HILBERT–POINCARÉ SERIES FOR SPACES OF COMMUTING ELEMENTS IN LIE GROUPS

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Abstract. In this article we study the homology of spaces $\mathrm{Hom}(\mathbb{Z}^n, G)$ of ordered pairwise commuting n-tuples in a Lie group G. We give an explicit formula for the Poincaré series of these spaces in terms of invariants of the Weyl group of G. By work of Bergeron and Silberman, our results also apply to $\mathrm{Hom}(F_n/\Gamma_n^m, G)$, where the subgroups Γ_n^m are the terms in the descending central series of the free group F_n. Finally, we show that there is a stable equivalence between the space $\mathrm{Comm}(G)$ studied by Cohen–Stafa and its nilpotent analogues.

Contents

1. Introduction

Let G be a compact and connected Lie group and let π be a discrete group generated by n elements. In this article we study the rational homology of the space of group homomorphisms $\mathrm{Hom}(\pi, G) \subseteq G^n$, endowed with the subspace topology from G^n. In particular, when π is free abelian or nilpotent we give an explicit formula for the Poincaré series of $\mathrm{Hom}(\pi, G)_1$, the connected component of the trivial representation, in terms of invariants of the Weyl group W of G.

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The topology of the spaces $\mathrm{Hom}(\pi, G)$ has been studied extensively in recent years, in particular when π is a free abelian group [2, 4, 5, 16, 22, 15, 14]; in this case $\mathrm{Hom}(\mathbb{Z}^n, G)$ is known as *the space of ordered commuting n-tuples in G*. The case in which π is a finitely generated nilpotent group was recently analyzed by Bergeron and Silberman [6, 7]. These spaces and variations thereon, such as the space of almost commuting elements [9], have been studied in various settings, including work of Witten and Kac–Smilga on supersymmetric Yang-Mills theory [31, 32, 20].

Our formula for the Poincaré series of the identity component $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ builds on work of Baird [4] and Cohen–Reiner–Stafa [15]. In fact, we give a formula for a more refined *Hilbert–Poincaré series*, which is a tri-graded version of the standard Poincaré series that arises from a certain cohomological description of these spaces due to Baird. Work of Bergeron and Silberman [7] then leads immediately to results for nilpotent groups. The formula we produce is obtained by comparing stable splittings of $\mathrm{Hom}(\mathbb{Z}^n, G)$ and of the space $\mathrm{Comm}(G)$ introduced in [15]. The latter space is an analogue of the James reduced product construction for commuting elements in G; see Section 2.2.

1.1. **Main results.** The main purpose of this paper is to give an explicit formula for the Poincaré series of the component $\mathrm{Hom}(\mathbb{Z}^n, G)_1$.

Theorem 1.1. *The Poincaré series of* $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ *is given by*

$$P(\mathrm{Hom}(\mathbb{Z}^n, G)_1; q) = \frac{\prod_{i=1}^r (1 - q^{2d_i})}{|W|} \left(\sum_{w \in W} \frac{\det(1 + qw)^n}{\det(1 - q^2 w)} \right),$$

where the integers $d_1, \ldots, d_r$ are the characteristic degrees of the Weyl group W.

A similar formula for the homology of the character variety $\mathrm{Hom}(\mathbb{Z}^n, G)/G$ appears in Stafa [29].

Some comments are in order regarding the above formula. Let $T \subset G$ be a maximal torus with lie algebra $\mathfrak{t}$. Then the Weyl group W acts on the dual space $\mathfrak{t}^*$ as a finite reflection group, and the determinants in the formula are defined in terms of this linear representation of W. The characteristic degrees of W arise by considering the induced action of W on the polynomial algebra $\mathbb{R}[x_1, \ldots, x_r]$, where the x_i form a basis for $\mathfrak{t}^*$ (so $r = \mathrm{rank}\,(G)$). It is a theorem of Shephard–Todd [25] and Chevalley [13] that the W–invariants $\mathbb{R}[x_1, \ldots, x_r]^W$ form a polynomial ring with r homogeneous generators. The characteristic degrees of W are then the degrees of the homogeneous generators for $\mathbb{R}[x_1, \ldots, x_r]^W$. These degrees are well-known, and are displayed in Table 1. For further discussion of these ideas, see Section 3.1.

The spaces $\mathrm{Hom}(\pi, G)$ are not path-connected in general; see for instance [26] where the path components of $\mathrm{Hom}(\mathbb{Z}^n, SO(3))$ are described. However, if there is only one conjugacy class of maximal abelian subgroups in G, namely the conjugacy class of maximal tori, then $\mathrm{Hom}(\mathbb{Z}^n, G)$ and $\mathrm{Comm}(G)$ are both path-connected. This is true, for instance, if $G = U(n)$, $SU(n)$, or $Sp(n)$; on the other hand, $\mathrm{Hom}(\mathbb{Z}^n, SO(2n+1))$ is disconnected for $n \geqslant 2$ and $\mathrm{Hom}(\mathbb{Z}^n, G_2)$ is disconnected for $n \geqslant 3$. In fact, Kac and Smilga have classified those compact, simple Lie groups for which $\mathrm{Hom}(\mathbb{Z}^n, G)$ is path connected [20]. Moreover, when G is semisimple and simply connected, it is a theorem of Richardson that $\mathrm{Hom}(\mathbb{Z}^2, G)$ is an irreducible algebraic variety, and hence is connected [24].

Theorem 1.1 can also be applied to nilpotent groups. Let $F_n \trianglerighteq \Gamma_n^2 \trianglerighteq \Gamma_n^3 \cdots$ be the descending central series of the free group F_n. Bergeron and Silberman [7] show that for each $m \geqslant 2$, the identity component $\mathrm{Hom}(F_n/\Gamma_n^m, G)_1$ consists entirely of abelian representations. In fact, they show that if N is a finitely generated nilpotent group, then the natural map

$$\mathrm{Hom}(N/[N,N], G) \longrightarrow \mathrm{Hom}(N, G)$$

restricts to a homeomorphism between the identity components. Since G admits a neighborhood U of the identity that contains no subgroup other than the trivial subgroup, every representation in the identity component of $\mathrm{Hom}(N/[N,N], G)$ kills the torsion subgroup of $N/[N,N]$. Thus our main result also yields the homology of the identity component in $\mathrm{Hom}(N, G)$.

The assumption that G is compact is not in fact very restrictive, since if G is the complex points of a connected reductive linear algebraic group over $\mathbb{C}$ (or the real points if G is defined over $\mathbb{R}$) and $K \leqslant G$ is a maximal compact subgroup, then Pettet and Souto [22] showed that $\mathrm{Hom}(\mathbb{Z}^n, G)$ deformation retracts onto $\mathrm{Hom}(\mathbb{Z}^n, K)$. For simplicity, we refer to such groups G simply as *reductive* Lie groups. Bergeron [6] generalized this result to finitely generated nilpotent groups (in fact, Bergeron's result also allows G to be disconnected). It should be emphasized, however, that for other discrete groups π it is known that the homotopy types of $\mathrm{Hom}(\pi, G)$ and $\mathrm{Hom}(\pi, K)$ can differ; examples appear in [2].

The above descending central series can be used to define a filtration of the James reduced product of G, denoted $J(G)$, which is also known as the free monoid generated by the based space G. The filtration is given by the spaces

$$\mathrm{Comm}(G) = \mathrm{X}(2, G) \subset \mathrm{X}(3, G) \subset \cdots \subset \mathrm{X}(\infty, G) = J(G)$$

defined in Section 2, and was studied by Cohen and Stafa [15]. Here we show that all the terms in the filtration have the same Poincaré series.

Theorem 1.2. *The inclusion*

$$P(\mathrm{Comm}(G)_1; q) \hookrightarrow P(\mathrm{X}(m, G)_1; q)$$

induces an isomorphism in homology for every $m \geq 2$ *.*

1.2. **Structure of the paper.** We start in Section 2 by giving some basic topological properties of the spaces of homomorphisms $\mathrm{Hom}(\mathbb{Z}^n, G)$ and we define the spaces $\mathrm{X}(m, G) \subset J(G)$ considered above. In particular, we explain how all these spaces decompose into wedge sums after a single suspension. We prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively. In Section 5, we consider the ungraded cohomology and the rational complex K–theory of $\mathrm{Hom}(\mathbb{Z}^n, G)_1$. Finally, we give examples of Hilbert–Poincaré series in Section 6, most notably for the exceptional Lie group G_2.

