Thickness Design for Ambiguous Cylinder Illusion

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Abstract

This paper proposes methods for giving as uniform a thickness as possible to a class of illusion solids called ambiguous cylinders. Ambiguous cylinders are solids that have two quite different appearances when seen from two specific viewpoints, and thus create the impression of impossible objects. In order to realize them as physical objects, we have to give them thickness. However, it is impossible to give a completely uniform thickness despite this being desirable. Instead we have to content ourselves with second-best methods. For this purpose, this paper proposes three alternative strategies for creating objects as uniform as possible. Each strategy has its own merits and demerits, and hence users can choose their method according to their priorities for the visual effects which they want to emphasize.

Keywords: Optical illusion, impossible object, anomalous cylinder, ambiguous object, uniform thickness.

1 Introduction

Optical illusions are perceptual phenomena in which what we see differs from reality. Historically, optical illusions were found using 2D pictures [6–8, 10, 11], but recently many new illusions have been discovered or invented using 3D objects [3,13,14,17]. Optical illusions can be mysterious, surprising, inspiring, and beautiful. Therefore, they are potential resources for new products in many industrial fields such as education, architecture, museum curation, entertainment, art, advertising, and tourism.

The author recently found a method for designing cylindrical objects that each have two desired appearances when seen from two specific viewpoints. This class of objects creates an optical illusion when viewed directly and in a mirror simultaneously. For example, a cylinder with a circular cross-section changes to a cylinder with a rectangular cross-section, a full moon cross-section to a star-shaped section, and a diamond-shaped cross-section to a heart-shaped cross-section. These are called *ambiguous cylinders* because their appearances

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are ambiguous [15]. This concept was also extended to “partly invisible objects” for which parts of the objects disappear in a mirror [16].

The ambiguous cylinder illusion can be considered as a variant of the well-known illusion called trompe l’œil or anamorphosis, in which pictures painted on planar surfaces or on the surfaces of 3D objects give meaningful appearances only when they are seen from a unique special viewpoint [11, 17]. Ambiguous cylinders, on the other hand, give meaningful appearances when seen from two special viewpoints. In this sense, the ambiguous cylinder illusion might be regarded as double trompe l’œil illusion.

However, the design method proposed in [15] only creates surfaces without thickness. This is okay conceptually, but not satisfactory if we want to construct physical models because thickness is necessary to make the objects rigid.

Ideally we want to make the cylinder surfaces such that they have uniform thickness. That is, we want to construct a cylinder so that its thickness appears to be uniform from both of the viewpoints. However, this requirement is not consistent with the nature of an ambiguous cylinder. There is a trade-off between uniform thickness and the quality of the appearances: If we want to achieve uniform thickness, we have to change the appearances of the cylinder from the desired ones, and if we want to keep the desired appearances, we have to give up having complete uniformness of thickness.

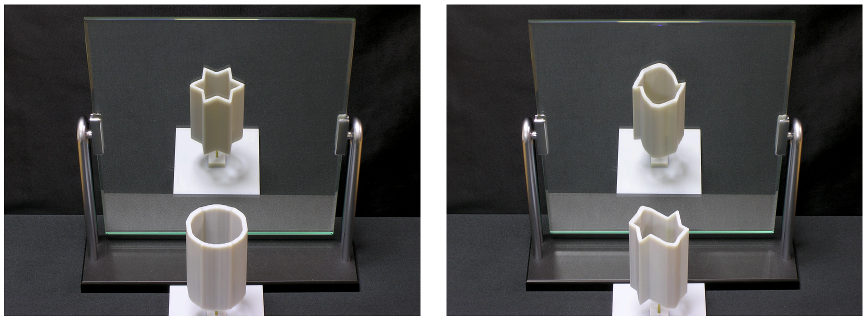
In this paper, we present several alternative methods for handling this inconsistency. Which method to use depends on the pair of appearances of the ambiguous cylinder and the preference of the designer. Hence, the contribution of this paper is to offer a menu from which designers can select the method that is most suitable for their aims and preferences.

We first review the method for constructing ambiguous cylinders without thickness (section 2) before presenting the problem of thickness design (section 3). Then, we define some concepts and procedures as preparation (section 4), propose three methods for handling the inconsistency (section 5), show additional examples (section 6), and give concluding remarks (section 7).

2 Review of Ambiguous Cylinders

An ambiguous cylinder is a cylindrical object that has two different appearances when seen from two different specific viewpoints. It creates an optical illusion in the sense that the two appearances are so different that it is almost unbelievable that they could come from the same object. In particular, if the cylinder is reflected in a mirror, the viewer has the impression that the object changes to another object in the mirror instead of being a normal mirror image.

Fig. 1 shows an example of an ambiguous cylinder. In this figure, (a) shows a scene with an object and a vertical mirror behind it. In the direct view of the object, it appears to be a cylinder whose cross-section is a circle, in other words, the outline of a full moon, while the mirror image appears to be a cylinder with a star-shaped cross-section. Both the full moon shape and the star shape appear to be planar curves obtained by cutting the cylinder with a horizontal plane, but actually the upper edge of the cylinder is a non-planar space curve. This is the trick that makes the two appearances quite different.



(a) (b)

Figure 1: Ambiguous cylinder "Full Moon and Star": (a) special view; (b) general view.

The method for designing an ambiguous cylinder is based on two observations, one mathematical and the other psychological. Mathematically, a 2D image of a 3D object lacks depth information and hence there are infinitely many 3D objects whose projections coincide with a given 2D image [12]. This property allows us to create an object that has two desired appearances when seen from two specific viewpoints.

