Quantification and Scaling of Multipartite Entanglement in Continuous Variable Systems

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We present a theoretical method to determine the multipartite entanglement between different partitions of multimode, fully or partially symmetric Gaussian states of continuous variable systems. For such states, we determine the exact expression of the logarithmic negativity and show that it coincides with that of equivalent two-mode Gaussian states. Exploiting this reduction, we demonstrate the scaling of the multipartite entanglement with the number of modes and its reliable experimental estimate by direct measurements of the global and local purities.

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The full understanding of the structure of multipartite quantum entanglement is a major scope in quantum information theory that is yet to be achieved. At the experimental level, it would be crucial to devise effective strategies to conveniently distribute the entanglement between different parties, depending on the needs of the addressed information protocol. Concerning the theory, the conditions of separability for generic bipartitions of Gaussian states of continuous variable (CV) systems have been derived and analyzed [1–3]. However, the quantification and scaling of entanglement for arbitrary states of multipartite systems remains in general a formidable task [4]. In this work, we present a theoretical scheme to exactly determine the multipartite entanglement of generic Gaussian symmetric states (pure or mixed) of CV systems.

We consider a CV system consisting of N canonical bosonic modes, associated with an infinite-dimensional Hilbert space and described by the vector $\hat{X}$ of the field quadrature operators. Quantum states of paramount importance in CV systems are the so-called Gaussian states, i.e., states fully characterized by first and second moments of the canonical operators. When addressing physical properties invariant under local unitary transformations, one can neglect first moments and completely characterize Gaussian states by the $2N \times 2N$ real covariance matrix (CM) $\boldsymbol{\sigma}$, whose entries are $\sigma_{ij} = 1/2\langle\{\hat{X}_i, \hat{X}_j\}\rangle - \langle\hat{X}_i\rangle\langle\hat{X}_j\rangle$. The CM $\boldsymbol{\sigma}$ must fulfill the uncertainty relation $\boldsymbol{\sigma} + i\Omega \geq 0$, with the symplectic form $\Omega = \oplus_{i=1}^{n}\omega$ and $\omega = \delta_{ij-1} - \delta_{ij+1}, i, j = 1, 2$. Symplectic operations (i.e., belonging to the group $Sp_{(2N,\mathbb{R})} = \{S \in SL(2N, \mathbb{R}){:}S^T\Omega S = \Omega\}$) acting by congruence on CMs in phase space amount to unitary operations on density matrices in Hilbert space. In phase space, any N-mode Gaussian state can be written as $\boldsymbol{\sigma} = S^T\boldsymbol{\nu}S$, with $\boldsymbol{\nu} = \mathrm{diag}\{n_1, n_1, \ldots n_N, n_N\}$. The set $\Sigma = \{n_i\}$ constitutes the symplectic spectrum of σ and its elements must fulfill the conditions $n_i \geq 1$, ensuring positivity of the density matrix ϱ associated to $\boldsymbol{\sigma}$. The symplectic eigenvalues n_i can be computed as the eigenvalues of the matrix $|i\Omega\boldsymbol{\sigma}|$. They are determined by N symplectic invariants associated to the characteristic polynomial of such a matrix; two global invariants which will be useful are the determinant $\mathrm{Det}\boldsymbol{\sigma} = \prod_i n_i^2$ and the quantity $\Delta = \sum_i n_i^2$, which is the sum of the determinants of all the 2×2 submatrices of σ related to each mode.