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2. Topology of commuting elements in Lie groups

Let G be a compact and connected Lie group. Fix a maximal torus $T \leqslant G$ and let $W = N_G(T)/T$ be the Weyl group of G. The map

$$G \times T \to G \tag{1}$$

conjugating elements of the maximal torus by elements of G has been studied as far back as Weyl's work, and can be used to show that the rational cohomology of G is the ring of invariants $[H^*(G/T) \otimes H^*(T)]^W$. To study the rational cohomology of $\mathrm{Hom}(\mathbb{Z}^n, G)$ we can proceed as follows. The action by conjugation of T on itself is trivial, so (1) descends to a map $G/T \times T \to G$, which is invariant under the W–action $([g], t) \cdot [n] = ([gn], n^{-1}tn)$, where $n \in N_G(T)$. In [4] Baird showed that the induced map

$$\begin{aligned} \theta_n : G/T \times_W T^n &\to \mathrm{Hom}(\mathbb{Z}^n, G) \\ [g, t_1, \ldots, t_n] &\mapsto (gt_1g^{-1}, \ldots, gt_ng^{-1}) \end{aligned} \tag{2}$$

surjects onto $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ and induces an isomorphism of rational cohomology groups

$$H^*(\mathrm{Hom}(\mathbb{Z}^n, G)_1; \mathbb{Q}) \cong [H^*(G/T; \mathbb{Q}) \otimes H^*(T^n; \mathbb{Q})]^W. \tag{3}$$

This recovers the above fact about the cohomology of G when $n = 1$. Baird in fact shows that all torsion in $H^*(\mathrm{Hom}(\mathbb{Z}^n, G)_1; \mathbb{Z})$ has order dividing $|W|$, but little else is known about the torsion in these spaces, beyond the case of $SU(2)$ [5] and the fact that $H_1(\mathrm{Hom}(\mathbb{Z}^n, G)_1; \mathbb{Z})$ is torsion-free [16].

We note that as an ungraded $\mathbb{Q}W$-module, the ring $H^*(G/T; \mathbb{Q})$ is simply the regular representation $\mathbb{Q}W$, a well-known fact that dates back to Borel [8] – a proof can be found for instance in the exposition by M. Reeder [23]. This fact implies that the ungraded cohomology of the homomorphism space is just a regraded version of the cohomology of T^n. As we will see, various topological constructions related to the maps θ_n enjoy a similar structure in their cohomology.

Adem and Cohen [2] showed that there is a homotopy decomposition of the suspension of $\mathrm{Hom}(\mathbb{Z}^n, G)$ into a wedge sum of *smaller* spaces as follows

$$\Sigma \mathrm{Hom}(\mathbb{Z}^n, G) \simeq \Sigma \bigvee_{1 \le k \le n} \bigvee_{\binom{n}{k}} \widehat{\mathrm{Hom}}(\mathbb{Z}^k, G), \tag{4}$$

where $\widehat{\mathrm{Hom}}(\mathbb{Z}^k, G)$ is the quotient of $\mathrm{Hom}(\mathbb{Z}^n, G)$ by the subspace consisting of all commuting n–tuples $(g_1, \ldots, g_n)$ such that $g_i = 1$ for at least one coordinate i. This decomposition, along with the analogous decomposition of $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ given in Lemma 2.1, will play a key role in our study of homology.

Recall that the descending central series of a group π is the sequence of subgroups of π given by $\pi = \Gamma^1 \trianglerighteq \Gamma^2 = [\pi, \pi] \trianglerighteq \cdots \trianglerighteq \Gamma^{k+1} \trianglerighteq \cdots$, where inductively $\Gamma^{k+1} = [\pi, \Gamma^k]$. Let Γ^k_n be the k-th stage in the descending central series of the free group F_n, and note that $\Gamma^\infty_n = \bigcap_{k=1}^\infty \Gamma^k_n = 1$. Then we obtain a filtration

$$\mathrm{Hom}(F_n/\Gamma^2_n, G) \subset \mathrm{Hom}(F_n/\Gamma^3_n, G) \subset \cdots \subset \mathrm{Hom}(F_n/\Gamma^\infty_n, G) = G^n \tag{5}$$

of the space G^n by subspaces of nilpotent n-tuples, where the first term of the filtration is the space of commuting n-tuples $F_n/\Gamma^2_n = F_n/[F_n, F_n] = \mathbb{Z}^n$. It should be noted that this filtration need not be exhaustive; that is, $\bigcup_{k=1}^\infty \mathrm{Hom}(F_n/\Gamma^k_n, G)$ is in general a proper subset of G^n. Also note that $T^n \subset F_n/\Gamma^2_n = \mathrm{Hom}(\mathbb{Z}^n, G)$, a fact that will be used later. We obtain the following stable decompositions of the connected components of the trivial representations for nilpotent n-tuples.

Proposition 2.1. *Let G be either a compact connected Lie group or a reductive connected Lie group. For each $m \geq 2$ there is a homotopy equivalence*

$$\Sigma \mathrm{Hom}(F_n/\Gamma_n^m, G)_1 \simeq \Sigma \bigvee_{1\leq k\leq n} \bigvee_{\binom{n}{k}} \widehat{\mathrm{Hom}}(F_k/\Gamma_k^m, G)_1.$$

In particular there is a homotopy equivalence

$$\Sigma \mathrm{Hom}(\mathbb{Z}^n, G)_1 \simeq \Sigma \bigvee_{1\leq k\leq n} \bigvee_{\binom{n}{k}} \widehat{\mathrm{Hom}}(\mathbb{Z}^k, G)_1.$$

Proof. This is a minor modification to the arguments in [30, Corollary 2.21], where the corresponding decompositions for the full representation spaces are obtained. The spaces $\{\mathrm{Hom}(F_n/\Gamma_n^m, G)\}_n$ form a simplicial space, which Villarreal shows is simplicially NDR in the sense defined in [1]. The face and degeneracy maps in these simplicial spaces preserve the identity components, so $\{\mathrm{Hom}(F_n/\Gamma_n^m, G)_1\}_n$ is also a simplicial space, and is again simplicially NDR. The decompositions now follow from the main result of [1, Theorem 1.6]. □

2.1. **The James reduced product.** The *James reduced product* $J(Y)$ can be defined for any CW-complex Y with basepoint $*$. In our discussion Y is usually a compact Lie group with basepoint the identity element. Define $J(Y)$ as the quotient space

$$J(Y) := \left(\bigsqcup_{n\geq 0} Y^n\right)/\sim$$

where $\sim$ is the relation $(\dots, *, \dots) \sim (\dots, \widehat{*}, \dots)$ omitting the coordinates equal to the basepoint. This can also be seen as the free monoid generated by the elements of Y with the basepoint acting as the identity element. It is a classical result that $J(Y)$ is weakly homotopy equivalent to $\Omega\Sigma Y$, the loops on the suspension of Y. Moreover, the suspension of $J(Y)$ is given by

$$\Sigma J(Y) \simeq \Sigma \bigvee_{n\geq 1} \widehat{Y^n},$$

where $\widehat{Y^n}$ is the n-fold smash product. It was first observed by Bott and Samelson [10] that the homology of $J(Y)$ is isomorphic as an algebra to the tensor algebra $\mathcal{T}[\widetilde{H}_*(Y;R)]$ generated by the reduced homology of Y, given that the homology of Y is a free R-module. This is a central result used in our calculation.

2.2. **The spaces** $\mathrm{X}(m, G)$**.** Now consider the case in which $Y = G$, a connected Lie group with basepoint the identity element $1 \in G$. A filtration of the free monoid $J(G)$ is given by

$$\mathrm{X}(2, G) \subset \mathrm{X}(3, G) \subset \mathrm{X}(4, G) \subset \cdots \subset \mathrm{X}(\infty, G) = J(G), \tag{6}$$

where each space is defined by

$$\mathrm{X}(m, G) := \left(\bigsqcup_{n\geq 0} \mathrm{Hom}(F_n/\Gamma_n^m, G)\right)/\sim$$

where $\sim$ is the same relation as in $J(G)$. The spaces $\mathrm{X}(m,G)$ and $\mathrm{Comm}(G) = \mathrm{X}(2,G)$ were studied in [15], where it was shown that $\mathrm{Comm}(G)$ carries important information about the spaces of commuting n-tuples $\mathrm{Hom}(\mathbb{Z}^n, G)$. However, note that in general the spaces $\mathrm{X}(m,G)$ do not have the structure of a monoid for any m. As in the case of spaces of homomorphisms, the spaces $\mathrm{X}(m,G)$ need not be path connected. For instance, the space $\mathrm{X}(2,SO(3))$ has infinitely many path components, as shown in [28]. We can define the connected component of the trivial representation for each space $\mathrm{X}(m,G)$ by

$$\mathrm{X}(m,G)_1 := \left(\bigsqcup_{n\geq 0} \mathrm{Hom}(F_n/\Gamma_n^m, G)_1\right)/\sim$$

with $\mathrm{X}(2,G)_1 = \mathrm{Comm}(G)_1$. Cohen and Stafa [15, Theorem 5.2] show that there is a stable decomposition of this space as follows:

$$\Sigma \mathrm{X}(m,G) \simeq \Sigma \bigvee_{k\geq 1} \widehat{\mathrm{Hom}}(F_k/\Gamma_k^m, G). \tag{7}$$

We have an analogous result for the identity components.