Psychologically, on the other hand, we usually perceive a specific 3D object from a 2D picture although there are infinitely many possible interpretations. It has been observed that the human vision system prefers rectangular structures, and hence tries to interpret a picture as an object that has as many 90-degree angles as possible [9]. Hence, we can cheat the vision system in such a way that the edge curve of a cylinder appears as if it were a planar curve perpendicular to the axis of the cylinder. Specifically, we can construct an ambiguous cylinder in the following way [15].

Let (x, y, z) be the Cartesian coordinate system where the xy plane is horizontal and the positive z axis directs upward. As shown in Fig. 2, let H be the xy plane, and E_1 and E_2 be two viewpoints that are above H. We assume that the line $\mathrm{E}_1\mathrm{E}_2$ is parallel to the y axis, where for any two points P and Q, the line passing through P and Q is referred to as the line PQ.

Let A and B be two curves on H. We assume that A and B are x-monotone and their start points share the same x coordinate and their end points also share the same x coordinate. Therefore, these curves can be represented in parametric forms with the common parameter x. Without loss of generality, we denote the two curves as $A(x)$ and $B(x)$, $0 \le x \le 1$. Our goal is to construct a space curve, say $C(x)$, that coincides with $A(x)$ when it is seen from E_1, and coincides with $B(x)$ when seen from E_2.

For any fixed x, $0 \le x \le 1$, the four points $\mathrm{E}_1, \mathrm{E}_2, A(x)$, and $B(x)$ are on a

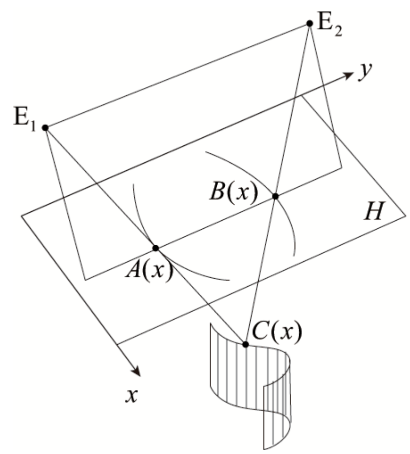


Figure 2: Space curve that has appearances of two desired planar curves.

common plane because the line E_1E_2 and the line $A(x)B(x)$ are parallel. Hence, the line $E_1A(x)$ and the line $E_2B(x)$ intersect. Let the point of intersection be denoted by $C(x)$. Note that the point $C(x)$ coincides with $A(x)$ when it is seen from E_1 and coincides with $B(x)$ when seen from E_2.

We call the obtained $C(x)$ the *ambiguous curve* generated from $A(x)$ and $B(x)$. Note that $C(x)$ is defined unambiguously from $A(x), B(x)$, E_1, and E_2. The term "ambiguous" is used simply because the curve can generate an ambiguous cylinder in the following way.

Note that the geometric fact that $C(x)$ coincides with $A(x)$ when seen from the first viewpoint E_1 does not necessarily mean that $C(x)$ appears to be $A(x)$. This is because a 2D image does not have depth information and hence the image of $C(x)$ taken at E_1 has infinitely many interpretations of the corresponding space curve, including $A(x)$ and $C(x)$. In order to force the vision system to choose $A(x)$ as the perceived interpretation, we create a surface swept by a line segment that is perpendicular to H in the following way.

Let $\boldsymbol{e}_z$ be a unit vector in the z direction, and h be a positive real number. We define surface S by

$$S = \{C(x) - u\boldsymbol{e}_z \mid 0 \leq x \leq 1, 0 \leq u \leq h\}. \tag{1}$$

Intuitively, S is obtained as the surface swept by a vertical line segment with length h when it moves in such a way that the upper terminal point travels along the curve $C(x)$.

Note that the vertical length of this surface is equal to h no matter where we measure it. Hence, if this surface is seen from E_1, we can expect it to appear to be a cylindrical surface and its upper edge to appear to be the planar curve on a plane perpendicular to the axis of the cylinder i.e., the curve $A(x)$ on H. For the same reason, if this surface is seen from E_2, we can expect the upper edge of the surface to appear to be the planar curve perpendicular to the axis, i.e., the curve $B(x)$.

A closed cylinder such as the one in Fig. 1 can be constructed by applying the above method twice, once to the upper pair of monotone curves and once more to the lower pair of monotone curves. More complicated shapes can also be handled if the boundaries of the two shapes can be decomposed into a finite number of pairs of monotone curves such that each pair spans the same x range; that is, the left terminal points share the same x coordinate and the right terminal points also share the same x coordinate.

3 Problem of Uniform Thickness

In order to understand the difficulty of attaining uniform thickness, let us consider the full moon shape and star shape shown in Fig. 3. First, let us concentrate on the outer boundaries of the two shapes. Each boundary can be divided into an upper curve and a lower curve, both of which are horizontally monotone. Moreover, the horizontal widths of the two shapes are the same, as shown by the rightmost and leftmost vertical dashed lines. Therefore, we can apply our method for creating an ambiguous cylinder surface S without thickness.

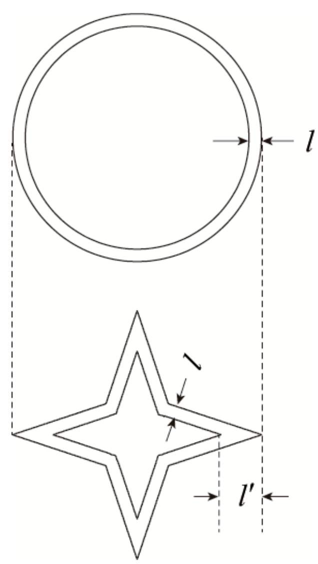


Figure 3: Uniform thickness given to two 2D figures.