The degree of mixedness of a quantum state ϱ is characterized by its purity $\mu = \mathrm{Tr}\varrho^2$. For a Gaussian state with CM $\boldsymbol{\sigma}$ one has simply $\mu = 1/\sqrt{\mathrm{Det}\boldsymbol{\sigma}}$. As for the entanglement, we recall that positivity of the partially transposed state $\tilde{\varrho}$, obtained by transposing the reduced state of only one of the subsystems, is a necessary and sufficient condition [positive partial transpose (PPT) criterion] of separability for $(N+1)$-mode Gaussian states of $1 \times N$-mode bipartitions [5,6]. In phase space, partial transposition amounts to a mirror reflection of one quadrature associated to the single-mode partition. If $\{\tilde{n}_i\}$ is the symplectic spectrum of the partially transposed CM $\tilde{\boldsymbol{\sigma}}$, then a $(N+1)$-mode Gaussian state with CM $\boldsymbol{\sigma}$ is separable if and only if $\tilde{n}_i \geq 1 \;\; \forall\, i$. A proper measure of CV entanglement is the logarithmic negativity $E_{\mathcal{N}}$ [7], which is readily computed in terms of the symplectic spectrum $\tilde{n}_i$ of $\tilde{\boldsymbol{\sigma}}$ as $E_{\mathcal{N}} = -\sum_{i:\tilde{n}_i<1} \ln\tilde{n}_i$. Such a measure quantifies the extent to which the PPT condition $\tilde{n}_i \geq 1$ is violated.

Let us first consider the $2N \times 2N$ CM $\boldsymbol{\sigma}_{\beta^N}$ of a fully symmetric N-mode Gaussian state (i.e,. a state invariant under the exchange of any two modes)

$$\boldsymbol{\sigma}_{\beta^N} = \begin{pmatrix} \boldsymbol{\beta} & \boldsymbol{\varepsilon} & \cdots & \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} & \boldsymbol{\beta} & \boldsymbol{\varepsilon} & \vdots \\ \vdots & \boldsymbol{\varepsilon} & \ddots & \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} & \cdots & \boldsymbol{\varepsilon} & \boldsymbol{\beta} \end{pmatrix}, \qquad (1)$$

where $\boldsymbol{\beta}$ and $\boldsymbol{\varepsilon}$ are 2×2 submatrices. Because of the symmetry of such a state, $\boldsymbol{\beta}$ and $\boldsymbol{\varepsilon}$ can be put by means of local (single-mode) symplectic operations in the form $\boldsymbol{\beta} = \mathrm{diag}\{b, b\}$, $\boldsymbol{\varepsilon} = \mathrm{diag}\{e_1, e_2\}$. The symplectic spectrum Σ_{β^N} of $\boldsymbol{\sigma}_{\beta^N}$ has then the structure (see the Appendix)

$$\Sigma_{\beta^N} = \{\underbrace{\nu_-, \ldots, \nu_-}_{N-1}, \nu_{+^{(N)}}\}, \quad (2)$$

$$\nu_-^2 = (b - e_1)(b - e_2),$$

$$\nu_{+^{(N)}}^2 = [b + (N-1)e_1][b + (N-1)e_2].$$

The $(N-1)$-degenerate eigenvalue ν_- is independent of N, while $\nu_{+^{(N)}}$ can be expressed as a function of the purity $\mu_\beta \equiv (\mathrm{Det}\boldsymbol{\beta})^{-1/2}$ of the single-mode reduced state and of the symplectic spectrum of the two-mode block $\boldsymbol{\sigma}_{\beta^2}$, $\Sigma_{\beta^2} = \{\nu_-, \nu_+ \equiv \nu_{+^{(2)}}\}$

$$\nu_{+^{(N)}}^2 = -\frac{N(N-2)}{\mu_\beta^2} + \frac{(N-1)}{2}[N\nu_+^2 + (N-2)\nu_-^2]. \quad (3)$$

The global purity of the fully symmetric state is

$$\mu_{\beta^N} \equiv (\mathrm{Det}\boldsymbol{\sigma}_{\beta^N})^{-1/2} = (\nu_-^{N-1}\nu_{+^{(N)}})^{-1}, \quad (4)$$

and, through Eq. (3), can be directly linked to the one- and two-mode parameters. In particular, the symplectic eigenvalues $\nu_\mp$ are determined in terms of the two $Sp_{(4,\mathbb{R})}$ invariants μ_{β^2} and Δ_{β^2} by the relation [8] $2\nu_\mp^2 = \Delta_{\beta^2} \mp \sqrt{\Delta_{\beta^2}^2 - 4/\mu_{\beta^2}^2}$.

