}$$

and the field operators read $\hat{\Phi}(x) = \frac{\hbar}{\sqrt{2m}}\left(\hat{a}_x^\dagger + \hat{a}_x\right)$, $\hat{\Pi}(x) = \mathrm{i}\sqrt{\frac{m}{2}}(\hat{a}_x^\dagger - \hat{\alpha}_x)$.

A.2 Momentum space

The creation and annihilation operators $a_k, a_k^\dagger$ associated with the harmonic oscillators in momentum space are defined such that

$$\hat{\Phi}_k = \frac{\sqrt{\hbar}}{2\beta}\left(a_k + a_{-k}^\dagger\right) \tag{A.2.1}$$
$$\hat{\Pi}_k = \mathrm{i}\beta\sqrt{\hbar}\left(-a_k + a_{-k}^\dagger\right) \tag{A.2.2}$$

where $\hat{\Phi}_k = \int \frac{\mathrm{d}k}{\sqrt{2\pi}} e^{-ikx}\hat{\Phi}(x)$ (wtmeans). Again the form of (A.3.1) is such that $[\Phi_k, \Pi_{k'}] = \mathrm{i}\hbar\delta(k+k')$ implies $[a_k, a_{k'}^\dagger] = \delta(k-k')$ and we pick the coefficient β to be $\beta = \sqrt{\omega_k/2c^2}$

$$\hat{\Phi}_k = \sqrt{\frac{\hbar c^2}{2\omega_k}}\left(a_k + a_{-k}^\dagger\right) \tag{A.2.3}$$
$$\hat{\Pi}_k = \mathrm{i}\sqrt{\frac{\hbar\omega_k}{2c^2}}\left(-a_k + a_{-k}^\dagger\right) \tag{A.2.4}$$

so that the Hamiltonian reads $\hat{H} = \int \frac{\mathrm{d}k}{\sqrt{2\pi}} \hbar\omega_k a_k^\dagger a_k$ where $\omega_k = c\sqrt{k^2 + 1/\lambda^2}$. Note that with this choice of units the usual field decomposition looks like

$$\hat{\Phi}(x) = \int \frac{\mathrm{d}k}{\sqrt{2\pi}} \hat{\Phi}(k) e^{\mathrm{i}kx} = \int \frac{\mathrm{d}k}{\sqrt{2\pi}} \sqrt{\frac{\hbar c^2}{2\omega_k}} \left(a_k + a_{-k}^\dagger\right) e^{\mathrm{i}kx} = \int \frac{\mathrm{d}k}{\sqrt{2\pi}} \sqrt{\frac{\hbar c^2}{2\omega_k}} \left(a_k e^{\mathrm{i}kx} + a_k^\dagger e^{-\mathrm{i}kx}\right).$$

A.3 The transformation

Now we want to find the transformation between the two sets of creation and annihilation operators in position and momentum space which we can work out from the definitions above

$$\begin{aligned}
a_x &= \sqrt{\frac{m}{2\hbar^2}} \hat{\Phi}(x) + \frac{\mathrm{i}}{\sqrt{2m}} \hat{\Pi}(x) \\
&= \sqrt{\frac{m}{2\hbar^2}} \int \frac{\mathrm{d}k}{\sqrt{2\pi}} e^{ikx} \hat{\Phi}(k) + \frac{\mathrm{i}}{\sqrt{2m}} \int \frac{\mathrm{d}k}{\sqrt{2\pi}} e^{ikx} \hat{\Pi}(k) \\
&= \sqrt{\frac{m}{2\hbar^2}} \int \frac{\mathrm{d}k}{\sqrt{2\pi}} e^{ikx} \sqrt{\frac{\hbar c^2}{2\omega_k}} \left(a_k + a_{-k}^\dagger\right) - \frac{1}{\sqrt{2m}} \int \frac{\mathrm{d}k}{\sqrt{2\pi}} e^{ikx} \sqrt{\frac{\hbar\omega_k}{2c^2}} \left(-a_k + a_{-k}^\dagger\right) \\
&= \frac{1}{2} \int \frac{\mathrm{d}k}{\sqrt{2\pi}} e^{ikx} \left(\sqrt{\frac{mc^2}{\hbar\omega_k}} + \sqrt{\frac{\hbar\omega_k}{mc^2}}\right) a_k + e^{-ikx} \left(\sqrt{\frac{mc^2}{\hbar\omega_k}} - \sqrt{\frac{\hbar\omega_k}{mc^2}}\right) a_k^\dagger
\end{aligned}$$