Next, we consider how to add uniform thickness to this cylinder. Here, what we want is not only to add uniform thickness to each of the two appearances but also to make their thicknesses the same. If we add thickness with these properties to the full moon and the star, we get the inner boundaries of the two shapes shown in Fig. 3. The thickness in this figure is uniform in the sense that the perpendicular distance between the outer and inner boundaries is l. We will define uniform thickness formally a little later; here let us accept that the inner curve of the star in Fig. 3 gives an intuitively uniform thickness. However, as shown by the distance l' in Fig. 3, the horizontal thickness of the rightmost point of the star is larger than l, because it forms a sharp corner. On the other hand, the horizontal thickness at the rightmost point of the full moon remains l because the tangent is vertical. Therefore, the horizontal widths of the two inner boundaries are not the same. The inner boundary of the full moon is larger than that of the star. This implies that we cannot establish a one-to-one correspondence between the two inner boundaries, and consequently we cannot construct an ambiguous cylinder surface from the pair of inner boundaries. Thus, attaining uniform thickness is not consistent with constructing the desired ambiguous cylinder. Therefore, we have to consider ways to compromise in this situation of having incompatible aims.

4 Thinning, Fattening, and Bridging

The following are some concepts and operations that are necessary for discussing our strategies.

Let C be a closed curve in 2D space, and let $\mathrm{In}(C)$ and $\mathrm{Out}(C)$ represent the regions interior and exterior, respectively, to C. For two points P and Q, let $d(\mathrm{P}, \mathrm{Q})$ denote the Euclidean distance between P and Q. We extend the distance d to the distance between point P and region X by

$$d(\mathrm{P}, X) = \inf_{\mathrm{Q} \in X} d(\mathrm{P}, \mathrm{Q}). \tag{2}$$

Let us define a thinning operation by

$$\mathrm{Thin}(C, h) = \{\mathrm{P} \mid \mathrm{P} \in \mathrm{In}(C), d(\mathrm{P}, C) \geq h\}, \tag{3}$$

where h is a positive constant. $\mathrm{Thin}(C, h)$ is the region composed of points that are in $\mathrm{In}(C)$ and whose distances from the boundary C are not less than h. We call the boundary of $\mathrm{Thin}(C, h)$ the *inner offset curve* of C by h [2, 5].

Similarly, we define a fattening operation by

$$\mathrm{Fat}(C, h) = \{\mathrm{P} \mid d(\mathrm{P}, \mathrm{In}(C)) \leq h\}. \tag{4}$$

$\mathrm{Fat}(C, h)$ is the region composed of points whose distances from $\mathrm{In}(C)$ are not greater than h. Note that if $\mathrm{P} \in \mathrm{In}(C), d(\mathrm{P}, \mathrm{In}(C)) = 0$ and hence $\mathrm{Fat}(C, h)$ includes $\mathrm{In}(C)$. We call the boundary of $\mathrm{Fat}(C, h)$ the *outer offset curve* of C by h.

Fig. 4 shows examples of the inner and outer offset curves. There are three closed curves; the middle is the original curve, and the other two are the inner and outer offset curves.

Let P be a corner point on the curve C. If the angle measured in $\mathrm{In}(C)$ is less than 180 degrees, the corner is called a *convex corner*, and if it is greater

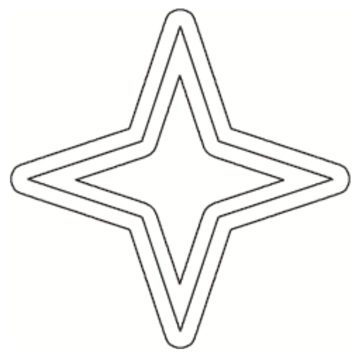


Figure 4: Inner and outer offset curves.

than 180 degrees, it is called a *reflex corner*. The sharpness of a corner is not necessarily preserved in the offset curves. As we can see in Fig. 4, a convex corner corresponds to a smooth curve in the outer offset curve, while a reflex corner corresponds to a smooth curve in the inner offset curve.

In order to preserve the sharpness of corners, we can modify the offset curves. As shone in Fig. 5, let $C(t)$ be a closed curve in 2D space and P be a convex corner. Let us denote by C_1 and C_2 the two local parts of $C(t)$ divided at P. We generate the outer offset curve of C_1 and that of C_2 by displacing points on $C(t)$ perpendicular to their tangents. Let P_1 and P_2 be the points displaced from P in the directions perpendicular to the tangents of C_1 and C_2, respectively. In the original offset curve, P_1 and P_2 are connected by a circular arc as shown by the dashed curve in Fig. 5. Instead, we connect P_1 and P_2 by the two line segments parallel to the tangents of C_1 and C_2 at P as shown by the solid lines in Fig. 5. In this way, we obtain a curve in which the sharpness of the convex corner is preserved. We call this curve the *sharpness-preserved outer offset curve*.

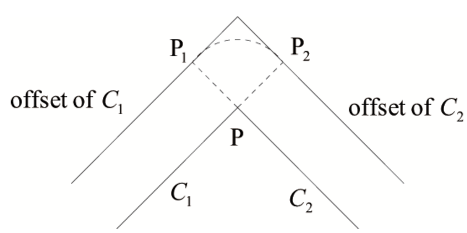


Figure 5: Sharpness-preserved offset curve.

We can apply the same procedure to reflex corners when constructing an inner offset curve. We call the resulting curve the *sharpness-preserved inner offset curve*.

Note that the thickness of the star in Fig. 3 is created by the sharpness-preserved offset curve.