Proposition 2.2. *Let G be either a compact connected Lie group or a reductive connected Lie group. For each $m \geq 2$ there is a homotopy equivalence*

$$\Sigma \mathrm{X}(m,G)_1 \simeq \Sigma \bigvee_{k\geq 1} \widehat{\mathrm{Hom}}(F_k/\Gamma_k^m, G)_1.$$

This is true in particular for $\mathrm{Comm}(G)_1$.

Proof. This follows from the proof of [15, Theorem 5.2]. □

3. Poincaré series of $\mathrm{Hom}(\mathbb{Z}^n, G)_1$

For a topological space X the (rational) *Poincaré series* is the series

$$P(X;q) := \sum_{k\geq 0} \mathrm{rank}_{\mathbb{Q}}(H_k(X;\mathbb{Q}))q^k.$$

In this section we describe the Poincaré series of $\mathrm{Hom}(\mathbb{Z}^n, G)_1$. Following [15], we will refine the usual grading of cohomology and introduce tri-graded *Hilbert–Poincaré series* for $X = \mathrm{Hom}(\mathbb{Z}^n, G)$, $\mathrm{Comm}(G)$, or $\mathrm{X}(m,G)$. These additional gradings will facilitate the computation of the Poincaré series itself.

For the remainder of this section, we will drop the coefficient group $\mathbb{Q}$ from our notation for (co)homology. The statements are true for any field with characteristic 0 or relatively prime to $|W|$.

The maps $\theta_n : G/T \times_W T^n \to \mathrm{Hom}(\mathbb{Z}^n, G)$ can be assembled to give a map

$$\Theta : G/T \times_W J(T) \to \mathrm{Comm}(G) \tag{8}$$

which surjects onto the connected component $\mathrm{Comm}(G)_1$. It was shown in [15] that Θ induces isomorphisms on the level of rational (co)homology, so rationally we obtain

$$H^*(\mathrm{Comm}(G)_1) \cong [H^*(G/T) \otimes H^*(J(T))]^W \cong [H^*(G/T) \otimes \mathcal{T}^*[\widetilde{H}_*(T)]]^W, \tag{9}$$

where $\mathcal{T}^*$ denotes the dual of the tensor algebra. This interpretation of the cohomology in terms of Weyl group invariants allows us to make the following definition.

Define the Hilbert–Poincaré series of $\mathrm{Comm}(G)_1$ as the tri-graded series

$$P(\mathrm{Comm}(G)_1; q, s, t) = \sum_{i,j,k\geq 0} \operatorname{rank} A(i,j,k)^W \; q^i s^j t^m,$$

where

$$A(i,j,k) := H^i(G/T) \otimes \mathcal{T}^*[\widetilde{H}_*(T)]_{j,m}$$

and $\mathcal{T}^*[\widetilde{H}_*(T)]_{j,m}$ is the dual of the submodule of $\mathcal{T}[\widetilde{H}_*(T)]$ generated by the m–fold tensors of total cohomological degree j.

To recover the ordinary Poincaré series we can set s equal to q and t equal to 1 since the tensor degree does not affect the (co)homological degree. In order to understand this tri-graded version of the Poincaré series, we take a short diversion to discuss the characteristic degrees of a finite reflection group.

3.1. **Finite reflection groups.** A finite reflection group is a finite subgroup $W \subset GL_k(\mathbf{k})$, with $\mathbf{k}$ a field of characteristic 0, such that W is generated by reflections. Equivalently, consider an n-dimensional vector space V over $\mathbf{k}$ equipped with the action of a finite subgroup $W \subset GL(V)$. There is a corresponding action on the symmetric algebra R of V, which is isomorphic to the polynomial algebra $R := \mathbf{k}[x_1, \dots, x_n]$[1]. It is a classical result of Chevalley [13] and Shephard–Todd [25] that when W is generated by reflections, the invariant elements of the W-action also form an algebra generated by n elements, and these generators can be chosen to be (algebraically independent) homogeneous polynomials $f_1, \dots, f_n$. Hence the W-invariant subalgebra is given by $R^W = \mathbf{k}[f_1, \dots, f_n]$. The degrees of the f_i are independent of the choice of the homogeneous generators. The degrees $d_i = \deg(f_i)$ are called the *characteristic degrees* of the reflection group W. See [27, 11] for a thorough exposition.

Let W be the Weyl group of a compact and connected Lie group G, which is finite. Then W is a unitary reflection group: W acts on the maximal torus T of G, and there is an induced action of W on the Cartan subalgebra $\mathfrak{t}$ of the Lie algebra $\mathfrak{g}$. The actions of W on $\mathfrak{t}$ and its dual $\mathfrak{t}^*$ are faithful, so W can be considered as a subgroup of $GL(\mathfrak{t}^*)$. Moreover, W is generated by reflections. This action of W has associated characteristic degrees $d_1, \dots, d_r$, where r is the rank of the maximal torus T, and it is a well-known fact that $|W| = \prod_i d_i$. Characteristic degrees of reflection groups have many other remarkable properties, outside the scope of this paper.

As an example consider the unitary group $U(n)$ with Weyl group the symmetric group Σ_n on n letters. The rank of $U(n)$ is n and the Σ_n acts on the maximal torus $T = (S^1)^n$ by permuting the coordinates, so it acts on $\mathfrak{t}$ by permuting the basis vectors. Therefore, as a subgroup of $GL(\mathfrak{t}^*)$ the Weyl group Σ_n consists of permutation matrices. The invariant subalgebra is then generated by the elementary symmetric polynomials $\epsilon_1, \dots, \epsilon_n$, with degrees $d_i = \deg(\epsilon_i) = i$ for $i = 1, \dots, n$.

Table 1 summarizes the Weyl groups and their associated characteristic degrees for families of simple Lie groups, including exceptional Lie groups. In the column for W, the group H_n is the kernel of the multiplication map $\mathbb{Z}_2^n = \{\pm 1\} \to \mathbb{Z}_2$ (so H_n consists of n–tuples containing an even number of -1's), D_n denotes the dihedral

[1]Some authors (e.g. Grove and Benson [17, Chapter 7], or Humphreys [19, Chapter 3]) replace V by its dual V^* in this discussion.

Table 1. Characteristic degrees of Weyl groups W

Type	Lie group	Rank	W	$\|W\|$	Characteristic degrees
A_n	$SU(n+1)$	$n \geq 1$	Σ_{n+1}	$(n+1)!$	$2, 3, ..., n+1$
B_n	$SO(2n+1)$	n	$\mathbb{Z}_2^n \rtimes \Sigma_n$	$n!2^n$	$2, 4, \ldots, 2n$
C_n	$Sp(n)$	n	$\mathbb{Z}_2^n \rtimes \Sigma_n$	$n!2^n$	$2, 4, \ldots, 2n$
D_n	$SO(2n)$	n	$H_n \rtimes \Sigma_n$	$n!2^{n-1}$	$2, 4, \ldots, 2n-2, n$
G_2	G_2	2	$D_{2^2.3}$	12	$2, 6$
F_4	F_4	4	$D_{2^7.3^2}$	1,152	$2, 6, 8, 12$
E_6	E_6	6	$O(6, \mathbb{F}_2)$	51,840	$2, 5, 6, 8, 9, 12$
E_7	E_7	7	$O(7, \mathbb{F}_2) \times \mathbb{Z}_2$	2,903,040	$2, 6, 8, 10, 12, 14, 18$
E_8	E_8	8	$\widehat{O(8, \mathbb{F}_2)}$	$2^{14}3^5 5^2 7$	$2, 8, 12, 14, 18, 20, 24, 30$

group of order n, and $\widehat{O(8, \mathbb{F}_2)}$ is a double cover of $O(8, \mathbb{F}_2)$. Similar information about characteristic degrees can also be found in [27, p. 175] and [19, p. 59].