We see that the annihilation operator in position space is a 'mixture' of the creation and annihilation operators in momentum space namely

$$a_x = \frac{1}{2} \int \frac{\mathrm{d}k}{\sqrt{2\pi}} \left(c_+(k) e^{ikx} a_k + c_-(k) e^{-ikx} a_k^\dagger\right) \tag{A.3.1}$$

$$c_\pm(k) = \sqrt{mc^2/\hbar\omega_k} \pm \sqrt{\hbar\omega_k/mc^2} \tag{A.3.2}$$

The inverse transformation is $a_k = \frac{1}{2} \int \mathrm{d}x e^{-\mathrm{i}kx} \left(c_+(k) a_x - c_-(k) a_x^\dagger\right)$.

A.4 Unitary inequivalence

In [12, 3, 4] one finds the following sufficient condition for unitary inequivalence of two representations. Two Fock-space representations are unitarily inequivalent if the vacuum

state of one representation has infinitely many particles in terms of the number operator of the other representation. Applying this criterion to the transformation (A.3.1) between the local and the global creation and annihilation operators we get

$$\langle 0_G| \int \mathrm{d}x a_x^\dagger a_x |0_G\rangle = \int \mathrm{d}k |c_-(k)|^2 \delta(0) \tag{A.4.1}$$

which is infinite because of the $\delta(0)$, and so it demonstrates unitary inequivalence of the two representations. We can also check for the global number operators over the local vacuum

$$\langle 0_L| a_k^\dagger a_k |0_L\rangle = |c_-(k)|^2 \int_{-\infty}^{\infty} \mathrm{d}x = \infty. \tag{A.4.2}$$

We conclude that $(\mathcal{F}_{\mathrm{L}}, \{a_x, a_x^\dagger\})$ and $(\mathcal{F}_{\mathrm{G}}, \{a_k, a_k^\dagger\})$ are unitarily inequivalent i.e. there is no unitry map that can relate the Hilbert spaces *and* the corresponding algebra. The notation $\mathcal{F}_{\mathrm{L}} \ncong \mathcal{F}_{\mathrm{G}}$ that we used in the text might be confusing because it does not explicitly refer to the operators but only the Hilbert spaces. Of course there is a unitary between the two Fock spaces, but it cannot preserve the algebra structure. In this sense, the local number operators are not well defined in the global vacuum and the global number operators are not well defined in the local vacuum. Nevertheless, this symmetry breaks in the bandlimitted where the local number operators are well defined in the global ground state

$$\begin{aligned}\langle 0_G| a_x^\dagger a_x |0_G\rangle &= \langle 0_G| \int_{-\Lambda}^{\Lambda} \mathrm{d}k \mathrm{d}k' e^{\mathrm{i}x(k-k')} c_-(k) c_-(k') a_k a_{k'}^\dagger |0_G\rangle \\ &= \langle 0_G| \int_{-\Lambda}^{\Lambda} \mathrm{d}k \mathrm{d}k' e^{\mathrm{i}x(k-k')} c_-(k) c_-(k') \delta(k-k') |0_G\rangle = \int_{-\Lambda}^{\Lambda} \mathrm{d}k |c_-(k)|^2 < \infty\end{aligned}$$

Appendix B

Continuously infinite tensor products

We have been contrasting continuous tensor product structures with Fock space structures all along. These are generally very different constructions. For example one might worry that a continuous tensor product structure would generally give a non separable space, while one might expect that a Fock space is separable. Eventhough it is mathematically counter intuitive one can suitably define an infinite tensor product $\otimes_x \mathcal{H}_x$ to be a Fock space, in the case that the $\mathcal{H}_x$'s are themselves Fock spaces, namely we can suitably define $\otimes_x \mathcal{F}[\mathcal{H}_x]$ to be the Fockspace of something. The question we are adressing in this appendix is whether we write the continuous tensor product of Fock spaces as a Fock space. Schematically, we are asking

$$\otimes_x \mathcal{F}[\mathcal{H}_x] \simeq \mathcal{F}[\,\text{of what?}\,] \tag{B.0.1}$$