For any point P in 3D space, we denote its Cartesian coordinates by $(\mathrm{P}_x, \mathrm{P}_y, \mathrm{P}_z)$ and the projection of P on the xy plane by P_{xy}, i.e., $\mathrm{P}_{xy} = (\mathrm{P}_x, \mathrm{P}_y)$. Let $P(s)$ and $Q(t)$ be two closed curves in 3D space parameterized by s and t, $0 \le s, t \le 1$, $P(0) = P(1)$, and $Q(0) = Q(1)$. We assume that their projections $\mathrm{P}_{xy}(s)$ and $\mathrm{Q}_{xy}(t)$ are simple non-intersecting curves and that $\mathrm{Q}_{xy}(t)$ is inside $\mathrm{P}_{xy}(s)$. Let R denote the 2D region bounded by $\mathrm{P}_{xy}(s)$ and $\mathrm{Q}_{xy}(t)$; that is, $\mathrm{P}_{xy}(s)$ is the outer boundary and $\mathrm{Q}_{xy}(t)$ is the inner boundary of R.

Suppose that $f(x, y)$ is a real-valued continuous function defined in R such that

$$f(x,y) = \begin{cases} \mathrm{P}_z(s) & \text{if } (x,y) = \mathrm{P}_{xy}(s), \\ \mathrm{Q}_z(t) & \text{if } (x,y) = \mathrm{Q}_{xy}(t). \end{cases} \tag{5}$$

We define

$$F = \{(x, y, f(x,y)) \mid (x,y) \in R\}. \tag{6}$$

F forms a surface connecting the two curves $P(s)$ and $Q(t)$. We call F a *bridging surface* of $P(s)$ and $Q(t)$. In what follows, F is used to cap the top and bottom of the cylindrical solid between the swept surfaces by the two curves $P(s)$ and $Q(t)$, so we want an F that does not have unnecessary undulation. One method for generating such a surface approximately is to first replace $P(s)$ and $Q(t)$ with point strings, next triangulate the region R using the projections of those points as vertices, and finally lift the resulting triangular mesh to 3D space. Methods for this purpose are well studied in surface modeling and computer graphics [1,4], and hence we assume that we can construct bridging surfaces as necessary and do not go into further detail.

5 Strategies for Uniform Thickness

In what follows, we mainly consider closed curves composed of upper and lower monotone curves. For that purpose, we adjust our notation in the following way. Let A be a closed curve in 2D space and $A_1(x)$ and $A_2(x)$ be its upper and lower monotone curves. We parameterize A by t as

$$A(t) = \begin{cases} A_1(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ A_2(2-2t) & \text{if } \frac{1}{2} \le t \le 1. \end{cases} \tag{7}$$

That is, the new parameter t moves from 0 to 1, and the first half $0 \le t \le \frac{1}{2}$ corresponds to the upper curve, and the second half $\frac{1}{2} \le t \le 1$ corresponds to the lower curve, parameterized in the reverse direction. Similarly, for the other closed curve B with the upper monotone curve $B_1(t)$ and the lower monotone curve $B_2(x)$, we define

$$B(t) = \begin{cases} B_1(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ B_2(2-2t) & \text{if } \frac{1}{2} \le t \le 1. \end{cases} \tag{8}$$

For each fixed t, the two points $A(t)$ and $B(t)$ share the same x coordinate, and hence we can apply our method for constructing an ambiguous cylinder. Let

$C(t), 0 \le t \le 1$, be the closed ambiguous curve constructed from $A(t)$ and $B(t)$, and S be the cylindrical surface defined by

$$S = \{C(t) - u\boldsymbol{e}_z \mid 0 \le t \le 1, 0 \le u \le h\}. \tag{9}$$

Our basic idea for generating an ambiguous cylinder with uniform thickness is as follows. We first create another pair of closed curves $A'(t)$ and $B'(t)$ inside or outside $A(t)$ and $B(t)$ so that $A(t)$ and $A'(t)$ give uniform thickness to the first shape and $B(t)$ and $B'(t)$ give uniform thickness to the second shape, and that the leftmost points of $A'(t)$ and $B'(t)$ share the same x coordinate, and the rightmost points of $A'(t)$ and $B'(t)$ also share the same x coordinate. Next we construct the ambiguous space curve $C'(t)$ from $A'(t)$ and $B'(t)$, and the associated cylindrical surface

$$S' = \{C'(t) - u\boldsymbol{e}_z \mid 0 \le t \le 1, 0 \le u \le h\}. \tag{10}$$

Finally, we pack material between S and S', and thus obtain a solid cylinder with finite thickness. These steps can be represented formally as the following algorithm.

Basic algorithm for adding thickness
Input: two closed curves $A(t)$ and $B(t)$, thickness parameter l, and height h.
Procedure:
Step 1. Create closed curves $A'(t)$ and $B'(t)$ inside or outside $A(t)$ and $B(t)$, respectively.
Step 2. Construct the ambiguous space curve $C(t)$ from $A(t)$ and $B(t)$.
Step 3. Construct the ambiguous space curve $C'(t)$ from $A'(t)$ and $B'(t)$.
Step 4. Generate bridging surface F of $C(t)$ and $C'(t)$.
Step 5. Generate cylindrical solid D by

$$D = \{\mathrm{P} - u\boldsymbol{e}_z \mid \mathrm{P} \in F, 0 \le u \le h\}. \tag{11}$$

The set D of points defined in (11) is the volume swept by the bridging surface F in the direction parallel to the z axis by length h. Hence, it is equivalent to the volume obtained by packing material between the two cylindrical surfaces S and S' defined by (9) and (10).

Note that there is freedom in the choice of $A'(t)$ and $B'(t)$ in Step 1, but once $A'(t)$ and $B'(t)$ are set, Steps 2 to 5 are performed without ambiguity. Hence we concentrate on Step 1. Note also that the input parameter l does not appear in the above procedure, but we will use this parameter in Step 1 as described in what follows.