As shown in [15], the information in Table 1 and the realization of the Weyl group W as a subgroup of $GL(\mathfrak{t}^*)$ suffice to describe the rational cohomology of $\mathrm{Comm}(G)_1$. This information will be used below to describe the corresponding Hilbert–Poincaré series for $\mathrm{Hom}(\mathbb{Z}^k, G)_1$.

3.2. **Hilbert–Poincaré series.** Suppose G has rank r. Let us denote by $A_W(q)$ the quantity

$$A_W(q) := \frac{\prod_{i=1}^r (1 - q^{2d_i})}{|W|},$$

where $d_1, \ldots, d_r$ are the characteristic degrees of W. It was shown by Cohen, Reiner and Stafa [15] that the Hilbert–Poincaré series of $\mathrm{Comm}(G)_1$ is given by the following infinite series.

Theorem 3.1. *Let G be a compact and connected Lie group with maximal torus T and Weyl group W. Then the Hilbert–Poincaré series of the connected component of the trivial representation in* $\mathrm{Comm}(G)$ *is given by*

$$P(Comm(G)_1; q, s, t) = A_W(q) \sum_{w \in W} \frac{1}{\det(1 - q^2 w)(1 - t(\det(1 + sw) - 1))}. \tag{10}$$

Using this theorem and stable decompositions of $\mathrm{Comm}(G)_1$ given above, we will now describe the Hilbert–Poincaré polynomial of the space of ordered pairwise commuting n-tuples. We begin with the following result, which is the fundamental step in our calculation of Poincaré polynomials for homomorphism spaces.

Proposition 3.2. *For $m \geqslant 1$, the reduced Hilbert–Poincaré series of $\widehat{\mathrm{Hom}}(\mathbb{Z}^m, G)_1$ is given by*

$$P(\widehat{\mathrm{Hom}}(\mathbb{Z}^m, G)_1; q, s) = A_W(q) \sum_{w \in W} \frac{(\det(1 + sw) - 1)^m}{\det(1 - q^2 w)}. \tag{11}$$

In particular, setting $s = q$ gives the reduced Poincaré series of $\widehat{\mathrm{Hom}}(\mathbb{Z}^m, G)_1$.

When $m = 0$, the same formulas yield the (unreduced) Hilbert–Poincaré and Poincaré series of the one-point space $\widehat{\mathrm{Hom}}(\mathbb{Z}^0, G)_1$.

When $m = 0$, this result asserts that the above series reduces to the constant series 1. An algebraic explanation of this fact, in terms of Molien's Theorem, is given at the end of this section.

The bigrading in this Hilbert–Poincaré series arises from applying the homology isomorphism (12) described in the proof, together with the Künneth Theorem. More specifically, let $\widehat{T^m}$ denote the m–fold smash product of the maximal torus $T \leqslant G$ with itself. Then the coefficient of $q^i s^j$ in the above Hilbert–Poincaré series is the rank of the subspace $[H^i(G/T) \otimes H^j(\widehat{T^m})]^W$ of W–invariant elements.

Proof. First rearrange the terms in the Hilbert–Poincaré series of $\mathrm{Comm}(G)_1$:

$$\begin{aligned}
P(\mathrm{Comm}(G)_1; q, s, t) &= A_W(q) \sum_{w \in W} \frac{1}{\det(1 - q^2 w)(1 - t(\det(1 + sw) - 1))} \\
&= A_W(q) \sum_{w \in W} \frac{\sum_{m=0}^{\infty} (t(\det(1 + sw) - 1))^m}{\det(1 - q^2 w)} \\
&= A_W(q) \sum_{w \in W} \sum_{m=0}^{\infty} \frac{(\det(1 + sw) - 1))^m t^m}{\det(1 - q^2 w)} \\
&= A_W(q) \sum_{m=0}^{\infty} \sum_{w \in W} \frac{(\det(1 + sw) - 1))^m t^m}{\det(1 - q^2 w)} \\
&= \sum_{m=0}^{\infty} \left(A_W(q) \sum_{w \in W} \frac{(\det(1 + sw) - 1))^m}{\det(1 - q^2 w)} \right) t^m.
\end{aligned}$$

We claim that after setting $s = q$, the coefficient of t^m in $P(\mathrm{Comm}(G)_1; q, s, t)$ is the Poincaré series of the stable wedge summand $\widehat{\mathrm{Hom}}(\mathbb{Z}^m, G)_1$ appearing in the decomposition of $\mathrm{Comm}(G)_1$ given by Proposition 2.2.

Recall that our tri-grading of the (co)homology of $\mathrm{Comm}(G)_1$ comes from the natural map

$$\Theta\colon\ G/T \times_W J(T) \longrightarrow \mathrm{Comm}(G)_1,$$

(see (8)) which induces isomorphisms in (rational) cohomology. On the left-hand side, we have

$$H^*(G/T \times_W J(T)) \cong \big(H^*(G/T) \otimes H^*(J(T))\big)^W \cong \big(H^*(G/T) \otimes \mathcal{T}^*[\widetilde{H}_*(T)]\big)^W.$$

Let $\mathcal{T}^*_m[\widetilde{H}_*(T)]$ denote the dual of the submodule

$$\mathcal{T}_m[\widetilde{H}_*(T)] \subset \mathcal{T}[\widetilde{H}_*(T)]$$

of m–fold tensors. The action of W preserves these submodules, so we obtain a decomposition

$$H^*(G/T \times_W J(T)) \cong \bigoplus_m \big(H^*(G/T) \otimes \mathcal{T}^*_m[\widetilde{H}_*(T)]\big)^W.$$

Note that for $m > 0$, the terms in this decomposition are in fact the reduced cohomology of $G/T \times_W \widehat{T^m}$, where $\widehat{T^m}$ denotes the m–fold smash product of T with itself, so the coefficient of t^m in $P(\mathrm{Comm}(G)_1; q, s, t)$ is the (bigraded, reduced) Hilbert–Poincaré series of $G/T \times_W \widehat{T^m}$. Similarly, the $m = 0$ term in this decomposition is unreduced cohomology of $G/T \times_W \widehat{T^0} = (G/T)/W$. Note that the rational cohomology of $G/T \times_W 1 \cong (G/T)/W$ is trivial, since the action of W on $H^*(G/T)$ is the regular representation.

To complete the proof, it will suffice to show that the map

$$G/T \times_W \widehat{T^m} \longrightarrow \widehat{\mathrm{Hom}}(\mathbb{Z}^m, G)_1 \tag{12}$$

is an isomorphism in (rational) cohomology. As shown in the proof of [15, Theorem 6.3], the induced map

$$\big(G/T \times_W \widehat{T^m}\big)/(G/T \times_W 1) \longrightarrow \widehat{\mathrm{Hom}}(\mathbb{Z}^m, G)_1$$

induces an equivalence in rational cohomology. But the map

$$G/T \times_W \widehat{T^m} \longrightarrow \big(G/T \times_W \widehat{T^m}\big)/(G/T \times_W 1)$$

is also an equivalence in rational cohomology, because the rational cohomology of $G/T \times_W 1 \cong (G/T)/W$ is trivial, as noted above. □

Baird's formula (3), together with the Künneth Theorem, provides a bigraded Hilbert–Poincaré series for $\mathrm{Hom}(\mathbb{Z}^n, G)_1$, in which the coefficient of $q^i s^j$ records the rank of $[H^i(G/T) \otimes H^j(T^n)]^W$. We now compute this series.

