On finite groups with many supersoluble subgroups

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Abstract

The solubility of a finite group with less than 6 non-supersoluble subgroups is confirmed in the paper. Moreover we prove that a finite insoluble group has exactly 6 non-supersoluble subgroups if and only if it is isomorphic to A_5 or $\mathrm{SL}_2(5)$. Furthermore, it is shown that a finite insoluble group has exactly 22 non-nilpotent subgroups if and only if it is isomorphic to A_5 or $\mathrm{SL}_2(5)$. This confirms a conjecture of Zarrin [*Arch. Math. (Basel)*, 99 (2012), 201–206].

Keywords: finite group; supersoluble subgroup; soluble group

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1 Introduction

Throughout this paper, G always denotes a finite group.

The results of the present article are motivated by a paper of Zarrin [13], where an extension of the classical result of Schmidt [10] about the solubility of a group with all proper subgroups nilpotent is proved. Zarrin showed that if a group G has at most 21 non-nilpotent subgroups, then G is soluble. He also proposed the following conjecture.

Conjecture 1.1. *Let G be an insoluble group. Then G has exactly 22 non-nilpotent subgroups if and only if it is isomorphic to A_5 or $\mathrm{SL}_2(5)$.*

Our first main result confirms that conjecture.

Theorem A. *Let G be an insoluble group. Then G has exactly 22 non-nilpotent subgroups if and only if it is isomorphic to A_5 or $\mathrm{SL}_2(5)$.*

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On the other hand, Huppert [7] proved that *nilpotent* in Schmidt's theorem can be replaced by *supersoluble* with the same conclusion. Therefore it seems natural to ask: What is the minimum number of non-supersoluble subgroups to guarantee solubility? Our second main result answers this question.

Theorem B. *A group with less than* 6 *non-supersoluble subgroups is soluble.*

Our last result shows that A_5 and $\mathrm{SL}_2(5)$ are the only insoluble groups with exactly 6 non-supersoluble subgroups.

Theorem C. *Let G be an insoluble group. Then G has exactly* 6 *non-supersoluble subgroups if and only if it is isomorphic to A_5 or $\mathrm{SL}_2(5)$.*

The notion that we use is standard and follows that in Doerk and Hawkes [3] or Huppert [8]. We use $\mathrm{SL}_m(q)$ and $\mathrm{PSL}_m(q)$ to denote the special linear group and the projective special linear group, respectively, of dimension m over the field with q elements, where q is a prime power.

2 Proofs

The proofs of our results depend on the following lemmas.

Lemma 2.1. *Let G be a group. The number of non-supersoluble subgroups of $G/\Phi(G)$ is less than or equal to the number of non-supersoluble subgroups of G.*

This follows from the fact that if $H/\Phi(G)$ is a non-supersoluble subgroup of $G/\Phi(G)$, then H is a non-supersoluble subgroup of G.

Recall that a minimal simple group is a simple group whose maximal subgroups are soluble. Suppose that N is a non-trivial proper normal subgroup of a group G such that $\Phi(G) = 1$ and that all maximal subgroups of G are soluble. Then there exists a maximal subgroup M of G such that $G = NM$. Since by hypothesis N and M are soluble, then G is soluble. This implies the following result.

Lemma 2.2. *Let G be a non-soluble group whose maximal subgroups are soluble. Then $G/\Phi(G)$ is a minimal simple group.*

We will use the symbol $\delta(n)$ to denote the number of natural divisors of the natural number n.

Lemma 2.3. *The number of non-supersoluble subgroups of a minimal simple group is at least* 6*. The only minimal simple group with exactly* 6 *non-supersoluble subgroups is A_5.*

Proof. By [12] (see also [8, Kapitel II, Bemerkung 7.5]), G is isomorphic to one of the following groups:

1. $\mathrm{PSL}_2(p)$, where $p > 3$ is a prime and $5 \nmid p^2 - 1$;

2. $\mathrm{PSL}_2(2^q)$, where q is a prime;

3. $\mathrm{PSL}_2(3^q)$, where q is an odd prime;

4. $\mathrm{PSL}_3(3)$;

5. a Suzuki group $\mathrm{Sz}(2^q)$, where q is an odd prime.