Our first strategy is to use the outer offset curves. Let $A'(t)$ and $B'(t)$ be the outer offset curves of $A(t)$ and $B(t)$, respectively, by l. Note that the rightmost point and the leftmost point of a closed curve are each either a smooth point or a convex corner. Therefore, the horizontal distance between $A(t)$ and $A'(t)$ at the rightmost or leftmost point is the same as l. The same is true of the horizontal distance between $B(t)$ and $B'(t)$. Hence, the horizontal widths of $A'(t)$ and $B'(t)$ are the same, and a one-to-one correspondence can be established between $A'(t)$ and $B'(t)$. Consequently, we can construct the associated ambiguous curve $C'(t)$. Thus, we have the following strategy.

Strategy 1 (fattening strategy)
Step 1-1. Construct the outer offset curve $A'(t)$ of $A(t)$ by l, and the outer offset curve $B'(t)$ of $B(t)$ by l.

This strategy gives two appearances with uniform thickness in the sense that the thickness is determined by the original curves and their outer offset curves. However, this thickness is a little different from our intuitive idea of uniformness because the sharpness of convex corners is lost.

Fig. 6 shows the two appearances of the ambiguous solid created by this strategy. (These are computer graphic images generated by the solid modeling software Rhinoceros version 5.) As we can see in the lower image, the sharpness of the star has been lost. If we use a small value for the thickness parameter l, we will get a thin cylinder with a sharp star as shown in Fig. 7. However, this can be physically realized only when we use a hard material such as metal.

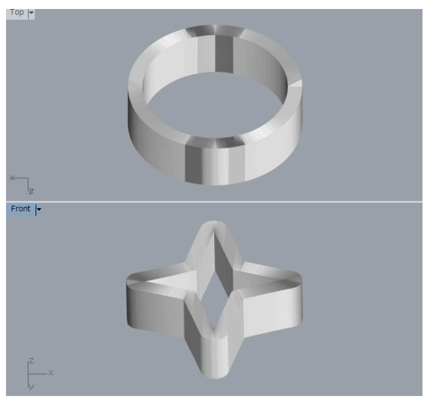


Figure 6: Solid created by Strategy 1 (outer offset strategy); sharpness of the star is lost.

The sharpness-preserved offset curve maintains the feature points of the original curve more faithfully, but the horizontal distance at the rightmost or the leftmost point may become larger. In that case, we can normalize the sharpness-preserved offset curve so that the horizontal distances become equal. Thus we have the following strategy.

Strategy 2 (shrink-and-normalize strategy)
Step 1-2. Construct the sharpness-preserved inner offset curves $A'(t)$ of $A(t)$

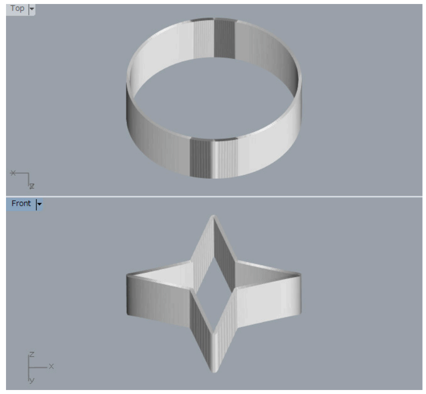


Figure 7: Thin solid created by Strategy 1 (outer offset strategy); it will be fragile unless made of a hard material.

by h, and the sharpness-preserved inner offset curves $B'(t)$ of $B(t)$ by h. Then, expand or shrink the curve $B'(t)$ in such a way that the leftmost and the rightmost points share the same x coordinates as $A'(t)$

Fig. 8 shows the two appearances of the ambiguous solid created by this strategy. We can see that each shape has uniform thickness and that sharp corners are preserved. However, the two shapes do not have the same thickness: the star is thinner than the full moon.

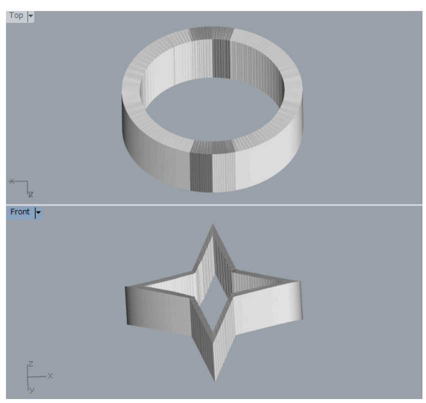


Figure 8: Solid created by Strategy 2 (horizontal width normalization strategy); the star is thinner than the full moon.

We should note that this strategy does not always work. An example is shown in Fig. 9. Suppose that the given curve and the sharpness-preserved inner offset are as shown in (a). Because the horizontal thickness at the rightmost corner is large, the normalized inner curve becomes the dashed line in (b), which intersects with the original curve. Hence, the desired ambiguous solid cannot be created.

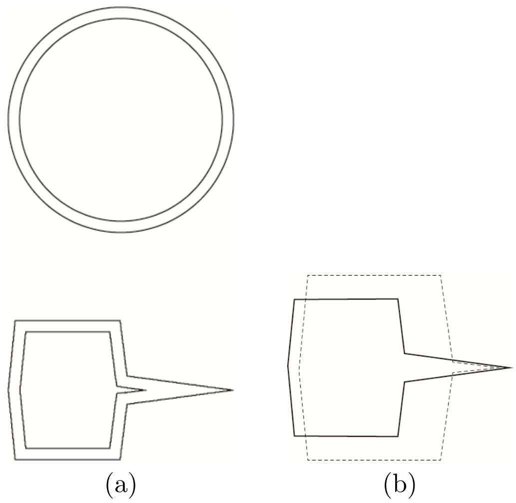


Figure 9: Example in which Strategy 2 does not work: (a) original curve and its inner offset curve; (b) normalized inner curve which intersects the original curve.