One could guess that we can follow our intuition from the finite case [12] where for example we know that $\mathcal{F}[\mathcal{H}_1] \otimes \mathcal{F}[\mathcal{H}_2] \simeq \mathcal{F}[\mathcal{H}_1 \oplus \mathcal{H}_2]$ to claim that

$$\otimes_x \mathcal{F}[\mathcal{H}_x] \simeq \mathcal{F}[\int^{\oplus} \mathcal{H}_x]. \tag{B.0.2}$$

This is basically what turns out to be, but it is non trivial to ensure that the inner product is well defined in the infinite case. This construction can be found in [14] and here I will reproduce the steps as I understand them, because the 'trick' is quite insightful and also applicable in the QFT case. First we need to define the 'exponentiation' of a Hilbert space

$$e^{\mathcal{H}} := \bigoplus_{n=0}^{\infty} (\otimes \mathcal{H})^n_S \tag{B.0.3}$$

where by $(\otimes\mathcal{H})^n_S$ we mean the spanned by vectors of the form $(\otimes|\phi\rangle)^n$, which are symmetric by construction. One might think that this should be a subspace of the usual $n-$particle sector of a Fock space, but actually it is the whole space. This is because one can always write symmetrized product vectors as linear combinations of elements of $(\otimes\mathcal{H})^n_S$. For example, if $n=2$ we have that

$$|\phi_1\rangle\otimes|\phi_2\rangle+|\phi_2\rangle\otimes|\phi_1\rangle=|\phi_1+\phi_2\rangle\otimes|\phi_1+\phi_2\rangle-|\phi_1\rangle\otimes|\phi_1\rangle-|\phi_2\rangle\otimes|\phi_2\rangle$$

So $e^{\mathcal{H}}$ is just another way to write the Fock space $\mathcal{F}_s(\mathcal{H})$. Let us consider the states in $e^{\mathcal{H}}$ that are of the form

$$|\exp\phi\rangle:=\sum_{n=0}^{\infty}\frac{(\otimes|\phi\rangle)^n}{\sqrt{n!}} \tag{B.0.4}$$

where $|\phi\rangle\in\mathcal{H}$. The inner product of such states is

$$\langle\exp\phi|\exp\psi\rangle_{e^{\mathcal{H}}}=\sum_{n=0}^{\infty}\frac{\langle\phi|\psi\rangle^n}{n!}=e^{\langle\phi|\psi\rangle_{\mathcal{H}}} \tag{B.0.5}$$

Note that this inner product is quite counter intuitive. For example if the vectors that we are exponentiating are orthogonal $\langle\phi|\psi\rangle_{\mathcal{H}}=0$ then $\langle\exp\phi|\exp\psi\rangle_{e^{\mathcal{H}}}=1$, and if $\langle\phi|\psi\rangle_{\mathcal{H}}=1$ then $\langle\exp\phi|\exp\phi\rangle_{e^{\mathcal{H}}}=e$ and so the states (B.0.4) are not normalized. Overall, they form an (over)complete basis for $e^{\mathcal{H}}$.

The reason why we introduce this kind of Hilbert space here, is because one can use the construction above to define the continuous tensor product of such spaces. In particular, given a family of Hilbert spaces $\mathcal{H}_x$ for some continuous index x, we can suitably define $\otimes_x e^{\mathcal{H}_x}$. If we try to define $\otimes_x\mathcal{H}_x$ instead we run into the following trouble. Motivated by the finite case where one defines an inner product in $\otimes_{i=1}^n\mathcal{H}_i$ as

$$\langle u_1\otimes...\otimes u_n|v_1\otimes...\otimes v_n\rangle=\prod_{i=1}^{n}\langle u_i|v_i\rangle_{\mathcal{H}_i}=\exp\sum_{i=1}^{n}\log\langle u_i,v_i\rangle_{\mathcal{H}_i} \tag{B.0.6}$$

one can attempt to define the inner product of $\otimes_x\mathcal{H}_x$ as

$$\langle\otimes_x u_x|\otimes_x v_x\rangle:=\exp\int\mathrm{d}x\log\langle u_x,v_x\rangle_{\mathcal{H}_x} \tag{B.0.7}$$

which is generally well defined because of the logarithm. But using the property (B.0.5) we can have a well defined inner product in $\otimes_x e^{\mathcal{H}_x}$ since

$$\langle \otimes_x \exp\phi_x | \otimes_x \exp\psi_x \rangle := \exp \int \mathrm{d}x \log \langle \exp\phi_x, \exp\psi_x \rangle_{e^{\mathcal{H}_x}} \tag{B.0.8}$$

$$= \exp \int \mathrm{d}x \log \left(e^{\langle \phi_x | \psi_x \rangle_{\mathcal{H}_x}} \right) \tag{B.0.9}$$

$$= \exp \int \mathrm{d}x \, \langle \phi_x | \psi_x \rangle_{\mathcal{H}_x} \tag{B.0.10}$$