Next, we consider the $(N+1)$-mode Gaussian states constituted by generic single-mode states with CM $\boldsymbol{\alpha}$ and fully symmetric N-mode states with CM $\boldsymbol{\sigma}_{\beta^N}$ of the form (1). The mode with CM $\boldsymbol{\alpha}$ is then coupled with all other N modes by the same 2×2 real matrix $\boldsymbol{\gamma}$. The CM $\boldsymbol{\sigma}$ of such $(N+1)$-mode states reads

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\Gamma} \\ \boldsymbol{\Gamma}^{\mathsf{T}} & \boldsymbol{\sigma}_{\beta^N} \end{pmatrix}, \boldsymbol{\Gamma} \equiv (\underbrace{\boldsymbol{\gamma} \ldots \boldsymbol{\gamma}}_{N}). \quad (5)$$

We will now show that the properties of mixedness and entanglement of these states are determined by a suitable, limited set of global and local invariants under symplectic (unitary) operations. Let us introduce the purity $\mu_\alpha \equiv (\mathrm{Det}\boldsymbol{\alpha})^{-1/2}$ of the single-mode party, the global purity $\mu_\sigma \equiv (\mathrm{Det}\boldsymbol{\sigma})^{-1/2}$ of the state (5), and the global $Sp_{(2N+2,\mathbb{R})}$ invariant $\Delta_\sigma \equiv \Delta_{\alpha\gamma} + \Delta_{\beta^N} = \sum_i n_i^2$, where the n_is constitute the symplectic spectrum $\Sigma = \{n_1, \ldots, n_{N+1}\}$ of the CM $\boldsymbol{\sigma}$, and

$$\Delta_{\alpha\gamma} \equiv \mathrm{Det}\boldsymbol{\alpha} + 2N\mathrm{Det}\boldsymbol{\gamma}, \quad (6)$$

$$\Delta_{\beta^N} \equiv N[\mathrm{Det}\boldsymbol{\beta} + (N-1)\mathrm{Det}\boldsymbol{\varepsilon}]. \quad (7)$$

We are now in the position to characterize and quantify the bipartite entanglement between the single-mode α and the N-mode block $\boldsymbol{\sigma}_{\beta^N}$, the multipartite entanglement between all the $N+1$ modes, and to provide an operational scheme for their experimental determination in terms of measurements of the global and local purities. To proceed, we must evaluate the logarithmic negativity by determining the partially transposed CM $\tilde{\boldsymbol{\sigma}}$, with respect to the partition $\alpha|\beta^N$, which is obtained by flipping the sign of $\mathrm{Det}\boldsymbol{\gamma}$. Mixedness and entanglement are encoded, respectively, in the symplectic spectrum of $\boldsymbol{\sigma}$, and of $\tilde{\boldsymbol{\sigma}}$. It is worth noting that, of the previously introduced parameters, only $\Delta_{\alpha\gamma}$ is affected by the operation of partial transposition $\Delta_{\alpha\gamma} \overset{\boldsymbol{\sigma}\to\tilde{\boldsymbol{\sigma}}}{\longrightarrow} \tilde{\Delta}_{\alpha\gamma}$, with

$$\tilde{\Delta}_{\alpha\gamma} \equiv \mathrm{Det}\boldsymbol{\alpha} - 2N\mathrm{Det}\boldsymbol{\gamma} \equiv -\Delta_{\alpha\gamma} + 2/\mu_\alpha^2. \quad (8)$$

The symplectic spectrum $\Sigma = \{n_i\}$ $(i = 1, \ldots, N+1)$ of the CM $\boldsymbol{\sigma}$ Eq. (5) is of the form (see Appendix)

$$\Sigma = \{\underbrace{\nu_-, \ldots, \nu_-}_{N-1}, n_-, n_+\}, \quad (9)$$

where ν_- is the lowest symplectic eigenvalue of the reduced two-mode state $\boldsymbol{\sigma}_{\beta^2}$. The eigenvalues $n_\mp$ can be evaluated observing that Eqs. (4), (7), and (9) impose the identity $\Delta_\sigma = \Delta_{\alpha\gamma} + (N-1)\nu_-^2 + (\nu_-^{N-1}\mu_{\beta^N})^{-2}$ which can be used to obtain

$$2n_\mp^2 = \Delta_{\alpha\gamma} + (\nu_-^{N-1}\mu_{\beta^N})^{-2} \mp \sqrt{[\Delta_{\alpha\gamma} + (\nu_-^{N-1}\mu_{\beta^N})^{-2}]^2 - \frac{4}{(\nu_-^{N-1}\mu_\sigma)^2}}. \quad (10)$$