It will be enough to show that in all these cases the number of non-supersoluble subgroups of G is at least 6.

The subgroups of $\mathrm{PSL}_2(p^f)$ have been studied in [2] (see also [8, Kapitel II, Satz 8.27]). These subgroups fall into the following classes:

1. elementary abelian p-groups;

2. cyclic p-groups of order z, where z divides $(p^f \pm 1)/k$ and $k = \gcd(p^f - 1, 2)$;

3. dihedral groups of order $2z$ where z is as in 2 above;

4. alternating groups A_4 for $p > 2$ or $p = 2$ and $f \equiv 0 \pmod 2$;

5. symmetric groups Σ_4 for $p^{2f} - 1 \equiv 0 \pmod{16}$;

6. alternating groups A_5 for $p = 5$ or $p^{2f} - 1 \equiv 0 \pmod 5$;

7. semidirect products of elementary abelian groups of order p^m with cyclic groups of order t; here $t \mid p^m - 1$ and $t \mid p^f - 1$;

8. groups $\mathrm{PSL}_2(p^m)$ for $m \mid f$ and $\mathrm{PGL}_2(p^m)$ for $2m \mid f$.

Recall that, by [8, Kapitel II, Hilfssatz 6.2],

$$|\mathrm{PSL}_2(p^f)| = p^f(p^f - 1)(p^f + 1)/\gcd(2, p^f - 1).$$

Assume that $G \cong \mathrm{PSL}_2(p)$ with $p > 3$ a prime and $5 \nmid p^2 - 1$. Since $\mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4)$, we can assume that $p > 5$. Therefore the only non-supersoluble proper subgroups of G are of the form A_4 for $p > 2$ and, when $p^2 - 1 \equiv 0 \pmod{16}$, Σ_4. If $p^2 - 1 \equiv 0 \pmod{16}$, there are two conjugacy classes of subgroups isomorphic to A_4 with normaliser isomorphic to Σ_4. In this case, the number of non-supersoluble proper subgroups isomorphic to A_4 or Σ_4 of G is $4p(p-1)(p+1)/(2 \cdot 24) = p(p-1)(p+1)/12$. Therefore the number of non-supersoluble subgroups is $p(p-1)(p+1)/12 + 1 \geq 7 \cdot 6 \cdot 8/12 + 1 = 29$. Note that, in the previous argument, we add 1 because we are counting the non-supersoluble subgroups of G, not only the proper non-supersoluble subgroups of G. Otherwise, there is a unique conjugacy class of self-normalising subgroups isomorphic to A_4. The number of such subgroups is the index of its normaliser, namely $p(p-1)(p-2)/24$. Hence the number of non-supersoluble subgroups is $p(p-1)(p+1)/24 + 1 \geq 11 \cdot 10 \cdot 12/24 + 1 = 56$.

Assume now that $G \cong \mathrm{PSL}_2(2^q)$, with q a prime number. If $q = 2$, then $G \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$ has 5 subgroups isomorphic to A_4 and so it has 6 non-supersoluble subgroups. Therefore we can suppose that $q \geq 3$. In this case, the only possibility for a proper non-supersoluble subgroup of G has the following structure: It must be a semidirect product of an elementary abelian group of order 2^m with a cyclic group of order t with $t \mid 2^q - 1$ and $t \mid 2^m - 1$. Since q is a prime, $m = q$. The normalisers of all these subgroups are semidirect products of an elementary abelian group of order 2^q with a cyclic subgroup of order $2^q - 1$. Each of these normalisers in G has order $2^q(2^q - 1)$ and index $2^q + 1$ in G. It follows that the number of all non-supersoluble proper subgroups of G is $(2^q+1)\big(\delta(2^q-1)-1\big)$. Hence the number of non-supersoluble subgroups of G is greater than or equal to $2^q + 1 + 1 = 2^q + 2 \geq 2^3 + 2 = 10$.

