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SOME GEOMETRIC ASPECTS OF RICCI FLOW'S ROLE IN POINCARÉ'S CONJECTURE

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Introduction

Last century was an exciting century for mathematics. There were some remarkable results through diverse areas and deep connections between different research areas that brought together new insights and helped develop new tools that are widely used today. Amongst all of them, differential geometry was a branch of mathematical research that was nourished profoundly from diverse insights and the work of exceptional mathematicians from all over the world. It is the aim of this text to provide a brief introduction to a tool, the Ricci flow, that held a distinguished position in this nourishing as provided important results concerning the relationship between the geometry and topology of a space in low dimensions. These results were mostly used to prove important connections between the geometry and topology of a space, connections that are being explored nowadays.

Chapter 1

History

In the beginning of the 20th century, the french mathematician Henri Poincaré was working on the idea to characterize topological properties that distinguish different spaces, among these, the sphere. His first attempt to characterize these spaces was a technique called homology which he built based on the work done by Enrico Betti and the idea was to obtain certain sequence of algebraic objects (groups, modules) and associate these sequences to other objects such a topological spaces. His initial claim was that this technique was enough to characterize the 3-sphere S^3. Despite his efforts, this technique was not effective as he would later show building a famous counter-example known as "Poincaré's homological sphere" which is a topological space with the same homology sequence (homology groups) as S^3 but unlike the latter one, it is not simply connected. After this, he introduced a new topological invariant called the fundamental group, a tool used to prove the difference between S^3 and his counter example. With this new tool Poincaré asked: "Consider a compact 3-dimensional manifold M without boundary. Is it possible that the fundamental group of M could be trivial, even though M is not homeomorphic to S^3?"

This question started to rumble across the mathematical community all over the 20th century and many attempts were made to prove it and prove it's generalizations. In these attempts, new fields of study were constructed, fields such as differential topology, high dimensional differential geometry, among others (see [16] and [17]) and many new concepts arised. In the 1980's two mathematicians gave a new insight to this question asked by Poincaré which was by then stated in it's modern form:

Conjecture. "Every simply connected, closed 3-manifold is homeomorphic to S^3".

William Thurston (1946-2012) was an American mathematician and a pioneer in the field of low dimensional topology. Thurston visualized all possible 3-dimensional geometries and how they could behave when attached to different topological spaces and arrived to a more general conjecture than the one by Poincaré in the context of topology of 3-manifolds and called "The geometrization conjecture" (which was G. Perelman's main achievement) and encoded Poincaré's conjecture as a particular case of this. On the other hand, Richard S. Hamilton (1943-present day) is another American mathematician who by the time (1980's) developed a technique and a program whose objective was proving Poincaré's conjecture (and later, it would be known as a key factor in proving the geometrization conjecture as well). This technique, called ricci rlow was developed by Hamilton in order to understand how continuous deformations of the geometry of a particular topological space would exhibit an underlying structure of topological invariants and therefore could led to a classification of such spaces.

In a more general sense, not only these conjectures but the general form of Poincaré's conjecture in higher dimensions, stated formally as: "Every homotopy sphere (a n-manifold with the same homotopy groups as the n-sphere) in the chosen category (topological manifolds, smooth manifolds, piecewise-linear manifolds) is isomorphic in the chosen category (homeomorphic, diffeomorphic, piecewise-linear isomorphic) to the standard n-sphere" have played a key role throughout the past century in the advance of many areas of mathematics. Nevertheless, this text will be focused on the ricci flow and some of the main achievements needed to prove Poincaré's conjecture, and all other conjectures stated here are mentioned as historical notes.

Chapter 2

Preliminaries

2.1 Topology

A strinking fact about Poincaré's conjecture and it's proof is the deep relationship between the tools used and the key argument in the last part of the argument. That is, despite that the argument to show that the initial space is indeed a sphere is topological, most of the technical results are geometrical. For that matter, let us walk through some important concepts in topology:

2.1.1 Topological 3-manifolds

The main object of interest are topological 3-manifolds.

Definition 2.1.1. Topological manifolds are topological Hausdorff spaces M with countable basis that are locally homeomorphic to $\mathbb{R}^n$. This means, there exists an open cover of X whose elements we denote by $U_i \subseteq X$ and a bijective continuous function $f_i : U_i \to \mathbb{R}^n$ whose inverse f^{-1} is also continuous and bijective and therefore f defines a homeomorphism. The pair $(X, (U_i, f_i)_i)$ where $\bigcup_{i=1}^{\infty} U_i = M$ and $f_i \restriction_{U_i}: U_i \to \mathbb{R}^n$ is a homeomorphism, is called a topological manifold.

One should note that X as a topological manifold has the dimension of the $\mathbb{R}^n$ that it is locally homeomorphic to. For the purpose of this dissertation, one is particularly interested in spaces with dimension 3, and one writes: X^3 to say that X has dimension 3. As examples there are:

1. $\mathbb{R}^n$ for every n is a topological manifold with the trivial homeomorphism being the identity.
2. S^n which is also a topological manifold (using the stereographic projection one can explicitly compute the homeomorphisms and the open sets U_i).
3. Real and complex projective spaces in any dimension $\mathbb{RP}^n$ and $\mathbb{CP}^n$.
4. The torus $\mathbb{T}^n = S^1 \times ... \times S^1$ the product of n copies of S^1.

In general, topological manifolds might behave quite oddly, so one would want them to satisfy additional conditions; namely:

Definition 2.1.2. Given X a topological manifold, if it cannot be represented (in terms of sets) as the union of two disjoint sets (meaning that if X is a topological manifold, there are no subsets U, V of M such that: $U \subseteq X$ open, $V \subseteq X$ open, $U \cap V = 0$ and $U \cup V = X$). Then X is called a connected topological manifold, and it means that it is connected as a topological space.

Furthermore, one could ask X to satisfy a stronger condition.

Definition 2.1.3. A path from a point $x \in X$ to a point $y \in X$ is a continuous function $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. Having this definition, one can define an equivalence relationship $\sim$ defined as: $x \sim y$ if there is a path as defined above between x and y, being points in the same topological manifold X. Furthermore, every point that can be connected by a path belongs to a "path-connected component" of the topological manifold X. One says that X is a path-connected space if there is only one connected component, this means that for every two points in X there is a path with those points as extreme values.

Definition 2.1.4. The starting point of a closed path, namely $f(0) = x_0 = f(1)$ is called the base point and we consider all the closed paths (or loops) in a given point x_0 of X.

Definition 2.1.5. A homotopy is a continuous deformation of one loop into the other. Formally, a homotopy between two loops γ_0 and γ_1 is a function: $F : X \times [0,1] \to X$ such that: If $x \in X$ then $F(x,0) = \gamma_0(x)$ and $F(x,1) = \gamma_1(x)$.

Such loops can be distinguished by homotopies and with this in mind one can define the set of all the homotopy-equivalent loops in a point and this forms a group under composition called the fundamental group of X in x_0.The fundamental group is a key concept to study the topological behaviour of a topological manifold. They were used by Poincaré to construct a counter example to the original formulation of it's own conjecture.

2.1.2 Poincaré's homological sphere

We now study briefly the counter example constructed by Poincaré himself to the initial statement of the conjecture:

This space, called Poincaré's homological sphere or the spherical dodecahedron space, arises from a dodecahedron twisting opposite lying pentagons by $\frac{\pi}{5}$ radians relative to one another and then identifying them. There are 5 non-equivalent vertices O, P, Q, R, S. There are 1 non-equivalent edges, formed by identifying three equivalent edges. It is easy to compute the Euler characteristic: $\chi(X) = -5 + 10 - 6 + 1 = 2$. (Compare this with the Euler characteristic of a sphere). And define O as an initial point of the closed paths, which we will call a, h, f^{-1}, $f^{-1}d$ and these will identify O to the other vertices P, Q, R, S. By looking at the edge network of the dodecahedron (the configuation drawing of the paths), one can then see the generating paths of the fundamental group are represented by the following (where the product $*$ is taken to be composition of paths):

- A $= a * a^{-1}$.
- B $= a * b * h^{-1}$.
- C $= h * e * f$.
- D $= f^{-1} * d * (d^{-1} * f)$.
- E $= (f^{-1} * d) * e$.
- F $= f^{-1} * f$.

- G $= (f^{-1*} * d)g * a^{-1}$.
- H $= h * h^{-1}$.
- J $= a * i * f$.
- K $= h * k * (d^{-1} * f)$.

Note one can obtain relations of certain type after noticing that $A = D = F = H = Id$ and the remaining relations become trivial when one writes the paths in capital letters and write the paths A, D, F, H as the identity:

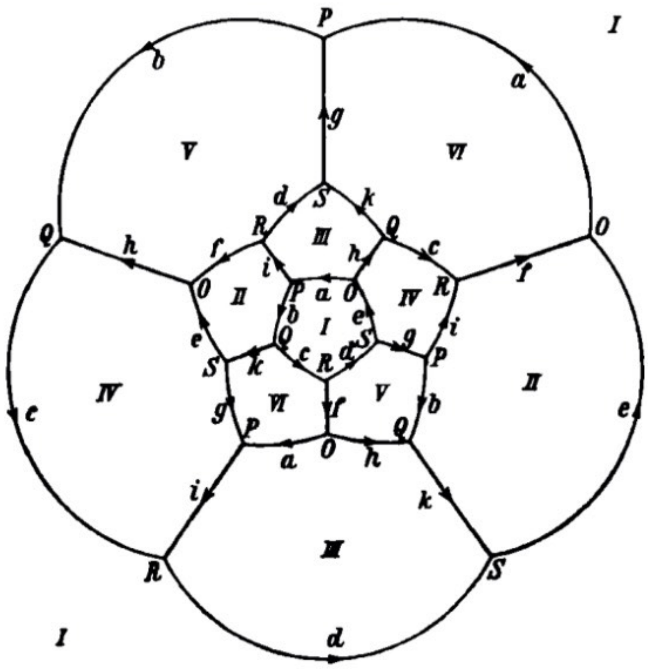


Figure 2.1: Configuration drawing [3]

[3]
After this, running through the pentagons:

- $B * C * E = I * D$.
- $B * K * E * J^{-1} = Id$.
- $J = K$.
- $C * J^{-1} * G^{-1} * E = Id$.
- $B = G^{-1}$.
- $G^{-1} * K^{-1} * C = Id$.

Therefore, from the relations for K and G, we have:

- $B * C * E = Id$.
- $B * J * E * J^{-1} = Id$.
- $C * J^{-1} * B * E = Id$.
- $B * J^{-1} * C = Id$.

From the first and fourth relations we get:

$$E = C^{-1} * B^{-1} \quad \text{and} \quad J = C * B.$$

Using these to eliminate E and J from the second and third relations, we get:

1. $B * C * B * C^{-1} * B^{-2} * C^{-1} = Id$ and,
2. $C * B^{-1} * C^{-1} * B * C^{-1} * B^{-1} = Id$.

It is worth mentioning that the first homology group can be obtained as the abelianization of the fundamental group and therefore, the first homology group is obtained by taking these previous relations abelian, and we obtain:

$$\overline{C} = 0,$$
$$-\overline{C} - \overline{B} = 0.$$

Thus $\overline{B} = \overline{C} = 0$. That is, the first homology group consist of the null element alone. Since the dodecahedron space is orientable, one can compute the values for the Betti numbers:

$$\rho^0 = 1\ ,\ \rho^1 = \rho^2 = 0,\ \rho^3 = 1.$$

These are the numerical invariants of the 3-sphere. Thus, homology is not enough to distinguish whether the 3-sphere does or does not coincide with the dodecahedron space. Now, we take a look at the fundamental group of these spaces: Let us replace (2) into (1) and obtain:

$$B^2 * C^{-1} * B^{-3} * C^{-1} = Id,$$

and introducing a new generator U into (1) and (2), where $C = U^{-1} * B$, we get:

$$B^5 = (B * U)^2 = U^3.$$

We recognize from these relations that the fundamental groups differ because the fundamental group of the sphere is trivial while the aforementioned relations show that the fundamental group of the spherical dodecahedron space is not. The relations before mentioned are satisfied by the icosahedral group, if one takes B as a rotation of $\frac{2\pi}{5}$ radians about a vertex and U as a rotation of $\frac{2\pi}{3}$ having the same orientation as B about the midpoint of a triangle adjoining that vertex. With this information, it is possible to show that these relations are satisfied by the binary icosahedral group whose order is 120, different from the order of the fundamental group of the sphere.

2.1.3 Milnor's comment

During the 20th century, topology was a very active area of mathematical research and during the second half of the last century crucial contributions were made in the study of low dimensional topology, the area of topology that focuses in spaces of dimension 3 and 4. The main categories where the study of 3-manifolds and 4-manifolds from the perspective of topology take place are the smooth manifolds, piecewise-linear manifolds and topological manifolds. A key fact proved independently (by Moise [22] and Munkres & Smale [21]) is that every topological 3-manifold has an essentially unique piecewise-linear structure and a unique differentiable structure, meaning that every topological 3-manifold can be triangulated as a simplicial complex whose combinatorial type is unique up to subdivision. And every triangulation of a 3 manifold can be taken to be a smooth triangulation in some differential structure on the manifold, unique up to diffeomorphism. Thus every topological 3 manifold has a unique smooth structure. (This will play a key role along the text). This remark was deeply used in the Ricci flow program proposed by Richard S. Hamilton in order to achieve a proof of the geometrization conjecture and Poincaré's conjecture.

2.1.4 Connected sum

In order to define the connected sum formally, we need the following result:

Theorem 2.1. *Suppose M is a manifold of dimension $n \geq 1$ and $B \subseteq M$ is a regular coordinate ball. Then $M \setminus B$ is an n-manifold with boundary, whose boundary is homeomorphic to S^{n-1}.*

Proof. $M \setminus \bar{B}$ is open in M and therefore is locally euclidean, showing that is a manifold. Now we are interested in the fact that ∂B is the boundary of $M \setminus B$.

Given that B is a coordinate ball, there exists a function $f : B' \to B_{r'}(x)$ where B' is a neighbourhood of B such that $f(B) = B_r(x)$ and $f(\bar{B}) = \bar{B}_r(x)$. Now we know that there is a neighbourhood of the boundary of B, namely $B' \cup M \setminus B$ which is isomorphic to a closure of a ball in $\mathbb{R}^n$. With this $M \setminus B$ is a manifold with boundary.

Furthermore,the boundary of the manifold is the boundary of B, which is isomorphic to S^{n-1} because of the following: $f(\partial B) = (\partial B_r(x))$ and is a well-known result (generalized Alexander's theorem, proven by Brown) that every n-sphere bounds a $(n+1)$-ball.

□

The connected sum process is an operation in topology used to obtain new surfaces and manifolds from existing ones. The procedure for manifolds is as follows: Suppose M and N are manifolds of dimension $n \geq 1$ and let $B_1 \subseteq M$ and $B_2 \subseteq N$ be regular balls in M and N respectively. Consider $M' = M \setminus B_1$ and $N' = N \setminus B_2$. Choose a homeomorphism $f : \partial M' \to \partial N'$ (this homeomorphism exists because of the previous theorem). Let $M \# N$ called the connected sum of M and N be the adjunction space $M' \cup_f N'$, where this means we glue M' and N' according to the action of f. It is easy to see that $M \# N$ is an n-manifold without boundary.

Connected sums preserve certain topological structures and properties, e.g. the connected sum of compact manifolds is compact and the connected sum of connected manifolds is again connected. Given two different manifolds, there are 2 different possibilities for new manifolds to arise according to their connected sum and those are when the homeomorphism preserves or reverses orientability. Additionally, this process has a neutral element (a consequence of Alexander's theorem stated above) and due to the fact that S^3 is prime in the sense of decomposition in connected sums (this means that S^3 decomposes only as a connected sum of itself with another S^3), therefore S^3 plays the role of a neutral element in the connected sum. (For a further explanation see example 6.6 in [13]) and from this one can state a notion of irreducibility in topological manifolds, namely, a topological manifold is irreducible if it cannot be decomposed in the connected sum of trivial elements (itself and copies of S^3). There are some other types of connected sums (related to gluing tori handles to a manifold and therefore changing it's genus) but those procedures change the topology completely and they will not be used in this text.

2.1.5 Surgery

Just like the connected sum process, the surgery process is a topological operations used to obtain new types of manifolds from existing ones. The surgery process is a much more difficult process than the one of connected sum and besides being highly non trivial it is not completely well understood. Its main achievements have been achieved in dimensions 5 or higher and therefore its applications to low dimensional manifold theory were quite shocking at the time. Milnor first introduced this technique in order to construct new manifolds as follows: Let us begin with a well understood manifold M and by attaching handles (curved copies of cylinders) [17] and since then, several processes have been constructed in different dimensions and with variations on the original procedure to construct new manifolds from an initial one.

We will focus on a special type of surgery which preserves orientation and controls the topological change. Indeed, in the context of 3-manifolds and 4-manifolds we are interested in cutting a spherical region around a point and then gluing a 3-ball (initially it is a topological 3-ball but we will be interested in equipping it with the standard Riemannian metric) on the region we cut. This process of cutting 3-balls along manifolds was developed by Kneser and it was a main tool used to develop Ricci flow with surgery. In fact, the procedure of cutting 3-balls around points in a manifold and gluing 3-balls we obtain the following theorem:

Theorem 2.2 (Kneser's descomposition)**.** *Every compact, orientable 3-manifold is a connected sum of 3-manifolds that are either homeomorphic to $S^2 \times S^1$ or irreducible. Moreover, the connected sum summands are unique up to ordering and orientation-preserving homeomorphism.*

A detailed proof of this theorem can be found in [17].

2.2 Riemannian geometry

Now we review some basic material on Riemannian geometry needed to understand some of the most important facts related to the geometric overview of Ricci flow. All of the material discussed here can be found in any standard Riemannian geometry textbook such as [12] or [**?**].