But now we notice that this is exactly the inner product in the exponentiated tenor sum $e^{\int^{\oplus} \mathcal{H}_x}$, which would establish an isomorphism between this space and $\otimes_x e^{\mathcal{H}_x}$ (note that in contrast to the continuous tensor products, continuous tensor sums are well defined as long as we make sure that the integrals in the tensor product converge). To see this, let us recall that for finite tensor sums the inner product is defined as

$$\langle \phi_1 \oplus ... \oplus \phi_n | \psi_1 \oplus ... \oplus \psi_n \rangle := \sum_{i=1}^{n} \langle \phi_i | \psi_i \rangle_{\mathcal{H}_i} \tag{B.0.11}$$

Let us denote as $|\phi(x)\rangle = |\oplus_x \phi_x\rangle$ a vector in the continuous tensor sum $\int^{\oplus} \mathcal{H}_x$. Then in analogy with the finite case one defines

$$\langle \phi(x) | \psi(x) \rangle = \int \mathrm{d}x \, \langle \phi_x, \psi_x \rangle_{\mathcal{H}_x} \tag{B.0.12}$$

and accordingly the inner product in $e^{\int^{\oplus} \mathcal{H}_x}$ is

$$\langle \exp \phi(x) | \exp \psi(x) \rangle = \exp \int \mathrm{d}x \, \langle \phi_x | \psi_x \rangle_{\mathcal{H}_x} \tag{B.0.13}$$

which is is exactly what we ended up with above. So we have that

$$\otimes_x e^{\mathcal{H}_x} \simeq e^{\int^{\oplus} \mathcal{H}_x} \tag{B.0.14}$$

which exactly confirms our initial guess (B.0.2). Finally, one can convince themselves that the familiar function space $\mathcal{L}^2$ can be written as the following tensor sum

$$\mathcal{L}^2(\mathbb{R}, \mathrm{d}x) \simeq \int^{\oplus} \mathrm{d}x \mathbb{C}_x. \tag{B.0.15}$$

Using this we can establish that

$$e^{\mathcal{L}^2(\mathbb{R})} \simeq e^{\int^{\otimes} \mathrm{d}x \mathbb{C}_x} \simeq \otimes_x e^{\mathbb{C}_x} \simeq \otimes_x \mathcal{L}^2_x(\mathbb{R}) \tag{B.0.16}$$

where we have used (B.0.14). Or in more familiar terms, we have that

$$\otimes_x \mathcal{L}^2_x(\mathbb{R}) \simeq \mathcal{F}[\mathcal{L}^2(\mathbb{R}, \mathrm{d}x)] \tag{B.0.17}$$

Quite magically, the continuous tensor product of copies of $\mathcal{L}^2$ turns out to be the Fock space over $\mathcal{L}^2$, which can be suitably applied in the QFT case when we wish to define continuously infinite tensor products. In particular, from the perspective of the local harmonic oscillators that we defined in the previous appendix, the field amplitude $\Phi(x)$ is the amplitude of the oscillator at each point x with $\mathcal{L}_2(\mathbb{R}, \mathrm{d}\Phi(x))$ being the corresponding Hilbert space. Then one can think of the total Hilbert space of the collection of oscillators as

$$\otimes_x \mathcal{L}_2(\mathbb{R}, \mathrm{d}\Phi(x)) \tag{B.0.18}$$

i.e. the continuously infinite tensor product of $\mathcal{H}_x := \mathcal{L}_2(\mathbb{R}, \mathrm{d}\Phi(x))$, the space where the local operators act. From (B.0.19) we can establish that

$$\otimes_x L_2(\mathbb{R}, \mathrm{d}\Phi(x)) \cong \mathcal{F}[L_2(\mathbb{R}, \mathrm{d}x)] := \mathcal{F}_L \tag{B.0.19}$$

to define the continuous tensor product as a separable Fock space, which in this thesis we denote as the local Fock space $\mathcal{F}_L$.