Stress analysis of finite anisotropic plates with cutouts under displacement boundary conditions

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ABSTRACT

The aim of this article is to provide an analytical tool to estimate the stress concentration in anisotropic plates weakened by a circular or elliptical hole; it is achieved by developing Lekhnitskii formalism in order to allow finite boundary effects to be added (original formulation considers infinite plates). Only membrane problem is herein considered, particularly when prescribed displacements are applied at external boundaries, which could simulate boundary conditions of a manhole structure between wing spars. A boundary collocation method in conjunction with a least squares approach is used to solve the problem.

KEYWORDS: Analytical procedure, cutout, open-hole, stress concentration, composite structures, anisotropic plates

1. INTRODUCTION

Composite structures are widely used in aeronautical industry, mostly due to the lightweight requirements of this sector. It is common to have structures of aircrafts weakened by cutouts due to maintenance and accessibility requirements. For instance, the so-called manholes are cutouts in the structure which provide human access to internal systems or structures. Cutouts are commonly circular or elliptical and cause a stress concentration around them, which have to be quantified for a safe design of the structure.

Commercial FE software packages are useful to quantify this stress concentration. Nevertheless, this way of estimation requires to solve the problem many times (most involving remeshing) in order to satisfy the parametric studies required in the design phase. A complex variable formulation, based on *Lekhnitskii [1]* formalism for anisotropic plates, seems a better choice for a most versatile resolution that could be useful for parametric studies, since remeshing is not necessary.

The objective of present article is to develop an analytical tool (based on Lekhnitskii formalism) to estimate the stress concentration produced by cutouts in anisotropic finite plates. Only membrane forces are considered in the model, since in-plane displacements are applied at the outer boundaries.

Finite plate theories have been developed by several authors, mostly considering outer load boundary conditions. There are two main methods which can be found in literature; on one hand there is a method based on the use of Boundary Collocation Points in conjunction with the Least Squares Method, e.g.: *Lin and Ko [2]*, *Xu et al. [3]* and *Hufenbach et al. [4]*. On the other hand, there is an energetic procedure based on the use of Minimum Total Potential Energy Theorem, e.g.: *Xiong [5]*. The Boundary Collocation Method leads to a simpler solution but is sensitive to point selection; even so this method will be used in what follows.

Methodology herein presented is implemented into mathematical software. The results obtained from the method show satisfactory agreement with respect to those obtained with FE, while reducing computational time and avoiding the need of renew the model every time that data change.

1.1 Geometry and boundary conditions

The problem herein considered is geometrically defined by the external dimensions of the rectangular plate and the open-hole geometry, as shown in Figure 1.

As mentioned before, the main interest of this article will be to apply prescribed displacements to external boundaries. Thus, the following configuration is chosen:

- The open-hole boundary is considered unloaded.
- Prescribed uniform displacements will be applied at the external boundaries, defined by $\tilde{u}$ and $\tilde{v}$ (Figure 1).

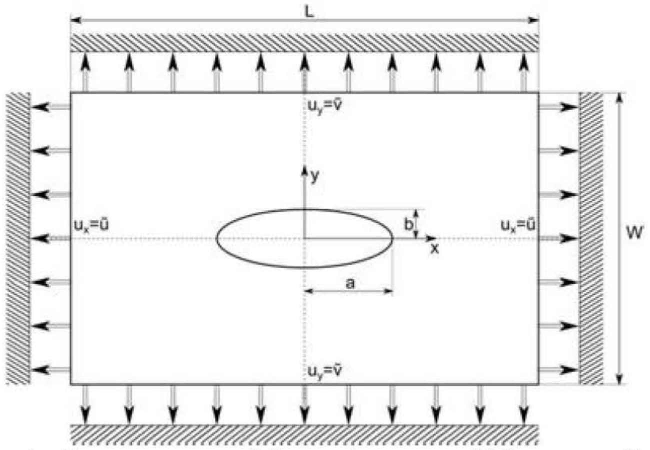

Fig. 1. Geometry and boundary conditions applied.

2. THEORETICAL MODEL

2.1 Lekhnitskii formalism

Lekhnitskii [1] proposed a complex variable formalism for the analytical estimation of the stress distribution near a cutout structure in infinite plates. It is based on the accomplishment of equilibrium equations by using an Airy stress function φ, i.e.:

$$N_x = \frac{\partial^2\varphi}{\partial y^2}\ ;\ N_y = \frac{\partial^2\varphi}{\partial x^2}\ ;\ N_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y} \tag{1}$$

Following this formalism, the derivative $\Phi_j = \frac{\partial \varphi_j}{\partial z_j}$ is taken as main variable, where $z_j = x + \mu_j y$ and (x, y) are the coordinates. As well as μ_j are the characteristic eigenvalues of laminate, i.e. the roots of the characteristic equation:

$$a_{11}\mu^4 - a_{13}\mu^3 + (2a_{12} + a_{33})\mu^2 - 2a_{23}\mu + 2a_{22} = 0 \quad (2)$$