2.2.1 Metric tensor

Now it is time to focus all the attention on certain special type of smooth manifolds, namely, Riemannian manifolds. A Riemannian metric g on a manifold M is a symmetric positive definite (2,0)-tensor field, meaning by this that $g \in \Gamma(\mathrm{Sym}^2\, T^*M)$ and g_p defines an inner product for each $p \in M$. We call a manifold M with a given Riemannian metric g a Riemannian manifold (M, g). In local coordinates (x^i) then g takes the form: $g = g_{ij} dx^i \otimes dx^j$. It is now a natural question to ask what happens when the manifold is built from the product of two (or possibly more) manifolds. If $(M_1^n,\, g^1)$ and $(M_2^m,\, g^2)$ are two Riemannian manifolds of dimension n and m respectively, using the following fact for the tangent spaces:

$$T_{p_1,p_2}(M_1 \times M_2) \cong T_{p_1}M_1 \times T_{p_2}M_2.$$

Then there is a natural metric acting on the product manifold, namely $g = g^1 \oplus g^2$ defined as follows:

$$g_{p_1,p_2}(u_1 + u_2, v_1 + v_2) = g^1_{p_1}(u_1 + u_2) + g^2_{p_2}(v_1 + v_2).$$

Where:

$$u_1, u_2 \in T_{p_1}M_1^n,$$

and

$$v_1, v_2 \in T_{p_2}M_2^m.$$

It's matrix form would then be:

$$g_{ij} = \begin{bmatrix} g^1 & 0 \\ 0 & g^2 \end{bmatrix}.$$

Starting from this, we define a curve in our space in the following way:

Definition 2.2.1. Given $\gamma_0 : [0,1] \to M$ a smooth curve, we define the length of γ_0 by:

$$\int_0^1 \|\dot{\gamma_0}(t)\|_g dt.$$

Where $\|.\|_g$ describes the norm induced by the Riemannian metric. Now we can define the distance between two different points in M as follows:

Definition 2.2.2. If a, b are points in M, we define:

$$d(a,b) = \inf\left\{\int_0^1 \|\dot{\gamma}(t)\|_g dt : \gamma : [0,1] \to M, \gamma(0) = a, \gamma(1) = b\right\}.$$

Curves at which this infimum is realized are called geodesics. In an analogous way, we can define geodesics as extremals of the following length functional:

$$L(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_g dt, \text{ where } \gamma \text{ runs amongst all curves } \gamma : [0,1] \to M, \gamma(0) = a \text{and} \gamma(1) = b.$$

We can define the Energy functional as follows:

$$E(\gamma) = \frac{1}{2}\int_0^1 \|\dot{\gamma}(t)\|_g^2 dt.$$

Noticing that (with the aid of Cauchy-Schwartz inequality):

$$L(\gamma)^2 \leq 2E(\gamma) \text{ then minimizing } L(\gamma) \text{ is equivalent to minimizing } E(\gamma).$$

When minimizing the energy functional in physics, we obtain the Euler-Lagrange equations and so:

$$\ddot{\gamma^i}(t) + \Gamma^i_{jk}(\gamma(t))\dot{\gamma^j}(t)\dot{\gamma^k}(t) = 0.$$

Therefore any $\gamma : [0,1] \to M$ that satisfies the equation is called a geodesic. Their short-time existence and uniqueness is guaranteed by the theory of Ordinary Differential Equations (Picard-Lindelöf theorem).

2.2.2 Connection and curvature

Connections provide a general way of thinking about directional derivatives. Then, if we wish to take the directional derivative of $X = X^i e_i = X^i \partial_i$ in the direction of v then we have: $Der_v X = v(X^i)\partial_i = v(X^i)e_i$.

Therefore a connection provides a "natural" way of connecting tangent spaces at different points of M. If E is a vector bundle over a manifold M, then the connection acts on elements of $\Gamma(E)$ as a module over $C^\infty M$.

$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ and this acts over elements: $\nabla : (X, \sigma) \to \nabla_X \sigma$ and satifies:

1. ∇ is C^∞(M)-linear in X, meaning: $\nabla_{f_1X_1+f_2X_2}\sigma = f_1\nabla_{X_1}\sigma + f_2\nabla_{X_2}\sigma$.
2. ∇ is ($\mathbb{R}$)-linear in σ, meaning: $\nabla_X(\lambda_1\sigma_1 + \lambda_2\sigma_2) = \lambda_1\nabla_X\sigma_1 + \lambda_2\nabla_X\sigma_2$.

3. $\nabla_X(f\sigma) = X(f)\sigma + f\nabla_X\sigma$.

And we call it covariant derivative of σ. One of the main features of the connection are the connection coefficients also known as Christoffel symbols which in local coordinates $(x^i)_{i=1}^n$ can be written as:

$$\Gamma_{ij}^k = dx^k(\nabla_{\partial_i}\partial_j),$$

and from this we can conclude:

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k.$$

Some important facts related to this special type of derivation are the following:

1. $\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G)$.
2. $\nabla_X(\operatorname{tr} F) = \operatorname{tr}(\nabla_X F)$.

Which basically state that the covariant derivative behaves as one would expect when dealing with tensor fields and operations with tensors (such as taking the trace of a tensor). In general, if F is a (k,l)-tensor, $Y, X \in \mathfrak{X}(M)$ and ω is a 1-form then we have:

$$(\nabla_X F)(\omega^1, ..., \omega^l, Y^1, ..., Y^k) = \nabla F(X, Y^1, ..., Y^k, \omega^1, ..., \omega^l).$$

Therefore ∇ acts as an operator which takes:

$$\mathbb{T}_l^k(M) \to \mathbb{T}_l^{k+1}(M).$$

Where $\mathbb{T}_l^k$ denotes the space of (k,l)-tensors. A natural question is the definition of the second covariant derivative, noted as ∇^2. For that matter, let $E = \mathbb{T}_l^k(M)$ be a vector bundle over a manifold M with it's respective connection and $F \in \Gamma(E)$ then:

$$\nabla F \in \Gamma(E \otimes T^*M) \text{ and so } \nabla^2 F \in \Gamma(E \otimes T^*M \otimes T^*M)$$

and acts as follows:

$$\begin{aligned}
\nabla^2 F(X,Y) &= (\nabla_X(\nabla F))(Y) \\
&= \nabla_X((\nabla F)(Y)) - (\nabla F)(\nabla_X Y) \\
&= \nabla_X(\nabla_Y F) - \nabla_{\nabla_X Y} F.
\end{aligned}$$

Once there is a description of the second covariant derivative, it is interesting to ask how the Laplacian operator acts with these derivatives, for that matter:

The simplest form of the Laplacian operator Δ for functions f in $C^\infty(M)$ is: $\Delta f = \operatorname{div}(\operatorname{grad} f)$ where (div) is the divergence operator and (grad) is the gradient operator acting on f.

Definition 2.2.3. For $F \in \mathbb{T}_l^k(M)$ the space of (k,l)-tensors, the connection Laplacian is: $\Delta F = \operatorname{tr} \nabla^2 F$.
In local coordinates:

$$\begin{aligned}
(\Delta F)_{i_1 \dots\dots i_k}^{j_1 \dots\dots j_l} (\operatorname{tr}_g \nabla^2 F)_{i_1 \dots\dots\dots i_k}^{j_1 \dots\dots\dots j_l} &= (\operatorname{tr}_{13} \operatorname{tr}_{24} g^{-1} \otimes \nabla^2 F)_{i_1 \dots i_k}^{j_1 \dots\dots j_l} \\
&= g^{pq}(\nabla_{\partial_p}\nabla_{\partial_q} F)(\partial_{j_1}, ..., \partial_{j_l}, dx^{i1}, ..., dx^{i_k}).
\end{aligned}$$

Let's take a look at an example of this:

Example 2.2.1. In $\mathbb{T}^0_0(M) = C^\infty(M)$ we have:

$$\begin{aligned}
\Delta f &= g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j} f) \\
&= g^{ij}(\nabla_{\partial_i}(\nabla f))(\partial_j) \\
&= g^{ij}(\partial_i((\nabla f)(\partial_j))) \\
&= g^{ij}\nabla_{\partial_i}((\nabla f)(\partial_j)) - \nabla f(\nabla_{\partial_i}\nabla_{\partial_j}) \\
&= g^{ij}(\partial_i\partial_j f) - \nabla_{\partial_k} f\Gamma^k_{ij} \\
&= g^{ij}(\partial_i\partial_j f) - \Gamma^k_{ij}\nabla_{\partial_k} f
\end{aligned}$$

and this is the expression for the Laplacian operator acting on f.

Now that the description of the connection is detailed enough, we can proceed in our analysis by defining the torsion associated to a connection:

$$\tau_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

and notice that τ is a $(2,1)$-tensor. Now it is of interest to see if there is a compatibility condition between the Riemannian metric g and the connection. For that matter, there is an important theorem which relates both structures:

Theorem 2.3. *(Fundamental Theorem of Riemannian Geometry) Given a Riemannian manifold there exists a unique connection that satisfies the aforementioned conditions: It is linear with respect to functions in the first entry, it is $\mathbb{R}$-linear in the second entry and satisfies the Leibnitz rule. Besides this, it has no torsion and is compatible with the metric tensor (meaning for all $\beta, \xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$ we have*

$$X(g(\beta,\xi)) = g(\nabla_X\beta,\xi) + g(\beta,\nabla_X\xi).$$

This connection is known as the Levi-Civita connection for Riemannian manifolds.

A direct consequence of the properties of the Levi-Civita connection is that its Christoffel symbols have symmetry in the lower indexes:

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

In local coordinates (x^i) the Christoffel symbols can be expressed in terms of the metric (due to the compatibility condition of the connection) as:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}).$$

Now we attend our attention to another expression of aspect of the geometry connections: Curvature.
In a general frame, given a vector bundle E (the reader should notice that E here stands for any vector bundle, not to confuse with the recurrent use of $E = TM$ in the Riemanian geometry setting) over a smooth manifold M one can define the curvature operator associated to a given connection as: $\mathrm{R}_\nabla \in \Gamma(T^*M \otimes T^*M \otimes E^* \otimes E)$ acting:

$$\mathrm{R}_\nabla(X,Y)(\sigma) = \nabla_Y(\nabla_X\sigma) - \nabla_X(\nabla_Y\sigma) + \nabla_{[X,Y]}\sigma$$

One important example is to study the curvature of $E_1^* \otimes E_2$ which are E_2-valued tensors acting on E_1 as follows:

If there is $S \in \Gamma(E_1^* \otimes E_2)$, $\varphi \in \Gamma(E_1)$ and $X, Y \in \mathfrak{X}M$ then:

$$(\mathrm{R}(X,Y)S)(\varphi) = \mathrm{R}_{\nabla_2}(X,Y)(S(\varphi)) - S(\mathrm{R}_{\nabla_1}(X,Y)\varphi).$$

As expected, the curvature operator we just defined satisfies Leibnitz rule for tensor products and conmmutes with taking the trace of tensors. Besides this, it can be shown with a few calculations that if ∇ is symmetric (meaning it has no torsion) then

$$\mathrm{R}(X,Y) = \nabla^2_{Y,X} - \nabla^2_{X,Y}.$$

Now, if we work with the Levi-Civita connection for a Riemannian manifold, we can give a coordinate description of the curvature operator, that is:

$$\mathrm{R} = \mathrm{R}_{ijk}^{l}\, dx^i \otimes dx^j \otimes dx^k \otimes \partial_l \text{ where: } \mathrm{R}(\partial_i, \partial_j)\partial_k = \mathrm{R}_{ijk}^{l}\, \partial_l$$

and using the compatibility condition between the connection and the metric tensor we can take the curvature tensor from a $(3,1)$-tensor to be a $(4,0)$-tensor as follows:

$$\mathrm{R}(X,Y,Z,W) = g(\mathrm{R}(X,Y)Z, W)$$

and in index notation:

$$\mathrm{R}_{ijkl} = g_{lp}\, \mathrm{R}_{ijk}^{p}$$

There are some symmetries hidden in the tensor which we now state:

1. $(\mathrm{R}_{ijkl} + \mathrm{R}_{jikl}) = (\mathrm{R}_{ijkl} + \mathrm{R}_{ijlk}) = 0$.
2. $\mathrm{R}_{ijkl} = \mathrm{R}_{jilk}$.
3. First Bianchi identity: $\mathrm{R}_{ijkl} + \mathrm{R}_{jkil} + \mathrm{R}_{kijl} = 0$.
4. Second Bianchi identity: $\nabla_m \mathrm{R}_{ijkl} + \nabla_k \mathrm{R}_{ijlm} + \nabla_l \mathrm{R}_{ijmk} = 0$.

Before we proceed, there is an important concept to discuss, the concept of taking trace. Just as there is a notion of taking "the trace of a matrix" $A = (a_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ given by $\mathrm{tr}(A) = \sum_i a_{ii}$. We can extend this definition to $\mathrm{End}(V)$ by taking trace over a matrix representation. With this, we can define a tensor contraction by pairing a finite dimensional vector space with it's dual. For any tensor $F \in \mathbb{T}^{k+1}_{l+1}$, where $\mathbb{T}^{k+1}_{l+1}$ denotes the tensor field of tensors F which acts on $(k+1)$ different vector fields and also acts on $(l+1)$ different 1-forms, we define the contraction of F by taking the trace of F as follows:

$$(\mathrm{tr}\, F)(\omega^1, ..., \omega^l, X_1, ..., X_k) = tr(F(\omega^1, ..., \omega^l, \cdot), X_1, ..., X_k, \cdot)$$

And so, component-wise, taking the trace of a tensor is just taking the sum of repeated indices over the indexed positions, e.g. if F_{ik}^{j} is a $\mathbb{T}^2_1$-tensor then we can take $\mathrm{tr}_{12} F = F(\cdot, \cdot, e_k) = F_{ik}^{i}$ or $\mathrm{tr}_{23} F = F(e_k, \cdot, \cdot) = F_{ki}^{i}$.

Despite the description we have of the curvature operator, it can be quite awful to handle such tensor. To avoid this treatment of curvature, some other curvatures can be defined to study the global behaviour of curvature in a Riemannian manifold that are, in a sense, easier to handle. Along with this spirit, one formally defines:

Definition 2.2.4. The Ricci curvature

$$\operatorname{Ric}(X,Y) = \operatorname{tr}_g(X,\cdot,Y,\cdot) = (\operatorname{tr}_{14}\operatorname{tr}_{26}(g^{-1}\otimes \mathrm{R}))(X,Y)$$

and this in local coordinates (x^i) takes the form:

$$\operatorname{Ric}(\partial_i,\partial_j)\,\mathrm{R}_{ij} = \mathrm{R}^k_{ikj} = g^{pq}\,\mathrm{R}_{ipjq}$$

It is worth mentioning that the Ricci curvature is also symmetric. Besides the Ricci curvature, there is another curvature one can define called the scalar curvature, formally:

Definition 2.2.5.

$$\mathcal{R} = \operatorname{Scal} = \operatorname{tr}_g \operatorname{Ric} = \operatorname{Ric}^i_i = g^{ij} R_{ij}.$$

Now using the expressions:

$$\mathrm{R}^l_{ijk} = \partial_j\Gamma^l_{ik} - \partial_i\Gamma^l_{jk} + \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im},$$

and

$$\mathrm{R}_{ijkl} = \tfrac{1}{2}(\partial_j\partial_k g_{il} + \partial_i\partial_l g_{jk} - \partial_i\partial_k g_{jl} - \partial_j\partial_l g_{ik}) + g_{lp}(\Gamma^m_{ik}\Gamma^p_{jm} - \Gamma^m_{jk}\Gamma^p_{im}),$$

we obtain a expression in local coordinates:

$$\operatorname{Ric}_{ik} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^j\partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i\partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i\partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j\partial x^l}\right) + \Gamma^m_{ik}\Gamma^j_{jm} - \Gamma^m_{jk}\Gamma^j_{im}.$$

Now we can state an important theorem:

Theorem 2.4. *Given that the metric tensor is parallel with respect to the Levi-Civita connection, then:*

1. $\nabla_k \operatorname{Ric}_{ij} = g^{pq}\nabla_k \mathrm{R}_{ipjq}$.
2. $\nabla^2_{k,l} \operatorname{Ric}_{ij} = g^{pq}\nabla^2_{k,l} \mathrm{R}_{ipjq}$.

Proof. 1. Let $X,Y,Z \in \mathfrak{X}(M)$, then: $\nabla_Z \operatorname{Ric}(X,Y) = (\nabla_Z \operatorname{tr}(g^{-1}\otimes \mathrm{R}))(X,Y)$ using that the covariant derivative commutes with respect to taking the trace and satisfies the Leibniz rule with respect to tensor product, then:

$$[\operatorname{tr}(\nabla_Z g^{-1}\otimes \mathrm{R}) + \operatorname{tr}(g^{-1}\otimes \nabla_Z \mathrm{R})](X,Y)$$

and because $\nabla g = 0$ then $\nabla_Z g^1 = 0$ thus we have:

$$\operatorname{tr}(g^{-1}\otimes \nabla_Z \mathrm{R})(X,Y) = \operatorname{tr}_g \nabla_Z \mathrm{R}(X,\cdot,Y,\cdot).$$

2. $\nabla^2 \operatorname{Ric} = \nabla^2(\operatorname{tr} g^{-1}\otimes \mathrm{R})$ and because g is parallel, then:

$$\operatorname{tr}(g^{-1}\otimes \nabla^2 \mathrm{R}) = \operatorname{tr}_g \nabla^2 \mathrm{R}\,.$$

□

We can also take some calculations and obtain an expression for the covariant derivative of the scalar curvature: $\nabla_l \mathcal{R} = 2g^{jk}\nabla_k \operatorname{Ric}_{ij}$. Finally there is another curvature we wish to mention, namely, the sectional curvature.

Definition 2.2.6. Let π be a 2-dim subspace of T_pM for M a Riemannian manifold of dimension $n \geq 3$ then we define K, the sectional curvature of π as:

$$K(\pi) = \frac{\mathrm{R}(x, y, x, y)}{\|x\|^2 \cdot \|y\|^2 - g(x, y)^2}$$

where x, y are a basis for π.

All of these curvatures have different meaning but there is an important fact in 3-manifold theory and is that all of these curvatures are closely related (in higher dimensions, in order to study curvature additional curvature expressions arise and to have the general picture it is not enough to study only the aforementioned ones) but in 3 dimensions we have this diagram:

Metric tensor $(g) \to$ Riemann's curvature tensor (R) $\to$ Ricci Curvature (Ric) $\to$ scalar curvature $(\mathcal{R})$.

Evenmore, using the sectional curvature, we can control the behaviour of the Ricci and scalar curvature as follows:

If (M, g) a Riemannian manifold has bounded sectional curvature, then we define:

$$\delta = \min_{X,Y \in T_pM} K(X, Y),$$

and

$$\Delta = \sup_{X,Y \in T_pM} K(X, Y),$$

with $\delta \geq 0$ then:

1. $\|R(X, Y, W, Y)\| \leq (\frac{(\Delta-\delta)}{2})$.
2. $\|R(X, Y, W, Z)\| \leq (\frac{2(\Delta-\delta)}{3})$.