Assume now that $G \cong \mathrm{PSL}_2(3^q)$ with q an odd prime. There is a unique conjugacy class of self-normalising subgroups isomorphic to A_4 and, since $3^{2q} - 1 \equiv 8 \pmod{16}$, there are no symmetric subgroups. The number of subgroups isomorphic to A_4 is $3^q(3^q - 1)(3^q + 1)/24 = 3^{q-1}(3^q - 1)(3^q + 1)/8 \geq 3^2(3^3 - 1)(3^3 + 1)/8 = 819$.

Assume now that $G \cong \mathrm{PSL}_3(3)$. A calculation with GAP [4] shows that G possesses 1 093 non-supersoluble subgroups.

Finally, assume that $G \cong \mathrm{Sz}(2^q)$ with q an odd prime. The order of G is $2^{2q}(2^{2q}+1)(2^q-1)$ by [11]. According to [1, Table 8.16], G has a unique conjugacy class of maximal subgroups of G of type $[E_{2^q}^{1+1}]C_{2^q-1}$ and order $2^{2q}(2^q-1)$. Hence G has $2^{2q} + 1$ subgroups of this type and, therefore, the number of non-supersoluble subgroups of G is at least $(2^{2\cdot 3} + 1) + 1 = 66$. □

Proof of Theorem A. If $G \cong A_5$ or $G \cong \mathrm{SL}_2(5)$, then it is routine to check that G has exactly 22 non-nilpotent subgroups.

Conversely, assume that G has exactly 22 non-nilpotent subgroups. Let H be a maximal subgroup of G. If H is nilpotent, then H is certainly soluble. If H is non-nilpotent, then H has less than 22 non-nilpotent subgroups. By [13, Theorem A], H is soluble. It follows that G is a minimal non-soluble group, and so $G/\Phi(G)$ is a minimal simple group. Then, according to [13, Theorem A], $G/\Phi(G) \cong A_5$ and $G/\Phi(G)$ has exactly 22 non-nilpotent subgroups, and every second maximal subgroup of $G/\Phi(G)$ is nilpotent. Hence every second maximal subgroup of G is nilpotent. By [9, Satz], $G \cong A_5$ or $\mathrm{SL}_2(5)$, as desired. □

Proof of Theorem B. Assume that the number of non-supersoluble subgroups of a group G is less than 6. We prove that G is soluble by induction on $|G|$. Clearly, we may assume that every maximal subgroup of G is soluble. If G were not soluble, $G/\Phi(G)$ would be a minimal simple group with less than 6 non-supersoluble subgroups by Lemmas 2.1 and 2.2. This would contradict Lemma 2.3. Therefore G is soluble, as desired. □

Proof of Theorem C. Assume that G has exactly 6 non-supersoluble subgroups. Suppose, arguing by contradiction, that G is not isomorphic to A_5 or $\mathrm{SL}(2,5)$. Let us choose G of least order. Since G is not soluble, G contains a minimal non-soluble subgroup S. By Lemma 2.2, $S/\Phi(S)$ is a minimal simple group. By Lemma 2.3, the only minimal simple group with at most 6 non-supersoluble subgroups is A_5. If $S < G$, then the number of non-supersoluble subgroups of S is less than the number of non-supersoluble subgroups of G, and so is the number of non-supersoluble subgroups of $S/\Phi(S)$. Hence $S = G$. By Lemma 2.3, $G/\Phi(G)$