Where $a_{ij} = A_{ij}^{-1}$ are the components of the inverse of the elastic stiffness matrix for generalized plain stress. Characteristic eigenvalues are conjugated two-by-two; hence it is possible to write:

$$\Phi = 2Re\left(\sum_{j=1}^{2}\left(\Phi_j(z_j)\right)\right) \quad (3)$$

The variable $\Phi_j(z_j)$ is expanded in series making use of C_{nj} unknown coefficients:

$$\Phi_j(z_j) = C_{0j} + \sum_{n=1}^{N}\left(C_{nj}\,\zeta_j^{-n}\right) \quad (4)$$

ζ_j is the conformal mapping of z_j;

$$\zeta_j = \frac{z_j \pm \sqrt{z_j^2 - a^2 - b^2\mu_j^2}}{a - ib\mu_j} \quad (5)$$

It transforms the external region of the elliptical hole into the external region of an unitary radius circle, and it is defined below, i.e. At the hole boundary $\zeta_1 = \zeta_2 = \sigma = e^{i\psi}$. A sign selection algorithm for the square root sign could be found in *Koussios [6]*.

Load boundary conditions in Lekhnitskii formalism are developed as:

$$\mp \int_0^s Y_n ds = 2Re\{\Phi_1(z_1) + \Phi_2(z_2)\} = \alpha_0 + \sum_{n=1}^{N}(\alpha_n\sigma^n + \overline{\alpha}_n\,\sigma^{-n}) \quad (6a)$$

$$\pm \int_0^s X_n ds = 2Re\{\mu_1\Phi_1(z_1) + \mu_2\phi_2(z_2)\} = \beta_0 + \sum_{n=1}^{N}\left(\beta_n\sigma^n + \overline{\beta}_n\,\sigma^{-n}\right) \quad (6b)$$

Where the right-hand side term is a Fourier series expansion that could be used when conditions are applied at the cutout boundary. Similarly for applied displacements:

$$\tilde{} = 2Re\{p_1\Phi_1(z_1) + p_2\Phi_2(z_2)\} = \alpha_0 + \sum_{n=1}^{N}(\alpha_n\sigma^n + \overline{\alpha}_n\,\sigma^{-n}) \quad (7a)$$

$$\tilde{} = 2Re\{q_1\Phi_1(z_1) + q_2\phi_2(z_2)\} = \beta_0 + \sum_{n=1}^{N}\left(\beta_n\sigma^n + \overline{\beta}_n\,\sigma^{-n}\right) \quad (7b)$$

Where p_j and q_j are:

$$\begin{cases} p_j = a_{11}\mu_j^2 + a_{12} - a_{13}\mu_j \\ q_j = a_{12}\mu_j + \frac{a_{22}}{\mu_j} - a_{26} \end{cases} \quad (7)$$

2.2 Application to finite plates

The problem of finite plates is solved on the basis of Lekhnitskii formalism by introducing a new term in the expression of $\phi_j(z_j)$ presented in (4):

$$\Phi_j(z_j) = C_{0j} + \sum_{n=1}^{N}\left(C_{nj}\,\zeta_j^{-n} + C_{nj}^{*}\,\zeta_j^{n}\right) \tag{8}$$

On the basis of Lekhnitskii formalism, the actual problem could be reduced by simplifying with a set of two sub-problems, i.e.:

1. A finite plate without cutout, with displacement boundary conditions applied to the external boundaries equal to those applied to the original problem.
2. A finite plate with the cutout and undisplaced external boundaries. A load is applied to the internal contour, in such a way that the sum of the first and second problem gives the solution of the original problem.

Solution to the first sub-problem is elementary and will not be discussed. The second one is solved by substituting boundary conditions into external and internal boundaries. On the first place, conditions at internal boundary have to satisfy conditions (9). Note that $\Phi_1(z_1)$ and $\Phi_2(z_2)$ will be replaced by $h_1(\zeta_1)$ and $h_2(\zeta_2)$ respectively according to the conformal mapping presented in (5).

$$h_1(\sigma) + \bar{h}_1(\bar{\sigma}) + h_2(\sigma) + \bar{h}_2(\bar{\sigma}) = \int_0^s Y_n\,ds \tag{9a}$$

$$\mu_1 h_1(\sigma) + \bar{\mu}_1\bar{h}_1(\bar{\sigma}) + \mu_2 h_2(\sigma) + \bar{\mu}_2\bar{h}_2(\bar{\sigma}) = -\int_0^s X_n\,ds \tag{9b}$$

Substituting (8) into expression (9) it is easy to obtain an expression which relates $h_2(\zeta_2)$ and $h_1(\zeta_1)$. Hence, unknowns are reduced to half since C_{n2} and C_{n2}^{*} coefficients are related to C_{n1} and C_{n1}^{*} coefficients. Additional equations are needed to solve the remaining coefficients; it can be achieved by collocation of points at the external boundaries, where null displacement conditions must be satisfied:

$$p_1 h_1(\zeta_1) + \bar{p}_1\bar{h}_1(\bar{\zeta}_1) + p_2 h_2(\zeta_2) + \bar{p}_2\bar{h}_2(\bar{\zeta}_2) = 0 \tag{10a}$$

$$q_1 h_1(\zeta_1) + \bar{q}_1\bar{h}_1(\bar{\zeta}_1) + q_2 h_2(\zeta_2) + \bar{q}_2\bar{h}_2(\bar{\zeta}_2) = 0 \tag{10b}$$

For each collocation point a set of two equations is obtained. Hence, at least $2N+1$ points are required to solve real and imaginary parts of remaining coefficients. A linear system of equations will be obtained, which is solved by the least squares method.