Theorem 3.3. *The homology of the component of the trivial representation in the space of commuting n-tuples in G is given by the following Hilbert–Poincaré series:*

$$P(\mathrm{Hom}(\mathbb{Z}^n, G)_1; q, s) = A_W(q) \sum_{w \in W} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{(\det(1 + sw) - 1)^k}{\det(1 - q^2 w)} \right).$$

The Poincaré series of $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ *is given by*

$$P(\mathrm{Hom}(\mathbb{Z}^n, G)_1; q) = A_W(q) \sum_{w \in W} \frac{\det(1 + qw)^n}{\det(1 - q^2 w)}. \tag{13}$$

Proof. Consider the bigraded series

$$P(\mathrm{Hom}(\mathbb{Z}^n, G)_1; q, s) = \sum_{k=0}^{n} \binom{n}{k} A_W(q) \left(\sum_{w \in W} \frac{(\det(1 + sw) - 1)^k}{\det(1 - q^2 w)} \right), \tag{14}$$

which in which the summands are the Hilbert–Poincaré series from Proposition 3.2. Since the terms in this sum match the terms from the stable decomposition of the space $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ in equation (4), we see that setting $q = s$ in (14) yields the Poincaré series of $\widehat{\mathrm{Hom}}(\mathbb{Z}^m, G)_1$:

$$\begin{aligned} P(\mathrm{Hom}(\mathbb{Z}^n, G)_1; q) =& A_W(q) \sum_{k=0}^{n} \binom{n}{k} \left(\sum_{w \in W} \frac{(\det(1 + qw) - 1)^k}{\det(1 - q^2 w)} \right) \\ =& A_W(q) \sum_{w \in W} \frac{\sum_{k=0}^{n} \binom{n}{k} (\det(1 + qw) - 1)^k}{\det(1 - q^2 w)}. \end{aligned}$$

Setting $x = \det(1 + qw) - 1$ in the binomial expansion

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

gives

$$\sum_{k=0}^{n} \binom{n}{k} (\det(1 + qw) - 1)^k = \det(1 + qw)^n,$$

yielding the simplified form (13).

Finally, we check that the bigrading in $P(\mathrm{Hom}(\mathbb{Z}^n, G)_1; q, s)$ agrees with the bigrading arising from the Künneth Theorem applied to (3). More precisely, we want to show that for each $i, j \geqslant 0$,

$$\text{rank}\,[H^i(G/T) \otimes H^j(T^n)]^W = \sum_{k=0}^{n} \binom{n}{k} \text{rank}\,[H^i(G/T) \otimes H^j(\widehat{T^k})]^W. \tag{15}$$

The spaces $G/T \times T^n$ form a simplicial space $G/T \times T^\bullet$ as n varies, where the simplicial structure arises from the bar construction on T (so the face and degeneracy maps are the identity on the G/T factors). The main result of [1] provides a stable splitting of the spaces $G/T \times T^n$:

$$\gamma \colon \Sigma(G/T \times T^n) \xrightarrow{\simeq} \bigvee_{k=0}^{n} \bigvee_{\binom{n}{k}} \Sigma(G/T \times \widehat{T^k}).$$

In fact, the stable splittings from [1] apply to any (sufficiently nice) simplicial space, and are natural with respect to simplicial maps. In particular, we can consider the simplicial maps

$$G/T \longleftarrow G/T \times T^\bullet \longrightarrow T^\bullet,$$

where G/T is viewed as a constant simplicial space, and $T^\bullet$ is the bar construction on T. Naturality of the splittings yields a commutative diagram

$$\begin{array}{ccccc} \Sigma(G/T) & \longleftarrow & \bigvee_{k=0}^{n} \bigvee_{\binom{n}{k}} \Sigma(G/T \times \widehat{T^k}) & \longrightarrow & \bigvee_{k} \bigvee_{\binom{n}{k}} \Sigma\widehat{T^k} \\ \uparrow & & \gamma\uparrow & & \uparrow \\ \Sigma(G/T) & \longleftarrow & \Sigma(G/T \times T^n) & \longrightarrow & \Sigma T^n, \end{array}$$

in which the vertical maps are weak equivalences. Commutativity implies that the map on cohomology induced by γ respects the Künneth decompositions of $H^p(\Sigma(G/T \times T^n))$ and $H^p(\Sigma(G/T \times \widehat{T^k}))$ $(p \geqslant 1)$, so that γ induces isomorphisms

$$\bigoplus_{k=0}^{n} \bigoplus_{\binom{n}{k}} (H^i(G/T) \otimes H^j(\widehat{T^k})) \xrightarrow{\cong} H^i(G/T) \otimes H^j(\widehat{T^k}) \tag{16}$$

for each $i, j \geqslant 0$. Moreover, W acts simplicially on $G/T \times T^\bullet$, so naturality implies that γ is W–equivariant, and hence the maps (16) induce isomorphisms when restricted to W–invariants. This establishes the desired equality (15). □

Remark on Molien's Theorem. Since $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ is path connected, the constant term in its Poincaré series must be 1. This can be understood in terms of a classical theorem of Molien [21] (also see [25, p. 289]). Let $R = \mathbf{k}[x_1, \ldots, x_r]$ and W be as above, with $x_1, \ldots, x_r$ in degree 1. Molien's Theorem states that the number of linearly independent elements in degree m in the invariant ring $R^W = \mathbf{k}[x_1, \ldots, x_r]^W$ is given by the coefficients of the generating function

$$\sum_{m=0}^{\infty} l_m q^m = \frac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1 - qw)}.$$

Moreover, Chevalley [13] and Shephard–Todd [25] give the following generating function for $R^W = \mathbf{k}[f_1, \ldots, f_r]$

$$\sum_{m=0}^{\infty} l_m q^m = \prod_{i=1}^{r} \frac{1}{(1-q^{d_i})},$$

Therefore, after doubling the degree of q one obtains the equation

$$1 = \frac{\prod_{i=1}^{r}(1-q^{2d_i})}{|W|} \sum_{w\in W} \frac{1}{\det(1-q^2 w)},$$

which corresponds to the constant term in the Hilbert–Poincaré series of the spaces of homomorphisms $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ in Theorem 3.3.

Remark on the rank of the fundamental group. By work of Gomez–Pettet–Souto [16],

$$\pi_1(\mathrm{Hom}(\mathbb{Z}^n, G)_1) \cong (\pi_1 G)^n.$$

It follows that

$$\mathrm{rank}\,(H^1(\mathrm{Hom}(\mathbb{Z}^n, G)_1)) = n \cdot \mathrm{rank}\,(H^1 G). \tag{17}$$

This can in fact be seen directly from the formula in Theorem 3.3 by analyzing the coefficient of q. Indeed, any non-zero coefficient of q must come from one of the terms $\det(1+qw)^n$. We have

$$\det(1+qw) = \prod(1+\lambda(w)q),$$

where $\lambda(w)$ ranges over the eigenvalues of w (counted with multiplicity). Hence the constant term of $\det(1+qw)$ is 1, and the coefficient of q is the trace of w (acting on $\mathfrak{t}^*$). It follows that the coefficient of q in $P(\mathrm{Hom}(\mathbb{Z}^n, G)_1; q)$ is precisely

$$\frac{n}{|W|} \sum_{w\in W} \mathrm{trace}\,(w) = n\langle \chi, 1\rangle = n \cdot \mathrm{rank}\,((\mathfrak{t}^*)^W),$$

where χ is the character of the representation of W on $\mathfrak{t}^*$ and $\langle \chi, 1\rangle$ is the inner product of this character with the trivial 1-dimensional character. Since this representation is isomorphic to the natural representation of W on $H^1(T;\mathbb{C})$, we find that $\mathrm{rank}\,(H^1(\mathrm{Hom}(\mathbb{Z}^n, G)_1)) = n \cdot \mathrm{rank}\,(H^1(T;\mathbb{C})^W)$. As discussed above, $H^*(G) \cong [H^*(G/T) \otimes H^*(T)]^W$, and the action of W on $H^*(G/T)$ is the regular representation. Hence

$$H^1(G) \cong (H^1(T))^W,$$

and combining the previous two formulas yields (17).

4. Poincaré series of $\mathrm{X}(m, G)_1$

The following theorem describes the Poincaré series of $\mathrm{X}(m, G)_1$ for all $m \geq 2$. Note that when G is a compact connected Lie group, it follows from Bergeron–Silberman [7] that $\mathrm{X}(m, G)_1 = \mathrm{Comm}(G)_1$.