For $X, Y, W, Z \in T_p\ M$ orthonormal. This is a famous result called Berger's lemma.

2.2.3 Pullback and Pushforward bundle structure

Definition 2.2.7. Given a smooth function $f : M \to N$, the pullback bundle of a vector bundle E on a manifold N by f, denoted by $f^*(E)$ is a smooth vector bundle over the manifold M, denoted as:

$$f^*(E) = \{(p, \xi) : p \in M, \quad \xi \in E, \quad \pi(\xi) = f(p)\}$$

.

If $\{\xi_i\}_{i=1}^k$ is a local frame for E near $f(p) \in N$ then $\Xi_i(p) = \xi_i(f(p))$ are a local frame for $f^*(E)$ near$p$. There are some important properties held by pullbacks, such as:

1. Commutes with taking duals: $(f^*(E))^* = f^*(E^*)$.
2. Commutes with tensor products: $f^*(E_1) \otimes f^*(E_2) = f^*(E_1 \otimes E_2)$.

Now we direct our attention to restrictions. A restriction $\xi_f \in \Gamma(f^*(E))$ of $\xi \in \Gamma(E)$ by f is defined as:

$$\xi_f(p) = \xi(f(p)) \in E_{f(p)} = (f^*(E))_p \text{ for every } p \in M.$$

Example 2.2.2. Define g a metric on E a smooth vector bundle over N, then: $g \in \Gamma(E^* \otimes E^*)$ and by restriction we obtain a metric $g_f \in \Gamma((f^*E)^* \otimes (f^*E)^*)$ which is a metric defined on the smooth vector bundle f^*E over M, meaning: If $\xi, \eta \in (f^*E)_p = E_{f(p)}$ then: $g_f(p)(\xi, \eta) = g(f(p))(\xi, \eta)$.

It is natural to ask if we can define a pullback operation for tensors and the answer is yes, if we can combine restrictions and duality, meaning: If we have a (k, l)-tensor on M $S \in \Gamma(\otimes^k T^*N)$ by restriction: $S_f \in \Gamma(\otimes^k (f^*T^*N))$ and therefore we define the pullback: $f^*S \in \Gamma(\otimes^k T^*M)$ by $f^*S(X_1, ..., X_k) = S_f(f^*X_1, ..., f^*X_k)$.

Another important operation we can define when thinking about bundles is the pushforward. Given a smooth map between manifolds: $f : M \to N$ for each $p \in M$ we have: $f_*(p) : T_pM \to T_{f(p)}N = (f^*TN)_p$ that is: $f_*(p) \in T_p^*M \otimes (f^*TN)_p$ so f_* is a smooth section of $T^*M \otimes f^*TN$. Meaning that: If we have a smooth vector field $X \in \Gamma(TM) = \mathfrak{X}(M)$, the pushforward of X is a smooth section $f_*X \in \Gamma(f^*TN)$ obtained by applying f_* to X.

After defining the pullback and pushforward operations, a question one might ask is how these operations relate to some structures such as connections and curvature. Consider then a connection ∇ on E, a vector bundle over a smooth manifold $N : \pi : E \to N$ and a smooth function $f : M \to N$, then how can we define something as $\nabla_v^f(\xi_f)$ a connection defined on f^*E if ∇ is a given connection on E? we can proceed, as follows: For a fixed point $p \in M$, $\xi \in \Gamma(f^*E)$ which can be written as: $\xi = \sum_{i=1}^n \xi^i(\sigma_i)_f$ where ξ^i are smooth functions defined near p and $\{\sigma_i\}_{i=1}^N$ a local frame for $f(p)$ in E we have:

$\nabla_v^f(\xi^i(\sigma_i)_f) = \xi^i \nabla_v^f(\sigma_i)_f$ and this is just: $\xi^i \nabla_{f_*v}\sigma_i + v(\xi^i)(\sigma_i)_f$. Now, we have:

Theorem 2.5. *Given a connection ∇ in E, a vector bundle $\pi : E \to N$ over a smooth manifold N, there exists a unique connection ∇^f on f^*E the pullback connection such that:*
$\nabla_v^f(\xi_f) = \nabla_{f_*v}(\xi)$ for every $v \in TM$ and $\xi \in \Gamma(E)$.

There are some important properties satisfied by the pullback connection:

1. If g is a metric on E and ∇ a connection on E compatible with g then ∇f compatible with g_f.

2. The curvature of the pullback connection is the pullback of the curvature of the original connection, meaning: $R_{\nabla^f}(X, Y)\xi_f = (f^*R_\nabla)(X, Y)\xi$.

Finally, before we proceed to study Ricci flow, there is a final definition we need to discuss:

Let $\Omega \subset \mathbb{R}^n$ an open domain then harmonic functions are solutions of the Laplace equation: $\Delta f = 0$ where Δ is the Laplacian operator defined in previous sections. In a more general setting: Harmonic functions are critical points of the Dirichlet functional:

$$E_\Omega(f) = \frac{1}{2} \int_\Omega \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i(x)} \right)^2 d^m x = \frac{1}{2} \int_\Omega \|df_x\|^2 d^m x.$$

In the Riemannian manifold setting: Let $\varphi : (M, g) \to (N, h)$ and $\Omega \subseteq M$ with boundary. Then, as previously done, we can define a geometric equivalent of the Dirichlet functional as follows: $E_\Omega(\varphi) = \int_\Omega \|d\varphi\|^2 \omega_g$ where ω_g is the volume measure defined by the metric $\omega_g = \sqrt{g}\, d^m x$ and $\|d\varphi\|^2 = g^{ij} h_{ab}(\varphi(x)) \partial_i \varphi^a \partial_j \varphi^b$. Some examples of harmonic maps between Riemannian manifolds include constant maps, identity maps, isometries, harmonic maps between euclidean spaces and harmonic maps from Riemannian manifolds to euclidean space, geodesics are examples as well of harmonic maps and holomorphic maps between Kähler manifolds among other examples. For a more detailed exposition on harmonic maps [26] is a very complete reference.

Chapter 3

Ricci flow

3.1 Basics of Ricci flow

Based on the work done by Eells and Sampson related to the harmonic map heat flow ([29]), Hamilton proposed that something could be done to the metric tensor g_{ij} to "improve it" by means of a heat type equation. By improving it one should specify improving it's curvature behaviour (for instance). Therefore, if one would encounter a one-parameter family of Riemannian metrics $g_{ij}(t)$ it should be expected that $\frac{\partial}{\partial t} g_{ij}(t)$ would be some "Laplacian" expression involving second derivatives of the metric tensor.

Recall that we have the following expression

$$\mathrm{Ric}_{ik} = \frac{1}{2} g^{jl} \left(\frac{\partial^2 g_{il}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l} \right) + \Gamma^m_{ik} \Gamma^j_{jm} - \Gamma^m_{jk} \Gamma^i_{im},$$

and notice that this expression involving the Ricci tensor also involves second order derivatives of the metric tensor as it was desired for the expression in the derivative. Notice, additionally that the last terms in the above mentioned expression look like a "Laplacian", and so we have: $\partial_t g(t) = -2 \,\mathrm{Ric}_{g(t)}$ Then what is the intuition behind how this expression works?
If (M, g) is a Riemannian manifold and with the aid of local existence theory, the assignment: $t \to g(t)$ is a family of Riemannian metrics, having $g(t)$ as sections of the fixed background bundle $\mathrm{Sym}^2 T^*M$, then the expression $\partial_t g(t)$ has a precise meaning: For every point p in the manifold we derive $g(t)$ in the vector space given by the fiber of $\mathrm{Sym}^2 T^*M$ at $p \in M$.

One may wonder then what the "t" variable means here and it stands for the "time" vector field, meaning a vector field that acts tangent to the copy of $\mathbb{R}$ that runs as a parameter for the family of Riemannian metrics according to the smooth assignment: $t \to g(t)$.

So the Ricci flow is an evolution equation that describes the relationship between the metric tensor and the Ricci curvature of a manifold. A natural question is to ask what does the change in the metric tensor means. To understand these changes we need some context for the topological discussion as well as for the geometric one: We are particularly interested in the evolution of Ricci flow on a 4-Manifold $\overline{M}$ that can be viewed as: $M \times I$ where M is a Riemannian 3-manifold and $I \subset \mathbb{R}$ is an open interval of $\mathbb{R}$ or the whole line (where the "time" parameter runs along) and therefore in each time t we can view the 4-Manifold $\overline{M}$ as a pair $(M, g(t))$ where M is a 3-manifold embedded in $\overline{M}$ and $g(t)$ is a metric tensor assigned in each time t. Hamilton proved uniqueness and short-time existence for compact manifolds in [**?**] using a technique called the inverse function theorem of Nash-Moser ([30]) but shortly after, De-Turck managed to skip this technical de-tour and built a more natural proof of the existence and parabolic nature using a technique called "De-Turck trick" which we will describe briefly. For now we present some important derivative estimates:

Proposition 3.1.1. *The time derivative of the inverse metric tensor is given by:*

$$\frac{\partial}{\partial t}g^{ij} = -g^{ik}g^{jl}\frac{\partial}{\partial t}g_{kl}$$

Proof. If we contract the metric tensor we arrive to the expression: $g_{al}g^{lb} = \delta_a^b$ and taking derivative we obtain:

$$\left(\frac{\partial}{\partial t}g_{al}\right)g^{lb} + g_{al}\left(\frac{\partial}{\partial t}g^{lb}\right) = 0.$$

Taking product with g^{ca} we arrive at:

$$g^{ca}g^{lb}\left(\frac{\partial}{\partial t}g_{al}\right) + \delta_c^l\left(\frac{\partial}{\partial t}g^{lb}\right) = 0,$$

from which:

$$g^{ca}g^{lb}\left(\frac{\partial}{\partial t}g^{cb}\right) = -\frac{\partial}{\partial t}g^{cb}.$$

□

It is useful sometimes to work in a coordinate system where our calculations can be highly simplified and inspired by that, we define:

Definition 3.1.1. A geodesic normal coordinate system (commonly known as normal coordinate system) is a coordinate system in which all the connection symbols vanish, meaning: $\Gamma_{ij}^k = 0$ at a fixed point p in a manifold M. This is equivalent to: $\frac{\partial}{\partial x^k}g_{ij} = 0$.

We already defined Christoffel symbols in page 11, let us recall the expression for them in local coordinates as:

$$\Gamma_{ij}^k = dx^k(\nabla_{\partial_i}\partial_j),$$

and we also have a detailed local description for them in terms of the metric tensor:

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}).$$

And from this last local description it should be clear that the the Symbols (and in general, every expression written in terms of the metric tensor) is not evaluated in t.

Proposition 3.1.2. *The time derivative of the Christoffel symbols in a normal coordinate system is given by:*

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = \frac{1}{2}g^{kl}\left(\left(\nabla_{\partial i}\frac{\partial}{\partial t}g\right)_{jl} + \left(\nabla_{\partial j}\frac{\partial}{\partial t}g\right)_{il} - \left(\nabla_{\partial l}\frac{\partial}{\partial t}g\right)_{ij}\right)$$

Proof. Recalling the expression for the Christoffel symbols in terms of derivatives of the metric tensor, taking it's time derivative, fixing p a point in M and working in normal coordinates we arrive at the following expression:

$$\frac{\partial}{\partial t}\Gamma_{ij}^k\big|_p = \frac{1}{2}\left(\frac{\partial}{\partial t}g^{kl}\right)(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})\big|_p + \frac{1}{2}g^{kl}(\partial_i\partial_t g_{jl} + \partial_j\partial_t g_{il} - \partial_l\partial_t g_{ij})\big|_p$$

Where the first term is zero because of the normal coordinate system condition. □

Proposition 3.1.3. *The time derivative of the Riemann curvature tensor in a normal coordinate system is:*

$$\frac{\partial}{\partial t}\mathrm{R}^l_{ijk} = \frac{1}{2}g^{ls}\left(\left(\nabla^2_{ij}\frac{\partial}{\partial t}g\right)_{ks} + \left(\nabla^2_{ik}\frac{\partial}{\partial t}g\right)_{js} - \left(\nabla^2_{is}\frac{\partial}{\partial t}g\right)_{jk} - \left(\nabla^2_{ji}\frac{\partial}{\partial t}g\right)_{ks} - \left(\nabla^2_{jk}\frac{\partial}{\partial t}g\right)_{is} + \left(\nabla^2_{js}\frac{\partial}{\partial t}g)_{ik}\right)\right.$$

Proof. Recall the expression for R^l_{ijk}:

$$\mathrm{R}^l_{ijk} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^s_{jk}\Gamma^l_{is} - \Gamma^s_{ik}\Gamma^l_{js},$$

taking the time derivative:

$$\frac{\partial}{\partial t}\mathrm{R}^l_{ijk} = \partial_i\left(\frac{\partial}{\partial t}\Gamma^l_{jk}\right) - \partial_j\left(\frac{\partial}{\partial t}\Gamma^l_{ik}\right) + \left(\frac{\partial}{\partial t}\Gamma^s_{jk}\right)\Gamma^l_{is} + \left(\frac{\partial}{\partial t}\Gamma^l_{is}\right)\Gamma^s_{jk} - \left(\frac{\partial}{\partial t}\Gamma^s_{ik}\right)\Gamma^l_{js} - \left(\frac{\partial}{\partial t}\Gamma^l_{js}\right)\Gamma^s_{ik}$$

Once again for a fixed $p \in M$, working in normal coordinates and using the previous proposition, the result follows. □

Proposition 3.1.4. *The time derivative of the Ricci curvature in a normal coordinate system is given by:*

$$\frac{\partial}{\partial t}\mathrm{Ric}_{ij} = \frac{1}{2}g^{ks}\left(\left(\nabla^2_{ki}\frac{\partial}{\partial t}g\right)_{js} + \left(\nabla^2_{kj}\frac{\partial}{\partial t}g\right)_{is} - \left(\nabla^2_{ks}\frac{\partial}{\partial t}g\right)_{ij} - \left(\nabla^2_{ij}\frac{\partial}{\partial t}g\right)_{ks}\right).$$

Proof. Contracting the pevious result with $i = l$ We find:

$$\frac{\partial}{\partial t}\mathrm{Ric}_{jk} = \frac{\partial}{\partial t}\mathrm{R}^i_{ijk}$$

$$\tfrac{\partial}{\partial t}\mathrm{R}^i_{ijk} = \tfrac{1}{2}g^{ls}\left(\left(\nabla^2_{ij}\tfrac{\partial}{\partial t}g\right)_{ks} + \left(\nabla^2_{ik}\tfrac{\partial}{\partial t}g\right)_{js} - \left(\nabla^2_{is}\tfrac{\partial}{\partial t}g\right)_{jk} - \left(\nabla^2_{jk}\tfrac{\partial}{\partial t}g\right)_{is}\right) + \tfrac{1}{2}g^{is}\left(\left(\nabla^2_{js}\tfrac{\partial}{\partial t}g\right)_{ik} - \left(\nabla^2_{ji}\tfrac{\partial}{\partial t}g\right)_{ks}\right)$$

Notice that:

$$\begin{aligned}
&= g^{is}\left(\nabla^2_{js}\frac{\partial}{\partial t}g\right)_{ik} - g^{is}\left(\nabla^2_{ji}\frac{\partial}{\partial t}g\right)_{ks} \\
&= g^{is}\left(\nabla^2_{js}\frac{\partial}{\partial t}g\right)_{ik} - g^{si}\left(\nabla^2_{js}\frac{\partial}{\partial t}g\right)_{ik} \\
&= 0
\end{aligned}$$

As g is symmetric and the result follows. □

Proposition 3.1.5. *The time derivative of the scalar curvature in a normal coordinate system is given by:*

$$\frac{\partial}{\partial t}\mathcal{R} = -\Delta(\mathrm{tr}_g\frac{\partial}{\partial t}g) + g^{ij}g^{pq}\nabla^2_{qj}\frac{\partial}{\partial t}g_{pi} - \langle\frac{\partial}{\partial t}g, \mathrm{Ric}_{ij}\rangle$$

Proof. Using proposition (3.1.1) and the previous proposition we find that:

$$\frac{\partial}{\partial t}\mathcal{R} = \mathrm{Ric}_{ik}\frac{\partial}{\partial t}g^{ik} + \frac{\partial}{\partial t}\mathrm{Ric}_{ik}\,g^{ik}.$$

Changing the index names and substituting, we arrive at:

$$\frac{\partial}{\partial t}\mathcal{R} = -g^{ij}g^{pq}\frac{\partial}{\partial t}g_{jq}\mathrm{Ric}_{ip} + g^{ij}g^{pq}\nabla^2_{qj}\frac{\partial}{\partial t}g_{ip} - g^{ij}g^{pq}\nabla^2_{ij}\frac{\partial}{\partial t}g_{pq},$$

and finally we have:

$$\frac{\partial}{\partial t}\mathcal{R} = -\frac{\partial}{\partial t}g_{jq}\mathrm{Ric}^{jq} + g^{ij}g^{pq}\nabla^2_{qj}\frac{\partial}{\partial t}g_{pi} - \Delta\,\mathrm{tr}_g\frac{\partial}{\partial t}g.$$

□

Proposition 3.1.6. *The time derivative of the volume form in a normal coordinate system is:*

$$\frac{\partial}{\partial t}\,\mathrm{Vol}_n = \frac{1}{2}\left(\mathrm{tr}_g\,\frac{\partial g}{\partial t}\right)\mathrm{Vol}_n$$

Proof.

$$\mathrm{Vol}_n = \sqrt{\det(g_{ij})}\;dx^1 \wedge ... \wedge dx^n$$

Now, using Jacobi's formula for the determinant, which states that given a family of invertible matrices $A(t)$ then:

$$\frac{d}{dt}\det A(t) = \det A(t) \quad \mathrm{tr}\left(A^{-1}\frac{dA}{dt}\right).$$

Using this:

$$\frac{\partial}{\partial t}\sqrt{\det g_{ij}} = \frac{1}{2\sqrt{\det g_{ij}}}\partial_t \det g_{ij} = \frac{1}{2}\sqrt{\det g_{ij}} \quad (\mathrm{tr}_g\,\partial_t g).$$

This means:

$$\partial_t\,\mathrm{Vol} = \frac{1}{2}\,\mathrm{tr}_g\,\partial_t g_{ij}\,\mathrm{Vol}_n\,.$$

□

Now, we provide here some important examples:

3.1.1 Einstein metrics

A Riemannian metric is called Einstein if: $\mathrm{Ric}(g_0) = \lambda g_0$ for $\lambda \in \mathbb{R}$, and in this case the family $g(t) = (1-2\lambda t)g_0$ is a family of solutions for the Ricci flow (Take the ∂_t derivative). $\lambda \geq 0$, $\lambda = 0$, $\lambda \leq 0$ describe shrinking, steady and expanding solutions to the Ricci flow respectively. The simplest shrinking solution is (S^n, g_0) and here the Ricci curvature takes the form: $\mathrm{Ric}_{g_0} = (n-1)g_o$ therefore $g(t) = (1-2(n-1))g_0$ is a solution for the Ricci flow defined up to a maximum time $T = \frac{1}{2(n-1)}$ where S^n shrinks to a point.