3. RESULTS AND COMPARISON

The methodology presented above is applied to a particular case in order to be able to validate it by comparing with respect to a FE analysis. Considered material is AS4/8552, its properties as well as boundary conditions and geometry are found in Table 1 and 2 respectively.

Table 1. Tested composite properties.

Fiber percentage			Thickness (mm)	E_{11}/E_{22}	E_{11}/G_{12}	ν_{12}
0%	±45%	90%				
25	60	15	23.92	15.55	30.1	0.3

Table 2. Geometry and boundary conditions.

L(m)	W(m)	a(m)	b(m)	$\tilde{u}$(mm)	$\tilde{v}$(mm)
1.5	1.2	0.3	0.2	2	0

Present methodology results are compared against FE results. Resultant forces (through-thickness stress) and displacements at the cutout boundary are shown in Figure 2 and 3 respectively. A great agreement between analytical and finite element model is observed, since relative errors of maximum values remain below 3 per cent (see Table 3). The approximation order N for the calculus has been taken equal to 10. And 40 collocation points have been used.

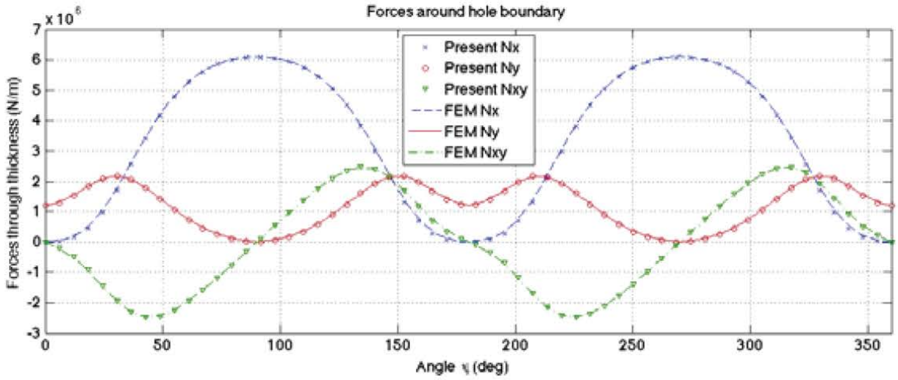

Fig. 2. Resultant forces comparison of the present methodology with respect to FE results

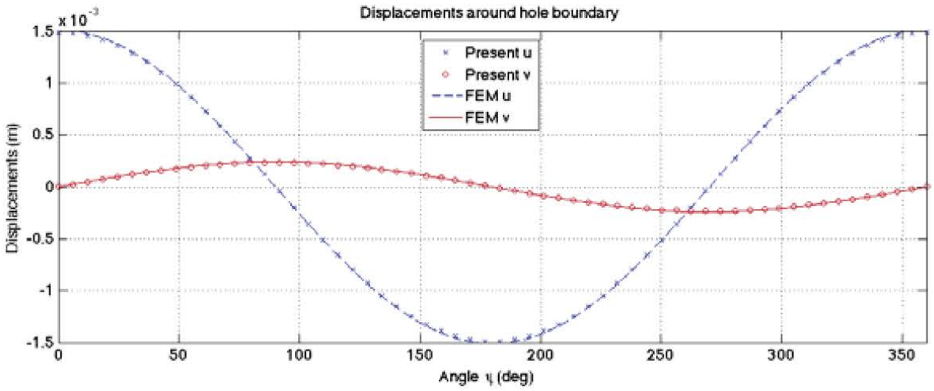

Fig. 3. Displacements comparison of the present methodology with respect to FE results

Table 3. Results comparison.

Resultant Force	Max. Present (MPa·m)	Max. FE (MPa·m)	Relative Error (%)
N_x	6.074	6.129	0.93
N_y	2.111	2.167	2.69
N_{xy}	2.428	2.451	0.93

3. SUMMARY AND CONCLUSIONS

An analytical method for the estimation of stress field of finite anisotropic plates with cutouts under uniform displacement boundary conditions has been presented.

Once the program is implemented, it could be executed on a standard PC leading to accurate results which are obtained in a few seconds (~2 *sec*). Results have shown excellent agreement with respect to finite element analysis, with the main benefit of having a simplified method for stress estimation, which could be easily repeated with different data for iterative design.

Further developments of the present methodology should be made to take into account more complex loading cases, such as mixed boundary conditions or complicated displacement distributions at the external edges. Similarly, bending problem of finite plates could be developed following *Lekhnitskii [1]* formalism for bending plates.

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