Since partial transposition leaves the N-mode symmetric block $\boldsymbol{\sigma}_{\beta^N}$ unchanged, the symplectic eigenvalues of $\tilde{\boldsymbol{\sigma}}$ are again of the form $\tilde{\Sigma} \equiv \{\tilde{n}_i\} = \{\nu_-, \ldots, \nu_-, \tilde{n}_-, \tilde{n}_+\}$, with $\tilde{n}_\mp$ defined as in Eq. (10), but with $\Delta_{\alpha\gamma}$ replaced by $\tilde{\Delta}_{\alpha\gamma}$ from Eq. (8). The logarithmic negativity $E_N^{\alpha|\beta^N}$, quantifying the bipartite entanglement between $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}_{\beta^N}$, is determined only by those symplectic eigenvalues of $\tilde{\boldsymbol{\sigma}}$ which satisfy $\tilde{n}_i < 1$. Since $\nu_- \geq 1$ (because it belongs to the symplectic spectrum of $\boldsymbol{\sigma}$), the entanglement is determined only by the eigenvalues $\tilde{n}_\mp$. On the other hand, the eigenvalues $n_\mp$ of Eq. (10) can be interpreted as the symplectic spectrum of an *equivalent* two-mode state of CM $\boldsymbol{\sigma}^{eq}$ with global purity μ^{eq} and invariant Δ^{eq} given by

$$\mu^{eq} \equiv \nu_-^{N-1}\mu_\sigma, \qquad \Delta^{eq} \equiv \Delta_{\alpha\gamma} + (\nu_-^{N-1}\mu_{\beta^N})^{-2}. \quad (11)$$

The corresponding $\tilde{\Delta}^{eq}$ associated to the partially transposed CM $\tilde{\boldsymbol{\sigma}}^{eq}$ reads then $\tilde{\Delta}^{eq} \equiv -\Delta^{eq} + 2/\mu_\alpha^2 + 2/(\nu_-^{N-1}\mu_{\beta^N})^2$. By comparison with the expression $\tilde{\Delta} = -\Delta + 2/\mu_1^2 + 2/\mu_2^2$, holding for a generic two-mode state with local purities μ_1 and μ_2 [8], one determines the local purities of the equivalent two-mode state $\boldsymbol{\sigma}^{eq}$:

$$\mu_1^{eq} = \mu_\alpha, \qquad \mu_2^{eq} = \nu_-^{N-1}\mu_{\beta^N}. \quad (12)$$

The two global invariants [Eq. (11)] and the two local invariants [Eq. (12)] determine uniquely the entanglement of the two-mode Gaussian state with CM $\boldsymbol{\sigma}^{eq}$. In particular, one can immediately see that the symplectic eigenvalues of the partially transposed CM $\tilde{\boldsymbol{\sigma}}^{eq}$ coincide with $\tilde{n}_\mp$, so that we obtain the crucial result that the logarithmic negativity of the equivalent two-mode state coincides with the logarithmic negativity $E_{\mathcal{N}}^{\alpha|\beta^N}$ of the $(N+1)$-mode state. Explicitly, one has

$$E_{\mathcal{N}}^{\alpha|\beta^N} = \max\{0, -\log \tilde{n}_-\}, \tag{13}$$

with $2\tilde{n}_-^2 \equiv \tilde{\Delta}^{eq} - \sqrt{\tilde{\Delta}^{2eq} - 4/\mu^{2eq}}$. Indeed, only the smallest symplectic eigenvalue $\tilde{n}_-$ enters in the determination of the multimode entanglement, since $\tilde{n}_+ > 1$ for two-mode states [8].