On the other hand if g_0 is hyperbolic, then: $Ric_{g_0} = -(n-1)g_0$ and we have: $g(t) = (1+2(n-1)t)g_0$ and this solution will expand for every positive time and we can define an initial time $T = -\frac{1}{2(n-1)}$ where the manifold explodes from a point and starts expanding.

3.1.2 Ricci solitons

Let's go into further detail with the Ricci flow. One is mainly interested in solutions of the Ricci flow as families of metrics; and a special type of solutions for the Ricci flow are called Ricci solitons, which satisfy a generalized condition of the Einstein metrics: Remembering that the Ricci flow equation is:

$$\frac{\partial g(t)}{\partial t} = -2Ric_{g(t)}$$

Then a Ricci soliton is a one-parameter family of Riemannian metrics which can be written as:

$$g(t) = c(t)\varphi_t^*(g(0)),$$

where, $c(t)$ is a constant for every t, φ_t a diffeomorphism and satisfies the initial condition: $c(0) = 1$, $g(0) = g_0$ and $\varphi_0 = Id$.

φ_t is a (possible time dependent) flow associated to a one-parameter of vector fields $X(t)$. If $X = \nabla f$ then we say is a gradient Ricci soliton.

Deriving the condition:

$$g(t) = c(t)\varphi_t^* g(0),$$

evaluating at zero and assuming the Ricci flow equation is satisfied, then:

$$-2\operatorname{Ric}_{g(0)} = \partial_t\big|_{t=0} g(t) = \dot{c}(0)\varphi_0^* g(0) + c(0)\mathcal{L}_{\nabla f}\varphi_0^* g(0)$$

and from this we obtain:

$$2\operatorname{Ric}_{g(t)} + \dot{c}(t)g + \mathcal{L}_{\nabla f}\varphi_t^* g(t) = 0$$

which is the Ricci soliton condition. Evenmore, if we are dealing with a gradient Ricci soliton, then the condition $\mathcal{L}_{\nabla f}\varphi_t^*(g(t))$ can be rewritten as:

$$\operatorname{Ric}_{g(t)} + \operatorname{Hess}_{g(t)} f + \frac{1}{2}\dot{c}(t)g(t) = 0$$

This is because we have φ_t the flow associated to a vector field $X(t)$ which is the gradient of a certain function f and then noticing that: $\mathcal{L}_{\nabla f}\varphi_t^* g(t) = 2\operatorname{Hess}_{g(t)} f$ and we call it a shrinking Ricci soliton if $\dot{c}(t) \geq 0$, a steady soliton if $\dot{c}(t)$ is zero, and an expanding soliton if $\dot{c}(t) \leq 0$.

To address the question on how can we construct the 1-parameter family of Riemannian metrics we consider the diffeomorphism φ such that in every point $x \in M$ we have $\varphi_t(p)$ is a solution of the ODE:

$$\frac{\partial}{\partial t}\varphi_t(p) = \frac{1}{1-2\rho t}X_{\varphi_t(p)}.$$

With the initial value $\varphi_0(p) = p$ (notice that in our previous equations the value of ρ is actually $\dot{c}(t)$ which is in fact a constant for every value of t). Then we define φ_t as the 1-parameter family of diffeomorphisms.

3.2 Ricci flow on surfaces

In the simplest setting, ricci flow can be regarded as a flow for metrics on surfaces.This method was used to prove a different version of the uniformization theorem [28] and Hamilton managed to show the behaviour of basic properties in the compact case [27]. Let us take a quick look at how does curvature evolve:

In 2 dimensions there is only one sectional curvature, namely, the Gaussian curvature K. Now:

$$R_{ijkl} = -K[g_{ik}g_{jl} - g_{il}g_{jk}],$$

This from the definition of sectional curvature and the fact that we are in dimension 2. Contracting with the metric tensor, we thus obtain:

$$\begin{aligned}
\operatorname{Ric}_{ij} &= g^{kl}R_{kijl} \\
&= -K[g^{kl}g_{kj}g_{il} - g^{kl}g_{kl}g_{ij}] \\
&= -K[\delta^i_j g_{il} - (tr_g g)g_{ij}] \\
&= -K(1-2)g_{ij} \\
&= Kg.
\end{aligned}$$

Therefore: $\mathrm{Ric} = Kg$ and contracting again we obtain:

$$S = \mathrm{tr}_g \mathrm{Ric} = K(\mathrm{tr}_g g) = 2K.$$

These expressions will be useful for us when dealing with the Ricci flow analysis in surfaces. On the other hand we can state a really important theorem:

Theorem 3.1. *(Gauss-Bonnet theorem) Let M be a compact and orientable 2-dimensional Riemannian manifold, then:*

$$\int_M d\,\mathrm{A} = 2\pi\chi(M)$$

Where $\chi(M)$ is the Euler characteristic of M and $d\,\mathrm{A}$ isthe Area differential.

Using this, if the Ricci flow equation is satisfied, then:

$$\begin{aligned}
\partial_t \mathrm{A}(M) &= \int_M \frac{\partial}{\partial t} d\,\mathrm{A} \\
&= -\int_M (\mathrm{tr}_g \mathrm{Ric}) d\,\mathrm{A} \\
&= -\int_M S d\,\mathrm{A} \\
&= -\int_M 2K d\,\mathrm{A} \\
&= -4\pi\chi(M)
\end{aligned}$$

The last equality by Gauss-Bonnet theorem.

3.2.1 Hamilton's cigar soliton

We now review with more detail one of the most important solutions for Ricci flow on surfaces, namely: Hamilton's cigar soliton ([27]). This is

$$(\mathbb{R}^2, g_t = \frac{1}{e^{4t} + x^2 + y^2} dx^2 + dy^2)$$

Let's use the Christoffel symbols to compute the information we need:

- $\Gamma^x_{xx} = \Gamma^y_{xy} = \Gamma^y_{yx} = \frac{-x}{e^{4t}+x^2+y^2}$,
- $\Gamma^y_{yy} = \Gamma^x_{xy} = \Gamma^x_{yx} = \frac{-y}{e^{4t}+x^2+y^2}$,
- $\Gamma^x_{yy} = \frac{x}{e^{4t}+x^2+y^2}$,
- $\Gamma^y_{xx} = \frac{y}{e^{4t}+x^2+y^2}$.

Using this we can compute the components of the Ricci tensor for the cigar soliton:

$$\mathrm{Ric}_{ij} = \partial_k \Gamma^k_{ji} - \partial_j \Gamma^k_{ki} + \Gamma^k_{kl}\Gamma^l_{ji} - \Gamma^k_{jl}\Gamma^l_{ki}.$$

And using this, we compute:

$$\mathrm{Ric}_{ij}(g(t)) = \frac{2e^{4t}}{(e^{4t} + x^2 + y^2)^2} dx^2 + dy^2.$$

Finally, it is of interest to find the φ_t that allow us to write:

$$g(t) = \varphi_t^* g(0).$$

For:

$$g(0) = g_0 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

Now, consider $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$, which acts as follows:

$$\varphi_t(X, Y) = (e^{-2t} X, e^{-2t} Y).$$

Computing:

$$\begin{aligned}
\varphi_t^*(g(0)) &= \frac{e^{-4t} dx^2 e^{-4t} dy^2}{1 + e^{-4t} x^2 + e^{-4t} y^2} \\
&= \frac{e^{-4t}(dx^2 + dy^2)}{1 + e^{-4t}(x^2 + y^2)} \\
&= \frac{dx^2 + dy^2}{e^{4t} + x^2 + y^2} \\
&= g(t)
\end{aligned}$$

Therefore, we have proved that $g(t) = \varphi_t^* g(0)$.

By a change of coordinates we can study Hamilton's cigar soliton in $\mathbb{C}$ in terms of complex coordinates $z = x + iy$ as:

$$g_C(z) = \frac{\|dz\|^2}{1 + \|z\|^2}$$

This corresponding to the previous metric we defined as g_0 and with its Gaussian curvature K:

$$K = \frac{2}{1 + \|z\|^2}.$$

Associating to this metric the cigar solution in complex coordinates $(g_{\mathbb{C}}(t))_{t \in \mathbb{R}}$:

$$g_C(t; z) = e^{2u(t,z)} \|dz\|^2 = \frac{\|dz\|^2}{e^{4t} + \|z\|^2}.$$

From this we obtain: $u(t, z) = -\frac{1}{2} \operatorname{Log}(e^{4t} + \|z\|^2)$ where we are running through the principal branch of the Log function, $\operatorname{Log} : \mathbb{C} \to \mathbb{C}$.
There is yet another way to analyze the cigar soliton, namely, in polar coordinates where we can re write g_0 as:

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2} = \frac{dr^2 + r^2 d\theta^2}{1 + r^2}.$$

Defining a new distance function in terms of the radius: $s = sinh^{-1}(r)$ which is equivalent to: $r = sinh(s)$ then the metric has the following expression:

$$g = ds^2 + \tanh{(s)}^2 d\theta^2$$

Then if we define the potential function as:

$$f = -2 \ln{(\cos s)})$$

we can explicitly compute the soliton equation:

$$\mathrm{Ric} + \mathrm{Hess}(f) = 0$$

Notice that the expression is a steady soliton. Also, the use of ln is reserved for the real-valued function and Log for the complex-multi-valued function.

Before we state a really important theorem for the Rici flow in surfaces, we will define some important concepts:

Definition 3.2.1. Given a linear connection ∇ on a Riemannian manifold M and $\phi^1, ..., \phi^n$ a local trame for T^*M we can define the connection 1-forms ω_i^j. We define the curvature 2-forms by:

$$\Omega_i^j = \frac{1}{2} R_{klij} \phi^k \wedge \phi^l$$

We have the following identities also known as Cartan's first and second structure equations:

1. $d\phi^j = \phi^i \wedge \omega_i^j + \tau^j$ where τ^j are the torsion 2-forms.
2. $\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j$.

The 1-forms ω_i^j define an affine connection on the tangent bundle and the first structure equation is an expression for the torsion tensor. The second structure equation is equivalent to the expression used to define the curvature tensor.
The main idea behind the proof of these equations is recursiveness of some sort. One starts with a co-frame and applies d, then express the result in terms of something known and something new but with values in the Lie algebra. Now we can apply d to both terms and express them in known "older" terms.

Let us mention an important result to prove uniqueness of the cigar soliton in surfaces:

Theorem 3.2. *(The cylinder to sphere rule) Let $0 \leq w \leq \infty$ and let g be a metric on the topological cylinder $((-w, w) \times S^n)$ of the form:*

$$g = \varphi^2(z) dz^2 + \psi^2(z) g_{can}$$

where φ and $\psi : (-w, w) \to \mathbb{R}_+$ and g_{can} is the canonical round metric of radius 1 on S^n. Then g extends to a smooth metric on S^{n+1} if and only if:

1. $\int_{-w}^{w} \varphi(s) ds \lneq \infty$.
2. $\lim_{z \to \pm w} \psi(z) = 0$.
3. $\lim_{z \to \pm w} \frac{\psi\prime(z)}{\varphi(z)} = 1$.
4. $\lim_{z \to \pm w} \frac{d^{2k}\psi}{ds^{2k}}(z) = 0$.

for every natural k, where ds is the element of arc length induced by φ.

Proof. Let's give a sketch of the proof. For a detailed proof, see ([?], pg 29.).

Let's restrict to the 2-dimensional situation where g takes the form:

$$g = \varphi^2(z)dz^2 + \psi^2(z)d\theta^2.$$

By an appropiate change of coordinates (cartesian coordinates), assuming boundedness of $\int_0^z \varphi(t)dt$ as z approaches the north pole, and defining the following function: $\rho(r) = \frac{\psi(z)}{r}$ we arrive at the following expression:

$$g = (1 + \frac{\rho^2 - 1}{r^2}y^2)dx^2 - (2xy\frac{\rho^2 - 1}{r^2})dxdy + (1 + \frac{\rho^2 - 1}{r^2}x^2)dy^2. \tag{3.2.1}$$

Then the metric extends to a smooth metric if $r = 0$ at $\frac{\rho^2 - 1}{r^2}$ extends to an even function (for this, we study it by means of the Taylor expansion) and this happens if and only if conditions 2 and 3 are satisfied.

□

Finally we can state an important theorem for surfaces:

Theorem 3.3. *Every rotationally symmetric metric in* $\mathbb{R}^2$ *is of the form:*

$$g = ds^2 + \varphi(s)d\theta^2.$$

For $\varphi(s) \gneq 0$. *In particular, up to homothety, the cigar is the unique rotationally symmetric gradient Ricci soliton of positive curvature on* $\mathbb{R}^2$.

Proof. We can write g in terms of local frames as $g = \delta_{ij}\omega^i \otimes \omega^j$ for a local frame $\{\omega^1, \omega^2\}$ for $\omega^1 = ds$; $\omega^2 = \varphi(s)d\theta$ and their duals: $\{e_1, e_2\}$ where: $e_1 = \partial_s$ and $e_2 = \frac{1}{\varphi(s)d\theta}$.

Now, for every vector field: $X = X^1\partial_s + X^2\partial_\theta$ we can compute the following:

$$\nabla_X e_1 = \frac{\varphi^{'}(s)}{\varphi(s)}X^2\partial_\theta = \varphi^{'}(s)\langle d\theta X, e_2\rangle$$

$$\nabla_X e_2 = -\varphi^{'}(s)X^2\partial_s = -\varphi^{'}(s)\langle d\theta X, e_1\rangle.$$

Now, the connection 1-forms associated to e_1, e_2 are:

$$\nabla_X e_i = \omega_i^j(X)e_j,$$

then we have:

$$\omega_1^2 = -\omega_2^1 = \varphi^{'}(s)ds.$$

Now: Remembering the expression for Cartan's structure equations: $d\omega^i = \omega^j \wedge \omega_j^i$ and $\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j$. In this particular scenario we have:

$$d\omega^1 = 0,$$

$$d\omega^2 = \frac{\varphi^{'}(s)}{\varphi(s)}\omega^1 \wedge \omega^2,$$

$$\Omega_1^2 = d\omega_1^2 = \varphi^{''}(s)ds \wedge d\theta = \frac{\varphi^{''}(s)}{\varphi(s)}\omega^1 \wedge \varphi^2.$$

Using this, we find the Gauss curvature is given by:

$$K = \Omega_1^2(e_2, e_1) = -\frac{\varphi''(s)}{\varphi(s)}.$$

Recall that g is a steady gradient soliton if there is a function f such that: $Kg = \nabla\nabla f$. Our assumptions are that the curvature K is positive and f is a radial function $f(s)$, then f is convex. (This is due a traditional theorem in convex theory which says that f such that $f \in C^2$ is convex if and only if $\nabla^2 f(x)$ is positive semi definite.)
Recalling the expression for the connection 1-forms, notice that:

$$\nabla_{e_1} e_1 = \omega_1^2(e_1)\dot{e}_2 = 0,$$

$$\nabla_{e_2} e_2 = \omega_2^1(e_2)\dot{e}_1 = \frac{\varphi'(s)}{\varphi(s)} e_1.$$

Using this and the fact that $g = \delta$ in this frame $\{e_1, e_2\}$.
Then we have that:

$$Kg = \nabla\nabla f,$$

is satisfied if we can express the Gaussian curvature as:

$$K = e_1(e_1 f) - (\nabla_{e_1} e_1) f = f''(s),$$

$$K = e_2(e_2 f) - (\nabla_{e_2} e_2) f = \frac{\varphi'(s) f'(s)}{\varphi(s)}.$$

And both expressions are satisfied simultaneously. Using, additionally that: $K = \frac{\varphi''(s)}{\varphi(s)}$ then we arrive at the expression:

$$-\frac{\varphi''(s)}{\varphi(s)} = f''(s) = \frac{\varphi'(s) f'(s)}{\varphi(s)}.$$

Now, solving the separable ODE: $\frac{f''}{f'} = \frac{\varphi'}{\varphi}$ we obtain:

$$(\log(f'))' = (\log(\varphi))',$$

and integrating we obtain:

$$f'(s) = C\varphi(s),$$

for $C \gneq 0$. This condition is because if $c = 0$ then g would be flat. Now, given that $\varphi \gneq 0$ then f' is positive (as f is convex).
Setting $C = 2a^2$, for a not zero, we have $f' = 2a^2\varphi$ and substituting this we arrive at:

$$-\frac{\varphi''(s)}{\varphi(s)} = f''(s).$$

We must then solve the following ODE: $0 = \varphi''(s) + 2a^2\varphi'(s)\varphi s$

In order to extend to a smooth metric at $s = 0$, $\varphi(0) = 0$ and $\varphi'(0) = 1$ by the cylinder to sphere rule this happens if and only if $b = 1$. Then solving $\varphi'(s) + a^2\varphi^2(s) = 1$ we find: $\varphi(s) = \frac{1}{a}\tanh(as)$ and so:

$$g = ds^2 + \frac{1}{a^2}\tanh^2(as)d\theta^2; \quad \text{with } a \gneq 0$$

We claim that this is a constant multiple of the cigar soliton metric obtained by substituting $\sigma = as$ and arriving:

$$g = \frac{1}{a^2}(d\sigma^2 + \tanh^2(\sigma)ds^2) = \frac{1}{a^2}g_{cigar}$$

Even more: If we have $f\prime(s) = C\varphi(s)$ then $f(s) = \frac{1}{a}\log(\cosh(as))$ then $X = -\operatorname{grad}(f) = -C\varphi(s)e_1 = -\tanh(as)\partial_s$.

□

3.2.2 Rosenau's solution

Let us begin with a connection between comformally related metrics and their curvatures:

Theorem 3.4. *If g and h are metrics on M^2 comformally related, meaning there is a positive real function u such that: $g = e^{2u}h$ then their curvatures are related by:*

$$R_g = e^{-2u}(-2\Delta_h u + R_h)$$

Where R_g and R_h denote the scalar curvatures associated to g and h respectively.