Another strategy for making the horizontal widths of the two curves $A'(t)$ and $B'(t)$ the same is to add horizontal line segments as shown in Fig. 10, which is based on the same pair of curves as before, i.e., a full moon $A(t)$ and a star $B(t)$. As shown, we let $A_1(t)$ and $A_2(t)$ [resp. $B_1(t)$ and $B_2(t)$] be the upper and lower monotone curves of $A(t)$ [resp. $B(t)$], and let $A'_1(t)$ and $A'_2(t)$ [resp. $B'_1(t)$ and $B'_2(t)$] be the inner offset curves of $A_1(t)$ and $A_2(t)$ [resp. $B_1(t)$ and $B_2(t)$] by l.

Because $B(t)$ has sharp corners at the rightmost and leftmost points, the horizontal width of the inner offset curve of $B(t)$ is smaller than that of $A(t)$. Instead of expanding it, we add horizontal line segment, say B_{l} and B_{r}, to the left and to the right of the offset curves so that the horizontal width becomes the same as that of the other offset curve $A'(t)$. In other words, we augment $B'_1(t)$ and $B'_2(t)$ by adding B_{l} to the left and B_{r} to the right. Then, the resulting horizontal widths are the same as those of $A'_1(t)$ and $A'_2(t)$, and hence we have one-to-one correspondences between $A'_1(t)$ and $B'_1(t)$, and between $A'_2(t)$ and $B'_2(t)$, enabling us to apply the method of ambiguous cylinder construction. Thus, we have the following strategy.

Strategy 3 (shrink-and-augment strategy)
Step 1-3. Construct the sharpness-preserved inner offset curves $A'(t)$ of $A(t)$

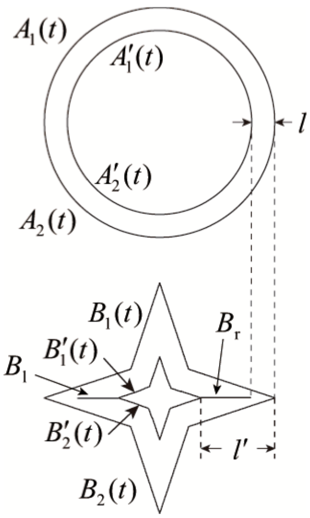


Figure 10: Normalization of the horizontal width by adding horizontal line segments.

and $B'(t)$ of $B(t)$ by h. Augment the offset curve with smaller horizontal width by adding horizontal line segments to the leftmost and rightmost points so that the horizontal widths become the same.

This strategy generates an ideal ambiguous cylinder in that the apparent thicknesses of the two curves are homogeneous and equal. Fig. 11 shows an example of the solid generated by this strategy illustrating these properties.

However, the solid created by this strategy is degenerate in the sense that the appearance changes drastically if we move the viewpoint even slightly. This is due to the added horizontal line segments. As shown in Fig. 12, we can see a sharp crack if we shift our viewpoint.

The three strategies that we have presented each have their own merits and demerits. Theses are summarized in Table 1.

Strategy 1 (fattening strategy) has many good properties. Its only demerit is that corner sharpness is lost. This demerit will decrease if the cylinder is made with a thinner surface, so this strategy is suitable for cases where we can make the cylinder from a thin hard material such as metal or the object can be treated carefully, such as the case where the object is protected by a showcase.

Strategy 2 (shrink-and-normalize strategy) can preserve the sharpness of corners, but cannot necessarily attain uniformity of thickness. Hence, this strategy is applicable only to limited cases.

Strategy 3 (shrink-and-augment strategy) has many good properties: it attains complete uniformity of thickness and preserves corner sharpness. However, the resulting cylinder has a crack structure; this crack is invisible from the cor-

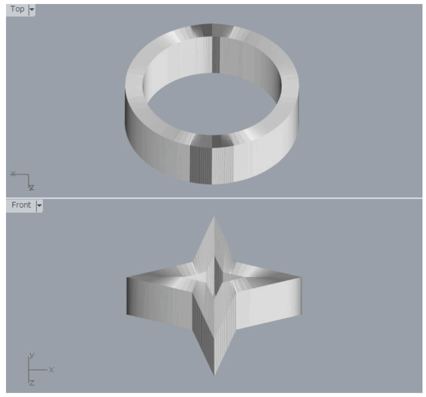


Figure 11: Solid created by Strategy 3 (shrink-and-augment strategy).

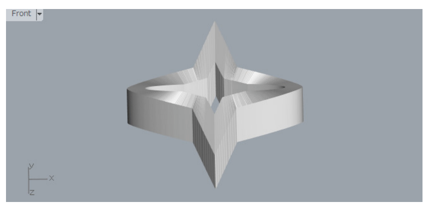


Figure 12: Appearance of the star of the solid in Fig. 9 from a slightly deviated viewpoint.

rect viewpoint if it can be strictly maintained, but it becomes visible if the viewpoint moves even slightly. Hence, this strategy is suitable for cases where we can control the viewpoint strictly, such as the case where the object is displayed to be seen through a view hole or where images are taken by a carefully placed camera.

6 Examples

We present further examples to better show the behaviors of the presented strategies.

First, let us consider a pair of shapes whose leftmost and rightmost points are smooth, i.e., have vertical tangents. An example of such a pair is the shape of "heart" and that of "club" of playing cards. The horizontal distances of the original curves and their offset curves are the same as the offset distance l. Therefore, we can apply Strategy 2. That is, we generate the sharpness-preserved offset curves and construct the ambiguous space curve directly; the normalization process can be skipped. The resulting solid is shown in Fig. 13.