The $1 \times N$ entanglement is completely quantified by measuring the two local purities μ_α and μ_{β^N}, the global purity μ_σ, the symplectic eigenvalue ν_-, and $\mathrm{Det}\boldsymbol{\gamma}$ (which together with μ_α determines $\Delta_{\alpha\gamma}$). For two-mode Gaussian states, a reliable quantitative estimate of the logarithmic negativity, yielding exact (and very narrow) lower and upper bounds on the entanglement, can be obtained by simply measuring the global and local purities of the state [8]. In the present instance, this fact implies that a reliable estimate of the $1 \times N$ entanglement does not require the knowledge of the correlation matrix $\boldsymbol{\gamma}$, while the remaining four quantities (the three purities and the eigenvalue ν_-) can be measured even without homodyning by direct single-photon detections [9]. Moreover, knowledge of these few quantities is also sufficient to determine the multimode, multipartite entanglement of the state $\boldsymbol{\sigma}$. In fact, the fully symmetric N-mode block $\boldsymbol{\sigma}_{\beta^N}$ can be again regarded as a state describing a mode with CM $\boldsymbol{\beta}$ coupled with a fully symmetric $(N-1)$-mode block $\boldsymbol{\sigma}_{\beta^{N-1}}$, and thus the $1 \times (N-1)$ entanglement within $\boldsymbol{\sigma}_{\beta^N}$ can again be computed by constructing the corresponding equivalent two-mode state and evaluating its entanglement. This scaling procedure can be iterated to determine all the multimode entanglements existing between each mode and each fully symmetric K-mode sub-block $\boldsymbol{\sigma}_{\beta^K}$ $(K = 1, \ldots, N)$. The $1 \times K$ entanglement between the single-mode $\boldsymbol{\alpha}$ and any fully symmetric K-mode partition $\boldsymbol{\sigma}_{\beta^K}$ of $\boldsymbol{\sigma}_{\beta^N}$ can be determined in a similar way. A crucial feature of this scaling structure of the multipartite entanglement is that, at every step of the cascade, the $1 \times K$ entanglement is always equivalent to a 1×1 entanglement, so that the quantum correlations between the different partitions of $\boldsymbol{\sigma}$ can be directly compared to each other; it is thus possible to establish a multimode entanglement hierarchy without any problem of ordering.