Proof. Let $\{ f_1, f_2 \}$ be an orthonormal frame for h and $\{ \xi^1, \xi^2\}$ it's dual co-frame and define the connection 1-forms as ξ^1_2, then we can write an orthonormal frame for g as:
$\{e_1, e_2\}$ where: $e_1 = e^{-u}f_1$ and $e_2 = e^{-u}f_2$ and equivalently the dual co-frame for g can be expressed as:
$\{ \omega^1, \omega^2 \}$ where: $\omega^i = e^u\xi^i$. Now, using Cartan's expression we obtain:

- $d\omega^1 = e^u(d\xi^1 + du \wedge \xi^1) = e^u(\xi^2 \wedge \xi^1_2 + f_2(u)\xi^1 \wedge \xi^2)$,
- $d\omega^2 = e^u(d\xi^2 + du \wedge \xi^2) = e^u(\xi 1 \wedge \xi^2_1 + f_1(u)\xi^1 \wedge \xi 2)$.

Using this we define the connection 1-forms associated to g as:

$$\omega^1_2 = d\omega^1(e_2, e_1)\omega^1 + d\omega^2(e_2, e_1)\omega^2.$$

And so:

$$\omega^1_2 = \xi^1_2 + f_2(u)\xi^1 - f_1(u)\xi^2.$$

Now, with the previous work and the second Cartan's structure equation, we arrive at the following expression for the curvature:

$$Rm[g]^1_2 = d\omega^1_2 = d\xi^1_2 + d[f_2(u)\xi^1 + f_1(u)\xi^2].$$

now, writing this in terms of h, we have:

$$Rm[g]^1_2 = Rm[h]^1_2 + f_2f_2(u)\xi^2 \wedge \xi^1 - f_1f_1(u)\xi^2 \wedge \xi^1 + f_2(u)\xi^1_2(f_1)\xi^2 \wedge \xi^1 - f_1(u)\xi^2_1(f_2)\xi^2 \wedge \xi^1.$$

This, using the expression for ω^1_2 and $d\xi^k = \xi^2_1(f_k)\xi 1 \wedge \xi^2$.

Finally, using that:

$$\Delta_h(u) = \nabla^2_{f_1,f_1}(u) + \nabla^2_{f_2,f_2}(u),$$

we have:

$$Rm[g]^1_2 = Rm[h]^1_2 - (\Delta_h(u))\xi^1 \wedge \xi^2.$$

From which we conclude:

$$R_g = 2kg = 2Rm[g]^1_2(e_1, e_2) = 2e^{-2u}(K_h - \Delta_h u) = e^{-2u}(R_h - 2\Delta_h(u)).$$

□

Now, if $g = uh$ for $u : M^2 \to \mathbb{R}_+$ then the curvature becomes:

$$R_g = \frac{-\Delta_h(\log(u)) + R_h}{u}.$$

Now if we fix h, then g satisfies the Ricci flow equation for surfaces if and only if

$$\frac{\partial u}{\partial t}h = (\frac{\Delta_h \log(u) - R_h}{u})g,$$

and using that $g = uh$ then the last equation is satisfied if and only if:

$$\frac{\partial u}{\partial t} = \Delta_h \log{(u)} - R_h.$$

And if (M^2, h) is flat then the last equation becomes

$$\frac{\partial u}{\partial t} = \Delta_h(u).$$

Let us talk a little bit about the motivation of this. Consider the following equation $\frac{\partial}{\partial \tau}u = \Delta u^m$ and notice that if $m = 1$ this is the heat equation. Now, if $0 \leq m \leq 1$ then there is a connection of this equation with plasma physics and the theory of difussion processes. Let us connect these two equations: By a simple variable change we have that if $t = m\tau$ then $\frac{\partial}{\partial \tau}u$ becomes: $\frac{\partial}{\partial t}u = \frac{1}{m}\Delta u^m$ and what is the use of this change? Given that there is this well known limit: $\lim_{m\to 0}[\frac{u^m - 1}{m}] = log(u)$, then we have:

$$\begin{aligned}
\lim_{m\to 0} \Delta \left(\frac{u^m - 1}{m}\right) &= \lim_{m\to 0}(u^{m-1}\Delta u + (m-1)u^{m-2}|\nabla u|^2) \\
&= \frac{\Delta u}{u} - \frac{|u|^2}{u^2} \\
&= \Delta log(u)
\end{aligned}$$

This connection between the porous media flow and the Ricci flow on surfaces was studied by Sigurd Angement. This lets us use Rosenau's result from diffusion processes: Where one studies a flow function on a porous media which prevents accumulation of information along time via fast diffusion processes (in the context of Ricci flow, the accumulation of information is the tendency of a region to develop high curvature along the flow and the diffusion process is to avoid high scalar curvature in the region) [18].
We now study the idea behind Rosenau's solution:

Let h be a flat metric on $M^2 = \mathbb{R} \times S^1$, $x \in \mathbb{R}$, $\theta \in S^1$. Rosenau's solution is defined as a ancient solution to Ricci flow, meaning it exists up to a time T. If $g = uh$ defined for $t \leq 0$:

$$u(x,t) = \frac{2\beta \sinh{(-\alpha\lambda t)}}{\cosh{(\alpha x)} + \cosh{(\alpha\lambda t)}},$$

for $\alpha, \beta, \lambda \geq 0$ to be determined. Given that u does not depend on the variable θ then:

$$\Delta_h \log{(u)} = \frac{\partial^2}{\partial x^2}\log{(u)}.$$

And so, computing we have:

- $\partial_t u = -2\alpha\beta\lambda \dfrac{\cosh{(\alpha\lambda t)}\cosh{(\alpha x)} + 1}{(\cosh{(\alpha x)} + \cosh{(\alpha\lambda t)})^2},$
- $\partial_x^2 \log{(u)} = -\alpha^2 \dfrac{\cosh{(\alpha\lambda t)}\cosh{(\alpha x)} + 1}{(\cosh{(\alpha x)} + \cosh{(\alpha\lambda t)})^2}.$

This shows u satisfies $\partial_t u = \Delta_h \log{(u)}$ if and only if $2\beta\lambda = \alpha$ and then, by the previous theorem relating the curvatures we have that g defined as: $g = uh$ satisfies the Ricci flow.

3.3 Ricci flow in 3 dimensions

Now we direct our attention to the 3-dimensional case where the conformal structure is not preserved in general and, unlike the 2-dimensional case, singularities and oddities happen with much ease. This difference between the surface situation and 3-dimensional one was not well understood before the work of G. Perelman, whose main achievement in this direction was to study the local behaviour of singularities.

Definition 3.3.1. A geometric structure on a topological manifold M^n is a complete locally homogeneous Riemannian metric g. More precisely: For every pair of points $x, y \in M^n$ there are open subsets $x \in U_x \subset M^n$ and $y \in U_y \subset M^n$ and a g-isometry $\varphi_{xy} : U_x \to U_y$ such that $\varphi_{xy}(x) = y$. If we can extend φ_{xy} to all of M^n then $(M^n,\, g)$ is called homogeneous

If $(M^n,\, g)$ is locally homogeneous, we say its geometry is given by the homogeneous model $(\widehat{M^n},\, \widehat{g})$ where $\widehat{M^n}$ is the universal cover of M^n and $\widehat{g}$ is the lifting of g. Therefore, it suffices to study homogeneous model. Now, following Klein's philosophy, a model geometry is a triple $(M^n,\, G,\, G_*)$ where M^n is simply-connected, G is a subgroup of the diffeomorphism group acting transitively and smoothly on M^n such that for each $x \in M^n$:

$$G_x = \{\varphi \in G : \varphi(x) = x\} \simeq G_*.$$

Where G_x is the isotropy subgroup (point stabilizer) of the action of G on M^n. We call a model geometry maximal if G is maximal among all subgroups of the diffeomorphism group of the base manifold that have compact isotropy groups. So, if we have $(M^n,\, g)$ a complete homogeneous space, we regard $(M^n,\, \text{Isom }(M^n,\, g),\, I_x(M^n, g))$ as a model geometry. In dimension 3, if we set $M^3 = G$ a Lie group acting transitively on itself by multiplication then there is a rich collection of Model geometries.

Definition 3.3.2. G is unimodular if the 1-parameter family of diffeomorphisms generated by any left-invariant vector field preserves its volume.

As we are interested only in closed manifolds on G and only unimodular groups admit compact quotients because only those groups admit quotients of finite volume. Milnor classified all 3-dimensional unimodular Lie groups and found hat the key to their classification is to take a closer look at the isotropy subgroup.
On the other hand, let G be a Lie group and g it is Lie algebra of left-invariant vector fields on G. Since any left-invariant metric on G is equivalent to a scalar product on $\mathfrak{g}$, the set of such metrics can be identified with S^3_+ which is the space of symmetric positive-definite 3×3 matrices. Noticing that topologically S^3_+ is an open and convex subset of $\mathbb{R}^6$ then for each g acting on G the Ricci flow can be thought as a path $t(t) \in S^3_+$ and so the Ricci flow can be reduced to a system of six ordinary differential equations. This shows, in particular, the interest of studying the Ricci flow on homogeneous geometries.

Now if we equip G with a moving frame F_i then the structure constants c^k_{ij} for the lie algebra $\mathfrak{g}$ are defined as:

$$[F_i, F_j] = c^k_{ij} F_k$$

and the adjoint representation of $\mathfrak{g}$ is:

$$ad : \mathfrak{g} \to gl(\mathfrak{g}) \simeq gl(n, \mathbb{R}),$$

which in components takes the form of:

$$(adV)W = [V, W] = V^i W^j c^k_{ij} F_k.$$

Additionally if g is a left-invariant metric on G, then $\mathrm{ad}^* : \mathfrak{g} \to Gl(\mathfrak{g})$ the adjoint with respect to g of the map ad is given by:

$$\langle (adX)^* Y, Z\rangle \;=\; \langle Y, (adX)Z\rangle \;=\; \langle Y, [X, Z]\rangle$$

In the particular case where the dimension of the Lie group is 3, then we may define a vector space isomorphism between $\mathfrak{g}$ and $\bigwedge^2 \mathfrak{g}$ by:

$$F_i \to F_{(i+1)\ mod\ 3} \wedge F_{(i+2)\ mod\ 3}.$$

Composing this with the commutator which sends $\bigwedge^2 \mathfrak{g} \to \mathfrak{g}$ via:

$$VW \to [V, W].$$

Yelds $\Gamma : \mathfrak{g} \to \mathfrak{g}$ whose matrix representation with respect to the ordered basis $\beta = \{F_1, F_2, F_3\}$ is:

$$\Gamma_\beta = \begin{bmatrix} c^1_{23} & c^1_{31} & c^1_{12} \\ c^2_{23} & c^2_{31} & c^2_{12} \\ c^2_{23} & c^3_{31} & c^3_{12} \end{bmatrix}.$$

If G is unimodular, then $\mathrm{tr}(\mathrm{ad}\, V) = 0$ for every $V \in \mathfrak{g}$ and then Γ is self-adjoint with respect to g.

So by an orthogonal change of basis $\beta \to \alpha = \{\widehat{F_1}, \widehat{F_2}, \widehat{F_3}\}$ we have:

$$\Gamma_\alpha = \begin{bmatrix} \widehat{c^1_{23}} & 0 & 0 \\ 0 & \widehat{c^2_{31}} & 0 \\ 0 & 0 & \widehat{c^3_{12}} \end{bmatrix} = \begin{bmatrix} 2\lambda & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & 2\nu \end{bmatrix}.$$

We can set $\lambda, \mu, \nu \in \{-1, 0, 1\}$ without changing the metric g if we want the frame F_i to be orthogonal. Evenmore, we call an orthogonal frame for which $\lambda \leq \mu \leq \nu \in \{-1, 0, 1\}$ a Milnor frame.

If $\{F_i\}$ is a Milnor frame, then there are constants A, B and C so that g may be written with respect to the set $\{\omega^i\}$ of 1-forms dual to the frame $\{F_i\}$ as:

$$g = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega 3.$$

Definition 3.3.3. (Homogeneous space)

Let M be a differentiable manifold, $G = \mathrm{Diif}(M)$ the group of diffeomorphisms acting transitively on M and H the isotropy group of G. Then one can define a differential structure on $\frac{G}{H}$ such that:

$$\left(\frac{G}{H}, \mathfrak{A}\right) \cong M.$$

And call $\left(\frac{G}{H}, \mathfrak{A}\right)$ a homogeneous space.

3.4 Berger spheres

Definition 3.4.1. (ϵ-collapsed manifold)

A Riemannian manifold $(M^n,\ g)$ is said to be ϵ-collapsed if its injectivity radius satisfies that $\mathrm{inj}(x) \leq \epsilon$ for any $x \in M^n$.

Intuitively, a collapsed manifold appears to be of lower dimension when viewed at large scales compared to ϵ.

Theorem 3.5. *M^n is said to be collapsed with bounded curvature if it admits a family of metrics $\{g_\epsilon : \epsilon \gneq 0\}$ such that:*

$$\sup{}_{x \in M^n} |Rm[g_\epsilon](x)|_{g_\epsilon},$$

is bounded but:

$$\lim_{\epsilon \to 0}(\sup_{x \in M^n} (\mathrm{inj}_{g_\epsilon}(x))) = 0.$$

Now, recall that:

$$SU(2) = \left\{ \begin{bmatrix} \omega & -\overline{z} \\ z & \overline{\omega} \end{bmatrix} : \omega, z \in \mathbb{C} : |\omega|^2 + |z|^2 = 1 \right\},$$

which we already identified topologically with S^3. Then the signature of the Milnor frame on SU(2) is $\lambda = \mu = \nu = -1$.
And the Hopf fibration $S^1 \hookrightarrow S^3 \hookrightarrow S^1$ is induced by:

$$\pi : S^3 \cong SU(2) \to \mathbb{CP}^1 \cong S^2$$

defined by: $\pi(\omega, z) = [\omega, z]$ A natural question that arises is how to collapse S^3 with bounded curvature?

One can obtain S^3 as the quotient: $\frac{SU(2)}{SU(1)} \simeq S^3$ and as $SU(1)$ is trivial, then it follows that $S^3 \simeq SU(2)$. Notice that: $T_e SU(2)$ can be identified with the set of trace-free matrices $\mathfrak{su}(2)$ and we can compute it's generators as follows:

$$e_1 = \begin{bmatrix} i & o \\ 0 & -i \end{bmatrix}, \, e_1 = \begin{bmatrix} 1 & o \\ 0 & -1 \end{bmatrix}, \, e_1 = \begin{bmatrix} i & o \\ 0 & i \end{bmatrix}.$$

and the inner produc t defined by:

$$\langle A, B \rangle = \mathrm{tr}(AB).$$

Then the family of metrics $g_\epsilon : \epsilon \gneq 0$ is defined:

$$g_\epsilon = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $\epsilon = 1$ then it is the standard metric on S^3 and if $\epsilon = 0$ then it is the standard metric on S^2. This family of Riemannian metrics on the sphere gives rise to Berger's spheres. Now, let us plot the Lie bracket for left-invariant vector fields:

$$[X_1, X_2] = 2X_3,$$
$$[X_2, X_3] = 2X_1,$$
$$[X_3, X_1] = 2X_2,$$
$$c_{ij}^k = \langle [X_i, X_j], X_k \rangle.$$

In the last equation we can them compute explicitely the components of the structure constants for the Lie algebra $\mathfrak{su}(2)$. In a general context, we can compute the Gaussian curvature with the aid of the last equation by:

$$K(X_i,X_j) = -\frac{3}{4}\sum_k (c_{ij}^k)^2 + \frac{1}{4}\Big(\sum_k (c_{ik}^j)^2 + \sum_k (c_{jk}^i)^2\Big) - \sum c_{ik}^i c_{jk}^j + \frac{1}{2}\Big(\sum_k c_{ij}^k (c_{ki}^j - c_{kj}^i) + \sum_k c_{jk}^i c_{ik}^j\Big)$$

And, using this we find:

$$K(e_1,e_1) = \epsilon,$$
$$K(e_2,e_2) = \epsilon,$$
$$K(e_3,e_3) = 4-3\epsilon.$$

And so we can write the components of the Ricci tensor as:

$$Ric_{ij} = \begin{bmatrix} 2\epsilon & o & o \\ 0 & 4-2\epsilon & 0 \\ 0 & 0 & 4-2\epsilon \end{bmatrix}.$$

In the last section we saw that, for a fixed Milnor frame, there are some constants related to their duals which allow the metric tensor to be expressed as: $g = \epsilon A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$ and using this one can define a general Riemannian metric for the Berger sphere setting $A = B = C = 1$.
In the spirit of our previous calculations, the components of the Ricci tensor in a general frame can be obtained as:

$$Ric(F_1,F_1) = \frac{2}{BC}[(\epsilon A)^2 - (B-C)^2],$$
$$Ric(F_2,F_2) = 4 - 2\frac{\epsilon A}{B} + 2\frac{B^2-C^2}{\epsilon AC},$$
$$Ric(F_3,F_3) = 4 - 2\frac{\epsilon A}{B} + 2\frac{C^2-B^2}{\epsilon AB}.$$

Theorem 3.6. *For any $\epsilon \in (0,1]$ and any choice of initial data $\epsilon A_0, B_0, C_0 \gneq 0$ the unique solution $g_\epsilon(t)$ to the Ricci flow equation for a fixed value of ϵ exist for a maximal finite time interval $0 \leq t \leq T \lneq \infty$ and becomes asymptotically round as t approaches T*

Proof. Let's define $D = \epsilon A$ for simplicity. Now, if the Ricci flow equation is satisfied,then we have:

$$\frac{d}{dt}B = -8 + 4\frac{C^2+B^2-D^2}{CD},$$
$$\frac{d}{dt}C = -8 + 4\frac{B^2+D^2-C^2}{BD},$$
$$\frac{d}{dt}D = -8 + 4\frac{B^2+C^2-D^2}{BC}.$$

We can assume without loss of generality, due to the symmetries of the previous equations, that for an initial set of data we have $D_0 \leq C_0 \leq B_0$ and then:

$$\frac{d}{dt}(B-D) = 4(B-D)\frac{C^2-(B+D)^2}{BCD}.$$

Then the inequality persist for as long as the solution exists. Notice that according to our assumptions the fraction is negative and then we can assume $B_0 - D_0 \gneq 0$ additionally.

$$\frac{d}{dt}(B) \leq -8 + 4\frac{D}{C} \leq -4$$

and this implies that the solution exists only on finite time interval, meaning there is a maximal time T such that $D \to 0$ as $t \to T$. Finally we can compute:

$$\frac{d}{dt}\left(\frac{B-D}{D}\right) = 8\left(\frac{B-D}{D}\right)\frac{C-D-B^2}{CD} \leq 0$$

Then, if we define $\frac{B-D}{D} \leq \delta$; where $\delta = \frac{B_0-D_0}{D_0} \gneq 0$ then we have $0 \leq B - D \leq \delta D$ for a time interval $0 \leq t \leq T$ and the result follows. □

Finally, despite being a solution to the Ricci flow equation, we can ask if the family of metrics for the Berger sphere (for any ϵ fixed constitute a soliton.) Now, for any left-invariant vector field $X = aX_1 + bX_2 + cX_3$ on $\mathfrak{su}(2)$ we have:

$$\nabla_{X_1}X = (\epsilon - 2)(cX_2 - bX_3), \tag{3.4.1}$$

$$\nabla_{X_2}X = cX_1 - a\epsilon X_3, \tag{3.4.2}$$

$$\nabla_{X_3}X = -bX_1 + a\epsilon X_2. \tag{3.4.3}$$

Besides this, we can compute the $\pounds_X g$ matrix:

$$\pounds_X g = \begin{bmatrix} 0 & 2(1-\epsilon)c & 2(1-\epsilon)b \\ 2(1-\epsilon)c & 0 & 0 \\ 2(1-\epsilon)b & 0 & 0 \end{bmatrix}.$$

If it were a Ricci soliton $\pounds_X g = \lambda g - Ric$, then, matching by components:

$$\begin{aligned} 2(1-\epsilon)c &= 0, \\ 2(1-\epsilon)b &= 0, \\ 2\epsilon^2 - \lambda\epsilon &= 0, \\ 4 - 2\epsilon - \lambda &= 0. \end{aligned}$$

If $\epsilon = 1$ then g_ϵ is Einstein on S^3 but if it is not 1, then from the first 2 equations we obtain that $b = c = 0$ and from the third equation we have that ϵ must be zero and $\lambda = 4$ is obtained from the last equation and so (S^3, g_ϵ) is not a homogeneous Ricci soliton agreeing with the classic result that there are no left-invariant Ricci solitons on 3-dimensional Lie groups (See [10]).