Table 1: Comparison of the three strategies

	Property	1. Fattening strategy	2. Shrink-and-normalize strategy	3. Shrink-and-augment strategy
(i)	Each appearance has uniform thickness	YES	NO	YES
(ii)	Two appearances have the same thickness	YES	NO	YES
(iii)	Sharp corners are preserved	NO	YES	YES
(iv)	Applicable to any shapes	YES	NO	YES
(v)	Stable against viewpoint deviation	YES	YES	NO

We can see that the two appearances have the same uniform thickness. Thus, Strategy 2 works well if both of the shapes have smooth leftmost and rightmost points.

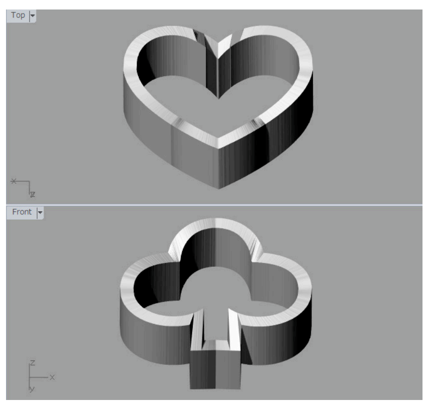


Figure 13: "Heart" and "club"; in both shapes, the leftmost and rightmost points are smooth points, and hence Strategy 2 can be applied directly.

The next example is a pair of dolls whose heads are on the right and on the left. Applying Strategy 1, we obtain the cylinder whose two appearances are as shown in Fig. 14. Uniformity of thickness is attained, but the apex at the legs is lost. If we apply Strategy 2, the normalized inner curve is displaced from the center, as shown in the lower diagram in Fig. 15, and consequently the thickness becomes non-uniform. The resulting solid will be as shown in Fig. 16. If we apply Strategy 3, on the other hand, we can construct a solid whose two appearances have the same uniform thickness, as shown in Fig. 17.

However, this solid has a crack around the apex at the legs, which can be seen if we slightly move our viewpoint, as shown in Fig. 18. Hence, Strategy 1 seems most suitable for this pair of shapes if we want to place the solid in a museum for exhibition.

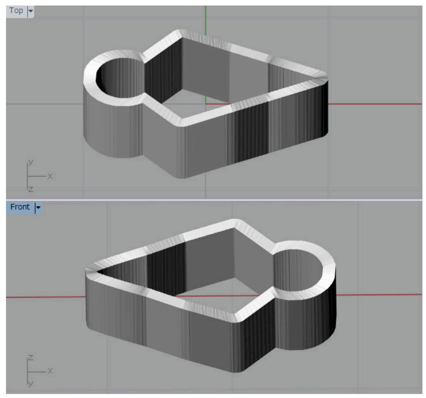


Figure 14: Left-oriented and right-oriented dolls generated by Strategy 1; the sharpness of the corner is lost.

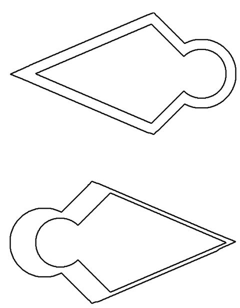


Figure 15: Normalization of the horizontal width according to Strategy 2.

Next, let us consider the pair of shapes shown in Fig. 19(a): a circle and a triangle-like closed curve. Our plan is to combine the ambiguous cylinder generated by this pair of shapes and its mirrored and/or rotated copies to form the crossing leaves shown in (b).

Fig. 20 shows the unit ambiguous cylinders generated by the three strategies. Strategy 1 results in (a), which seems to have good appearances if we see this unit alone. Strategy 2 results in (b), which is not of good quality because the thickness is not uniform: the leftmost edge is wider than the other two edges. Strategy 3 results in (c), which preserves sharp corners but has unstable appearances because the cylinder has a small crack at the right corner, as shown

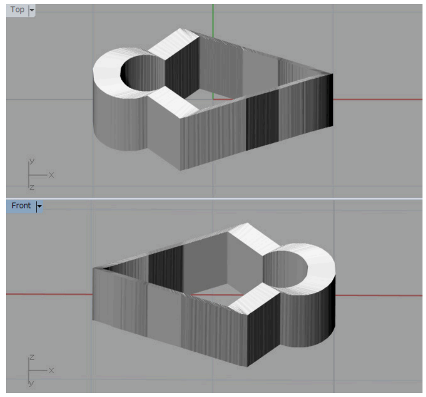


Figure 16: Left-oriented and right-oriented dolls generated by Strategy 2; the thickness is far from uniform.

in Fig. 21.

Combining the unit ambiguous cylinder and its mirrored and/or rotated copies, we get four circles and crossing leaves. The object generated by Strategy 1 is shown in Fig. 22. This object is not satisfactory because the edges are not smooth at the crossing points. Figs. 22(b) and (c) show close-up views of the bottom and right crossing points in (a). In both cases, the edges are not smoothly connected because sharpness is lost at the corner of the unit cylinder.

In contrast, the object generated by Strategy 3, shown in Fig. 23, is neater than that in Fig. 22 in that the edges are smooth at the crossing points. Indeed, looking at the close-up views in Figs. 23(b) and (c) of the bottom and right crossing points, the associated edges are smooth; this is because the sharpness of the corners in the unit cylinder is not lost.

Thus, Strategy 3 is the best among the three for this object.

As we have seen in the presented examples, the three strategies each have their own merits and demerits. Therefore, which strategy is suitable much depends on individual objects, and hence we have to choose the strategy carefully.

7 Concluding Remarks

We considered the problem of how to give uniform thickness to ambiguous cylinders. The requirement that the appearances of an ambiguous cylinder be both uniform and equal in thickness is not consistent with the illusion effect. Hence, we had to search for second-best solutions. We have presented three strategies, each of which has merits and demerits, and therefore the user needs to choose one according to the individual object and the desired visual effect.

