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**On the existence of connections
with a prescribed skew-symmetric Ricci tensor**

ABSTRACT. We study the so-called inverse problem. Namely, given a prescribed skew-symmetric Ricci tensor we find (locally) a respective linear connection.

1. Introduction. All manifolds and maps between manifolds considered in the paper are assumed to be smooth (i.e. of class C^∞).

The concept of a linear connection ∇ on a manifold M and its Ricci tensor S can be found in the fundamental monograph [4].

In the present paper, we study the so-called inverse problem.

More detailed, under some assumption on a tensor field r of type $(0, 2)$ on M , we prove the existence of a local solution of the equation

$$(1) \quad S = r$$

with unknown linear connection ∇ on M .

In particular, we deduce that any 2-form ω on a manifold M with $\dim(M) \geq 2$ is locally the Ricci tensor S of some linear connection ∇ on M .

In the analytic situation, the inverse problem was studied in many papers, e.g. [1, 2, 3, 5]. For example, in [5], using the Cauchy–Kowalevski theorem, the authors found (locally) all analytic linear connections for a prescribed analytic Ricci tensor. In the C^∞ situation, we can not apply the Cauchy–Kowalevski theorem.

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From now on, x^1, \dots, x^n denote the usual coordinates on \mathbf{R}^n and $\partial_1, \dots, \partial_n$ denote the usual canonical vector fields on \mathbf{R}^n . Given a map $f : \mathbf{R}^n \rightarrow \mathbf{R}$ let $(f)_i := \partial_i(f) = \frac{\partial f}{\partial x^i}$ for $i = 1, \dots, n$.

2. The main result. The main result of the paper is the following

Theorem 1. *Let M be a manifold such that $\dim(M) \geq 2$ and let $x_o \in M$. Let r be a tensor field of type $(0, 2)$ on M such that $r(X, X) = 0$ around x_o for some vector field $X \in \mathcal{X}(M)$ with $X_{x_o} \neq 0$. Then there is a linear connection ∇ on M such that r is the Ricci tensor S of ∇ on some neighborhood of x_o .*

Proof. We may assume that $M = \mathbf{R}^n$, $x_o = 0$ and $X = \partial_1$.

Let r be the tensor field of type $(0, 2)$ on \mathbf{R}^n and denote $r_{ij} = r(\partial_i, \partial_j)$ for $i, j = 1, \dots, n$. Then

$$(2) \quad r_{11} = 0.$$

The Ricci tensor S of a linear connection ∇ has the following rather well-known coordinate expression

$$(3) \quad S(\partial_i, \partial_j) = \sum_{k=1}^n [(\Gamma_{ij}^k)_k - (\Gamma_{kj}^k)_i] + \sum_{k,l=1}^n [\Gamma_{ij}^l \Gamma_{kl}^k - \Gamma_{kj}^l \Gamma_{il}^k], \quad i, j = 1, \dots, n,$$

where Γ_{jk}^i are the Christoffel symbols of ∇ , see [4].

It is sufficient to show that under assumption (2), equation (1) has a local solution (defined on some neighborhood of 0) $\nabla = (\Gamma_{bc}^a)$ such that

$$(4) \quad \begin{aligned} \Gamma_{bc}^a &= 0 \text{ for } a = 3, \dots, n, \quad b, c = 1, \dots, n, \\ \Gamma_{bc}^2 &= 0 \text{ for } b, c = 2, \dots, n, \\ \Gamma_{b1}^2 &= 0 \text{ for } b = 1, \dots, n, \\ \Gamma_{1b}^1 &= 0 \text{ for } b = 1, \dots, n. \end{aligned}$$

In other words, we put $\Gamma_{bc}^a = 0$ for $a, b, c = 1, \dots, n$ except for Γ_{1j}^2 with $j = 2, \dots, n$ and Γ_{ij}^1 with $i = 2, \dots, n$ and $j = 1, \dots, n$.

Using (4) and the coordinate expression (3), we get

$$(5) \quad S(\partial_i, \partial_j) = \sum_{k=1}^2 (\Gamma_{ij}^k)_k - \sum_{\substack{k,l \in \{1,2\} \\ k \neq l}} \Gamma_{kj}^l \Gamma_{il}^k = (\Gamma_{ij}^1)_1 + (\Gamma_{ij}^2)_2 - \Gamma_{2j}^1 \Gamma_{i1}^2 - \Gamma_{1j}^2 \Gamma_{i2}^1$$

as $\Gamma_{bc}^a = 0$ if $a = 3, \dots, n$ and $b, c = 1, \dots, n$, and $\Gamma_{ac}^a = 0$ if $a, c = 1, \dots, n$.