3.5 Properties and details

3.5.1 Canonical neighbourhoods and collapse

Definition 3.5.1. Pointed convergence.

A pointed Riemannian manifold is a Riemannian manifold M together with a choice of basepoint at $p \in M$. If g is a complete Riemannian metric then the triple (M, g, p) is said to be a complete pointed manifold. Moreover, if $(M, g(t))$ is a solution to the Ricci flow then we say that $(M, g(t), p)$ is a pointed solution to the Ricci flow

Definition 3.5.2. Cheeger-Gromov convergence in C^∞ A sequence of complete, pointed Riemannian manifolds (M_k, g_k, p_k) converges to a complete, pointed Riemannian manifold $(M_\infty, g_\infty, p_\infty)$ if there exists:

1. An exhaustion (Compact sets such that $U_k \subset U_{k+1}$ and whose union is the whole space) (U_k) of M_∞ with $p_\infty \in U_k$;

2. A sequence of diffeomorphisms $\{\phi_k\}$: $U_k \to V_k \subset M_k$ with $\phi(p_\infty) = p_k$ such that: $\{(\phi_k^* g_k)\}$ converges in C^∞ to g_∞ on compact sets in M_∞.

For the Ricci flow setting, we have the same changing g_k and g_∞ for $g_k(t)$ and $g_\infty(t)$ respectively and the convergence is on compact sets in $M_\infty \times (a, b)$

Example 3.5.1. Consider a sequence of pointed Riemannian manifolds (S_k^n, g_k, N) where S_k^n is the standard sphere of radius k with it's canonical metric and basepoint N, the north pole. This sequence has limiting pointed manifold $(\mathbb{R}^n, \delta_{ij}, 0)$ centered in the origin.
To see this, take an exhaustion $U_k = B_k(0)$ of topological balls centered at 0 and radius k with a sequence of diffeomorphisms:

$$\phi_k : B_k(0) \to S_k^n \subset \mathbb{R}^{n+1},$$

defined as:

$$\phi_k : x \to (x, \sqrt{k^2 - |x|^2}).$$

For ϕ_k we can compute the pullback metric as:

$$(\phi_k^* g_k)_{ij} = \langle \frac{\partial \phi_k}{\partial x^i}, \frac{\partial \phi_k}{\partial x^j} \rangle_{\mathbb{R}^{n+1}} = \delta_{ij} + \frac{x^i x^j}{k^2 - |x|^2}.$$

And this happens because: $\frac{\partial \phi_k}{\partial x^i} = \left(e^i, \frac{x^i}{\sqrt{k^2 - |x|^2}} \right)$; being e^i the canonical vector in the i-th direction.
So it suffices to show that:

$$\left(B_k(0), \delta_{ij} + \frac{x^i x^j}{k^2 - |x|^2} \right) \xrightarrow{C^\infty} (\mathbb{R}^n, \delta_{ij})$$

converges uniformly on compact sets, which is achieved by showing that: $\frac{1}{1 - |\frac{x}{k}|^2}$ converges to 1 in C^∞ on compact sets of $\mathbb{R}^n$

Now let's define singularities and canonical neighbourhoods for Ricci flow:

Definition 3.5.3. (Finite-time singularities) Let $(M, g(t))$ be a solution to the Ricci flow on a compact manifold M on a maximal time interval $[0, T)$. If $T \lneq \infty$, then:

$$\lim_{t \to T^-} \max_{x \in M} |\operatorname{Rm}(t)|^2_{g(t)} = \infty$$

Uniqueness is used for there to exist a maximal time interval on which Ricci flow has a well-defined solution. If there exist a solution on $[0, T_1)$ and another solution on $[T_1 - \epsilon, T_2)$ then they must agree on the overlap and one can consider the flow on $[0, T_2)$.

Definition 3.5.4. Consider a Riemannian manifold (M, g), we say M has a bounded geometry if it exists a positive constant $C \in \mathbb{R}$ such that the diameter, the ratio of curvatures at any two distinct points in M and the volume of M are bounded by C

Now we will describe some regions which will help us define canonical neighbourhoods:

Definition 3.5.5. (neck) An ϵ-neck in a Riemannian 3-manifold centred at a point x of the manifold is a submanifold $N \subset M$ and a diffeomorphism: $\psi : S^2 \times (-\epsilon^{-1}, \epsilon^{-1}) \to N$ such that $x \in \psi(S^2 \times \{0\})$ and the pull-back of the rescaled metric $\psi^*(\mathcal{R}(x)g)$ is within ϵ distance in the Fréchet topology to the product of the round metric of scalar curvature 1 on S^2 with the usual metric on the interval $(-\epsilon^{-1}, \epsilon^{-1})$.
Furthermore, An ϵ-tube is a sub-manifold of M diffeomorphic to $S^2 \times (0, 1)$ with the property that it is a union of ϵ-necks and every point in the tube is in the center of an ϵ-neck in M.

In the study of Ricci flow, sometimes to study 3-manifolds along the flow we need an additional dimensional parameter to run the flow as long as it is defined, making the product of the time parameter and the manifold a 4-manifold $\overline{M}$ and we must therefore consider a time function: $\tau : \overline{M} \to \mathbb{R}$ and a vector field χ such that: For every $x \in \overline{M}$ has a neighbourhood of the form: $U \times J$ where U is open in $\mathbb{R}^3$ and J is an open interval in the real line such that: The time function τ is a projection over J and the vector field χ is tangent to the 1-dimensional foliation $U \times J$ pointing in the direction of increasing values of τ. We say $\tau(t)$ the t-time slice is a smooth 3-manifold.

$\tau(M)$ is embedded in a connected component of the real line (possibly the whole line) and it preserves boundaries, meaning, the boundaries of the manifold are the boundaries of J under the projection. $\tau^{-1}(t)$ are level sets from a co-dimension one foliation of $\overline{M}$ called the horizontal foliation and the boundary components are the leaves. There is a natural Riemannian metric G on the horizontal distribution (distribution in the sense of tangent geometric distribution) tangent to the level sets defined by τ and induces a Riemannian metric on each τ-time slice varying smoothly as we vary the time slice. We can study the curvature of G in $\overline{M}$ by studying the curvature of g, where g is the induced metric by G on a particular time slice in a point x of the space-time manifold. Having this clear and the study of the derivative of the metric tensor, then we can write the Ricci flow as: $\mathcal{L}_\chi(G) = -2Ric(G)$.

In this context of space-time there is an additional type of canonical neighbourhood: The one of a strong ϵ-neck. Consider a diffeomorpism $\psi : (S^2 \times (-\epsilon^{-1}, \epsilon^{-1})) \times (0, 1] \to \overline{M}$ such that the time function τ pulls-back to the projection on $(0, 1]$ and χ pulls-back to ∂_t. If there is an embedding that makes the pull-back of the rescaled horizontal metric to be ϵ apart from the product of a shrinking S^2 with the euclidean metric on $(-\epsilon^{-1}, \epsilon^{-1})$ in the sense of $C^{\frac{1}{\epsilon}}$-topology then we say that the domain of ψ is a strong ϵ-neck.

Definition 3.5.6. (cap) Let (M^3, g) a Riemannian manifold. A (C, ϵ)-cap in (M^3, g) is a non compact submanifold $(X, g|_X)$ with an open submanifold $N \subset X$ such that:

1. X is diffeomorphic to an open 3-ball or a punctured $\mathbb{RP}^3$ (The projective space minus a point),
2. N is a ϵ-neck,
3. $Y = X \setminus N$ is a compact sub-manifold with boundary,
4. X has bounded geometry.

Definition 3.5.7. (C-component) A compact Riemannian manifold (M, g) is called a C-component if:

1. M is diffeomorphic to S^3 or $\mathbb{RP}^3$,
2. (M, g) has positive sectional curvature,
3. (M, g) has bounded geometry.

With the aid of these definitions we will stablish the conditions to define a canonical neighbourhood, which are of interest as these will be the regions where we will perform surgery.

Definition 3.5.8. A point $p \in M$ is in a (C, ϵ)-canonical neighbourhood if one of the following holds:

1. p is contained in a C-component.
2. p is contained in an open set with a metric which is ϵ apart from the round metric in the $C^{\frac{1}{\epsilon}}$-topology.
3. p is contained in the core of a (C, ϵ)-cap.
4. p is in the center of a strong ϵ-neck.

Hamilton managed to prove that the singularities of a Ricci flow are all contained in unions of canonical neighbourhoods with respect to the metrics at nearby earlier times $t \leq T$. There is one important result concerning canonical neighbourhoods that is extremely important and useful: Any complete 3-manifold of positive curvature does not admit ϵ-necks of arbitrarily high curvature. In particular, if M is a complete Riemannian 3-manifold with the property that every point of scalar curvature greater than r_0^{-2} has a canonical neighbourhood, then M has bounded curvature.

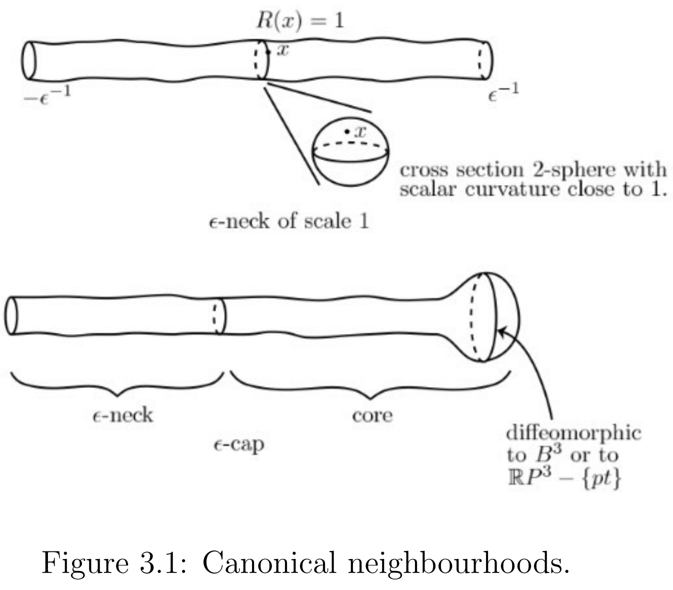


Figure 3.1: Canonical neighbourhoods.

Besides the previous regions, we can build up another family of canonical neighbourhoods:

An important fact is that any Riemannian 3-manifold with positive sectional curvature dos not admit ϵ-necks of arbitrarily high curvature [24] and evenmore: If M is a complete Riemannian 3-manifold with the property that every point with scalar curvature greater than r_0^{-2} (where r_0 is the injectivity radius) has a canonical neighbourhood then M has bounded curvature.

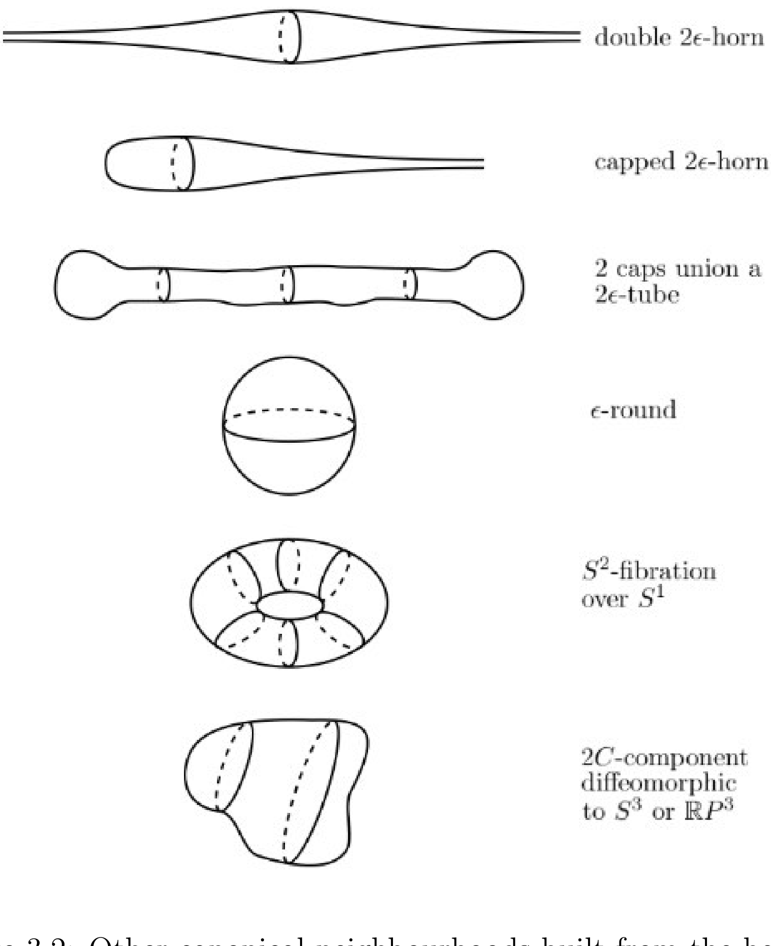


Figure 3.2: Other canonical neighbourhoods built from the basic ones.

Being a PDE, one starts the Ricci flow with an initial condition on $(M_t, g(t))$ and a solution of the Ricci flow is therefore a Riemannian manifold. As an example, let us consider $(M, g(t))$ a complete Riemannian manifold (complete in the sense that geodesics can be defined along all the real line) and suppose that there is a constant $\lambda \geq 0$ such that: $Ric(g(t)) = \lambda \cdot g(t)$. In this case it is easy to see that there is a Ricci flow given by: $g(t) = (1 - 2\lambda \cdot t)g_0$ where g_0 is the initial metric under which the flow starts. Note that all the metrics in this flow differ by a constant and a factor depending on time and the metric is a decreasing function. These solutions are called solitons as we previously stated.

Recall from definition (3.6.3) that at singularities $|\operatorname{Rm}|^2$ is going to infinity. This is not a desired property to study Ricci flow around a singularity and in order to be able to obtain information we can scale the metric g to $c\,\mathrm{g}$ and we get: $|\,\mathrm{cg}\,|^2 = \frac{1}{c^2}|\,\mathrm{g}\,|^2$. If we wish to bound Rm so it doesn't go to infinity we must choose the appropriate bound: $L_n^{-2} = max_{x\in M}|\operatorname{Rm}(g_{t_n})|$ and thus resale the metric by L_n^{-2}.

Then a sequence of solutions to the Ricci flow would be rescaled as: $g_n(t) = \frac{1}{L_n^2} g(t_n + tL_n^2)$. If there is a Ricci flow defined on $[0, T)$ then the rescaled solutions exist on $[-\frac{t_n}{L_n^2}, \frac{T-t_n}{L_n^2})$. In this scenario if we can extract a limit (with convergence in the Cheeger-Gromov sense) the limit metrics for every time slice will be ancient, i.ei., exist for a time interval: $(-\infty, t)$. The main goal of this is to improve the metric as we approach the limi and restrict the geometry of our region (restrict it in a sense which makes it easier to study).

Now we can define collapse:

Definition 3.5.9. A pointed sequence of Riemannian manifolds (M_n, g_n, p_n) is collapsing if $\operatorname{inj}_{p_n} \to 0$ as $n \to \infty$. Where inj_{p_n} is the injectivity radius a the point p_n

We may want to rescale the manifolds (M_n, g_n) to bound the behaviour of Rm say by making $\operatorname{Rm} \leq 1$, then the rescaled collapsing becomes: $|\operatorname{Rm}(p_n)|^{\frac{1}{2}} \operatorname{inj}_{p_n} \to 0$

Theorem 3.7. *(Cheeger) Suppose that* $|\operatorname{Rm}|_g \leq \frac{C}{r_0^2}$ *on* $B((r_0, p)) \subset M^d$ *and* $\operatorname{Vol}(B(r_0, p)) \geq \delta r_0^d$ *for* $\delta \geq 0$.
Then, with these assumptions the injectivity radius of p is at least $\operatorname{inj}_p \geq cr_0$ *for a constant* $c = c(C, \delta, d) \geq 0$

To sum up, collapsing generally means the injectivity radius going to zero. Chegeer's theorem tells us that when there is bounded curvature, then volume of balls getting smaller and injectivity radius getting smaller are essentially the same thing. This motivates the following:

Definition 3.5.10. A Riemannian manifold (M^n, g) is k-collapsed at $p \in M$ at a scale r_0 if:

1. There is bounded curvature: $|\operatorname{Rm}|_g \leq r_0^2$ for all $x \in B(r_0, p)$ and,

2. There is collapsed volume: $\operatorname{Vol}(B(r_0, p)) \leq kr_0^d$.

If these are not satisfied, then we say the manifold is k-noncollapsed at p at the scale r_0.