is isomorphic to A_5. If $\Phi(G) = 1$, then $G \cong A_5$, contrary our supposition. Hence $\Phi(G) \neq 1$. Let $\Phi(G)/K$ be a chief factor of G. By [8, Kapitel III, Satz 3.6 and Satz 3.8], $\Phi(G)$ is a nilpotent $\{2, 3, 5\}$-group. By a result of Gaschütz [5], G/K is a quotient of a universal Frattini extension with elementary abelian kernel. Suppose that $\Phi(G)/K$ is a 3-group. Then $\Phi(G)/K$ is an irreducible module of dimension 4 for A_5 by [6, Example 1]. In this case, given a Sylow 5-subgroup C/K of G/K, $\Phi(G)C/K$ is a non-supersoluble subgroup of G/K. On the other hand, let $T/\Phi(G)$ be one of the 6 non-supersoluble subgroups of $G/\Phi(G)$. Then T/K is also a non-supersoluble subgroup of G/K. Moreover $\Phi(G)C/K$ cannot be obtained in this way because $\Phi(G)C/\Phi(G)$ is supersoluble. Hence G/K has more than 6 non-supersoluble subgroups, and the same can be said about G. Suppose that $\Phi(G)/K$ is a 5-group. Then $\Phi(G)/K$ is an irreducible module of dimension 3 for A_5 by [6, Example 1], namely, the head of the corresponding Frattini module. A Sylow 3-subgroup T/K of G/K does not centralise $\Phi(G)/K$ since $\Phi(G)/K$ is acted on faithfully by $G/\Phi(G)$. Therefore, by [3, Chapter A, Proposition 12.5], $[T/K, \Phi(G)/K]$ is a non-trivial normal subgroup of $\Phi(G)T/K$ on which T/K acts faithfully. In particular, $[T/K, \Phi(G)/K](T/K)$ cannot be supersoluble, since 3 does not divide $5 - 1$. Arguing as above, we conclude that G/K has more than 6 non-supersoluble subgroups. In particular, we can assume that $\Phi(G)$ is a 2-group. By [6, Example 1], the only possibility for $\Phi(G)/K$ is that $\Phi(G)/K$ has order 2, the head of the corresponding Frattini module. Therefore $G/K \cong \mathrm{SL}_2(5)$. Suppose that K/L is a chief 2-factor of G. Note that K/L is an irreducible module for $\mathrm{SL}_2(5)$ and so for A_5, by [3, Chapter B, Proposition 3.12]. By [8, Kapitel V, Satz 25.5], the Schur multiplier of $\mathrm{SL}_2(5)$ is trivial. It follows that K/L is not cyclic and so it has dimension 4 ([6, Example 1]). By considering a Sylow 5-subgroup C of G, we obtain that $\Phi(G)C/L$ is not supersoluble. As above, G/L has more than 6 non-supersoluble subgroups, and the same can be said about G, contrary to assumption. We conclude that $G \cong A_5$ or $G \cong \mathrm{SL}_2(5)$.

The converse is clear by Lemma 2.3. □

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References

[1] J. Bray, D. Holt, and C. Roney-Dougal. *The maximal subgroups of the low-dimensional finite classical groups*, volume 407 of *London Math. Soc. Lect. Note Ser.* Cambridge Univ. Press, Cambridge, UK, 2013.

[2] L. E. Dickson. *Linear groups: With an exposition of the Galois field theory.* Dover Publications Inc., New York, 1958.

[3] K. Doerk and T. Hawkes. *Finite Soluble Groups*, volume 4 of *De Gruyter Expositions in Mathematics.* Walter de Gruyter, Berlin, New York, 1992.

[4] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.8.6*, November 2016. `http://www.gap-system.org`.

[5] W. Gaschütz. Über modulare Darstellungen endlicher Gruppen, die von freien Gruppen induziert werden. *Math. Z.*, 60:274–286, 1954.

[6] R. L. Griess and P. Schmid. The Frattini module. *Arch. Math.*, 30:256–266, 1978.

[7] B. Huppert. Normalteiler und maximale Untergruppen endlicher Gruppen. *Math. Z.*, 60:409–434, 1954.

[8] B. Huppert. *Endliche Gruppen I*, volume 134 of *Grund. Math. Wiss.* Springer Verlag, Berlin, Heidelberg, New York, 1967.

[9] Z. Janko. Endliche Gruppen mit lauter nilpotenten zweitmaximalen Untergruppen. *Math. Z.*, 79:422–424, 1962.

[10] O. J. Schmidt. Über Gruppen, deren sämtliche Teiler spezielle Gruppen sind. *Mat. Sbornik*, 31:366–372, 1924.

[11] M. Suzuki. A new type of simple groups of finite order. *Proc. Natl. Acad. Sci. USA*, 46:868–870, 1960.

[12] J. G. Thompson. Nonsolvable finite groups all of whose local subgroups are solvable. *Bull. Amer. Math. Soc.*, 74:383–437, 1968.

[13] M. Zarrin. A generalization of Schmidt's theorem on groups with all subgroups nilpotent. *Arch. Math. (Basel)*, 99:201–206, 2012.