Then using (5) and (4), we get

$$(6) \quad S(\partial_1, \partial_1) = 0,$$

$$(7) \quad S(\partial_1, \partial_j) = (\Gamma_{1j}^2)_2 \text{ for } j = 2, \dots, n,$$

$$(8) \quad S(\partial_i, \partial_1) = (\Gamma_{i1}^1)_1 \text{ for } i = 2, \dots, n,$$

$$(9) \quad S(\partial_i, \partial_j) = (\Gamma_{ij}^1)_1 - \Gamma_{1j}^2 \Gamma_{i2}^1 \text{ for } i, j = 2, \dots, n.$$

More precisely, to obtain (6) we use (5) with $(i, j) = (1, 1)$ and the assumed (in (4)) conditions $\Gamma_{11}^1 = \Gamma_{11}^2 = 0$. To obtain (7), we use (5) with $(i, j) = (1, j)$ and the assumed (in (4)) conditions $\Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{1j}^1 = 0$. To obtain (8), we use (5) with $(i, j) = (i, 1)$ and the assumed (in (4)) conditions $\Gamma_{11}^2 = \Gamma_{i1}^2 = 0$. To obtain (9), we use (5) with $i, j = 2, \dots, n$ and the assumed (in (4)) conditions $\Gamma_{i1}^2 = \Gamma_{ij}^2 = 0$.

Then, by (2), (4) and (6)–(9), the equation (1) with unknown ∇ satisfying (4) is equivalent to the system of systems of differential equations

$$(10) \quad (\Gamma_{1j}^2)_2 = r_{1j} \text{ for } j = 2, \dots, n,$$

$$(11) \quad (\Gamma_{i1}^1)_1 = r_{i1} \text{ for } i = 2, \dots, n,$$

$$(12) \quad (\Gamma_{ij}^1)_1 = \Gamma_{1j}^2 \Gamma_{i2}^1 + r_{ij} \text{ for } i, j = 2, \dots, n.$$

It remains to observe that the system (10)–(12) has a solution of class C^∞ .

We see that the solution of (10) is

$$\Gamma_{1j}^2(x) = \int_0^{x^2} r_{1j}(x^1, t, x^3, \dots, x^n) dt + a_j(x^1, x^3, \dots, x^n)$$

for $j = 2, \dots, n$, and that the solution of (11) is

$$\Gamma_{i1}^1(x) = \int_0^{x^1} r_{i1}(t, x^2, \dots, x^n) dt + b_i(x^2, \dots, x^n)$$

for $i = 2, \dots, n$, where a_j, b_i are arbitrary maps in $n - 1$ variables.

Substituting the obtained Γ_{1j}^2 into (12), we get the system of ordinary first order differential equations with parameters x^2, \dots, x^n .

Such obtained system (12) has a solution of class C^∞ according to the well-known theory of differential equations. We can even solve it explicitly as follows.

Each of the equations

$$(\Gamma_{i2}^1)_1 = \Gamma_{12}^2 \Gamma_{i2}^1 + r_{i2} \text{ for } i = 2, \dots, n$$

(from the system (12)) is linear non-homogeneous with parameters. Solving them separately (using the well-known method), we obtain

$$\begin{aligned} & \Gamma_{i2}^1(x^1, \dots, x^n) \\ &= \left(\int_0^{x^1} r_{i2}(t, x^2, \dots, x^n) e^{-\int_0^t \Gamma_{12}^2(\tau, x^2, \dots, x^n) d\tau} dt + c_{i2}(x^2, \dots, x^n) \right) \\ & \quad \times e^{\int_0^{x^1} \Gamma_{12}^2(t, x^2, \dots, x^n) dt} \end{aligned}$$

for $i = 2, \dots, n$, where c_{i2} are arbitrary maps in $n - 1$ variables. Then the other equations of (12) (with Γ_{i2}^1 as above) have solutions given by

$$\begin{aligned} & \Gamma_{ij}^1(x^1, \dots, x^n) \\ &= \int_0^{x^1} (\Gamma_{1j}^2(t, x^2, \dots, x^n) \Gamma_{i2}^1(t, x^2, \dots, x^n) + r_{ij}(t, x^2, \dots, x^n)) dt \\ & \quad + d_{ij}(x^2, \dots, x^n), \end{aligned}$$

where d_{ij} are arbitrary maps in $n - 1$ variables.

The proof of Theorem 1 is now complete. \square

We have the following interesting corollary of Theorem 1.

Corollary 1. *Let M be a manifold such that $\dim(M) \geq 2$ and let $x_o \in M$. Let ω be a 2-form on M . Then there is a linear connection ∇ on M such that ω is the Ricci tensor S of ∇ on some neighborhood of x_o .*

Proof. For any vector field X (in particular with $X_{x_o} \neq 0$) we have $\omega(X, X) = 0$. Then we apply Theorem 1 with ω playing the role of r . \square

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