We can make some observations:

- If it is k-noncollapsed, Chegeer theorem asserts that there is a lower bound on the injectivity radius.

- Balls with large curvature imply k-noncollapsed at that scale.

- Every manifold is k-noncollapsed at a small enough scale.

- This definition of k-collapsed is scale-independent . If we consider $\overline{g} = r_0^{-2} g$ then: $|\operatorname{Rm}(\overline{g})|_{\overline{g}} \leq 1$ and $V(B_{\overline{g}}(p,1)) \leq k$

Perelman adapted these ideas to Ricci flow as follows:

Definition 3.5.11. Perelman's k-collapsed Let $(M^n, g(t))$ be a solution to a Ricci flow and we have that $k \geq 0$. That Ricci flow is k-collapsed at (t_0, x_0) in the t_0-slice of the spacetime manifold at a scale r_0 if:

1. $|Rm(t,x)|_{g(t)} \leq r_0^{-2}$ for all $(t,x) \in [t_0 - r_0^2, t_0] \times B_{g_{t_0}}(x_0, r_0)$, and
2. $\operatorname{Vol}(B_{g_{t_0}}(x_0, r_0)) \leq k r_0^n$

Otherwise, we say that the solution is k-noncollapsed at p at scale r_0

Theorem 3.8. *(Perelman's noncollapsing theorem) Let $(M, g(t))$ be a solution to a Ricci flow on a compact M^3 for a time interval $[0, T_0]$ such that a $t = 0$ we have:*

$$|\operatorname{Rm}(p)|_{g_0} \leq 1,$$

$$\operatorname{Vol}(B_{g_0}(p,1)) \geq \alpha.$$

For every $p \in M$ and $\alpha \geq 0$ fixed. The there exist $k_1 = k(\alpha T_0) \geq 0$ such that the Ricci flow is k_1-noncollapsed for all $(t_0, x_0) \in [0, T_0] \times M$ and scales $0 \leq r_0 \leq \sqrt{t_0}$.

A natural question that arises is whether the cigar soliton can be immersed in $\mathbb{R}^3$ or not. One can prove with the aid of Perelman's collapsed theorem that the cigar metric immersed in $\mathbb{R}^3$ does not satisfy the k-noncollapsed condition and therefore it is not of interest. For higher dimensional analogues of this phenomena, the Bryant soliton and rotationally symmetric solitons appear.

3.5.2 Ricci De-Turck flow and parabolicity

De-Turck's trick is a technique used to visualize the Ricci flow as a heat-type equation (a parabolic PDE) but this at first fails to happen as the Ricci flow is not strictly parabolic but weakly parabolic. To avoid this we make use of something called De-Turck's trick which proves short-time existence and uniqueness. This is important because it allows all the theory from partial differential equations to be applied accordingly to the Ricci flow, and evenmore, to understand what the Ricci flow does over time on a manifold M. For that matter, consider a smooth map between Riemannian manifolds: $f : (M, g) \to (N, h)$ then one can consider the laplacian of f, defined as: $\Delta_{g,h} f = \sum_k (\nabla_{e_k} df)(e_k)$ and it is important to notice that the Laplacian is invariant under the action of diffeomorphisms from M to itself. To formalize this, we have:

Fact. Let f be a smooth map between Riemannian manifolds (M, g) and (N, h) and let φ a diffeomorphism from M to itself. Then: $(\Delta_{\varphi^*(g),h}(f\varphi))\|_p = (\Delta_{g,h} f)\|_{\varphi(p) \in T_{f(\varphi(p))}N}$ for all points $p \in M$.

To prove uniqueness and short-time existence for Ricci flow, we prove it for a strictly parabolic equation called the Ricci De-Turck flow, defined as: Being M a compact manifold and h a background metric. Moreover, suppose $\overline{g}(t), t \in (0, T]$ is a 1-parameter family of Riemannian metrics on M. We say this family of metrics is a solution to the Ricci De-Turck flow if: $\frac{\partial}{\partial t}\overline{g}(t) = -2Ric_{\overline{g}(t)} - \mathcal{L}_{\xi_t}\overline{g}(t)$ where $\xi_t = \Delta_{\overline{g}(t),h} id$. For a detailed discussion on the analytic aspects of this matter, see [8].

Theorem 3.9. *(Shi's theorem) Let $B(x, t_0, r)$ be the metric ball in $(M, g(t_0)$ centred at x and of radius r. If we can control the norm of the Riemannian curvature tensor on a backwards neighbourhood of the form $B(x, t_0, r) \times [0, t_0]$, then for each $k \geq 0$ we can control the k^{th} covariant derivative of the curvature on $B(x, t_0, \frac{r}{2^k}) \times [0, t_0]$ by a constant divided by $t^{\frac{k}{2}}$*

Proof. See [14] □

Using Shi's theorem, Hamilton managed to prove that if there is a Ricci flow defined up to some time T and the Riemannian curvature is uniformly bounded for the entire flow (for all the times it is defined) then one can extend this Ricci flow past this time T.

3.5.3 Ricci flow with surgery

The main obstruction in Hamilton's program to use Ricci flow was the existence of singularities and how these singularities can be removed in a controlled way in order to keep studying the evolution of the manifold. Then one starts with the Ricci flow and encounters singularities but there are several obstructions: How to know if the singularities are finite and do not acumulate? and an even more substantial question is how to define on the singularities the regions where to apply surgery?

Theorem 3.10. *The space of based k-solutions at points $(x, 0)$ with $R(x, 0) = 1$ is compact.*

Now we define a set on the foliations of the time-space manifold $M \times \mathbb{R}$, called Ω where each point of the set has bounded curvature by C, a positive constant and we study the canonical neighbourhoods on these points. Then we proceed as follows:

1. Through the analysis of k-noncollapsed solutions, we study those points that do not belong to the k-noncollapsed solution (so we cannot guarantee the geometry around the point will behave as it is supposed along the flow).
2. If the solution behaves asymptotically (here meaning when the time parameter goes to infinity) like a cylinder or a shrinking sphere, then one does nothing (as we discussed previously on our canonical neighbourhoods section). Let us call the points where this is the situation Ω_C, where C is the constant we described above and due to the compactness hypothesis, the set Ω/Ω_C is finite and therefore, there are finitely many points where singularities can happen.
3. As we mentioned before, these singularities occur along ϵ-necks or compositions of these (particularly, ϵ-horns that are ϵ with an extreme tending to infinity). On each of these components the Ricci flow will collapse.
4. Once it collapses, one takes the injectivity radius around a point x in a component of this type (future singularity region) and the metric-ball defined by x and the injectivity radius.
5. The injectivity radius contains the boundary of the non-infinity part of the ϵ-horn and therefore we can cut a component diffeomorphic to one of the canonical neighbourhoods previously mentioned along the ball defined by this radius.
6. After this, one proceeds to glue a 3-ball with the standard metric cut in half along a maximal boundary circle (each half 3-ball equipped with the standard metric is called a standard cap):

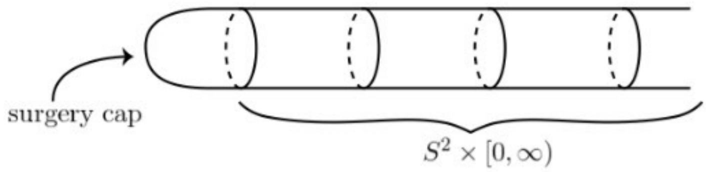


Figure 3.3: ϵ-cap

7. And then glue each standard cap preserving orientation as suggested in the method by Kneser [19]. The idea to glue a 3-ball with the standard metric is to let the Ricci flow evolve without the problem of the metric collapsing in an uncontrolled way (and extinguing that component in finite time as a condition in theorem 3.3).

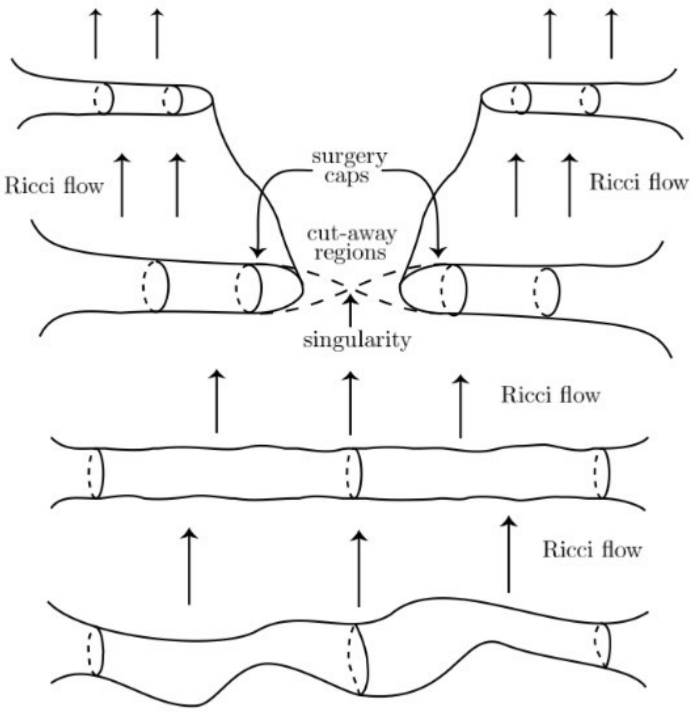


Figure 3.4: Surgery process of cutting the injectivity radiuscomponent and gluing caps [2]

Chapter 4

Perelman's program

4.1 Deep tools

This is a rather informal section aimed to expose briefly some of the main techniques developed by G. Perelman towards proving the Poincaré conjecture. Some of the main achievements include the connection between the local behaviour of Ricci flow and it's singularities with the topological behaviour of a space concerning it's fundamental group.

4.1.1 Perelman's Functionals.

Let $\mathcal{M}$ denote the space of smooth Riemanian metrics g on a smooth manifold M. The set $\mathcal{M}$ is formally an infinite dimensional manifold. $T_g\mathcal{M}$consists of the symmetric covariant 2-tensors v_{ij} on M. Additionally $C^\infty(M)$ is also an infinite dimensional manifold with $T_f C^\infty(M) \cong C^\infty(M)$. The diffeomorphism group $\mathrm{Diff}(M)$ acts on both of these spaces by pullback. Now we will take a quick look at some of the properties and utilities of the functionals used to analyze Ricci flow as an operator on $\mathcal{M}$ the space of Riemannian metrics.

Definition 4.1.1. $\mathcal{F}$-functional The $\mathcal{F}$-functional $\mathcal{F} : \mathcal{M} \times C^\infty(M) \to \mathbb{R}$ is given by:

$$\mathcal{F}(g, f) = \int_M (\mathrm{R} + |\nabla f|^2) e^{-f} d\,\mathrm{Vol}\,.$$

Where R stands for the scalar curvature.

Proposition 4.1.1. *The variation of $\mathcal{F}$-functional, written as $\delta\mathcal{F}$ is given by:*

$$\delta\mathcal{F}(v_{ij}, h) = \int_M e^{-f}\left[-v_{ij}(R_{ij} + \nabla_i\nabla_j f) + (\frac{v}{2} - h)(2\Delta f - |\nabla f|^2 + R)\right] d\,\mathrm{Vol},$$

for $v_{ij} \in T_g(\mathcal{M})$, $h \in T_f(C^\infty(M))$, $v = g^{ij}v_{ij}$, Ric_{ij} the Ricci tensor and R *the scalar curvature.*

Proof. We will give a brief sketch of the proof. Recall that δR is as in proposition (3.1.5), $\delta|\nabla f|^2 = -v^{ij}\nabla_i\nabla_j f + 2\langle\nabla f, \nabla h\rangle$ and $\delta(dVol) = \frac{v}{2} dVol$ which agrees with proposition (3.1.6) and finally: $\delta(e^{-f}dVol) = (\frac{v}{2} - h)e^{-f}dVol$. Bringing together all of these expressions and using the fact that the variation operator δ is distributive along each term of $\mathcal{F}$ the result follows. $\square$

We may wish to work with a simpler form of this variation, for this purpose we can consider: $e^{-f}d\,\mathrm{Vol}$ fixed so the second term of the variation goes to zero. Finally we arrive at an expression with looks familiar. Now, if we define $dm = e^{-f}d\,\mathrm{Vol}$ and defining a smooth section s :

$\mathcal{M} \to \mathcal{M} \times C^\infty(M)$ by $s(g) = (g, \ln(\frac{d\,\mathrm{Vol}}{dm}))$, we can define a function from $\mathcal{M}$ to $\mathbb{R}$ as follows: $\mathcal{F}^m = \mathcal{F} \circ s$ and we can calculate its differential:

$$\delta\mathcal{F}^m(v_{ij}) = \delta(\int_M (\mathrm{R} + |\nabla \ln(\frac{d\,\mathrm{Vol}}{dm})|^2))e^{-\ln\frac{d\,\mathrm{Vol}}{dm}} = \delta e^{-f}[-v_{ij}(\mathrm{Ric}_{ij} + \nabla_i\nabla_j f)]d\,\mathrm{Vol}\,.$$

From this, we see that the gradient flow of $\mathcal{F}^m$ on $\mathcal{M}$:

1. $\frac{\partial}{\partial t} g_{ij} = -2(\mathrm{Ric}_{ij} + \nabla_i\nabla_j f)$,
2. and the induced flow for f: $\frac{\partial}{\partial t} f = \frac{\partial}{\partial t}\ln(\frac{d\,\mathrm{Vol}}{dm}) = \frac{1}{2}g^{ij}(g_{ij})_t = -\mathrm{R} - \Delta f$.

Finally, let us perform a transformation through a time dependent diffeomorphism on the gradient flow of $\mathcal{F}^m$ to transform it into the Ricci flow equation. If $V(t)$ is the time-dependent generating vector field of the diffeomorphism then:

1. $\frac{\partial}{\partial t} g_{ij} = -2(\mathrm{Ric} + \nabla_i\nabla_j f) + \mathcal{L}_V(g)$ and,
2. $\frac{\partial}{\partial t} f = -\Delta f - \mathrm{R} + \mathcal{L}_V(f)$.

If we take V as a gradient vector field for f, then: The gradient flow equations become:

1. $\frac{\partial}{\partial t} g_{ij} = -2\,\mathrm{Ric}_{ij}$,
2. $\frac{\partial}{\partial t} f = -\Delta f - \mathrm{R} + |\nabla f|^2$.

And so we arrive, finally, at:

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_M |\,\mathrm{Ric}_{ij} + \nabla_i\nabla_j f|^2 e^{-f} d\,\mathrm{Vol}\,. \tag{4.1.1}$$

Notice that (4.1.1) is monotone under Ricci flow. Despite this, we want to avoid dealing with $\frac{d}{dt}\mathcal{F}$ we take then the infimun over f:

$$\lambda(M, g) = \inf\{\mathcal{F}(M, g, f) : \int_M e^{-f} d\,\mathrm{Vol} = 1\}.$$

(By means of technical details, one could show that the minimun is realized and that λ behaves in a controlled way under Ricci flow). We will define $\overline{f}$ the minimizer and define the λ-functional by: $\lambda(g) = \mathcal{F}(g, \overline{f})$. We can consider a rescaled version of the λ-functional, namely: $\overline{\lambda}(g) = \lambda(g)Vol^{\frac{2}{n}}(g)$ which is the scale invariant version. Given this, we would like a scale-invariant version of the $\mathcal{F}$-functional. Let us define:

Definition 4.1.2. The $\mathcal{W}$-functional: $\mathcal{W} : \mathcal{M} \times C^\infty(M) \times \mathbb{R}^+ \to \mathbb{R}$ given by:

$$\mathcal{W}(fg, f, \tau) = \int_M [\tau(|\nabla f|^2 + \mathrm{R}) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} d\,\mathrm{Vol}$$

This is a scale-invariant variant of $\mathcal{F}$. It is an even a more powerful technique to analyze gradient Ricci solitons than the $\mathcal{F}$-functional. Just like before we would like to obtain a minimizer and a minimizer-functional for this (which we can achieve by means of log-Sobolev type inequalities but this goes beyond the scope of this text).

Finally, there is another important functional we want to describe: Consider the space time manifold $M \times I$ where I is a subset of the real line. Suppose $I = (0, T]$ and fix a point in the space time $(x, t) \in M \times (0, T]$. We consider $\gamma(\tau)$ for $0 \leq \tau \leq \overline{\tau}$ such that for every $\tau \leq \overline{\tau}$ we have $\gamma(t) \in M \times \{-t, t\}$ and $\gamma(0) = x$. These paths are said to be parametrized by backward time. But what is the interest of studying these paths? It is common to study the energy functional when studying Riemannian geometry, yet, G. Perelman managed to work out an analogous functional called the $\mathcal{L}$ functional:

Definition 4.1.3. The $\mathcal{L}$-functional:

$$\mathcal{L}(\gamma) = \int_0^{\overline{\tau}} \sqrt{\tau}(\mathrm{R}(\gamma(\tau)) + \|\gamma'(\tau)^2\|)d\tau,$$

where the derivative on γ is the spatial derivative. And with this we define:
Reduced length:

$$\frac{\mathcal{L}(\gamma)}{2\sqrt{\tau}},$$

and use this new length definition to study a notion of $\mathcal{L}$-geodesic:
The $\mathcal{L}$-geodesic equation is:

$$\nabla_X X - \frac{1}{2}\nabla R + \frac{1}{2\tau}X + 2Ric(X,\cdot) = 0$$

The idea behind these definitions is to achieve a precise notion of $\mathcal{L}$-exponential and $\mathcal{L}$-volume to analyze Ricci flow near singularities in a controlled way so one can stablish canonical neighbourhoods with high precision, and describe the correct times to perform surgery in order to define continuity for a Ricci flow with surgery.