To illustrate the scaling structure of multipartite entanglement in CV systems let us consider a pure, $(N+1)$-mode fully symmetric Gaussian state of the form of Eq. (1). Imposing the constraint of pure state ($\mu = 1 \Leftrightarrow \nu_- = \nu_{+(N+1)} = 1$), one obtains $e_i = \{1 + b^2(N-1) - N - (-1)^i \sqrt{(b^2-1)[b^2(N+1)^2 - (N-1)^2]}\}/2bN$. Such a state belongs to the class of CV Greenberger-Horne-Zeilinger (GHZ)-type states discussed in Ref. [3]. These multipartite entangled states are the outputs of a sequence of N beam splitters with $N+1$ single-mode squeezed inputs [3,10]. In the limit of infinite squeezing, these states reduce to the proper GHZ states of CV systems [3]. The CM $\boldsymbol{\sigma}_{\beta^{N+1}}^{\mathrm{GHZ}}$ of this class of pure states, for a given number of modes, depends only on the parameter $b \equiv 1/\mu_\beta \geq 1$, which is an increasing function of the single-mode squeezing. Correlations between the modes are induced according to the above expression for the covariances e_i. Exploiting our previous analysis, we can compute the entanglement between a single-mode with reduced CM $\boldsymbol{\beta}$ and any K-mode partition of the remaining modes $(1 \leq K \leq N)$, by determining the equivalent two-mode CM $\boldsymbol{\sigma}_{\beta|\beta^K}^{eq}$. The $1 \times K$ entanglement quantified by the logarithmic negativity $E_{\mathcal{N}}^{\beta|\beta^K}$ is determined by the smallest symplectic eigenvalue $\tilde{n}_-^{(K,N)}$ of the partially transposed CM $\tilde{\boldsymbol{\sigma}}_{\beta|\beta^K}^{eq}$. For any nonzero squeezing (i.e., $b > 1$) one has that $\tilde{n}_-^{(K,N)} < 1$, meaning that the state exhibits genuine multipartite entanglement; each mode is entangled with any other K-mode block, as first remarked in Ref. [3]. Further, the genuine multipartite nature of the entanglement can be precisely quantified by observing that $E_{\mathcal{N}}^{\beta|\beta^K} \geq E_{\mathcal{N}}^{\beta|\beta^{K-1}}$, as shown in Fig. 1. The 1×1 entanglement between two modes is weaker than the 1×2 one between a mode and other two modes, which is in turn weaker than the $1 \times K$ one, and so on with increasing K in this typical cascade structure. From an operational point of view, the signature of genuine multipartite entanglement is revealed by the fact that performing, e.g., a local measurement on a single-mode will affect *all* the other N modes. This means that the quantum correlations contained in the state with CM $\boldsymbol{\sigma}_{\beta^{N+1}}^{\mathrm{GHZ}}$ can be fully recovered only when considering the $1 \times N$ partition. In particular, the pure-state $1 \times N$ logarithmic negativity is, as expected, independent of N, being a simple monotonic function of the entropy of entanglement E_V (defined as the von Neumann entropy of the reduced single-mode state with CM $\boldsymbol{\beta}$). It is worth noting that, in the limit of infinite squeezing ($b \to \infty$), only the $1 \times N$ entanglement diverges while all the other $1 \times K$ quantum correlations remain finite (see Fig. 1). Namely, $E_{\mathcal{N}}^{\beta|\beta^K}(\boldsymbol{\sigma}_{\beta^{N+1}}^{\mathrm{GHZ}}) \overset{b\to\infty}{\longrightarrow} -(1/2)\log\{1 - 4K/[N(K+1) - K(K-3)]\}$, which can-

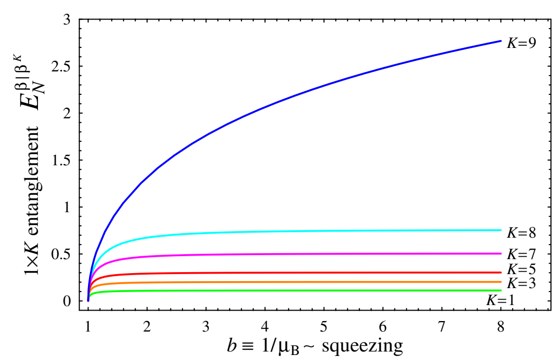

FIG. 1 (color online). Entanglement hierarchy for $(N+1)$-mode GHZ-type states ($N = 9$).

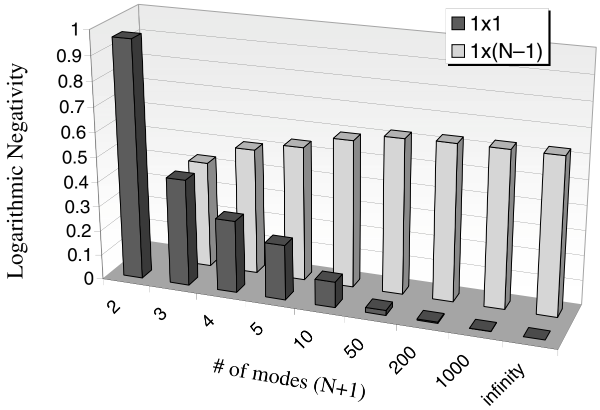

FIG. 2. Scaling as a function of N of the 1×1 and of the $1 \times (N-1)$ entanglement for a $(N+1)$-mode GHZ-type CV pure-state ($b = 1.5$).