Theorem 4.1. *Let G be a reductive connected Lie group. Then the natural inclusion maps*

$$X(2, G)_1 \hookrightarrow X(3, G)_1 \hookrightarrow \cdots \hookrightarrow X(m, G)_1 \hookrightarrow \cdots$$

all induce homotopy equivalences after one suspension. In particular, the Hilbert–Poincaré series of $\mathrm{X}(m,G)_1$*, for all* $m \geq 2$*, is given by*

$$P(\mathrm{X}(m,G)_1;q,s,t) = A_W(q)\sum_{w\in W}\frac{1}{\det(1-q^2w)(1-t(\det(1+sw)-1))}, \tag{18}$$

where W *is the Weyl group of a maximal compact subgroup* $K \leqslant G$*.*

Proof. By Proposition 2.2, there is a stable decomposition of $\mathrm{X}(m,G)_1$ into a wedge sum

$$\Sigma\mathrm{X}(m,G)_1 \simeq \Sigma\bigvee_{k\geq 1}\widehat{\mathrm{Hom}}(F_k/\Gamma_k^m,G)_1. \tag{19}$$

Consider the commutative diagram of cofibrations

$$\begin{array}{ccccc}
S_{n,2}(G) & \longrightarrow & \mathrm{Hom}(\mathbb{Z}^n,G)_1 & \longrightarrow & \widehat{\mathrm{Hom}}(\mathbb{Z}^n,G)_1 \\
\downarrow & & \downarrow{\scriptstyle i} & & \downarrow \\
S_{n,m}(G) & \longrightarrow & \mathrm{Hom}(F_n/\Gamma_n^m,G)_1 & \longrightarrow & \widehat{\mathrm{Hom}}(F_n/\Gamma_n^m,G)_1,
\end{array}$$

where $S_{n,m}(G)$ is the subspace of $\mathrm{Hom}(F_n/\Gamma_n^m,G)_1$ consisting of n-tuples with at least one coordinate the identity, and $m \geq 2$. The middle vertical map

$$i\colon\ \mathrm{Hom}(\mathbb{Z}^n,G)_1 \hookrightarrow \mathrm{Hom}(F_n/\Gamma_n^m,G)_1$$

is a homotopy equivalence: by [6], up to homotopy we can replace G by a maximal compact subgroup, and by [7], the map i is a homeomorphism in the compact case. The first vertical map

$$S_{n,2}(G) \hookrightarrow S_{n,m}(G)$$

is a homotopy equivalence by the Gluing Lemma [12], since these spaces can be built up inductively as pushouts of subspaces of the form

$$\{(g_1,\ldots,g_n)\ :\ g_i = 1 \text{ for all } i \in I\}$$

for various $I \subset \{1,\ldots n\}$, and on these subspaces the results from [6] and [7] apply. Applying the Gluing Lemma again, the third vertical map

$$\widehat{\mathrm{Hom}}(\mathbb{Z}^n,G)_1 \to \widehat{\mathrm{Hom}}(F_n/\Gamma_n^m,G)_1$$

is a homotopy equivalence as well, and the theorem follows from the decompositions (19). □

5. Ungraded cohomology and K–theory

The *ungraded cohomology* $H^u(X;R)$ of a space X with coefficients in R refers to the ungraded direct sum of all the cohomology groups of X as an R–module. It is a classical result [8] that the ungraded cohomology of G/T with rational coefficients, viewed as a W–module, is the regular representation $\mathbb{Q}W$. This alone yields some interesting consequences. Let M be a graded $\mathbb{Q}W$-module. Then it follows that $(H^u(G/T;\mathbb{Q})\otimes M)^W \cong M$. Applying this principle to formulas (3) and (9) yields the following result.

Proposition 5.1. *Let* G *be a compact and connected Lie group. Then*

(1) the ungraded rational cohomology of the compact and connected Lie group G is the same as the ungraded rational cohomology of its maximal torus:

$$(H^u(G/T;\mathbb{Q}) \otimes H^u(T))^W \cong H^u(T);$$

(2) the ungraded cohomology of $\mathrm{Hom}(F_n/\Gamma_n^m, G)_1$ is given by

$$H^u(\mathrm{Hom}(F_n/\Gamma_n^m, G)_1;\mathbb{Q}) \cong H^u(T^n;\mathbb{Q})$$

for all integers $m \geq 2$.

It is quite interesting, although not a surprise, that the maximal torus $T \subset G$ plays a fundamental role in the topology of nilpotent representations into G, similar to the role it plays in the topology of G from the classical theory of Lie groups. Recall that the rational cohomology of Lie groups can be described by the cohomology of a product of as many spheres of odd dimension as the rank of G [23]. It would however be very compelling to understand, in a topological manner, the regrading process that produces the cohomology of $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ and $\mathrm{Comm}(G)_1$ from the cohomology of T^n and $J(T)$, respectively.

Corollary 5.2. *Let G be a compact and connected Lie group of rank r. Then*

$$\sum_{k\geq 0} \mathrm{rank}\,(H^k(\mathrm{Hom}(\mathbb{Z}^n, G)_1;\mathbb{Q})) = \sum_{k\geq 0} \mathrm{rank}\,(H^k(T^n;\mathbb{Q})) = 2^{nr}.$$

Remark 5.3. Having identified the total rank of the cohomology of these spaces, one can ask if they satisfy Halperin's Toral Rank Conjecture, which states that if a topological space X has an almost free action of a torus of rank k, then the rank of the total cohomology of X is at least 2^k; see [18, Problem 1.4]. In this setting, the conjecture predicts that if a torus T' acts almost freely on the space of homomorphisms $\mathrm{Hom}(\mathbb{Z}^n, G)_1$, then the rank of T' must be at least nr. Hence it would be interesting to understand almost-free torus actions on these spaces.

Having identified the total rank of the cohomology of $\mathrm{Hom}(\mathbb{Z}^n, G)_1$, we can also identify its rational complex K–theory.

Corollary 5.4. *Let G be a compact and connected Lie group of rank r. Then*

$$\mathrm{rank}\,(K^i(\mathrm{Hom}(\mathbb{Z}^n, G)_1)) = 2^{nr-1}.$$

for every i.

Proof. Since $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ is a finite CW complex, the Chern character provides an isomorphism from $K^i(\mathrm{Hom}(\mathbb{Z}^n, G)_1) \otimes \mathbb{Q}$ to the sum of the rational cohomology groups of $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ in dimensions congruent to i (mod 2). The fibration sequence

$$G/T \times T^n \longrightarrow G/T \times_W T^n \longrightarrow BW$$

implies that the Euler characteristic of $G/T \times_W T^n$ is zero, and the same follows for $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ by Baird's result (3). Hence

$$\sum_{k \text{ even}} \mathrm{rank}\,(H^k(\mathrm{Hom}(\mathbb{Z}^n, G)_1)) = \sum_{k \text{ odd}} \mathrm{rank}\,(H^k(\mathrm{Hom}(\mathbb{Z}^n, G)_1)),$$

and the result follows from Corollary 5.2. □

Question 5.5. When G is a product of groups of the form $SU(r)$, $U(q)$, and $Sp(k)$, Adem and Gomez [3, Corollary 6.8] showed that the G–equivariant K–theory ring $K^*_G(\mathrm{Hom}(\mathbb{Z}^n, G))$ is free as module of rank 2^{nr} over the representation ring $R(G)$ (for such G we have $\mathrm{Hom}(\mathbb{Z}^n, G)_1 = \mathrm{Hom}(Z^n, G)$). In light of Corollary 5.4, it is natural to ask whether the map $R(G) \otimes K^*(\mathrm{Hom}(\mathbb{Z}^n, G)) \to K^*_G(\mathrm{Hom}(\mathbb{Z}^n, G))$ is an isomorphism for these groups.

Remark 5.6. We now explain how to compute $\mathrm{rank}\,(H^u(\mathrm{Hom}(\mathbb{Z}^n, G)_1))$, and the Euler characteristic $\chi(\mathrm{Hom}(\mathbb{Z}^n, G)_1)$, directly from Theorem 3.3, by setting $q = 1$ or -1 in the formula

$$P(\mathrm{Hom}(\mathbb{Z}^n, G)_1; q) = A_W(q) \sum_{w \in W} \frac{\det(1 + qw)^n}{\det(1 - q^2 w)}. \tag{20}$$

To do so, we must compute the multiplicities of ± 1 as roots of $A_W(q)$ and of $\det(1 - q^2 w)$ $(w \in W)$. We have

$$A_W(q) = \frac{1}{|W|} \prod_{i=1}^{r} (1 - q^{d_i})(1 + q^{d_i}),$$

so the multiplicity of ± 1 as a root of $A_W(q)$ is $r = \mathrm{rank}\,(G)$. On the other hand,

$$\det(1 - q^2 w) = \prod_i (1 - q^2 \lambda_i(w))^{n_i},$$

where the numbers $\lambda_i(w)$ are the eigenvalues of w (acting on $\mathfrak{t}^*$) and n_i is the dimension of the corresponding eigenspace. So the multiplicity of ± 1 is the dimension of the eigenspace for $\lambda_i(w) = 1$, which is strictly less than $\mathrm{rank}\,(G)$ unless $w = 1$, in which case it is exactly $\mathrm{rank}\,(G)$. Canceling factors of $1 \pm q$ in

$$\frac{\prod_{i=1}^{r} (1 - q^{d_i})(1 + q^{d_i})}{\det(1 - q^2 w)} \det(1 + qw)^n$$

and plugging in $q = \pm 1$, we see that all terms for $w \neq 1$ are zero.