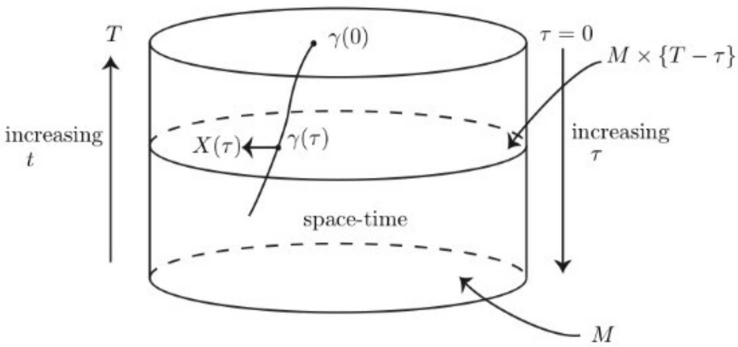


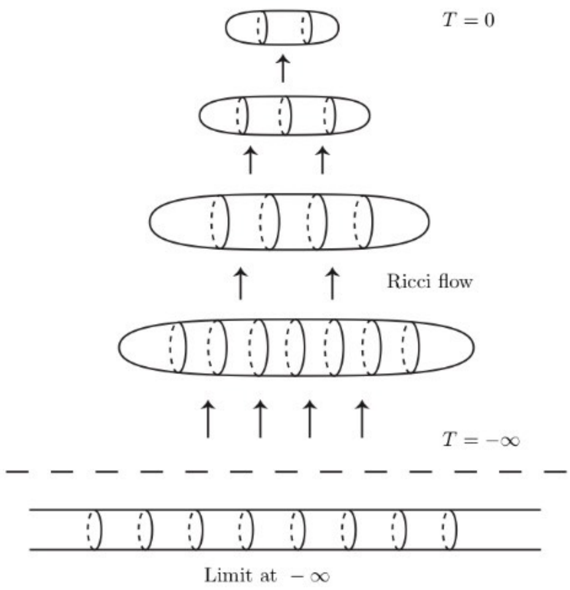
Figure 4.1: Canonical neighbourhoods.
[2]

In a detailed way, these tools are developed to study non-collapsing results needed in a Ricci flow with surgery in the following sense: Using the geodesics for this new functional, one can establish non-collapsing result in different areas of the manifold, ensuring it will not collapse at the same time in different ways. It is also a way to study non-collapsing results in order to control the way the volume changes in a neighbourhood and therefore establish geometric conclusions on the region (such as a geometric bound) after a surgery process.

4.1.2 Perelman's main theorems.

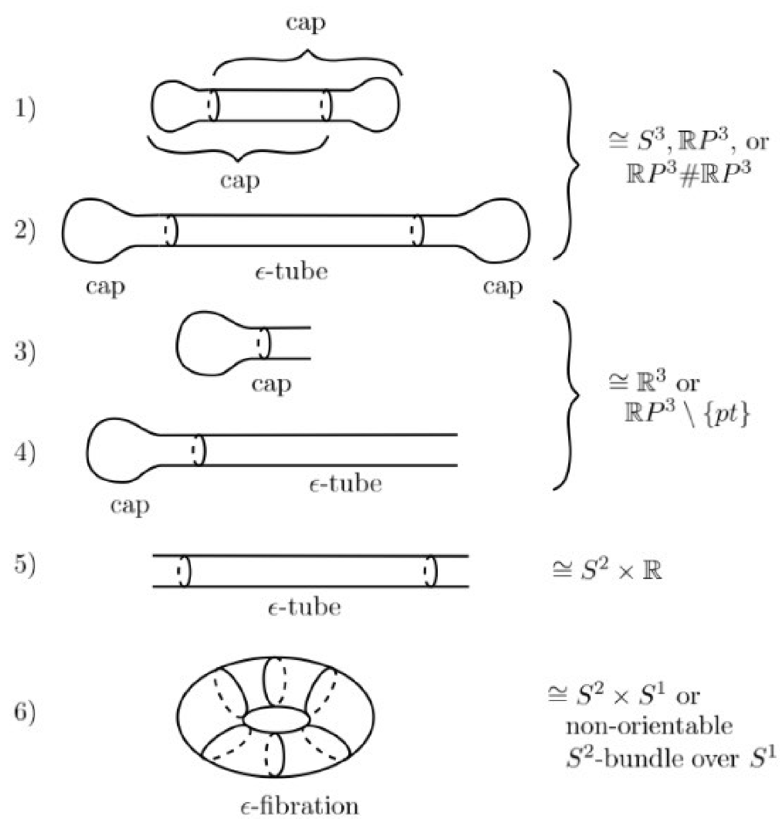
We begin here our exposition of Perelman's main results which led him to prove Poincaré's conjecture. These results canbe found in a detailed manner in [23], [24], [25]:

Theorem 4.1. *Let (M, g_0) be a closed Riemannian 3-manifold- Suppose that there is no embeded, locally separating $\mathbb{RP}^2$ (this to consider only orientable manifolds) contained in M. Then there is a Ricci flow with surgery defined for all positive time with initial metric g_0. The set of discontinuity times for this Ricci flow with surgery is a discrete subset of [0, ∞). The topological change in the 3-manifold in a surgery time is a connected sum decomposition together with removal of connected components, each of which is diffeomorphic to one of $S^2 \times S^1, \mathbb{RP}^3\#\mathbb{RP}^3$, the non orientable 2-sphere bundle over S^1 or a manifold admitting a metric of constant positive curvature.*



[2]

Figure 4.2: Visualization of ancient solutions to Ricci flow.



[2]

Figure 4.3: Components to be removed.

An M-bundle over S^1 is called a mapping torus and is classified (classified in this context means fiber-preserving diffeomorphism) by its isotopy class of a diffeomorphism on M. If $M = S^2$, there are two isotopy classes of maps, one given by the identity and one given by the antipodal map. Therefore the unoriented S^2-bundle is: $S^2 \times \mathbb{R}/(x,t) \sim (-x, t+1)$ and it has $S^2 \times \mathbb{R}$ geometry and therefore it has positive sectional curvature. What we mean by "...Topological change... in a surgery time is.." is that after one applies surgery at a singularity time, then one can recover the initial manifold by gluing one of the removed components or stating the initial manifold as a connected sum of other components and re-starting the Ricci flow in each of these components until one finishes or encounters another singularity.

Theorem 4.2. *Let M be a closed 3-manifold whose fundamental group is a free product of finite groups and infinite cyclic groups. Let g_0 be any Riemannian metric on M. Then M is orientable and there is a Ricci flow with surgery defined for every positive time with g_0 as initial metric. This Ricci flow with surgery becomes extinct after some finite time T, meaning that the manifolds M_t are empty for all $t \geq T$.*

We see from these two theorems that after a finite number of operations, each of which is a removal of a component as stated in Theorem 3.2 or a connected sum, then the manifold becomes empty. Clearly this states that the original manifold is a decomposition of each of the components mentioned in Theorem 3.2 and therefore, a connected sum of manifolds with positive Ricci curvature. This result is deeply important as one has Kneser's descomposition and:

Theorem 4.3. *Let M be a closed 3-manifold and suppose that the fundamental group of M is a free product of finite groups and infinite cyclic groups. Then M is diffeomorphic to a connected sum of spherical space forms (see appendix A), copies of $S^2 \times S^1$, and copies of the unique (up to diffeomorphism) non-orientable* 2*-sphere bundle over S^1*

These theorems allow to state the main goal of Perelman's work to classify geometric properties relying on asumptions on the fundamental group (topological asumptions). Another way to state these asumptions for the context of the geometrization conjecture is:

Theorem 4.4. *Let M be a closed, orientable, irreducible, atoroidal (atoroidal meaning that there is not an embedded torus) 3-manifold. Then:*
1. If $\pi_1(M)$ is finite, then M is spherical.
2. If $\pi_1(M)$ is infinite, then M is hyperbolic or M is seifert fibred.

Finally, as a particular case of all of these theorems (based on the hypothesis of closedness, connectedness on a 3-manifold) the assertion for Poincaré's conjecture follows as: "A closed, simply connected 3-manifold is diffeomorphic to S^3" and Milnor's comment ensures the topological implication of the above statement.

4.1.3 Sketch of the proof

After building all of these prerequisites, one can describe the idea behind Perelman's proof of the Conjecture by steps as follows:

1. One starts with a space-time manifold $\overline{M}^4$ and the time-like vector field ∂_t and a compact, closed, simply connected Riemannian 3-manifold (M^3, g_0) and one applies the generalized Ricci flow with M^3 as a foliation of the space-time manifold.

2. The Ricci flows evolves until a moment where it finds a singularity.

3. One studies the injectivity radius around the singularity and determines the component to be removed embebbed in the ball determined by the injectivity radius.

4. After removing it, we glue the standard caps (half of 3-balls with the standard metric) along the region we cut and then re-start the Ricci flow.

5. As the set of singularities is finite, one finishes the procedure and for a time T the Ricci flow converges to a metric of constant curvature and all of the components become extinct.

6. Due to the hypothesis of simply connectedness, and finite extinction, then one has that each component is a connected sum of spherical space forms (finite extinction condition) and as the initial manifold has trivial fundamental group (simply-connected hypothesis) then the quotient of all of these spherical space forms is trivial and one has that the manifold is, therefore, a connected sum of spheres which is a sphere.

And so, we see that most of the technical details and machinery are geometrical but the idea behind the arguments remains topological. Never the less it is important to point out that G. Perelman's achievement goes beyond the proof of this conjecture and Thurston's Geometrization conjecture. The most important idea, besides providing the gradient formalism to analyze solutions to Ricci flow, is to develop a sort of theory of entropy that can be applied to certain geometric quantities such as volume and curvature. From this point of view there is very little known about Ricci solitons in the general frame. Yet, some advances in the direction of Lie groups and Homogeneous spaces can be found in [10] and [31]. On the other hand,recent progress has been done in the context of Kähler manifolds, for details see [32].

Chapter 5

Appendix

5.1 spherical space forms

We will recall briefly what a space form (and what a spherical space form is, for our particular purpose).
In general, a space form is defined to be a manifold admitting a Riemannian metric of constant sectional curvature. Cartan managed to prove that a manifold is a space form if and only if it is a quotient of $S^n, \mathbb{R}^n, \mathbb{H}^n$ with their usual metrics by a discrete group of isometries Γ acting properly discontinuosly. Γ is isomorphic to the fundamental group of the space form and in the case of $S^n \setminus \Gamma$ we say it is a spherical space form. For a more detailed discussion on the subject and a detailed proof of Cartan's theorem exposed here [9]

5.2 PDE's and maximum principle

We will briefly describe the different versions of the maximum principle and state how this applies to the Ricci flow case (and some consequences, such as De-Turck trick).

We are interested in the following boundary problem:

- $Lu = f$ in U, where U is an open and bounded subset of $\mathbb{R}^n$
- $u = 0$ in ∂U.

Where u: $U \to \mathbb{R}$ is the unknown and L is a second order partial differential operator which can be written in one of two possible ways:

1. $Lu = -\sum_{i,j=1}^{n}(a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u$ or
2. $Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u$

Where a, b, c are component functions. We say that the PDE $Lu = f$ is in divergence form if it's written as in (1) and it is in non divergence form if it is written as in (2). We are mainly interested in the expression taken by L in (2)

Definition 5.2.1. L is uniformly elliptic if there exists a constant $\theta \geqslant 0$ such that:

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta\|\xi\|^2.$$

for every x in U and every ξ vector in $\mathbb{R}^n$

Therefore, ellipticity implies that for every point x in U the symmetric matrix $n \times n$: $A(x) = (a^{ij}(x))$ is positive definite and it's smallest eigenvalue is greater or equal than θ. If $a^{ij} = \delta_{ij}$ and the others are zero, then L is $-\Delta$. Furthermore, solutions to the general elliptic partial differential equation $Lu = 0$ are similar to harmonic functions (such as many of the functionals in Riemannian geometry, hence the interest). These methods for the maximum principle are built upon the observation that if a C^2-function u reaches it's maximum over an open subset U of $\mathbb{R}$ in $x_0 \in U$ then:

$$Du(x_0) = 0 \tag{5.2.1}$$

$$D^2u(x_0) \leq 0 \tag{5.2.2}$$

therefore (5.2.2) tells us that Hess (u) is definite non-positive in x_0. So conclusions of these observations can be applied only locally. Generally, we want functions u to be C^2 to make sense when evaluating D y D^2, even though from regularity theory we already know weak solutions are smooth if the coefficients functions are regular enough. Recalling the non divergent form of L: $Lu = -\sum_{i,j}^n a^{ij}u_{x_ix_j} + \sum_{i=1}^n b^i u_{x_i} + cu$,

where the coefficients are continuous and the uniform ellipticity condition is satisfied and we have an additional symmetry: $a^{ij} = a^{ji}$.

Theorem 5.1. *Suppose $u \in C^2(U) \cap C(\overline{U})$ and $c = 0$ in U, then:*
(i). If $Lu \leq 0$ in U then $\max_{\mathring{U}} u = \max_{\partial U} u$

(ii). If $Lu \geq 0$ in U then $\min_{\mathring{U}} u = \min_{\partial U} u$

A function that satisfies (i) is called a subsolution and a function that satisfies (ii) is called a supersolution. We are interested in subsolutions with maximum values in the boundary and supersolutions with minimum values in the boundary.

2. *Proof.* Suppose $Lu \lneq 0$ in U y and there exist a point x_0 in U such that: $u(x_0) = \max_{\mathring{U}} u$, then as it is a maximum it satisfies:

$$Du(x_0) = 0$$

and

$$D^2u(x_0) \leq 0.$$

Given $A = (a^{ij}(x_0))$ symmetric and positive definite, there exists P orthogonal matrix such that:
$PAP^T = diag(p_1, ..., p_n)$, where $P \cdot P^T = Id$ and all the $d?`p_k$'s are positive. Writing $y = x_0 + OP(x - x_0)$ we have: $x - x_0 = P^T(y - x_0)$ and so:
$u_{x_i} = \sum_{i,j=1}^n u_{y_k} p_{i_k}$.
$u_{x_ix_j} = \sum_{k,l}^n u_{y_ky_l} p_{i_k} \cdot p_{j_l}$
and in a point x_0 we have

$\sum_{i,j=1}^n a^{ij}u_{x_ix_j} = \sum_{k,l=1}^n \sum_{i,j=1}^n a^{ij}u_{y_ky_l} o_{i_k} o_{j_l} = \sum_{k=1}^n d_k u_{y_ky_k}$ this by definition of P. then in x_0 we can write Lu

$$Lu = -\sum_{i,j=1}^n a^{ij}u_{x_ix_j} + \sum_{i=1}^n b^i u_{x_i} \geq 0$$

so the initial supposition dos not hold and we have a contradiction.

In the general case: $Lu \leq 0$, we write:

$$u^{\epsilon}(x) = u(x) + \epsilon e^{\lambda x_1}$$

donde $\lambda, \epsilon \geq 0$. Given that uniformly elliptic implies $a^{ii}(x) \geq \theta$, then:
$Lu^{\epsilon} = Lu + \epsilon L(e^{\lambda x_1}) \leq \epsilon e^{\lambda x_1}[-\lambda^2 a^{11} + \lambda b^1] \leq \epsilon e^{\lambda x_1}[-\lambda^2\theta + \|b\|_{L^\infty}\lambda] \leq 0$ in U if taking λ sufficiently large. Assuming the existence of a maximum in the interior and P as in the first steps of this proof, one finds the maximum and making $\epsilon \to 0$ we find the first part of the theorem. Finally $-u$ is subsolution if u is a supersolution, the second part of the theorem follows. □

Now if c is positive and not zero and: $u^+ = max(u, 0)$, $u^- = -min(u, 0)$:

Theorem 5.2. *Let u as in the previous theorem and $c \geq 0$ in U, then:*

1. *If $Lu \leq 0$ in U then we have that $max_{\mathring{U}} u \leq max_{\partial U} u^+$.*
2. *If $Lu \geq 0$ in U then we have that $min_{\mathring{U}} u \geq -m_{\partial U} u^-$.*
3. *If $Lu = 0$ inn U, Then the maximum inside and the one in the boundary are the same.*

Proof. Let u be a subsoltuion and let $V = \{x \in U : u(x) \geq 0\}$. Then, define: $Ku := Lu - cu \leq -cu \leq 0$ in V.
K has no zeroth order term and we can apply the previous theoremta to ob n$imax_{\mathring{V}} u = max_{\partial V} u = max_{\partial U} u^+$ and noticing that $(-u)^+ = u^-$ the rest of the theorem follows. □

A more general statement of the maximum principle can be obtained if we have a technical lemma first:

Hopf's Lemma:

Suppose $u \in C^2 \cap C^1(\overline{U})$ and $c = 0$ in U and let $Lu \leq 0$ in U and there is a point $x_0 \in \partial U$ such that $u(x_0) \geq u(x)$ for every x in U. Additionally U satisfies the interior ball condition in x_0, meaning there is $B(x, r) \subset U$ open ball with $x_0 \in \partial U$ such that: $\frac{\partial u}{\partial v}(x_0) \geqslant 0$ where v is the unitary normal exterior vector to B in x_0.
If $c \geq 0$ in U, then $u(X_0) \geq 0$ y $\frac{\partial u}{\partial v}(x_0) \geqslant 0$

Proof. A detailed proof can be found in the pages (330-332) of [20] □

Finally:

Theorem 5.3. *Suppose $u \in C^2(U) \cap C(\overline{U})$ and $c = 0$ in U an open, connected and bounded subset of R^n.*
1.$If Lu \leq 0$ in U and u has a maximum in $\overline{U}$ at an interior point then u is constant within U
2. If $Lu \geq 0$ in U and has a minimuum in $\overline{U}$ at an interior point then u is constant within U.

Proof. Let's define $M := max_{\overline{U}} u$ and let C:=$\{x \in U : u(x) = M\}$, so if u is not in M we have $V := \{x \in U : u(x)M\}$
take y in V such that: $d(y, C) \leq d(y, \partial U)$ and let B the ball of biggest possible radius with center in y and whose interior is totally contained in V. Then there is p in V such that this point is in the boundary of B. Clearly V satisfies the condition of the interior ball and therefore if we apply Hopf's lemma $\frac{\partial u}{\partial v}(p) \geqslant 0$. But this is a contradiction as if u has a maximum in p then $Du(p) = 0$ □

If c is positive, there is also a version of the maximum principle:

Theorem 5.4. *Suppose $u \in C^2(U) \cap C(\overline{U})$ and $c \geq 0$ in U open, connected and bounded subset of R^n.*
1.*$If\, Lu \leq 0$ in U and u has a non negative maximum in $\overline{U}$ in an interior point then u is consant within U*
2. If $Lu \geq 0$ in U and u has a non positive minimum in $\overline{U}$ in an interior point then u is constant within U.

Proof. Is the same as the previous theorem but we apply the second part of Hopf's lemma, not the first one.

□

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