not exceed $\log\sqrt{5} \simeq 0.8$ for any N and for any $K < N$. At fixed squeezing, the scaling with N of the $1 \times (N-1)$ entanglement compared to the 1×1 entanglement is shown in Fig. 2 (we recall that the $1 \times N$ entanglement is independent on N). Notice how, with increasing number of modes, the multipartite entanglement increases to the detriment of the two-mode one which becomes distributed between all the modes. We remark that this scaling occurs in any Gaussian states, either fully or partially symmetric, pure or mixed. For instance, this is the case for a single-mode squeezed state coupled with a N-mode symmetric thermal squeezed state. The simplest example of a mixed state with genuine multipartite entanglement is obtained from $\boldsymbol{\sigma}^{\rm GHZ}_{\beta^{N+1}}$ by tracing out some of the modes. Figure 2 can then also be seen as a demonstration of the scaling in such a N-mode mixed state, where the $1 \times (N-1)$ entanglement is the strongest one. Thus, with increasing N, the global mixedness can limit but not destroy the genuine multipartite entanglement between all the modes. This entanglement is experimentally accessible by all-optical means [3], and it also allows for a reliable (i.e., with fidelity $\mathcal{F} > 1/2$) quantum teleportation between any two parties [10]. Therefore, the quantification of multipartite entanglement by measurements of purity, which, as we have already remarked, can be experimentally implemented even without homodyning, leads to an accurate estimate of the multiparty teleportation efficiency and to direct control on the transfer of quantum information.

In conclusion, we have shown that multipartite quantum correlations of Gaussian states of $1 \times N$ bipartitions under symmetry are endowed with a scaling structure that reduces the problem to the analysis of the entanglement of equivalent two-mode Gaussian states. Thanks to this reduction, it is possible to determine exactly the logarithmic negativity of the multimode states and to allow for a reliable experimental estimate of the multipartite entanglement by direct measurements of global and local purities, without the need for the full reconstruction of the covariance matrix. Our results apply to many cases of practical interest. For instance, the entire class of bi-symmetric—i.e., invariant under the exchange of two given modes—three-mode Gaussian states [2] has its multipartite entanglement completely quantified by the present analysis. The generalization of the present approach for the quantification of multipartite CV entanglement to states with weaker symmetry constraints and to $M \times N$-mode bipartitions (with $M > 1$) awaits further study.

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Appendix: Proof of the symplectic degeneracy.—We prove here the multiplicity of the symplectic eigenvalue ν_- for the CMs $\boldsymbol{\sigma}_{\beta^N}$ and $\boldsymbol{\sigma}$, asserted in Eqs. (2) and (9). We first recall that, if $\Sigma = \{\nu_1, \ldots, \nu_N\}$ is the symplectic spectrum of the CM $\boldsymbol{\sigma}$, then the $2N$ eigenvalues of the matrix $i\Omega\boldsymbol{\sigma}$ are given by the set $\{\mp\nu_i\}$. Let us focus next on the CM $\boldsymbol{\sigma}_{\beta^2}$: in the linear space on which the matrix $i\Omega\boldsymbol{\sigma}_{\beta^2}$ acts, the eigenvector v_- corresponding to the eigenvalue ν_- reads $v_- = (-i\frac{b-e_1}{\nu_-}, -1, i\frac{b-e_1}{\nu_-}, 1)^{\mathsf{T}}$. Because of the symmetry of $\boldsymbol{\sigma}_{\beta^N}$, any $2N$-dimensional vector v of the form

$$v = (0, \ldots 0, \underbrace{-i\frac{b-e_1}{\nu_-}, -1}_{\text{mode } i}, 0 \ldots 0, \underbrace{i\frac{b-e_1}{\nu_-}, 1}_{\text{mode } j}, 0, \ldots 0)^{\mathsf{T}} \tag{1a}$$

(i.e., any vector obtained by taking v_- in a couple of modes ij and appending to it 0 elements for all the other modes) is an eigenvector of $i\Omega\boldsymbol{\sigma}_{\beta^N}$ with eigenvalue ν_-. It can be seen immediately that one can construct $N-1$ linear independent vectors of the above form, proving Eq. (2). Clearly, an analogous reasoning holds for the matrix $\boldsymbol{\sigma}$, proving Eq. (9).

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