Now consider what happens when we plug in $q = -1$ into (20). The term for $w = 1$ contains the determinant of $I + qI = I - I = 0$ as a factor, so it too vanishes. This gives another proof that $\chi(\mathrm{Hom}(\mathbb{Z}^n, G)_1) = 0$.

To calculate $\mathrm{rank}\,(H^u(\mathrm{Hom}(\mathbb{Z}^n, G)_1))$, we must analyze the $w = 1$ term of (20) more closely. This term has the form

$$\begin{aligned}
\frac{\prod_{i=1}^{r} (1 + q^{d_i})(1 - q^{d_i})}{|W|} \cdot \frac{\det((1+q)I)^n}{\det((1-q^2)I)}
&= \frac{\prod_{i=1}^{r} (1 + q^{d_i})(1 - q^{d_i})}{|W|} \cdot \frac{(1+q)^{rn}}{(1-q^2)^r} \\
&= \frac{\prod_{i=1}^{r} (1 + q^{d_i})(1 + q + q^2 + \cdots + q^{d_i - 1})}{|W|} \cdot \frac{(1+q)^{rn}}{(1+q)^r}.
\end{aligned}$$

Plugging in $q = 1$, we find that

$$\mathrm{rank}\,(H^u(\mathrm{Hom}(\mathbb{Z}^n, G)_1)) = \frac{\left(\prod_{i=1}^{r} 2d_i\right) 2^{rn}}{|W| \cdot 2^r} = 2^{rn},$$

where we have used the equation $\prod_{i=1}^{r} d_i = |W|$.

6. Examples of Hilbert–Poincaré series

Using Theorem 3.3 and Table 1, one can obtain explicit formulas for the Hilbert–Poincaré and Poincaré series described in this article. We demonstrate this for some low-dimensional Lie groups and for the exceptional Lie group G_2. We give only the formulas for the Poincaré series. The Hilbert–Poincaré series can then be deduced similarly from Theorem 3.3 and are left to the reader.

Example 6.1 ($G = SU(2)$)**.** The maximal torus of $SU(2)$ has rank 1 and the Weyl group is isomorphic to $W = \mathbb{Z}_2$. The dual space $\mathfrak{t}$ is 1-dimensional, and W is represented as $\{1, -1\} \subset GL(\mathfrak{t}^*)$. The only characteristic degree of W is $d_1 = 2$. Therefore, we have $A_W(q) = (1-q^4)/2$ and $\det(1+qw)$ equals $1+q$ and $1-q$, for w equal to 1 and -1, respectively, and

$$\frac{\det(1+qw)^n}{\det(1-q^2w)} = \begin{cases} \dfrac{(1+q)^n}{1-q^2} & \text{if } w = 1, \\ \\ \dfrac{(1-q)^n}{1+q^2} & \text{if } w = -1. \end{cases}$$

We know the space of commuting n-tuples in $SU(n)$ is path connected, so it equals the component of the trivial representation.

$$\begin{aligned} P(\mathrm{Hom}(\mathbb{Z}^n, SU(2)); q) &= A_W(q) \sum_{w \in \mathbb{Z}_2} \frac{\det(1+qw)^n}{\det(1-q^2w)} \\ &= \frac{1}{2}\Big((1+q)^n(1+q^2) + (1-q)^n(1-q^2) \Big), \end{aligned}$$

which agrees with calculations in [4].

Example 6.2 ($G = U(2)$)**.** The maximal torus of $U(2)$ has rank 2, the Weyl group $W \cong \mathbb{Z}_2$ acts on $\mathfrak{t}^*$ via the matrices

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

and the characteristic degrees of W are $d_1 = 1, d_2 = 2$. We know the space of commuting n-tuples in $U(n)$ is path connected, so again it equals the component of the trivial representation. We have $A_W(q) = (1-q^2)(1-q^4)/2$ and

$$\frac{\det(1+qw)^n}{\det(1-q^2w)} = \begin{cases} \dfrac{(1+q)^{2n}}{(1-q^2)^2} & \text{if } w = 1, \\ \\ \dfrac{(1-q^2)^n}{1-q^4} & \text{if } w \neq 1. \end{cases}$$

Therefore, we get the following Poincaré series

$$P(\mathrm{Hom}(\mathbb{Z}^n, U(2)); q) = \frac{1}{2}\Big((1+q)^{2n}(1+q^2) + (1-q^2)^{n+1} \Big).$$

For example, we get

$$
\begin{aligned}
P(\mathrm{Hom}(\mathbb{Z}^2,U(2));q) &= 1+2q+2q^2+4q^3+5q^4+2q^5,\\
P(\mathrm{Hom}(\mathbb{Z}^3,U(2));q) &= 1+3q+6q^2+13q^3+18q^4+13q^5+6q^6+3q^7+q^8,\\
P(\mathrm{Hom}(\mathbb{Z}^4,U(2));q) &= 1+4q+12q^2+32q^3+54q^4+56q^5+44q^6+32q^7\\
&\quad +17q^8+4q^9.
\end{aligned}
$$

Example 6.3 ($G=U(3)$)**.** The maximal torus has rank 3 and the Weyl group is the symmetric group on 3 letters

$$W=\Sigma_3=\{e,(12),(13),(23),(123),(132)\}.$$

The characteristic degrees of W are 1, 2, and 3, so

$$A_W(q)=\frac{1}{6}(1-q^2)(1-q^4)(1-q^6).$$

The matrix representations $W\leqslant GL(\mathfrak{t}^*)$ can be obtained by applying each permutation in Σ_3 to the rows of the 3×3 identity matrix $I_{3\times 3}$. This can be done in general for the Weyl group Σ_n of $U(n)$. For the transpositions $w=(12),(13),(23)\in W$ and for the 3-cycles we obtain the same determinants, respectively, since they are in the same conjugacy class. Hence we get

$$
\frac{\det(1+qw)^n}{\det(1-q^2w)}=\begin{cases}
\dfrac{(1+q)^{3n}}{(1-q^2)^3} & \text{if } w=e,\\[2ex]
\dfrac{(1+q)^n(1-q^2)^n}{(1-q^2)(1-q^4)} & \text{if } w=(12),(13),(23),\\[2ex]
\dfrac{(1+q^3)^n}{1-q^6} & \text{if } w=(123),(132).
\end{cases}
$$

Therefore, the Poincaré series is given by

$$
\begin{aligned}
P(\mathrm{Hom}(\mathbb{Z}^n,U(3)),q)=\frac{1}{6}\Big(&(1+q^2)(1+q^2+q^4)(1+q)^{3n}\\
&+3(1-q^6)(1+q)^n(1-q^2)^n+2(1-q^2)(1-q^4)(1+q^3)^n\Big).
\end{aligned}
$$

In particular, the following are the Poincaré series for pairwise commuting pairs, triples, and quadruples in $U(3)$, respectively:

$$
\begin{aligned}
P(\mathrm{Hom}(\mathbb{Z}^2,U(3)),q) =&1+2q+2q^2+4q^3+7q^4+10q^5+11q^6+8q^7+8q^8\\
&+8q^9+3q^{10}\\
P(\mathrm{Hom}(\mathbb{Z}^3,U(3)),q) =&\,1+3q+6q^2+14q^3+30q^4+54q^5+73q^6+75q^7+75q^8\\
&+73q^9+54q^{10}+30q^{11}+14q^{12}+6q^{13}+3q^{14}+q^{15},\\
P(\mathrm{Hom}(\mathbb{Z}^4,U(3)),q) =&\,1+4q+12q^2+36q^3+96q^4+212q^5+357q^6+472q^7\\
&+555q^8+604q^9+574q^{10}+468q^{11}+330q^{12}+204q^{13}\\
&+113q^{14}+48q^{15}+10q^{16}.
\end{aligned}
$$