Although in this paper we have concentrated on the apparent thickness from

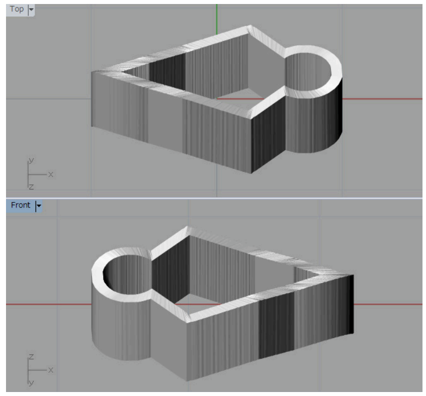


Figure 17: Left-oriented and right-oriented dolls generated by Strategy 3.

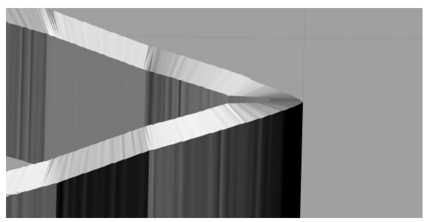
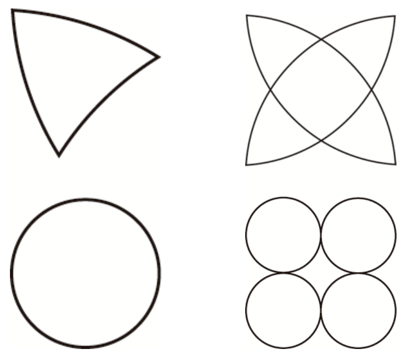


Figure 18: Crack structure around the apex of the legs of the solid in Fig. 17.

the two special viewpoints, the thickness control problem also includes other aspects. From the physical rigidity point of view, we also have to consider the actual thickness of the cylinder. Empirically, if we make the apparent thickness as uniform as possible, the physical thickness of the resulting cylinder will tend to be relatively uniform. However, this is not theoretically guaranteed. Thus, one task for future study is to understand the relation between the apparent thickness and the physical thickness of an ambiguous cylinder. If there exists a trade-off between these two, then we will also need to develop methods for balancing them.

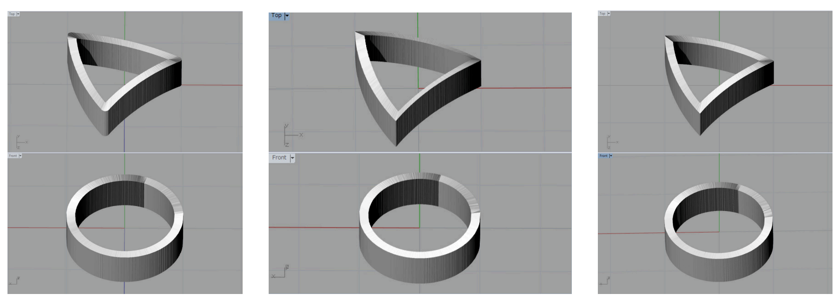
Another direction for future work is the intentional control of nonuniform thickness. From a visual artistic point of view, we might want to make the apparent thickness nonuniform according to the details of the shape. In that case, we need a quite different approach for thickness control.

Still another related problem is filling the inside of the cylinder with material. Sometimes we want to make short ambiguous cylinders. In that case, one natural way to make physical models is to fill the cylinder by covering the top and the bottom with smooth surfaces. It is not difficult to do this, but there are infinitely many possible surfaces. On the other hand, the resulting surface



(a) (b)

Figure 19: Four circles and crossing leaves: (a) unit pair of shapes; (b) four circles and crossing leaves created by combination of (a) and its mirrored and/or rotated copies.



(a) (b) (c)

Figure 20: Unit ambiguous cylinders generated by three strategies: (a) Strategy 1; (b) Strategy 2; (c) Strategy 3.

creates shading effects, and different surfaces give different shading and consequently give different visual effects, so we need to find the smooth surface such that the resulting shading enhances (or at least does not diminish) the visual effect of the ambiguous cylinder.

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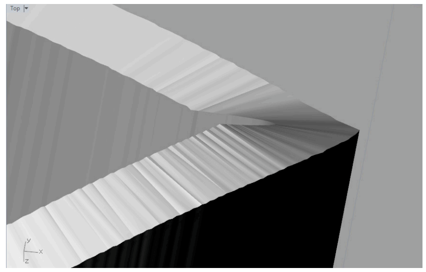
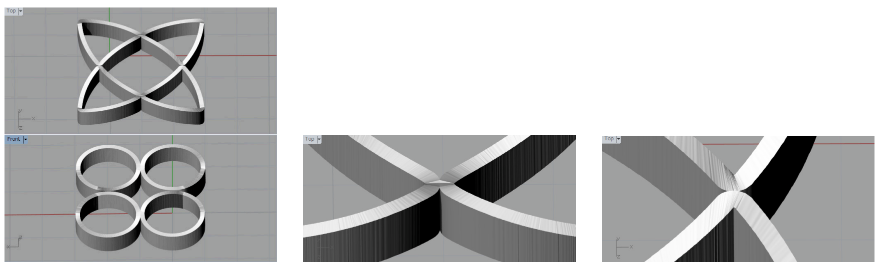


Figure 21: Crack created by Strategy 3.



(a) (b) (c)

Figure 22: Four circles and crossing leaves generated by Strategy 1: (a) full view of appearance; (b) close-up view of the bottom crossing point; (c) close-up view of the right crossing point.

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(a) (b) (c)

Figure 23: Four circles and crossing leaves generated by Strategy 3: (a) full view of appearance; (b) close-up view of the bottom crossing point; (c) close-up view of the right crossing point.

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