Example 6.4 ($G = G_2$)**.** Now consider the exceptional Lie group G_2, a 14 dimensional submanifold of $SO(7)$, which has rank 2 and Weyl group the dihedral group $W = D_{12}$ of order 12 with presentation $\langle s,t|s^2,t^6,(st)^2\rangle$. We can write $W = \{1,t,t^2,t^3,t^4,t^5,s,st,st^2,st^3,st^4,st^5\}$ as a subgroup of $GL(\mathfrak{t}^*)$ by setting

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } t = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

The characteristic degrees of W are 2 and 6 as given in Table 1. The space $\mathrm{Hom}(\mathbb{Z}^n, G_2)$ is not path-connected since it has an elementary abelian 2–subgroup of rank 3, which is non-toral. Setting $t = 1$ and $q = s$ the Poincaré series of $\mathrm{Hom}(\mathbb{Z}^n, G_2)_1$ is calculated using Equation 13:

$$\begin{aligned} P(\mathrm{Hom}(\mathbb{Z}^n, G_2)_1; q) &= 1 + \frac{1}{12}\big[(2q^{14} - 2q^{12} - 2q^2 + 2)(-q^2+1)^{n-1} \\ &+ (2q^{12} - 2q^{10} - 2q^8 + 4q^6 - 2q^4 - 2q^2 + 2)(q^2 - q + 1)^n \\ &+ (2q^{12} + 2q^{10} - 2q^8 - 4q^6 - 2q^4 + 2q^2 + 2)(q^2 + q + 1)^n \\ &+ (q^{12} - 2q^{10} + 2q^8 - 2q^6 + 2q^4 - 2q^2 + 1)(-1+q)^{2n} + (-4q^{12} + 4)(-q^2+1)^n \\ &+ (q^{12} + 2q^{10} + 2q^8 + 2q^6 + 2q^4 + 2q^2 + 1)(q+1)^{2n}\big]. \end{aligned}$$

For example, for $n = 1,2,3$ we obtain:

$$\begin{aligned} P(\mathrm{Hom}(\mathbb{Z}^1, G_2)_1; q) =& 1 + q^3 + q^{11} + q^{14} = P(G_2; q), \\ P(\mathrm{Hom}(\mathbb{Z}^2, G_2)_1; q) =& 1 + q^2 + 2q^3 + q^4 + 2q^5 + q^6 + q^{10} + 2q^{11} + 2q^{13} + 3q^{14}, \\ P(\mathrm{Hom}(\mathbb{Z}^3, G_2)_1; q) =& 1 + 3q^2 + 3q^3 + 6q^4 + 9q^5 + 3q^6 + 3q^7 + 3q^8 + 2q^9 + 3q^{10} \\ &+ 3q^{11} + 3q^{12} + 9q^{13} + 6q^{14} + 3q^{15} + 3q^{16} + q^{18}. \end{aligned}$$

It can be observed from the above formula for the Poincaré series that the *rational homological dimension* of the spaces of commuting $(2k-1)$-tuples in G_2 is the same as that for commuting $2k$-tuples, namely $12 + 4k$. However, it is not clear if there is a topological reason for this phenomenon.

Remark 6.5. The above formulas suggests that for odd n, $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ is a *rational Poincaré duality space*; in particular, the coefficients of the above Poincaré series are palindromic for n odd. A geometric proof of this fact was provided to us by Antolín, Gritschacher, and Villarreal (private communication). Briefly, Baird's theorem (as discussed in Section 2) reduces us to showing that for n odd, the action of W on the manifold $G/T \times T^n$ is *orientation preserving*. From the case $n = 1$, where $H^*(G)$ is known, we can see that for each $w \in W$ the action of w on G/T is orientation-preserving if and only if the action of w on T is orientation preserving. Since W acts diagonally on T^n, the result follows.

References

[1] A. Adem, A. Bahri, M. Bendersky, F. R. Cohen, and S. Gitler. On decomposing suspensions of simplicial spaces. *Bol. Soc. Mat. Mexicana (3)*, 15(1):91–102, 2009.

[2] A. Adem and F. R. Cohen. Commuting elements and spaces of homomorphisms. *Math. Ann.*, 338(3):587–626, 2007.

[3] Alejandro Adem and José Manuel Gómez. Equivariant K-theory of compact Lie group actions with maximal rank isotropy. *J. Topol.*, 5(2):431–457, 2012.

[4] T. Baird. Cohomology of the space of commuting n-tuples in a compact Lie group. *Algebr. Geom. Topol.*, 7:737–754, 2007.

[5] T. Baird, L. C Jeffrey, and P. Selick. The space of commuting n-tuples in SU(2). *Illinois J. Math.*, 55(3):805–813, 2011.

[6] M. Bergeron. The topology of nilpotent representations in reductive groups and their maximal compact subgroups. *Geom. Topol.*, 19:1383–1407, 2015.

[7] M. Bergeron and L. Silberman. A note on nilpotent representations. *J. Group Theory*, 19(1):125–135, 2016.

[8] A. Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groupes de Lie compacts. *Ann. Math.*, 57(1):115–207, 1953.

[9] A Borel, R. Friedman, and J. Morgan. *Almost Commuting Elements in Compact Lie Groups*. Number 747 in Mem. Amer. Math. Soc. AMS, 2002.

[10] R. Bott and H. Samelson. On the Pontryagin product in spaces of paths. *Comment. Math. Helv.*, 27(1):320–337, 1953.

[11] M. Broué. *Introduction to complex reflection groups and their braid groups*, volume 1988 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010.

[12] Ronald Brown. *Topology and groupoids*. BookSurge, LLC, Charleston, SC, 2006.

[13] C. Chevalley. Invariants of finite groups generated by reflections. *American J. Math.*, 77(4):778–782, 1955.

[14] F. R. Cohen and M. Stafa. A survey on spaces of homomorphisms to Lie groups. In *Configurations Spaces: Geometry, Topology and Representation Theory*, volume 14 of *Springer INdAM series*, pages 361–379. Springer, 2016.

[15] F. R. Cohen and M. Stafa. On spaces of commuting elements in Lie groups. *Math. Proc. Cambridge Philos. Soc.*, 161(3):381–407, 2016.

[16] J. Gómez, A. Pettet, and J. Souto. On the fundamental group of $\mathrm{Hom}(\mathbb{Z}^k, G)$. *Math. Z.*, 271(1-2):33–44, 2012.

[17] L. C. Grove and C. T. Benson. *Finite reflection groups*, volume 99 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1985.

[18] S. Halperin. *Rational homotopy and torus actions*, pages 293–306. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge 1985.

[19] J. E. Humphreys. *Reflection groups and Coxeter groups*, volume 29. Cambridge University Press, 1992.

[20] V. Kac and A. Smilga. Vacuum structure in supersymmetric Yang-Mills theories with any gauge group. *The many faces of the superworld*, pages 185–234, 2000.

[21] Th. Molien. Ueber die Invarianten der linearen Substitutionsgruppen. *Berl. Ber.*, 1897:1152–1156, 1897.

[22] A. Pettet and J. Souto. Commuting tuples in reductive groups and their maximal compact subgroups. *Geom. Topol.*, 17(5):2513–2593, 2013.

[23] M. Reeder. On the cohomology of compact Lie groups. *Enseign. Math.*, 41:181–200, 1995.

[24] R. W. Richardson. Commuting varieties of semisimple Lie algebras and algebraic groups. *Compositio Math.*, 38(3):311–327, 1979.

[25] G. C. Shephard and J. A. Todd. Finite unitary reflection groups. *Canad. J. Math*, 6(2):274–301, 1954.

[26] D. Sjerve and E. Torres-Giese. Fundamental groups of commuting elements in Lie groups. *Bull. Lond. Math. Soc.*, 40(1):65–76, 2008.

[27] T. A. Springer. Regular elements of finite reflection groups. *Invent. Math.*, 25(2):159–198, 1974.

[28] M. Stafa. *On Polyhedral Products and Spaces of Commuting Elements in Lie Groups*. PhD thesis, University of Rochester, 2013.

[29] Mentor Stafa. Poincaré series of character varieties for nilpotent groups. *arXiv preprint arXiv:1705.01443*, 2017.

[30] Bernardo Villarreal. Cosimplicial groups and spaces of homomorphisms. *Algebr. Geom. Topol.*, 17(6):3519–3545, 2017.

[31] E. Witten. Constraints on supersymmetry breaking. *Nuclear Phys. B*, 202:253–316, July 1982.

[32] Edward Witten. Toroidal compactification without vector structure. *J. High Energy Phys.*, (2):Paper 6, 43 pp., 1998.

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